1 Introduction

This paper counts permutations which are restricted in the length of their longest consecutive subsequence (which we shall call *consecutivity*), for four naturally arising definitions of *consecutivity*, using a recursive method. Permutations are mapped to sequences of vertices (paths) in a polygon for a very visual approach, and categorized into five classes (with mutual overlap). Recursive steps describe the removal or merging of sides and/or vertices, which reduces the problem to counting shorter paths in smaller polygons.

2 The Four Types of Consecutivity

Let $\{p_k\} = p_1, p_2, ..., p_k$ be a permutation of k elements from $[1, ..., n]$. For positive integers $x, y \leq k$, let

$$\{p_k\}_x^y = \begin{cases} p_x, ..., p_y & x \leq y \\ p_x, ..., p_k, p_1, ..., p_y & x > y \end{cases}$$

be a *proper* and *improper subsequence* of $\{p_k\}$ respectively.

We say $\{p_k\}_x^y$ is *consecutive* if $x \leq y$ and $p_h - p_i = \pm 1$ for all $h, i \in dom(\{p_k\}_x^y)$ satisfying $h - i = 1$. We say $\{p_k\}_x^y$ is *wrap-consecutive* if $x \leq y$ and $p_h - p_i \equiv \pm 1 \pmod{n}$ for all $h, i \in dom(\{p_k\}_x^y)$ satisfying $h - i = 1$. We say $\{p_k\}_x^y$ is *cyclic-consecutive* if $p_h - p_i = \pm 1$ for all $h, i \in dom(\{p_k\}_x^y)$ satisfying $h - i \equiv 1 \pmod{k}$. We say $\{p_k\}_x^y$ is *wrap-cyclic-consecutive* if $p_h - p_i \equiv \pm 1 \pmod{n}$ for all $h, i \in dom(\{p_k\}_x^y)$ satisfying $h - i \equiv 1 \pmod{k}$.

Let * specify each of the four types of consecutiveness respectively: ‘’, ‘*wrap-*’, ‘*cyclic-*’, and ‘*wrap-cyclic-*’. Let the **consecutivity* of $\{p_k\}$ be the length of the longest **consecutive* subsequence it contains. When determining *wrap-consecutivity*, 1 and n are considered consecutive because n ‘wraps’ around to 1. When determining *cyclic-consecutivity*, p_1 and p_k are considered adjacent because p_k ‘cycles’ back to p_1. When determining *wrap-cyclic-consecutivity*, 1 and n are considered consecutive and p_1 and p_k are considered adjacent.

If the **consecutivity* of $\{p_k\}$ is less than j, we say $\{p_k\}$ has “no j **consecutive*”. For any class of partial permutations $\mathbb{A}$, let $\mathbb{A}_j$, $\mathbb{A}_j^w$, $\mathbb{A}_j^c$, $\mathbb{A}_j^{wc}$ be the subclass of $\mathbb{A}$ whose partial permutations have no j **consecutive*; let $\mathbb{A}_j(n,k)$, $\mathbb{A}_j^w(n,k)$, $\mathbb{A}_j^c(n,k)$, $\mathbb{A}_j^{wc}(n,k)$ be the set of $\{p_k\}$ belonging to $\mathbb{A}_j$, $\mathbb{A}_j^w$, $\mathbb{A}_j^c$, $\mathbb{A}_j^{wc}$ respectively (alternatively $\mathbb{A}(n, k, j, wrap, cyclic)$, where *wrap* and *cyclic* are boolean parameters specifying the type of consecutiveness); finally, let the single-stroke counterpart $\mathrm{A}_j(n,k)$, $\mathrm{A}_j^w(n,k)$, $\mathrm{A}_j^c(n,k)$, $\mathrm{A}_j^{wc}(n,k)$ denote the cardinality of $\mathbb{A}_j(n,k)$, $\mathbb{A}_j^w(n,k)$, $\mathbb{A}_j^c(n,k)$, $\mathbb{A}_j^{wc}(n,k)$ respectively.

Let $\mathbb{P}$ be the class of unrestricted partial permutations. This paper develops recursive formulas for n permute k with no j **consecutive* - $\mathrm{P}_j(n,k)$, $\mathrm{P}_j^w(n,k)$, $\mathrm{P}_j^c(n,k)$, and $\mathrm{P}_j^{wc}(n,k)$.

3 Permutations as Polygonal Paths

Consider an n-sided polygon with vertices labelled $[1,...,n]$. By convention, we label the bottom right vertex as 1 and label the remaining vertices with increasing index in the counter-clockwise direction. Every k-permutation $\{p_k\}$ can be represented by a unique, directed length-k path in this polygon through vertices $p_1, p_2, ..., p_k$. Regular convex polygons are used for aesthetic purposes.

