ASYMPTOTIC STRUCTURE OF TRANSLATING SOLITONS

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ABSTRACT. In this paper, we prove that each end of a noncompact translating soliton in $\mathbb{R}^3$ with bounded area ratio, finite topology, and asymptotic to self-shrinking cylinders, is smoothly asymptotic to a self-shrinking cylinder.

1. INTRODUCTION

A translating soliton is a hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ such that

$$t \mapsto \Sigma - te_{n+1}$$

is a mean curvature flow, i.e., the normal component of the velocity at each point is equal to the mean curvature at that point:

$$H = -\langle \nu, e_{n+1} \rangle . \tag{1.1}$$

Translating solitons from a special class of eternal solutions for the mean curvature flow that besides their own intrinsic interest, are models of slow singularity formation. Important examples of translating solitons in $\mathbb{R}^{n+1}$ includes: bowl soliton – a graphical, rotationally symmetric translating soliton; $\triangle$-wings – a graphic, non-rotationally symmetric translating soliton defined over any given slab of width $> \pi$; and "winglike" translating solutions – nonconvex translating solutions, made up of the union of two graphs.

Definition 1.1. We say a hypersurface $\Sigma \subset \mathbb{R}^3$ has *bounded area ratio* if

$$\text{Area}(B_R(x_0) \cap \Sigma) \leq \Lambda R^2$$

for all $x_0 \in \mathbb{R}^3$ and all $R > 0$.

In this paper, we give a geometric description of the asymptotic structure of translating solitons in $\mathbb{R}^3$ with bounded area ratio, finite topology, and asymptotic to self-shrinking cylinders. In particular we prove the following " multiplicity one theorem."

Theorem 1.2. *If Σ is an end of a noncompact translating soliton in $\mathbb{R}^3$ with bounded area ratio and finite topology. Moreover, Σ is asymptotic to self-shrinking cylinder, i.e., there* *exists a sequence* $\{\tau_i\}$ *with* $\tau_i \to \infty$ *such that*

$$\frac{\Sigma + \tau_i e_3}{\sqrt{\tau_i}} \to \mathcal{S} \text{ as varifolds},$$

where $\mathcal{S}$ *is a self-shrinking cylinder. Then* $\mathcal{S}$ *has multiplicity one.*

Remark 1.3. (1). Unlike self-shrinkers, translating solitons do not necessarily have Euclidean volume growth; see the example constructed in [20].
(2). By Huisken's monotonicity formula [13], White's stratification theorem [26], and the classification theorem of Abresh-Langer [1], we know that when $n = 2$, the translating soliton Σ must be asymptotic to $\mathbb{R} \times \mathbb{R}_{e_3}$ or $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}_{e_3}$. However, when Σ is asymptotic to $\mathbb{R} \times \mathbb{R}_{e_3}$, the multiplicity is not necessarily one; see $\triangle$-wings for example.

The classification problem for translating solitons is a central topic in the study of mean curvature flow. There are many significant results on characterizing translating solitons; cf. [10], [12], [25], [23], [2], [4], [11], [3], [5]. However, the abundance of glueing constructions for translating solitons with high genus (see [18],[19],[20], [9], [7], [21]), suggests a general classification is unlikely.

Theorem 1.2 serves an important step towards a thorough understanding of the nature of noncompact translating solitons in $\mathbb{R}^3$ with bounded area ratio and finite topology. It's an analogue to the result of the first author on self-shrinkers in $\mathbb{R}^3$ (see [24]).We will follow the idea in [24] to prove Theorem 1.2. In particular, first we will pick out the "good" pieces of Σ, i.e., pieces with bounded total curvature and very small (tends to 0 at infinity) principle curvatures; then we will show that the blowdown of those "good" pieces can be written as normal exponential multivalue graphs over $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}_{e_3}$; finally we will show that those graphs of "good" pieces can be extended to infinity and remain disjoint. Combining this fact with the assumption on the topology of Σ proves Theorem 1.2.

The organization of the paper is as follows. In Section 2, we recall some well known results of graphical mean curvature flow and translating solitons, these results will be needed in later sections. Section 3 is the most important section of this paper, in which we show that the blowdown of those "good" pieces can be written as normal exponential multivalue graphs over $\mathbb{S}^1(\sqrt{2}) \times \mathbb{R}_{e_3}$. Lemma 3.1 plays a key role in this section, which is based on a delicate proof by contradiction. In Section 4, we prove a curvature bound on pieces with bounded total curvature (Lemma 4.1). This result will be used to find "good" pieces in Section 5. In Section 6, we put everything together and prove Theorem 1.2.

2. PRELIMINARY

The main purpose of this section is to recall some important estimates for graphical mean curvature flow and translating solitons. We will use these estimates in later sections. More specifically, estimates for graphical mean curvature flow will be used in the proof of Lemma 3.1; while the curvature estimates for translating solitons (see Subsection 2.2) tell us on pieces with bounded total curvature, there are at most finite singularities, this fact will be used in the proof of Lemma 4.1.

2.1. **Mean curvature evolution of graphs.** We consider the immersion

$$\mathbf{F} : M^n \to \mathbb{R}^{n+1}$$

of n-dimensional hypersurfaces in $\mathbb{R}^{n+1}$. We say that M^n moves by mean curvature if there is a one parameter family $\mathbf{F}_t = \mathbf{F}(\cdot, t)$ of immersions with corresponding image $M_t = \mathbf{F}_t(M)$ such that

$$\frac{d}{dt}\mathbf{F}(p,t) = \mathbf{H}(p,t),\ \ p \in M. \tag{2.1}$$

If we assume that M can be written as a graph, then up to tangential diffeomorphisms, equation (2.1) corresponds to the following quasilinear equation

$$\frac{d}{dt}u = (\delta_{ij} - \frac{u_i u_j}{w^2})u_{ij}, \tag{2.2}$$

where $w = \sqrt{1+|Du|^2}$. We will need the following well known results about mean curvature flow of graphs in Section 3.

Lemma 2.1. *(See* [6]*) If $\rho \geq (2n+1)^{1/2}$ and the graph of $u : B_{\rho r} \times [0, r^2] \to \mathbb{R}$ flows by mean curvature, then*

$$\max_{B_r \times [0,r^2]} |u(x,t)| \leq \frac{(2n+1)r}{\rho} + \max_{B_{\rho r}} |u(x,0)|.$$

Lemma 2.2. *(See* [8]*) Let $R > 0$ and $0 \leq \theta < 1$. Then we have for $t \in [0,T]$ the estimate*

$$\sup_{B_{\theta R}(y_0)} |A|^2(t) \leq c(n)(1-\theta^2)^{-2}\left(\frac{1}{R^2} + \frac{1}{t}\right) \sup_{B_R(y_0)\times[0,t]} (1+|Du|^2)^2,$$

where $B_R(y_0) \subset \mathbb{R}^n$ denotes a ball in the hyperplane orthogonal to e_{n+1}.

Lemma 2.3. *(See* [8]*) Let $R > 0$, $0 \leq \theta < 1$ and $m \geq 0$. Then we have for $t \in [0,T]$ the estimate*

$$\sup_{B_{\theta R}(y_0)} |\nabla^m A|^2(t) \leq c_m \left(\frac{1}{R^2} + \frac{1}{t}\right)^{m+1}$$

where $B_R(y_0) \subset \mathbb{R}^n$ *denotes a ball in the hyperplane orthogonal to* e_{n+1} *and* $c_m = c_m(\theta, n, m, \sup_{B_R(y_0)\times[0,t]}(1+|Du|^2))$.

2.2. **A priori bounds for translating solitons.** In this subsection we will show that under our assumptions, the set of singularities is finite on any compact set of Σ.

