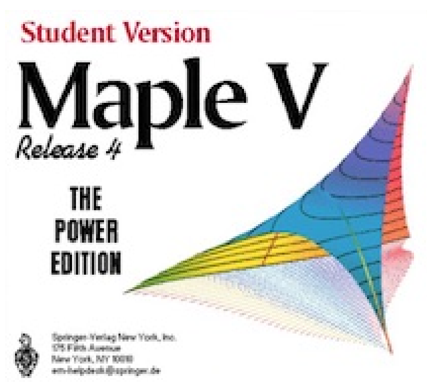
Exploring Math with EIGENMATH

Elementary Differential Geometry of Surfaces

Maple Vr4 Solutions



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2023

Contents

Preface

This solution manual presents calculations of

- the surface metric tensor g_{ij}
- the 2^{nd} fundamental form h_{ij}
- the shape operator A
- the CHRISTOFFEL symbols Γ of first and 2nd kind
- the GAUSS curvature K
- the mean curvature H
- the principal curvatures κ_i

for most of the surfaces, which are used in the examples and exercises of the main text [2]. Instead of ϕ the name F (german: "Fläche" = surface) is used to denote a parameterization.

These calculations were done using the Maple Vr4 packet `diffgeo.m` form H. RECKZIEGEL, M. KRIENER & K. PAVEL [4]. The calculation of the RIEMANN curvature tensors R of 1^{st} and 2^{nd} kind or the RICCI tensor Ric and the RICCI curvature scalar R are not implemented in that packet, but can be calculated with EIGENMATH and the functions of the main text.

I hope that this collection is helpful to check your calculations done with EIGENMATH, but: without guarantee for the correctness ;)

Wolfgang Lindner
Leichlingen, Germany
January 2023

1 Saddle surface $\phi(u,v) = (u,v,uv)$

saddle surface (maple Vr4 1996) Dr. W. Lindner 4/2023

```
STUDENT > F:=(u,v)->[u,v,u*v];
```

$$F := (u,v) \to [u, v, u\,v]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[v^2+1, u\,v], [u\,v, u^2+1]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[0, \frac{1}{\sqrt{u^2+v^2+1}}\right], \left[\frac{1}{\sqrt{u^2+v^2+1}}, 0\right]\right]$$

```
STUDENT >
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to \left[\left[0, \frac{v}{u^2+v^2+1}\right], \left[\frac{v}{u^2+v^2+1}, 0\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to \left[\left[0, \frac{u}{u^2+v^2+1}\right], \left[\frac{u}{u^2+v^2+1}, 0\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to \left[\left[\left[0, \frac{v}{u^2+v^2+1}\right], \left[\frac{v}{u^2+v^2+1}, 0\right]\right], \left[\left[0, \frac{u}{u^2+v^2+1}\right], \left[\frac{u}{u^2+v^2+1}, 0\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to \left[\left[-\frac{v\,u}{(1+u^2+v^2)^{3/2}}, \frac{1+u^2}{(1+u^2+v^2)^{3/2}}\right], \left[\frac{1+v^2}{(1+u^2+v^2)^{3/2}}, -\frac{v\,u}{(1+u^2+v^2)^{3/2}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u,v) \to -\frac{1}{(u^2+v^2+1)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u,v) \to -\frac{u\,v}{(u^2+v^2+1)^{3/2}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$(u,v) \to -\frac{1}{(u^2+v^2+1)^2}$$

```
STUDENT > princ_curvature[1](F);
```

$$(u,v) \to -\frac{u\,v + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}\,v^2 + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}\,u^2 + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}}{(u^2+v^2+1)^{3/2}}$$

```
STUDENT > princ_curvature[1](F);
```

$$(u,v) \to -\frac{u\,v + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}\,v^2 + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}\,u^2 + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}}{(u^2+v^2+1)^{3/2}}$$

```
STUDENT >
STUDENT > princ_curvature[2](F);
```

$$(u,v) \to \frac{-u\,v + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}\,v^2 + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}\,u^2 + \sqrt{\frac{(v^2+1)(u^2+1)}{(u^2+v^2+1)^3}}\sqrt{u^2+v^2+1}}{(u^2+v^2+1)^{3/2}}$$

```
STUDENT > plot3d( F(u,v),u=-2..2, v=-2..2, scaling=constrained);
```

- $\triangleright$ Galloway: saddle.
- $\triangleright$ Shifrin: p.49.

2 Cylinder $\phi(u,v) = (a\cos(v), a\sin(v), u)$

```
STUDENT > assume(a>0);
STUDENT > F:=(u,v)->[a *cos(v),a*sin(v),u ];
```

$$F := (u, v) \to [a\cos(v), a\sin(v), u]$$

```
STUDENT > gik(F);
```

$$(u, v) \to [[1, 0], [0, a^2]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to [[0, 0], [0, a]]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to [[0, 0], [0, 0]]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to [[0, 0], [0, 0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to [[[0, 0], [0, 0]], [[0, 0], [0, 0]]]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[[0, 0], \left[0, \frac{1}{a}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$0$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to \frac{1}{2}\frac{1}{a}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

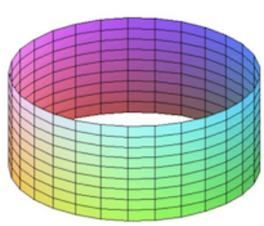
```
STUDENT > princ_curvature[1](F);
```

$$0$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to \frac{1}{a}$$

```
STUDENT > plot3d(Zylinder(2,1)(t,s),t=0..2*Pi,s=0..3,style=PATCH,
                 grid=[30,10],orientation=[90,30],scaling=constrained);
```



More info:

- ▷ Wiki: cylinder
- ▷ MathWorld: cylinder

3 Cone $\phi(u, v) = (u\cos(v), u\sin(v), u)$

```
STUDENT > assume(u>0):assume(x>-Pi/2,x<Pi/2):
STUDENT > assume(u>0);
STUDENT > F:=(u,v)->[u *cos(v),u*sin(v),u ];
```

$$F := (u, v) \to [u\cos(v), u\sin(v), u]$$

```
STUDENT > gik(F);
```

$$(u, v) \to [[2, 0], [0, u^2]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[[0, 0], \left[0, \frac{1}{2} u \sqrt{2}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to [[0, 0], [0, 0]]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to [[0, 0], [0, 0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to [[[0, 0], [0, 0]], [[0, 0], [0, 0]]]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[[0, 0], \left[0, \frac{1}{2} \frac{\sqrt{2}}{u}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$0$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to \frac{1}{4} \frac{\sqrt{2}}{u}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

```
STUDENT > princ_curvature[1](F);
```

$$0$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to \frac{1}{2} \frac{\sqrt{2}}{u}$$

More info:

- ▷ MathWorld: Cone
- ▷ Wiki: Cone

4 Elliptic paraboloid $\phi(u,v) = (u,v,u^2+v^2)$

```
STUDENT > F:=(u,v)->[u,v,u^2+v^2 ];
```

$$F := (u,v) \to [u, v, v^2 + u^2]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[1+4u^2, 4uv], [4uv, 1+4v^2]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[\frac{2}{\sqrt{4u^2+4v^2+1}}, 0\right], \left[0, \frac{2}{\sqrt{4u^2+4v^2+1}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to \left[[0,0], \left[0, 4\frac{u}{4u^2+4v^2+1}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to \left[[0,0], \left[0, 4\frac{v}{4u^2+4v^2+1}\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to \left[\left[[0,0], \left[0, 4\frac{u}{4u^2+4v^2+1}\right]\right], \left[[0,0], \left[0, 4\frac{v}{4u^2+4v^2+1}\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to \left[\left[2\frac{1+4v^2}{(1+4v^2+4u^2)^{3/2}}, -8\frac{uv}{(1+4v^2+4u^2)^{3/2}}\right], \left[-8\frac{uv}{(1+4v^2+4u^2)^{3/2}}, 2\frac{1+4u^2}{(1+4v^2+4u^2)^{3/2}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u,v) \to \frac{4}{(4u^2+4v^2+1)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u,v) \to 2\frac{1+2u^2+2v^2}{(4u^2+4v^2+1)^{3/2}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

```
STUDENT > princ_curvature[1](F);
```

$$(u,v) \to \frac{2}{(1+4v^2+4u^2)^{3/2}}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u,v) \to \frac{2}{\sqrt{1+4v^2+4u^2}}$$

More info:

- ▷ MathWorld: Elliptic Helicoid
- ▷ Wiki: Paraboloid

5 Hyperbolic paraboloid $\phi(u,v) = (u,v,u^2 - v^2)$

hyperbolic paraboloid - (maple Vr4 1996) Dr. W. Lindner 4/2023

```
STUDENT >
STUDENT > F:=(u,v)->[u,v,u^2-v^2 ];
```

$$F := (u,v) \to [u, v, u^2 - v^2]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[1+4u^2, -4uv], [-4uv, 1+4v^2]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[\frac{2}{\sqrt{4u^2+4v^2+1}}, 0\right], \left[0, -\frac{2}{\sqrt{4u^2+4v^2+1}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to \left[[0,0], \left[0, -4\frac{u}{4u^2+4v^2+1}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to \left[[0,0], \left[0, 4\frac{v}{4u^2+4v^2+1}\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to \left[\left[[0,0], \left[0, -4\frac{u}{4u^2+4v^2+1}\right]\right], \left[[0,0], \left[0, 4\frac{v}{4u^2+4v^2+1}\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to \left[\left[2\frac{1+4v^2}{(1+4v^2+4u^2)^{3/2}}, -8\frac{uv}{(1+4v^2+4u^2)^{3/2}}\right], \left[8\frac{uv}{(1+4v^2+4u^2)^{3/2}}, -2\frac{1+4u^2}{(1+4v^2+4u^2)^{3/2}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u,v) \to -\frac{4}{(1+4v^2+4u^2)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u,v) \to 4\frac{-u^2+v^2}{(1+4v^2+4u^2)^{3/2}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

