ECE212 Homework 8

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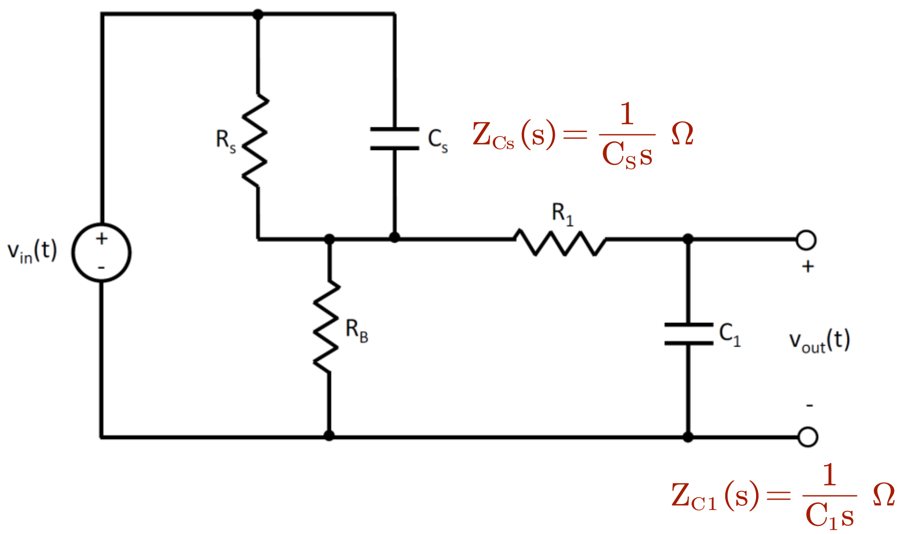
Q1.



a) Assume that the initial voltage of both capacitors is zero. Find $v_{out}(s)/v_{in}(s)$, expressing the numerator and denominator polynomials in expanded form with coefficients that are functions of the circuit elements in Fig. 1

As the circuit has zero initial conditions for all dynamic elements, its circuit diagram remains as it is with its elements expressed with their impedances in s domains (see above diagram). By inspections, R_s and C_s are connected in parallel (this circuit equivalent is denoted as Z_{eq1}) while R_B and C_1 with R_2 are also connected in parallel (this circuit equivalent is denoted as Z_{eq2}). Z_{eq1} and Z_{eq2} are then connected in series with the input voltage source. Therefore, the output voltage can be calculated using the principle of voltage division.

$$Z_{eq1} = Z_{Rs}||Z_{Cs} = \frac{Z_{Rs} \cdot Z_{Cs}}{Z_{Rs} + Z_{Cs}} = \frac{R_s \frac{1}{C_s s}}{R_s + \frac{1}{C_s s}}$$

$$Z_{eq2} = Z_{R_B}||\left(Z_{Cs} + Z_{R_1}\right) = R_B||\left(\frac{1}{C_1 s} + R_1\right) = \frac{R_B\left(\frac{1}{C_1 s} + R_1\right)}{\frac{1}{C_1 s} + R_1 + R_B}$$

$$V_{out}(s) = \frac{Z_{eq2}}{Z_{eq1}+Z_{eq2}} \cdot \frac{Z_{C1}}{Z_{R1}+Z_{C1}} \cdot V_{in}(s)$$

$$= \frac{\dfrac{R_B\left(\frac{1}{C_1 s}+R_1\right)}{\frac{1}{C_1 s}+R_1+R_B}}{\dfrac{R_s\frac{1}{C_s s}}{R_s+\frac{1}{C_s s}}+\dfrac{R_B\left(\frac{1}{C_1 s}+R_1\right)}{\frac{1}{C_1 s}+R_1+R_B}} \cdot \frac{\frac{1}{C_1 s}}{R_1+\frac{1}{C_1 s}} \cdot V_{in}(s)$$

$$= \frac{\dfrac{R_B\left(\frac{1}{C_1 s}+R_1\right)}{\frac{1}{C_1 s}+R_1+R_B}}{\dfrac{R_s\frac{1}{C_s s}\left(\frac{1}{C_1 s}+R_1+R_B\right)+R_B\left(\frac{1}{C_1 s}+R_1\right)\left(R_s+\frac{1}{C_s s}\right)}{\left(R_s+\frac{1}{C_s s}\right)\left(\frac{1}{C_1 s}+R_1+R_B\right)}} \cdot \frac{\frac{1}{C_1 s}}{R_1+\frac{1}{C_1 s}} \cdot V_{in}(s)$$

