FREE BOUNDARY REGULARITY ON THE FOCUSING PROBLEM FOR THE Q_k CURVATURE FLOW WITH FLAT SIDES I

XUMIN JIANG AND LING XIAO

Abstract. In this paper, we consider the motion of a compact, weakly convex hypersurface of revolution $\Sigma_0 \subset \mathbb{R}^{n+1}$ under the Q_k curvature flow. Assume that Σ_0 has a flat side, under a certain non-degeneracy initial condition, we show that Σ_t is smooth up to the flat side for $t > 0$. Moreover, the interface separating the flat side from the strictly convex side, moves by the Q_{k-1} flow until the flat side disappears. We also show that at the focusing time T, i.e., the time when the flat side disappears, the pressure function g is of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$ depends on n and k.

1. Introduction

We consider, in this paper, the evolution of a compact convex hypersurface $\Sigma_0 \in \mathbb{R}^{n+1}$ by Q_k curvature flow for $2 \leq k \leq n$, namely, the equation

$$\frac{\partial X}{\partial t} = -Q_k(\kappa)\nu, \tag{1.1}$$

where each point $X \in \Sigma_t$ moves in the direction of its outer normal vector ν by a speed

$$Q_k(\kappa) = \frac{S_k^n(\kappa)}{S_{k-1}^n(\kappa)}.$$

Here,

$$S_k^n(\kappa) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \kappa_{i_1} \cdots \kappa_{i_k}$$

is the k-th elementary symmetric polynomial of the principle curvatures, and $S_0^n(\kappa) = 1$.

Andrews [1] proved that for any strictly convex hypersurface in $\mathbb{R}^{n+1}$, the solution to (1.1) exists up to some finite time T^*, at which it shrinks to a point in an asymptotically spherical manner. Dieter [7] considered the flow (1.1) of convex hypersurface with additional assumption that $S_{k-1}^n(\kappa) > 0$. Caputo, Daskalopoulos, and Sesum [4] studied the existence and uniqueness of a $C^{1,1}$ solution of (1.1) in the viscosity sense for compact convex hypersurfaces Σ_t embedded in $\mathbb{R}^{n+1}, n \geq 2$. The solution exists up to the time $T^* < \infty$ at which the enclosed volume becomes zero. In particular, they showed for compact convex hypersurfaces with flat sides, under a certain non-degeneracy initial condition, the interface Γ_t separating the flat from the strictly convex side becomes smooth, and it moves by the Q_{k-1} flow at least for a short time.

In this paper, we study the regularity of the interface up to its *focusing time* $T \in (0, T^*)$, that is the time when the flat side shrinks to a point. We conjecture that, under certain assumptions on the initial hypersurface Σ_0, the free boundary Γ_t will be smooth for all time $0 < t < T$. Moreover, denote the height function of Σ_t by f, then the *pressure function* $g = \sqrt{f}$ is smooth up to the interface. However, in this paper, we will only treat the case when Σ_0 is a hypersurface of revolution; in an upcoming paper [8], we will prove the general case.

In [5] and [6], Daskalopoulos and Lee considered the corresponding problem for the Gauss curvature flow with flat sides in $\mathbb{R}^3$. They showed that under certain assumptions for the initial surface, the C^∞ regularity of Σ_t is preserved up to the focusing time of the flat side. Moreover, when the initial surface is rotationally symmetric, they showed that, at the focusing time, the pressure function g is of class $C^{1,\beta}$ for $\beta < 1/4$ and is no better than $C^{1,2/5}$. For the Q_k flow, when the initial hypersurface is rotationally symmetric, we can prove that at the focusing time, the pressure function g is of class $C^{1,\alpha}$, for some $\alpha \in (0,1)$ only depends on n and k. In [8] we will show this is also true for the non-rotationally symmetric case.