```
STUDENT > princ_curvature[1](F);
```

$$(u,v) \to 2\frac{-2u^2-8u^4+2v^2+8v^4-\sqrt{4u^4-8u^2v^2+4v^4+1+4v^2+4u^2}-4\sqrt{4u^4-8u^2v^2+4v^4+1+4v^2+4u^2}\,v^2-4\sqrt{4u^4-8u^2v^2+4v^4+1+4v^2+4u^2}\,u^2}{(1+4v^2+4u^2)^{5/2}}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u,v) \to 2\frac{-2u^2-8u^4+2v^2+8v^4+\sqrt{4u^4-8u^2v^2+4v^4+1+4v^2+4u^2}+4\sqrt{4u^4-8u^2v^2+4v^4+1+4v^2+4u^2}\,v^2+4\sqrt{4u^4-8u^2v^2+4v^4+1+4v^2+4u^2}\,u^2}{(1+4v^2+4u^2)^{5/2}}$$

More info:

- ▷ Wiki: Paraboloid
- ▷ WOLFRAM|*alpha*: hyperbolic+paraboloid

6 ? surface $\phi(u,v) = (u+v, u-v, uv)$

```
STUDENT > F:=(u,v)->[u+v,u-v,u*v];
```

$$F := (u, v) \to [u+v, u-v, u\,v]$$

```
STUDENT > gik(F);
```

$$(u, v) \to [[2+v^2, u\,v], [u\,v, 2+u^2]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[\left[0, -\frac{2}{\sqrt{2\,u^2+2\,v^2+4}}\right], \left[-\frac{2}{\sqrt{2\,u^2+2\,v^2+4}}, 0\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[0, \frac{v}{u^2+v^2+2}\right], \left[\frac{v}{u^2+v^2+2}, 0\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to \left[\left[0, \frac{u}{u^2+v^2+2}\right], \left[\frac{u}{u^2+v^2+2}, 0\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[0, \frac{v}{u^2+v^2+2}\right], \left[\frac{v}{u^2+v^2+2}, 0\right]\right], \left[\left[0, \frac{u}{u^2+v^2+2}\right], \left[\frac{u}{u^2+v^2+2}, 0\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[\frac{u\,v}{(u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}}, -\frac{2+u^2}{(u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}}\right], \left[-\frac{2+v^2}{(u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}}, \frac{u\,v}{(u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \to -\frac{1}{(u^2+v^2+2)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to \frac{u\,v}{(u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$(u, v) \to -\frac{1}{(u^2+v^2+2)^2}$$

STUDENT > **princ_curvature[1](F);**

$$(u,v) \to -\frac{1}{2}\left(-2\,u\,v + \sqrt{2}\,\sqrt{\frac{(2+v^2)(2+u^2)}{(u^2+v^2+2)^3}}\,\sqrt{2\,u^2+2\,v^2+4}\,u^2 + \sqrt{2}\,\sqrt{\frac{(2+v^2)(2+u^2)}{(u^2+v^2+2)^3}}\,\sqrt{2\,u^2+2\,v^2+4}\,v^2 + 2\sqrt{2}\,\sqrt{\frac{(2+v^2)(2+u^2)}{(u^2+v^2+2)^3}}\,\sqrt{2\,u^2+2\,v^2+4}\right) \Big/ \left((u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}\right)$$

STUDENT > **princ_curvature[2](F);**

$$(u,v) \to \frac{1}{2}\left(2\,u\,v + \sqrt{2}\,\sqrt{\frac{(2+v^2)(2+u^2)}{(u^2+v^2+2)^3}}\,\sqrt{2\,u^2+2\,v^2+4}\,u^2 + \sqrt{2}\,\sqrt{\frac{(2+v^2)(2+u^2)}{(u^2+v^2+2)^3}}\,\sqrt{2\,u^2+2\,v^2+4}\,v^2 + 2\sqrt{2}\,\sqrt{\frac{(2+v^2)(2+u^2)}{(u^2+v^2+2)^3}}\,\sqrt{2\,u^2+2\,v^2+4}\right) \Big/ \left((u^2+v^2+2)\sqrt{2\,u^2+2\,v^2+4}\right)$$

7 Unit sphere $\phi(u,v) = (\cos(u)\sin v, \sin(u)\sin(v), \cos(v))$

```
STUDENT > F:=(u,v)->[cos(u)*sin(v),sin(u)*sin(v),cos(v)];
```

$$F := (u, v) \to [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)]$$

```
STUDENT > assume(u>0); assume(v>0);
STUDENT > gik(F);
```

$$(u, v) \to [[1 - \cos(v)^2, 0], [0, 1]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to [[\mathrm{csgn}(\sin(v)) - \mathrm{csgn}(\sin(v))\cos(v)^2, 0], [0, \mathrm{csgn}(\sin(v))]]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[0, -\frac{\cos(v)\sin(v)}{-1+\cos(v)^2}\right], \left[-\frac{\cos(v)\sin(v)}{-1+\cos(v)^2}, 0\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to [[-\cos(v)\sin(v), 0], [0, 0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[0, -\frac{\cos(v)\sin(v)}{-1+\cos(v)^2}\right], \left[-\frac{\cos(v)\sin(v)}{-1+\cos(v)^2}, 0\right]\right], [[-\cos(v)\sin(v), 0], [0, 0]]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to [[\mathrm{csgn}(\sin(v)), 0], [0, \mathrm{csgn}(\sin(v))]]$$

```
STUDENT > gauss_curvature(F);
```

$$1$$

```
STUDENT > mean_curvature(F);
```

$$csgn@((u, v) \to \sin(v))$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$1$$

```
STUDENT > princ_curvature[1](F);
```

$$csgn@((u, v) \to \sin(v))$$

```
STUDENT > princ_curvature[2](F);
```

$$csgn@((u, v) \to \sin(v))$$

Remark. `csgn` is a Maple function fo calculate the sign $^{+1}_{-1}$ of an real or complex expression.

More info:

- ▷ Wiki: Sphere
- ▷ MathWorld: Sphere
- ▷ WOLFRAM|*alpha*: Sphere
- ▷ Cox: R
- ▷ Wheeler: p.15
- BANCHOFF [1, p.226] .

8 Helicoid $\phi(u,v) = (v\cos(u), v\sin(u), u)$

```
STUDENT > assume(v>0):
STUDENT > F:=(u,v)->[v *cos(u),v*sin(u),u ];
```

$$F := (u,v) \to [v\cos(u), v\sin(u), u]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[v^2+1, 0], [0, 1]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[0, \frac{1}{\sqrt{v^2+1}}\right], \left[\frac{1}{\sqrt{v^2+1}}, 0\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to [[0,0],[0,0]]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to [[0,0],[0,0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to [[[0,0],[0,0]],[[0,0],[0,0]]]$$

```
STUDENT > gauss_curvature(F);
```

$$(u,v) \to -\frac{1}{(v^2+1)^2}$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to \left[\left[0, \frac{1}{(v^2+1)^{3/2}}\right], \left[\frac{1}{\sqrt{v^2+1}}, 0\right]\right]$$

```
STUDENT > mean_curvature(F);
```

$$0$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

```
STUDENT > princ_curvature[1](F);
```

$$(u,v) \to -\frac{1}{v^2+1}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u,v) \to \frac{1}{v^2+1}$$

monkey1

More info:

- ▷ Wiki: Helicoid
- ▷ MathWorld: Helicoid
- ▷ WOLFRAM|*alpha*: Helicoid

9 Catenoid $\phi(u,v) = (c\cosh\frac{v}{c}\cos u, c\cosh\frac{v}{c}\sin u, v)$

```
STUDENT > F:=(u,v)->[cosh(v)*cos(u), cosh(v)*sin(u), v];
```

$$F := (u, v) \to [\cos(u)\cosh(v), \sin(u)\cosh(v), v]$$

```
STUDENT > assume(u>0); assume(v>0);
STUDENT > gik(F);
```

$$(u, v) \to [[\cosh(v)^2, 0], [0, \cosh(v)^2]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to [[-1, 0], [0, 1]]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to [[0, 0], [0, 0]]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to [[0, 0], [0, 0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to [[[0, 0], [0, 0]], [[0, 0], [0, 0]]]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[-\frac{1}{\cosh(v)^2}, 0\right], \left[0, \frac{1}{\cosh(v)^2}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \to -\frac{1}{\cosh(v)^4}$$

```
STUDENT > mean_curvature(F);
```

$$0$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

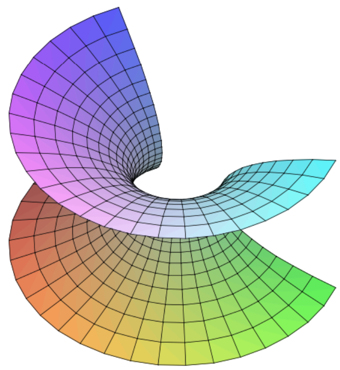
```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -\frac{1}{\cosh(v)^2}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to \frac{1}{\cosh(v)^2}$$

```
STUDENT > plot3d(F(t,s),t=-2..2,s=-2..2,style=PATCH,scaling=constr
          ained);
```



More info:

- ▷ MathWorld: catenoid
- ▷ Wiki: catenoid
- ▷ WOLFRAM|*alpha*: catenoid

10 Monkey saddle $\phi(u,v) = (u,v,u^3-3uv^2)$

```
STUDENT > F:=(u,v)->[u,v,u^3-3*u*v^2];
```

$$F := (u,v) \to [u, v, u^3 - 3\,u\,v^2]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[1 + 9\,u^4 - 18\,u^2\,v^2 + 9\,v^4, 18\,(-u^2+v^2)\,u\,v], [18\,(-u^2+v^2)\,u\,v, 1 + 36\,u^2\,v^2]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[6\,\frac{u}{\sqrt{9\,u^4+18\,u^2\,v^2+9\,v^4+1}}, -6\,\frac{v}{\sqrt{9\,u^4+18\,u^2\,v^2+9\,v^4+1}}\right], \left[-6\,\frac{v}{\sqrt{9\,u^4+18\,u^2\,v^2+9\,v^4+1}}, -6\,\frac{u}{\sqrt{9\,u^4+18\,u^2\,v^2+9\,v^4+1}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to \left[\left[-18\,\frac{u\,(-u^2+v^2)}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 18\,\frac{(-u^2+v^2)\,v}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right], \left[18\,\frac{(-u^2+v^2)\,v}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 18\,\frac{u\,(-u^2+v^2)}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to \left[\left[-36\,\frac{v\,u^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 36\,\frac{u\,v^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right], \left[36\,\frac{u\,v^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 36\,\frac{v\,u^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to \left[\left[\left[-18\,\frac{u\,(-u^2+v^2)}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 18\,\frac{(-u^2+v^2)\,v}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right], \left[18\,\frac{(-u^2+v^2)\,v}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 18\,\frac{u\,(-u^2+v^2)}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right]\right], \left[\left[-36\,\frac{v\,u^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 36\,\frac{u\,v^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right], \left[36\,\frac{u\,v^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}, 36\,\frac{v\,u^2}{9\,u^4+18\,u^2\,v^2+9\,v^4+1}\right]\right]\right]$$