Proposition 2.4. *(Choi-Schoen type curvature bound see* [23]*) Let* $\Sigma \subset \mathbb{R}^3$ *be a two-sided immersed translating soliton and let* $\mathcal{B}_\rho(P) \subset \Sigma$ *be disjoint from the cut locus of* P*. Then there exists* $\hat{\epsilon}, \tau < \frac{\sqrt{\hat{\epsilon}}}{2\pi} < \rho$ *such that if for all* $r_0 \leq \tau$*, there holds* $\int_{\mathcal{B}_{r_0}(P)} |A|^2 \leq \hat{\epsilon}$*, then for all* $0 < \sigma \leq r_0$*,* $y \in \mathcal{B}_{r_0-\sigma}(P)$ *we have* $|A|^2(y) \leq \sigma^{-2}$*.*

For any compact set K, in order to find the singular set $\mathfrak{S}$ on $K \cap \Sigma$, we define the measure ν by

$$\nu(U) = \int_{U\cap\Sigma} |A|^2 \leq C.$$

Applying the bounded area ratio and finite topology assumptions with the local Gauss-Bonnet estimate (see [15]) we get

$$\nu(K) \leq C. \tag{2.3}$$

We define the set $\mathfrak{S} = \{x \in K \cap \Sigma | \nu(x) \geq \hat{\epsilon}\}$. It follows immediately from (2.3) that $\mathfrak{S}$ contains at most $C\hat{\epsilon}^{-1}$ points.

Now given any $y \in (\Sigma \cap K)\backslash\mathfrak{S}$ we have $\nu(y) < \hat{\epsilon}$. Since ν is a radon measure and hence Borel regular, there exists some $0 < s < r_0$ such that $\nu(\mathcal{B}_s(y)) < \hat{\epsilon}$, therefore by Proposition 2.4 we have $|A|^2(y) \leq \frac{1}{s^2}$.

3. Conditional graphical property for translating solitons

This section is devoted to show an improvement on the oscillation of unit normal to a "good" piece of translating solitons (see Theorem 3.3). In order to establish Theorem 3.3, first, in Lemma 3.1 we show that along the translating direction e_3, the oscillation of unit normal is very small; then, combining Lemma 3.1 with the assumption that the total curvature is bounded on the "good" piece of translating solitons, we prove that for a specifically chosen direction e_2 which is perpendicular to e_3, the oscillation of unit normal is very small as well (see Proposition 3.2); finally, by a bootstrap machinery developed in [24], we prove Theorem 3.3.

In the following, we will denote the geodesic distance in the surface Σ by d_Σ. Given any unit vector $v \in \mathbb{R}^3\backslash\{0\}$ and $a, b \in \mathbb{R}$ with $a < b$, we let

$$I_v(a,b) = \{x \in \mathbb{R}^3 : a < x \cdot v < b\}.$$

Lemma 3.1. *There exist universal constant* $0 < \varepsilon_0, \eta_0 \ll 1$ *such that given* $\varepsilon \in (0, \varepsilon_0)$ *and* $\hat{z} \in \mathbb{R}^3$, *if* Σ *is a translating soliton in* $B^3_r(\hat{z})$ *with* $\hat{z} \in \Sigma$, $r = \frac{1}{\varepsilon^2}$, *and it satisfies*

$$\sup_{\hat{x}\in\Sigma} |\langle \nu(\hat{x}), e_3\rangle| < \varepsilon_0 \tag{3.1}$$

and

$$\sup_{\substack{\hat{x},\hat{x}'\in\Sigma \\ d_\Sigma(\hat{x},\hat{x}')<\varepsilon r}} |\nu(\hat{x}) - \nu(\hat{x}')| < \varepsilon_0. \tag{3.2}$$

Then Σ *contains the graph of a function* u *on the rectangle*

$$\eta_0 r\{I_{e_3}(-1,1) \times I_{e_2}(-\varepsilon,\varepsilon)\} + \hat{z}$$

with $|Du| \le 1$, *where* $e_2 = \nu(\hat{z}) \times e_3, N = e_3 \times e_2$. *Here we assume* Σ *satisfies* $H = -\langle \nu, e_3\rangle$.

Proof. We shall assume $\varepsilon_0 \le 10^{-12}$ in this proof.

Step 1. For any $\hat{x} \in \Sigma$ and $d_\Sigma(\hat{x},\hat{z}) < \frac{1}{\varepsilon}$, by equations (3.1) and (3.2) we have

$$\langle \nu(\hat{x}), N\rangle = \langle \nu(\hat{x}) - \nu(\hat{z}), N\rangle + \langle \nu(\hat{z}), N\rangle > 1 - 2\varepsilon_0.$$

Therefore, Σ can be written as a graph over the plane

$$\left\{ I_{e_3}[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}] \times I_{e_2}[-\frac{1}{2\varepsilon}, \frac{1}{2\varepsilon}] \right\} + \hat{z}$$

with $|Du| < 3\varepsilon_0^{1/2}$.

Step 2. Consider $v(x,y,t) = u(x+te_3, y)$ on $B^2_{\frac{1}{8\varepsilon}}(0)\times\left[-\frac{1}{8\varepsilon}, \frac{1}{8\varepsilon}\right]$, where $x = \langle \hat{x} - \hat{z}, e_3\rangle$, $y = \langle \hat{x} - \hat{z}, e_2\rangle$, and $B^2_{\frac{1}{8\varepsilon}}(0) \subset e_3e_2$ plane. By a straightforward calculation we get

$$v_t = \langle Du, e_3\rangle = -w\langle \nu, e_3\rangle = wH,$$

where $\nu = \frac{(-Du,1)}{\sqrt{1+|Du|^2}}$ and $w = \sqrt{1+|Du|^2}$. Thus

$$v_t = \left(\delta_{ij} - \frac{v_iv_j}{w^2}\right) v_{ij},$$

which yields that graph(v) is moving by mean curvature.

Applying Lemma 2.2 and 2.3, we conclude that in $B^3_{\frac{1}{16\varepsilon}}(\hat{z}) \cap \Sigma$,

$$|A| < C_0\varepsilon^{1/2} \text{ and } |\nabla A| < C_0\varepsilon.$$

Notice that our estimates are independent of $\hat{z}$, therefore we have

$$|A| < C_0\varepsilon^{1/2} \text{ and } |\nabla A| < C_0\varepsilon \text{ in } B^3_{r-\frac{2}{\varepsilon}}(\hat{z}).$$

Step 3. Let τ be the largest number such that Σ can be written as a graph of u over the domain

$$\Omega = \left\{ I_{e_3}(-\tau,\tau) \times I_{e_2}(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}) \right\} + \hat{z}$$

with $|Du| < 1$. We shall show that $\tau \geq \frac{\varepsilon_0^{1/2}}{20\varepsilon^2}$.

