ENTIRE SPACELIKE HYPERSURFACES WITH CONSTANT σ_{n-1} CURVATURE IN MINKOWSKI SPACE

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ABSTRACT. We prove that, in the Minkowski space, if a spacelike, $(n-1)$-convex hypersurface $\mathcal{M}$ with constant σ_{n-1} curvature has bounded principal curvatures, then $\mathcal{M}$ is convex. Moreover, if $\mathcal{M}$ is not strictly convex, after an $\mathbb{R}^{n,1}$ rigid motion, $\mathcal{M}$ splits as a product $\mathcal{M}^{n-1}\times\mathbb{R}$. We also construct nontrivial examples of strictly convex, spacelike hypersurface $\mathcal{M}$ with constant σ_{n-1} curvature and bounded principal curvatures.

1. INTRODUCTION

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2=\sum_{i=1}^{n}dx_i^2-dx_{n+1}^2.$$

In this paper, we will study spacelike hypersurfaces with positive constant σ_{n-1} curvature in Minkowski space $\mathbb{R}^{n,1}$. Any such hypersurface can be written locally as the graph of a function $x_{n+1}=u(x), x\in\mathbb{R}^n$ satisfying the spacelike condition

$$|Du|<1. \tag{1.1}$$

Before stating our main results, we need the following definition:

Definition 1. *A C^2 regular hypersurface $\mathcal{M}\subset\mathbb{R}^{n,1}$ is k-convex, if the principal curvatures of $\mathcal{M}$ at $X\in\mathcal{M}$ satisfy $\kappa[X]\in\Gamma_k$ for all $X\in\mathcal{M}$, where Γ_k is the Gårding cone*

$$\Gamma_k=\{\kappa\in\mathbb{R}^n|\sigma_m(\kappa)>0, m=1,\cdots,k\}.$$

We will investigate the $(n-1)$-convex, spacelike hypersurface $\mathcal{M}_u:=\{(x,u(x))|x\in\mathbb{R}^n\}$ satisfying

$$\sigma_{n-1}(\kappa[\mathcal{M}_u])=\sigma_{n-1}(\kappa_1,\cdots,\kappa_n)=1, \tag{1.2}$$

where $\kappa[\mathcal{M}_u]=(\kappa_1,\cdots,\kappa_n)$ are the principal curvatures of $\mathcal{M}_u$ and σ_{n-1} is the $(n-1)$-th elementary symmetric polynomial, i.e.,

$$\sigma_{n-1}(\kappa)=\sum_{1\leqslant i_1<\cdots<i_{n-1}\leqslant n}\kappa_{i_1}\cdots\kappa_{i_{n-1}}.$$

Research of the first author is supported by NSFC Grant No. 11871243 and the second author is supported by NSFC Grant No.11871161 and 11771103.

In contrast to the Euclidean case, where the existence of an entire zero mean curvature graph implies that the graph is a hyperplane only for dimensions $n \leqslant 7$, Cheng-Yau [7] showed that an entire spacelike maximal hypersurface (zero mean curvature) in Minkowski space is a hyperplane for all dimensions. This raised the question of whether the only entire spacelike hypersurface of constant mean curvature (CMC) in Minkowski space is the hyperboloid. In [25], Treibergs answered this question by showing that for an arbitrary C^2 perturbation of the light cone in Minkowski space, one can construct a spacelike CMC hypersurface which is asymptotic to this perturbation. Moreover, Treibergs also showed that every entire spacelike CMC hypersurface is convex and has bounded principal curvatures. Later, Choi-Treibergs [10] further studied the Guass map of the CMC hypersurfaces. They proved that the Gauss map of a spacelike CMC hypersurface in Minkowski space is a harmonic map to hyperbolic space. Furthermore, they showed, given an arbitrary closed set in the ideal boundary at infinity of hyperbolic space, there are many complete entire spacelike CMC hypersurfaces whose Gauss maps are diffeomorphisms onto the interior of the convex hull of the corresponding set in the unit ball.

It may be traced back to Liebmann [20] who showed that every compact embedded 2-dimensional surface with constant Gauss curvature is a sphere. Hsiung [17] extended this result to all dimensions. Later, a classical result of Aleksandrov [1] states that a compact embedded hypersurface with constant mean curvature is a sphere. One can investigate what happens if we replace the " constant mean curvature" by " constant σ_k curvature." In case $k = 2$, Cheng-Yau [9] showed the Aleksandrov result still holds. In [23], A. Ros extended Cheng-Yau's result, showing that the sphere is the only embedded compact k-convex hypersurface with constant σ_k curvature. Around the same time, Ecker-Huisken [11] proved a similar result using different approaches.

Inspired by the similarity of constant σ_k curvature hypersurfaces and CMC hypersurfaces in Euclidean space, it is natural to ask, do constant σ_k curvature hypersurfaces also share similar properties as CMC hypersurfaces in Minkowski space? More specifically, can we show that every entire spacelike constant σ_k curvature hypersurface is convex and has bounded principal curvatures?

In [13], Guan-Jian-Schoen considered the existence and regularity of entire spacelike constant Gauss curvature hypersurfaces with prescribed tangent cone at infinity. In addition, they showed that there do exist entire spacelike constant Gauss curvature hypersurfaces with unbounded principal curvatures. Despite so, there still are some nice properties for entire spacelike constant Gauss curvature hypersurfaces. In [22], by studying the Legendre transform of entire spacelike constant Gauss curvature hypersurfaces, Anmin Li showed that one can construct spacelike constant Gauss curvature hypersurfaces with bounded principal curvatures whose Guass map image is the unit ball.

Due to technical reasons, the study of entire spacelike constant σ_k curvature hypersurface remains wide open. The main difficulties are the following: first, we can't show that the entire spacelike constant σ_k curvature hypersurfaces are convex; second, the principal curvatures of the entire spacelike constant σ_k curvature hypersurfaces are not necessarily bounded; lastly, in the process of constructing entire spacelike constant σ_k curvature hypersurfaces, we need to solve a corresponding Dirichlet problem (see [3] Appendix B for example). Unfortunately, we do not have the existence result for such Dirichlet problems in general. In this paper, we will overcome these difficulties and study the convexity and existence of the entire spacelike constant σ_{n-1} curvature hypersurfaces.

We will divide this paper into two parts. In the first part, we will prove that every entire spacelike constant σ_{n-1} curvature hypersurface with bounded principal curvatures is convex. In the second part, we will construct nontrivial examples of strictly convex, spacelike hypersurfaces that have bounded principal curvatures and satisfy equation (1.2). In particular, we will prove

Theorem 2. *Let $\mathcal{M}$ be an $(n-1)$-convex, spacelike hypersurface with bounded principal curvatures, and $\mathcal{M}$ satisfies equation* (1.2). *Then $\mathcal{M}$ is convex. Moreover, if $\mathcal{M}$ is not strictly convex, then after an $\mathbb{R}^{n,1}$ rigid motion, $\mathbb{R}^{n,1}$ splits as a product $\mathbb{R}^{n-1,1}\times\mathbb{R}$ such that $\mathcal{M}$ also splits as a product $\mathcal{M}^{n-1}\times\mathbb{R}$. Here $\mathcal{M}^{n-1}\subset\mathbb{R}^{n-1,1}$ is a strictly convex, $(n-1)$-dimensional graph whose Gauss curvature is equal to* 1.

Remark 3. *One may compare this theorem with constant rank theorems in Euclidean space (see* [5] *and* [14] *for example). Recall that constant rank theorems in Euclidean space only study compact hypersurfaces, which are essentially assuming the principal curvatures are bounded. In this sense, our theorem is similar to constant rank theorems in Euclidean space. On the other hand, we don't need to assume the convexity of $\mathcal{M}$ at any point. Therefore, our theorem is much stronger than constant rank theorems.*

In order to construct entire, spacelike, constant σ_{n-1} curvature hypersurfaces with bounded principal curvatures, we will use Anmin Li's idea (see [22]) and consider the Legendre transform of the solution to equation (1.2). We will show in Section 7 that the study of complete, spacelike, convex hypersurfaces $\mathcal{M}_u$ with bounded principal curvatures and satisfying $\sigma_{n-1}(\kappa[\mathcal{M}_u])=1$ can be reduced to the study of the following equation:

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj})=1, \text{ in } B_1\\ \qquad\qquad u^*=\varphi, \text{ on } \partial B_1,\end{cases} \tag{1.3}$$

where $B_1=\{\xi\in\mathbb{R}^n||\xi|<1\}$, u^* is the Legendre transform of u, $\varphi\in C^2(\partial B_1)$,

$$w^*=\sqrt{1-|\xi|^2},\;\; \gamma^*_{ik}=\delta_{ik}-\frac{\xi_i\xi_k}{1+w^*},\;\; u^*_{kl}=\frac{\partial^2u^*}{\partial\xi_k\partial\xi_l},$$

and

$$F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj})=\left(\frac{\sigma_n}{\sigma_1}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}])\right)^{\frac{1}{n-1}}.$$

Here, $\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}]=(\kappa^*_1,\cdots,\kappa^*_n)$ are the eigenvalues of the matrix $(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj})$.

Theorem 4. *Given a C^2 function φ on B_1, there is a unique strictly convex solution $u^* \in C^\infty(B_1)\cap C^0(\bar{B}_1)$ to the equation* (1.3)*. Moreover, the Legendre transform of u^*, which we will denote by u satisfies*

$$\sigma_{n-1}(\kappa[\mathcal{M}_u]) = 1 \text{ and } \kappa[\mathcal{M}_u] \leqslant C.$$

Here, $\mathcal{M}_u = \{(x, u(x)) | \, x \in \mathbb{R}^n\}$ is the spacelike graph of u, $\kappa[\mathcal{M}_u]$ denotes the principal curvatures of $\mathcal{M}_u$, and the constant C only depends on $|\varphi|_{C^2}$.

We can also state the above theorem using geometric terminologies:

Corollary 5. *For any given C^2 function φ defined on the $(n-1)$-dimensional sphere, there exits an unique entire graphical hypersurface $\mathcal{M} = \{(x, u(x))|x \in \mathbb{R}^n\}$ with constant σ_{n-1} curvature and bounded principal curvatures. Moreover, the ideal boundary of $\mathcal{M}$ is the sphere, and on the ideal boundary, the Legendre transform of u is equal to φ.*

Remark 6. *We can generalize the result of Theorem 4 to spacelike constant σ_k curvature hypersurfaces for all $1 \leqslant k \leqslant n$. We will include this result in an upcoming paper, where we will focus on studying properties of spacelike constant σ_k curvature hypersurfaces for all $1 \leqslant k \leqslant n$.*

An outline of the paper is as follows. In Section 2, we introduce some basic formulas, notations, as well as properties of the k-th elementary symmetric function that will be used in later sections. Sections 3, 4, 5, and 6 are devoted to proving Theorem 2. We will see (for details see Section 6 Lemma 25) that the key step in proving Theorem 2 is to prove Theorem 8 (see Section 3), which is carried out in Sections 3, 4, and 5. More specifically, in Section 3, we reduce the proof of Theorem 8 to the proof of the semi-positivity of a symmetric matrix S (see the last 2 paragraphs of Section 3 for the definition of S). In Sections 4 and 5, we show that S is indeed semi-positive. Since these two sections involve very delicate and complicated calculations, first-time readers may want to skip this part. We prove the splitting theorem and complete the proof of Theorem 2 in Section 6. Sections 7, 8, 9 10, and 11 are devoted to proving Theorem 4. In this part, we use Legendre transform to construct many examples of strictly convex solutions with bounded principal curvatures to equation (1.2). In particular, in Section 7, we investigate spacelike hypersurfaces under the Gauss map and the Legendre transform respectively. The reason we look at two models in Section 7 is that each model has its own advantages in the study of the corresponding Dirichlet problem (see Section 8 and 9) and convergence result (see Section 10). We prove that the solution to equation (1.3) exists in Section 8, 9 and 10. In Section 11, we show that the Legendre transform of this solution satisfies equation (1.2) and has bounded principal curvatures. This completes the proof of Theorem 4.

ACKNOWLEDGEMENTS

Part of this work was done while Z.W. was visiting Jilin University, and he would like to thank their hospitality. Part of this work was done while L.X. was visiting the School of Mathematical

Science at Fudan University, and she gratefully acknowledges their hospitality. L.X. would also like to thank Matthew McGonagle for helpful conversations on various aspects of this work.

2. Preliminaries

We first recall some basic formulas for the geometric quantities of spacelike hypersurfaces in Minkowski space $\mathbb{R}^{n,1}$, which is $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$ds^2 = dx_1^2 + \cdots dx_n^2 - dx_{n+1}^2.$$

Throughout this paper, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n,1}$. The corresponding Levi-Civita connection is denoted by $\bar{\nabla}$.

A spacelike hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$ is a codimension-one submanifold whose induced metric is Riemannian. Locally $\mathcal{M}$ can be written as a graph

$$\mathcal{M}_u = \{X = (x, u(x)) | x \in \mathbb{R}^n\}$$

satisfying the spacelike condition (1.1). Let $E = (0, \cdots, 0, 1)$, then the height function of $\mathcal{M}$ is $u(x) = -\langle X, E \rangle$. It's easy to see that the induced metric and second fundamental form of $\mathcal{M}$ are given by

$$g_{ij} = \delta_{ij} - D_{x_i} u D_{x_j} u, \;\; 1 \leqslant i, j \leqslant n,$$

and

$$h_{ij} = \frac{u_{x_i x_j}}{\sqrt{1 - |Du|^2}},$$

while the timelike unit normal vector field to $\mathcal{M}$ is

$$\nu = \frac{(Du, 1)}{\sqrt{1 - |Du|^2}},$$

where $Du = (u_{x_1}, \cdots, u_{x_n})$ and $D^2 u = \left(u_{x_i x_j}\right)$ denote the ordinary gradient and Hessian of u, respectively.

One important example of the spacelike hypersurface with constant mean curvature is the hyperboloid

$$u(x) = \left(\frac{n^2}{H^2} + \sum_{i=1}^n x_i^2\right)^{1/2},$$

which is umbilic, i.e., it satisfies $\kappa_1 = \kappa_2 = \cdots = \kappa_n = \frac{H}{n}$. Other examples of spacelike CMC hypersurfaces include hypersurfaces of revolution, in which case the graph takes the form $u(x) = \sqrt{f(x_1)^2 + |\bar{x}|^2}, \; x = (x_1, \bar{x}) = (x_1, \cdots, x_n) \in \mathbb{R}^n$, where f is a function only depending on x_1.

Now, let $\{\tau_1, \tau_2, \cdots, \tau_n\}$ be a local orthonormal frame on $T\mathcal{M}$. We will use ∇ to denote the induced Levi-Civita connection on $\mathcal{M}$. For a function v on $\mathcal{M}$, we denote $v_i = \nabla_{\tau_i} v$, $v_{ij} =$

$\nabla_{\tau_i}\nabla_{\tau_j}v$, etc. In particular, we have

$$|\nabla u| = \sqrt{g^{ij}u_{x_i}u_{x_j}} = \frac{|Du|}{\sqrt{1-|Du|^2}}.$$

We also need the following well known fundamental equations for a hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$:

$$\begin{aligned} X_{ij} = &\quad h_{ij}\nu \quad \text{(Gauss formula)} \\ (\nu)_i = &\quad h_{ij}\tau_j \quad \text{(Weigarten formula)} \\ h_{ijk} = &\quad h_{ikj} \quad \text{(Codazzi equation)} \\ R_{ijkl} = &\quad -(h_{ik}h_{jl}-h_{il}h_{jk}) \quad \text{(Gauss equation)}, \end{aligned} \tag{2.1}$$

where R_{ijkl} is the $(4,0)$-Riemannian curvature tensor of $\mathcal{M}$, and the derivative here is covariant derivative with respect to the metric on $\mathcal{M}$. It is clear that the Gauss formula and the Gauss equation in (2.1) are different from those in Euclidean space. Therefore, the Ricci identity becomes,

$$\begin{aligned} h_{ijkl} = &\quad h_{ijlk} + h_{mj}R_{imlk} + h_{im}R_{jmlk} \\ = &\quad h_{klij} - (h_{mj}h_{il} - h_{ml}h_{ij})h_{mk} - (h_{mj}h_{kl} - h_{ml}h_{kj})h_{mi}. \end{aligned} \tag{2.2}$$

Although in this paper we only study constant σ_{n-1} curvature hypersurface, we will need to use other elementary sysmetric polynomials in the process. In the following, we will introduce notations and properties for general curvature functions.

Recall that the k-th elementary symmetric polynomial is defined by,

$$\sigma_k(\kappa) = \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} \kappa_{i_1}\cdots\kappa_{i_k},$$

where $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n) \in \mathbb{R}^n$ and $1 \leqslant k \leqslant n$. We also set $\sigma_0(\kappa) = 1$ and $\sigma_k(\kappa) = 0$ for $k > n$. It's well known that, a suitable domain of definition for σ_k is the Gårding cone Γ_k (see [6]). By the definition of Γ_k, we can see that

$$\Gamma_n \subset \cdots \subset \Gamma_k \cdots \subset \Gamma_1.$$

Moreover, Korevaar [19] showed that the Gårding cone Γ_k can also be characterized as

$$\left\{\kappa\in\mathbb{R}^n; \sigma_k(\kappa)>0, \frac{\partial\sigma_k(\kappa)}{\partial\kappa_{i_1}}>0, \cdots, \frac{\partial^k\sigma_k(\kappa)}{\partial\kappa_{i_1}\cdots\partial\kappa_{i_k}}>0, \text{ for all } 1\leqslant i_1<\cdots<i_k\leqslant n\right\}. \tag{2.3}$$

This characterization will be used throughout this paper. It's particularly useful in analyzing equation (3.10).

Let $\mathcal{S}$ be the vector space of $n\times n$ symmetric matrices and

$$\mathcal{S}_K = \{W\in\mathcal{S} : \kappa[W]\in K\},$$

where $\kappa[W] = (\kappa_1, \cdots, \kappa_n)$ denotes the eigenvalues of W, and K is the admissable set, for example, Γ_k. We let $\kappa[W]$ represent the eigenvalues of the matrix $W = (w_{ij})$. Define a function F by

$$f(\kappa[W]) = F(W).$$

Throughout this paper we denote,

$$F^{pq} = \frac{\partial F}{\partial w_{pq}}, \text{ and } F^{pq,rs} = \frac{\partial^2 F}{\partial w_{pq} \partial w_{rs}}.$$

The matrix $\left(F^{ij}(W)\right)$ is symmetric and has eigenvalues $f_1, \cdots, f_n$, where $f_i = \frac{\partial f}{\partial \kappa_i}$, $1 \leqslant i \leqslant n$. Moreover, if f is a concave function in K, then F is concave as well. That is,

$$F^{ij,kl}(W)\xi_{ij}\xi_{kl} \leqslant 0, \forall\, (\xi_{ij}) \in \mathcal{S}, W \in \mathcal{S}_K.$$

In particular, we should keep in mind that both $\sigma_k^{1/k}(\kappa)$ and $\left(\frac{\sigma_k(\kappa)}{\sigma_l(\kappa)}\right)^{1/(k-l)}$, $l < k$, are concave functions in Γ_k, for $1 \leqslant k \leqslant n$. Let's recall the following well known Lemma (see [2]) which will be needed in the proof of Theorem 4.

Lemma 7. *Denote $Sym(n)$ the set of all $n \times n$ symmetric matrices. Let F be a C^2 symmetric function defined in some open subset $\Psi \subset Sym(n)$. At any diagonal matrix $W \in \Psi$ with distinct eigenvalues, let $\ddot{F}(B, B)$ be the second derivative of C^2 symmetric function F in direction $B \in Sym(n)$, then*

$$\ddot{F}(B, B) = \sum_{j,k=1}^{n} f_{jk} B_{jj} B_{kk} + 2 \sum_{j<k} \frac{f_j - f_k}{\kappa_j - \kappa_k} B_{jk}^2,$$

where $f_j = \frac{\partial f}{\partial \kappa_j}$ and $f_{jk} = \frac{\partial^2 f}{\partial \kappa_j \partial \kappa_k}$.

From the discussion above, we can see that the definition of the k-th elementary symmetric polynomial can be extended to symmetric matrices. Suppose W is an $n \times n$ symmetric matrix and $\kappa[W] \subset \Gamma_k$. We define

$$\sigma_k(W) = \sigma_k(\kappa[W]).$$

In the following, we list some algebraic identities and properties of σ_k that will be used later. For $1 \leqslant l \leqslant n$, we define $\sigma_l(\kappa|a)$ the l-th elementary symmetric polynomial of $\kappa_1, \kappa_2, \cdots, \kappa_n$ with $\kappa_a = 0$, $\sigma_l(\kappa|ab)$ the l-th elementary symmetric polynomial of $\kappa_1, \kappa_2, \cdots, \kappa_n$ with $\kappa_a = \kappa_b = 0$, and similarly, we can define $\sigma_l(\kappa|abc\cdots)$. Thus, we have

(i) $\sigma_k^{pp}(\kappa) := \dfrac{\partial \sigma_k(\kappa)}{\partial \kappa_p} = \sigma_{k-1}(\kappa|p)$ for any $p = 1, \cdots, n$;

(ii) $\sigma_k^{pp,qq}(\kappa) := \dfrac{\partial^2 \sigma_k(\kappa)}{\partial \kappa_p \partial \kappa_q} = \sigma_{k-2}(\kappa|pq)$ for any $p, q = 1, \cdots, n$ and $\sigma_k^{pp,pp}(\kappa) = 0$;

(iii) $\sigma_k(\kappa) = \kappa_i \sigma_{k-1}(\kappa|i) + \sigma_k(\kappa|i)$ for any fixed $1 \leqslant i \leqslant n$;

(iv) $\displaystyle\sum_{i=1}^{n} \kappa_i \sigma_{k-1}(\kappa|i) = k\sigma_k(\kappa)$.

Moreover, for a Codazzi tensor $W = (w_{ij})$, if W is diagonal, then we have

(v) $-\sum_{p,q,r,s}\sigma_k^{pq,rs}w_{pql}w_{rsl}=\sum_{p,q}\sigma_k^{pp,qq}w_{pql}^2-\sum_{p,q}\sigma_k^{pp,qq}w_{ppl}w_{qql}$,
where w_{pql} is the covariant derivative of w_{pq} and $\sigma_k^{pq,rs}=\frac{\partial^2\sigma_k(W)}{\partial w_{pq}\partial w_{rs}}$. The definition of the Codazzi tensor can be found in [16].
For $\kappa\in\Gamma_k$, if we assume $\kappa_1\geqslant\cdots\geqslant\kappa_n$, then we have
(vi) $\sigma_{k-1}(\kappa|n)\geqslant\cdots\geqslant\sigma_{k-1}(\kappa|1)>0$;
(vii) $\kappa_1\sigma_{k-1}(\kappa|1)\geqslant C_{n,k}\sigma_k(\kappa)$,
where $C_{n,k}$ is a positive constant depending only on n,k. Details of the proof of these formulas can be found in [18] and [27].