STUDENT > **shapeoperator(F);**

$$(u, v) \to \left[\left[6\frac{u(1+18u^2v^2+18v^4)}{(9u^4+18u^2v^2+9v^4+1)^{3/2}}, -6\frac{v(1+18u^2v^2+18u^4)}{(9u^4+18u^2v^2+9v^4+1)^{3/2}}\right], \left[-6\frac{v(-9u^4+1+9v^4)}{(9u^4+18u^2v^2+9v^4+1)^{3/2}}, 6\frac{u(9v^4-1-9u^4)}{(9u^4+18u^2v^2+9v^4+1)^{3/2}}\right]\right]$$

STUDENT > **gauss_curvature(F);**

$$(u, v) \to -36\frac{u^2+v^2}{(9u^4+18u^2v^2+9v^4+1)^2}$$

STUDENT > **mean_curvature(F);**

$$(u, v) \to 27\frac{u(-u^4+2u^2v^2+3v^4)}{(9u^4+18u^2v^2+9v^4+1)^{3/2}}$$

STUDENT > **intrinsic_curvature(gik(F));**

$$(u, v) \to -36\frac{u^2+v^2}{(9u^4+18u^2v^2+9v^4+1)^2}$$

STUDENT > **princ_curvature[1](F);**

$$\begin{aligned}(u, v) \to 3\,(&-9u^5+18u^3v^2+27uv^4-9\,((81u^{10}-324u^8v^2-162u^6v^4+972u^4v^6\\
&+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\Big/\\
&(9u^4+18u^2v^2+9v^4+1)^3)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\;u^4-18\,((81u^{10}-324u^8v^2\\
&-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\Big/\\
&(9u^4+18u^2v^2+9v^4+1)^3)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\;u^2v^2-9\,((81u^{10}-324u^8v^2\\
&-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\Big/\\
&(9u^4+18u^2v^2+9v^4+1)^3)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\;v^4-((81u^{10}-324u^8v^2\\
&-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\Big/\\
&(9u^4+18u^2v^2+9v^4+1)^3)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\,)\Big/\\
&(9u^4+18u^2v^2+9v^4+1)^{3/2}\end{aligned}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u,v)\to 3\Big(-9u^5+18u^3v^2+27uv^4+9\big((81u^{10}-324u^8v^2-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\big/ (9u^4+18u^2v^2+9v^4+1)^3\big)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\,u^4+18\big((81u^{10}-324u^8v^2-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\big/ (9u^4+18u^2v^2+9v^4+1)^3\big)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\,u^2v^2+9\big((81u^{10}-324u^8v^2-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\big/ (9u^4+18u^2v^2+9v^4+1)^3\big)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\,v^4+\big((81u^{10}-324u^8v^2-162u^6v^4+972u^4v^6+729u^2v^8+36u^6+108u^4v^2+108u^2v^4+4u^2+36v^6+4v^2)\big/ (9u^4+18u^2v^2+9v^4+1)^3\big)^{1/2}\sqrt{9u^4+18u^2v^2+9v^4+1}\Big)\Big/ (9u^4+18u^2v^2+9v^4+1)^{3/2}$$

More info:

- ▷ MathWorld: MonkeySaddle
- ▷ Wiki: Monkey saddle
- ▷ WOLFRAM|*alpha*: Monkey saddle

11 Torus $\phi(u,v) = ((b{+}a\cos(v))\cos(u), (b{+}a\cos(v))\sin(u), a\sin(v))$

```
STUDENT > F:=(u,v)->[((2+1*cos(v))*cos(u),(2+1*cos(v))*sin(u),1*sin(v))];
```

$$F := (u, v) \to [(2 + \cos(v))\cos(u), (2 + \cos(v))\sin(u), \sin(v)]$$

```
STUDENT > gik(F);
```

$$(u, v) \to [[4 + 4\cos(v) + \cos(v)^2, 0], [0, 1]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to [[-2\cos(v)\,\mathrm{csgn}(2 + \cos(v)) - \cos(v)^2\,\mathrm{csgn}(2 + \cos(v)), 0], [0, -\mathrm{csgn}(2 + \cos(v))]]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[0, -\frac{\sin(v)}{2 + \cos(v)}\right], \left[-\frac{\sin(v)}{2 + \cos(v)}, 0\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to [[2\sin(v) + \cos(v)\sin(v), 0], [0, 0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[0, -\frac{\sin(v)}{2 + \cos(v)}\right], \left[-\frac{\sin(v)}{2 + \cos(v)}, 0\right]\right], [[2\sin(v) + \cos(v)\sin(v), 0], [0, 0]]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[-\frac{\cos(v)\,\mathrm{csgn}(2 + \cos(v))}{2 + \cos(v)}, 0\right], [0, -\mathrm{csgn}(2 + \cos(v))]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \to \frac{\cos(v)}{2 + \cos(v)}$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to -\frac{(\cos(v) + 1)\,\mathrm{csgn}(2 + \cos(v))}{2 + \cos(v)}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

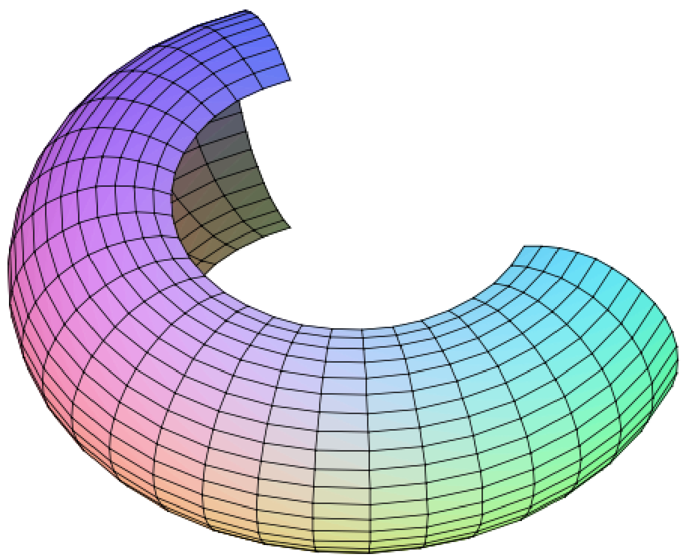
$$(u, v) \to \frac{\cos(v)}{2 + \cos(v)}$$

```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -\frac{\cos(v)\,\mathrm{csgn}(2 + \cos(v)) + \mathrm{csgn}(2 + \cos(v)) + \mathrm{csgn}(2 + \cos(\bar{v}))}{2 + \cos(v)}$$

$$(u, v) \to -\frac{\cos(v)\,\mathrm{csgn}(2+\cos(v)) + \mathrm{csgn}(2+\cos(v)) - \mathrm{csgn}(2+\cos(v))}{2+\cos(v)}$$

```
STUDENT > plot3d(F(t,s),t=-2..2,s=-2..2,style=PATCH,scaling=constr
          ained);
```



Remark. `csgn` is a Maple function fo calculate the sign $^{+1}_{-1}$ of an real or complex expression.
More info:

- ▷ MathWorld: Torus
- ▷ Wiki: Torus
- ▷ WOLFRAM|*alpha*: Torus
- ▷ Cox: R
- KREYSZIG: *Differentialgeometrie.* p. 165.
- BANCHOFF [1, p.182] .

12 Function graph $\phi(u,v) = (u,v,f(u,v))$.

function graph - (maple Vr4 1996) Dr. W. Lindner 4/2023

```
STUDENT > F:=(u,v)->[u,v, f(u,v)];
```

$$F := (u,v) \to [u, v, f(u,v)]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[1 + D_1(f)(u,v)^2, D_1(f)(u,v)\,D_2(f)(u,v)], [D_1(f)(u,v)\,D_2(f)(u,v), 1 + D_2(f)(u,v)^2]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[\frac{D_{1,1}(f)(u,v)}{\sqrt{D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1}}, \frac{D_{1,2}(f)(u,v)}{\sqrt{D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1}}\right], \left[\frac{D_{1,2}(f)(u,v)}{\sqrt{D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1}}, \frac{D_{2,2}(f)(u,v)}{\sqrt{D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to [[0,0],[0,0]]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to [[0,0],[0,0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to [[[0,0],[0,0]],[[0,0],[0,0]]]$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to \left[\left[-\frac{-D_{1,1}(f)(u,v) - D_{1,1}(f)(u,v)\,D_2(f)(u,v)^2 + D_1(f)(u,v)\,D_2(f)(u,v)\,D_{1,2}(f)(u,v)}{(D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1)^{3/2}}, \frac{D_{1,2}(f)(u,v) + D_{1,2}(f)(u,v)\,D_2(f)(u,v)^2 - D_1(f)(u,v)\,D_2(f)(u,v)\,D_{2,2}(f)(u,v)}{(D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1)^{3/2}}\right], \left[\frac{-D_1(f)(u,v)\,D_2(f)(u,v)\,D_{1,1}(f)(u,v) + D_{1,2}(f)(u,v) + D_{1,2}(f)(u,v)\,D_1(f)(u,v)^2}{(D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1)^{3/2}}, -\frac{D_1(f)(u,v)\,D_2(f)(u,v)\,D_{1,2}(f)(u,v) - D_{2,2}(f)(u,v) - D_{2,2}(f)(u,v)\,D_1(f)(u,v)^2}{(D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1)^{3/2}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u,v) \to \frac{D_{1,1}(f)(u,v)\,D_{2,2}(f)(u,v) - D_{1,2}(f)(u,v)^2}{(D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u,v) \to \frac{1}{2}\,\frac{D_{2,2}(f)(u,v)\,D_1(f)(u,v)^2 + D_{2,2}(f)(u,v) - 2\,D_1(f)(u,v)\,D_2(f)(u,v)\,D_{1,2}(f)(u,v) + D_{1,1}(f)(u,v) + D_{1,1}(f)(u,v)\,D_2(f)(u,v)^2}{(D_1(f)(u,v)^2 + D_2(f)(u,v)^2 + 1)^{3/2}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$0$$