$$= \frac{R_B\left(R_s+\frac{1}{C_s s}\right)\frac{1}{C_1 s}}{R_s\frac{1}{C_s s}\left(\frac{1}{C_1 s}+R_1+R_B\right)+R_B\left(\frac{1}{C_1 s}+R_1\right)\left(R_s+\frac{1}{C_s s}\right)} \cdot V_{in}(s)$$

$$= \frac{R_B\left(R_s s+\frac{1}{C_s}\right)\frac{1}{C_1}}{R_s\frac{1}{C_s}\left(\frac{1}{C_1}+R_1 s+R_B s\right)+R_B\left(\frac{1}{C_1}+R_1 s\right)\left(R_s s+\frac{1}{C_s}\right)} \cdot V_{in}(s)$$

Therefore,

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R_B\left(R_s s+\frac{1}{C_s}\right)\frac{1}{C_1}}{R_s\frac{1}{C_s}\left(\frac{1}{C_1}+R_1 s+R_B s\right)+R_B\left(\frac{1}{C_1}+R_1 s\right)\left(R_s s+\frac{1}{C_s}\right)}$$

b) Substitute in R_s = 1.5 kΩ, C_s = 220 nF, R_B = 500 Ω, R_1 = 10 kΩ and C_1 = 22 nF, and find the poles and zeros of $v_{out}(s)/v_{in}(s)$

Given the definition that zeros are roots of the numerator, the zeros of $\frac{V_{out}(s)}{V_{in}(s)}$ *can be calculated when the numerator equals 0, e.g.,* $R_B\left(R_s s+\frac{1}{C_s}\right)\frac{1}{C_1}=0$:

$$R_B\left(R_s s+\frac{1}{C_s}\right)\frac{1}{C_1}=0 \quad \Rightarrow 500\left(1500\cdot s+\frac{1}{220\cdot 10^{-9}}\right)\frac{1}{22\cdot 10^{-9}}=0$$

$$s=-\frac{100000}{33}=-3030.3$$

Also, noticed that the denominator has a higher degree for variable s (degree of 2) while the numerator only has a degree of 1, this means that when s approaches infinity, the value of $\frac{V_{out}(s)}{V_{in}(s)}$ *will approach 0.*

$$\text{Given that } \lim_{s\to\infty}\frac{V_{out}(s)}{V_{in}(s)}=0 \text{ , another zero can be found at } s=\infty.$$

On the other hand, given the definition that poles are roots of the denominator, the poles of $\frac{V_{out}(s)}{V_{in}(s)}$ *can be calculated when the denominator equals 0, e.g.,* $R_s\frac{1}{C_s}\left(\frac{1}{C_1}+R_1s+R_Bs\right)+R_B\left(\frac{1}{C_1}+R_1s\right)\left(R_ss+\frac{1}{C_s}\right)=0$:

$$R_s\frac{1}{C_s}\left(\frac{1}{C_1}+R_1s+R_Bs\right)+R_B\left(\frac{1}{C_1}+R_1s\right)\left(R_ss+\frac{1}{C_s}\right)=0$$

$$1500\frac{1}{220\cdot10^{-9}}\left(\frac{1}{22\cdot10^{-9}}+10000s+500s\right)+500\left(\frac{1}{22\cdot10^{-9}}+10000s\right)\left(1500s+\frac{1}{220\cdot10^{-9}}\right)=0$$

$$7500000000s^2+\frac{14125000000000000s}{11}+\frac{500000000000000000000}{121}=0$$

$$s_1=-12825,\ s_2=-4295.9$$

Therefore, for the function $\frac{V_{out}(s)}{V_{in}(s)}$, two zeros can be found at $s=-3030.3$ and $s=\infty$, and two poles can be found at $s=-12825,\ s=-4295.9$.

c) If $v_{in}(t)$ = 10u(t), find $v_{out}(t)$.