Let's assume that the initial hypersurface Σ_0 is a hypersurface of revolution around the z-axis, Σ_0 has only one flat side, and its flat side lies on the plane $\{z = 0\}$; while the strictly convex side has $z > 0$. Namely, at time $t = 0$ we have

$$\Sigma_0 = \Sigma_1 \cup \Sigma_2,$$

where Σ_1 is the flat side and Σ_2 is the strictly convex part of the hypersurface. The junction between the two sides is an $n-1$ dimensional hypersurface $\Gamma_0 = \Sigma_1 \cap \Sigma_2$. Then the lower part of the hypersurface can be represented as the graph of a radial function $z = f(r)$, satisfying

$$f(r) \equiv 0, \text{ for } 0 \le r \le r_0, \text{ and } \lim_{r \to r_1^-} f_r(r) = +\infty. \tag{1.2}$$

Set $g = \sqrt{f}$, our main assumption on the initial hypersurface Σ_0 is that it's of class $C^{1,1}$, and the function g is smooth up to Γ_0. Moreover, at $t = 0$, g satisfies the following *nondegenerate condition*:

$$g_r(r_0) \ge \lambda > 0. \tag{1.3}$$

Since f satisfies (1.2), it's easy to see that g satisfies

$$g(r) \equiv 0, \text{ for } 0 \le r \le r_0, \text{ and } \lim_{r \to r_1^-} g_r(r) = +\infty. \tag{1.4}$$

Following [4] we define

Definition 1.1. We define $\mathfrak{S}$ to be the class of convex compact hypersurfaces Σ in $\mathbb{R}^{n+1}$ so that $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is a hypersurface contained in the hyperplane $\{z = 0\}$ with smooth boundary Γ, and Σ_2 is a strictly convex hypersurface, smooth up to its boundary Γ which lies above the hyperplane $\{z = 0\}$.

Recall Theorem 1.5 of [4],

Theorem 1.2. *Assume that at time $t = 0$, Σ_0 is a weak convex compact hypersurface in $\mathbb{R}^{n+1}$ which belongs to the class $\mathfrak{S}$ so that the pressure function g is smooth up to the interface Γ_0 and it satisfies the condition* (1.3). *Let Σ_t be the unique viscosity solution of* (1.1) *for $2 \le k \le n$ with initial data Σ_0. Then, there exists a time $\tau > 0$ such that* *the pressure function $g(\cdot,t)$ is smooth up to the interface $\{z=0\}$ and satisfies condition (1.3) for all $t\in[0,\tau)$. In particular, the interface Γ_t between the flat side and the strictly convex side is smooth hypersurface for all t in $0<t\le\tau$ and it moves by Q_{k-1} flow.*

Now, suppose that $T_0\le T$ is the largest time such that for any $t\in[0,T_0)$ the pressure function $g(\cdot,t)$ is smooth up to the interface $\{z=0\}$, and

$$g_r(\cdot,t)>0 \text{ on } \Gamma_t.$$

We will prove $T_0=T$ by a contradiction argument. To this end, we prove that if $T_0<T$, then at $t=T_0$, $g(\cdot,T_0)$ is smooth up to the interface $\{z=0\}$, and (1.3) holds for some $\lambda(T_0)>0$. Then, the openness result in [4] contradicts the assumption that T_0 is the largest number such that $g(\cdot,t)$ is smooth up to $\{z=0\}$ and $g_r(\Gamma_t,t)>0$ in $[0,T_0)$.

Throughout this paper, by a simple rescaling, we assume

$$\max g(\cdot,t)\ge 2,\ \text{for } 0\le t\le T, \tag{1.5}$$

where T is the focusing time of the flat side.

Our main results are listed below.

Theorem 1.3. *Assume that Σ_0 is an compact, convex, n dimensional hypersurface of revolution around z-axis with a flat side. Moreover, the lower part of Σ_0 is the graph of a function $z=f(r,0)\ge 0$, $0\le r\le r_1$, with $z=0$ being the flat side for $0\le r\le r_0$. Assume also that $g(\cdot,0)=\sqrt{f(\cdot,0)}$ is smooth on $[r_0,r_1)$ and satisfies conditions (1.3) and (1.4). Then the function $g=\sqrt{f}$ will be smooth up to the interface $g=0$ for all $0<t<T$, with T denoting the focusing time of the flat side. In particular, the interface $\gamma(t):=\partial\{g(r,t)=0\}$ will be smooth.*

Our second result describes the behavior of g at the focusing time:

Theorem 1.4. *Under the hypotheses of Theorem 1.3, the pressure function g satisfies the following derivative estimates*

$$C_1 r^{\frac{B}{A}M_{k,1}}\le g_r(r,t)\le C_2 r, \text{ and } |r^N g_{rr}|<C_3, \text{ for } 0\le t<T$$

near the interface $\gamma(t)$, where C_i, $i=1,2,3$, only depends on n, k, and Σ_0. In addition, at the focusing time T of the flat side, the function g is of class $C^{1,\frac{1}{1+N}}$, where $N=\max\{\frac{B}{A}M_{k,1}+1,\frac{B}{A}(M_{k,2}-3M_{k,1})\}$. Here, $M_{k,1}$ and $M_{k,2}$ are positive constants given in Section 3.