```
STUDENT > princ_curvature[1](F);
```

Remark. The expressions for `princ-curvature[1](gik(F))` are too complicated to be useful to print.

More info:

- ▷ Deserno: p.12.
- ▷ Galloway: p.123.
- ▷ Gluck: p.22 ff.
- BANCHOFF [1, p.201].
- KREYSZIG: *Differentialgeometrie.* p. 383.
- PRESSLEY [3, p.149].

13 surface of Revolution $\phi(u,v)=(f(u)\cos(v),f(u)\sin(v),g(u))$

```
STUDENT > F:=(u,v)->[ f(u)*cos(v), f(u)*sin(v), g(u)];
```

$$F := (u,v) \to [\mathrm{f}(u)\cos(v), \mathrm{f}(u)\sin(v), \mathrm{g}(u)]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2, 0], [0, \mathrm{f}(u)^2]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[-\frac{\mathrm{f}(u)\,((D^{(2)})(f)(u)\,\mathrm{D}(g)(u) - (D^{(2)})(g)(u)\,\mathrm{D}(f)(u))}{\sqrt{\mathrm{f}(u)^2\,(\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2)}}, 0\right],\right.$$
$$\left.\left[0, \frac{\mathrm{f}(u)^2\,\mathrm{D}(g)(u)}{\sqrt{\mathrm{f}(u)^2\,(\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2)}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to$$
$$\left[\left[\frac{\mathrm{D}(f)(u)\,(D^{(2)})(f)(u) + \mathrm{D}(g)(u)\,(D^{(2)})(g)(u)}{\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2}, 0\right], \left[0, -\frac{\mathrm{D}(f)(u)\,\mathrm{f}(u)}{\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to \left[\left[0, \frac{\mathrm{D}(f)(u)}{\mathrm{f}(u)}\right], \left[\frac{\mathrm{D}(f)(u)}{\mathrm{f}(u)}, 0\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to \Bigg[$$
$$\left[\left[\frac{\mathrm{D}(f)(u)\,(D^{(2)})(f)(u) + \mathrm{D}(g)(u)\,(D^{(2)})(g)(u)}{\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2}, 0\right], \left[0, -\frac{\mathrm{D}(f)(u)\,\mathrm{f}(u)}{\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2}\right]\right],$$
$$\left[\left[0, \frac{\mathrm{D}(f)(u)}{\mathrm{f}(u)}\right], \left[\frac{\mathrm{D}(f)(u)}{\mathrm{f}(u)}, 0\right]\right]\Bigg]$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to \left[\left[\frac{\mathrm{f}(u)\,(-(D^{(2)})(f)(u)\,\mathrm{D}(g)(u) + (D^{(2)})(g)(u)\,\mathrm{D}(f)(u))}{(\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2)\,\sqrt{\mathrm{f}(u)^2\,(\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2)}}, 0\right],\right.$$
$$\left.\left[0, \frac{\mathrm{D}(g)(u)}{\sqrt{\mathrm{f}(u)^2\,(\mathrm{D}(f)(u)^2 + \mathrm{D}(g)(u)^2)}}\right]\right]$$

STUDENT > **gauss_curvature(F);**

$$(u,v) \to -\frac{((D^{(2)})(f)(u)\,D(g)(u) - (D^{(2)})(g)(u)\,D(f)(u))\,D(g)(u)}{f(u)\,(D(f)(u)^2 + D(g)(u)^2)^2}$$

STUDENT > **mean_curvature(F);**

$$(u,v) \to \frac{1}{2}\,\frac{D(g)(u)\,D(f)(u)^2 + D(g)(u)^3 - f(u)\,(D^{(2)})(f)(u)\,D(g)(u) + f(u)\,(D^{(2)})(g)(u)\,D(f)(u)}{\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,(D(f)(u)^2 + D(g)(u)^2)}$$

STUDENT > **intrinsic_curvature(gik(F));**

$$(u,v) \to \frac{D(g)(u)\,(-(D^{(2)})(f)(u)\,D(g)(u) + (D^{(2)})(g)(u)\,D(f)(u))}{f(u)\,(D(f)(u)^2 + D(g)(u)^2)^2}$$

STUDENT > **princ_curvature[1](F);**

$$(u,v) \to -\frac{1}{2}(-D(g)(u)\,D(f)(u)^2 - D(g)(u)^3 + f(u)\,(D^{(2)})(f)(u)\,D(g)(u)$$
$$- f(u)\,(D^{(2)})(g)(u)\,D(f)(u) + ($$
$$(f(u)\,(D^{(2)})(f)(u)\,D(g)(u) + D(g)(u)\,D(f)(u)^2 + D(g)(u)^3 - f(u)\,(D^{(2)})(g)(u)\,D(f)(u))\hat{}$$
$$2 \Big/ (f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)^3))^{1/2}\,\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,D(f)(u)^2 + ($$
$$(f(u)\,(D^{(2)})(f)(u)\,D(g)(u) + D(g)(u)\,D(f)(u)^2 + D(g)(u)^3 - f(u)\,(D^{(2)})(g)(u)\,D(f)(u))\hat{}$$
$$2 \Big/ (f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)^3))^{1/2}\,\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,D(g)(u)^2) \Big/ ($$
$$\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,(D(f)(u)^2 + D(g)(u)^2))$$

STUDENT > **princ_curvature[2](F);**

$$(u,v) \to \frac{1}{2}(D(g)(u)\,D(f)(u)^2 + D(g)(u)^3 - f(u)\,(D^{(2)})(f)(u)\,D(g)(u)$$
$$+ f(u)\,(D^{(2)})(g)(u)\,D(f)(u) + (($$
$$-D(g)(u)\,D(f)(u)^2 - D(g)(u)^3 - f(u)\,(D^{(2)})(f)(u)\,D(g)(u) + f(u)\,(D^{(2)})(g)(u)\,D(f)(u))\hat{}$$
$$2 \Big/ (f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)^3))^{1/2}\,\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,D(f)(u)^2 + (($$
$$-D(g)(u)\,D(f)(u)^2 - D(g)(u)^3 - f(u)\,(D^{(2)})(f)(u)\,D(g)(u) + f(u)\,(D^{(2)})(g)(u)\,D(f)(u))\hat{}$$
$$2 \Big/ (f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)^3))^{1/2}\,\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,D(g)(u)^2) \Big/ ($$
$$\sqrt{f(u)^2\,(D(f)(u)^2 + D(g)(u)^2)}\,(D(f)(u)^2 + D(g)(u)^2))$$

More info:

- ▷ MathWorld: Surface of revolution
- ▷ Wiki: Surface of revolution
- ▷ Wheeler: p.16
- ▷ Shifrin: p.52.
- Banchoff [1, p.196, p.205] .
- Pressley [3, p.149] .
- Lipschutz: *Differential Geometry.* p. 202, Example 10.1.

14 Ruled surface $\phi(u, v) = c(u) + v \cdot E(u)$

More info:

- ▷ Wiki: Ruled surface
- ▷ WOLFRAM|*alpha*: Ruled surface
- ▷ Wheeler: p.18
- LIPSCHUTZ: *Differential Geometry.* p. 225, Ex. 10.30.
- BANCHOFF [1, p.210] .

15 Plücker conoid $\phi(u,v) = (v \cdot \cos(u), v \cdot \sin(u), 2\cos(u) \cdot \sin(u))$

```
STUDENT > F:=(u,v)->[ v*cos(u), v*sin(u),  2*cos(u)*sin(u)];
```

$$F := (u, v) \to [v\cos(u), v\sin(u), 2\cos(u)\sin(u)]$$

```
STUDENT > gik(F);
```

$$(u, v) \to [[v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4, 0], [0, 1]]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[\left[8\frac{\cos(u)\sin(u)\,v}{\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}}, 2\frac{2\cos(u)^2 - 1}{\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}}\right], \left[2\frac{2\cos(u)^2 - 1}{\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}}, 0\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[-16\frac{\cos(u)\sin(u)(2\cos(u)^2 - 1)}{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}, \frac{v}{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}\right], \left[\frac{v}{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}, 0\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to [[-v, 0], [0, 0]]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[-16\frac{\cos(u)\sin(u)(2\cos(u)^2 - 1)}{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}, \frac{v}{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}\right], \left[\frac{v}{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}, 0\right]\right], [[-v, 0], [0, 0]]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[8\frac{\cos(u)\sin(u)\,v}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^{3/2}}, 2\frac{2\cos(u)^2 - 1}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^{3/2}}\right], \left[2\frac{2\cos(u)^2 - 1}{\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}}, 0\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \to -4\,(4\cos(u)^4 - 4\cos(u)^2 + 1) \Big/ (v^4 + 32\,v^2\cos(u)^4 - 32\,v^2\cos(u)^2 + 8\,v^2 + 256\cos(u)^8 - 512\cos(u)^6 + 384\cos(u)^4 - 128\cos(u)^2 + 16)$$

STUDENT > **mean_curvature(F);**

$$(u, v) \to 4\,\frac{\cos(u)\sin(u)\,v}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^{3/2}}$$

STUDENT > **intrinsic_curvature(gik(F));**

$$(u, v) \to -4\,(4\cos(u)^4 - 4\cos(u)^2 + 1) \Big/ (v^4 + 32\,v^2\cos(u)^4 - 32\,v^2\cos(u)^2 + 8\,v^2 + 256\cos(u)^8 - 512\cos(u)^6 + 384\cos(u)^4 - 128\cos(u)^2 + 16)$$

STUDENT > **princ_curvature[1](F);**

$$(u, v) \to -2\left(-2\cos(u)\sin(u)\,v + \sqrt{\frac{v^2 - 128\cos(u)^6 + 96\cos(u)^4 + 4 - 32\cos(u)^2 + 64\cos(u)^8}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^3}}\,\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}\;v^2 + 16\,\sqrt{\frac{v^2 - 128\cos(u)^6 + 96\cos(u)^4 + 4 - 32\cos(u)^2 + 64\cos(u)^8}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^3}}\,\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}\;\cos(u)^4 - 16\,\sqrt{\frac{v^2 - 128\cos(u)^6 + 96\cos(u)^4 + 4 - 32\cos(u)^2 + 64\cos(u)^8}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^3}}\,\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}\;\cos(u)^2 + 4\,\sqrt{\frac{v^2 - 128\cos(u)^6 + 96\cos(u)^4 + 4 - 32\cos(u)^2 + 64\cos(u)^8}{(v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^3}}\,\sqrt{v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4}\right) \Big/ (v^2 + 16\cos(u)^4 - 16\cos(u)^2 + 4)^{3/2}$$

More info:

- ▷ WOLFRAM|*alpha*: Pluecker conoid

16 Banchoff funnel $\phi(u,v) = (v\cos(u), v\sin(u), \ln(v))$

Banchoff funnel p.155 - (maple Vr4 1996) Dr. W. Lindner 4/2023