Since we cannot draw all polygon sizes $n \in \mathbb{N}$ in a class, we will use abbreviated polygons with ...... representing the omitted sides. A class of partial permutations is denoted by an abbreviated polygon containing various indicators. The cardinality $\mathrm{A}_j^*(n,k)$ is denoted by an abbreviated polygon containing various indicators. A solid dot $\bullet$ indicates the starting vertex p_1. A hollow dot $\circ$ indicates the ending vertex p_k. A list of vertex numbers enclosed by square brackets denotes a side or undirected path section. A side or undirected path section is used *independently* if no sides are used immediately before or after it. Bolded lines indicate a path section that is used *independently*, in either forward or reverse order. The symbol $||$ on $[1,n]$ indicates *wrap* is false and thus $[1,n]$ can be used freely. The symbol $\circlearrowleft$ around $\bullet$ indicates *cyclic* is true.

Let $\mathbb{Z}$ be the class of partial permutations which start from one of $\{1,n\}$ and end at the other. Let $\overset{x\to y}{\mathbb{Z}}$ be the subclass of $\mathbb{Z}$ whose partial permutations start and end with exactly x and y consecutive sides respectively. Let $\mathbb{S}$ be the class of partial permutations which start from one of $\{1,n\}$. Let $\overset{x}{\mathbb{S}}$ be the subclass of $\mathbb{S}$ whose partial permutations start with exactly x consecutive sides in the direction opposite to $[1,n]$. Let $\mathbb{O}$ be the class of partial permutations which use $[1,n]$. Let $\overset{x\_y}{\mathbb{O}}$ be the subclass of $\mathbb{O}$ whose partial permutations use exactly x and y consecutive sides to the left and right of $[1,n]$ respectively.

Definition 3.0.1. *If $x < 0$, let $\mathrm{S}(n,k,j,x) = 0$. If $x < 0$ or $y < 0$, let $\mathrm{Z}(n,k,j,x,y) = \mathrm{O}(n,k,j,x,y) = \mathrm{U}(n,k,j,x,y) = 0$.*

Remark. Negative side counts may arise from summation. Since paths cannot use a negative number of sides, the functions are defined to be 0 in this case.

Visual representations of these classes are shown below. See the supplementary Python program to generate images showing all possible paths.

4 Counting Polygonal Paths $(n \geq 2)$

The formulas developed in this section only hold for non-degenerate polygons ($n > 2$ in all summands). Base cases, exceptions, and values will be given in the next section.

4.1 Vertex Removal Theorem Z. $\mathrm{Z}(n, k, j, [j-2], [j-2]) = \mathrm{S}(n-1, k-1, j, [j-2]) - \mathrm{Z}(n-1, k-1, j, j-2, [j-2])$.

Proof. We first find half of $\mathrm{Z}(n, k, j, [j-2], [j-2])$.

Starting from 1, the next vertex can be $[2...n]$. Since 1 cannot be used again, cut it away by constructing side $(n, 2)$, then count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from $[2...n]$ and end at n. There are $\frac{1}{2}\mathrm{S}(n-1, k-1, j, [j-2])$ paths from $[2...n]$ to n. The paths starting from 2 and using $j-2$ consecutive sides $((2, 3, ..., j-1, j))$ have been overcounted since the side $(1, 2)$ was used, so subtract $\mathrm{Z}(n-1, k-1, j, j-2, [j-2])$ such paths. Finally, multiply by 2 choices for the starting vertex (1 and n). $\square$

4.2 Side Removal Theorem Z. $\mathrm{Z}(n, k, j, x, y) = \mathrm{Z}(n-a-b, k-a-b, j, x-a, y-b)$ *for* $a \leq x,\ b \leq y$.

Remark. Since the a sides and b sides next to $(1, n)$ must be used, they can be "cut" away to form a smaller polygon without affecting path count.

4.2.1 Corollary. *Take* $a = x,\ b = y$*; then* $\mathrm{Z}(n, k, j, x, y) = \mathrm{Z}(n-x-y, k-x-y, j, 0, 0)$.

4.3 Square Theorem. $\mathrm{Z}(n, k, j, 0, 0) = \mathrm{Z}(n, k, j, [j-2], [j-2]) - \sum_{i=1}^{j-2} \mathrm{Z}(n, k, j, [j-2], i) - \sum_{i=1}^{j-2} \mathrm{Z}(n, k, j, i, [j-2]) + \mathrm{Z}(n-2, k-2, j, [j-3], [j-3])$.