2. Preliminary

The main purpose of this section is to rewrite equation (1.1) in cylindrical coordinates. We will construct the explicit solution of (1.1) in Section 3 using this coordinates. Recall that under the cylindrical coordinates, the Euclidean metric can be expressed as

$$ds^2=dr^2+r^2dS^{n-1}+dz^2.$$

Denote the position vector $X(\theta, z, t) = r(z, t)\theta + ze_{n+1}$, where $e_{n+1} = (0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$ and θ is the angle vector in S^{n-1}. At a fixed point $P \in \Sigma_0$, we assume $\{\theta_i\}_{1\leq i\leq n-1}$ are normal coordinates of S^{n-1} at $\theta(P)$. Then at P, for $1 \leq i \leq n-1$,

$$X_i = r(z,t)\theta_i, \text{ and } X_{n+1} = r_z\theta + e_{n+1}.$$

Hence the unit outer normal vector at P is

$$\nu = \frac{\theta - r_z e_{n+1}}{w},$$

where

$$w = \sqrt{1 + r_z^2}.$$

Since at P, for $1 \leq i, j \leq n-1$,

$$\begin{aligned} h_{ij} &= -\langle X_{ii}\delta_{ij}, \nu\rangle = -\left\langle -r\theta\delta_{ij}, \frac{\theta - r_z e_{n+1}}{w}\right\rangle = \frac{r\delta_{ij}}{w}, \\ h_{i,n+1} &= -\langle X_{i,n+1}, \nu\rangle = -\left\langle r_z\theta_i, \frac{\theta - r_z e_{n+1}}{w}\right\rangle = 0, \\ h_{n+1,n+1} &= -\langle X_{n+1,n+1}, \nu\rangle = -\left\langle r_{zz}\theta, \frac{\theta - r_z e_{n+1}}{w}\right\rangle = \frac{-r_{zz}}{w}, \end{aligned}$$

and

$$\begin{aligned} g_{ij} &= \langle X_i, X_j\rangle = r^2\delta_{ij}, \\ g_{i,n+1} &= \langle X_i, X_{n+1}\rangle = 0, \\ g_{n+1,n+1} &= \langle X_{n+1}, X_{n+1}\rangle = 1 + r_z^2. \end{aligned}$$

By a straightforward calculation we get the principal curvatures are

$$\kappa_i = \frac{1}{rw}, \quad \text{for } 1 \leq i \leq n-1, \quad \text{and } \kappa_n = \frac{-r_{zz}}{w^3}, \tag{2.1}$$

which are independent of the choice of normal coordinates $\{\theta_i\}_{1\leq i\leq n-1}$.

Multiplying (1.1) by ν on both sides we get

$$\langle x_t, \nu\rangle = -\frac{S_k^n(\kappa)}{S_{k-1}^n(\kappa)},$$

which implies

$$\begin{aligned} \frac{r_t}{w} &= -\frac{C_{n-1}^{k-1}(\frac{1}{rw})^{k-1} \cdot \frac{-r_{zz}}{w^3} + C_{n-1}^{k}(\frac{1}{rw})^{k}}{C_{n-1}^{k-2}(\frac{1}{rw})^{k-2} \cdot \frac{-r_{zz}}{w^3} + C_{n-1}^{k-1}(\frac{1}{rw})^{k-1}} \\ &= -\frac{-k(n-k+1)rr_{zz} + (n-k+1)(n-k)w^2}{-k(k-1)r^2r_{zz}w + k(n-k+1)rw^3}, \end{aligned} \tag{2.2}$$

where $C_{n-1}^k = 0$ if $k = n$.

Let $s = z^{\frac{1}{2}}$, then we have

$$r_z = \frac{r_s}{2s}, \quad r_{zz} = -\frac{r_s}{4s^3} + \frac{r_{ss}}{4s^2}.$$

Therefore, equation (2.2) is equivalent to

$$Q(r) = 0, \tag{2.3}$$

where

$$\begin{aligned} Q(r) := {} & rr_t(4As^3 + Asr_s^2 + Brr_s - Bsrr_{ss}) \\ & - A(srr_{ss} - rr_s) + D(4s^3 + sr_s^2). \end{aligned}$$

Here and throughout this paper, we denote $A = k(n-k+1), B = k(k-1)$, and $D = (n-k+1)(n-k)$.