```
STUDENT > F:=(u,v)->[ v*cos(u), v*sin(u), ln(v)];
```

$$F := (u, v) \to [v\cos(u), v\sin(u), \ln(v)]$$

```
STUDENT > gik(F);
```

$$(u, v) \to \left[[v^2, 0], \left[0, \frac{v^2+1}{v^2}\right]\right]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[\left[-\frac{v}{\sqrt{v^2+1}}, 0\right], \left[0, \frac{1}{v\sqrt{v^2+1}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[0, \frac{1}{v}\right], \left[\frac{1}{v}, 0\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to \left[\left[-\frac{v^3}{v^2+1}, 0\right], \left[0, -\frac{1}{v(v^2+1)}\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[0, \frac{1}{v}\right], \left[\frac{1}{v}, 0\right]\right], \left[\left[-\frac{v^3}{v^2+1}, 0\right], \left[0, -\frac{1}{v(v^2+1)}\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[-\frac{1}{v\sqrt{v^2+1}}, 0\right], \left[0, \frac{v}{(v^2+1)^{3/2}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \to -\frac{1}{(v^2+1)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to -\frac{1}{2}\frac{1}{v(v^2+1)^{3/2}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$(u, v) \to -\frac{1}{(v^2+1)^2}$$

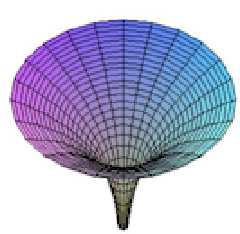
```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -\frac{1}{2}\frac{1 + \sqrt{\frac{(2v^2+1)^2}{v^2(v^2+1)^3}}\,v^3\sqrt{v^2+1} + \sqrt{\frac{(2v^2+1)^2}{v^2(v^2+1)^3}}\,v\sqrt{v^2+1}}{v(v^2+1)^{3/2}}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to \frac{1}{2}\frac{-1 + \sqrt{\frac{(2v^2+1)^2}{v^2(v^2+1)^3}}\,v^3\sqrt{v^2+1} + \sqrt{\frac{(2v^2+1)^2}{v^2(v^2+1)^3}}\,v\sqrt{v^2+1}}{v(v^2+1)^{3/2}}$$

```
STUDENT > plot3d(F(t,s),t=0..2*Pi,s=-0..2,style=PATCH,scaling=constrained);
```



17 Scherk surface $\phi(u,v) = (u, v, \ln(\frac{\cos v}{\cos u}))$

```
STUDENT > F:=(u,v)->[ u, v, ln(cos(v)/cos(u))];
```

$$F := (u, v) \to \left[u, v, \ln\left(\frac{\cos(v)}{\cos(u)}\right) \right]$$

```
STUDENT > gik(F);
```

$$(u, v) \to \left[\left[\frac{1}{\cos(u)^2}, -\frac{\sin(u)\sin(v)}{\cos(u)\cos(v)} \right], \left[-\frac{\sin(u)\sin(v)}{\cos(u)\cos(v)}, \frac{1}{\cos(v)^2} \right] \right]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[\left[\frac{1}{\cos(u)^2 \sqrt{-\frac{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}{\cos(u)^2\cos(v)^2}}}, 0 \right], \left[0, -\frac{1}{\cos(v)^2 \sqrt{-\frac{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}{\cos(u)^2\cos(v)^2}}} \right] \right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[-\frac{\sin(u)\cos(v)^2}{\cos(u)(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)}, 0 \right], \left[0, \frac{\sin(u)\cos(u)}{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2} \right] \right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to \left[\left[\frac{\sin(v)\cos(v)}{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}, 0 \right], \left[0, -\frac{\cos(u)^2\sin(v)}{\cos(v)(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)} \right] \right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[-\frac{\sin(u)\cos(v)^2}{\cos(u)(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)}, 0 \right], \left[0, \frac{\sin(u)\cos(u)}{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2} \right] \right], \left[\left[\frac{\sin(v)\cos(v)}{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}, 0 \right], \right.$$

$$\left[0, -\frac{\cos(u)^2 \sin(v)}{\cos(v)\,(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)}\right]\Bigg]\Bigg]$$

STUDENT > **shapeoperator(F);**

$$(u, v) \to \Bigg[\Bigg[$$

$$-\frac{1}{\sqrt{-\frac{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}{\cos(u)^2\cos(v)^2}}\,(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)},$$

$$\left.\frac{\cos(u)\sin(u)\sin(v)}{\cos(v)\sqrt{-\frac{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}{\cos(u)^2\cos(v)^2}}\,(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)}\right]$$

$$,\Bigg[-$$

$$\frac{\sin(u)\sin(v)\cos(v)}{\cos(u)\sqrt{-\frac{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}{\cos(u)^2\cos(v)^2}}\,(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)},$$

$$\frac{1}{\sqrt{-\frac{-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2}{\cos(u)^2\cos(v)^2}}\,(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)}\Bigg]\Bigg]$$

STUDENT > **gauss_curvature(F);**

$$(u, v) \to -\cos(u)^2\cos(v)^2 \Big/ (\cos(v)^4 + 2\cos(u)^2\cos(v)^2 - 2\cos(v)^4\cos(u)^2$$
$$- 2\cos(u)^4\cos(v)^2 + \cos(u)^4\cos(v)^4 + \cos(u)^4)$$

STUDENT > **mean_curvature(F);**

$$0$$

STUDENT > **intrinsic_curvature(gik(F));**

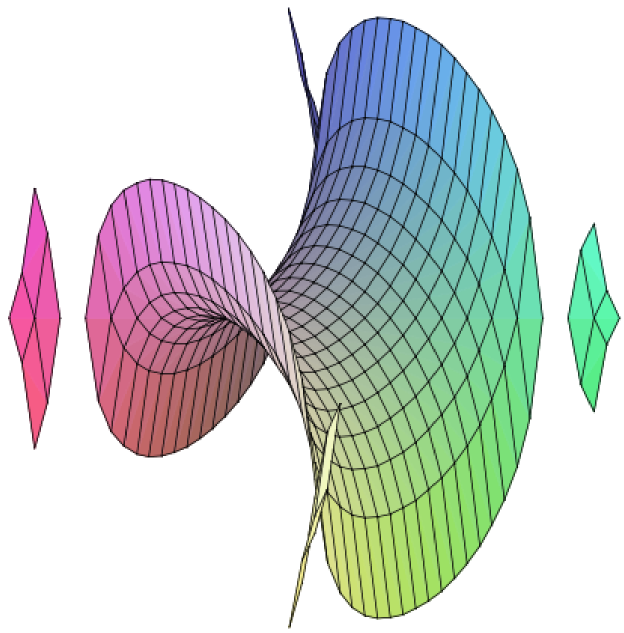
$$(u, v) \to -\cos(u)^2\cos(v)^2 \Big/ (\cos(v)^4 + 2\cos(u)^2\cos(v)^2 - 2\cos(v)^4\cos(u)^2$$
$$- 2\cos(u)^4\cos(v)^2 + \cos(u)^4\cos(v)^4 + \cos(u)^4)$$

STUDENT > **princ_curvature[1](F);**

$$(u, v) \to -\sqrt{\frac{\cos(u)^2\cos(v)^2}{(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)^2}}$$

STUDENT > **princ_curvature[2](F);**

$$(u, v) \to \sqrt{\frac{\cos(u)^2\cos(v)^2}{(-\cos(v)^2 + \cos(u)^2\cos(v)^2 - \cos(u)^2)^2}}$$



More info:

- ▷ MathWorld: Scherk surface
- ▷ Wiki: Scherk surface
- ▷ WOLFRAM|*alpha*: Scherk surface