Proof. This follows from Inclusion-Exclusion. $\square$

4.4 Column Theorem Z1. $\mathrm{Z}(n, k, j, x, [j-2]) = \mathrm{Z}(n-a, k-a, j, x-a, [j-2])$ *for* $a \leq x$.

Proof.

$$\begin{aligned}
\mathrm{Z}(n,k,j,x,[j-2]) &= \sum_{i=0}^{j-2} \mathrm{Z}(n,k,j,x,i) \\
&= \sum_{i=0}^{j-2} \mathrm{Z}(n-a,k-a,j,x-a,i) \qquad (4.2)\\
&= \mathrm{Z}(n-a,k-a,j,x-a,[j-2])
\end{aligned}$$

□

4.4.1 Corollary. *Take* $a = x-1$*; then* $\mathrm{Z}(n,k,j,x,[j-2]) = \mathrm{Z}(n-x+1,k-x+1,j,1,[j-2])$.

4.4.2 Corollary. $\mathrm{Z}(n,k,j,0,0) = \mathrm{Z}(n+1,k+1,j,1,[j-2]) - \sum_{i=1}^{j-2} \mathrm{Z}(n,k,j,0,i)$.

4.5 Column Theorem Z2. $\mathrm{Z}(n,k,j,1,[j-2]) = \mathrm{Z}(n-1,k-1,j,[j-2],[j-2]) - \sum_{i=1}^{j-2} \mathrm{Z}(n-1,k-1,j,i,[j-2])$.

Proof.

$$\begin{aligned}
\mathrm{Z}(n,k,j,1,[j-2]) &= \mathrm{Z}(n-1,k-1,j,0,[j-2]) \qquad (4.4)\\
&= \mathrm{Z}(n-1,k-1,j,[j-2],[j-2]) \\
&\quad - \sum_{i=1}^{j-2} \mathrm{Z}(n-1,k-1,j,i,[j-2])
\end{aligned}$$

□

4.6 Symmetry Theorem Z. $\mathrm{Z}(n,k,j,x,y) = \mathrm{Z}(n,k,j,y,x)$.

Proof. This follows from symmetry. □

4.7 Complement Theorem. $\mathrm{Z}(n,k,j,x,0) = \mathrm{S}(n-x-1,k-x-1,j,0) - \mathrm{Z}(n-1,k-1,j,x,[j-2])$.

Proof. We first consider half of $\mathrm{Z}(n,k,j,x,0)$, the paths starting from 1.

Since the path ends at n with 0 consecutive sides, construct side $(1,n-1)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from 1, use x consecutive sides, and end anywhere except $n-1$. There are $\frac{1}{2}\mathrm{S}(n-1,k-1,j,x)$ paths which start from 1, use x consecutive sides, and end at $[1+x...n-1]$. Subtract $\frac{1}{2}\mathrm{Z}(n-1,k-1,j,x,[j-2])$, the number of over-counted paths ending at $n-1$. Finally, multiply by two choices for the starting vertex (1 and n). □

4.8 Vertex Removal Theorem S1. $\mathrm{S}(n,k,j,0) = 2\mathrm{P}(n-1,k-1,j,false,false) - \mathrm{S}(n-1,k-1,j,[j-2])$.

Proof. Again, we first consider half of $\mathrm{S}(n,k,j,0)$, the paths starting from 1.

The second vertex can be $[3...n]$. Since 1 cannot be used again, cut the vertex away by constructing side $(n,2)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from $[3...n]$ and can use $(n,2)$ freely. There are $\mathrm{P}(n-1,k-1,j,false,false)$ paths starting from $[2...n]$ which can use $(n,2)$ freely. Subtract $\frac{1}{2}\mathrm{S}(n-1,k-1,j,[j-2])$ paths starting from 2. Finally, multiply by two choices for the starting vertex. □

4.9 Vertex Removal Theorem S2. $\mathrm{S}(n,k,j,[j-2]) = 2\mathrm{P}(n-1,k-1,j,false,false) - \mathrm{S}(n-1,k-1,j,j-2)$.

Proof.

Case 1. $j=2$

$\mathrm{S}(n,k,j,0) = 2\mathrm{P}(n-1,k-1,2,false,false) - \mathrm{S}(n-1,k-1,2,0)$. This is equivalent to $j=2$ in Theorem 4.8.

Case 2. $j>2$

First consider half of $\mathrm{S}(n,k,j,[j-2])$, the paths starting from 1.

The second vertex can be $[2...n]$. Since 1 cannot be used again, cut the vertex away by constructing side $(n,2)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from $[2...n]$ and can use $(n,2)$ freely. There are $\mathrm{P}(n-1,k-1,j,false,false)$ paths starting from $[2...n]$ which can use $(n,2)$ freely. The paths starting at 2 with $j-2$ consecutive sides $((2,3,...,j-1,j))$ have been over-counted since $(1,2)$ was used, so subtract $\frac{1}{2}\mathrm{S}(n-1,k-1,j,j-2)$ such paths. Finally, multiply the expression by two choices for the starting vertex.