We prove by contradiction. In the following, we assume $\tau < \frac{\varepsilon_0^{1/2}}{20\varepsilon^2}$. By the maximality of τ, without loss of generality, we may assume $|Du(\tau,y_0)| = 1$, for some $y_0 \in (-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon})$. This implies that $\langle \nu, N\rangle^2(\tau,y_0) = 1/2$. Then, by equation (3.2), for any $\hat{x} \in \Sigma$ and $d_\Sigma(\hat{x},(\tau,y_0,u(\tau,y_0))) < \frac{5}{\varepsilon}$, we have

$$\frac{1}{\sqrt{2}} - 5\varepsilon_0 < \langle \nu(\hat{x}), N\rangle < \frac{1}{\sqrt{2}} + 5\varepsilon_0.$$

In other words, for any $(x,y) \in \left\{ I_{e_3}(\tau-\frac{2}{\varepsilon},\tau) \times I_{e_2}(-\frac{1}{2\varepsilon},\frac{1}{2\varepsilon}) \right\} + \hat{z}$ we have

$$\sqrt{2} - 10\varepsilon_0 < w(x,y) = \langle \nu, N\rangle^{-1} < \sqrt{2} + 10\varepsilon_0.$$

Step 4. Recall that in Step 2 we showed that the graph of $v(x,y,t) = u(x+te_3,y)$ is moving by mean curvature. Applying Lemma 2.1 with $r = \sqrt{\tau}$ and $\rho = \frac{\sqrt{5}}{\varepsilon_0^{1/4}}$, we get

$$|v(x,y,t)| \leq \frac{5\sqrt{\tau}}{\rho} + 3\varepsilon_0^{1/2}\rho\sqrt{\tau} = 4\varepsilon_0^{1/2}\rho\sqrt{\tau} \tag{3.3}$$

in $B^2_{\sqrt{\tau}} \times [0,\tau]$. Note that, here by the choice of r and ρ, we have $\rho\sqrt{\tau} < \frac{1}{2\varepsilon}$; the second term in the inequality comes from the gradient estimate in Step 1 and the assumption that $u(0) = 0$. Therefore, we conclude that when $u(0) = 0$ in $\{I_{e_3}[0,\tau] \times I_{e_2}[-\sqrt{\tau},\sqrt{\tau}]\} + \hat{z}$, we have

$$|u| < 4\varepsilon_0^{1/2}\rho\sqrt{\tau}. \tag{3.4}$$

In other words, the oscillation of u is less than $8\varepsilon_0^{1/2}\rho\sqrt{\tau}$.

Step 5. Let $g = \langle \nu, N\rangle$, by a straightforward calculation we can see that g satisfies

$$\triangle g = \langle \nabla g, e_3\rangle - |A|^2 g.$$

Let $w = g^{-1}$, then w satisfies

$$\triangle w = \langle \nabla w, e_3\rangle + |A|^2 w + \frac{2|\nabla w|^2}{w}.$$

For any cutoff function η, let $h = \eta w$ and $\mathcal{L} := \triangle - \langle \nabla, e_3 \rangle$, at the interior maximum point of h we have

$$\mathcal{L}h = w\left(|A|^2\eta + \mathcal{L}\eta\right) \leq 0.$$

Step 6. Consider $\hat{p} = (\tau - \frac{1}{\varepsilon}, 0, u(\tau - \frac{1}{\varepsilon}, 0))$ and let $u_0 = u(\tau - \frac{1}{\varepsilon}, 0) = 8\varepsilon_0^{1/2}\rho\sqrt{\tau}$, here we move the origin so that we have $u > 0$ in $U_p := \{I_{e_3}[\tau - \frac{2}{\varepsilon}, \tau] \times I_{e_2}[-\sqrt{\tau}, \sqrt{\tau}]\} + \hat{z}$, the neighborhood of the projection of $\hat{p}$.

Now let $\phi = \frac{-u}{2u_0} + 1 - \varepsilon^2\tilde{x}^2 - \frac{1}{\tau}\tilde{y}^2$, where $\tilde{x} = \langle \hat{x} - \hat{p}, e_3 \rangle$ and $\tilde{y} = \langle \hat{x} - \hat{p}, e_2 \rangle$; define $\eta = \left(e^{K\phi} - 1\right)_+$ where $K > 0$ to be determined. We want to point out that $\eta = 0$ on ∂U_p, hence $h = \eta w$ achieves its maximum at an interior point of U_p. By a direct calculation we get

$$\mathcal{L}\eta = Ke^{K\phi}\left\{K|\nabla\phi|^2 + \mathcal{L}\phi\right\}.$$

Since $\nabla\phi = \frac{-\nabla u}{2u_0} - \varepsilon^2\nabla\tilde{x}^2 - \frac{1}{\tau}\nabla\tilde{y}^2$, we get

$$|\nabla\phi|^2 \geq \frac{\langle \tau_i, N \rangle^2}{4u_0^2} + \frac{2\varepsilon^2\tilde{x}\langle \tau_i, N \rangle \langle \tau_i, e_3 \rangle}{u_0} + \frac{2\tilde{y}\langle \tau_i, N \rangle \langle \tau_i, e_2 \rangle}{\tau u_0}.$$

Note that

$$N = \langle N, \tau_1 \rangle \tau_1 + \langle N, \tau_2 \rangle \tau_2 + \langle N, \nu \rangle \nu,$$

and

$$0 = \langle N, e_i \rangle = \sum_{j=1}^{2} \langle N, \tau_j \rangle \langle \tau_j, e_i \rangle + \langle N, \nu \rangle \langle \nu, e_i \rangle.$$

Therefore we have,

$$|\nabla\phi|^2 \geq \frac{\left(1 - \frac{1}{w^2}\right)}{4u_0^2} - \frac{2\varepsilon|H|}{u_0 w} - \frac{2}{\sqrt{\tau}u_0 w}\langle \nu, e_2 \rangle.$$

Moreover, since

$$\langle \nu, e_2 \rangle^2 + \langle \nu, e_3 \rangle^2 + \langle \nu, N \rangle^2 = 1,$$

we obtain

$$\langle \nu, e_2 \rangle^2 \leq 1 - \frac{1}{w^2}.$$

Combining with the estimates on principal curvatures from Step 2 we get

$$|\nabla\phi|^2 \geq \frac{\left(1 - \frac{1}{w^2}\right)}{4u_0^2} - \frac{2C_0\varepsilon^{3/2}}{u_0 w} - \frac{2}{\sqrt{\tau}u_0 w}\sqrt{1 - \frac{1}{w^2}}.$$

Step 7. By our choice of u_0, ρ, and the assumption on $\sqrt{\tau}$, we can see that

$$\frac{1 - \frac{1}{w^2}}{20u_0^2} - \frac{2C_0\varepsilon^{3/2}}{u_0 w} > 0.$$

Moreover, by Step 3 we know that in the neighborhood under consideration we have

$$w^2 - 1 > 1 - 40\varepsilon_0.$$

Hence we have,

$$\frac{1-\frac{1}{w^2}}{20u_0^2} > \frac{2}{\sqrt{\tau}u_0 w}\sqrt{1-\frac{1}{w^2}},$$

which yields

$$|\nabla\phi|^2 \geq \frac{3\left(1-\frac{1}{w^2}\right)}{20u_0^2} > \frac{1}{16u_0^2}.$$

Since

$$\begin{aligned}
\mathcal{L}\phi &= -\frac{\mathcal{L}u}{2u_0} - \varepsilon^2\mathcal{L}\tilde{x}^2 - \frac{1}{\tau}\mathcal{L}\tilde{y}^2 \\
&= -2\varepsilon^2\tilde{x}\mathcal{L}\tilde{x} - 2\varepsilon^2\sum_i \langle\tau_i, e_3\rangle^2 - \sum_i \frac{2\langle\tau_i, e_2\rangle^2}{\tau} \\
&= 2\varepsilon^2\tilde{x} - 2\varepsilon^2\sum_i \langle\tau_i, e_3\rangle^2 - \sum_i \frac{2\langle\tau_i, e_2\rangle^2}{\tau} \\
&\geq -3\varepsilon,
\end{aligned} \tag{3.5}$$

here we used $\mathcal{L}u = \mathcal{L}\tilde{y} = 0$ and $\mathcal{L}\tilde{x} = -1$. Therefore we obtain that

$$\mathcal{L}\eta = Ke^{K\phi}\{K|\nabla\phi|^2 + \mathcal{L}\phi\} > Ke^{K}\phi\left\{\frac{K}{16u_0^2} - 3\varepsilon\right\}.$$