18 ENNEPER surface $\phi(u,v) = (\frac{1}{3}u(1-\frac{1}{3}u^2+v^2), \frac{1}{3}v(1-\frac{1}{3}v^2+u^2), \frac{1}{3}(u^2-v^2))$

```
STUDENT > F:=(u,v)->[ u-1/3*u^3+u*v^2, v-1/3*v^3+v*u^2, u^2-v^2];
```

$$F := (u,v) \to \left[u - \frac{1}{3}u^3 + u\,v^2,\ v - \frac{1}{3}v^3 + v\,u^2,\ u^2 - v^2 \right]$$

```
STUDENT > gik(F);
```

$$(u,v) \to [[1+2\,u^2+2\,v^2+u^4+2\,v^2\,u^2+v^4,\ 0],\ [0,\ 1+2\,u^2+2\,v^2+u^4+2\,v^2\,u^2+v^4]]$$

```
STUDENT > hik(F);
```

$$(u,v) \to [[2\,\mathrm{csgn}((u^2+v^2+1)^2),\ 0],\ [0,\ -2\,\mathrm{csgn}((u^2+v^2+1)^2)]]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u,v) \to \left[\left[2\frac{u}{u^2+v^2+1},\ 2\frac{v}{u^2+v^2+1}\right],\ \left[2\frac{v}{u^2+v^2+1},\ -2\frac{u}{u^2+v^2+1}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u,v) \to \left[\left[-2\frac{v}{u^2+v^2+1},\ 2\frac{u}{u^2+v^2+1}\right],\ \left[2\frac{u}{u^2+v^2+1},\ 2\frac{v}{u^2+v^2+1}\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u,v) \to \left[\left[\left[2\frac{u}{u^2+v^2+1},\ 2\frac{v}{u^2+v^2+1}\right],\ \left[2\frac{v}{u^2+v^2+1},\ -2\frac{u}{u^2+v^2+1}\right]\right],\right.$$
$$\left.\left[\left[-2\frac{v}{u^2+v^2+1},\ 2\frac{u}{u^2+v^2+1}\right],\ \left[2\frac{u}{u^2+v^2+1},\ 2\frac{v}{u^2+v^2+1}\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u,v) \to$$
$$\left[\left[2\frac{\mathrm{csgn}((u^2+v^2+1)^2)}{1+2\,u^2+2\,v^2+u^4+2\,v^2\,u^2+v^4},\ 0\right],\ \left[0,\ -2\frac{\mathrm{csgn}((u^2+v^2+1)^2)}{1+2\,u^2+2\,v^2+u^4+2\,v^2\,u^2+v^4}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u,v) \to -\frac{4}{(1+2\,u^2+2\,v^2+u^4+2\,v^2\,u^2+v^4)^2}$$

```
STUDENT > mean_curvature(F);
```

$$0$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$(u,v) \to -\frac{4}{(1+2\,u^2+2\,v^2+u^4+2\,v^2\,u^2+v^4)\,(u^2+v^2+1)^2}$$

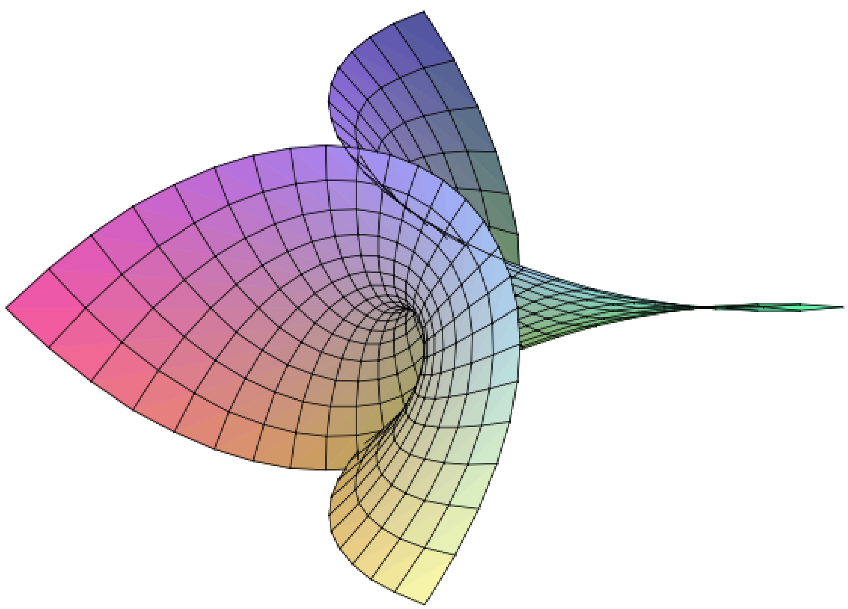
```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -2\,\frac{\operatorname{csgn}\left(\left(1+\overline{u^2+v^2}\right)^2\right)}{(u^2+v^2+1)^2}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to 2\,\frac{\operatorname{csgn}\left(\left(1+\overline{u^2+v^2}\right)^2\right)}{(u^2+v^2+1)^2}$$

```
STUDENT > plot3d(F(t,s),t=-2..2,s=-2..2,style=PATCH,scaling=constr
          ained);
```



Remark. `csgn` is a Maple function fo calculate the sign $^{+1}_{-1}$ of an real or complex expression.

More info:

- ▷ MathWorld: Enneper's Surface
- ▷ WOLFRAM|*alpha*: Enneper's Surface
- ▷ Deserno: p.22.
- ▷ Earl: p.23.
- Banchoff [1, p.197].
- Pressley [3, p.214].

19 Beltrami pseudosphere $\phi(u,v) = (\sin u \cos v, \ldots)$

```
STUDENT > F:=(u,v)->[ sin(u)*cos(v), sin(u)*sin(v),
          cos(u)+ln(sin(u)/(cos(u)+1))];
```

$$F := (u, v) \to \left[\sin(u)\cos(v),\ \sin(u)\sin(v),\ \cos(u) + \ln\left(\frac{\sin(u)}{\cos(u)+1}\right) \right]$$

```
STUDENT > gik(F);
```

$$(u, v) \to \left[\left[-\frac{\cos(u)^2}{-1+\cos(u)^2},\ 0 \right],\ [0,\ 1-\cos(u)^2] \right]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[\left[\frac{\sin(u)\operatorname{csgn}(\cos(u))\cos(u)}{-1+\cos(u)^2},\ 0 \right],\ [0,\ \sin(u)\operatorname{csgn}(\cos(u))\cos(u)] \right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[\frac{\sin(u)}{\cos(u)(-1+\cos(u)^2)},\ 0 \right],\ \left[0,\ \frac{\sin(u)(-1+\cos(u)^2)}{\cos(u)} \right] \right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to \left[\left[0,\ -\frac{\cos(u)\sin(u)}{-1+\cos(u)^2} \right],\ \left[-\frac{\cos(u)\sin(u)}{-1+\cos(u)^2},\ 0 \right] \right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[\frac{\sin(u)}{\cos(u)(-1+\cos(u)^2)},\ 0 \right],\ \left[0,\ \frac{\sin(u)(-1+\cos(u)^2)}{\cos(u)} \right] \right], \left[\left[0,\ -\frac{\cos(u)\sin(u)}{-1+\cos(u)^2} \right],\ \left[-\frac{\cos(u)\sin(u)}{-1+\cos(u)^2},\ 0 \right] \right] \right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[-\frac{\sin(u)\operatorname{csgn}(\cos(u))}{\cos(u)},\ 0 \right],\ \left[0,\ -\frac{\sin(u)\operatorname{csgn}(\cos(u))\cos(u)}{-1+\cos(u)^2} \right] \right]$$

```
STUDENT > gauss_curvature(F);
```

$$-1$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to -\frac{1}{2}\,\frac{\sin(u)\operatorname{csgn}(\cos(u))(2\cos(u)^2-1)}{\cos(u)(-1+\cos(u)^2)}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

-1

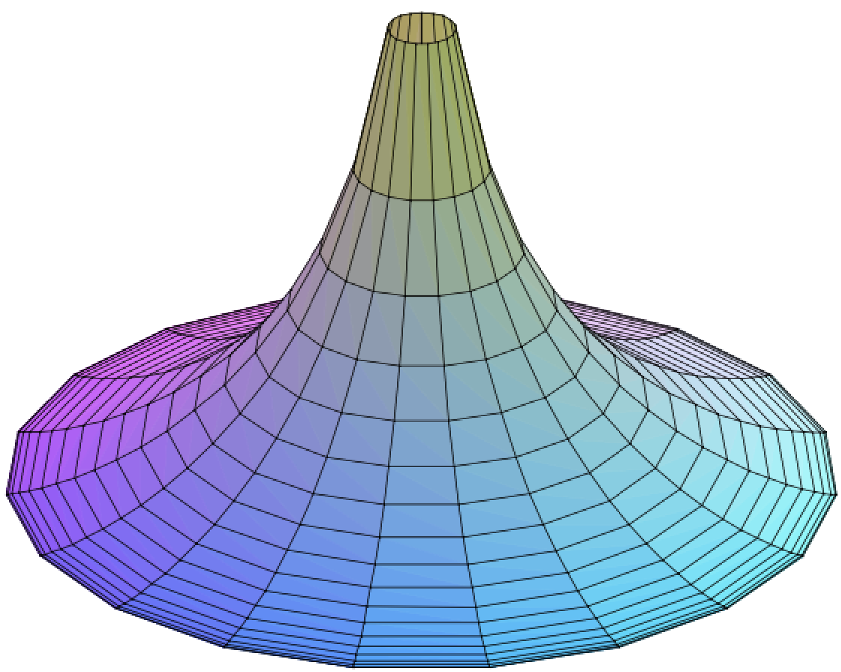
```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -\frac{1}{2}\left(2\sin(u)\operatorname{csgn}(\cos(u))\cos(u)^2 - \sin(u)\operatorname{csgn}(\cos(u)) - \sqrt{\frac{1}{\sin(u)^2\cos(u)^2}}\cos(u) + \sqrt{\frac{1}{\sin(u)^2\cos(u)^2}}\cos(u)^3\right) \Big/ \left(\cos(u)\left(-1 + \cos(u)^2\right)\right)$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to \frac{1}{2}\left(-2\sin(u)\operatorname{csgn}(\cos(u))\cos(u)^2 + \sin(u)\operatorname{csgn}(\cos(u)) - \sqrt{\frac{1}{\sin(u)^2\cos(u)^2}}\cos(u) + \sqrt{\frac{1}{\sin(u)^2\cos(u)^2}}\cos(u)^3\right) \Big/ \left(\cos(u)\left(-1 + \cos(u)^2\right)\right)$$

```
STUDENT > plot3d(Pseudosphaere(t,s),t=0..Pi/2,s=0..2*Pi,style=PATC
          H,grid=[20,20],orientation=[0,-130]);
```



```
STUDENT >
```

More info:

- ▷ MathWorld: Pseudosphere
- ▷ Wiki: Pseudosphere
- ▷ WOLFRAM|*alpha*: Pseudosphere
- ▷ Wheeler: p.23
- Banchoff [1, p.202] .