□

4.10 Side Removal Theorem S. $\mathrm{S}(n,k,j,x) = \mathrm{S}(n-x,k-x,j,0)$.

Proof. As in 4.2, since the x sides next to the starting vertex must be used, cut them away by constructing side $(n,1+x)$ (or $(1,n-x)$) and count the number of length $k-x$ paths in the resulting $(n-x)$-gon which start at $1+x$ (or $n-x$) with 0 consecutive sides. □

4.11 Vertex Merge Theorem O. $\mathrm{O}(n,k,j,0,0) = 2\binom{k-1}{n-1}\mathrm{P}(n-1,k-1,j,true,false) - 2\sum_{x=0}^{j-2}\sum_{y=1-x}^{j-2-x}\mathrm{O}(n,k,j,x,y) + \sum_{x=0}^{j-1}\sum_{y=1-x}^{j-1}\mathrm{O}(n,k,j,x,y)$

Proof. WLOG, $(1,n)$ must be used in a subsequence of $[n-j+1,n-j+2,...,n-1,1,n,2,...,j-1,j]$ of length $l \geq 2$.

Case 1. $l=2$, $(1,n)$ used independently

Merge 1 and n into a single vertex N and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which use N. There are $\mathrm{P}(n-1,k-$

$1, j, true, false)$ length $k-1$ paths with no j *wrap-consecutive* in an $(n-1)$-gon, cumulatively using a vertex $(k-1)\mathrm{P}(n-1, k-1, j, true, false)$ times. Since $\mathbb{P}(n-1, k-1, j, true, false)$ is rotationally symmetric, each particular vertex in the $(n-1)$-gon is used by $\left(\frac{k-1}{n-1}\right)\mathrm{P}(n-1, k-1, j, true, false)$ paths.

The paths in which N has up to $j-2$ consecutive sides (isn't used independently) have been over-counted. Let there be x and y consecutive sides to the left and right of N respectively. There are $\mathrm{O}(n, k, j, x, y)$ over-counted paths for every pair x, y satisfying $1 \leq x+y \leq j-2$, so subtract $\sum_{x=0}^{j-2} \sum_{y=1-x}^{j-2-x} \mathrm{O}(n, k, j, x, y)$ paths.

Then multiply the expression by 2 since each length $k-1$ path using N independently maps to two length k paths containing $(1, n)$ (either 1 or n can be visited first).

Case 2. $l > 2$

Let the subsequence of $[n-j+1, n-j+2, ..., n-1, 1, n, 2, ..., j-1, j]$ contain x and y vertices to the left and right of 1 and n respectively. There is a distinct subsequence with $l > 2$ for every pair x, y satisfying $x \leq j-1$, $y \leq j-1$, and $x+y \geq 1$, so there are $\sum_{x=0}^{j-1} \sum_{y=1-x}^{j-1} \mathrm{O}(n, k, j, x, y)$ paths.

Remark. Some of the paths subtracted in case 1 are re-added in case 2.

□

4.12 Side Removal Theorem O. $\mathrm{O}(n, k, j, x, y) = \mathrm{O}(n-x-y, k-x-y, j, 0, 0)$.

Remark. As in 4.2 and 4.10, since the x and y sides next to n and 1 must be used, they can be "cut" away to form a smaller polygon without affecting path count.

4.13 Symmetry Theorem O. $\mathrm{O}(n, k, j, x, y) = \mathrm{O}(n, k, j, y, x)$.

Proof. This follows from symmetry. □

4.14 Vertex Merge Theorem U. $\mathrm{U}(n, k, j, 0, 0) = 2\left(\frac{k-1}{n-1}\right)\mathrm{P}(n-1, k-1, j, true, true) - 2\sum_{x=0}^{j-2} \sum_{y=1-x}^{j-2-x} [\mathrm{U}(n, k, j, x, y) - \sum_{d=j-x-y-1}^{j-x-2} \mathrm{Z}(n-x-1, k-x-1, j, y, d-1) - \sum_{d=j-x-y-1}^{j-y-2} \mathrm{Z}(n-y-1, k-y-1, j, x, d-1)] + \sum_{x=0}^{j-1} \sum_{y=1-x}^{j-1} [\mathrm{U}(n-x-y, k-x-y, j, 0, 0) + \sum_{d=j-1}^{j-1-x} \mathrm{Z}(n-y-1, k-y-1, j, 0, d-1) + \sum_{d=j-1}^{j-1-y} \mathrm{Z}(n-x-1, k-x-1, j, 0, d-1) - \sum_{d=j-x}^{j-2} \mathrm{Z}(n-x-y-1, k-x-y-1, j, 0, d-1) - \sum_{d=j-y}^{j-2} \mathrm{Z}(n-x-y-1, k-x-y-1, j, 0, d-1)] + 2\mathrm{Z}(n-1, k-1, j, 0, j-2)$.