Choosing $K = 16 \times 64 \times 5 \times \varepsilon_0^{1/2} \times \tau \times 3\varepsilon$ we have $\mathcal{L}\eta > 0$. This combining with the analysis in Step 5 implies that the function ηw doesn't have interior maximum in U_p, which is impossible. Therefore, we can conclude that $\tau \geq \frac{\varepsilon_0^{1/2}}{20\varepsilon^2}$ and prove the Lemma 3.1. □

Proposition 3.2. *Let $\varepsilon_0 \leq 10^{-12}$ be an universal constant. If for any $\varepsilon \in (0, \varepsilon_0)$, Σ is a translating soliton in $B^3_\sigma(\hat{x}_1)$ with $\hat{x}_1 \in \Sigma$ and $\sigma = \varepsilon^{-2}$. Moreover, Σ satisfies that*

$$\sup_{\hat{x}\in\Sigma} |\langle\nu(\hat{x}), e_3\rangle| < \varepsilon_0, \tag{3.6}$$

$$\sup_{\substack{\hat{x},\hat{x}'\in\Sigma \\ d_\Sigma(\hat{x},\hat{x}')<\varepsilon\sigma}} |\nu(\hat{x}) - \nu(\hat{x}')| < \varepsilon_0, \tag{3.7}$$

and

$$\int_\Sigma |A_\Sigma|^2 d\mathcal{H}^2 < k. \tag{3.8}$$

Then for any given $\beta \in (0,1)$, there exists $\eta_1 = \eta_1(\beta, k)$ such that

$$\sup_{\substack{\hat{x}\in\Sigma \\ d_\Sigma(\hat{x},\hat{x}_1)<\eta_1\sigma}} |\nu(\hat{x}) - \nu(\hat{x}_1)| < \beta\varepsilon_0. \tag{3.9}$$

Proof. Step 1. By Lemma 3.1, Lemma 2.2, and Lemma 2.3 we have

$$|A| \leq C_1\varepsilon \text{ and } |\nabla A| \leq C_1\varepsilon^2 \text{ in } B^3_{7/8\sigma}(\hat{x}_1) \cap \Sigma.$$

Since $H = -\langle \nu, e_3\rangle$ and $|\nabla H| = |\langle \nabla\nu, e_3\rangle| \leq |\nabla A|$, we have

$$|\nu(x + se_3, y) - \nu(x, y)| \leq 10s \times C_1\varepsilon^2 := C_2 s\varepsilon^2.$$

Here and in the following, we use $\nu(x, y)$ to denote the normal direction at the point $(x, y, u(x, y))$. Moreover, we can see that when $|x_1 - x_2| \leq \frac{\beta\varepsilon_0}{8C_2\varepsilon^2}$ we have

$$|\nu(x_1, y) - \nu(x_2, y)| < \frac{\beta\varepsilon_0}{8}.$$

Step 2. Let $\zeta = \frac{\beta\varepsilon_0}{8C_2}\sigma$ and define τ to be the largest number so that on

$$I_{e_3}(-\zeta, \zeta) \times I_{e_2}(-\zeta\tau, \zeta\tau) + \hat{x}_1$$

we have

$$\sup |\nu(\hat{x}) - \nu(\hat{x}_1)| < \beta\varepsilon_0.$$

By the maximality of τ we get

$$\inf_{a\in(-\zeta,\zeta)} |\nu(a, \zeta\tau) - \nu(a, 0)| > \frac{\beta\varepsilon_0}{4}$$

or

$$\inf_{a\in(-\zeta,\zeta)} |\nu(a, -\zeta\tau) - \nu(a, 0)| > \frac{\beta\varepsilon_0}{4},$$

Without loss of generality, we assume the former inequality holds. This implies

$$\begin{aligned} \frac{1}{2}\zeta\beta\varepsilon_0 &< \int_{-\zeta}^{\zeta} |\nu(a, \zeta\tau) - \nu(a, 0)| da \\ &\leq \sqrt{2}\int_{\text{graph}(u)} |A_\Sigma| d\mathcal{H}^2 \\ &\leq \sqrt{2}\left(\int_{\text{graph}(u)} |A_\Sigma|^2 d\mathcal{H}^2\right)^{1/2} \left(\int_{\text{graph}(u)} d\mathcal{H}^2\right)^{1/2} \\ &\leq 4\zeta(\tau k)^{1/2}, \end{aligned} \tag{3.10}$$

which yields $\tau \geq \frac{\beta^2\varepsilon_0^2}{64k}$. Therefore, the Proposition holds with $\eta_1 = \frac{1}{C_2 k}\left(\frac{\beta\varepsilon_0}{8}\right)^3$. □

Theorem 3.3. *Let $\varepsilon_0 \leq 10^{-12}$ and $\tilde{\epsilon}_0 = \min\{2^{-4}\eta_1, 2^{-2}\varepsilon_0\}$ be universal constants. Given $k > 0$ and $\delta \in (0,1)$, there exist constants $0 < \alpha, \rho < 1$ depending only on k and δ, such that given a translating soliton $\Sigma \subset B^3_R(\hat{x}_0)$ with $\hat{x}_0 \in \Sigma$ and $R > \frac{1}{\tilde{\epsilon}_0^2}$, if*

$$\sup_{\hat{x}\in\Sigma} |A_\Sigma| < \alpha \tag{3.11}$$

and

$$\int_\Sigma |A_\Sigma|^2 d\mathcal{H}^2 < k, \tag{3.12}$$

then the connected component of $\Sigma \cap B^3_{\rho R}(\hat{x}_0)$ can be written as a graph over $T_{x_0}\Sigma$ with gradient bounded by δ.

Proof. Step 1. Choosing $\beta = \min\left\{2^{-1}, \varepsilon_0^{-1}\left(2 - 2\left(1+\delta^2\right)^{-1/2}\right)^{1/2}\right\}$ and $\alpha = \frac{\tilde{\epsilon}_0\varepsilon_0}{2}$. Let $r_0 = \frac{1}{\tilde{\epsilon}_0^2}$, by our assumptions we have

$$\sup_{\hat{x}\in\Sigma} |\langle \nu(\hat{x}), e_3\rangle| = \sup_{\hat{x}\in\Sigma}|H| \leq \sup_{\hat{x}\in\Sigma}\sqrt{2}|A| < \varepsilon_0, \tag{3.13}$$

and

$$\sup_{\substack{\hat{x},\hat{x}'\in\Sigma \\ d_\Sigma(\hat{x},\hat{x}')<\tilde{\epsilon}_0 r_0}} |\nu(\hat{x}) - \nu(\hat{x}')| < \tilde{\epsilon}_0 r_0 \sup_{\hat{x}\in\Sigma}|A| < \varepsilon_0. \tag{3.14}$$

Step 2. Let $\tilde{\epsilon}_i = 2^{-i}\tilde{\epsilon}_0$ and $r_i = 4^i r_0$. Define $\rho_0 = R$ and $\rho_{i+1} = \rho_i - 2^{-1}r_i$. We will show for $i \in \mathbb{N}$, if $r_i < \frac{\rho_0}{2}$, then

$$\sup_{\substack{\hat{x},\hat{x}'\in\Sigma\cap B^3_{\rho_i}(\hat{x}_0) \\ d_\Sigma(\hat{x},\hat{x}')<\tilde{\epsilon}_i r_i}} |\nu(\hat{x}) - \nu(\hat{x}')| < \varepsilon_0. \tag{3.15}$$

For $i = 0$ by Step 1, we know the claim is true. Inductively, assume the claim holds for $i \in \mathbb{N}$. If $r_{i+1} > \frac{\rho_0}{2}$, then $r_i > \frac{\rho_0}{8}$, applying Proposition 3.2 the Theorem is proved. Therefore, we assume $r_{i+1} < \frac{\rho_0}{2}$.