20 Hexenhut $\phi(u,v) = (\alpha \cdot \frac{\cos v}{\sqrt{u}}, \alpha \cdot \frac{\sin v}{\sqrt{u}}, u)$ where $\alpha^2 = \frac{2}{3\sqrt{3}}$

```
STUDENT > C := 2/(3*sqrt(3));
```

$$C := \frac{2}{9}\sqrt{3}$$

```
STUDENT > F:=(u,v)->[ C/sqrt(u)*cos(v), C/sqrt(u)*sin(v), u];
```

$$F := (u, v) \to \left[\frac{C\cos(v)}{\sqrt{u}}, \frac{C\sin(v)}{\sqrt{u}}, u\right]$$

```
STUDENT > gik(F);
```

$$(u, v) \to \left[\left[\frac{1}{27}\frac{1+27u^3}{u^3}, 0\right], \left[0, \frac{4}{27}\frac{1}{u}\right]\right]$$

```
STUDENT > hik(F);
```

$$(u, v) \to \left[\left[-\frac{3}{2}\frac{1}{u^3\sqrt{\frac{1+27u^3}{u^4}}}, 0\right], \left[0, \frac{2}{u\sqrt{\frac{1+27u^3}{u^4}}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \to \left[\left[-\frac{3}{2}\frac{1}{u(1+27u^3)}, 0\right], \left[0, 2\frac{u}{1+27u^3}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \to \left[\left[0, -\frac{1}{2}\frac{1}{u}\right], \left[-\frac{1}{2}\frac{1}{u}, 0\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \to \left[\left[\left[-\frac{3}{2}\frac{1}{u(1+27u^3)}, 0\right], \left[0, 2\frac{u}{1+27u^3}\right]\right], \left[\left[0, -\frac{1}{2}\frac{1}{u}\right], \left[-\frac{1}{2}\frac{1}{u}, 0\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \to \left[\left[-\frac{81}{2}\frac{1}{(1+27u^3)\sqrt{\frac{1+27u^3}{u^4}}}, 0\right], \left[0, \frac{27}{2}\frac{1}{\sqrt{\frac{1+27u^3}{u^4}}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \to -\frac{2187}{4}\frac{u^4}{(1+27u^3)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \to \frac{27}{4} \frac{-2 + 27\,u^3}{\sqrt{\frac{1 + 27\,u^3}{u^4}}\,(1 + 27\,u^3)}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$(u, v) \to -\frac{2187}{4} \frac{u^4}{(1 + 27\,u^3)^2}$$

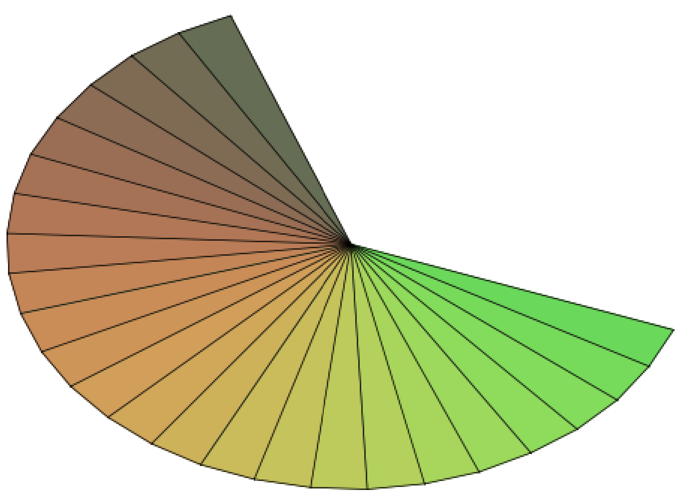
```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -\frac{27}{4} \frac{2 - 27\,u^3 + \sqrt{\frac{u^4\,(4 + 27\,u^3)^2}{(1 + 27\,u^3)^3}}\,\sqrt{\frac{1 + 27\,u^3}{u^4}} + 27\sqrt{\frac{u^4\,(4 + 27\,u^3)^2}{(1 + 27\,u^3)^3}}\,\sqrt{\frac{1 + 27\,u^3}{u^4}}\,u^3}{\sqrt{\frac{1 + 27\,u^3}{u^4}}\,(1 + 27\,u^3)}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to \frac{27}{4} \left(-2 + 27\,u^3 + \sqrt{\frac{u^4\,(4 + 27\,u^3)^2}{(1 + 27\,u^3)^3}}\,\sqrt{\frac{1 + 27\,u^3}{u^4}} + 27\sqrt{\frac{u^4\,(4 + 27\,u^3)^2}{(1 + 27\,u^3)^3}}\,\sqrt{\frac{1 + 27\,u^3}{u^4}}\,u^3 \right) \Big/ \left(\sqrt{\frac{1 + 27\,u^3}{u^4}}\,(1 + 27\,u^3) \right)$$

```
STUDENT > plot3d(F(t,s),t=-2..2,s=-2..2,style=PATCH,scaling=constr
          ained);
```



More info: • ▷ Wheeler: p.21 ff

21 Hexe $\phi(u,v) = (\sqrt{u}\cdot \cos v, \sqrt{u}\cdot \sin v, u)$

```
STUDENT > F:=(u,v)->[ sqrt(u)*cos(v), sqrt(u)*sin(v), u];
```

$$F := (u, v) \rightarrow [\sqrt{u}\cos(v), \sqrt{u}\sin(v), u]$$

```
STUDENT > gik(F);
```

$$(u, v) \rightarrow \left[\left[\frac{1}{4}\frac{1+4u}{u}, 0\right], [0, u]\right]$$

```
STUDENT > hik(F);
```

$$(u, v) \rightarrow \left[\left[\frac{1}{2}\frac{1}{u\sqrt{1+4u}}, 0\right], \left[0, 2\frac{u}{\sqrt{1+4u}}\right]\right]$$

```
STUDENT > christoffel_list[1](gik(F));
```

$$(u, v) \rightarrow \left[\left[-\frac{1}{2}\frac{1}{u(1+4u)}, 0\right], \left[0, -2\frac{u}{1+4u}\right]\right]$$

```
STUDENT > christoffel_list[2](gik(F));
```

$$(u, v) \rightarrow \left[\left[0, \frac{1}{2}\frac{1}{u}\right], \left[\frac{1}{2}\frac{1}{u}, 0\right]\right]$$

```
STUDENT > christoffel_list(gik(F));
```

$$(u, v) \rightarrow \left[\left[\left[-\frac{1}{2}\frac{1}{u(1+4u)}, 0\right], \left[0, -2\frac{u}{1+4u}\right]\right], \left[\left[0, \frac{1}{2}\frac{1}{u}\right], \left[\frac{1}{2}\frac{1}{u}, 0\right]\right]\right]$$

```
STUDENT > shapeoperator(F);
```

$$(u, v) \rightarrow \left[\left[\frac{2}{(1+4u)^{3/2}}, 0\right], \left[0, \frac{2}{\sqrt{1+4u}}\right]\right]$$

```
STUDENT > gauss_curvature(F);
```

$$(u, v) \rightarrow \frac{4}{(1+4u)^2}$$

```
STUDENT > mean_curvature(F);
```

$$(u, v) \rightarrow 2\frac{2u+1}{(1+4u)^{3/2}}$$

```
STUDENT > intrinsic_curvature(gik(F));
```

$$(u, v) \rightarrow \frac{4}{(1+4u)^2}$$

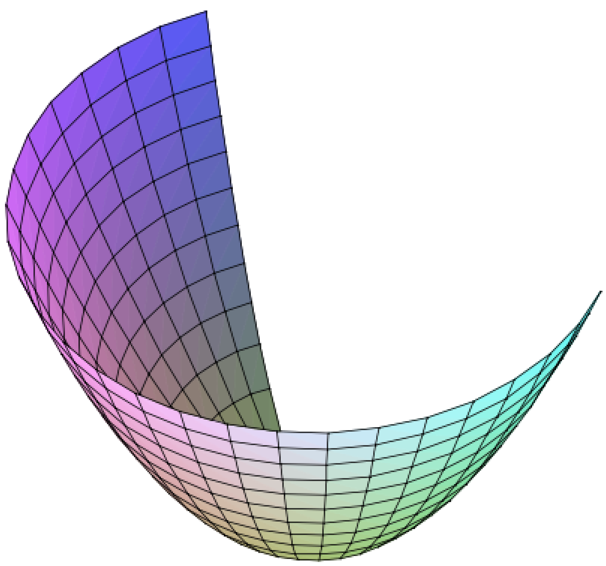
```
STUDENT > princ_curvature[1](F);
```

$$(u, v) \to -2\,\frac{-1 - 2u + 2\sqrt{\frac{u^2}{(1+4u)^3}}\sqrt{1+4u} + 8\sqrt{\frac{u^2}{(1+4u)^3}}\sqrt{1+4u}\,u}{(1+4u)^{3/2}}$$

```
STUDENT > princ_curvature[2](F);
```

$$(u, v) \to 2\,\frac{1 + 2u + 2\sqrt{\frac{u^2}{(1+4u)^3}}\sqrt{1+4u} + 8\sqrt{\frac{u^2}{(1+4u)^3}}\sqrt{1+4u}\,u}{(1+4u)^{3/2}}$$

```
STUDENT > plot3d(F(t,s),t=-2..2,s=-2..2,style=PATCH,scaling=constr
          ained);
```



More info: no, it's a little variant of 'hexenhut' given by my.

22 Bianchi surface $\phi(u,v) = (\frac{2\cdot\sqrt{v^2+1}\cdot\sin(u)\cdot\cos(-v+\arctan(v))}{1+v^2\sin(u)^2}, \ldots$

```
STUDENT > F:=(u,v)->[ (
          2*sqrt(v^2+1)*sin(u)*cos(-v+arctan(v)))/(1+v^2*sin(u)^2)
          ,

          (-2*sqrt(v^2+1)*sin(u)*sin(-v+arctan(v)))/(1+v^2*sin(u)^
          2),
                       ln(tan(1/2*u)) + (2*cos(u))/( 1+v^2*sin(u)^2
          ) ];
```

$$F := (u,v) \to \left[2\,\frac{\sqrt{v^2+1}\,\sin(u)\cos(-v+\arctan(v))}{1+v^2\sin(u)^2},\right.$$
$$\left.-2\,\frac{\sqrt{v^2+1}\,\sin(u)\sin(-v+\arctan(v))}{1+v^2\sin(u)^2}, \ln\left(\tan\left(\frac{1}{2}u\right)\right)+2\,\frac{\cos(u)}{1+v^2\sin(u)^2}\right]$$