3. Convexity estimates of the hypersurface

In this section, we will start to study the convexity of the spacelike hypersurface $\mathcal{M}_u=\{(x,u(x))|x\in\mathbb{R}^n,|Du|<1\}$ that satisfies the following conditions:

- $\kappa[\mathcal{M}_u]\in\Gamma_{n-1}$;
- $\sigma_{n-1}(\kappa[\mathcal{M}_u])=1$.

If a spacelike hypersurface $\mathcal{M}_u$ satisfies these conditions, then we say $\mathcal{M}_u$ is *admissible*. Note that here we don't require the principal curvatures of $\mathcal{M}_u$ are bounded.

Next, we will state one of our main theorems. This theorem plays a key role in the proof of Theorem 2.

Theorem 8. *Let $\mathcal{M}_u$ be an admissible hypersurface and $\kappa[\mathcal{M}_u]$ be its principal curvatures, then we have*

$$\sigma_{n-1}^{ij}(\sigma_n(\kappa[\mathcal{M}_u]))_{ij}\leqslant\sigma_1\sigma_{n-1}\sigma_n-n^2\sigma_n^2.\tag{3.1}$$

Since the proof of Theorem 8 is very complicated, we will split it into 3 sections. In this section, we will simplify our equation and reduce the proof of this theorem into the proof of the semi-positivity of a matrix. In the next two sections, we will confirm that the matrix we obtain here is indeed semi-positive.

Lemma 9. *Inequality* (3.1) *holds if the following inequality holds on $\mathcal{M}_u$:*

$$\begin{aligned}&\sum_{j\neq1}\sigma_{n-2}^2(\kappa|1j)\left[2\sigma_{n-2}(\kappa|1)+2\sigma_{n-2}(\kappa|j)\right]h_{jj1}^2\\&+\sum_{p,q\neq1;p\neq q}\sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq)\left[\sigma_{n-2}(\kappa|1)+(\kappa_p+\kappa_q-2\kappa_1)\sigma_{n-3}(\kappa|1pq)\right]h_{pp1}h_{qq1}\geqslant0.\end{aligned}\tag{3.2}$$

where h_{ijk} is the covariant derivative of the second fundamental form h_{ij} .

Proof. For an arbitrary $X_0 \in \mathcal{M}_u$, we can choose an orthonormal local frame $\tau_1, \cdots, \tau_n$ around X_0 on $T\mathcal{M}_u$, such that at X_0,

$$h_{ij} = \kappa_i \delta_{ij},$$

where $\kappa_1, \cdots, \kappa_n$ are the principal curvatures of $\mathcal{M}_u$ at X_0. Our calculation below is done at the point X_0. We will consider the test function

$$\phi = \sigma_n(h).$$

Differentiating ϕ twice, we get

$$\phi_i = \sigma_n^{jj} h_{jji} \tag{3.3}$$

and

$$\phi_{ii} = \sigma_n^{jj} h_{jjii} + \sigma_n^{pq,rs} h_{pqi} h_{rsi}. \tag{3.4}$$

Contracting with σ_{n-1}^{ii} on both sides we have,

$$\sigma_{n-1}^{ii}\phi_{ii} = \sigma_{n-1}^{ii}\sigma_n^{jj} h_{jjii} + \sigma_{n-1}^{ii}\sigma_n^{pq,rs} h_{pqi} h_{rsi}. \tag{3.5}$$

Now, let's differentiate equation (1.2) twice, then we obtain

$$\sigma_{n-1}^{ii} h_{iij} = 0, \tag{3.6}$$

and

$$\sigma_{n-1}^{ii} h_{iijj} + \sigma_{n-1}^{pq,rs} h_{pqj} h_{rsj} = 0. \tag{3.7}$$

By (2.2), we can see that at X_0 we have

$$h_{jjii} = h_{iijj} + h_{ii}^2 h_{jj} - h_{ii} h_{jj}^2.$$

Thus, we get

$$\begin{aligned}
&\sigma_{n-1}^{ii}\phi_{ii} \\
=& -\sigma_n^{jj}\sigma_{n-1}^{pq,rs} h_{pqj} h_{rsj} + \sigma_{n-1}^{ii}\sigma_n^{pq,rs} h_{pqi} h_{rsi} + \sigma_{n-1}^{ii}\sigma_n^{jj}(h_{ii}^2 h_{jj} - h_{ii} h_{jj}^2) \\
=& (\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq}) h_{ppi} h_{qqi} - (\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq}) h_{pqi}^2 \\
&+ \sigma_1\sigma_{n-1}\sigma_n - n^2\sigma_n^2,
\end{aligned} \tag{3.8}$$

where we used $\sigma_{n-1}^{pq,rs} h_{pqi} h_{rsi} = \sigma_{n-1}^{pp,qq} h_{ppi} h_{qqi} - \sigma_{n-1}^{pp,qq} h_{pqi}^2$.

In order to prove inequality (3.1) we only need to prove

$$(\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq}) h_{ppi} h_{qqi} - (\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq}) h_{pqi}^2 \leqslant 0. \tag{3.9}$$

First, when the indices i, p, q are not equal to each other, by a straightforward calculation we get,

$$
\begin{aligned}
&\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq} \\
=& \sigma_{n-2}(\kappa|i)\sigma_{n-2}(\kappa|pq) - \sigma_{n-1}(\kappa|i)\sigma_{n-3}(\kappa|pq) \\
=& [\kappa_p\kappa_q\sigma_{n-4}(\kappa|ipq) + (\kappa_p+\kappa_q)\sigma_{n-3}(\kappa|ipq)]\kappa_i\sigma_{n-3}(\kappa|ipq) \\
&-\kappa_p\kappa_q\sigma_{n-3}(\kappa|ipq)[\kappa_i\sigma_{n-4}(\kappa|ipq) + \sigma_{n-3}(\kappa|ipq)] \\
=& \sigma_{n-3}^2(\kappa|ipq)(\kappa_i\kappa_p + \kappa_i\kappa_q - \kappa_p\kappa_q).
\end{aligned} \tag{3.10}
$$

Since $\kappa \in \Gamma_{n-1}$, by (2.3) we have

$$\sigma_2(\kappa_i, \kappa_p, \kappa_q) = \kappa_i\kappa_p + \kappa_i\kappa_q + \kappa_p\kappa_q > 0.$$

Also note that

$$\sigma_{n-3}(\kappa|ipq) = \sigma_{n-3}(\kappa|piq) = \sigma_{n-3}(\kappa|iqp).$$

Therefore, by rotating i, p, q and summing them up we obtain

$$-\sum_{i\neq p\neq q}(\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq})h_{pqi}^2 \leqslant 0. \tag{3.11}$$

In view of (3.9), now we only need to prove, for any fixed i, $1 \leqslant i \leqslant n$,

$$L_i := 2\sum_{j\neq i}(\sigma_{n-1}^{jj}\sigma_n^{ii,jj} - \sigma_n^{jj}\sigma_{n-1}^{ii,jj})h_{jji}^2 - \sum_{p\neq q}(\sigma_{n-1}^{ii}\sigma_n^{pp,qq} - \sigma_n^{ii}\sigma_{n-1}^{pp,qq})h_{ppi}h_{qqi} \geqslant 0. \tag{3.12}$$

Next, without loss of generality, we will consider the case when $i = 1$. Namely, we will show that $L_1 \geqslant 0$.

From equation (3.6),we have

$$h_{111} = -\sum_{j\neq 1}\frac{\sigma_{n-1}^{jj}}{\sigma_{n-1}^{11}}h_{jj1}. \tag{3.13}$$

Plugging it into equation (3.12) we get,

$$
\begin{aligned}
L_1 =&2\sum_{j\neq 1}(\sigma_{n-1}^{jj}\sigma_n^{11,jj}-\sigma_n^{jj}\sigma_{n-1}^{11,jj})h_{jj1}^2-\sum_{p\neq q,p,q\neq 1}(\sigma_{n-1}^{11}\sigma_n^{pp,qq}-\sigma_n^{11}\sigma_{n-1}^{pp,qq})h_{pp1}h_{qq1}\\
&+\sum_{q\neq 1}(\sigma_{n-1}^{11}\sigma_n^{11,qq}-\sigma_n^{11}\sigma_{n-1}^{11,qq})\sum_{r\neq 1}\frac{\sigma_{n-1}^{rr}}{\sigma_{n-1}^{11}}h_{rr1}h_{qq1}\\
&+\sum_{p\neq 1}(\sigma_{n-1}^{11}\sigma_n^{11,pp}-\sigma_n^{11}\sigma_{n-1}^{11,pp})\sum_{s\neq 1}\frac{\sigma_{n-1}^{ss}}{\sigma_{n-1}^{11}}h_{pp1}h_{ss1}\\
=&\sum_{j\neq 1}\left(4\sigma_{n-1}^{jj}\sigma_n^{11,jj}-2\sigma_n^{jj}\sigma_{n-1}^{11,jj}-2\sigma_n^{11}\sigma_{n-1}^{11,jj}\frac{\sigma_{n-1}^{jj}}{\sigma_{n-1}^{11}}\right)h_{jj1}^2\\
&+\sum_{p\neq q,p,q\neq 1}\Big[\sigma_{n-1}^{pp}\sigma_n^{11,qq}+\sigma_{n-1}^{qq}\sigma_n^{11,pp}-\sigma_{n-1}^{11}\sigma_n^{pp,qq}+\sigma_n^{11}\sigma_{n-1}^{pp,qq}\\
&-\frac{\sigma_n^{11}}{\sigma_{n-1}^{11}}\left(\sigma_{n-1}^{11,pp}\sigma_{n-1}^{qq}+\sigma_{n-1}^{11,qq}\sigma_{n-1}^{pp}\right)\Big]h_{pp1}h_{qq1}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\sigma_{n-1}^{11}L_1 =&\sum_{j\neq 1}\left(4\sigma_{n-1}^{11}\sigma_{n-1}^{jj}\sigma_n^{11,jj}-2\sigma_{n-1}^{11}\sigma_n^{jj}\sigma_{n-1}^{11,jj}-2\sigma_n^{11}\sigma_{n-1}^{jj}\sigma_{n-1}^{11,jj}\right)h_{jj1}^2\\
&+\sum_{p\neq q,p,q\neq 1}\big(\sigma_{n-1}^{11}\sigma_{n-1}^{pp}\sigma_n^{11,qq}+\sigma_{n-1}^{11}\sigma_{n-1}^{qq}\sigma_n^{11,pp}+\sigma_{n-1}^{11}\sigma_n^{11}\sigma_{n-1}^{pp,qq}\\
&-\sigma_{n-1}^{11}\sigma_{n-1}^{11}\sigma_n^{pp,qq}-\sigma_n^{11}\sigma_{n-1}^{11,pp}\sigma_{n-1}^{qq}-\sigma_n^{11}\sigma_{n-1}^{11,qq}\sigma_{n-1}^{pp}\big)h_{pp1}h_{qq1}
\end{aligned}
\tag{3.14}
$$

Finally, we want to simplify equation (3.14). It is straightforward to verify

$$
\begin{aligned}
&\sigma_{n-1}^{jj}\sigma_n^{11,jj}-\sigma_n^{jj}\sigma_{n-1}^{11,jj}\\
=&\sigma_{n-2}(\kappa|j)\sigma_{n-2}(\kappa|1j)-\sigma_{n-1}(\kappa|j)\sigma_{n-3}(\kappa|1j)\\
=&\big(\kappa_1\sigma_{n-3}(\kappa|1j)+\sigma_{n-2}(\kappa|1j)\big)\sigma_{n-2}(\kappa|1j)-\kappa_1\sigma_{n-2}(\kappa|1j)\sigma_{n-3}(\kappa|1j)\\
=&\sigma_{n-2}^2(\kappa|1j).
\end{aligned}
$$

Similarly, we get

$$
\sigma_{n-1}^{11}\sigma_n^{11,jj}-\sigma_n^{11}\sigma_{n-1}^{11,jj}=\sigma_{n-2}^2(\kappa|1j).
\tag{3.15}
$$

Therefore,

$$
\begin{aligned}
&4\sigma_{n-1}^{11}\sigma_{n-1}^{jj}\sigma_n^{11,jj}-2\sigma_{n-1}^{11}\sigma_n^{jj}\sigma_{n-1}^{11,jj}-2\sigma_n^{11}\sigma_{n-1}^{jj}\sigma_{n-1}^{11,jj}\\
&=2\sigma_{n-2}^2(\kappa|1j)\big(\sigma_{n-2}(\kappa|1)+\sigma_{n-2}(\kappa|j)\big).
\end{aligned}
\tag{3.16}
$$

Moreover, by (3.15) we obtain

$$\begin{aligned}(3.17)\qquad &\sigma_{n-1}^{11}\sigma_{n-1}^{pp}\sigma_n^{11,qq}-\sigma_n^{11}\sigma_{n-1}^{11,qq}\sigma_{n-1}^{pp}\\ &=\sigma_{n-1}^{pp}\left(\sigma_{n-1}^{11}\sigma_n^{11,qq}-\sigma_n^{11}\sigma_{n-1}^{11,qq}\right)\\ &=\sigma_{n-2}(\kappa|p)\kappa_p^2\sigma_{n-3}^2(\kappa|1pq)\\ &=[\sigma_{n-1}-\sigma_{n-1}(\kappa|p)]\kappa_p\sigma_{n-3}^2(\kappa|1pq)\\ &=\kappa_p\sigma_{n-1}\sigma_{n-3}^2(\kappa|1pq)-\sigma_n\sigma_{n-3}^2(\kappa|1pq).\end{aligned}$$

Similarly, we have

$$(3.18)\qquad \sigma_{n-1}^{11}\sigma_{n-1}^{qq}\sigma_n^{11,pp}-\sigma_n^{11}\sigma_{n-1}^{11,pp}\sigma_{n-1}^{qq}=\kappa_q\sigma_{n-1}\sigma_{n-3}^2(\kappa|1pq)-\sigma_n\sigma_{n-3}^2(\kappa|1pq).$$

Finally, we compute

$$\begin{aligned}(3.19)\qquad &\sigma_{n-1}^{11}\sigma_n^{11}\sigma_{n-1}^{pp,qq}-\sigma_{n-1}^{11}\sigma_{n-1}^{11}\sigma_n^{pp,qq}\\ &=\sigma_{n-2}(\kappa|1)\big(\sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|pq)-\sigma_{n-2}(\kappa|1)\sigma_{n-2}(\kappa|pq)\big)\\ &=\sigma_{n-2}(\kappa|1)\big\{\kappa_p\kappa_q\sigma_{n-3}(\kappa|1pq)\big[\kappa_1\sigma_{n-4}(\kappa|1pq)+\sigma_{n-3}(\kappa|1pq)\big]\\ &\quad-\big[\kappa_p\kappa_q\sigma_{n-4}(\kappa|1pq)+(\kappa_p+\kappa_q)\sigma_{n-3}(\kappa|1pq)\big]\kappa_1\sigma_{n-3}(\kappa|1pq)\big\}\\ &=\sigma_{n-2}(\kappa|1)\big[\kappa_p\kappa_q\sigma_{n-3}^2(\kappa|1pq)-\kappa_1(\kappa_p+\kappa_q)\sigma_{n-3}^2(\kappa|1pq)\big]\\ &=\sigma_{n-2}(\kappa|1)\sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq)-\big(\sigma_{n-1}-\sigma_{n-1}(\kappa|1)\big)(\kappa_p+\kappa_q)\sigma_{n-3}^2(\kappa|1pq).\end{aligned}$$

Thus, we conclude

$$\begin{aligned}(3.20)\qquad &\sigma_{n-1}^{11}\sigma_{n-1}^{pp}\sigma_n^{11,qq}+\sigma_{n-1}^{11}\sigma_{n-1}^{qq}\sigma_n^{11,pp}+\sigma_{n-1}^{11}\sigma_n^{11}\sigma_{n-1}^{pp,qq}-\sigma_{n-1}^{11}\sigma_{n-1}^{11}\sigma_n^{pp,qq}\\ &-\sigma_n^{11}\sigma_{n-1}^{11,pp}\sigma_{n-1}^{qq}-\sigma_n^{11}\sigma_{n-1}^{11,qq}\sigma_{n-1}^{pp}\\ &=\sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq)\big(\sigma_{n-2}(\kappa|1)+(\kappa_p+\kappa_q-2\kappa_1)\sigma_{n-3}(\kappa|1pq)\big).\end{aligned}$$

Substituting equation (3.16) and (3.20) into (3.14) we obtain,

$$\begin{aligned}&\sigma_{n-1}^{11}L_1\\ &=\sum_{j\neq 1}\sigma_{n-2}^2(\kappa|1j)\Big[2\sigma_{n-2}(\kappa|1)+2\sigma_{n-2}(\kappa|j)\Big]h_{jj1}^2\\ &\quad+\sum_{p\neq q,p,q\neq 1}\sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq)\Big[\sigma_{n-2}(\kappa|1)+(\kappa_p+\kappa_q-2\kappa_1)\sigma_{n-3}(\kappa|1pq)\Big]h_{pp1}h_{qq1}.\end{aligned}$$

This completes the proof of Lemma 9. □

Now notice that if we can prove the $(n-1)\times(n-1)$ matrix $R=(r_{pq})$, where

$$r_{pq}=\begin{cases}\sigma_{n-2}^2(\kappa|1p)[2\sigma_{n-2}(\kappa|1)+2\sigma_{n-2}(\kappa|p)] & \text{for } p=q\\ \sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq)[\sigma_{n-2}(\kappa|1)+(\kappa_p+\kappa_q-2\kappa_1)\sigma_{n-3}(\kappa|1pq)] & \text{for } p\neq q\end{cases}$$

is semi-positive definite, then we will be done.

Observe that R can be written as the Hadamard product of matrix $T = (t_{pq})$ and $S = (s_{pq})$, where

$$t_{pq} = \begin{cases} \sigma_{n-2}^2(\kappa|1p) & \text{for } p = q \\ \sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq) & \text{for } p \neq q \end{cases}$$

and

$$s_{pq} = \begin{cases} 2\sigma_{n-2}(\kappa|1) + 2\sigma_{n-2}(\kappa|p) & \text{for } p = q \\ \sigma_{n-2}(\kappa|1) + (\kappa_p + \kappa_q - 2\kappa_1)\sigma_{n-3}(\kappa|1pq) & \text{for } p \neq q. \end{cases}$$

Since when $p \neq q$ we have,

$$\sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq) = \sigma_{n-2}(\kappa|1p)\sigma_{n-2}(\kappa|1q).$$

This implies the nonnegativity of the quadratic form that matrix T corresponding to,

$$\begin{aligned} &\sum_{j\neq 1} \sigma_{n-2}^2(\kappa|1j)h_{jj1}^2 + \sum_{p\neq q, p,q\neq 1} \sigma_{n-1}(\kappa|1)\sigma_{n-3}(\kappa|1pq)h_{pp1}h_{qq1} \\ = &\left[\sum_{j\neq 1} \sigma_{n-2}(\kappa|1j)h_{jj1}\right]^2 \geqslant 0. \end{aligned} \tag{3.21}$$

Thus, the matrix T is a semi-positive definite matrix. By the Schur product Theorem, if we can prove the matrix S is also a semi-positive definite matrix, then we would obtain R is semi-positive definite. This would complete the proof of Theorem 8.

We will devote the next two sections to proving the matrix S is semi-positive. In particular, in Section 4, we will show S is semi-positive by assuming $\kappa_1 \leqslant 0$; while in Section 5, we will prove the case when $\kappa_1 > 0$.

4. The case when $\kappa_1 \leqslant 0$

Let

$$\begin{aligned} Q_S := &\sum_{j\neq 1} \left[2\sigma_{n-2}(\kappa|1) + 2\sigma_{n-2}(\kappa|j)\right]\xi_j^2 \\ &+ \sum_{p\neq q, p,q\neq 1} \left[\sigma_{n-2}(\kappa|1) + (\kappa_p + \kappa_q - 2\kappa_1)\sigma_{n-3}(\kappa|1pq)\right]\xi_p\xi_q \end{aligned} \tag{4.1}$$

be the quadratic form of S. In this section, we will prove $Q_S \geqslant 0$ for $\kappa_1 \leqslant 0$.