Step 3. For any $\hat{x} \in B^3_{\rho_{i+1}}(\hat{x}_0)$ we have

$$B^3_{\frac{1}{4}r_i}(\hat{x}) = B^3_{\frac{1}{4\tilde{\epsilon}_i^2}}(\hat{x}) \subset B^3_{\rho_i}(\hat{x}_0).$$

Now let $\sigma = \frac{1}{4\tilde{\epsilon}_i^2}$, by inductive assumption we have

$$\sup_{\hat{x}\in\Sigma\cap B^3_\sigma(\hat{x})} |\langle \nu(\hat{x}), e_3\rangle| < \varepsilon_0$$

and

$$\sup_{\substack{\hat{x},\hat{x}'\in\Sigma\cap B^3_{\rho_i}(x_0)\\ d_\Sigma(\hat{x},\hat{x}')<\tilde{\epsilon}_i\sigma}} |\nu(\hat{x})-\nu(\hat{x}')|<\varepsilon_0.$$

Applying Proposition 3.2, let $\eta_1=\frac{1}{C_2k}\left(\frac{\beta\varepsilon_0}{8}\right)^3$ we have

$$\sup_{\substack{\hat{x}'\in\Sigma\cap B^3_{\sigma}(\hat{x})\\ d_\Sigma(\hat{x},\hat{x}')<\eta_1\sigma}} |\nu(\hat{x})-\nu(\hat{x}')|<\beta\varepsilon_0.$$

Since $\eta_1\sigma=\frac{\eta_1}{4\tilde{\epsilon}_i^2}\geq\tilde{\epsilon}_{i+1}r_{i+1}$ and $\hat{x}$ is an arbitrary point in $\Sigma\cap B^3_{\rho_{i+1}}(\hat{x}_0)$, the claim holds for $i+1$.

Step 4. Finally, let $i_0=\max\{i\in\mathbb{N}:r_i<\frac{\rho_0}{2}\}$, then we have

$$\frac{\rho_0}{8}<r_{i_0}<\frac{\rho_0}{2},$$

and

$$\rho_{i_0}=\rho_0-\frac{1}{2}\sum_{i=0}^{i_0}r_i=\rho_0-\frac{r_{i_0}}{2}\sum_{i=0}^{i_0}4^{-i}>\frac{1}{2}\rho_0.$$

Let $\tilde{\Sigma}=\Sigma\cap B^3_{r_{i_0}}(\hat{x}_0)$, then by Proposition 3.2 we have

$$\sup_{\substack{\hat{x}\in\tilde{\Sigma}\\ d_{\tilde{\Sigma}}(\hat{x},\hat{x}_0)<\eta_1 r_{i_0}}} |\nu(\hat{x})-\nu(\hat{x}_0)|<\beta\varepsilon_0.$$

Therefore Σ contains the graph of a function u on $B^2_{\frac{\eta_1}{16}\rho_0}(\hat{x}_0)\subset T_{x_0}\Sigma$ with $|Du|\leq\delta$. □

4. C^2 ESTIMATES

In this section, we will show that, if the total curvature is bounded on a "big" piece of Σ, then there is no singularity on this "big" piece. Here, "big" means that after blowdown, the portion doesn't disappear. More precisely, we prove the following Lemma.

Lemma 4.1. *Let Σ be a translating soliton with finite topology and bounded area ratio. Moreover, Σ is asymptotic to a family of shrinker cylinders, i.e.,*

$$\frac{\Sigma+\tau_i e_3}{\sqrt{\tau_i}}\to\mathbb{S}^1(\sqrt{2})\times\mathbb{R}_{e_3} \text{ as varifolds.} \tag{4.1}$$

If

$$\int_{\Sigma\cap B^3_{l\sqrt{\tau_i}+1}(-\tau_i e_3)}|A_\Sigma|^2d\mathcal{H}^2<C_l, \tag{4.2}$$

then

$$\sup_{x\in\Sigma\cap B^3_{l\sqrt{\tau_i}(-\tau_i e_3)}} |A_\Sigma(x)| < C$$

for i sufficiently large and C is independent of i.

Proof. Follow the idea of [22], we argue by contradiction. We assume $\Sigma\cap B^3_1(0)\neq\emptyset$ and Σ is homeomorphic to $\bar{B}^2_1(0)\backslash\{0\}$. Suppose there exists a sequence of points $x_k\in\Sigma\cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)$ so that $|A_\Sigma(x_k)|\to+\infty$. We define on $\bar{B}^3_1(x_k)\cap\Sigma$,

$$f_k(x) = (1-|x-x_k|)|A_\Sigma(x)|.$$

Thus f_k attains its maximum at some point $z_k\in B^3_1(x_k)\cap\Sigma$. Let $\sigma_k=\frac{1}{2}(1-|z_k-x_k|)$, by the triangle inequality, for any $x\in B^3_{\sigma_k}(z_k)$, we have

$$1-|x-x_k|\geq(1-|z_k-x_k|)-|x-z_k|>\sigma_k,$$

thus

$$\sup_{x\in B^3_{\sigma_k}(z_k)\cap\Sigma}|A_\Sigma(x)|<2|A_\Sigma(z_k)|.$$

By our assumption (4.2) we have

$$\int_{B^3_{\sigma_k}(z_k)\cap\Sigma}|A_\Sigma|^2 d\mathcal{H}^2 < C_l$$

Moreover, we have $f_k(z_k)=2\sigma_k|A_\Sigma(z_k)|>|A_\Sigma(x_k)|\to+\infty$. Define the rescaled surfaces $N_k=|A_\Sigma(z_k)|(\Sigma-z_k)$, we can see that

$$H_{N_k}\to 0,\ \text{ as } k\to\infty.$$

After passing to a subsequence and relabeling, N_k converges locally smoothly (possibly with multiplicity) to a properly embedded minimal surface $N\in\mathbb{R}^3$. Moreover, by our assumptions we know N has finite total curvature and genus 0. In addition, N satisfies

$$|A_N(0)|=1 \text{ and } \sup_{x\in N}|A_N(x)|\leq 2.$$

It follows from [17] that N must be a Catenoid in $\mathbb{R}^3$ which rotates around a straight line of direction $\mathbf{v}$.

Next, we choose R to be a sufficiently large constant so that $N\cap(B^3_{4R}\backslash B^3_R)$ is close to two parallel planes with normal $\mathbf{v}$. Let $M_k=|A_\Sigma(z_k)|^{-1}\tilde{N}_k+z_k$, where $\tilde{N}_k$ is the connected component of $N_k\cap B^3_{4R}$ containing origin. For k sufficiently large, $\tilde{N}_k$ is given by the normal exponential graph of a small function over some subset of N. Let $\gamma_k=|A_\Sigma(z_k)|^{-1}\tilde{\gamma}_k+z_k$, where $\tilde{\gamma}_k$ is the image of the closed geodesic, which encircles the neck of N, via the graphical representation of $\tilde{N}_k$. Thus γ_k for k sufficiently large is a simple closed planar curve in M_k. Denote by M_k^+, M_k^- the two connected components of $M_k \backslash \gamma_k$. We divide the rest of the proof into two cases according to whether $\mathbf{v}$ is parallel to e_3 or not.

Suppose first $\mathbf{v}$ is not parallel to e_3. Assume now that $p_1 \in \hat{P}_{\tilde{z}_k} \cap M_k^+$ and $p_2 \in \hat{P}_{\tilde{z}_k} \cap M_k^-$, then there exists a simply embedded curve connects p_1 and p_2 through z_k we denote it by γ_0. Furthermore, there exists $\gamma^+ \subset M_k^+ (\gamma^- \subset M_k^-)$ emanating from p_1 (p_2) and extending to ∞ with $\gamma^+ \cap \gamma^- = \emptyset$. By a standard smoothing process, we may assume $\gamma = \gamma_0 \cup \gamma^+ \cup \gamma^-$ is a smooth embedded curve.