Proof. As in 4.11, WLOG, $(1, n)$ must be used in a subsequence of $[n-j+1, n-j+2, ..., n-1, 1, n, 2, ..., j-1, j]$ of length $l \geq 2$.

Case 1. $l = 2$, $(1, n)$ used independently

Merge 1, n into a single vertex N and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which use N and have no j *wrap-cyclic-consecutive*. There are $\mathrm{P}(n-1, k-1, j, true, true)$ length $k-1$ paths with no j *wrap-cyclic-consecutive* in an $(n-1)$-gon, cumulatively using a vertex $(k-1)\mathrm{P}(n-1, k-1, j, true, true)$ times. Since $\mathbb{P}(n-1, k-1, j, true, true)$ is rotationally symmetric, each particular vertex in the $(n-1)$-gon is used by $\left(\frac{k-1}{n-1}\right) \mathrm{P}(n-1, k-1, j, true, true)$ paths.

The paths in which N has up to $j-2$ consecutive sides (isn't used independently) have been over-counted. There are $\mathrm{U}(n, k, j, x, y)$ paths with x and y consecutive sides to the left and right of N respectively. Those which satisfy $1 \leq x+y \leq j-2$ *and* have no j *wrap-cyclic-consecutive* after merging 1 and n have been over-counted in $\left(\frac{k-1}{n-1}\right) \mathrm{P}(n-1, k-1, j, true, true)$.

For those starting $n-x$ and ending $n-x-1$, the first and last vertices connect to form a consecutive section that may exceed $j-1$ vertices in length after 1 and n merge. Let the path end at $n-x-1$ with d consecutive vertices ($d-1$ consecutive sides). After merging, the joint section will exceed $j-1$ vertices if $d+x+y+1 \geq j$. The paths with $d \geq j-x-1$ were not counted in $\mathbb{U}(n, k, j, x, y)$, since the d vertices would connect with $[n-x, n-x+1, ..., n-1, n]$ to exceed $j-1$ consecutive vertices. Thus there are $\frac{1}{2}\sum_{d=j-x-y-1}^{j-x-2} \mathrm{Z}(n-x-1, k-x-1, j, y, d-1)$ paths in $\mathbb{U}(n, k, j, x, y)$ which start at $n-x$ and end at $n-x-1$ and exceed $j-1$ consecutive vertices after merging. There are $\frac{1}{2}\sum_{d=j-x-y-1}^{j-x-2} \mathrm{Z}(n-x-1, k-x-1, j, y, d-1)$ more which start at $n-x-1$ and end at $n-x$. Similarly, there are $\sum_{d=j-x-y-1}^{j-y-2} \mathrm{Z}(n-y-1, k-y-1, j, x, d-1)$ more such paths with endpoints $1+y$, $2+y$.

Thus there are $\mathrm{U}(n, k, j, x, y) - \sum_{d=j-x-y-1}^{j-x-2} \mathrm{Z}(n-x-1, k-x-1, j, y, d-1) - \sum_{d=j-x-y-1}^{j-y-2} \mathrm{Z}(n-y-1, k-y-1, j, x, d-1)$ over-counted paths in $\left(\frac{k-1}{n-1}\right) \mathbb{P}(n-1, k-1, j, true, true)$ for every pair x, y satisfying $1 \leq x+y \leq j-2$, so subtract $\sum_{x=0}^{j-2}\sum_{y=1-x}^{j-2-x}[\mathrm{U}(n, k, j, x, y) - \sum_{d=j-x-y-1}^{j-x-2} \mathrm{Z}(n-x-1, k-x-1, j, y, d-1) - \sum_{d=j-x-y-1}^{j-y-2} \mathrm{Z}(n-y-1, k-y-1, j, x, d-1)]$ paths.

Then multiply the expression by 2 since each length $k-1$ path using N independently maps to two length k paths containing $(1, n)$ (either 1 or n can be visited first).

There is a special case which the merge method has not counted. The paths starting with $[1, n]$ and not using $(n, 2)$ and ending at $n-1$ with $j-2$ consecutive sides, although having no j *cyclic-consecutive*, end up with a *cyclic-consecutivity* of j after the merge, and thus were not counted in $\left(\frac{k-1}{n-1}\right) \mathrm{P}(n-1, k-1, j, true, true)$. Since $(n, 2)$ isn't used, we can remove n and shift the starting vertex to 1 in the reduced polygon without affecting path count. Cut n away by constructing side $(n-1, 1)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which *start* at 1 and end at $n-1$ with $j-2$ consecutive sides. Add $\frac{1}{2}\mathrm{Z}(n-1,k-1,j,0,j-2)$ such paths. Add $\frac{1}{2}\mathrm{Z}(n-1,k-1,j,0,j-2)$ more for paths in the opposite direction (from $n-1$ to 1). Add $\mathrm{Z}(n-1,k-1,j,0,j-2)$ more for the right hand side.