Lemma 10. *If $\kappa_1 \leqslant 0$, then we have $Q_S \geqslant 0$. Therefore, the matrix S is a semi-positive definite matrix.*

Proof. Since $\kappa \in \Gamma_{n-1}$ and we assumed $\kappa_1 \leqslant 0$, it is clear that in this case we have $\kappa_2, \cdots, \kappa_n > 0$. Thus, we can define

$$\mu_2 = \frac{1}{\kappa_2}, \mu_3 = \frac{1}{\kappa_3}, \cdots, \mu_n = \frac{1}{\kappa_n}.$$

We can rewrite equation (1.2) as follows:

$$1 = \kappa_1 \sigma_{n-2}(\kappa|1) + \sigma_{n-1}(\kappa|1).$$

This gives

$$\kappa_1 = \frac{1 - \sigma_{n-1}(\kappa|1)}{\sigma_{n-2}(\kappa|1)}. \tag{4.2}$$

Moreover, for any given $j \neq 1$ we have,

$$\sigma_{n-2}(\kappa|1) = \kappa_j \sigma_{n-3}(\kappa|1j) + \sigma_{n-2}(\kappa|1j),$$

and

$$\sigma_{n-1}(\kappa|1) = \kappa_j \sigma_{n-2}(\kappa|1j).$$

Therefore, we get

$$\sigma_{n-3}(\kappa|1j) = \frac{1}{\kappa_j}\sigma_{n-2}(\kappa|1) - \frac{1}{\kappa_j^2}\sigma_{n-1}(\kappa|1). \tag{4.3}$$

Applying (4.2) and (4.3) we can derive the following equalities,
(4.4)

$$\begin{aligned}
&2\sigma_{n-2}(\kappa|1) + 2\sigma_{n-2}(\kappa|j) \\
=&2\sigma_{n-2}(\kappa|1) + 2\kappa_1\sigma_{n-3}(\kappa|1j) + 2\sigma_{n-2}(\kappa|1j) \\
=&2\sigma_{n-2}(\kappa|1) - 2\frac{\sigma_{n-1}(\kappa|1)}{\sigma_{n-2}(\kappa|1)}\sigma_{n-3}(\kappa|1j) + 2\sigma_{n-2}(\kappa|1j) + \frac{2\sigma_{n-3}(\kappa|1j)}{\sigma_{n-2}(\kappa|1)} \\
=&2\sigma_{n-2}(\kappa|1) - 2\frac{\sigma_{n-1}(\kappa|1)}{\sigma_{n-2}(\kappa|1)}\left(\frac{1}{\kappa_j}\sigma_{n-2}(\kappa|1) - \frac{1}{\kappa_j^2}\sigma_{n-1}(\kappa|1)\right) + 2\sigma_{n-2}(\kappa|1j) + \frac{2\sigma_{n-3}(\kappa|1j)}{\sigma_{n-2}(\kappa|1)} \\
=&2\sigma_{n-2}(\kappa|1) + 2\frac{\sigma_{n-1}^2(\kappa|1)}{\kappa_j^2\sigma_{n-2}(\kappa|1)} - 2\frac{\sigma_{n-1}(\kappa|1)}{\kappa_j} + 2\sigma_{n-2}(\kappa|1j) + \frac{2\sigma_{n-3}(\kappa|1j)}{\sigma_{n-2}(\kappa|1)} \\
=&2\frac{\sigma_1(\mu)}{\sigma_{n-1}(\mu)} + 2\frac{\mu_j^2}{\sigma_1(\mu)\sigma_{n-1}(\mu)} + \frac{2\sigma_{n-3}(\kappa|1j)}{\sigma_{n-2}(\kappa|1)} \\
=&\frac{2\sigma_1^2(\mu) + 2\mu_j^2}{\sigma_1(\mu)\sigma_{n-1}(\mu)} + \frac{2\sigma_{n-3}(\kappa|1j)}{\sigma_{n-2}(\kappa|1)},
\end{aligned}$$

and

$$
\begin{aligned}
&\sigma_{n-2}(\kappa|1)+(\kappa_p+\kappa_q-2\kappa_1)\sigma_{n-3}(\kappa|1pq)\\
=&\sigma_{n-2}(\kappa|1)+\left(\kappa_p+\kappa_q+2\frac{\sigma_{n-1}(\kappa|1)}{\sigma_{n-2}(\kappa|1)}\right)\sigma_{n-3}(\kappa|1pq)-\frac{2\sigma_{n-3}(\kappa|1pq)}{\sigma_{n-2}(\kappa|1)}\\
=&\frac{\sigma_1(\mu)}{\sigma_{n-1}(\mu)}+\left(\frac{1}{\mu_p}+\frac{1}{\mu_q}+2\frac{1}{\sigma_1(\mu)}\right)\frac{\sigma_{n-1}(\kappa|1)}{\kappa_p\kappa_q}-\frac{2\sigma_{n-3}(\kappa|1pq)}{\sigma_{n-2}(\kappa|1)}\\
=&\frac{\sigma_1(\mu)}{\sigma_{n-1}(\mu)}+\frac{(\mu_p+\mu_q)\sigma_1(\mu)+2\mu_p\mu_q}{\mu_p\mu_q\sigma_1(\mu)}\frac{1}{\kappa_p\kappa_q\sigma_{n-1}(\mu)}-\frac{2\sigma_{n-3}(\kappa|1pq)}{\sigma_{n-2}(\kappa|1)}\\
=&\frac{\sigma_1^2(\mu)+(\mu_p+\mu_q)\sigma_1(\mu)+2\mu_p\mu_q}{\sigma_1(\mu)\sigma_{n-1}(\mu)}-\frac{2\sigma_{n-3}(\kappa|1pq)}{\sigma_{n-2}(\kappa|1)}.
\end{aligned}
\tag{4.5}
$$

Now, let's consider the quadratic form Q_S that is corresponding to the matrix S,

$$
\begin{aligned}
Q_S=&\sum_{j\neq 1}\Big[2\sigma_{n-2}(\kappa|1)+2\sigma_{n-2}(\kappa|j)\Big]\xi_j^2\\
&+\sum_{p\neq q,p,q\neq 1}\Big[\sigma_{n-2}(\kappa|1)+(\kappa_p+\kappa_q-2\kappa_1)\sigma_{n-3}(\kappa|1pq)\Big]\xi_p\xi_q.
\end{aligned}
$$

By equations (4.4), (4.5), and the Lemma 10 in [24], we obtain

$$
\begin{aligned}
&\sigma_1(\mu)\sigma_{n-1}(\mu)Q_S\\
\geqslant&\sum_{j\neq 1}\Big[2\sigma_1^2(\mu)+2\mu_j^2\Big]\xi_j^2+\sum_{p\neq q,p,q\neq 1}\Big[\sigma_1^2(\mu)+(\mu_p+\mu_q)\sigma_1(\mu)+2\mu_p\mu_q\Big]\xi_p\xi_q\\
=&\sigma_1^2(\mu)\sum_{j\neq 1}\xi_j^2+\sigma_1^2(\mu)\left(\sum_{j\neq 1}\xi_j\right)^2+2\left(\sum_{j\neq 1}\mu_j\xi_j\right)^2+2\sigma_1(\mu)\sum_{p\neq 1}\mu_p\xi_p\sum_{q\neq 1,p}\xi_q,
\end{aligned}
\tag{4.6}
$$

where we used Lemma 10 in [24] to show

$$
\sum_{j\neq 1}\sigma_{n-3}(\kappa|1j)\xi_j^2-\sum_{p,q\neq 1}\sigma_{n-3}(\kappa|1pq)\xi_p\xi_q\geqslant 0.
$$

It is easy to see that

$$
\sum_{p\neq 1}\mu_p\xi_p\sum_{q\neq 1,p}\xi_q=\sum_{p\neq 1}\mu_p\xi_p\sum_{q\neq 1}\xi_q-\sum_{p\neq 1}\mu_p\xi_p^2.
\tag{4.7}
$$

Moreover, we have

$$
\begin{aligned}
&\sigma_1^2(\mu)\left(\sum_{j\neq 1}\xi_j\right)^2+\left(\sum_{j\neq 1}\mu_j\xi_j\right)^2+2\sigma_1(\mu)\sum_{p\neq 1}\mu_p\xi_p\sum_{q\neq 1}\xi_q\\
=&\left(\sigma_1(\mu)\sum_{j\neq 1}\xi_j+\sum_{j\neq 1}\mu_j\xi_j\right)^2.
\end{aligned}
\tag{4.8}
$$

Combining (4.7) and (4.8) with (4.6), we get

$$
\begin{aligned}
&\sigma_1(\mu)\sigma_{n-1}(\mu)Q_S \\
\geqslant\ & \sigma_1^2(\mu)\sum_{j\neq 1}\xi_j^2+\left(\sum_{j\neq 1}\mu_j\xi_j\right)^2-2\sigma_1(\mu)\sum_{j\neq 1}\mu_j\xi_j^2+\left(\sigma_1(\mu)\sum_{j\neq 1}\xi_j+\sum_{j\neq 1}\mu_j\xi_j\right)^2 \\
\geqslant\ & \sigma_1^2(\mu)\sum_{j\neq 1}\xi_j^2+\left(\sum_{j\neq 1}\mu_j\xi_j\right)^2-2\sigma_1(\mu)\sum_{j\neq 1}\mu_j\xi_j^2 \\
=\ & \sum_{j\neq 1}\sigma_1^2(\mu)\xi_j^2-2\sum_{j\neq 1}\sigma_1(\mu)\xi_j\mu_j\xi_j+\sum_{j\neq 1}\mu_j^2\xi_j^2+\sum_{p\neq q,p,q\neq 1}\mu_p\xi_p\mu_q\xi_q \\
=\ & \sum_{j\neq 1}\left(\sigma_1(\mu)-\mu_j\right)^2\xi_j^2+\sum_{p\neq q,p,q\neq 1}\mu_p\xi_p\mu_q\xi_q \\
\geqslant\ & \sum_{j\neq 1}\left(\sum_{s\neq 1,j}\mu_s^2\right)\xi_j^2+\sum_{p\neq q,p,q\neq 1}\mu_p\xi_p\mu_q\xi_q \\
=\ & \sum_{p\neq q,p,q\neq 1}\mu_p^2\xi_q^2+\sum_{p\neq q,p,q\neq 1}\mu_p\xi_q\mu_q\xi_p \\
=\ & \frac{1}{2}\sum_{p\neq q,p,q\neq 1}\left(\mu_p\xi_q+\mu_q\xi_p\right)^2 \\
\geqslant\ & 0.
\end{aligned} \tag{4.9}
$$

Since $\sigma_1(\mu)\sigma_{n-1}(\mu)$ is positive, we have $Q_S\geqslant 0$. This completes the proof of Lemma 10. □

5. The case when $\kappa_1>0$

In this section, we will prove the following Lemma and complete the proof of Theorem 8.

Lemma 11. *If $\kappa_1>0$, then for any $1\leqslant m\leqslant n-1$, the sum of all m-th principal minors of matrix S is nonnegative. Therefore, the matrix S is a semi-positive definite matrix.*

Before starting the proof of Lemma 11, we want to recall an important Lemma from [24]. We will use this Lemma many times throughout this section.

Lemma 12. *(Lemma 9 in* [24]*) Suppose $2\leqslant i_1<i_2<\cdots<i_m\leqslant n$ are m ordered indices. Let $D_m(i_1\cdots i_m)$ denote the m-th principal minor of the matrix $(c_{pq})_{2\leqslant p,q\leqslant n}$, where*

$$c_{pq}=\begin{cases}\sigma_{n-3}(\kappa|1p) & \textit{for } p=q\\ -\sigma_{n-3}(\kappa|1pq) & \textit{for } p\neq q.\end{cases}$$

Then we have

$$D_m(i_1\cdots i_m) = \det\begin{bmatrix} c_{i_1i_1} & c_{i_1i_2} & \cdots & c_{i_1i_m} \\ c_{i_2i_1} & a_{i_2i_2} & \cdots & c_{i_2i_m} \\ \vdots & \vdots & \ddots & \vdots \\ c_{i_mi_1} & c_{i_mi_2} & \cdots & c_{i_mi_m} \end{bmatrix} = \sigma_{n-2}^{m-1}(\kappa|1)\sigma_{n-(m+2)}(\kappa|1i_1\cdots i_m). \tag{5.1}$$

Moreover, after deleting the l-th row and k-th column, where $l \neq k$, we get,

$$B_{m-1} = \det\begin{bmatrix} c_{i_1i_1} & c_{i_1i_2} & \cdots & c_{i_2i_{k-1}} & c_{i_2i_{k+1}} & \cdots & c_{i_2i_m} \\ c_{i_2i_1} & b_{i_2i_2} & \cdots & c_{i_2i_{k-1}} & c_{i_2i_{k+1}} & \cdots & c_{i_2i_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{i_{l-1}i_1} & c_{i_{l-1}i_2} & \cdots & c_{i_{l-1}i_{k-1}} & c_{i_{l-1}i_{k+1}} & \cdots & c_{i_{l-1}i_m} \\ c_{i_{l+1}i_1} & c_{i_{l+1}i_2} & \cdots & c_{i_{l+1}i_{k-1}} & c_{i_{l+1}i_{k+1}} & \cdots & c_{i_{l+1}i_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{i_mi_1} & c_{i_mi_2} & \cdots & c_{i_mi_{k-1}} & c_{i_mi_{k+1}} & \cdots & c_{i_mi_m} \end{bmatrix} = (-1)^{l+k}\sigma_{n-2}^{m-2}(\kappa|1)\sigma_{n-m-1}(\kappa|1i_1\cdots i_m). \tag{5.2}$$

Next, in order to simplify our calculations, we will decompose S into three $(n-1)\times(n-1)$ matrices: $A = (a_{pq})_{2\leqslant p,q\leqslant n}$, $B = (b_{pq})_{2\leqslant p,q\leqslant n}$, and $C = \sigma_{n-2}(\kappa|1)Id_{n-1}$. Here

$$a_{pq} = \begin{cases} 2\sigma_{n-2}(\kappa|1p) + \sigma_{n-2}(\kappa|1) & \text{for } p = q \\ (\kappa_p + \kappa_q)\sigma_{n-3}(\kappa|1pq) + \sigma_{n-2}(\kappa|1) & \text{for } p \neq q \end{cases},$$

$$b_{pq} = \begin{cases} 2\kappa_1\sigma_{n-3}(\kappa|1p) & \text{for } p = q \\ -2\kappa_1\sigma_{n-3}(\kappa|1pq) & \text{for } p \neq q \end{cases},$$

and Id_{n-1} is the $(n-1)\times(n-1)$ identity matrix. By the equality

$$\sigma_{n-2}(\kappa|p) = \kappa_1\sigma_{n-3}(\kappa|1p) + \sigma_{n-2}(\kappa|1p),$$

we can see that

$$S = A + B + \sigma_{n-2}(\kappa|1)Id_{n-1}. \tag{5.3}$$

One of the key reasons that the above decomposition (5.3) can simplify our calculation is the following.

Lemma 13. *The rank of the matrix A is at most two.*

Proof. For any $2 \leqslant j \leqslant n$, let

$$m_j = \sigma_{n-2}(\kappa|1j) + \frac{\sigma_{n-2}(\kappa|1)}{2}.$$

Then, it is easy to see that $a_{pq} = m_p + m_q$. Therefore, we have

$$A = \begin{bmatrix} m_2 & 1 \\ m_3 & 1 \\ \cdots & \\ m_n & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ m_2 & m_3 & \cdots & m_n \end{bmatrix}.$$

Hence, the rank of A is at most two. $\square$

For our convenience, we will introduce some notations. We will denote the set of multiple indices

$$\mathcal{I}_k = \{(i_2, \cdots, i_k) | 2 \leqslant i_2 < \cdots < i_k \leqslant n\},$$

and we will use I_k to denote an element in $\mathcal{I}_k$. Moreover, if $I_k = (i_2, i_3, \cdots, i_k)$, then $|I_k|$ denotes the set $\{i_2, i_3, \cdots, i_k\}$. For example, we have $I_n = (2, \cdots, n) \in \mathcal{I}_n$ and $|I_n| = \{2, 3, \cdots, n\}$. We also need the following definition.

Definition 14. *Suppose $A = (a_{pq}), B = (b_{pq})$ are two $(n-1) \times (n-1)$ matrices and $I_k = (i_2, i_3 \cdots, i_k) \in \mathcal{I}_k$ is a multiple index. We define the following principal minors $D_A(I_k), D_B(I_k)$ of matrices A and B respectively:*

$$D_A(I_k) = \det \begin{bmatrix} a_{i_2 i_2} & a_{i_2 i_3} & \cdots & a_{i_2 i_k} \\ a_{i_3 i_2} & a_{i_3 i_3} & \cdots & a_{i_3 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_k i_2} & a_{i_k i_3} & \cdots & a_{i_k i_k} \end{bmatrix}, \quad D_B(I_k) = \det \begin{bmatrix} b_{i_2 i_2} & b_{i_2 i_3} & \cdots & b_{i_2 i_k} \\ b_{i_3 i_2} & b_{i_3 i_3} & \cdots & b_{i_3 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{i_k i_2} & b_{i_k i_3} & \cdots & b_{i_k i_k} \end{bmatrix}.$$

For indices $i_l, i_p, i_q \in |I_k|$ and $i_p < i_q$, we also define the following "mixed" principal minors $D_{B,A}(I_k; i_l)$, $D_{B,A}(I_k; i_p i_q)$ of A, B:

$$D_{B,A}(I_k; i_l) = \det \begin{bmatrix} b_{i_2 i_2} & b_{i_2 i_3} & \cdots & b_{i_2 i_l} & \cdots & b_{i_2 i_k} \\ b_{i_3 i_2} & b_{i_3 i_3} & \cdots & b_{i_3 i_l} & \cdots & b_{i_3 i_k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_l i_2} & a_{i_l i_3} & \cdots & a_{i_l i_l} & \cdots & a_{i_l i_k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i_k i_2} & b_{i_k i_3} & \cdots & b_{i_k i_l} & \cdots & b_{i_k i_k} \end{bmatrix}.$$

and

$$D_{B,A}(I_k; i_p i_q) = \det \begin{bmatrix} b_{i_2 i_2} & b_{i_2 i_3} & \cdots & b_{i_2 i_p} & \cdots & b_{i_2 i_q} & \cdots & b_{i_2 i_k} \\ b_{i_3 i_2} & b_{i_3 i_3} & \cdots & b_{i_3 i_p} & \cdots & b_{i_3 i_q} & \cdots & b_{i_3 i_k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_p i_2} & a_{i_p i_3} & \cdots & a_{i_p i_p} & \cdots & a_{i_p i_q} & \cdots & a_{i_p i_k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i_q i_2} & a_{i_q i_3} & \cdots & a_{i_q i_p} & \cdots & a_{i_q i_q} & \cdots & a_{i_q i_k} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{i_k i_2} & b_{i_k i_3} & \cdots & b_{i_k i_p} & \cdots & b_{i_k i_q} & \cdots & b_{i_k i_k} \end{bmatrix}.$$

Finally, we are ready to prove Lemma 11. In the following, we will start with computing the principal minors of the matrix $A+B$. Then, we will compute the sum of all m-th principal minors of S for any $1 \leqslant m \leqslant n-1$.

By Lemma 13 we know that the rank of A is at most two. Thus, for $k \geqslant 4$ and any multiple index $I_k = (i_2, \cdots, i_k) \in \mathcal{I}_k$, we have

$$D_{A+B}(I_k) = D_B(I_k) + \sum_{i_l \in |I_k|} D_{B,A}(I_k; i_l) + \sum_{i_p, i_q \in |I_k|, i_p < i_q} D_{B,A}(I_k; i_p i_q). \tag{5.4}$$

For a given $I_k = (i_2, \cdots, i_k) \in \mathcal{I}_k$ and any integer $2 \leqslant s \leqslant k$, let's denote

$$\mathcal{J}_s(I_k) = \{(j_2, j_3, \cdots, j_s) | j_2 < j_3 < \cdots < j_s, \text{ where } j_l \in |I_k| \text{ for } l = 2, 3, \cdots, s\}.$$

Then by a straightforward calculation we get,

$$\begin{aligned} &\sum_{I_k \in \mathcal{I}_k} D_S(I_k) \\ =& \sum_{I_k \in \mathcal{I}_k} D_{A+B}(I_k) + \sum_{I_k \in \mathcal{I}_k} \sum_{J_{k-1} \in \mathcal{J}_{k-1}(I_k)} D_{A+B}(J_{k-1}) \sigma^1_{n-2}(\kappa|1) \\ &+ \sum_{I_k \in \mathcal{I}_k} \sum_{J_{k-2} \in \mathcal{J}_{k-2}(I_k)} D_{A+B}(J_{k-2}) \sigma^2_{n-2}(\kappa|1) + \cdots + \sum_{I_k \in \mathcal{I}_k} \sum_{J_s \in \mathcal{J}_s(I_k)} D_{A+B}(J_s) \sigma^{k-s}_{n-2}(\kappa|1) \\ &+ \cdots + \sum_{I_k \in \mathcal{I}_k} \sum_{J_2 \in \mathcal{J}_2(I_k)} D_{A+B}(J_2) \sigma^{k-2}_{n-2}(\kappa|1) + \sum_{I_k \in \mathcal{I}_k} \sigma^{k-1}_{n-2}(\kappa|1) \end{aligned} \tag{5.5}$$

From Lemma 15 to Lemma 21, we will compute terms involved in equations (5.4) and (5.5).

Lemma 15. *For a given $I_k = (i_2, \cdots, i_k) \in \mathcal{I}_k$, we have*

$$D_B(I_k) = (2\kappa_1)^{k-1} \sigma^{k-2}_{n-2}(\kappa|1) \sigma_{n-k-1}(\kappa|1 i_2 \cdots i_k).$$

Proof. By our definition of matrix B, we can see that for any $2 \leqslant p, q \leqslant n$,

$$\frac{b_{pq}}{2\kappa_1} = c_{pq}.$$

Thus the result follows from Lemma 12 directly. □

In next Lemma, we will compute the summation of principal minors of matrix B.

Lemma 16. *For any integer $2 \leqslant s \leqslant k$, we have*

$$\sum_{I_k \in \mathcal{I}_k} \sum_{J_s \in \mathcal{J}_s(I_k)} D_B(J_s) = \frac{s(n-s)!}{(k-s)!(n-k)!} (2\kappa_1)^{s-1} \sigma^{s-2}_{n-2}(\kappa|1) \sigma_{n-(s+1)}(\kappa|1).$$

Proof. Applying Lemma 15, we get

$$\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)} D_B(J_s)\\
=&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)} (2\kappa_1)^{s-1}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-(s+1)}(\kappa|1j_2\cdots j_s)\\
=&\frac{(n-s)C_{k-1}^{s-1}C_{n-1}^{k-1}}{C_{n-1}^{n-(s+1)}}(2\kappa_1)^{s-1}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-(s+1)}(\kappa|1)\\
=&\frac{s(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-1}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-(s+1)}(\kappa|1).
\end{aligned}$$

□

Before we continue with the calculation of principal minors, we need the following Lemma.

Lemma 17. *For any ordered indices $2\leqslant i_2<i_3<\cdots<i_k\leqslant n$, we have*

$$\begin{aligned}
&\sigma_{n-1}(\kappa|1)\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)+\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{l=2}^k\sigma_{n-2}(\kappa|1i_l)\\
&=\sigma_{n-2}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots i_k).
\end{aligned}\tag{5.6}$$

Proof. An inductive calculation shows,

$$\begin{aligned}
&\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^k\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\sigma_{n-1}(\kappa|1)\\
=&\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^k\sigma_{n-2}(\kappa|1i_s)+\kappa_{i_k}\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\sigma_{n-2}(\kappa|1i_k)\\
=&\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^k\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-k}(\kappa|1i_2\cdots i_{k-1})\sigma_{n-2}(\kappa|1i_k)\\
&-\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sigma_{n-2}(\kappa|1i_k)\\
=&\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^{k-1}\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-k}(\kappa|1i_2\cdots i_{k-1})\sigma_{n-2}(\kappa|1i_k)\\
=&\kappa_{i_k}\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^{k-1}\sigma_{n-3}(\kappa|1i_si_k)+\kappa_{i_{k-1}}\sigma_{n-k}(\kappa|1i_2\cdots i_{k-1})\sigma_{n-3}(\kappa|1i_{k-1}i_k)\\
=&\sigma_{n-k+1}(\kappa|1i_2\cdots i_{k-1})\sum_{s=2}^{k-1}\sigma_{n-3}(\kappa|1i_si_k)+\sigma_{n-k+1}(\kappa|1i_2\cdots i_{k-2})\sigma_{n-3}(\kappa|1i_{k-1}i_k)\\
&-\sigma_{n-k+1}(\kappa|1i_2\cdots i_{k-1})\sigma_{n-3}(\kappa|1i_{k-1}i_k)
\end{aligned}$$

$$
\begin{aligned}
&=\sigma_{n-k+1}(\kappa|1i_2\cdots i_{k-1})\sum_{s=2}^{k-2}\sigma_{n-3}(\kappa|1i_si_k)+\sigma_{n-k+1}(\kappa|1i_2\cdots i_{k-2})\sigma_{n-3}(\kappa|1i_{k-1}i_k)\\
&=\cdots\\
&=\sigma_{n-2}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots i_k).
\end{aligned}
$$

□

Now, let's get back to compute some more complicated principal minors.

Lemma 18. *For any multiple index $I_k=(i_2,i_3,\cdots,i_k)\in\mathcal{I}_k$, we have*

$$
\begin{aligned}
&\sum_{i_l\in|I_k|}D_{B,A}(I_k;i_l)\\
=&(2\kappa_1)^{k-2}\sigma_{n-2}^{k-2}(\kappa|1)\left[k(k-1)\sigma_{n-k}(\kappa|1i_2\cdots i_k)+\sum_{i_s\in|I_k|}\sigma_{n-k}(\kappa|1i_2\cdots\hat{i}_s\cdots i_k)\right].
\end{aligned}
$$

Here $\hat{i}_s$ means that the index i_s does not appear.