3. Formal computations

In this section, our goal is to solve for $u_N(s,t) = \varphi(t) + c_1(t)s + \cdots + c_N(t)s^N$, such that

$$Q(u_N) = O(s^{N+1}).$$

In this sense, we call the corresponding series $u_\infty(s,t)$ a formal solution of equation (2.3).

First, by letting the coefficient of $O(s^0)$ term of $Q(u_N)$ equals 0, we get

$$B\varphi^2(t)\varphi'(t)c_1(t) + A\varphi(t)c_1(t) = 0.$$

This yields

$$\varphi'(t) = -\frac{A}{B\varphi(t)}.$$

Solving this ODE we obtain

$$\varphi(t) = \sqrt{a_0^2 - \frac{2A}{B}t}, \tag{3.1}$$

where $a_0 = \varphi(0)$ is the radius of the initial flat side Σ_1. In particular, when $n = k = 2$, we have $A = B = 2$, and Γ_t is moved by curve shortening flow. From now on, we will denote

$$T = \frac{Ba_0^2}{2A}, \tag{3.2}$$

then $\varphi(T) = 0$.

Next, we look at the coefficient of $O(s)$ term of $Q(u_N)$, and derive

$$\begin{aligned} & \varphi\varphi' \cdot (Ac_1^2 + Bc_1^2 + B2sc_2\varphi - B2sc_2\varphi)s + \varphi c_1's \cdot B\varphi c_1 + c_1 s\varphi' \cdot B\varphi c_1 \\ & \quad - A(2sc_2\varphi - c_1^2 s - 2c_2 s\varphi) + Csc_1^2 = 0. \end{aligned}$$

Notice that in this process, c_2 terms are canceled. We obtain

$$\begin{aligned} c_1' &= -\frac{(A+2B)\varphi\varphi' c_1 + Ac_1 + Dc_1}{B\varphi^2} \\ &= \frac{A^2 + AB - BD}{B^2} \cdot \frac{c_1}{\varphi^2} := M_{k,1}\frac{c_1}{\varphi^2}. \end{aligned}$$

Solving the above ODE we get

$$c_1(t) = c_1(0)a_0^{\frac{BM_{k,1}}{A}}(a_0^2 - \frac{2A}{B}t)^{-\frac{BM_{k,1}}{2A}}, \tag{3.3}$$

where $M_{k,1} = \frac{A^2+AB-BD}{B^2}$ is a positive number depending on n, k.

For computing $c_l(t), l \geq 2$, we check the $O(s^l)$ terms. The coefficients of $O(s^l)$ terms satisfy

$$\begin{aligned}\varphi\varphi' \cdot &[(2lAc_1c_l + B((l+1)c_1c_l + (l+1)\varphi c_{l+1})\\ &- B(l(l-1)c_lc_1 + l(l+1)\varphi c_{l+1})]\\ &+ (\varphi c_1' + \varphi' c_1) \cdot (B\varphi l c_l - Bl(l-1)c_l\varphi)\\ &+ (\varphi c_l' + \varphi' c_l) \cdot B\varphi c_1\\ &- A[l(l-1)c_lc_1 + l(l+1)\varphi c_{l+1})\\ &- ((l+1)c_lc_1 + (l+1)\varphi c_{l+1})] + D \cdot 2lc_1c_l\\ &- P(\varphi, c_1, \cdots, c_{l-1}) = 0,\end{aligned}$$

where P is a polynomial in its arguments. Applying $\varphi(t)\varphi'(t) = -\frac{A}{B}$, we can see that c_{l+1} terms get cancelled. Therefore, we derive an ODE for $c_l(t)$

$$c_l' = \frac{M_{k,l}}{\varphi^2}c_l + \frac{1}{Bc_1\varphi^2}\Phi(\varphi, c_1, \cdots, c_{l-1}) \tag{3.4}$$

where Φ is a function smooth in its arguments when $\varphi > 0$, and $M_{k,l}$ is a number only depending on n, k, and l. In fact, we have for $l \geq 2$,