Case 2. $l > 2$

Let the subsequence of $[n-j+1, n-j+2, ..., n-1, 1, n, 2, ..., j-1, j]$ contain x and y vertices to the left and right of 1 and n respectively, where $x \leq j-1$, $y \leq j-1$, and $x+y \geq 1$. Cut the subsequence away by constructing side $(n-x, 1+y)$ and count the number of length $k-x-y$ paths in the resulting $(n-x-y)$-gon which use $(n-x, 1+y)$ independently and have no j *cyclic-consecutive* prior to reducing. As in 4.12, there are $\mathrm{U}(n-x-y, k-x-y, j, 0, 0)$ paths which have no j *cyclic-consecutive* in the reduced polygon.

$n-x$ is treated as a vertex (length 1) in the reduced polygon even if $x=0$, in which case $n-x$ connects to 2 and cannot form a consecutive chain with $n-1$. The paths starting at $1+y$ with 0 consecutive sides and ending at $n-1$ with $j-2$ consecutive sides (or starting at $n-1$ with $j-2$ consecutive sides and ending at $1+y$ with 0 consecutive sides) meet the consecutivity restriction but have not been counted in $\mathrm{U}(n-x-y, k-x-y, j, 0, 0)$, so add $\mathrm{Z}(n-y-1, k-y-1, j, 0, j-2)$ if $x=0$. Similarly, add $\mathrm{Z}(n-x-1, k-x-1, j, 0, j-2)$ if $y=0$.

As in 4.15, for paths with endpoints $n-x$ and $n-x-1$, the endpoints connect to form a consecutive section that may exceed $j-1$ vertices in length (since $x-1$ consecutive sides have been used next to $n-x$). Let the path end at $n-x-1$ with d consecutive vertices ($d-1$ consecutive sides). The joint start-end section will exceed $j-1$ vertices if $d+x \geq j$. Only the paths with $d+1 < j$ have been counted in $\mathrm{U}(n-x-y, k-x-y, j, 0, 0)$. Thus we subtract $\sum_{d=j-x}^{j-2} \mathrm{Z}(n-x-y-1, k-x-y-1, j, 0, d-1)$ over-counted paths. Similarly, subtract $\sum_{d=j-y}^{j-2} \mathrm{Z}(n-x-y-1, k-x-y-1, j, 0, d-1)$ for the right hand side.

There is a distinct subsequence with $l > 2$ for every pair x, y satisfying $x \leq j-1$, $y \leq j-1$, and $x+y \geq 1$, thus there are $\sum_{x=0}^{j-1} \sum_{y=1-x}^{j-1} \mathrm{U}(n-x-y, k-x-y, j, 0, 0) + \sum_{d=j-1}^{j-1-x} \mathrm{Z}(n-y-1, k-y-1, j, 0, d-1) + \sum_{d=j-1}^{j-1-y} \mathrm{Z}(n-x-1, k-x-1, j, 0, d-1) - \sum_{d=j-x}^{j-2} \mathrm{Z}(n-x-y-1, k-x-y-1, j, 0, d-1) - \sum_{d=j-y}^{j-2} \mathrm{Z}(n-x-y-1, k-x-y-1, j, 0, d-1)$ paths.

$\square$

4.15 Side Removal Theorem U. $\mathrm{U}(n,k,j,x,y) = \mathrm{U}(n-x-y, k-x-y, j, 0, 0) - \sum_{d=j-x-1}^{j-2} \mathrm{Z}(n-x-1, k-x-1, j, y, d-1) - \sum_{d=j-y-1}^{j-2} \mathrm{Z}(n-y-1, k-y-1, j, x, d-1)$.

Proof. Since x and y sides next to n and 1 must be used, cut them away by constructing side $(n-x, 1+y)$ and count the number of length $k-x-y$ paths in the resulting $(n-x-y)$-gon which use $(n-x, 1+y)$ independently and have no j *cyclic-consecutive* prior to reducing. As in 4.12, there are $\mathrm{U}(n-x-y, k-x-y, j, 0, 0)$ paths which have no j *cyclic-consecutive* in the reduced polygon.