By Laplace transform:

$$V_{in}(t)=10u(t)\ \Rightarrow\ V_{in}(s)=10\mathscr{L}\{u(t)\}=\frac{10}{s}$$

Using result from part a and part b:

$$V_{out}(s)=\frac{R_B\left(R_ss+\frac{1}{C_s}\right)\frac{1}{C_1}}{R_s\frac{1}{C_s}\left(\frac{1}{C_1}+R_1s+R_Bs\right)+R_B\left(\frac{1}{C_1}+R_1s\right)\left(R_ss+\frac{1}{C_s}\right)}\cdot V_{in}(s)$$

$$=\frac{\frac{125000000000000}{121}(33s+100000)}{7500000000(s+12825)(s+4295.9)}\cdot\frac{10}{s}$$

$$=1377.41\frac{(33s+100000)}{(s+12825)(s+4295.9)}\cdot\frac{1}{s}$$

$$= 1377.41\left(0.00182 \cdot \frac{1}{s} - 0.00295 \cdot \frac{1}{s+12825} + 0.00114 \cdot \frac{1}{s+4295.9}\right)$$

$$= 2.50 \cdot \frac{1}{s} - 4.07 \cdot \frac{1}{s+12825} + 1.57 \cdot \frac{1}{s+4295.9}$$

Using inverse Laplace transform to covert from s-domain to time domain:

$$V_{out}(t) = \mathscr{L}^{-1}\{V_{out}(s)\}$$

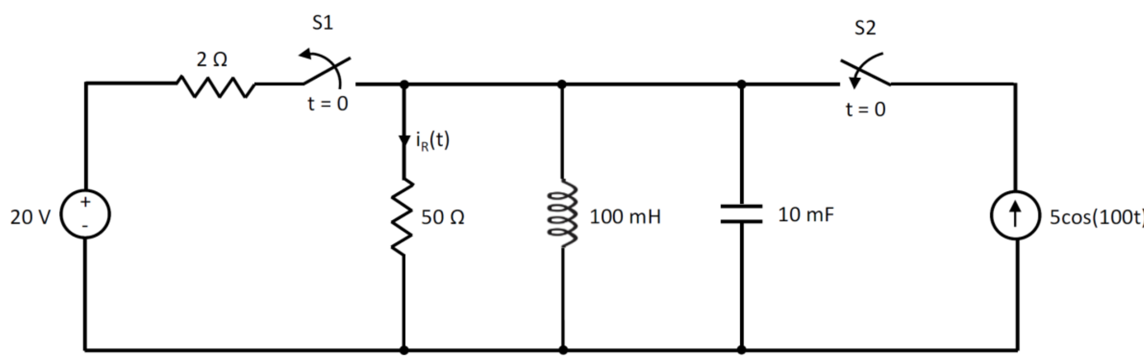
$$= \mathscr{L}^{-1}\left\{2.50 \cdot \frac{1}{s} - 4.07 \cdot \frac{1}{s+12825} + 1.57 \cdot \frac{1}{s+4295.9}\right\}$$

$$= 2.50\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 4.07\mathscr{L}^{-1}\left\{\frac{1}{s+12825}\right\} + 1.57\mathscr{L}^{-1}\left\{\frac{1}{s+4295.9}\right\}$$

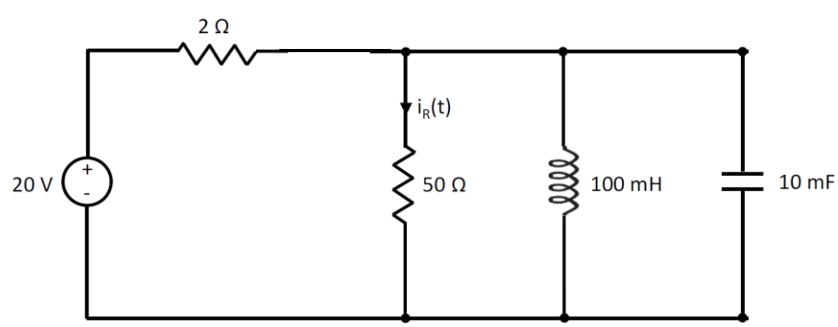
$$= 2.50 \cdot u(t) - 4.07 \cdot e^{-12825t} \cdot u(t) + 1.57 \cdot e^{-4295.9t} \cdot u(t)$$

$$= \left(2.50 - 4.07 \cdot e^{-12825t} + 1.57 \cdot e^{-4295.9t}\right)u(t)$$

Q2.



a) **Draw the circuit for t < 0 and find the initial conditions of inductor current and capacitor voltage.**



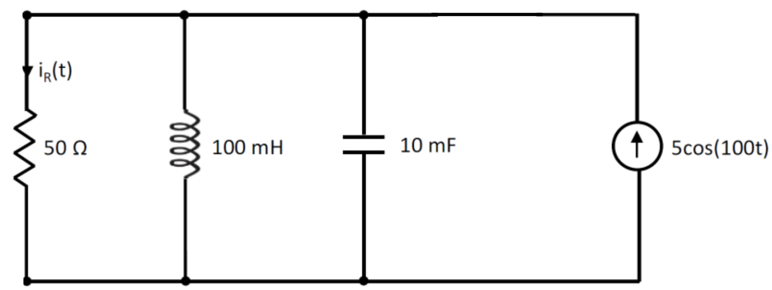
At steady state, capacitor can be seen as an open circuit, and inductor can be seen as a short circuit. As a result, in the redrawn circuit, the 50Ω *resister is shorted by the inductor at steady state. Therefore:*

$$V_C(0^-) = 0V$$

$$I_L(0^-) = \frac{20}{2} = 10A$$

b) Draw the circuit for t > 0 in the s-domain.