Proof. Given a index $i_l\in|I_k|$, for computing the "mixed" principal minor $D_{B,A}(I_k;i_l)$, we expand the determinant according to its i_l-th row,

$$
D_{B,A}(I_k;i_l)=a_{i_li_l}M_l+\sum_{s\neq l,s=2}^{k}(-1)^{s-1+l-1}a_{i_li_s}M_s. \tag{5.7}
$$

Here the minors M_s for $s=2,3\cdots,k$ are defined by

$$
M_s=\det\begin{bmatrix}
b_{i_2i_2} & b_{i_2i_3} & \cdots & b_{i_2i_{s-1}} & b_{i_2i_{s+1}} & \cdots & b_{i_2i_k}\\
b_{i_3i_2} & b_{i_3i_3} & \cdots & b_{i_3i_{s-1}} & b_{i_3i_{s+1}} & \cdots & b_{i_3i_k}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
b_{i_{l-1}i_2} & b_{i_{l-1}i_3} & \cdots & b_{i_{l-1}i_{s-1}} & b_{i_{l-1}i_{s+1}} & \cdots & b_{i_{l-1}i_k}\\
b_{i_{l+1}i_2} & b_{i_{l+1}i_3} & \cdots & b_{i_{l+1}i_{s-1}} & b_{i_{l+1}i_{s+1}} & \cdots & b_{i_{l+1}i_k}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
b_{i_ki_2} & b_{i_ki_3} & \cdots & b_{i_ki_{s-1}} & b_{i_ki_{s+1}} & \cdots & b_{i_ki_k}
\end{bmatrix}.
$$

By Lemma 12 we have,

$$
M_l=(2\kappa_1)^{k-2}\sigma_{n-2}^{k-3}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots\hat{i}_l\cdots i_k); \tag{5.8}
$$

and when $s\neq l$

$$
M_s=(-1)^{l-1+s-1}(2\kappa_1)^{k-2}\sigma_{n-2}^{k-3}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots i_k). \tag{5.9}
$$

Combing (5.8) and (5.9) with (5.7) we obtain,

(5.10)
$$\begin{aligned}&\sum_{i_l\in|I_k|} D_{B,A}(I_k;i_l)\\ =&(2\kappa_1)^{k-2}\sigma_{n-2}^{k-3}(\kappa|1)\sum_{l=2}^{k}\left(a_{i_li_l}\sigma_{n-k}(\kappa|1i_2\cdots\hat{i}_l\cdots i_k)+\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2,s\neq l}^{k}a_{i_li_s}\right).\end{aligned}$$

A straightforward calculation gives

$$\sigma_{n-k}(\kappa|1i_2\cdots\hat{i}_l\cdots i_k)=\kappa_{i_l}\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)+\sigma_{n-k}(\kappa|1i_2\cdots i_k). \tag{5.11}$$

Using the definition of a_{pq}, we get

$$\begin{aligned}&a_{i_li_l}\sigma_{n-k}(\kappa|1i_2\cdots\hat{i}_l\cdots i_k)+\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2,s\neq l}^{k}a_{i_li_s}\\
=&a_{i_li_l}\kappa_{i_l}\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)+\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^{k}a_{i_li_s}\\
=&\Big(2\sigma_{n-2}(\kappa|1i_l)+\sigma_{n-2}(\kappa|1)\Big)\kappa_{i_l}\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\\
&+\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^{k}\Big((\kappa_{i_l}+\kappa_{i_s})\sigma_{n-3}(\kappa|1i_li_s)+\sigma_{n-2}(\kappa|1)\Big)\\
=&\Big(2\sigma_{n-1}(\kappa|1)+\kappa_{i_l}\sigma_{n-2}(\kappa|1)\Big)\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\\
&+(k-1)\sigma_{n-2}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots i_k)\\
&+\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^{k}\Big[\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-2}(\kappa|1i_l)\Big].\end{aligned}\tag{5.12}$$

Since

$$\sum_{l=2}^{k}\sum_{s=2}^{k}\Big[\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-2}(\kappa|1i_l)\Big]=2(k-1)\sum_{s=2}^{k}\sigma_{n-2}(\kappa|1i_s), \tag{5.13}$$

equations (5.10), (5.12), and (5.13) yield

$$
\begin{aligned}
&\sum_{i_l\in|I_k|} D_{B,A}(I_k;i_l)\\
=&(2\kappa_1)^{k-2}\sigma_{n-2}^{k-3}(\kappa|1)\left\{(k-1)^2\sigma_{n-2}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots i_k)\right.\\
&+\left[2(k-1)\sigma_{n-1}(\kappa|1)+\sigma_{n-2}(\kappa|1)\sum_{l=2}^{k}\kappa_{i_l}\right]\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\\
&\left.+2(k-1)\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s=2}^{k}\sigma_{n-2}(\kappa|1i_s)\right\}.
\end{aligned}
\tag{5.14}
$$

Using Lemma 17 we obtain,

$$
\begin{aligned}
&\sum_{i_l\in|I_k|} D_{B,A}(I_k;i_l)\\
=&(2\kappa_1)^{k-2}\sigma_{n-2}^{k-3}(\kappa|1)\Big\{(k-1)(k+1)\sigma_{n-2}(\kappa|1)\sigma_{n-k}(\kappa|1i_2\cdots i_k)\\
&+\sigma_{n-2}(\kappa|1)\sum_{l=2}^{k}\kappa_{i_l}\sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\Big\}\\
=&(2\kappa_1)^{k-2}\sigma_{n-2}^{k-2}(\kappa|1)\left[k(k-1)\sigma_{n-k}(\kappa|1i_2\cdots i_k)+\sum_{s=2}^{k}\sigma_{n-k}(\kappa|1i_2\cdots\hat{i}_s\cdots i_k)\right].
\end{aligned}
$$

□

Lemma 19. *For any integer $2\leqslant s\leqslant k$, we have*

$$
\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{l=2}^{s} D_{B,A}(J_s;j_l)\\
=&\frac{(n+1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-2}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s}(\kappa|1).
\end{aligned}
$$

Proof. By Lemma 18, we have

$$
\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{l=2}^{s} D_{B,A}(J_s;j_l)\\
=&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}(2\kappa_1)^{s-2}\sigma_{n-2}^{s-2}(\kappa|1)\\
&\times\left[s(s-1)\sigma_{n-s}(\kappa|1j_2\cdots j_s)+\sum_{l=2}^{s}\sigma_{n-s}(\kappa|1j_2\cdots\hat{j}_l\cdots j_s)\right]\\
=&\frac{(n+1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-2}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s}(\kappa|1).
\end{aligned}
$$

□

In Lemma 20 and 21, we are going to compute the "mixed" principal minors of type $D_{B,A}(I_k;i_pi_q)$.

Lemma 20. *For the multiple index $I_k=(i_2,i_3,\cdots,i_k)\in\mathcal{I}_k$ and $k\geqslant 3$, we have*

$$\begin{aligned}&D_{B,A}(I_k;i_pi_q)\\=&-(2\kappa_1)^{k-3}\sigma_{n-2}^{k-3}(\kappa|1)\left(\kappa_{i_p}-\kappa_{i_q}\right)^2\sigma_{n-3}(\kappa|1i_pi_q)\sigma_{n-k}(\kappa|1i_2\cdots i_k),\end{aligned}$$

where $i_p,i_q\in|I_k|$ and $i_p<i_q$.

Proof. For any $2\leqslant s<t\leqslant k$, let's denote

$\det M_{st}=$

$$\det\begin{bmatrix}b_{i_2i_2}&b_{i_2i_3}&\cdots&b_{i_2i_{s-1}}&b_{i_2i_{s+1}}&\cdots&b_{i_2i_{t-1}}&b_{i_2i_{t+1}}&\cdots&b_{i_2i_k}\\b_{i_3i_2}&b_{i_3i_3}&\cdots&b_{i_3i_{s-1}}&b_{i_3i_{s+1}}&\cdots&b_{i_3i_{t-1}}&b_{i_3i_{t+1}}&\cdots&b_{i_3i_k}\\\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\b_{i_{p-1}i_2}&b_{i_{p-1}i_3}&\cdots&b_{i_{p-1}i_{s-1}}&b_{i_{p-1}i_{s+1}}&\cdots&b_{i_{p-1}i_{t-1}}&b_{i_{p-1}i_{t+1}}&\cdots&b_{i_{p-1}i_k}\\b_{i_{p+1}i_2}&b_{i_{p+1}i_3}&\cdots&b_{i_{p+1}i_{s-1}}&b_{i_{p+1}i_{s+1}}&\cdots&b_{i_{p+1}i_{t-1}}&b_{i_{p+1}i_{t+1}}&\cdots&b_{i_{p+1}i_k}\\\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\b_{i_{q-1}i_2}&b_{i_{q-1}i_3}&\cdots&b_{i_{q-1}i_{s-1}}&b_{i_{q-1}i_{s+1}}&\cdots&b_{i_{q-1}i_{t-1}}&b_{i_{q-1}i_{t+1}}&\cdots&b_{i_{q-1}i_k}\\b_{i_{q+1}i_2}&b_{i_{q+1}i_3}&\cdots&b_{i_{q+1}i_{s-1}}&b_{i_{q+1}i_{s+1}}&\cdots&b_{i_{q+1}i_{t-1}}&b_{i_{q+1}i_{t+1}}&\cdots&b_{i_{q+1}i_k}\\\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots\\b_{i_ki_2}&b_{i_ki_3}&\cdots&b_{i_ki_{s-1}}&b_{i_ki_{s+1}}&\cdots&b_{i_ki_{t-1}}&b_{i_ki_{t+1}}&\cdots&b_{i_ki_k}\end{bmatrix}.$$

Then we have

$$\begin{aligned}&D_{B,A}(I_k;i_pi_q)\\=&\sum_{s=2}^k(-1)^{q-1+s-1}a_{i_qi_s}\left(\sum_{t<s,2\leqslant t\leqslant k}(-1)^{p-1+t-1}a_{i_pi_t}\det M_{ts}\right.\\&\left.+\sum_{t>s,2\leqslant t\leqslant k}(-1)^{p-1+t-2}a_{i_pi_t}\det M_{st}\right).\end{aligned}$$

Therefore, in order to calculate $D_{B,A}(I_k;i_pi_q)$, we need to figure out the value of $\det M_{st}$ first.

In the following, we will calculate $\det M_{st}$ for different values of s,t.

(1) If $t<p$, we can see that the $(s-1)$-th row and $(t-1)$-th row of M_{st} are

$$\begin{aligned}&(b_{i_si_2},b_{i_si_3},\cdots,b_{i_si_{s-1}},b_{i_si_{s+1}},\cdots,b_{i_si_{t-1}},b_{i_si_{t+1}},\cdots,b_{i_si_k})\\=&-2\kappa_1\Big(\sigma_{n-3}(\kappa|1i_si_2),\sigma_{n-3}(\kappa|1i_si_3),\cdots,\sigma_{n-3}(\kappa|1i_s,i_{s-1}),\sigma_{n-3}(\kappa|1i_si_{s+1}),\cdots,\\&\sigma_{n-3}(\kappa|1i_si_{t-1}),\sigma_{n-3}(\kappa|1i_si_{t+1}),\cdots,\sigma_{n-3}(\kappa|1i_si_k)\Big),\end{aligned}\tag{5.15}$$

and

$$
\begin{aligned}
&(b_{i_ti_2}, b_{i_ti_3}, \cdots, b_{i_ti_{s-1}}, b_{i_ti_{s+1}}, \cdots, b_{i_ti_{t-1}}, b_{i_ti_{t+1}}, \cdots, b_{i_ti_k})\\
&= -2\kappa_1\Big(\sigma_{n-3}(\kappa|1i_ti_2), \sigma_{n-3}(\kappa|1i_ti_3), \cdots, \sigma_{n-3}(\kappa|1i_ti_{s-1}), \sigma_{n-3}(\kappa|1i_ti_{s+1}), \cdots,\\
&\quad \sigma_{n-3}(\kappa|1i_ti_{t-1}), \sigma_{n-3}(\kappa|1i_ti_{t+1}), \cdots, \sigma_{n-3}(\kappa|1i_ti_k)\Big).
\end{aligned} \tag{5.16}
$$

We note that the vector in (5.15) multiplying by κ_s is equal to the vector in (5.16) multiplying by κ_t. Thus, the $(s-1)$-th row and the $(t-1)$-th row of M_{st} are linearly dependent, which implies $\det M_{st} = 0$.

(2) If $t = p$, by Lemma 12 we have

$$\det M_{st} = (-1)^{q-2+s-1}(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots i_k).$$

(3) If $p < t < q$ and $s \neq p$, similar to the case (1), we have $\det M_{st} = 0$.

(4) If $p < t < q$ and $s = p$, similar to case (2), we have

$$\det M_{st} = (-1)^{q-2+t-2}(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots i_k).$$

(5) If $t = q$ and $s \neq p$, by Lemma 12, we have

$$\det M_{st} = (-1)^{p-1+s-1}(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_q\cdots i_k).$$

(6) If $t = q$ and $s = p$, by Lemma 12, we have

$$\det M_{st} = (2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots\hat{i}_q\cdots i_k),$$

(7) If $t > q$ and $s \neq p$ or q, similar to the case (1), we have $\det M_{st} = 0$.

(8) If $t > q$ and $s = p$, similar to the case (2), we have

$$\det M_{st} = (-1)^{q-2+t-2}(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots i_k).$$

(9) If $t > q$ and $s = q$, similar to the case (5), we have

$$\det M_{st} = (-1)^{p-1+t-2}(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_q\cdots i_k).$$

In view of the above calculation, if $\{s,t\} \cap \{i_p, i_q\} = \emptyset$, we have $\det M_{st} = 0$. Therefore, the expansion of $D_{B,A}(I_k; i_pi_q)$ becomes

(5.17)
$$
\begin{aligned}
&D_{B,A}(I_k;i_pi_q)\\
=&\sum_{s=2}^{k}(-1)^{q-1+s-1}a_{i_qi_s}\left(\sum_{t<s,2\leqslant t\leqslant k}(-1)^{p-1+t-1}a_{i_pi_t}\det M_{ts}\right.\\
&\left.\qquad+\sum_{t>s,2\leqslant t\leqslant k}(-1)^{p-1+t-2}a_{i_pi_t}\det M_{st}\right)\\
=&(-1)^{p+q}\sum_{s=2}^{k}\left(\sum_{t<s}(-1)^{s+t}a_{i_qi_s}a_{i_pi_t}M_{ts}-\sum_{t>s}(-1)^{s+t}a_{i_qi_s}a_{i_pi_t}\det M_{st}\right)\\
=&(-1)^{p+q}\sum_{s<t,2\leqslant s,t\leqslant k}(-1)^{s+t}\left(a_{i_pi_s}a_{i_qi_t}-a_{i_qi_s}a_{i_pi_t}\right)\det M_{st}\\
=&-\sum_{s<t=p}\left(a_{i_pi_s}a_{i_qi_p}-a_{i_qi_s}a_{i_pi_p}\right)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots i_k)\\
&+\sum_{s\neq p,s<t=q}\left(a_{i_pi_s}a_{i_qi_q}-a_{i_qi_s}a_{i_pi_q}\right)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_q\cdots i_k)\\
&+\sum_{s=p,t=q}\left(a_{i_pi_p}a_{i_qi_q}-a_{i_qi_p}a_{i_pi_q}\right)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots\hat{i}_q\cdots i_k)\\
&+\sum_{s=p<t,t\neq q}\left(a_{i_pi_p}a_{i_qi_t}-a_{i_qi_p}a_{i_pi_t}\right)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots i_k)\\
&-\sum_{s=q,t>q}\left(a_{i_pi_q}a_{i_qi_t}-a_{i_qi_q}a_{i_pi_t}\right)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_q\cdots i_k)\\
=&(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\Big\{\sum_{s\neq p,q}\left(a_{i_qi_s}a_{i_pi_p}-a_{i_pi_s}a_{i_qi_p}\right)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots i_k)\\
&+\sum_{s\neq p,q}\left(a_{i_pi_s}a_{i_qi_q}-a_{i_qi_s}a_{i_pi_q}\right)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_q\cdots i_k)\\
&+\left(a_{i_pi_p}a_{i_qi_q}-a_{i_qi_p}a_{i_pi_q}\right)\sigma_{n-k+1}(\kappa|1i_2\cdots\hat{i}_p\cdots\hat{i}_q\cdots i_k)\Big\}.
\end{aligned}
$$

Using the definition of a_{pq}, for $s\neq p,q$, we have

$$
\begin{aligned}
&a_{i_qi_s}a_{i_pi_p}-a_{i_pi_s}a_{i_qi_p}\\
=&\Big(\sigma_{n-2}(\kappa|1i_q)+\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-2}(\kappa|1)\Big)\Big(2\sigma_{n-2}(\kappa|1i_p)+\sigma_{n-2}(\kappa|1)\Big)\\
&-\Big(\sigma_{n-2}(\kappa|1i_p)+\sigma_{n-2}(\kappa|1i_s)+\sigma_{n-2}(\kappa|1)\Big)\\
&\times\Big(\sigma_{n-2}(\kappa|1i_p)+\sigma_{n-2}(\kappa|1i_q)+\sigma_{n-2}(\kappa|1)\Big)\\
=&\Big[\sigma_{n-2}(\kappa|1i_q)-\sigma_{n-2}(\kappa|1i_p)\Big]\Big[\sigma_{n-2}(\kappa|1i_p)-\sigma_{n-2}(\kappa|1i_s)\Big].
\end{aligned}
\tag{5.18}
$$

Similarly we can compute,

$$
\begin{aligned}
&a_{i_pi_s}a_{i_qi_q} - a_{i_qi_s}a_{i_pi_q} \\
=&\Big[\sigma_{n-2}(\kappa|1i_p) - \sigma_{n-2}(\kappa|1i_q)\Big]\Big[\sigma_{n-2}(\kappa|1i_q) - \sigma_{n-2}(\kappa|1i_s)\Big].
\end{aligned}
\tag{5.19}
$$

Combining equation (5.18) and (5.19) we have,

$$
\begin{aligned}
&\kappa_{i_p}\Big[a_{i_qi_s}a_{i_pi_p} - a_{i_pi_s}a_{i_qi_p}\Big] + \kappa_{i_q}\Big[a_{i_pi_s}a_{i_qi_q} - a_{i_qi_s}a_{i_pi_q}\Big] \\
=& \sigma_{n-2}(\kappa|1i_s)(\kappa_{i_q} - \kappa_{i_p})\Big[\sigma_{n-2}(\kappa|1i_q) - \sigma_{n-2}(\kappa|1i_p)\Big] \\
=& -\sigma_{n-2}(\kappa|1i_s)(\kappa_{i_q} - \kappa_{i_p})^2\sigma_{n-3}(\kappa|1i_pi_q).
\end{aligned}
\tag{5.20}
$$

For the case when s is equal to p or q we have

$$
\begin{aligned}
&a_{i_pi_p}a_{i_qi_q} - a_{i_qi_p}a_{i_pi_q} \\
=& \Big(2\sigma_{n-2}(\kappa|1i_p) + \sigma_{n-2}(\kappa|1)\Big)\Big(2\sigma_{n-2}(\kappa|1i_q) + \sigma_{n-2}(\kappa|1)\Big) \\
&-\Big(\sigma_{n-2}(\kappa|1i_p) + \sigma_{n-2}(\kappa|1i_q) + \sigma_{n-2}(\kappa|1)\Big)^2 \\
=& -\Big(\sigma_{n-2}(\kappa|1i_p) - \sigma_{n-2}(\kappa|1i_q)\Big)^2 \\
=& -(\kappa_{i_q} - \kappa_{i_p})^2\sigma_{n-3}^2(\kappa|1i_pi_q).
\end{aligned}
\tag{5.21}
$$

Therefore, by (5.17), (5.20) and (5.21), we obtain

$$
\begin{aligned}
&D_{B,A}(I_k; i_pi_q) \\
=&(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)\Big\{\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s\neq p,q}\Big[\kappa_{i_p}\left(a_{i_qi_s}a_{i_pi_p} - a_{i_pi_s}a_{i_qi_p}\right) \\
&+ \kappa_{i_q}\left(a_{i_pi_s}a_{i_qi_q} - a_{i_qi_s}a_{i_pi_q}\right)\Big] \\
&+ \left(a_{i_pi_p}a_{i_qi_q} - a_{i_qi_p}a_{i_pi_q}\right)\left(\kappa_{i_p}\kappa_{i_q}\sigma_{n-k-1}(\kappa|1i_2\cdots i_k) + (\kappa_{i_p} + \kappa_{i_q})\sigma_{n-k}(\kappa|1i_2\cdots i_k)\right)\Big\} \\
=& -(2\kappa_1)^{k-3}\sigma_{n-2}^{k-4}(\kappa|1)(\kappa_{i_q} - \kappa_{i_p})^2\sigma_{n-3}(\kappa|1i_pi_q)\Big\{\sigma_{n-k}(\kappa|1i_2\cdots i_k)\sum_{s\neq p,q}\sigma_{n-2}(\kappa|1i_s) \\
&+ \sigma_{n-k-1}(\kappa|1i_2\cdots i_k)\sigma_{n-1}(\kappa|1) + \left(\sigma_{n-2}(\kappa|1i_p) + \sigma_{n-2}(\kappa|1i_q)\right)\sigma_{n-k}(\kappa|1i_2\cdots i_k))\Big\} \\
=& -(2\kappa_1)^{k-3}\sigma_{n-2}^{k-3}(\kappa|1)(\kappa_{i_q} - \kappa_{i_p})^2\sigma_{n-3}(\kappa|1i_pi_q)\sigma_{n-k}(\kappa|1i_2\cdots i_k).
\end{aligned}
$$

Here in the last step, we used Lemma 17. $\square$

Lemma 21. *For the multiple index $I_k = (i_2, i_3, \cdots, i_k) \in \mathcal{I}_k,\ k \geqslant 3$ and any integer $3 \leqslant s \leqslant k$, we have*

$$\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{j_p,j_q\in|J_s|,j_p<j_q} D_{B,A}(J_s;j_pj_q)\\
=&\frac{(n-1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-3}(\kappa|1)\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1)\\
&-\frac{(n-s+1)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s+1}(\kappa|1).
\end{aligned}$$

Proof. By a straightforward calculation we get

$$\begin{aligned}
&\sum_{j_p,j_q\in|J_s|,j_p<j_q}(\kappa_{j_p}-\kappa_{j_q})^2\sigma_{n-3}(\kappa|1j_pj_q)\\
=&\sum_{j_p,j_q\in|J_s|,j_p<j_q}(\kappa_{j_p}^2+\kappa_{j_q}^2-2\kappa_{j_p}\kappa_{j_q})\sigma_{n-3}(\kappa|1j_pj_q)\\
=&\sum_{j_p\in|J_s|}\kappa_{j_p}\sum_{j_q\in|J_s|,j_q\neq j_p}\sigma_{n-2}(\kappa|1j_q)-\sum_{j_q\neq j_p}\sigma_{n-1}(\kappa|1)\\
=&\sum_{j_p\in|J_s|}\kappa_{j_p}\sum_{j_q\in|J_s|}\sigma_{n-2}(\kappa|1j_q)-(s-1)^2\sigma_{n-1}(\kappa|1).
\end{aligned} \tag{5.22}$$

Moreover, we have

$$\begin{aligned}
&\sigma_{n-s}(\kappa|1j_2\cdots j_s)\sum_{j_p\in|J_s|}\kappa_{j_p}\sum_{j_q\in|J_s|}\sigma_{n-2}(\kappa|1j_q)\\
=&\sum_{j_p\in|J_s|}\sigma_{n-s+1}(\kappa|1j_2\cdots\hat{j}_p\cdots j_s)\sum_{j_q\in|J_s|}\sigma_{n-2}(\kappa|1j_q)\\
=&\sum_{j_p\in|J_s|}\sigma_{n-s+1}(\kappa|1j_2\cdots\hat{j}_p\cdots j_s)\Big(\sigma_{n-2}(\kappa|1)-\sum_{j_q\in|I_n|\setminus|J_s|}\sigma_{n-2}(\kappa|1j_q)\Big)\\
=&\sigma_{n-2}(\kappa|1)\sum_{j_p\in|J_s|}\sigma_{n-s+1}(\kappa|1j_2\cdots\hat{j}_p\cdots j_s)\\
&-\sum_{j_p\in|J_s|}\sum_{j_q\in|I_n|\setminus|J_s|}\sigma_{n-s}(\kappa|1j_2\cdots\hat{j}_p\cdots j_sj_q)\sigma_{n-1}(\kappa|1).
\end{aligned} \tag{5.23}$$

Thus, using equation (5.22) and (5.23) we get

$$
\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{j_p,j_q\in|J_s|,j_p<j_q}\sigma_{n-s}(\kappa|1j_2\cdots j_s)(\kappa_{j_p}-\kappa_{j_q})^2\sigma_{n-3}(\kappa|1j_pj_q)\\
=&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sigma_{n-2}(\kappa|1)\sum_{j_p\in|J_s|}\sigma_{n-s+1}(\kappa|1j_2\cdots\hat{j}_p\cdots j_s)\\
&-\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{j_p\in|J_s|}\sum_{j_q\in|I_n|\setminus|J_s|}\sigma_{n-s}(\kappa|1j_2\cdots\hat{j}_p\cdots j_sj_q)\sigma_{n-1}(\kappa|1)\\
&-\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}(s-1)^2\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1j_2\cdots j_s)\\
=&\frac{(s-1)C_{k-1}^{s-1}C_{n-1}^{k-1}}{C_{n-1}^{n-s+1}}\sigma_{n-2}(\kappa|1)\sigma_{n-s+1}(\kappa|1)\\
&-\frac{(n-s)(s-1)C_{k-1}^{s-1}C_{n-1}^{k-1}}{C_{n-1}^{n-s}}\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1)\\
&-\frac{(s-1)^2C_{k-1}^{s-1}C_{n-1}^{k-1}}{C_{n-1}^{n-s}}\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1)\\
=&\frac{(n-s+1)!}{(k-s)!(n-k)!}\sigma_{n-2}(\kappa|1)\sigma_{n-s+1}(\kappa|1)\\
&-\frac{(n-1)(s-1)(n-s)!}{(k-s)!(n-k)!}\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1).
\end{aligned}\tag{5.24}
$$

Lemma 21 follows from equation (5.24) and Lemma 20 directly. □

Now, let's come back to the matrix S and prove Lemma 11.

