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Math 132
HW8: Finite Differences

Problem1a:
Find the constants c1 and c2 such that y(x)=c1*x+c2/x^2 - 3/10*sin(ln(x))-1/10*cos(ln(x)) is a solution to the ODE y''=-2/xy' + 2/x^2y + sin(ln(x))

```
y(1)=1; 1 = c_1+c_2 -1/10; c_1=1+1/10-c_2;
y(2)=2; 2 = c_1*2+c_2/4 -0.3*float(sin(ln(2)))-0.1*float(cos(ln(2)));
c_2 = 4*(2-2*c_1+0.268612273);
c_1 = 1.1392; c_2 = -0.0392;
```

$$y(1) = 1$$
$$1 = c_1 + c_2 - \frac{1}{10}$$
$$c_1 = \frac{11}{10} - c_2$$
$$y(2) = 2$$
$$2 = 2\,c_1 + \frac{c_2}{4} - 0.268612273$$
$$c_2 = 9.074449092 - 8\,c_1$$
$$c_1 = 1.1392$$
$$c_2 = -0.0392$$

Problem 1b: use your linear finite difference code to solve the problem above and show the results as in table 11.5. See attachment for figure.

```
Results:
  j  x(j)   w(j)   y(j)     err
  1 1.0000 1.0000 1.0000 0.0000e+000
  2 1.1000 1.0926 1.0926 2.6859e-005
  3 1.2000 1.1870 1.1871 3.8166e-005
  4 1.3000 1.2833 1.2834 4.0527e-005
  5 1.4000 1.3814 1.3814 3.7665e-005
  6 1.5000 1.4811 1.4812 3.1752e-005
  7 1.6000 1.5824 1.5824 2.4084e-005
  8 1.7000 1.6850 1.6850 1.5448e-005
  9 1.8000 1.7889 1.7889 6.3293e-006
 10 1.9000 1.8939 1.8939 2.9722e-006
 11 2.0000 2.0000 2.0000 1.2273e-005
```

Problem 2: See my code for my implimentation of NonLinearFiniteDifferences

Problem 3: 11.3 exercise 9 prove that the jacobian has a unique solution if h < 2/L where L=max{p(x_i)}

```
Theorem 6.31 says that if (I) |a_i_i|>=|a_i_i+1|+|a_i_i-1|, and
(II) |a_i_i|>|a_i_i-1| and (III)a_i_i+1 & a_i_i-1 are nonzero
then matrix A is non-singular and thus has a unique solution.
To make our Jacobian Matrix fit the requirements of theorem6.31
we can restrict h < 2/L where L=max{p(x_i)} that means
```

```
h < 2/L; h/2*abs(p(x_i) )< 1;
```

$$h < \frac{2}{L}$$
$$\frac{h\,|p(x_i)|}{2} < 1$$

```
(I) In our Jacobian the super and sub diagonals are non-zero
(II) |a_i_i|>|a_i+1_i| since the main diagonal is always > 2
(III) |a_i_i|>=|a_i_i+1|+|a_i+1_i| since |a_i_i+1|+|a_i+1_i| is
equivalent to
```

```
abs(-1 - h/2*p(x_i)) + abs(-1+h/2*p(x_i))
```

$$\left|\frac{h\,p(x_i)}{2}-1\right|+\left|\frac{h\,p(x_i)}{2}+1\right|$$

```
but that is always < 2
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```
abs(-1 - h/2*p(x_i)) + abs(-1+h/2*p(x_i)) < 2
```

$$\left|\frac{h\,p(x_i)}{2}-1\right|+\left|\frac{h\,p(x_i)}{2}+1\right|<2$$

```
and the main diagonal is > 2. So all 3 requirements for the
Jacobian to have a unique solution, are satisfied. Meaning it
the Jacobian has a unique solution if h/2*abs(p(x_i) )< 1
Thus Theorem 11.3 is proved.
```

Problem 4: Use Corolaary 11.2 to show that the ODE has a unique solution, with boundary conditions y(1)=0.5 and y(2)=ln(2)
y" = -2/x*y' + 1/x^2*y - (2 + x - x^2 + x^2*ln(x))/x^4   1<=x<=2

```
The requirements of Corollary 11.2 are
(I) p(x),q(x),r(x) are continuous on x=[1,2]
    which is true since p,q,r have no singularities on
    x=[1,2]
(II) q(x)=1/x^2 > 0 for x=[1,2]
     also true because q = 2/x^2*ln(x). The denominator
     is squared so it must be positive for the real values
     [1,2]. Further ln(x) is > 0 on the interval [1,2] so q > 0
     on the same interval.
Then if y(x)=x^-2 - 2*x^-2 + ln(x) is a solution to the ODE, it
must be a unique solution to the ODE by Corollary 11.2
```

Problem 5 Original : Use linear finite differences to approximate the solution and compare to the analytic solution. See attachment for figure.

```
Results:
  j  x(j)   w(j)   y(j)     err
  1 1.0000 0.5000 0.5000 0.0000e+000
  2 1.0500 0.5505 0.5443 6.2524e-003
  3 1.1000 0.5901 0.5788 1.1271e-002
  4 1.1500 0.6210 0.6057 1.5217e-002
  5 1.2000 0.6450 0.6268 1.8225e-002
  6 1.2500 0.6636 0.6431 2.0413e-002
  7 1.3000 0.6777 0.6559 2.1881e-002
  8 1.3500 0.6884 0.6657 2.2715e-002
  9 1.4000 0.6962 0.6732 2.2990e-002
 10 1.4500 0.7017 0.6789 2.2771e-002
 11 1.5000 0.7054 0.6832 2.2115e-002
 12 1.5500 0.7075 0.6864 2.1072e-002
 13 1.6000 0.7084 0.6888 1.9684e-002
 14 1.6500 0.7084 0.6904 1.7992e-002
 15 1.7000 0.7076 0.6915 1.6028e-002
 16 1.7500 0.7061 0.6923 1.3822e-002
 17 1.8000 0.7041 0.6927 1.1401e-002
 18 1.8500 0.7018 0.6930 8.7881e-003
 19 1.9000 0.6991 0.6931 6.0050e-003
```

```
 20 1.9500 0.6962 0.6931 3.0701e-003
 21 2.0000 0.6931 0.6931 1.1102e-016
```