Redrawn the schematic as below (in time domain):



Converting time domain circuit elements to s-domain:

Current source:

$$I_{sc}(s) = \mathcal{L}\{I_{sc}(t)\} = 5\mathcal{L}\{\cos(100t)\} = \frac{5s}{s^2 + 10000}\ A$$

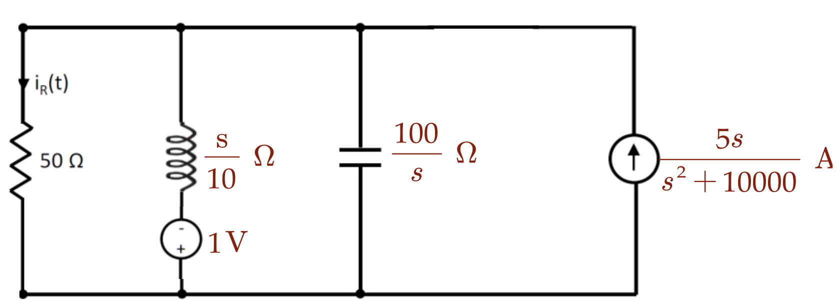
Inductor impedance:

$$Z_L(s) = Ls = 0.1s = \frac{s}{10}\ \Omega$$

Capacitor impedance:

$$Z_C(s) = \frac{1}{Cs} = \frac{1}{0.01s} = \frac{100}{s}\ \Omega$$

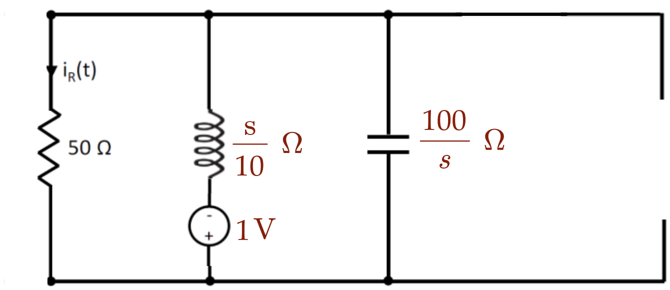
Also, as seen in part a, the initial condition for inductor is non-zero. Therefore, in s-domain, a new voltage source that has value of $V_{L0} = Li_L(0^-) = \frac{1}{10} \cdot 10 = 1V$ *must be added in the schematics in s-domain as follows:*



c) Find the value of $i_R(t)$ for t > 0.

Using superposition to calculate both the zero-input response and the zero-state response:

Zero-Input response (where all original sources are turned off):



$$By\ KVL:\ V_{R||C} + V_L = -1$$

$$Z_{eq} = \frac{50 \cdot \frac{100}{s}}{50 + \frac{100}{s}} = \frac{\frac{5000}{s}}{\frac{50s+100}{s}} = \frac{100}{s+2}\ \Omega$$

$$I_R(s) = \frac{V_R(s)}{R}$$

$$= -1 \cdot \frac{\frac{100}{s+2}}{\frac{100}{s+2} + \frac{s}{10}} \cdot \frac{1}{50}$$

$$= -\frac{1}{50} \cdot \frac{1000}{1000 + s^2 + 2s}$$

$$= -\frac{20}{s^2 + 2s + 1000} = -\frac{20}{3\sqrt{111}} \cdot \frac{3\sqrt{111}}{(s+1)^2 + 999}$$