For the paths starting $n-x$ and ending $n-x-1$, the first and last vertices connect to form a consecutive section that may exceed $j-1$ vertices in length (since x consecutive sides have been used next to n). Let the path end at $n-x-1$ with d consecutive vertices ($d-1$ consecutive sides). The joint start-end section will exceed $j-1$ vertices if $d \geq j-x-1$. The paths with $d \geq j-1$ have not been over-counted in $\mathrm{U}(n-x-y, k-x-y, j, 0, 0)$ since they already connect with $n-x-1$ to exceed $j-1$ consecutive vertices in the reduced polygon. Thus we subtract $\sum_{d=j-x-1}^{j-2} \frac{1}{2}\mathrm{Z}(n-x-1, k-x-1, j, y, d-1)$ over-counted paths. Subtract another $\sum_{d=j-x-1}^{j-2} \frac{1}{2}\mathrm{Z}(n-x-1, k-x-1, j, y, d-1)$ for paths in the opposite direction (starting $n-x-1$ and ending $n-x$). Similarly, some of the paths with endpoints $1+y$, $2+y$ have been over-counted, so subtract $\sum_{d=j-y-1}^{j-2} \mathrm{Z}(n-y-1, k-y-1, j, x, d-1)$ paths for the other side. □

4.16 Symmetry Theorem U. $\mathrm{U}(n, k, j, x, y) = \mathrm{U}(n, k, j, y, x)$.

Proof. This follows from symmetry. □

4.17 Restricted Consecutivity Theorem. $\mathrm{P}(n, k, j, false, false) = \mathrm{P}(n, k, j, true, false) + \sum_{x=0}^{j-2} \sum_{y=j-2-x}^{j-2} \mathrm{O}(n, k, j, x, y)$.

Proof. To find the number of paths where *wrap* is false ($(1, n)$ can be used without restriction), add the paths where $(1, n)$ *is not* in a consecutive section exceeding $j-1$ vertices and the paths where $(1, n)$ *is* in a consecutive section exceeding $j-1$ vertices. □

4.18 Restricted Wrap-Consecutivity Theorem. $\mathrm{P}(n, k, j, true, false) = n(\mathrm{P}(n-1, k-1, j, false, false) - \mathrm{S}(n-1, k-1, j, j-2))$.

Proof.

Case 1. $j = 2$

Starting from vertex s, the next vertex can be any of $[s+2, ..., n, 1, ..., s-2]$. Since s cannot be used again, cut it away by constructing side $(s-1, s+1)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from $[s+2, ..., s-2]$ and can use $(s-1, s+1)$ without restriction. There are $\mathrm{P}(n-1, k-1, 2, false, false)$ length $k-1$ paths in the reduced polygon which start from $[s+1, ..., s-1]$ and can use $(s-1, s+1)$ without restriction.

The paths starting at $s-1$ and $s+1$ have been over-counted, so subtract $\mathrm{S}(n-1, k-1, 2, 0)$ such paths. Finally, multiply by n choices for s.

Case 2. $j > 2$

Starting from vertex s, the next vertex can be any of $[s+1, ..., n, 1, ..., s-1]$. Since s cannot be used again, cut it away by constructing side $(s-1, s+1)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from $[s+1, ..., s-1]$, can use $(s-1, s+1)$ without restriction, and have no j *wrap-consecutive* prior to reducing. There are $\mathrm{P}(n-1, k-1, j, false, false)$ length $k-1$ paths in the reduced polygon which start at $[s+1, ..., s-1]$ and can use $(s-1, s+1)$ without restriction. The paths starting at $s-1$ or $s+1$ and using $j-2$ consecutive sides in the direction opposite to $(s-1, s+1)$ have been over-counted (since s has been used), so subtract $\mathrm{S}(n-1, k-1, j, j-2)$ such paths. Finally, multiply by n choices for s.

□

4.19 Restricted Cyclic-Consecutivity Theorem. $\mathrm{P}(n, k, j, false, true) = \mathrm{P}(n, k, j, true, true) + \sum_{x=0}^{j-2} \sum_{y=j-2-x}^{j-2} [\mathrm{U}(n, k, j, x, y) + \mathrm{Z}(n, k, j, x, y)] + \sum_{x=0}^{j-3} \sum_{y=0}^{j-3-x} [\sum_{d=j-x-y-2}^{j-2-x} \mathrm{Z}(n-x-y, k-x-y, j, 0, d) + \sum_{d=j-x-y-2}^{j-2-y} \mathrm{Z}(n-x-y, k-x-y, j, 0, d)]$.

Proof. If 1 and n belong to the same *wrap-cyclic-consecutive* section, let l be the length of that section, otherwise let $l = 0$. To find the number of paths with no j *cyclic-consecutive* where $(1, n)$ can be used without restriction, add the paths where $l < j$ and where $l \geq j$.

Remark. Since we are considering *wrap-cyclic-consecutivity*, $(1, n)$ can belong to a consecutive section without being used in the path.

Case 1. $l < j$

There are $\mathrm{P}(n, k, j, true, true)$ paths where $(1, n)$ is not in a *cyclic-consecutive* section exceeding $j-1$ vertices in length.

Case 2. $l \geq j$

Count the paths with *wrap-cyclic-consecutivity* greater than or equal to j (but *cyclic-consecutivity* less than j). We divide further based on whether or not the path uses $(1, n)$.