On the other hand, we assumed that the blowdown of Σ is a cylinder. Furthermore, by our assumption (4.2) and Proposition 2.4 we know Σ only has finite number of singularities. This implies that there exists a plane $\hat{P}_{\tilde{z}_k} \perp e_3$, where $\tilde{z}_k$ is in a small neighborhood of z_k, such that $\Sigma \cap \hat{P}_{\tilde{z}_k}$ are smooth curves. Therefore, there exist a smooth closed curve in $\Sigma \cap \hat{P}_{\tilde{z}_k}$ passes p_1, we denote this curve by $\tilde{\gamma}$. Moreover, $\tilde{\gamma}$ and γ intersects transversally at p_1 which contradicts to the assumption that Σ is homeomorphic to $\bar{B}^2 \backslash \{0\}$.

Suppose next that $\mathbf{v}$ is parallel to e_3. Without loss of generality we assume M_k^+ is on the $-e_3$ side of M_k^-. Furthermore, by (4.1), we know that when i sufficiently large the rescaling $\tau_i^{-1/2}\left(\Sigma \cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)(0)\right)$ looks like a cylinder. Therefore, same as before there exists $\tilde{z}_k$ close to z_k such that $\hat{P}_{\tilde{z}_k} \cap M_k^+$ are smooth curves. In particular, there exists a loop $\tilde{\gamma}$ such that *diam*$\tilde{\gamma} > \sqrt{\tau_i}$. Choosing any point $p_0 \in \tilde{\gamma}$, we can see that there is a smooth curve $\tilde{\gamma}_0$ emanating from p_0 and extending to $\Sigma \cap \partial B^3_1(0)$ with $\tilde{\gamma}_0 \cap \gamma_k = \emptyset$. Therefore, γ_k doesn't separate Σ, which leads to a contradiction. □

5. Local estimates on $|A|$ and $\int_\Sigma |A|^2 d\mathcal{H}^2$

In this section, we will show that there exists " good" pieces on Σ. In the following, we denote

$$\rho = \rho(x,t) = \frac{1}{-4\pi t} e^{\frac{|x|^2}{4t}}, \ t < 0.$$

Lemma 5.1. *If Σ is a translating soliton in $\mathbb{R}^3$ with bounded area ration, finite topology, and asymptotic to self-shrinking cylinder. Then there exists a sequence $\{\tau_i\}$ with $\tau_{i+1}/\tau_i \to 1$ and $\tau_i \to \infty$, such that for any $l \in \mathbb{Z}^+$*

$$\int_{\Sigma \cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)} |A_\Sigma|^2 d\mathcal{H}^2 < C_l \tag{5.1}$$

and

$$\sup_{x\in\Sigma\cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)} |A_\Sigma| \to 0, \ \textit{as } i\to\infty. \tag{5.2}$$

Proof. Recall the local monotonicity formula (see [15] pg13)

$$\frac{d}{dt}\int \psi\rho d\mu_t \le \int -\psi\rho\left|H-\frac{D(\psi\rho)^\perp}{\psi\rho}\right|^2 d\mu_t + \int Q(\psi\rho)d\mu_t,$$

where ψ is a cutoff function and $Q(\phi)=\frac{(D\phi^\perp)^2}{\phi}+(D^2\phi)^\top+\frac{\partial}{\partial t}\phi$ for any $\phi=\phi(x,t)\in C^1_c(\mathbb{R}^n\times\mathbb{R})$. This yields

$$-\frac{d}{dt}\int\psi\rho d\mu_t+\int Q(\psi\rho)d\mu_t \ge \int \psi\rho\left|H-\frac{x^\perp}{2t}-\frac{(D\psi)^\perp}{\psi}\right|^2 d\mu_t. \tag{5.3}$$

By a straightforward calculation we have

$$\rho_t=\rho\left(-\frac{1}{t}-\frac{|x|^2}{4t^2}\right),$$

$$\frac{(D\rho^\perp)^2}{\rho}=\frac{\rho|x^\perp|^2}{4t^2},$$

and

$$(D^2\rho)^\top=\rho\left(\frac{t^{-1}}{4}D\langle x,x\rangle^\top\right)^2+\frac{\rho}{4t}D^2\langle x,x\rangle^\top=\frac{\rho}{4t^2}|x^\top|^2+\frac{\rho}{t},$$

which yields $Q(\rho)=0$. Therefore,

$$Q(\psi\rho)=\rho Q(\psi)+2D\psi\cdot D\rho.$$

Next let $\psi:=\phi(0,|t|)$ be a cutoff function around the origin:

$$\psi=\begin{cases}1 & |x|\le|t|\\ \frac{(2|t|-|x|)^2}{t^2} & |t|<|x|\le 2|t|\\ 0 & |x|>2|t|\end{cases} \tag{5.4}$$

By a straightforward calculation we have

$$\begin{aligned}&\int Q(\psi\rho)d\mu_t\\ &\le C\int_{-\infty}^{-1}\int_{\Sigma-te_3}\left(\frac{1}{t^2}+\frac{1}{|t|}\right)\chi_{B_{2|t|}(0)\setminus B_{|t|}(0)}\rho d\mathcal{H}^2 dt\\ &\le C(\Lambda)\int_{-\infty}^{-1}\frac{1+|t|}{4\pi|t|}e^{-\frac{|t|}{4}}dt<\infty,\end{aligned} \tag{5.5}$$

where Λ is the bound of area ratio.

Now consider

$$
\begin{aligned}
\int \psi\rho d\mu_t &= \int_{\Sigma - te_3} \psi\rho d\mathcal{H}^2 \\
&= \int_{\Sigma} \phi(te_3, |t|)\frac{1}{-4\pi t} e^{\frac{|z-te_3|^2}{4t}} d\mathcal{H}^2 \\
&\leq \int_{\Sigma\cap B_{2|t|}(te_3)} \frac{1}{4\pi|t|} e^{-\frac{|z-te_3|^2}{4|t|}} d\mathcal{H}^2 \\
&\leq \frac{\Lambda}{\pi}(1-e^{-|t|}).
\end{aligned} \tag{5.6}
$$

Thus by equation (5.3) we have

$$
\int_{-\infty}^{-1}\int_{\Sigma_t} \psi\rho \left| H - \frac{x^{\perp}}{2t} - \frac{(D\psi)^{\perp}}{\psi}\right|^2 d\mathcal{H}^2 dt < \infty,
$$

where $\Sigma_t = \Sigma - te_3$. Notice that

$$
\left| H - \frac{x^{\perp}}{2t} - \frac{(D\psi)^{\perp}}{\psi}\right|^2 \geq \frac{1}{2}\left| H - \frac{x^{\perp}}{2t}\right|^2 - \left(\frac{|D\psi|^{\perp}}{\psi}\right)^2
$$

and

$$
\begin{aligned}
&\int_{-\infty}^{-1}\int_{\Sigma_t} \frac{\rho|D\psi^{\perp}|^2}{\psi} d\mathcal{H}^2 dt \\
&\leq C\int_{-\infty}^{-1}\int_{\Sigma_t\cap B_{2|t|}\setminus B_{|t|}} \frac{1}{4\pi|t|} e^{-\frac{|x|^2}{4|t|}} \frac{1}{t^2} d\mathcal{H}^2 dt \\
&\leq C\int_{-\infty}^{-1} \frac{1}{|t|} e^{-\frac{|t|}{4}} dt < \infty.
\end{aligned} \tag{5.7}
$$

Therefore, we obtain

$$
\int_{-\infty}^{-1}\int_{\Sigma_t} \psi\rho \left| H - \frac{x^{\perp}}{2t}\right|^2 d\mathcal{H}^2 dt < \infty. \tag{5.8}
$$