$$\begin{aligned}M_{k,l} &= \frac{2lA^2}{B^2} - \frac{A}{B}(l^2 - 2l - 1) + \frac{A^2 + AB - BD}{B^2}l(l-2) - \frac{2lD}{B}\\ &= \frac{A^2l^2 + AB - l^2BD}{B^2},\end{aligned}$$

which are all positive as $A^2 > BD$. In particular, when $l = 2$, (3.4) can be written as

$$c_2' = \frac{M_{k,2}}{\varphi^2}c_2 - \frac{A^3 + 2A^2B + AB^2 - ABD - 2B^2D}{B^3}\frac{c_1^2}{\varphi^3}.$$

Solving this ODE we get

$$c_2 = A_1\varphi^{-2\frac{B}{A}M_{k,1}-1} + A_2\varphi^{-\frac{B}{A}M_{k,2}}, \tag{3.5}$$

where A_1 and A_2 are constants only depending on n, k and Σ_0.

4. Basic derivative estimates

In this section we will prove some basic estimates on the first spacial derivative of g and the first time derivative of the junction $\gamma(t) := \partial\{g(r,t) = 0\}$.

Lemma 4.1. *Under the hypotheses of Theorem 1.3 and condition* (1.5), *there exists a constant $C < \infty$ such that*

$$0 \le g_r(r,t) \le C \text{ on } \{g \le 1, 0 \le t \le T\}. \tag{4.1}$$

Proof. We first notice that $g_r \ge 0$, because the function $f(\cdot,t) = g^2(\cdot,t)$ is increasing in r. Now when $g = 1$, by [4] we know $f \in C^{1,1}$, we have $f_r \le C$. This implies

$$g_r \le C \text{ on } \{g = 1\} \text{ for } 0 \le t \le T.$$

For the interior estimate, we first observe that we can approximate $\Sigma_0 = \Sigma_1 \cup \Sigma_2$ by a family of smooth strictly convex surfaces Σ_0^ϵ, where $\Sigma_0^\epsilon = \Sigma_1^\epsilon(0) \cup \Sigma_2^\epsilon(0)$ and $\Sigma_1^\epsilon(0)$ is below $\{z = \epsilon\}$, $\Sigma_2^\epsilon(0)$ is above $\{z = \epsilon\}$. Moreover, $\Sigma_2^\epsilon(0) \to \Sigma_2$ and $\Sigma_1^\epsilon(0) \to \Sigma_1$ as $\epsilon \to 0$. Then Σ_0^ϵ corresponding to a decreasing sequence of positive smooth increasing, rotationally symmetric, and strictly convex solution f^ϵ. We can take $g^\epsilon = \sqrt{f^\epsilon}$, it's easy to see that g^ϵ satisfies

$$g_r^\epsilon(r,0) \le C \text{ in } \{g^\epsilon(\cdot,0) \le 1\},$$

and

$$g_r^\epsilon \le C \text{ on } \{g^\epsilon = 1, 0 \le t \le T\}$$

with C being independent of ϵ.

Now since $\Sigma^\epsilon(t)$ are rotationally symmetric, by a straightforward calculation we have

$$\kappa_1 = \kappa_2 = \cdots = \kappa_{n-1} = \frac{f_r^\epsilon}{rW}, \text{ where } W = \sqrt{1 + |Df^\epsilon|^2},$$

and

$$\kappa_n = \frac{f_{rr}^\epsilon}{W^3} = \frac{2(g_r^\epsilon)^2 + 2g^\epsilon g_{rr}^\epsilon}{W^3}.$$

In the rest of this paper, for our convenience we will denote

$$\lambda_1 = \kappa_1 = \cdots = \kappa_{n-1} \text{ and } \lambda_2 = \kappa_n.$$

By Theorem 2.2 of [4] we know $\lambda_2 \le C$ in $[0,T]$, where C is independent of ϵ. If g_r^ϵ achieves its interior maximum, then at this point we have $g_{rr}^\epsilon = 0$. Therefore $\lambda_2 = \frac{2(g_r^\epsilon)^2}{W^3} \le C$, which gives $g_r^\epsilon \le C$. (Note that by condition (1.5), W is bounded.) $\square$

Lemma 4.2. *Under the hypotheses of Theorem 1.3 and condition* (1.5) *we have*

$$\gamma(t)' \ge -\frac{A}{B\gamma(t)}, \text{ for } 0 \le t < T, \tag{4.2}$$

where $A = k(n-k+1)$ and $B = k(k-1)$.