Proof. (Proof of Lemma 11) By basic Linear Algebra we know, given any multiple index $I_k=(i_2,i_3,\cdots,i_k)\in\mathcal{I}_k$ we have,

$$
\begin{aligned}
&D_S(I_k)\\
=&D_{A+B}(I_k)+\sum_{J_{k-1}\in\mathcal{J}_{k-1}(I_k)}D_{A+B}(J_{k-1})\sigma_{n-2}^1(\kappa|1)\\
&+\sum_{J_{k-2}\in\mathcal{J}_{k-2}(I_k)}D_{A+B}(J_{k-2})\sigma_{n-2}^2(\kappa|1)+\cdots+\sum_{J_s\in\mathcal{J}_s(I_k)}D_{A+B}(J_s)\sigma_{n-2}^{k-s}(\kappa|1)\\
&+\cdots+\sum_{J_2\in\mathcal{J}_2(I_k)}D_{A+B}(J_2)\sigma_{n-2}^{k-2}(\kappa|1)+\sigma_{n-2}^{k-1}(\kappa|1).
\end{aligned}\tag{5.25}
$$

By (5.4), for any multiple index $J_s=(j_2,\cdots,j_s)\in\mathcal{J}_s(I_k)$, we have

$$D_{A+B}(J_s)=D_B(J_s)+\sum_{l=2}^{s}D_{B,A}(J_s;j_l)+\sum_{j_2\leqslant j_p<j_q\leqslant j_s}D_{B,A}(J_s;j_pj_q).$$

Thus, for $3 \leqslant s \leqslant k$, using Lemma 16, Lemma 19, and Lemma 21 we obtain,

$$\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)} D_{A+B}(J_s) \\
=&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)} D_B(J_s)+\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{j_l\in|J_s|} D_{B,A}(J_s;j_l) \\
&+\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{j_p<j_q,j_pj_q\in|J_s|} D_{B,A}(J_s;j_pj_q) \\
=&\frac{s(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-1}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s-1}(\kappa|1) \\
&+\frac{(n+1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-2}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s}(\kappa|1) \\
&+\frac{(n-1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-3}(\kappa|1)\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1) \\
&-\frac{(n-s+1)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s+1}(\kappa|1).
\end{aligned}\tag{5.26}$$

For $s = 2$ and $3 \leqslant k \leqslant n$, the third term of $D_{A+B}(J_s)$ does not appear. Thus, we have

$$\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)} D_{A+B}(J_s) \\
=&\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)} D_B(J_s)+\sum_{I_k\in\mathcal{I}_k}\sum_{J_s\in\mathcal{J}_s(I_k)}\sum_{j_l\in|J_s|} D_{B,A}(J_s;j_l) \\
=&\frac{s(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-1}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s-1}(\kappa|1) \\
&+\frac{(n+1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-2}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s}(\kappa|1).
\end{aligned}\tag{5.27}$$

We want to rewrite $\sum_{I_k\in\mathcal{I}_k} D_S(I_k)$ as a polynomial of the variable $2\kappa_1$. Let's calculate the coefficient of $(2\kappa_1)^i$ for $i = 0, 1, \cdots, k-1$.

By equation (5.26), the coefficient of $(2\kappa_1)^{s-3}$ for $3 \leqslant s \leqslant k$ is

$$\begin{aligned}
&\frac{(s-2)(n-s+2)!}{(k-s+2)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-4}(\kappa|1)\sigma_{n-s+1}(\kappa|1)\cdot\sigma_{n-2}^{k-s+2}(\kappa|1)\\
&+\frac{(n+1)(s-2)(n-s+1)!}{(k-s+1)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-3}(\kappa|1)\sigma_{n-s+1}(\kappa|1)\cdot\sigma_{n-2}^{k-s+1}(\kappa|1)\\
&+\frac{(n-1)(s-1)(n-s)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-3}(\kappa|1)\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1)\cdot\sigma_{n-2}^{k-s}(\kappa|1)\\
&-\frac{(n-s+1)!}{(k-s)!(n-k)!}(2\kappa_1)^{s-3}\sigma_{n-2}^{s-2}(\kappa|1)\sigma_{n-s+1}(\kappa|1)\cdot\sigma_{n-2}^{k-s}(\kappa|1)\\
=&(2\kappa_1)^{s-3}\sigma_{n-2}^{k-3}(\kappa|1)\Big[P(s-3)\sigma_{n-2}(\kappa|1)\sigma_{n-s+1}(\kappa|1)+Q(s-3)\sigma_{n-1}(\kappa|1)\sigma_{n-s}(\kappa|1)\Big],
\end{aligned}\tag{5.28}$$

where the functions $P(s-3)$ and $Q(s-3)$ are defined by

$$P(s-3)=\frac{(s-2)(n-s+2)!}{(k-s+2)!(n-k)!}+\frac{(n+1)(s-2)(n-s+1)!}{(k-s+1)!(n-k)!}-\frac{(n-s+1)!}{(k-s)!(n-k)!},$$

and

$$Q(s-3)=\frac{(n-1)(s-1)(n-s)!}{(k-s)!(n-k)!}.$$

The coefficient of $(2\kappa_1)^{k-2}$ is

$$\begin{aligned}
&(k-1)(n-k+1)(2\kappa_1)^{k-2}\sigma_{n-2}^{k-3}(\kappa|1)\sigma_{n-k}(\kappa|1)\sigma_{n-2}(\kappa|1)\\
&+(n+1)(k-1)(2\kappa_1)^{k-2}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k}(\kappa|1)\\
=&(k-1)(2n+2-k)(2\kappa_1)^{k-2}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k}(\kappa|1).
\end{aligned}\tag{5.29}$$

The coefficient of $(2\kappa_1)^{k-1}$ is

$$k(2\kappa_1)^{k-1}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k-1}(\kappa|1).\tag{5.30}$$

We substitute (5.28), (5.29), and (5.30) into (5.25), then sum over $I_k\in\mathcal{I}_k, k\geqslant 3$, and get,

$$\begin{aligned}
&\sum_{I_k\in\mathcal{I}_k}D_S(I_k)\\
=&\sum_{s=0}^{k-3}(2\kappa_1)^s\sigma_{n-2}^{k-3}(\kappa|1)\Big[P(s)\sigma_{n-2}(\kappa|1)\sigma_{n-s-2}(\kappa|1)+Q(s)\sigma_{n-1}(\kappa|1)\sigma_{n-s-3}(\kappa|1)\Big]\\
&+(k-1)(2n+2-k)(2\kappa_1)^{k-2}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k}(\kappa|1)\\
&+k(2\kappa_1)^{k-1}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k-1}(\kappa|1).
\end{aligned}\tag{5.31}$$

Since we assume $\kappa_1 > 0$, the last two terms are non negative. We only need to analyze the first term. Note that

$$\begin{aligned}
P(s) &\geqslant \frac{(n+1)(s+1)(n-s-2)!}{(k-s-2)!(n-k)!} - \frac{(n-s-2)!}{(k-s-3)!(n-k)!} \\
&= \frac{(n-s-2)!}{(k-s-2)!(n-k)!}[(n+1)(s+1)-k+s+2] \\
&\geqslant 0.
\end{aligned}$$

If $\sigma_{n-1}(\kappa|1) \geqslant 0$, then we obtain for $k \geqslant 3$

$$\sum_{I_k \in \mathcal{I}_k} D_S(I_k) \geqslant 0.$$

If $\sigma_{n-1}(\kappa|1) < 0$, using the identity

$$\kappa_1 \sigma_{n-2}(\kappa|1) = \sigma_{n-1} - \sigma_{n-1}(\kappa|1) \geqslant -\sigma_{n-1}(\kappa|1) > 0,$$

(5.31) becomes

$$\begin{aligned}
&\sum_{I_k \in \mathcal{I}_k} D_S(I_k) \\
&= \sigma_{n-2}^{k-3}(\kappa|1)\left[\sum_{s=1}^{k-3} 2(2\kappa_1)^{s-1} P(s)\Big(\kappa_1\sigma_{n-2}(\kappa|1)\Big)\sigma_{n-s-2}(\kappa|1)\right. \\
&\quad \left.+\sum_{s=0}^{k-3}(2\kappa_1)^s Q(s)\sigma_{n-1}(\kappa|1)\sigma_{n-s-3}(\kappa|1)\right] \\
&\quad +2(k-1)(2n+2-k)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-3}(\kappa|1)\Big(\kappa_1\sigma_{n-2}(\kappa|1)\Big)\sigma_{n-k}(\kappa|1) \\
&\quad +k(2\kappa_1)^{k-1}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k-1}(\kappa|1) \\
&\geqslant \sigma_{n-2}^{k-3}(\kappa|1)\left[\sum_{s=1}^{k-3} 2(2\kappa_1)^{s-1} P(s)\Big(-\sigma_{n-1}(\kappa|1)\Big)\sigma_{n-s-2}(\kappa|1)\right. \\
&\quad \left.+\sum_{s=0}^{k-3}(2\kappa_1)^s Q(s)\sigma_{n-1}(\kappa|1)\sigma_{n-s-3}(\kappa|1)\right] \\
&\quad +2(k-1)(2n+2-k)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-3}(\kappa|1)\Big(-\sigma_{n-1}(\kappa|1)\Big)\sigma_{n-k}(\kappa|1) \\
&\quad +k(2\kappa_1)^{k-1}\sigma_{n-2}^{k-2}(\kappa|1)\sigma_{n-k-1}(\kappa|1)
\end{aligned} \tag{5.32}$$

$$
\begin{aligned}
&\geqslant \sigma_{n-2}^{k-3}(\kappa|1)\sum_{s=0}^{k-4}(2\kappa_1)^s\Big(2P(s+1)-Q(s)\Big)\Big(-\sigma_{n-1}(\kappa|1)\Big)\sigma_{n-s-3}(\kappa|1)\\
&\quad+\Big(2(k-1)(2n+2-k)-Q(k-3)\Big)(2\kappa_1)^{k-3}\sigma_{n-2}^{k-3}(\kappa|1)\\
&\quad\times\Big(-\sigma_{n-1}(\kappa|1)\Big)\sigma_{n-k}(\kappa|1).
\end{aligned}
$$

It's easy to see that

$$
\begin{aligned}
&2(k-1)(2n+2-k)-Q(k-3)\\
=&(k-1)\Big(2(2n+2-k)-(n-1)\Big)\\
\geqslant&\, 0.
\end{aligned}
\tag{5.33}
$$

Moreover, for $0 \leqslant s \leqslant k-4$, we have

$$
\begin{aligned}
&2P(s+1)-Q(s)\\
=&\frac{(n-s-3)!(s+2)}{(k-s-3)!(n-k)!}\left[2\left(\frac{n-s-2}{k-s-2}+(n+1)-\frac{k-s-3}{s+2}\right)-(n-1)\right]\\
\geqslant&\frac{(n-s-3)!(s+2)}{(k-s-3)!(n-k)!}\left[2(n+1)-\frac{2(k-s-3)}{s+2}-(n-1)\right]\\
\geqslant&\frac{(n-s-3)!(s+2)}{(k-s-3)!(n-k)!}\Big[(n+1)-(k-s-3)\Big]\\
\geqslant&\, 0.
\end{aligned}
\tag{5.34}
$$

Combining (5.32) with (5.33) and (5.34), we obtain if $\sigma_{n-1}(\kappa|1) \leqslant 0$ and $k \geqslant 3$,

$$
\sum_{I_k\in\mathcal{I}_k} D_S(I_k) \geqslant 0.
$$

Therefore, we have proved for $2 \leqslant m \leqslant n-1$ the sum of all m-th principal minors of matrix S is nonnegative. When $m = 1$, by the definition of S, we get $\sum\limits_{I_2\in\mathcal{I}_2} D_S(I_2) = \sum\limits_{p=2}^{n} s_{pp} > 0$ directly. This completes the proof of Lemma 11. □

Lemma 10 and Lemma 11 proved that the matrix S is a semi-positive matrix. This together with our analysis in Section 3 yields Theorem 8.

6. Bounded principal curvatures implies convexity

In this section, we will study the convexity of the admissible hypersurface $\mathcal{M}_u$ with bounded principal curvatures. More precisely, we will prove that every spacelike hypersurfaces $\mathcal{M}_u$ that satisfies $\kappa[\mathcal{M}_u] \in \Gamma_{n-1}$, $\sigma_{n-1}(\kappa[\mathcal{M}_u]) = 1$, and $|\kappa[\mathcal{M}_u]| < C$ must be convex.

Following, Cheng-Yau [7], we first prove the induced metric on $\mathcal{M}_u$ is complete. Due to our assumption on the principal curvatures, the proof here is much easier than it is in Cheng-Yau [7]. For readers' convenience, we will include it here.

Recall that the Minkowski distance is defined by

$$2z(X) = \|X\|^2 = \langle X, X\rangle^2 = \sum_{i=1}^{n} x_i^2 - x_{n+1}^2, \ X \in \mathbb{R}^{n,1}.$$

Cheng-Yau (see Proposition 1 in [7]) have shown the following: For a spacelike hypersurface $\mathcal{M}$ in $\mathbb{R}^{n,1}$ which is closed with respect to the Euclidean topology, if the origin $\mathbf{0} \in \mathcal{M}$, then z is a proper function defined on $\mathcal{M}$. Here being "proper" means that for any given constant $c > 0$, the set $\{X \in \mathcal{M} \subset \mathbb{R}^{n,1} | z(X) \leqslant c\}$ is compact. In general, if $\mathbf{0} \notin \mathcal{M}$, without loss of generality, we may assume $P = (0, \xi) \in \mathcal{M}$. Then, we can modify the function z to be

$$2z(X) = \|\tilde{X}\|^2 = \|X - \xi E\|^2,$$

and show the set $\{X \in \mathcal{M} \subset \mathbb{R}^{n,1} | z(X) \leqslant c\}$ is compact. Therefore, in the following, we will always assume $\mathbf{0} \in \mathcal{M}$.

Proposition 22. *Let $\mathcal{M} \in \mathbb{R}^{n,1}$ be a spacelike hypersurface with bounded principal curvatures, i.e., $|\kappa[\mathcal{M}]| \leqslant C_0$. Then there is a constant C only depending on C_0 such that*

$$|\nabla z(X)|^2 \leqslant C(z(X) + 1)^2, \ \textit{for } X \in \mathcal{M}. \tag{6.1}$$

Proof. In the following, for any $c > 0$, we denote $\mathcal{M}_c := \{X \in \mathcal{M} | z(X) \leqslant \frac{c}{2}\}$. Note that by earlier discussion we know that $\mathcal{M}_{2c}$ is compact. Considering an auxiliary function

$$\phi(X) = (c - z)^2 \frac{|\nabla z|^2}{(z + 1)^2}.$$

It is obvious that ϕ achieves its maximum value at some interior point $P_0 \in \mathcal{M}_{2c}$. Let $\{\tau_1, \cdots, \tau_n\}$ be an orthonormal frame at P_0. Now, we differentiate $\log \phi$ at P_0 and get,

$$2\frac{-z_i}{c - z} + \frac{2\sum_k z_k z_{ki}}{|\nabla z|^2} - 2\frac{z_i}{z + 1} = 0. \tag{6.2}$$

By a straightforward calculation we have

$$z_i = \langle X, \tau_i\rangle, \quad z_{ij} = \delta_{ij} - h_{ij}\langle X, \nu\rangle. \tag{6.3}$$

Moreover, since $z \geqslant 0$, we obtain

$$\langle X, \nu\rangle^2 \leqslant \sum_i \langle X, \tau_i\rangle^2. \tag{6.4}$$

We may choose an orthonormal coordinate at P_0 such that

$$z_1 = |\nabla z|, \text{ and } z_i = 0 \text{ for } i \neq 1.$$

We may also rotate $\{\tau_2, \cdots, \tau_n\}$ such that

$$h_{ij} = h_{ii}\delta_{ij} \text{ for } i, j \geqslant 2.$$

Thus, using (6.2) we get ,

$$2\frac{-z_1}{c-z} + \frac{2z_1 z_{11}}{|\nabla z|^2} - 2\frac{z_1}{z+1} = 0 \tag{6.5}$$

and

$$\frac{2z_1 z_{1i}}{|\nabla z|^2} = 0, \quad \text{for } i \geqslant 2. \tag{6.6}$$

This implies

$$z_{11} = \frac{|\nabla z|^2}{c-z} + \frac{|\nabla z|^2}{z+1}. \tag{6.7}$$

Without loss of generality we may assume $z_1 = |\nabla z| > 1$ at P_0. Since we are working on hypersurfaces with bounded curvatures, using (6.3), we have

$$\frac{|\nabla z|^2}{z+1} \leqslant z_{11} \leqslant 1 + |h_{11}||\langle X, \nu\rangle| \leqslant 1 + C|\langle X, \nu\rangle|.$$

By (6.4) we know $|\langle X, \nu\rangle| \leqslant |\nabla z|$, thus at P_0 we have

$$\frac{|\nabla z|}{z+1} \leqslant C. \tag{6.8}$$

This yields that

$$(c-z)^2 \frac{|\nabla z|^2}{(z+1)^2} \leqslant c^2 C^2.$$

Therefore, on M_c we have

$$|\nabla z|^2 \leqslant 4C^2|z+1|^2.$$

Since $c > 0$ is arbitrary, we proved (6.1). □

Now by the same argument as in [7] and [25], we have

Corollary 23. *Let $\mathcal{M} \in \mathbb{R}^{n,1}$ be a spacelike hypersurface which is closed with respect to the Euclidean topology. Suppose $\mathcal{M}$ has bounded principal curvatures. Then, $\mathcal{M}$ is complete with respect to the induced metric.*

Remark 24. *Proposition 22 and Corollary 23 give a different proof of the completeness of spacelike hypersurfaces with constant Gauss-Kronecker curvature and bounded principal curvatures (see Proposition 5.2 in* [22]*).*

Lemma 25. *Let $\mathcal{M}$ be an $(n-1)$-convex, spacelike hypersurface with bounded principal curvatures, and $\mathcal{M}$ satisfies equation* (1.2)*. Then $\mathcal{M}$ is convex.*

Proof. Recall Theorem 8 we have,

$$\sigma_{n-1}^{ij}(\sigma_n)_{ij} \leqslant \sigma_1\sigma_{n-1}\sigma_n - n^2\sigma_n^2. \tag{6.9}$$

Given a point $P \in \mathcal{M}$, we can define the distance function on $\mathcal{M}$

$$r(X) = d(P, X),$$

where $X \in \mathcal{M}$. By Corollary 23 we know that $\mathcal{M}$ is complete. Therefore, for any $a > 0$, let $\mathcal{B}_a := \{X \in \mathcal{M} | r(X) < a\}$ be the geodesic ball centered at P with radius a, then $\mathcal{B}_a$ is compact.