Next, changing variables in (5.8) by setting $e^s = -t$, then (5.8) can be rewritten as

$$
\int_0^{\infty}\int_{\Sigma_s} \psi\frac{1}{4\pi} e^{-\frac{|x|^2}{4e^s}} \left| H + \frac{x^{\perp}}{2e^s}\right| d\mathcal{H}^2 ds < \infty, \tag{5.9}
$$

where $\Sigma_s = \Sigma + e^s e_3$. Denote $y_s = -e^s e_3$, then equation (5.9) is equivalent to

$$
\int_0^{+\infty}\int_{\Sigma} \phi(y_s, e^s) e^{-\frac{|z-y_s|^2}{e^s}} \left| H + \frac{(z-y_s)^{\perp}}{2e^s}\right|^2 d\mathcal{H}^2 ds < \infty. \tag{5.10}
$$

Therefore, there is an increasing unbounded sequence s_i so that

$$\int_{s_i}^{\infty}\int_{\Sigma}\phi(y_s,e^s)e^{-\frac{|z-y_s|^2}{e^s}}\left|H+\frac{(z-y_s)^{\perp}}{2e^s}\right|^2 d\mathcal{H}^2 ds<i^{-2}.$$

By the mean value theorem, for each i, there exists

$$s_{i,m}\in[s_i+i^{-1}(m-1),s_i+i^{-1}m],m<i(s_{i+1}-s_i)$$

so that

$$\int_{\Sigma}\phi(y_{s_{i,m}},e^{s_{i,m}})e^{-\frac{|z-y_{s_{i,m}}|^2}{e^{s_{i,m}}}}\left|H+\frac{(z-y_{s_{i,m}})^{\perp}}{2e^{s_{i,m}}}\right|^2 d\mathcal{H}^2<i^{-1}.$$

Therefore, after relabeling $s_{i,m}$, we get an increasing sequence $\bar{s}_i\to\infty$ satisfies

$$\int_{\Sigma}\phi(y_{\bar{s}_i},e^{\bar{s}_i})e^{-\frac{|z-y_{\bar{s}_i}|^2}{e^{\bar{s}_i}}}\left|H+\frac{(z-y_{\bar{s}_i})^{\perp}}{2e^{\bar{s}_i}}\right|^2 d\mathcal{H}^2\to 0, \tag{5.11}$$

and $|\bar{s}_{i+1}-\bar{s}_i|\to 0$. Combining equation (5.11) with the local Gauss-Bonnet estimate we have, given $l\in\mathbb{Z}^+$, there exists an $i_l\in\mathbb{Z}^+$ such that for all $i>i_l$,

$$\int_{\Sigma\cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)}|A_\Sigma|^2 d\mathcal{H}^2<C_l,$$

thus we proved the equation (5.1).

Next, set $\tau_i=e^{\bar{s}_i}$, we want to show for all $l\in\mathbb{Z}^+$

$$\sup_{x\in\Sigma\cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)}|A_\Sigma(x)|\to 0,\ \text{ as } i\to\infty. \tag{5.12}$$

We will argue by contradiction. First note that from equation (5.11) we have

$$\int_{\Sigma\cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)}\left|H+\frac{(z-y_{\bar{s}_i})^{\perp}}{2e^{\bar{s}_i}}\right|^2 d\mathcal{H}^2\to 0. \tag{5.13}$$

If (5.12) was not true then there exists a subsequence $x_{i_k}\in\Sigma\cap B^3_{l\sqrt{\tau_{i_k}}}(-\tau_{i_k}e_3)$ so that $|A_\Sigma(x_{i_k})|>\zeta>0$. Let $N_{i_k}=\Sigma-x_{i_k}$, then we have $|A_{N_{i_k}}(0)|>\zeta>0$. By Lemma 4.1 we have $|A_{N_{i_k}\cap B_1(0)}|<C$. Therefore, $N_{i_k}\cap B_1(0)$ converges locally smoothly to a surface N in $B_1(0)$, and $H_N=-e_3^{\perp}$. From equation (5.13) we can obtain

$$\int_{N_{i_k}\cap B_1(0)}\left|-e_3^{\perp}+\frac{1}{2\sqrt{\tau_{i_k}}}\right|^2 d\mathcal{H}^2\to 0,$$

which implies

$$\int_{N\cap B_1(0)}|e_3^{\perp}|^2 d\mathcal{H}^2=0,$$

thus N is flat. This leads to a contradiction. $\square$

Proposition 5.2. *Let Σ be an end of a noncompact translating soliton in $\mathbb{R}^3$ with finite topology, bounded area ratio, and asymptotic to self-shrinking cylinder. Then there exist*
• *a positive integer L, a sufficiently large integer i_0,*
• *two increasing unbounded sequences R_i and τ_i, with $\tau_{i+1}\tau_i^{-1} \to 1$,*
• *a sequence of $S^{\tau_i} = \mathbb{S}^1(\sqrt{2}) \times \mathbb{R}_{e_3}$,*
such that for each $i > i_0$, we have
(1) $\frac{\Sigma+\tau_i e_3}{\sqrt{\tau_i}} \cap B^3_{2R_i}$ decomposes into L connected components, $\tilde{\Sigma}_1^{\tau_i}, \cdots, \tilde{\Sigma}_L^{\tau_i}$;
(2) each $\tilde{\Sigma}_j^{\tau_i}$ can be written as the normal exponential graph of a function $f_j^{\tau_i}$ over some subset $\Omega_j^{\tau_i}$ of S^{τ_i}. Furthermore, for each j

$$\lim_{i\to\infty} \sup_{\Omega_j^{\tau_i}} |f_j^{\tau_i}| + |\nabla_{S^{\tau_i}} f_j^{\tau_i}| = 0.$$

Proof. Combining Lemma 5.1 and Theorem 3.3 we conclude that for any positive integers k and l there is $i_{k,l} \in \mathbb{Z}^+$ and $\rho_{k,l} \in (0,1)$ such that for all $i > i_{k,l}$ and $X \in \Sigma \cap B^3_{l\sqrt{\tau_i}}(-\tau_i e_3)$, the connected component of $\tilde{\Sigma}_i \cap B^3_{\rho_{k,l}}(\tilde{x})$, where $\tilde{\Sigma}_i = \frac{\Sigma+\tau_i e_3}{\sqrt{\tau_i}}$ and $\tilde{x} = \frac{x+\tau_i e_3}{\sqrt{\tau_i}}$ can be written as the graph of a function over some subset of $T_{\tilde{x}}\tilde{\Sigma}_i$ with its gradient bounded by $1/k$.

By our assumption that $\tilde{\Sigma}_i \to \mathbb{R}_{e_3} \times \mathbb{S}^1(\sqrt{2})$ as varifolds, we know there exists a positive integer L such that the varifold distance between $\tilde{\Sigma}_i$ and LS^{τ_i} converges to 0. This together with the discussion before and the Aszela-Ascoli Theorem implies: given $l \in \mathbb{Z}^+$ there exists $i'_l \in \mathbb{Z}^+$ such that for all $i > i'_l$, $\tilde{\Sigma}_i \cap B^3_{l-1}(0)$ has L connected components. Moreover, each component is given by the exponential normal graph of a function over some subset of S^{τ_i} with its C^1-norm bounded by $1/l$. Without loss of generality, let's assume that i'_l is increasing in l, then choosing $i_0 = i'_1$, and $R_1 = 1/8$ if $i \le i'_1$; and $R_i = l/4$ if $i'_l < i \le i'_{l+1}$, the Proposition follows immediately. $\square$

6. Proof of the main Theorem

We restate for the readers convenience our main result.