Subcase 2.1. $(1, n)$ not used

For $(1, n)$ to be in a *cyclic-consecutive* section without being used, the path must have endpoints 1 and n. Let x and y consecutive sides be used next to n and 1 respectively. There are $\mathrm{Z}(n, k, j, x, y)$ paths where $l \geq j$ for every pair x, y satisfying $x \leq j-2$, $y \leq j-2$, and $x + y + 2 \geq j$, so $\sum_{x=0}^{j-2} \sum_{y=j-2-x}^{j-2} \mathrm{Z}(n, k, j, x, y)$ paths total.

Subcase 2.2. $(1, n)$ used

Again, let x and y sides be used next to n and 1 respectively.

Subsubcase 2.2.1. $x + y \geq j - 2$

There are $\mathrm{U}(n, k, j, x, y)$ paths where $l \geq j$ for every pair x, y satisfying $x \leq j - 2$, $y \leq j - 2$, and $x + y + 2 \geq j$, so $\sum_{x=0}^{j-2} \sum_{y=j-2-x}^{j-2} \mathrm{U}(n, k, j, x, y)$ paths total.

Subsubcase 2.2.2. $x + y < j - 2$

For the consecutive section containing $(1, n)$ to exceed $j-1$ vertices when $x+y < j-2$, the path must have endpoints $n-x$, $n-x-1$ or $1+y$, $2+y$. Consider the paths ending at $n - x - 1$ with d consecutive vertices ($d - 1$ consecutive sides). *Wrap-cyclic-consecutivity* will exceed $j - 1$ if $d + x + y + 2 \geq j$. *Cyclic-consecutivity* will be less than j if $d + x + 1 < j$. Thus add $\sum_{d=j-x-y-2}^{j-2-x} \mathrm{Z}(n - x - y - 1, k - x - y - 1, j, 0, d - 1)$ (or equivalently, $\sum_{d=j-x-y-2}^{j-2-x} \mathrm{Z}(n-x-y, k-x-y, j, 0, d))$ for every pair x, y satisfying $x+y < j-2$. Similarly, add $\sum_{d=j-x-y-2}^{j-2-y} \mathrm{Z}(n-x-y, k-x-y, j, 0, d)$ for every pair x, y satisfying $x + y < j - 2$.

□

4.20 Restricted Wrap-Cyclic-Consecutivity Theorem. $\mathrm{P}(n, k, j, true, true) = n(\mathrm{P}(n-1, k-1, j, false, false) - 2\mathrm{Z}(n, k, j, 0, j-2) - \sum_{x=0}^{j-2} \sum_{y=j-3-x}^{j-2} \mathrm{Z}(n - 1, k - 1, j, x, y))$.

Proof. Starting from s, the next vertex can be $[s+1, ..., n, 1, ..., s-1]$. Since s cannot be used again, cut it away by constructing side $(s-1, s+1)$ and count the number of length $k-1$ paths in the resulting $(n-1)$-gon which start from $[s+1, ..., s-1]$, can use $(s-1, s+1)$ without restriction, and have no j *wrap-cyclic-consecutive* prior to reducing. There are $\mathrm{P}(n-1, k-1, j, false, false)$ length $k-1$ paths in the reduced polygon which start at $[s+1, ..., s-1]$ and can use $(s-1, s+1)$ without restriction.

The paths starting with 0 consecutive sides (from s) and ending at one of $s-1$, $s+1$ with $j-2$ consecutive sides have been overcounted since $s \pm 1$ joins with s to form a consecutive section of length j, so subtract $\mathrm{Z}(n, k, j, 0, j-2)$ paths. Subtract another $\mathrm{Z}(n, k, j, 0, j-2)$ for paths in the opposite direction.

In the reduced polygon, the paths starting $s - 1$ with x consecutive sides and ending $s+1$ with y consecutive sides where $x \leq j-2$, $y \leq j-2$, and $x + y+3 \geq j$ have been overcounted since $s+1$ joins with s to form a consecutive section that is too long, so subtract $\sum_{x=0}^{j-2} \sum_{y=j-3-x}^{j-2} \frac{1}{2}\mathrm{Z}(n - 1, k - 1, j, x, y)$ paths. Subtract another $\sum_{x=0}^{j-2} \sum_{y=j-3-x}^{j-2} \frac{1}{2}\mathrm{Z}(n - 1, k - 1, j, x, y)$ for paths in the opposite direction.

Finally, multiply by n choices for s. □

5 Base Cases, Exceptions, Computation

$\mathrm{U}(2,2,j,0,0) = 2$ for $j > 2$. However, case 2 of Theorem 4.14 expects $\mathrm{U}(2,2,j,0,0) = 0$ if $x + y \geq j$.

prove why the base cases obtained are correct (since formulas do not hold for $n \leq 2$). use inequalities to determine the bounds of summation