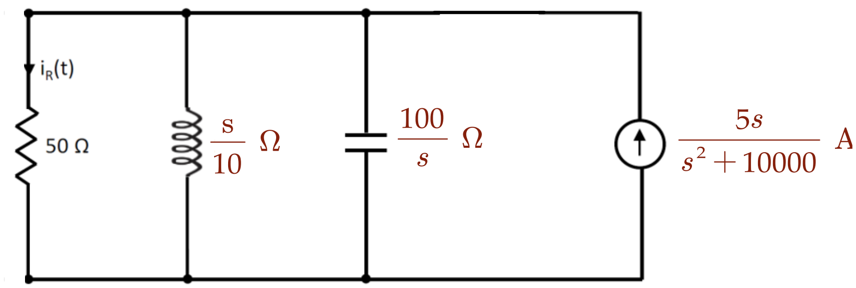
$$i_R(t) = \mathscr{L}^{-1}\{I_R(s)\}$$

$$= \mathscr{L}^{-1}\left\{-\frac{20}{3\sqrt{111}} \cdot \frac{3\sqrt{111}}{(s+1)^2 + 999}\right\}$$

$$= -\frac{20}{3\sqrt{111}} \cdot \mathscr{L}^{-1}\left\{\frac{3\sqrt{111}}{(s+1)^2 + 999}\right\}$$

$$= -0.633e^{-t}\sin(31.607t)\ A = -0.633e^{-t}\cos(-31.607t + 90°)\ A$$

Zero-State response (where all sources resulting from initial conditions are turned off):



Using current division principle:

$$Z_{L||C} = \frac{\frac{s}{10}\cdot\frac{100}{s}}{\frac{s}{10}+\frac{100}{s}} = \frac{10}{\frac{s^2+1000}{10s}} = \frac{100s}{s^2+1000}$$

$$I_R(s) = \frac{Z_{L||C}}{Z_{L||C}+Z_R}\cdot I_{SC}(s) = \frac{\frac{100s}{s^2+1000}}{\frac{100s}{s^2+1000}+50}\cdot I_{SC}(s) = \frac{2s}{(s+1)^2+999}\cdot I_{SC}(s)$$

Calculating the sinusoidal steady-state response:

$$T(s) = \frac{I_R(s)}{I_{SC}(s)} = \frac{2s}{(s+1)^2+999}$$

$$\Rightarrow \quad T(j\omega) = T(j100) = \frac{j200}{(j100+1)^2+999} = 0.0222\angle -88.73°$$

$$\Rightarrow \quad i_R(t) = 5\cdot 0.0222\cos(100t+0°-88.73°)A = 0.111\cos(100t-88.73°)A$$

Therefore, by superposition principle, the total response is:

$$i_R(t) = -0.633e^{-t}\cos(-31.607t+90°) \ +0.111\cos(100t-88.73°) \ A$$

d) Use phasor analysis to verify the magnitude and phase of the steady-state component of $i_R(t)$ which you found in part c.

Converting circuit into phasor domains:

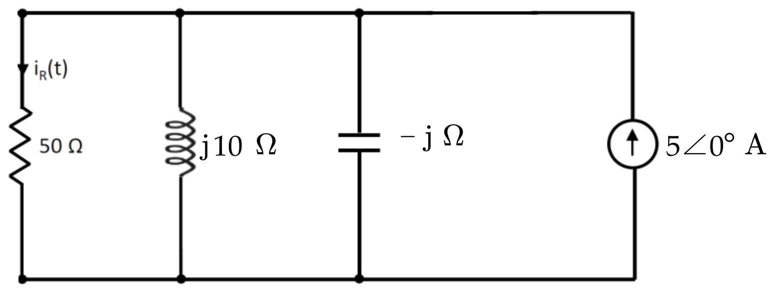
Current source:

$$I_{sc} = 5\angle 0°$$

Impedances:

$$Z_L = j\omega L = j\cdot 100\cdot 0.1 = j10 \ \Omega$$

$$Z_C = -j\frac{1}{\omega C} = -j\cdot\frac{1}{100\cdot 0.01} = -j \ \Omega$$



Calculating $I_R(t)$:

$$Z_{RLC} = Z_R||Z_L||Z_C \;\Rightarrow\; \frac{1}{Z_{RLC}} = \frac{1}{50} + \frac{1}{j10} + \frac{1}{-j} \;\Rightarrow\; Z_{RLC} = 1.11\angle -88.73°\ \Omega$$

$$V_R = I_{SC} \cdot Z_{RLC} = 5\angle 0° \cdot 1.11\angle -88.73° = 5.55\angle -88.73°\ V$$

$$I_R = \frac{V_R}{Z_R} = \frac{5.55\angle -88.73°}{50} = 0.111\angle -88.73°\ A$$

Converting from phasor domain to time domain:

$$I_R = 0.111\angle -88.73°\ A \;\Rightarrow\; i_R(t) = 0.111\cos(100t - 88.73°)\ A$$

The magnitude and phase of the steady-state component of $i_R(t)$ calculated in phasor domain aligns with the s-domain analysis from part c.