Now, we define an open subdomain of $\mathcal{M}$

$$\Omega = \{X \in \mathcal{M} | \sigma_n(\kappa[\mathcal{M}(X)]) < 0\}.$$

Without loss of generality we assume $\Omega \neq \emptyset$, otherwise, we would be done. Considering the auxiliary function

$$\varphi = -\eta^2(X)\sigma_n(X)$$

on Ω, where $\eta = a^2 - r^2(X)$ is the cutoff function and $\sigma_n(X) = \sigma(\kappa[\mathcal{M}(X)])$. It is obvious that the function φ achieves its maximum at an interior point X_0 in $\Omega \cap \mathcal{B}_a$. Moreover, use the same argument as [8], we can assume η is differentiable near X_0. Now, we choose a local orthonormal frame near X_0 such that at X_0, $h_{ij} = \kappa_i\delta_{ij}$. Differentiating $\log\varphi$ at X_0 twice we get,

$$\frac{(\sigma_n)_i}{\sigma_n} + 2\frac{\eta_i}{\eta} = 0; \tag{6.10}$$

$$\frac{(\sigma_n)_{ii}}{\sigma_n} - \frac{(\sigma_n)_i^2}{\sigma_n^2} + 2\frac{\eta_{ii}}{\eta} - 2\frac{\eta_i^2}{\eta^2} \leqslant 0. \tag{6.11}$$

Contracting (6.11) with σ_{n-1}^{ii} and applying (6.10) yields,

$$\frac{\sigma_{n-1}^{ii}(\sigma_n)_{ii}}{\sigma_n} \leqslant -2\frac{\sigma_{n-1}^{ii}\eta_{ii}}{\eta} + 6\frac{\sigma_{n-1}^{ii}\eta_i^2}{\eta^2}.$$

Combining with (6.9), we have

$$\begin{aligned} \eta^2\sigma_1\sigma_{n-1} + n^2\varphi &\leqslant -2\eta\sigma_{n-1}^{ii}\eta_{ii} + 6\sigma_{n-1}^{ii}\eta_i^2 \\ &= 4r\eta\sigma_{n-1}^{ii}r_{ii} + (4\eta + 24r^2)\sigma_{n-1}^{ii}r_i^2. \end{aligned} \tag{6.12}$$

Since $|\nabla r| = 1$, by our assumption that $\mathcal{M}$ has bounded principal curvatures, we can see in $\mathcal{B}_a$,

$$(4\eta + 24r^2)\sigma_{n-1}^{ii}r_i^2 \leqslant Ca^2,$$

where the constant C depends on $\kappa[\mathcal{M}]$. To deal with the term r_{ii}, we will use the Hessian comparison theorem. Since the sectional curvature of $\mathcal{M}$ satisfies

$$R_{ijij} = -h_{ii}h_{jj} \geqslant -C,$$

we have

$$r_{ii} \leqslant \frac{n-1}{r}(1 + Cr).$$

This implies in $\mathcal{B}_a$,

$$4r\eta\sigma_{n-1}^{ii}r_{ii} \leqslant Ca^3,$$

where the constant C depends on $\kappa[\mathcal{M}]$. Thus, we obtain for any $a > 0$ large in $\Omega \cap \mathcal{B}_a$,

$$\varphi(X) \leqslant Ca^3. \tag{6.13}$$

Now, for any given point $Y \in \Omega \subset \mathcal{M}$, we can take $a > 0$ sufficiently large such that $Y \in \Omega \cap \mathcal{B}_{a/2}$. Then, by (6.13) we have

$$-\sigma_n(Y) \leqslant \frac{C}{a}.$$

Let a go to infinity, we obtain

$$\sigma_n(Y) = 0.$$

Hence, we conclude that Ω is an empty set. This proves Lemma 25. □

Now that we have proved the convexity of $\mathcal{M}$, we are in the position to prove Theorem 2 of the introduction.

Proof. (proof of Theorem 2) In view of the formula (6.9), we know that for a convex hypersurface $\mathcal{M}$ satisfying $\sigma_{n-1}(\kappa[\mathcal{M}]) = 1$, if there is a degenerate point on $\mathcal{M}$, i.e., $\sigma_n = 0$, then $\sigma_n \equiv 0$ on $\mathcal{M}$. We will show in this case $\mathcal{M} = \mathcal{M}^{n-1} \times \mathbb{R}$.

Let τ_1 be the principal direction corresponding to the minimum principal curvature $\kappa_1 = 0$. Then τ_1 is a smooth vector field on $\mathcal{M}$. Let $\gamma(s)$ be the integral curve of τ_1, and Span$\{\tau_1, \cdots, \tau_n\} = T\mathcal{M}$. Then we have

$$\left\langle \bar{\nabla}_{\tau_1}\nu, \tau_i \right\rangle = 0 \text{ for } 1 \leqslant i \leqslant n,$$

where ν is the timelike unite normal of $\mathcal{M}$. Therefore, ν is a constant vector along $\gamma(s)$. This implies that $\gamma(s)$ lies in the hyperplane $\mathbb{P}$ that is perpendicular to ν.

Now we can choose a coordinate such that

$$\mathbb{P} = \{x | x_{n+1} = \langle X, E\rangle = 0\}$$

and $-\langle X, E\rangle \geqslant 0$ for any $X \in \mathcal{M}$, where $E = (0, \cdots, 0, 1)$. We claim $\gamma(s)$ is a straight line. If not, we can choose $p, q \in \gamma(s)$. Since the straight line connects p, q is in the convex hull of $\mathcal{M}$, we conclude that the flat region that is enclosed by the straight line connects p, q and $\gamma(s)$ is part of $\mathcal{M}$, i.e., $\mathcal{M}$ has a flat side. This leads to a contradiction.

Therefore, $\gamma(s)$ is a straight line. By Cheeger-Gromoll splitting theorem (see Theorem 2 in [4]), we complete the proof of Theorem 2. □

7. THE GAUSS MAP AND LEGENDRE TRANSFORM

In this section, we will discuss properties of the Gauss map and the Legendre transform. We will use these properties in later sections.

7.1. **The Gauss map.** Let $\mathcal{M}$ be a spacelike hypersurface, $\nu(X)$ be the timelike unit normal vector to $\mathcal{M}$ at X. It's well known that the hyperbolic space $\mathbb{H}^n(-1)$ is canonically embedded in $\mathbb{R}^{n,1}$ as the hypersurface

$$\langle X, X\rangle = -1,\ x_{n+1} > 0.$$

By parallel translating to the origin we can regard $\nu(X)$ as a point in $\mathbb{H}^n(-1)$. In this way, we define the Gauss map:

$$G : \mathcal{M} \to \mathbb{H}^n(-1);\ X \mapsto \nu(X).$$

If we take the hyperplane $\mathbb{P} := \{X = (x_1, \cdots, x_n, x_{n+1}) | \, x_{n+1} = 1\}$ and consider the projection of $\mathbb{H}^n(-1)$ from the origin into $\mathbb{P}$. Then $\mathbb{H}^n(-1)$ is mapped in a one-to-one fashion onto an open unit ball $B_1 := \{\xi \in \mathbb{R}^n | \sum \xi_k^2 < 1\}$. The map P is given by

$$P : \mathbb{H}^n(-1) \to B_1;\ (x_1, \cdots, x_{n+1}) \mapsto (\xi_1, \cdots, \xi_n),$$

where $x_{n+1} = \sqrt{1 + x_1^2 + \cdots + x_n^2}$, $\xi_i = \frac{x_i}{x_{n+1}}$. We will call the map $P \circ G : \mathcal{M} \to B_1$ the Gauss map and denote it by G for the sake of simplicity.

Next, let's consider the support function of $\mathcal{M}$. We denote

$$v := \langle X, \nu\rangle = \frac{1}{\sqrt{1 - |Du|^2}} \left(\sum_i x_i \frac{\partial u}{\partial x_i} - u \right).$$

Let $\{e_1, \cdots, e_n\}$ be an orthonormal frame on $\mathbb{H}^n$. We will also denote $\{e_1^*, \cdots, e_n^*\}$ the push-forward of e_i by the Gauss map G. Similar to the convex geometry case, we denote

$$\Lambda_{ij} = v_{ij} - v\delta_{ij}$$

the hyperbolic Hessian. Here v_{ij} denote the covariant derivatives with respect to the hyperbolic metric.

Let $\bar{\nabla}$ be the connection of the ambient space. Then, we have

$$v_i = \bar{\nabla}_{e_i^*} X \cdot \nu + X \cdot \bar{\nabla}_{e_i} \nu = X \cdot e_i,$$

this implies

$$X = \sum_i v_i e_i - v\nu.$$

Note that $\langle \nu, \nu \rangle = -1$, thus we have,

$$\begin{aligned}\bar{\nabla}_{e_j^*} X &= \sum_k (e_j(v_k) e_k + v_k \bar{\nabla}_{e_j} e_k) - v_j \nu - v \bar{\nabla}_{e_j} \nu \\ &= \sum_k (e_j(v_k) e_k + v_k \nabla_{e_j} e_k + v_k \delta_{kj} \nu) - v_j \nu - v e_j \\ &= \sum_k \Lambda_{kj} e_k,\end{aligned} \tag{7.1}$$

$$g_{ij} = \bar{\nabla}_{e_i^*} X \cdot \bar{\nabla}_{e_j^*} X = \sum_k \Lambda_{ik} \Lambda_{kj}, \tag{7.2}$$

$$h_{ij} = \bar{\nabla}_{e_i^*} X \cdot \bar{\nabla}_{e_j} \nu = \Lambda_{ij}. \tag{7.3}$$

This implies that the eigenvalues of the hyperbolic Hessian are the curvature radius of $\mathcal{M}$. That is, if the principal curvatures of $\mathcal{M}$ are $(\kappa_1, \cdots, \kappa_n)$, then the eigenvalues of the hyperbolic Hessian are $\left(\kappa_1^{-1}, \cdots, \kappa_n^{-1}\right)$.

Moreover, it is clear that

$$\bar{\nabla}_{e_j} \bar{\nabla}_{e_i} \nu = \delta_{ij} \nu, \tag{7.4}$$

this yields, for $k = 1, 2 \cdots, n+1$,

$$\bar{\nabla}_{e_j} \bar{\nabla}_{e_i} x_k = x_k \delta_{ij}, \tag{7.5}$$

where x_k is the coordinate function. These properties will be used in Section 9.

7.2. **Legendre transform.** Suppose $\mathcal{M}$ is a complete, noncompact, locally stictly convex, space-like hypersurface. Then $\mathcal{M}$ is the graph of a convex function

$$x_{n+1} = -\langle X, E \rangle = u(x_1, \cdots, x_n),$$

where $E = (0, \cdots, 0, 1)$. Introduce the Legendre transform

$$\xi_i = \frac{\partial u}{\partial x_i},\ \ u^* = \sum x_i \xi_i - u.$$

From the theory of convex bodies we know that

$$\Omega = \left\{ (\xi_1, \cdots, \xi_n) | \xi_i = \frac{\partial u}{\partial x_i}(x), x \in \mathbb{R}^n \right\}$$

is a convex domain.

In particular, let $u(x) = \sqrt{1 + |x|^2}$, $x \in \mathbb{R}^n$, be a hyperboloid with principal curvatures being equal to 1, then it's Legendre transform is $u^*(\xi) = -\sqrt{1 - |\xi|^2}$, $\xi \in B_1$.

Next, we calculate the first and the second fundamental forms in terms of ξ_i. Since

$$x_i = \frac{\partial u^*}{\partial \xi_i},\ \ u = \sum \xi_i \frac{\partial u^*}{\partial \xi_i} - u^*,$$

and it is well known that

$$\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = \left(\frac{\partial^2 u^*}{\partial \xi_i \partial \xi_j}\right)^{-1}.$$

We have, using the coordinate $\{\xi_1, \xi_2, \cdots, \xi_n\}$, the first and second fundamental forms can be rewritten as:

$$g_{ij} = \delta_{ij} - \xi_i \xi_j, \text{ and } h_{ij} = \frac{u^{*ij}}{\sqrt{1 - |\xi|^2}},$$

where $\left(u^{*ij}\right)$ denotes the inverse matrix of (u^*_{ij}) and $|\xi|^2 = \sum_i \xi_i^2$. Now, let W denote the Weingarten matrix of $\mathcal{M}$, then

$$(W^{-1})_{ij} = \sqrt{1 - |\xi|^2} g_{ik} u^*_{kj}.$$

From the discussion above we can see that if $\mathcal{M}_u = \{(x, u(x)) | x \in \mathbb{R}^n\}$ is a complete, strictly convex, spacelike hypersurface satisfies $\sigma_{n-1}(\kappa[\mathcal{M}]) = 1$, then the Legendre transform of u denoted by u^*, satisfies $\frac{\sigma_n}{\sigma_1}(\kappa^*[w^* \gamma^*_{ik} u^*_{kl} \gamma^*_{lj}]) = 1$. Here $w^* = \sqrt{1 - |\xi|^2}$ and $\gamma^*_{ij} = \delta_{ij} - \frac{\xi_i \xi_j}{1 + w^*}$ is the square root of the matrix g_{ij}.

8. Construction of $\sigma_{n-1} = 1$ convex hypersurfaces

Sections 8, 9, 10, and 11 will be devoted to the construction of complete, strictly convex, spacelike $\sigma_{n-1} = 1$ hypersurfaces with bounded principal curvatures. There are a few difficulties we need to conquer in this construction process. First, we need to make sure the hypersurface we construct is strictly convex. Second, we need to show that the hypersurface we construct has bounded principal curvatures. In order to overcome these difficulties, we will apply Anmin Li's idea (see [22]) to study the Legendre transform of the solution.

Let's recall Theorem 3.1 in [22].

Theorem 26. *(Theorem 3.1 in* [22]*) Let $\mathcal{M}$ be a closed, noncompact, spacelike, strictly convex hypersurface. If there exists a constant $d > 0$ such that $\kappa_i \geqslant d$ for all $i = 1, 2, \cdots, n$ everywhere on $\mathcal{M}$, then*
1. The Gauss map $G : \mathcal{M} \to B_1$ is a diffeomorphism;
2. $\varphi \in C^0(\partial B_1)$, where $\varphi = \lim_{\xi \to \partial B_1} u^*(\xi)$.
Here u^* is the Legendre transform of the height function of $\mathcal{M}$.

From Theorem 26 and the discussion in Subsection 7.2, we know that for a closed, noncompact, spacelike, strictly convex hypersurface $\mathcal{M}$ with principal curvatures bounded from below by a positive constant, and satisfies

$$\sigma_{n-1}(\kappa[\mathcal{M}]) = 1,$$

its Legendre transform u^* must satisfy the following equation:

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = 1, \text{ in } B_1 \\ u^* = \varphi, \text{ on } \partial B_1, \end{cases} \tag{8.1}$$

where $\varphi \in C^0(\partial B_1)$, $w^* = \sqrt{1-|\xi|^2}$, $\gamma^*_{ik} = \delta_{ik} - \frac{\xi_i\xi_k}{1+w^*}$, $u^*_{kl} = \frac{\partial^2 u^*}{\partial\xi_k\partial\xi_l}$, and $F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \left(\frac{\sigma_n}{\sigma_1}(\kappa^*[w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}])\right)^{\frac{1}{n-1}}$.

Due to technical issues, we cannot solve the Dirichlet problem with C^0 boundary data. In the following, we will study the existence of solutions to the following equation instead:

$$\begin{cases} F(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = 1, \text{ in } B_1 \\ u^* = \varphi, \text{ on } \partial B_1, \end{cases} \tag{8.2}$$

where $\varphi \in C^2(\partial B_1)$.

Notice that equation (8.2) is degenerate on ∂B_1. Therefore, we will consider the approximate problem:

$$\begin{cases} F(w^*\gamma^*_{ik}u^{r*}_{kl}\gamma^*_{lj}) = 1, \text{ in } B_r \\ u^{r*} = \varphi, \text{ on } \partial B_r, \end{cases} \tag{8.3}$$

where $0 < r < 1$.

9. Existence of solutions to equation (8.3)

In this section, we will show that for each $0 < r < 1$, there exists a solution to equation (8.3).

9.1. C^0 estimates.

Since u^{r*} is a convex function we have

$$\max_{B_r} u^{r*} \leqslant \max_{\partial B_r} \varphi.$$

In order to show that u^{r*} is bounded from below, similar to [22], we consider a special subsolution of (8.2)

$$\underline{u}^* = -n^{\frac{1}{n-1}}\sqrt{1-|\xi|^2} + a_1\xi_1 + \cdots + a_n\xi_n + c,$$

where $a_1, \cdots, a_n, c$ are constants such that

$$\underline{u}^*|_{\partial B_1} < \inf_{\partial B_1} \varphi.$$

Note that $\underline{u}^*$ is the linear translation of the Legendre transform of a standard Hyperboloid whose principal curvatures are equal to $n^{-\frac{1}{n-1}}$. Then the maximum principle implies $u^{r*} > \underline{u}^*$ for any $0 < r < 1$.

9.2. C^1 **estimates.** By Section 2 of [6], for any $0 < r < 1$, we can construct a subsolution $\underline{u}^{r*}$ such that

$$\begin{cases} F(w^*\gamma^*_{ik}\underline{u}^{r*}_{kl}\gamma^*_{lj}) \geqslant 1, \text{ in } B_r \\ \underline{u}^{r*} = \varphi, \text{ on } \partial B_r. \end{cases} \tag{9.1}$$

Then by the convexity of u^{r*} we have

$$|Du^{r*}| \leqslant \max_{\partial B_r} |D\underline{u}^{r*}|.$$

9.3. C^2 **boundary estimates.** For our convenience, in this subsection we will use the hyperbolic model (see Subsection 7.1), and write equation (8.3) as follows:

$$\begin{cases} F(v_{ij} - v\delta_{ij}) = 1, \text{ in } U_r \\ v = \dfrac{\varphi}{\sqrt{1-r^2}}, \text{ on } \partial U_r. \end{cases} \tag{9.2}$$

where $U_r = P^{-1}(B_r) \subset \mathbb{H}^n(-1)$. Here we want to point out that $v = \frac{u^*}{\sqrt{1-|\xi|^2}}$ and $\partial U_r \subset \mathbb{P} = \{x_{n+1} = \frac{1}{\sqrt{1-r^2}}\}$.

Equation of this type has been studied by Bo Guan in [12]. However, our function F is slightly different from functions in [12]. More precisely, our function F doesn't satisfy the assumption (1.7) in [12]. Therefore, in order to obtain the C^2 boundary estimates, we need to give a different proof of Lemma 6.2 in [12].

Lemma 27. *There exist some uniform positive constant t, δ, ϵ such that*

$$h = (v - \underline{v}) + t\left(\frac{1}{\sqrt{1-r^2}} - x_{n+1}\right)$$

satisfies

$$\mathfrak{L}h \leqslant -a(1 + \sum_i F^{ii}), \text{ in } U_{r\delta},$$

and

$$h \geqslant 0, \text{ on } \partial U_{r\delta}.$$

Here $a > 0$ is some positive constant, $\underline{v}$ is a subsolution, $\mathfrak{L}f := F^{ij}\nabla_{ij}f - f\sum_i F^{ii}$, and $U_{r\delta} := \left\{x \in U_r \,\middle|\, \frac{1}{\sqrt{1-r^2}} - x_{n+1} < \delta\right\}$.

Proof. When t large, $\delta > 0$ small, it's easy to see that we have $h \geqslant 0$ on $\partial U_{r\delta}$. Moreover, by equation (7.5) we get

$$\mathfrak{L}h \leqslant -t\frac{1}{\sqrt{1-r^2}}\sum F^{ii} - C,$$

where C depends on $\underline{v}$. Therefore, we are done. □

We want to point out that the existence of subsolution $\underline{v}$ has been proved in Theorem 1.2 of [12]. The rest of C^2 boundary estimates follows from [12] directly.

9.4. **Global C^2 estimates.** Just like before, since we don't have the assumption (1.7) of [12], we cannot apply the global C^2 estimates there. We need another approach to prove the C^2 estimate of (8.3). In particular, we will study the Legendre transform of u^{r*}, which we will denote by u^r. We will also denote $\Omega_r = Du^{r*}(B_r)$. Then, it's easy to see that u^r satisfies

$$\sigma_{n-1}(\kappa[a_{ij}]) = 1, \text{ in } \Omega_r, \tag{9.3}$$

where $a_{ij} = \frac{\gamma^{ik}u^r_{kl}\gamma^{lj}}{w}$, $\gamma^{ik} = \delta_{ik} + \frac{u^r_i u^r_k}{w(1+w)}$, and $w = \sqrt{1-|Du^r|^2}$.

Since the principal curvature lower bound of $\kappa[a_{ij}]$ implies the curvature radius upper bound of $\kappa^*[w^*\gamma^*_{ik}u^{r*}_{kl}\gamma^*_{lj}]$. We will consider

$$\phi = -\log \sigma_n(\kappa_1, \cdots, \kappa_n) - N\langle \nu, E\rangle . \tag{9.4}$$

If ϕ achieves its maximum at an interior point $x_0 \in \Omega_r$. Let $\{\tau_1, \cdots, \tau_n\}$ be the orthonormal frame such that $h_{ij} = \kappa_i\delta_{ij}$ at $X_0 = (x_0, u^r(x_0))$. Then at this point we have,

$$\phi_i = -\frac{(\sigma_n)_i}{\sigma_n} - Nh_{im}\langle \tau_m, E\rangle = 0, \tag{9.5}$$

and

$$\begin{aligned} 0 \geqslant \sigma^{ii}_{n-1}\phi_{ii} &= -\frac{\sigma^{ii}_{n-1}(\sigma_n)_{ii}}{\sigma_n} + \frac{\sigma^{ii}_{n-1}(\sigma_n)^2_i}{\sigma_n^2} - N\sigma^{ii}_{n-1}\kappa_i^2\langle \nu, E\rangle \\ &= -\frac{\sigma^{ii}_{n-1}(\sigma_n)_{ii}}{\sigma_n} + \sigma^{ii}_{n-1}N^2\kappa_i^2\langle \tau_i, E\rangle^2 - N\sigma^{ii}_{n-1}\kappa_i^2\langle \nu, E\rangle \\ &\geqslant n^2\sigma_n - \sigma_1\sigma_{n-1} - N\sigma_1\sigma_{n-1}\langle \nu, E\rangle \\ &\geqslant n^2\sigma_n - \sigma_1\sigma_{n-1} + N\sigma_1\sigma_{n-1}\frac{1}{\sqrt{1-|Du|^2}}, \end{aligned} \tag{9.6}$$

where we have used $\sigma^{ii}_{n-1}h_{iik} = 0$ and Theorem 8.

Choosing $N = 2$ leads to a contradiction. Therefore, we conclude that ϕ achieves its maximum at the boundary $\partial\Omega_r$. Combining with the boundary C^2 estimates in Subsection 9.3, we obtain $\sigma_n(\kappa^*)$ is bounded from above. Since $\frac{\sigma_n(\kappa^*)}{\sigma_1(\kappa^*)} = 1$, we have $\sigma_1(\kappa^*)$ is bounded from above. Therefore, we obtain the C^2 estimates for $|D^2u^{r*}|$. By the standard continuity argument, we know that equation (8.3) is solvable for any $0 < r < 1$.

10. Convergence of solutions to a strictly convex hypersurface

In this section we want to construct the solution to equation (8.2).

10.1. **Barrier function.** First, recall section 4 of [22] we know that there exists a strictly convex solution $\bar{u}_0^* \in C^\infty(B_1) \cap C^0(\bar{B}_1)$ satisfies

$$\begin{cases} \det\left(w^*\gamma_{ki}^* u_{kl}^* \gamma_{lj}^*\right) = n^{\frac{n}{n-1}}, \text{ in } B_1 \\ \qquad\qquad\qquad u^* = \varphi, \text{ on } \partial B_1. \end{cases} \tag{10.1}$$

By Maclaurin's inequality, we know that $\bar{u}_0^*$ is a supersolution of equation (8.2). On the other hand, consider the function

$$\underline{u}_0^* = -A\sqrt{1-|\xi|^2} + \varphi,$$

by a straightforward calculation we can see that, when $A > 0$ sufficiently large, $\underline{u}_0^*$ is a subsolution of (8.2).