Problem 5 revised: The professor changed the ODE, but I did the work, so I wanted to include the original and the edited problem 5 data

```
Results:
   j  x(j)   w(j)     y(j)     err
   1 1.0000 -1.0000 -1.0000 0.0000e+000
   2 1.0500 -0.8130 -0.8129 6.6555e-005
   3 1.1000 -0.6486 -0.6485 1.1017e-004
   4 1.1500 -0.5031 -0.5030 1.3737e-004
   5 1.2000 -0.3734 -0.3732 1.5275e-004
   6 1.2500 -0.2570 -0.2569 1.5958e-004
   7 1.3000 -0.1520 -0.1518 1.6023e-004
   8 1.3500 -0.0567 -0.0565 1.5642e-004
   9 1.4000 0.0302 0.0303 1.4940e-004
  10 1.4500 0.1098 0.1100 1.4010e-004
  11 1.5000 0.1831 0.1832 1.2920e-004
  12 1.5500 0.2508 0.2510 1.1722e-004
  13 1.6000 0.3136 0.3138 1.0454e-004
  14 1.6500 0.3721 0.3722 9.1427e-005
  15 1.7000 0.4267 0.4268 7.8103e-005
  16 1.7500 0.4779 0.4780 6.4719e-005
  17 1.8000 0.5260 0.5261 5.1392e-005
  18 1.8500 0.5713 0.5714 3.8204e-005
  19 1.9000 0.6141 0.6142 2.5217e-005
  20 1.9500 0.6547 0.6547 1.2473e-005
  21 2.0000 0.6931 0.6931 0.0000e+000
```

Problem6: Use your NonLinear Finite difference code to solve 11.4 exercise 3a.
See attachment for figure.

```
Results:
   j  x(j)  w(j)    y(j)      err
   1 1.0000 0.0000 0.0000 0.0000e+000
   2 1.1000 0.0952 0.0953 7.6296e-005
   3 1.2000 0.1822 0.1823 1.1943e-004
   4 1.3000 0.2622 0.2624 1.3986e-004
   5 1.4000 0.3363 0.3365 1.4422e-004
   6 1.5000 0.4053 0.4055 1.3684e-004
   7 1.6000 0.4699 0.4700 1.2065e-004
   8 1.7000 0.5305 0.5306 9.7647e-005
   9 1.8000 0.5877 0.5878 6.9247e-005
  10 1.9000 0.6418 0.6419 3.6455e-005
  11 2.0000 0.6931 0.6931 0.0000e+000
```

Problem 7: sec 11.4 exercise 4b. See attachment for figure.

```
Results:
   j  x(j)  w(j)    y(j)     err
   1 1.0000 2.0000 2.0000 0.0000e+000
   2 1.1000 2.0092 2.0091 1.4066e-004
   3 1.2000 2.0335 2.0333 1.8175e-004
   4 1.3000 2.0694 2.0692 1.7621e-004
   5 1.4000 2.1144 2.1143 1.5141e-004
   6 1.5000 2.1668 2.1667 1.2117e-004
   7 1.6000 2.2251 2.2250 9.1881e-005
   8 1.7000 2.2883 2.2882 6.5871e-005
   9 1.8000 2.3556 2.3556 4.3151e-005
  10 1.9000 2.4263 2.4263 2.2190e-005
  11 2.0000 2.5000 2.5000 0.0000e+000
```

Problem 8: (BONUS) 11.4 Exercise 7

```
Again we can satisfy the requirements of theorem 6.31
Notice that for the NonLinear case the Jacobian is structured
a_i_i   =  2+h^2*f_y(x,y,yp)
a_i_i+1 = -1+h/2*f_yp(x,y,yp)
a_i+1_i = -1-h/2*f_yp(x,y,yp)
Then notice that if f(x,y,yp) was linear the tridiagonal
entries listed above reduce to
a_i_i   =  2+h^2*q(x)
a_i_i+1 = -1+h/2*p(x)
a_i+1_i = -1-h/2*p(x)
Also notice that in the beginning of section 11.4 L is
redefined as L = max{ f_yp(x,y,yp) }, in the linear case
reduces to max{ p(x) } as expected. Similar to the proof of
the unique solution in the linear case, if  h < 2/L
we can write

h < 2/L   thus  h/2*f_yp(x,y,yp) < 1

also since f_y(x,y,yp) > 0 on the interval [a,b] the main
diagonal entries a_i_i   =  2+h^2*f_y(x,y,yp) are always
greater than 2.

Next, the main diagonal is >= the largest off diagonal
-1+h/2*f_yp(x,y,yp)

Last, the main diagonal is >= the sum of the two corresponding
off diagonals entries since
|-1+h/2*f_yp(x,y,yp)| + |-1-h/2*f_yp(x,y,yp)| < 2
because h/2*f_yp(x,y,yp) < 1

The requirements of theorem 6.31 are satisfied, thus
the Jacobian is non-singular.
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