```
STUDENT > assume(u>0); assume(v>0);
STUDENT > gik(F);
```

$$(u,v) \to \left[[-(v^4\cos(u)^4 - 2v^4\cos(u)^2 + 2v^2\cos(u)^2 - 2v^2 + v^4 + 1) \middle/ (\right.$$
$$v^4\cos(u)^6 - 3v^4\cos(u)^4 - 2v^2\cos(u)^4 + 3v^4\cos(u)^2 + 4v^2\cos(u)^2 + \cos(u)^2 - v^4 - 2v^2 - 1$$
$$\left.), 0], \left[0, -4\,\frac{(-1+\cos(u)^2)\,v^2}{1+2v^2-2v^2\cos(u)^2+v^4-2v^4\cos(u)^2+v^4\cos(u)^4}\right]\right]$$

```
STUDENT > hik(F);
```

$$(u,v) \to \left[\left[\left[2\,(v^4\cos(u)^4 - 2v^4\cos(u)^2 + 2v^2\cos(u)^2 - 2v^2 + v^4 + 1)\,v^2\sin(u) \middle/ \left((\right.\right.\right.\right.$$
$$v^8\cos(u)^{10} - 5v^8\cos(u)^8 + 10v^8\cos(u)^6 - 10v^8\cos(u)^4 + 5v^8\cos(u)^2 - v^8 - 4v^6\cos(u)^8$$
$$+ 16v^6\cos(u)^6 - 24v^6\cos(u)^4 + 16v^6\cos(u)^2 - 4v^6 + 6v^4\cos(u)^6 - 18v^4\cos(u)^4$$
$$+ 18v^4\cos(u)^2 - 6v^4 - 4v^2\cos(u)^4 + 8v^2\cos(u)^2 - 4v^2 + \cos(u)^2 - 1)$$
$$\left.\left.\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\right), 0\right], \left[0, 2\right.$$
$$(v^4\cos(u)^4 - 2v^4\cos(u)^2 + 2v^2\cos(u)^2 - 2v^2 + v^4 + 1)\sin(u)\,v^2 \middle/ \left((6v^4 + 12v^6\cos(u)^4\right.$$

$$- 12\, v^6 \cos(u)^2 - 4\, v^8 \cos(u)^6 + 6\, v^8 \cos(u)^4 - 4\, v^8 \cos(u)^2 - 4\, v^6 \cos(u)^6 + v^8 \cos(u)^8 + 4\, v^6 + v^8 + 6\, v^4 \cos(u)^4 - 4\, v^2 \cos(u)^2 - 12\, v^4 \cos(u)^2 + 1 + 4\, v^2) \sqrt{\frac{(-v^2 + v^2 \cos(u)^2 + 1)^2\, v^2}{(v^2 \cos(u)^2 - v^2 - 1)^4}} \Bigg) \Bigg] \Bigg]$$

STUDENT > `christoffel_list[1](gik(F));`

$$(u, v) \to \left[\left[\frac{\sin(u)\cos(u)\,(v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + 4\,v^2\cos(u)^2 + v^4 - 4\,v^2 - 1)}{v^4\cos(u)^6 - 3\,v^4\cos(u)^4 + 3\,v^4\cos(u)^2 - v^4 - \cos(u)^2 + 1}, -4\,\frac{v\,(-1+\cos(u)^2)}{v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + v^4 - 1} \right], \left[-4\,\frac{v\,(-1+\cos(u)^2)}{v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + v^4 - 1}, -4\,\frac{\sin(u)\cos(u)\,(-1+\cos(u)^2)\,v^2}{v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + v^4 - 1} \right] \right]$$

STUDENT > `christoffel_list[2](gik(F));`

$$(u, v) \to \left[\left[\frac{-v^2 + v^2\cos(u)^2 + 1}{v\,(1 + v^2 - \cos(u)^2 - 2\,v^2\cos(u)^2 + v^2\cos(u)^4)}, \frac{(-v^2 + v^2\cos(u)^2 + 1)\cos(u)\sin(u)}{1 + v^2 - \cos(u)^2 - 2\,v^2\cos(u)^2 + v^2\cos(u)^4} \right], \left[\frac{(-v^2 + v^2\cos(u)^2 + 1)\cos(u)\sin(u)}{1 + v^2 - \cos(u)^2 - 2\,v^2\cos(u)^2 + v^2\cos(u)^4}, -\frac{-v^2 + v^2\cos(u)^2 + 1}{v\,(v^2\cos(u)^2 - v^2 - 1)} \right] \right]$$

STUDENT > `christoffel_list(gik(F));`

$$(u, v) \to \left[\left[\left[\frac{\sin(u)\cos(u)\,(v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + 4\,v^2\cos(u)^2 + v^4 - 4\,v^2 - 1)}{v^4\cos(u)^6 - 3\,v^4\cos(u)^4 + 3\,v^4\cos(u)^2 - v^4 - \cos(u)^2 + 1}, -4\,\frac{v\,(-1+\cos(u)^2)}{v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + v^4 - 1} \right], \left[-4\,\frac{v\,(-1+\cos(u)^2)}{v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + v^4 - 1}, -4\,\frac{\sin(u)\cos(u)\,(-1+\cos(u)^2)\,v^2}{v^4\cos(u)^4 - 2\,v^4\cos(u)^2 + v^4 - 1} \right] \right], \left[\left[\frac{-v^2 + v^2\cos(u)^2 + 1}{v\,(1 + v^2 - \cos(u)^2 - 2\,v^2\cos(u)^2 + v^2\cos(u)^4)}, \frac{(-v^2 + v^2\cos(u)^2 + 1)\cos(u)\sin(u)}{1 + v^2 - \cos(u)^2 - 2\,v^2\cos(u)^2 + v^2\cos(u)^4} \right], \left[\frac{(-v^2 + v^2\cos(u)^2 + 1)\cos(u)\sin(u)}{1 + v^2 - \cos(u)^2 - 2\,v^2\cos(u)^2 + v^2\cos(u)^4}, -\frac{-v^2 + v^2\cos(u)^2 + 1}{v\,(v^2\cos(u)^2 - v^2 - 1)} \right] \right] \right]$$

STUDENT > `shapeoperator(F);`

$$-2\frac{\sin(u)\,v^2}{\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}\,(1+2\,v^2-2\,v^2\cos(u)^2+v^4-2\,v^4\cos(u)^2+v^4\cos(u)^4)}},$$

$$0\Bigg],\Bigg[0, -\frac{1}{2}\sin(u)\,(v^4\cos(u)^4-2\,v^4\cos(u)^2+2\,v^2\cos(u)^2-2\,v^2+v^4+1)\Bigg/\Bigg(\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,(v^4\cos(u)^6-3\,v^4\cos(u)^4-2\,v^2\cos(u)^4+3\,v^4\cos(u)^2+4\,v^2\cos(u)^2+\cos(u)^2-v^4-2\,v^2-1)\Bigg)\Bigg]\Bigg]$$

STUDENT > gauss_curvature(F);

$$-1$$

STUDENT > mean_curvature(F);

$$(u,v)\to -\frac{1}{4}\sin(u)\,(v^4\cos(u)^4-2\,v^4\cos(u)^2+6\,v^2\cos(u)^2+v^4-6\,v^2+1)\Bigg/\Bigg(\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,(v^4\cos(u)^6-3\,v^4\cos(u)^4-2\,v^2\cos(u)^4+3\,v^4\cos(u)^2+4\,v^2\cos(u)^2+\cos(u)^2-v^4-2\,v^2-1)\Bigg)$$

STUDENT > intrinsic_curvature(gik(F));

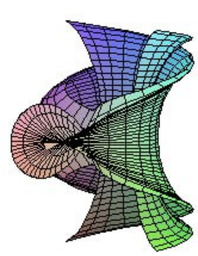
$$-1$$

STUDENT > princ_curvature[1](F);

$$(u,v)\to -\frac{1}{4}\Bigg(\sin(u)\,v^4\cos(u)^4-2\sin(u)\,v^4\cos(u)^2+\sin(u)\,v^4+6\sin(u)\,v^2\cos(u)^2-6\sin(u)\,v^2+\sin(u)+\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\,\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^4\cos(u)^6$$

$$
\begin{aligned}
& -3\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^4\cos(u)^4 \\
& -2\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^2\cos(u)^4 \\
& +3\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^4\cos(u)^2 \\
& +4\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^2\cos(u)^2 \\
& +\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,\cos(u)^2 \\
& -\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^4 \\
& -2\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,v^2 \\
& -\sqrt{\frac{(v^2\cos(u)^2-v^2-1)^4}{\sin(u)^2\,(-v^2+v^2\cos(u)^2+1)^2\,v^2}}\sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\Bigg) \Bigg/ \Bigg(\\
& \sqrt{\frac{(-v^2+v^2\cos(u)^2+1)^2\,v^2}{(v^2\cos(u)^2-v^2-1)^4}}\,(\\
& v^4\cos(u)^6-3\,v^4\cos(u)^4-2\,v^2\cos(u)^4+3\,v^4\cos(u)^2+4\,v^2\cos(u)^2+\cos(u)^2-v^4-2\,v^2-1 \\
&)\Bigg)
\end{aligned}
$$

Plot of BIANCHI surface with Maple Vr4.



23 Bibliography

References

[1] Banchoff, T. F. & Lovett, S. (^{3}2023): *Differential Geometry of Curves and Surfaces.* London: Chapman and Hall.

[2] Lindner, W. (2023): *Elementary Differential Geometry using* Eigenmath.
url: `ttps://georgeweigt.github.io/Lindner/Elementary-Differential-Geometry-of-Surfaces.pdf`

[3] Pressley, A. (2001): *Elementary Differential Geometry.* London: Springer.

[4] Reckziegel, H. et al. (1998): *Elementare Differentialgeometrie mit Maple.* Braunschweig: Vieweg.

[5] Rejbrand, A. (2023): *The Rejbrand Encyclopaedia of Curves and Surfaces.*
url:`https://trecs.se/surfaces.php`