In this section we will consider the convergence of functions u^{r*}, where u^{r*} satisfies

$$\begin{cases} F(w^*\gamma_{ik}^* u_{kl}^{r*} \gamma_{lj}^*) = 1, \text{ in } B_r \\ \qquad\qquad u^{r*} = \bar{u}_0^*, \text{ on } \partial B_r. \end{cases} \tag{10.2}$$

Note that the existence of the solution to equation (10.2) has been proved in Section 9. In the following, we will denote u^r as the Legendre transform of u^{r*}, $\underline{u}_0$ as the Legendre transform of $\underline{u}_0^*$, and $\bar{u}_0$ as the Legendre transform of $\bar{u}_0^*$. We will also denote $\Omega_r = Du^{r*}(B_r)$ as the domain of u^r. We will show that there exists a subsequence of $\{u^r\}$ which converges locally smoothly to a strictly convex function u. Moreover, u satisfies $\sigma_{n-1}(\kappa[\mathcal{M}_u]) = 1$ and $\kappa[\mathcal{M}_u] \leqslant C$.

10.2. **Local C^0 estimates.** By the maximum principle we know that for any $r > 0$ we have $\underline{u}_0^* \leqslant u^{r*} \leqslant \bar{u}_0^*$ everywhere. Therefore,

$$|u^{r*}| \leqslant C_0.$$

Moreover, since u^{r*} is convex we have, for any $r > 1/2$

$$|Du^{r*}(0)| \leqslant 2\left(\max u^{r*} - \min u^{r*}\right) \leqslant C_1.$$

Note also that at the point where $\min u^r$ is achieved we have $Du^r = 0$. Thus, $\min u^r$ is achieved in $B_{C_1}(0) \subset \mathbb{R}^n$. On the other hand, when $r > 1/2$ we have,

$$|\min u^r| = |0 \cdot Du^{r*}(0) - u^{r*}(0)| \leqslant C_0.$$

These together with the fact that $|Du^r| < 1$ yield in a ball of radius $R > C_1$ we have

$$|u^r| \leqslant C_0 + R,$$

for any $r > r_0 > 1/2$. Furthermore, from the discussion above, we know that by a coordinate transformation, we may always assume $2C_0 + 1 \geqslant u^r(0) \geqslant 1$ and $Du^r(0) = 0$.

10.3. **Local C^1 estimates.** Before we start to work on the derivative estimates, we need the following Lemma.

Lemma 28. *Let u^{r*} be the solution of* (10.2) *and u^r be the Legendre transform of u^{r*}. Then, $u^r|_{\partial\Omega_r} \to +\infty$ as $r \to 1$.*

Proof. By Lemma 5.6 in [22] and the maximum principle we have

$$\begin{aligned} u^r|_{\partial\Omega_r} &= [\xi \cdot Du^{r*}(\xi) - u^{r*}(\xi)]|_{\partial B_r} \\ &\geqslant [\xi \cdot D\bar{u}_0^*(\xi) - \bar{u}_0^*(\xi)]|_{\partial B_r} \\ &\geqslant \frac{d_1}{\sqrt{1-r^2}}. \end{aligned} \tag{10.3}$$

Therefore, it's easy to see that $u^r|_{\partial\Omega_r} \to +\infty$ as $r \to 1$. □

Next, we will prove the local C^1 estimates

Lemma 29. *Let u^{r*} be the solution of* (10.2) *and u^r be the Legendre transform of u^{r*}. Then, $u^r\sqrt{1-|Du^r|^2} \geqslant \min\{C_2, \min_{\partial\Omega_r} u^r\sqrt{1-|Du^r|^2}\}$.*

Proof. Let $\phi = u^r\sqrt{1-|Du^r|^2}$, we will consider $\log\phi$. In this proof, for our convenience, we will omit the superscript r. Suppose $\log\phi$ achieves its minimum at an interior point $x_0 \in \Omega_r$. We may choose a local orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that at x_0, we have $h_{ij} = \kappa_i\delta_{ij}$. Differentiating $\log\phi$ twice we get

$$\frac{\phi_i}{\phi} = \frac{u_i}{u} - \frac{u_k u_{ki}}{1-|Du|^2}, \tag{10.4}$$

and

$$\begin{aligned} \frac{\phi_{ii}}{\phi} - \frac{\phi_i^2}{\phi^2} &= \frac{u_{ii}}{u} - \frac{u_i^2}{u^2} \\ &\quad - \frac{u_{ki}^2}{1-|Du|^2} - \frac{u_k u_{kii}}{1-|Du|^2} - \frac{2(u_k u_{ki})^2}{(1-|Du|^2)^2}. \end{aligned} \tag{10.5}$$

Since $u = -\langle X, E\rangle$ we have

$$u_{ij} = -h_{ij}\langle \nu, E\rangle$$

and

$$u_{kii} = -h_{kii}\langle \nu, E\rangle - h_{ki}h_{im}\langle \tau_m, E\rangle.$$

Thus,

$$\begin{aligned} \sigma_{n-1}^{ii}u_k u_{kii} &= -\sigma_{n-1}^{ii}u_k h_{kii}\langle \nu, E\rangle - \sigma_{n-1}^{ii}u_k h_{ki}h_{im}\langle \tau_m, E\rangle \\ &= \sigma_{n-1}^{ii}u_k h_{ki}h_{im}u_m \geqslant 0, \end{aligned} \tag{10.6}$$

where we used $\sigma_{n-1}^{ii}h_{kii}=0$. Combining (10.6) with (10.5) we obtain, at x_0

$$
\begin{aligned}
0\leqslant \frac{\sigma_{n-1}^{ii}\phi_{ii}}{\phi}&\leqslant \frac{\sigma_{n-1}^{ii}u_{ii}}{u}-\frac{\sigma_{n-1}^{ii}h_{ii}^2}{(1-|Du|^2)^2}\\
&\leqslant \frac{n-1}{u\sqrt{1-|Du|^2}}-\frac{c_0}{(1-|Du|^2)^2}\\
&\leqslant \frac{n-1}{\sqrt{1-|Du|^2}}-\frac{c_0}{(1-|Du|^2)^2},
\end{aligned}\tag{10.7}
$$

where we used $\sigma_{n-1}^{ii}\kappa_i^2=\sigma_1\sigma_{n-1}-n\sigma_n\geqslant c_0$, and the assumption that $\min u\geqslant 1$. Therefore, we can see that at x_0,

$$1-|Du|^2\geqslant C_2,$$

which leads to our result. □

Lemma 30. *Let u^{r*} be the solution of* (10.2) *and u^r be the Legendre transform of u^{r*}. Then we have* $\min_{\partial\Omega_r}u^r\sqrt{1-|Du^r|^2}\geqslant C_3$.

Proof. Assume at $x_0\in\partial\Omega_r$, $u^r\sqrt{1-|Du^r|^2}(x_0)=\min_{\partial\Omega_r}u^r\sqrt{1-|Du^r|^2}$. Denote $\xi_0=Du^r(x_0)$, by the definition of Ω_r we know that $\xi_0\in\partial B_r$. Applying the maximum principle we obtain, at x_0

$$
\begin{aligned}
u^r\sqrt{1-|Du^r|^2}&=\left[\xi_0\cdot\frac{\partial u^{r*}}{\partial\xi}(\xi_0)-u^{r*}(\xi_0)\right]\sqrt{1-|\xi_0|^2}\\
&\geqslant\left[\xi_0\cdot\frac{\partial \bar u_0^{*}}{\partial\xi}(\xi_0)-\bar u_0^{*}(\xi_0)\right]\sqrt{1-|\xi_0|^2}\geqslant d_1,
\end{aligned}\tag{10.8}
$$

where the last inequality comes from Lemma 5.6 in [22]. This finishes the proof of this Lemma. □

Combining Lemma 28 and Lemma 29 with Lemma 30 we conclude

Corollary 31. *For any $s>2C_0+1$, there exists $r_s>0$ such that when $r>r_s$, $u^r|_{\partial\Omega_r}>s$. Moreover, in the domain $\{x\in\Omega_r|\, u^r(x)<s\}$ we have*

$$\sqrt{1-|Du^r|^2}\geqslant\frac{C_4}{s}.\tag{10.9}$$

10.4. **Local C^2 estimates.**

Lemma 32. *Let u^{r*} be the solution of* (10.2) *and u^r be the Legendre transform of u^{r*}. For any given $s>2C_0+1$, let $r_s>0$ be a positive number such that when $r>r_s$, $u^r|_{\partial\Omega_r}>s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u^r}$ at x, where $\mathcal{M}_{u^r}=\{(x,u^r(x))|\, x\in\Omega_r\}$. Then, for $r>r_s$ we have*

$$\max_{\mathcal{M}_{u^r}}(s-u^r)\kappa_{\max}\leqslant C_5.$$

Here, C_5 depends on the local C^1 estimates of u^r and s.

Proof. For our convenience, in this proof we will omit the superscript r. Let's consider

$$\varphi = m\log(s-u) + \log P_m - mN\left\langle \nu, E\right\rangle,$$

where $P_m = \sum_j \kappa_j^m$ and N, m are some constants to be determined. Suppose that the function φ achieves its maximum on $\mathcal{M}$ at some point x_0, we may choose a local orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that at x_0, $h_{ij} = \kappa_i\delta_{ij}$ and $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$. Differentiating φ twice at x_0 we get

$$\frac{\sum_j \kappa_j^{m-1}h_{jji}}{P_m} - Nh_{ii}\left\langle X_i, E\right\rangle + \frac{\left\langle X_i, E\right\rangle}{s-u} = 0 \tag{10.10}$$

and

$$\begin{aligned} 0 \geqslant\ & \frac{1}{P_m}\left[\sum_j \kappa_j^{m-1}h_{jjii} + (m-1)\sum_j \kappa_j^{m-2}h_{jji}^2 + \sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}h_{pqi}^2\right] \\ & - \frac{m}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 - Nh_{iji}\left\langle X_j, E\right\rangle - Nh_{ii}^2\left\langle \nu, E\right\rangle \\ & + \frac{h_{ii}\left\langle \nu, E\right\rangle}{s-u} - \frac{u_i^2}{(s-u)^2}, \end{aligned} \tag{10.11}$$

where $X_i = \nabla_{\tau_i}X = \tau_i$, for $1 \leqslant i \leqslant n$. Now, let's differentiate equation (1.2) twice

$$\sigma_{n-1}^{ii}h_{iij} = 0, \text{ and } \sigma_{n-1}^{ii}h_{iijj} + \sigma_{n-1}^{pq,rs}h_{pqj}h_{rsj} = 0. \tag{10.12}$$

Recall that in Minkowski space we have

$$h_{jjii} = h_{iijj} + h_{ii}^2h_{jj} - h_{ii}h_{jj}^2. \tag{10.13}$$

By (10.11) - (10.13) we find

(10.14)

$$\begin{aligned} 0 \geqslant \frac{1}{P_m}\Bigg\{ & \sum_j \kappa_j^{m-1}\left[-(n-1)h_{jj}^2 + K(\sigma_{n-1})_j^2 - \sigma_{n-1}^{pq,rs}h_{pqj}h_{rsj}\right] + (m-1)\sigma_{n-1}^{ii}\sum_j \kappa_j^{m-2}h_{jji}^2 \\ & + \sigma_{n-1}^{ii}\sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}h_{pqi}^2\Bigg\} - \frac{m\sigma_{n-1}^{ii}}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 \\ & - N\sigma_{n-1}^{ii}\kappa_i^2\langle \nu, E\rangle + \frac{(n-1)\left\langle \nu, E\right\rangle}{s-u} - \frac{\sigma_{n-1}^{ii}u_i^2}{(s-u)^2}, \end{aligned}$$

where K is a constant. We denote,

$$A_i = \frac{\kappa_i^{m-1}}{P_m}[K(\sigma_{n-1})_i^2 - \sum_{p,q}\sigma_{n-1}^{pp,qq}h_{ppi}h_{qqi}],$$

$$B_i = \frac{2\kappa_j^{m-1}}{P_m}\sum_j \sigma_{n-1}^{jj,ii}h_{jji}^2,$$

$$C_i = \frac{m-1}{P_m}\sigma_{n-1}^{ii}\sum_j \kappa_j^{m-2}h_{jji}^2,$$

$$D_i = \frac{2\sigma_{n-1}^{jj}}{P_m}\sum_{j\neq i}\frac{\kappa_j^{m-1}-\kappa_i^{m-1}}{\kappa_j-\kappa_i}h_{jji}^2,$$

and

$$E_i = \frac{m\sigma_{n-1}^{ii}}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2.$$

Since

$$-\sigma_{n-1}^{pq,rs}h_{pql}h_{rsl} = -\sigma_{n-1}^{pp,qq}h_{ppl}h_{qql} + \sigma_{n-1}^{pp,qq}h_{pql}^2,$$

equation (10.14) can be written as

$$\begin{aligned} 0 \geqslant\ & A_i + B_i + C_i + D_i - E_i - \frac{(n-1)\sum\kappa_j^{m+1}}{P_m} \\ & - N\sigma_{n-1}^{ii}\kappa_i^2\langle\nu, E\rangle + \frac{(n-1)\langle\nu, E\rangle}{s-u} - \frac{\sigma_{n-1}^{ii}u_i^2}{(s-u)^2}. \end{aligned} \tag{10.15}$$

By Lemma 8 and 9 in [21] we can assume the following claim holds.

Claim 1. *For any $i = 1, 2, \cdots, n$ we have*

$$A_i + B_i + C_i + D_i - \left(1+\frac{1}{m}\right)E_i \geqslant 0, \tag{10.16}$$

where $m > 0$ sufficiently large.

Here we note that, by Lemma 8 of [21], for $i = 2, 3, \cdots, n$, inequality (10.16) always holds; for $i = 1$, if (10.16) doesn't hold, by Lemma 9 of [21], we would obtain an upper bound for κ_1 at x_0 directly, then we would be done.

Therefore equation (10.15) and (10.16) implies

$$\begin{aligned} 0 \geqslant\ & \frac{1}{P_m}\sum_j \kappa_j^{m-1}(-C\kappa_1^2) + \sum_{i=2}^n \frac{\sigma_{n-1}^{ii}}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 \\ & - N\sigma_{n-1}^{ii}\kappa_i^2\langle\nu, E\rangle + \frac{(n-1)\langle\nu, E\rangle}{s-u} - \frac{\sigma_{n-1}^{ii}u_i^2}{(s-u)^2}. \end{aligned} \tag{10.17}$$

Recall equation (10.10) we obtain

$$-\frac{\sigma_{n-1}^{ii}u_i^2}{(s-u)^2} = -\frac{\sigma_{n-1}^{ii}}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 + N^2\sigma_{n-1}^{ii}\kappa_i^2u_i^2 - 2N\frac{\sigma_{n-1}^{ii}\kappa_i u_i^2}{s-u}. \tag{10.18}$$

Hence, (10.17) becomes

$$
\begin{aligned}
0 \geqslant & -C\kappa_1 + \sum_{i=2}^{n}\left(N^2\sigma_{n-1}^{ii}u_i^2\kappa_i^2 - 2N\frac{\sigma_{n-1}^{ii}u_i^2\kappa_i}{s-u}\right)\\
& - N\sum_{i=1}^{n}\sigma_{n-1}^{ii}\kappa_i^2\left\langle \nu, E\right\rangle + \frac{(n-1)\left\langle \nu, E\right\rangle}{s-u} - \frac{\sigma_{n-1}^{11}u_1^2}{(s-u)^2}.
\end{aligned} \tag{10.19}
$$

It's easy to see that there exists some positive constant c_0 such that $\sigma_{n-1}^{11}\kappa_1 \geqslant c_0$. Moreover, for any fixed $i = 1, \ldots, n$ we have $\sigma_{n-1} \geqslant \sigma_{n-1}^{ii}\kappa_i$ (no summation here). Thus, we get

$$
\begin{aligned}
0 \geqslant & \left(-\frac{c_0 N\left\langle \nu, E\right\rangle}{2} - C\right)\kappa_1 - 2N\sum_{i=2}^{n}\frac{\sigma_{n-1}^{ii}u_i^2\kappa_i}{s-u}\\
& - \frac{N}{2}\sigma_{n-1}^{11}\kappa_1^2\left\langle \nu, E\right\rangle + \frac{(n-1)\left\langle \nu, E\right\rangle}{s-u} - \frac{\sigma_{n-1}^{11}u_1^2}{(s-u)^2}.
\end{aligned} \tag{10.20}
$$

Note that

$$
\sum_{i=1}^{n}u_i^2 = \sum_{i=1}^{n}\left\langle X_i, E\right\rangle^2 = \left\langle \nu, E\right\rangle^2 - 1 \leqslant \left\langle \nu, E\right\rangle^2.
$$

When we choose $N > 0$ large enough such that $-N\left\langle \nu, E\right\rangle \geqslant \frac{4C}{c_0}$, (10.20) yields

$$
\frac{c_0 N}{4}\kappa_1(-\left\langle \nu, E\right\rangle) + \frac{N}{2}\sigma_{n-1}^{11}\kappa_1^2(-\left\langle \nu, E\right\rangle) \leqslant \frac{3N}{s-u}\left\langle \nu, E\right\rangle^2 + \frac{\sigma_{n-1}^{11}}{(s-u)^2}\left\langle \nu, E\right\rangle^2. \tag{10.21}
$$

If at x_0 we have $s - u \geqslant \sigma_{n-1}^{11}$, then (10.21) implies

$$
(s-u)\kappa_1 \leqslant C_5.
$$

If at x_0 we have $s - u < \sigma_{n-1}^{11}$, then from (10.21) we get

$$
\frac{N}{2}\sigma_{n-1}^{11}\kappa_1^2(s-u)^2 \leqslant -3N(s-u)\left\langle \nu, E\right\rangle - \sigma_{n-1}^{11}\left\langle \nu, E\right\rangle,
$$

this gives

$$
(s-u)\kappa_1 \leqslant C_5.
$$

Therefore, we obtain the desired Pogorelov type C^2 interior estimates. □

10.5. **Convergence of u^r to the solution u.** Combining estimates in Subsections 10.2 - 10.4 with the classic regularity theorem, we know that there exists a subsequence of $\{u^r\}$, which we will denote by $\{u^{r_i}\}$, $r_i \to 1$ as $i \to \infty$, converging locally smoothly to a convex function u defined over $\mathbb{R}^n$, and u satisfies

$$
\sigma_{n-1}(\kappa[\mathcal{M}_u]) = 1.
$$

Lemma 33. *u is a strictly convex function and $Du(\mathbb{R}^n) = B_1$.*

Proof. We will denote $u^* = x \cdot Du(x) - u(x)$. By Theorem 2 we know that to prove Lemma 33 is equivalent to prove u^* is defined on B_1.

For any $\xi_0 \in B_1$ and $i \geqslant i_0$, there exists a point $x_0^{r_i}$ such that

$$Du^{r_i}(x_0^{r_i}) = \xi_0.$$

In other words, we have $Du^{r_i*}(\xi_0) = x_0^{r_i}$. Denote $\eta_{r_i} = dist(\xi_0, \partial B_{r_i})$, since u^{r_i*} is convex, we get

$$|Du^{r_i*}(\xi_0)| \leqslant \frac{1}{\eta_{r_i}} (\max u^{r_i*} - \min u^{r_i*}) \leqslant \frac{2C_0}{\eta_{r_i}}.$$

This implies, when $i > i_0$ we have $x_0^{r_i} \in \bar{B}_{C_6}(0) \subset \mathbb{R}^n$, where $C_6 = \frac{2C_0}{\eta_{r_{i_0}}}$ is a constant. Therefore, there exists a subsequence of $\{x_0^{r_i}\}$ which we still denote as $\{x_0^{r_i}\}$ such that

$$\lim_{i \to \infty} x_0^{r_i} = x_0 \in \bar{B}_{C_6}(0).$$

Moreover, by the local C^1 estimates we conclude that

$$Du(x_0) = \xi_0.$$

Since $\xi_0 \in B_1$ is arbitrary, we proved this Lemma. □

An immediate consequence of Lemma 33 is the following

Corollary 34. *Let $\{u^{r_i*}\}$ be the Legendre transform of $\{u^{r_i}\}$, where $\{u^{r_i}\}$ is the same convergent sequence we chose earlier. Then, $u^* = \lim_{i\to\infty} u^{r_i*}$ solves equation* (8.2). *Moreover, u^* is the Legendre transform of $u = \lim_{i\to\infty} u^{r_i}$.*

11. A UNIFORM BOUND FOR $\kappa[\mathcal{M}_u]$

In this section, we will show that the function u we obtained in Section 10 satisfies $\kappa[\mathcal{M}_u] \leqslant C$. In other words, the noncompact, strictly convex, spacelike hypersurface we constructed in Section 10 has bounded principal curvatures.

Now let $S^{n-1}(1) = \{\xi \in \mathbb{R}^n | \sum \xi_i^2 = 1\}$. We denote the angular derivative $\xi_k \frac{\partial}{\partial \xi_l} - \xi_l \frac{\partial}{\partial \xi_k}$ on $S^{n-1}(1)$ by $\partial_{k,l}$, or simply by ∂, when no confusion arises. Then we have following Lemmas.

Lemma 35. *Let u^* be the solution of equation* (8.2). *Then, $|\partial u^*|$ is bounded by a constant depends on $|\varphi|_{C^1}$.*

Proof. Without loss of generality, we assume that $\partial = \xi_1 \frac{\partial}{\partial \xi_2} - \xi_2 \frac{\partial}{\partial \xi_1}$. By our equation (8.2) we have

$$F(\gamma^*_{ik} u^*_{kl} \gamma^*_{lj}) = \frac{1}{w^*}.$$

Differentiating it with respect to ∂ we get

$$F^{kl} \partial(\gamma^*_{ik} u^*_{ij} \gamma^*_{lj}) = 0. \tag{11.1}$$

Since

$$\partial(\gamma^*_{ki}u^*_{ij}\gamma^*_{jl}) = (\partial\gamma^*_{ki})u^*_{ij}\gamma^*_{jl} + \gamma^*_{ki}u^*_{ij}(\partial\gamma^*_{jl}) + \gamma^*_{ki}(\partial u^*_{ij})\gamma^*_{jl}, \tag{11.2}$$

we will compute these terms one by one.