Theorem 6.1. *If Σ is an end of a noncompact translating soliton in $\mathbb{R}^3$ with bounded area ratio and finite topology. Moreover, Σ is asymptotic to self-shrinking cylinder, i.e., there exists a sequence $\{\tau_i\}$ with $\tau_i \to \infty$ such that*

$$\frac{\Sigma + \tau_i e_3}{\sqrt{\tau_i}} \to \mathcal{S} \text{ as varifolds},$$

where $\mathcal{S}$ is a self-shrinking cylinder. Then $\mathcal{S}$ has multiplicity one.

Proof. Applying Proposition 5.2 and Brakke's regularity Theorem, it suffices to show that for all i sufficiently large, the connected components of $\Sigma \cap B^3_{R_i\sqrt{\tau_i}}(-\tau_i e_3)$ extend to infinity and remain disjoint in the solid half-cylinder $I_{e_3}(-\infty, -\tau_i e_3) \times B^2_{R_i\sqrt{\tau_i}}$. We can see that if this is true, as Σ is connected , the conclusion follows immediately.

For $i, k \in \mathbb{Z}^+$ and $j \in \{1, \cdots, L\}$ denote

$$\bigcup_{j=1}^{j=L} \gamma_j^{i,k} = \Sigma \cap B^3_{R_i\sqrt{\tau_k}}(-\tau_k e_3) \cap \{x \in \mathbb{R}^3 : x \cdot e_3 = -\tau_k\}.$$

Step 1. We'll show that given i sufficiently large, for any $k \geq i$ and any connected component

$$\Sigma^{i,k} \subset \Sigma \cap (I_{e_3}(-\tau_{k+1}, -\tau_k)) \times B^2_{R_i\sqrt{\tau_k}}$$

the two sets

$$\partial\Sigma^{i,k} \cap \{x \in \mathbb{R}^3 : \langle x, e_3 \rangle = -\tau_k\}$$

and

$$\partial\Sigma^{i,k} \cap \{x \in \mathbb{R}^3 : \langle x, e_3 \rangle = -\tau_{k+1}\}$$

are unions of the same number of elements of $\{\gamma_j^{i,k}\}$ and $\{\gamma_j^{i,k+1}\}$ respectively.

Fix any $k \geq i$, we assume $R_i\sqrt{\tau_k} < (\tau_{k+1} - \tau_k)$; otherwise the claim follows from Proposition 5.2. Now denote $\Sigma_s^{i,k} = e^{-s/2}\Sigma^{i,k} + e^{s/2}e_3$, then we have $X(s) = e^{-s/2}x + e^{s/2}e_3 \in \Sigma_s^{i,k}$. Therefore,

$$\begin{aligned}(\partial_s X(s))^\perp &= -\frac{1}{2}e^{-s/2}x^\perp + \frac{1}{2}e^{s/2}e_3^\perp \\ &= -\frac{X(s)^\perp}{2} - H_{\Sigma_s^{i,k}}.\end{aligned} \tag{6.1}$$

Define $s_k^+ = \ln(\tau_k + R_i\sqrt{\tau_k})$ and $s_{k+1}^- = \ln(\tau_{k+1} - R_i\sqrt{\tau_{k+1}})$. Let ψ be a cutoff function satisfies $\psi \equiv 1$ in $B^3_{R_i/4}$, $\psi \equiv 0$ outside of $B^3_{R_i/2}$, and $|D\psi| \leq \frac{8}{R_i}$. By the first variation formula (see [16] page 5) we have

$$\begin{aligned}&\frac{d}{ds}\int_{\Sigma_s^{i,k}} \psi^2 e^{-\frac{|x|^2}{4}} d\mathcal{H}^2 \\ &= \int_{\Sigma_s^{i,k}} \left|H + \frac{x^\perp}{2}\right|^2 \psi^2 e^{-\frac{|x|^2}{4}} d\mathcal{H}^2 \\ &+ 2\int_{\Sigma_s^{i,k}} D\psi \cdot \left(-H - \frac{x^\perp}{2}\right) \psi e^{-\frac{|x|^2}{4}} d\mathcal{H}^2.\end{aligned} \tag{6.2}$$

Therefore

$$
\begin{aligned}
&\left|\int_{\Sigma^{i,k}_{s_k^+}}\psi^2 e^{-\frac{|x|^2}{4}}d\mathcal{H}^2-\int_{\Sigma^{i,k}_{s_{k+1}^-}}\psi^2 e^{-\frac{|x|^2}{4}}d\mathcal{H}^2\right|\\
&\le 2\int_{s_k^+}^{s_{k+1}^-}\int_{\Sigma^{i,k}_s}\left|H_{\Sigma^{i,k}_s}+\frac{x^\perp}{2}\right|^2\psi^2 e^{-\frac{|x|^2}{4}}d\mathcal{H}^2ds\\
&+\int_{s_k^+}^{s_{k+1}^-}\int_{\Sigma^{i,k}_s}|D\psi|^2 e^{-\frac{|x|^2}{4}}d\mathcal{H}^2ds\\
&:= D_1+D_2
\end{aligned}
\tag{6.3}
$$

By equation (5.13) we have $\lim_{i\to\infty} D_1 = 0$. Moreover, by our assumption on bounded area ratio we have $D_2 \le Ce^{-\frac{R_i^2}{32}}(s_{k+1}^- - s_k^+)$. Thus, by our choice of s_{k+1}^- and s_k^+ we have

$$
\left|\int_{\Sigma^{i,k}_{s_k^+}}\psi^2 e^{-\frac{|x|^2}{4}}d\mathcal{H}^2-\int_{\Sigma^{i,k}_{s_{k+1}^-}}\psi^2 e^{-\frac{|x|^2}{4}}d\mathcal{H}^2\right|\to 0,\ \text{as } i\to\infty.
$$

Hence the claim follows.

Step 2. Next we fix any i large enough so that the preceding claim holds. Suppose that, for some $k_0 \ge i$ and some connected component Σ^{i,k_0}

$$
\partial\Sigma^{i,k_0}\cap\{x\in\mathbb{R}^3:\langle x,e_3\rangle=-\tau_{k_0+1}\}=\bigcup_{l=1}^m\gamma_{j_l}^{i,k_0+1},
$$

where $m \ge 2$. Picking points $z_l \in \gamma_{j_l}^{i,k_0+1}$ for $l = \{1,2\}$. Let γ_0 be a smooth embedded curve in Σ^{i,k_0} joining z_1 and z_2. By the preceding claim and our assumption on bounded area ratio, for $l\in\{1,2\}$, there is a smooth embedded curve $\gamma_l \subset \Sigma\cap\big(I_{e_3}(-\infty,-\tau_{k_0+1})\times B^2_{R_i}\big)$ emanating from z_l and extending to infinity. Moreover, we may assume $\gamma = \bar\gamma_0\cup\bar\gamma_1\cup\bar\gamma_2$ is a smooth embedded curve and $\gamma\cap\gamma_{j_l}^{i,k_0+1} = z_l$. Since $S^{\tau_{k_0+1}} = \mathbb{R}_{e_3}\times S^1(\sqrt{2})$, by Proposition 5.2 we have $\gamma_{j_l}^{i,k_0+1}, l = 1,2$ are simple closed smooth curves. As $\bar\Sigma$ is connected at infinity, there exist $z_1' \in \gamma_1$ and $z_2' \in \gamma_2$ so that they can be joint by a smooth embedded curve γ' outside a large ball, we may assume $\gamma'\cap\gamma_{j_1}^{i,k_0+1} = \emptyset$. Denote γ'' the part of γ joining z_1' and z_2' that contains γ_0. Therefore, $\overline{\gamma'\cup\gamma''}$ can be made as simple closed curve in M that transversally intersects $\gamma_{j_1}^{i,k_0+1}$ only at z_1. This leads to a contradiction.

$\square$

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