First, we can see that

$$\begin{aligned}(\partial u^*)_{ij} &= (\xi_1\partial_2 u^* - \xi_2\partial_1 u^*)_{ij} \\ &= \delta_{1i}u^*_{2j} + \delta_{1j}u^*_{2i} - \delta_{2i}u^*_{1j} - \delta_{2j}u^*_{1i} + \xi_1 u^*_{2ij} - \xi_2 u^*_{1ij}.\end{aligned} \tag{11.3}$$

Thus, we have

$$\gamma^*_{ki}(\partial u^*)_{ij}\gamma^*_{jl} = \gamma^*_{k1}u^*_{2j}\gamma^*_{jl} + \gamma^*_{ki}u^*_{2i}\gamma^*_{1l} - \gamma^*_{k2}u^*_{1j}\gamma^*_{jl} - \gamma^*_{ki}u^*_{1i}\gamma^*_{2l} + \gamma^*_{ki}\partial(u^*_{ij})\gamma^*_{jl}. \tag{11.4}$$

Next, we differentiate γ^*_{ki} and get

$$\begin{aligned}\partial\gamma^*_{ki} &= (\xi_1\partial_2 - \xi_2\partial_1)\left(\delta_{ki} - \frac{\xi_k\xi_i}{1+w^*}\right) \\ &= -\frac{1}{1+w^*}(\xi_1\delta_{k2}\xi_i + \xi_1\xi_k\delta_{i2} - \xi_2\delta_{k1}\xi_i - \xi_2\xi_k\delta_{1i}) \\ &= \delta_{k2}\gamma^*_{1i} + \delta_{i2}\gamma^*_{1k} - \delta_{k1}\gamma^*_{2i} - \delta_{1i}\gamma^*_{2k}.\end{aligned} \tag{11.5}$$

Hence, we have

$$\begin{aligned}(\partial\gamma^*_{ki})u^*_{ij}\gamma^*_{jl} &= \delta_{k2}\gamma^*_{1i}u^*_{ij}\gamma^*_{jl} + \gamma^*_{1k}u^*_{2j}\gamma^*_{jl} \\ &\quad - \delta_{k1}\gamma^*_{2i}u^*_{ij}\gamma^*_{jl} - \gamma^*_{jl}u^*_{1j}\gamma^*_{2k}.\end{aligned} \tag{11.6}$$

Similarly, we have

$$\gamma^*_{ki}u^*_{ij}(\partial\gamma^*_{jl}) = \delta_{l2}\gamma^*_{1j}u^*_{ij}\gamma^*_{ik} + \gamma^*_{1l}u^*_{2i}\gamma^*_{ik} - \delta_{l1}\gamma^*_{2j}u^*_{ij}\gamma^*_{ik} - \gamma^*_{ik}u^*_{1i}\gamma^*_{2l}. \tag{11.7}$$

Therefore we conclude that,

$$\begin{aligned}\partial(\gamma^*_{ki}u^*_{ij}\gamma^*_{jl}) &= \delta_{k2}\gamma^*_{1i}u^*_{ij}\gamma^*_{jl} - \delta_{k1}\gamma^*_{2i}u^*_{ij}\gamma^*_{jl} \\ &\quad + \gamma^*_{ki}(\partial u^*)_{ij}\gamma^*_{jl} + \delta_{l2}\gamma^*_{1j}u^*_{ij}\gamma^*_{ik} - \delta_{1l}\gamma^*_{2j}u^*_{ij}\gamma^*_{ik}\end{aligned} \tag{11.8}$$

Plugging equation (11.8) into (11.1) we get,

$$F^{kl}\gamma^*_{li}(\partial u^*)_{ij}\gamma^*_{jk} = 0. \tag{11.9}$$

Using the maximum principle we obtain

$$|\partial u^*| \leqslant |\varphi|_{C^1}.$$

□

Lemma 36. *Let u^* be the solution of equation* (8.2)*. Then, $\partial^2 u^*$ is bounded above by a constant depends on $|\varphi|_{C^2}$.*

Proof. We still assume $\partial = \xi_1\frac{\partial}{\partial\xi_2} - \xi_2\frac{\partial}{\partial\xi_1}$. For our convenience, we will also denote $a_{kl} = \gamma^*_{ki}u^*_{ij}\gamma^*_{jl}$, and $\tilde{a}_{kl} = \gamma^*_{ki}(\partial u^*)_{ij}\gamma^*_{jl}$. Note that F depends only on the eigenvalues of the matrix $A = (a_{kl})$, it is invariant under rotation of coordinates. Therefore, during our calculations we can always assume A is diagonal, i.e., $a_{kl} = \kappa_k\delta_{kl}$. Following the notation in section 2, we define

$$f(\kappa[A]) = F(A).$$

Differentiating (11.1) with respect to ∂ we get

$$F^{kl}(\partial^2 a_{kl}) + F^{pq,rs}(\partial a_{pq})(\partial a_{rs}) = 0. \tag{11.10}$$

By (11.8) we get

$$\partial a_{kl} = \delta_{k2}a_{1l} - \delta_{k1}a_{2l} + \delta_{l2}a_{1k} - \delta_{l1}a_{2k} + \tilde{a}_{kl}, \tag{11.11}$$

and

$$\begin{aligned}\partial^2 a_{kl} &= \delta_{k2}(\partial a_{1l}) - \delta_{k1}(\partial a_{2l}) + \delta_{l2}(\partial a_{1k}) - \delta_{l1}(\partial a_{2k}) + \partial\tilde{a}_{kl}\\ &= \delta_{k2}(\partial a_{1l}) - \delta_{k1}(\partial a_{2l}) + \delta_{l2}(\partial a_{1k}) - \delta_{l1}(\partial a_{2k})\\ &+ \delta_{k2}\tilde{a}_{1l} - \delta_{k1}\tilde{a}_{2l} + \delta_{l2}\tilde{a}_{1k} - \delta_{l1}\tilde{a}_{2k} + \gamma^*_{ki}(\partial^2 u^*)_{ij}\gamma^*_{jl}.\end{aligned} \tag{11.12}$$

Contracting (11.12) with F^{kl} we obtain,

$$\begin{aligned}F^{kl}(\partial^2 a_{kl}) &= 2\left(-F^{22}a_{22} + F^{22}a_{11} - F^{11}a_{11} + F^{11}a_{22} + F^{22}\tilde{a}_{12} - F^{11}\tilde{a}_{21}\right)\\ &+ 2\left(F^{22}\tilde{a}_{12} - F^{11}\tilde{a}_{21}\right) + F^{kl}\gamma^*_{ki}(\partial^2 u^*)_{ij}\gamma^*_{jl}\\ &= -2(f_2 - f_1)(\kappa_2 - \kappa_1) + 4(f_2 - f_1)\tilde{a}_{12} + F^{kl}\gamma^*_{ki}(\partial^2 u^*)_{ij}\gamma^*_{jl}.\end{aligned} \tag{11.13}$$

On the other hand by Lemma 7 we have,

$$\begin{aligned}&F^{pq,rs}(\partial a_{pq})(\partial a_{rs})\\ &= F^{pq,rs}\left(\delta_{p2}a_{1q} - \delta_{p1}a_{2q} + \delta_{q2}a_{1p} - \delta_{q1}a_{2p} + \tilde{a}_{pq}\right)\\ &\times\left(\delta_{r2}a_{1s} - \delta_{r1}a_{2s} + \delta_{s2}a_{1r} - \delta_{s1}a_{2r} + \tilde{a}_{sr}\right)\\ &= F^{pp,rr}\left(\delta_{p2}a_{1p} - \delta_{p1}a_{2p} + \delta_{p2}a_{1p} - \delta_{p1}a_{2p} + \tilde{a}_{pp}\right)\\ &\times\left(\delta_{r2}a_{1r} - \delta_{r1}a_{2r} + \delta_{r2}a_{1r} - \delta_{r1}a_{2r} + \tilde{a}_{rr}\right)\\ &+ \sum_{p\neq r}\frac{f_p - f_r}{\kappa_p - \kappa_r}\left(\delta_{p2}a_{1r} - \delta_{p1}a_{2r} + \delta_{r2}a_{1p} - \delta_{r1}a_{2p} + \tilde{a}_{pr}\right)^2\\ &= F^{pq,rs}\tilde{a}_{pq}\tilde{a}_{rs} + 2\frac{f_2 - f_1}{\kappa_2 - \kappa_1}\kappa_1^2 + 2\frac{f_1 - f_2}{\kappa_1 - \kappa_2}\kappa_2^2\\ &- 4\frac{f_2 - f_1}{\kappa_2 - \kappa_1}\kappa_1\kappa_2 + 4\frac{f_2 - f_1}{\kappa_2 - \kappa_1}\kappa_1\tilde{a}_{21} - 4\frac{f_1 - f_2}{\kappa_1 - \kappa_2}\kappa_2\tilde{a}_{12}\\ &= 2(f_2 - f_1)(\kappa_2 - \kappa_1) - 4(f_2 - f_1)\tilde{a}_{12} + F^{pq,rs}\tilde{a}_{pq}\tilde{a}_{rs}.\end{aligned} \tag{11.14}$$

Equations (11.13) and (11.14) yields

$$0 = F^{kl}(\partial^2 a_{kl}) + F^{pq,rs}(\partial a_{pq})(\partial a_{rs}) = F^{kl}\gamma^*_{ki}(\partial^2 u)_{ij}\gamma^*_{jl} + F^{pq,rs}\tilde{a}_{pq}\tilde{a}_{rs} \tag{11.15}$$

Since F is concave we conclude

$$F^{kl}\gamma^*_{ki}(\partial^2 u)_{ij}\gamma^*_{jl} \geqslant 0. \tag{11.16}$$

By the maximum principle, we prove this Lemma. □

Following [22], using Lemma 35 and Lemma 36 we can also prove the following Lemmas. Since the proof of these Lemmas are almost identical to [22] (see Lemma 5.3-Lemma 5.6), we will omit them here.

Lemma 37. *Let u^* be the solution of equation* (8.2)*. Then, there is a positive constant b such that*

$$\sqrt{1-|\xi|^2}\left|\partial^2 u^*\right| < b.$$

Lemma 38. *Let u^* be the solution of equation* (8.2)*, $\frac{1}{2} < r < 1$, and $S^{n-1}(r) = \{\xi \in \mathbb{R}^n | \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in S^{n-1}(r)$, there is a function*

$$\overline{u}^* = -n^{\frac{1}{n-1}}\sqrt{1-|\xi|^2} + b_1\xi_1 + \cdots + b_n\xi_n + b$$

such that

$$\overline{u}^*(\hat{\xi}) = u^*(\hat{\xi}),$$

and

$$\overline{u}^*(\hat{\xi}) > u^*(\xi), \text{ for any } \xi \in S^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Here $b_1, \cdots, b_n$ are constants depending on $\hat{\xi}$, and b is a positive constant independent of $\hat{\xi}$ and r.

Similarly we have

Lemma 39. *Let u^* be the solution of equation* (8.2)*, $\frac{1}{2} < r < 1$, and $S^{n-1}(r) = \{\xi \in \mathbb{R}^n | \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in S^{n-1}(r)$, there is a function*

$$\underline{u}^* = -n^{\frac{1}{n-1}}\sqrt{1-|\xi|^2} + a_1\xi_1 + \cdots + a_n\xi_n - a$$

such that

$$\underline{u}^*(\hat{\xi}) = u^*(\hat{\xi}),$$

and

$$\underline{u}^*(\hat{\xi}) < u^*(\xi), \text{ for any } \xi \in S^{n-1}(r) \setminus \{\hat{\xi}\}.$$

Here $a_1, \cdots, a_n$ are constants depending on $\hat{\xi}$, and a is a positive constant independent of $\hat{\xi}$ and r.

Using Lemma 38 and Lemma 39 we can show

Lemma 40. *Let u^* be the solution of equation* (8.2) *and u be the Legendre transform of u^*. There are positive constants $d_2 > d_1$ such that*

$$0 < d_1 \leqslant u\sqrt{1-|Du|^2} \leqslant d_2. \tag{11.17}$$

Finally, we are ready to estimate the principal curvatures of $\mathcal{M}_u$.

Proposition 41. *Let u^* be the solution of equation* (8.2) *and u be the Legendre transform of u^*. Then the hypersurface $\mathcal{M}_u = \{(x, u(x)) |\, x \in \mathbb{R}^n\}$ has bounded principal curvatures.*

Proof. We will use the idea of [22] to obtain a Pogorelov type interior estimate. For any $s > 0$, consider

$$\phi = e^{-\frac{ms}{s-u}} \left(\frac{-\langle \nu, E\rangle}{u}\right)^{mN} P_m,$$

where $m, N > 0$ are constants to be determined later and $u \geqslant 1$ (see Subsection 10.2). It's easy to see that ϕ achieves its local maximum at an interior point of $U_s = \{x \in \mathbb{R}^n |\, u(x) < s\}$, we will assume this point is x_0. We can choose a local orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that at x_0, $h_{ij} = \kappa_i \delta_{ij}$ and $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$.

Differentiating $\log \phi$ at x_0 we get,

$$\frac{\phi_i}{\phi} = \frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m} + N h_{ii} \frac{\langle \tau_i, E\rangle}{\langle \nu, E\rangle} - N\frac{u_i}{u} - \frac{s u_i}{(s-u)^2} = 0, \tag{11.18}$$

and

(11.19)

$$\begin{aligned}
\frac{\phi_{ii}}{\phi} - \frac{\phi_i^2}{\phi^2} = &\frac{1}{P_m}[\sum_j \kappa_j^{m-1} h_{jjii} + (m-1)\sum_j \kappa_j^{m-2} h_{jji}^2 + \sum_{p\neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} h_{pqi}^2] \\
&- \frac{m}{P_m^2}(\sum_j \kappa_j^{m-1} h_{jji})^2 + N h_{imi}\frac{\langle \tau_m, E\rangle}{\langle \nu, E\rangle} + N h_{ii}^2 - N h_{ii}^2 \frac{u_i^2}{\langle \nu, E\rangle^2} \\
&+ N\frac{h_{ii}\langle \nu, E\rangle}{u} + N\frac{u_i^2}{u^2} + s\frac{h_{ii}\langle \nu, E\rangle}{(s-u)^2} - 2s\frac{u_i^2}{(S-u)^3} \leqslant 0.
\end{aligned}$$

Following the same arguments as Lemma 32, we immediately derive

(11.20)

$$\begin{aligned}
0 \geqslant \sigma_{n-1}^{ii}\frac{\phi_{ii}}{\phi} = &\frac{\sigma_{n-1}^{ii}}{P_m}[\sum_j \kappa_j^{m-1} h_{jjii} + (m-1)\sum_j \kappa_j^{m-2} h_{jji}^2 + \sum_{p\neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} h_{pqi}^2] \\
&- \frac{m\sigma_{n-1}^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1} h_{jji})^2 + N\sigma_{n-1}^{ii} h_{imi}\frac{\langle \tau_m, E\rangle}{\langle \nu, E\rangle} + N\sigma_{n-1}^{ii} h_{ii}^2 - N\sigma_{n-1}^{ii} h_{ii}^2 \frac{u_i^2}{\langle \nu, E\rangle^2} \\
&+ N\sigma_{n-1}^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} + N\sigma_{n-1}^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_{n-1}^{ii} h_{ii}\langle \nu, E\rangle}{(s-u)^2} - 2s\frac{\sigma_{n-1}^{ii} u_i^2}{(s-u)^3} \\
\geqslant &-C\kappa_1 + \frac{\sigma_{n-1}^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1} h_{jji})^2 + N\sigma_k^{ii}\kappa_i^2 - N\sigma_{n-1}^{ii} h_{ii}^2 \frac{u_i^2}{\langle \nu, E\rangle^2} \\
&+ N\sigma_{n-1}^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} + N\sigma_{n-1}^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_{n-1}^{ii} h_{ii}\langle \nu, E\rangle}{(s-u)^2} - 2s\frac{\sigma_{n-1}^{ii} u_i^2}{(s-u)^3}.
\end{aligned}$$

From equation (11.18) we obtain

$$\left(\frac{\sum_j \kappa_j^{m-1} h_{jji}}{P_m}\right)^2 = N^2\frac{\kappa_i^2 u_i^2}{\langle \nu, E\rangle^2} + N^2\frac{u_i^2}{u^2} + \frac{s^2 u_i^2}{(s-u)^4} + 2N^2\frac{\kappa_i u_i^2}{u\langle \nu, E\rangle} + 2Ns\frac{\kappa_i u_i^2}{\langle \nu, E\rangle (s-u)^2} + 2Ns\frac{u_i^2}{u(s-u)^2}. \tag{11.21}$$

Inserting (11.21) into (11.20), we have

$$\begin{aligned}0 \geqslant& -C\kappa_1 + N\sigma_{n-1}^{ii}\kappa_i^2 + \frac{s^2\sigma_{n-1}^{ii}u_i^2}{(s-u)^4}\\ &+2N^2\frac{\sigma_{n-1}^{ii}\kappa_i u_i^2}{u\langle \nu, E\rangle} + 2Ns\frac{\sigma_{n-1}^{ii}\kappa_i u_i^2}{\langle \nu, E\rangle(s-u)^2} + 2Ns\frac{\sigma_{n-1}^{ii}u_i^2}{u(s-u)^2}\\ &+N\sigma_{n-1}^{ii}\frac{h_{ii}\langle \nu, E\rangle}{u} + N(N+1)\sigma_{n-1}^{ii}\frac{u_i^2}{u^2} + s\frac{\sigma_{n-1}^{ii}h_{ii}\langle \nu, E\rangle}{(s-u)^2} - 2s\frac{\sigma_{n-1}^{ii}u_i^2}{(s-u)^3}.\end{aligned}$$

By Lemma 40 we get

$$|\nabla u| = \frac{|Du|}{\sqrt{1-|Du|^2}} < -\langle \nu, E\rangle \leqslant \frac{u}{d_1}.$$

Moreover, notice that $\sigma_{n-1}^{ii}\kappa_i \leqslant 1$ (no summation), by a simple calculation we can see

$$\frac{s^2\sigma_{n-1}^{ii}u_i^2}{(s-u)^4} + 2Ns\frac{\sigma_{n-1}^{ii}u_i^2}{u(s-u)^2} - 2s\frac{\sigma_{n-1}^{ii}u_i^2}{(s-u)^3} \geqslant 0.$$

Thus, for $N > 1$ large, we have

$$0 \geqslant -C\kappa_1 + N\sigma_{n-1}^{ii}\kappa_i^2 - \frac{3N^2}{d_1} - 2Ns\frac{u}{d_1(s-u)^2} - s\frac{(n-1)u}{d_1(s-u)^2}.$$

This yields, at x_0

$$\kappa_1 \leqslant C(N, d_1)\frac{s^2}{(s-u)^2}.$$

Therefore, in U_s we have

$$\phi^{\frac{1}{m}} \leqslant C(N, d_1)e^{-\frac{s}{s-u}}\frac{s^2}{(s-u)^2}.$$

Now note that for $t \in [0, s]$,

$$\varphi(t) = e^{-\frac{s}{s-t}}\frac{s^2}{(s-t)^2} \leqslant 4e^{-2}.$$

Hence, we obtain that at any point $x \in U_s$,

$$\phi^{\frac{1}{m}} \leqslant C(N, d_1). \tag{11.22}$$

Now, for any $x \in \mathbb{R}^n$, we can choose s large such that $x \in U_{s/2}$. Then by (11.22) and Lemma 40, we conclude

$$\kappa_1(x) \leqslant C(N, d_1, d_2).$$

Since x is arbitrary, we finish proving Proposition 41. □

We conclude that Theorem 4 is proved. More specifically, in Section 8 we showed that in order to find a strictly convex, spacelike hypersurface with constant σ_{n-1} curvature, we only need to show there exists a solution to equation (8.2). Since equation (8.2) is degenerate, we considered the solvability of the approximate problem (8.3) in Section 9 instead. In Section 10, we proved that there exists a subsequence of solutions to (8.3) that converges to the solution of (8.2). Finally, in Section 11 we proved that the solution we obtained indeed has bounded principal curvatures.

REFERENCES

[1] A. D. Aleksandrov, *Uniqueness theorems for surfaces in the large,* Vestnik Leningrad Univ. Math. 13 (1958) 5–8.

[2] J.M. Ball, *Differentiability properties of symmetric and isotropic functions,* Duke Math. J. 51 (1984), no. 3, 699 –728.

[3] P. Bayard and O.C. Schnürer, *Entire spacelike hypersurfaces of constant Gauss curvature in Minkowski space*, J. Reine Angew. Math. 627 (2009), 1–29.

[4] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature,* J. Differential Geometry 6 (1971/72), 119–128.

[5] L. Caffarelli, P. Guan, and X.-N. Ma, *A constant rank theorem for solutions of fully nonlinear elliptic equations,* Comm. Pure Appl. Math. 60 (2007), no. 12, 1769–1791.

[6] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations. III. Functions of the eigenvalues of the Hessian,* Acta Math. 155 (1985), no. 3-4, 261–301.

[7] S.-Y. Cheng and S.-T. Yau, *Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces*, Ann. of Math. 104, 407–419.

[8] S.-Y. Cheng and S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric application*, Comm. Pure Appl. Math., 28, 337–354 (1975).

[9] S.-Y. Cheng and S.T. Yau, *Hypersurfaces with constant scalar curvature*, Math. Ann. 247, 81–93 (1980).

[10] H.I. Choi and A.E. Treibergs, *Gauss maps of spacelike constant mean curvature hypersurfaces of Minkowski space*, J. Diff. Geo. 32 (1990), 775–817.

[11] K. Ecker and G. Huisken, *Immersed hypersurfaces with constant Weingarten curvature*, Math. Ann. 283 (1989), no. 2, 329–332.

[12] B. Guan, *The Dirichlet problem for Hessian equations on Riemannian manifolds,* Calc. Var. Partial Differential Equations 8 (1999), no. 1, 45–69.

[13] B. Guan, H.-Y. Jian and R. M. Schoen, *Entire spacelike hypersurfaces of prescribed Gauss curvature in Minkowski space*, J. Reine Angew. Math. 595 (2006), 167–188.

[14] P. Guan and X.-N. Ma, *The Christoffel-Minkowski problem. I. Convexity of solutions of a Hessian equation,* Invent. Math. 151 (2003), no. 3, 553–577.

[15] J. Hano and K. Nomizu, *On isometric immersions of hyperbolic plane into the Lorentz-Minkowski space and the Monge-Ampere equation of a certain type,* Math. Ann., 262(1983), no.1, 245–253.

[16] P. Guan, C. Ren and Z. Wang, *Global C^2 estimates for curvature equation of convex solution,* Comm. Pure Appl. Math. LXVIII(2015) 1287–1325.

[17] C. C. Hsiung, *Some integral formulas for closed hypersurfaces,* Math. Scand. 2 (1954) 286–294.

[18] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. 183 (1999), no. 1, 45–70.

[19] N. J. Korevaar, *A priori interior gradient bounds for solutions to elliptic Weingarten equations*, Ann. Inst. H. Poincaré, Anal. Non Linéaire 4(1987), 405–421.
[20] H. Liebmann, *Eine neue Eigenschaft der Kugel, Nachr. Kg. Ges. Wiss. Gottingen,* Math.-Pys. Kl., 1899,44–55.
[21] M. Li, C. Ren and Z. Wang, *An interior estimate for convex solutions and a rigidity theorem,* J. Funct. Anal. 270 (2016), no. 7, 2691–2714.
[22] A.-M. Li, *Spacelike hypersurfaces with constant Gauss-Kronecker curvature in the Minkowski space*, Arch. Math., Vol.64, 534–551 (1995).
[23] A. Ros, *Compact hypersurfaces with constant scalar curvature and a congruence theorem. With an appendix by Nicholas J. Korevaar*, J. Diff. Geom. 27(1988), no. 2, 215–223.
[24] C. Ren and Z. Wang, *On the curvature estimates for Hessian equations*, American Journal of Mathematic, Vol.141, No.5, 1281–1315, 2019.
[25] A. E. Treibergs, *Entire spacelike hypersurfaces of constant mean curvature in Minkowski space*, Invent. Math. 66, 39–56 (1982).
[26] N. S. Trudinger, *The Dirichlet problem for the prescribed curvature equations*, Arch. Rational Mech. Anal., 111(1990), 153–170.
[27] X. Wang, *The k-Hessian equation. Geometric analysis and PDEs*, 177–252, Lecture Notes in Math., (1977), Springer, Dordrecht, (2009).

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