Proof. Fix a number $t_0 \in [0, T)$.

Case1. $g_r(\gamma(t_0)+, t_0) = 0$: For any $\epsilon > 0$ there exists a function $h_0 = h_0(r)$ which is linear on $\gamma(t_0) \leq r \leq \gamma(t_0) + \epsilon$ with slope $\beta > 0$ such that

$$\{r : h_0(r) = 0\} = \{r : g(r, t_0) = 0\} = \gamma(t_0),$$

and $h_0 \geq g$ for all r. Let h^2 be a solution of the flow equation (1.1) and $\eta(t)$ denote the free boundary of h, namely $\partial\{h = 0\}$. By Proposition 3.11 of [4] we have $\eta'(t_0) = -\frac{A}{B\eta(t_0)}$. Moreover, by the maximum principle we have

$$\gamma'(t_0) \geq \eta'(t_0) = -\frac{A}{B\eta(t_0)} = -\frac{A}{B\gamma(t_0)}.$$

Case2. $g_r(\gamma(t_0)+, t_0) > 0$: Similar to case 1, we can choose functions $h^{\pm}$ such that $h^+ \geq g \geq h^-$ and $h^+_r, h^-_r > 0$. By the maximum principle we have

$${\eta^+}'(t_0) \leq \gamma'(t_0) \leq {\eta^-}'(t_0),$$

where η^+, η^- are the free boundaries of h^+, h^- respectively. Again, applying Proposition 3.11 of [4] we have ${\eta^+}'(t_0) = {\eta^-}'(t_0) = -\frac{A}{B\gamma(t_0)}$. This completes the proof of Lemma 4.2. $\square$

5. Higher order estimates

In following sections, without loss of generality, we always assume that $T_0 < T$ is the largest time such that for any $t \in [0, T_0)$ the pressure function $g(\cdot, t)$ is smooth up to the interface $\{z = 0\}$, and

$$g_r(\cdot, t) > 0 \text{ on } \Gamma_t.$$

We will show that $g(\cdot, T_0)$ is smooth up to the interface $\{z = 0\}$ and

$$g_r(\cdot, T_0) > 0 \text{ on } \Gamma_{T_0}.$$

Then, we can apply Theorem 1.2 to Σ_{T_0}, which leads to a contradiction to the maximality of T_0. Therefore, we prove Theorem 1.3.

In this section, we will show the formal solution we obtained in Section 3 is a good approximation of the real solution of (1.1) in $[0, T_0)$.

First we derive the following estimate.

Lemma 5.1. *Under the hypotheses of Theorem 1.3 and condition* (1.5)*, we have*

$$|r(s,t) - \varphi(t)| \leq Cs \text{ in } [0,1] \times [0, T_0), \tag{5.1}$$

where $C = C(n, k, \Sigma_0, T_0) > 0$.

Proof. By Case 2 of Lemma 4.2 and the convexity of $\Sigma(t)$, we know that $r(s,t) > r(0,t) = \varphi(t)$, for $s > 0$ and $t \in [0, T_0)$. Therefore, it's sufficient to prove that there is a $\delta > 0$ small, such that $r(s,t) - \varphi(t) \leq Cs$ holds in $(s,t) \in [0, \delta] \times [0, T_0)$.

Consider the test function $M(s,t) = \varphi(t) + He^{Lt}s$, with undetermined coefficients $H, L > 0$. We first set H large such that

$$M(s,0) = \varphi(0) + Hs \geq r(s,0) \text{ for } s \in [0,\delta],$$

and

$$M(\delta,t) = \varphi(t) + He^{Lt}\delta > r(\delta,t)$$

for $t \in [0,T_0)$.

We compute

$$\begin{aligned}
Q(M) &= (\varphi(t) + He^{Lt}s)(\varphi' + LHe^{Lt}s)[4As^3 + As(He^{Lt})^2 + B(\varphi(t) + He^{Lt}s)He^{Lt}] \\
&\quad + A(\varphi(t) + He^{Lt}s)He^{Lt} + D(4s^3 + s(He^{Lt})^2) \\
&= (\varphi^2 BL - \frac{A^2 + AB - BD}{B})(He^{Lt})^2 s \\
&\quad + (\varphi'(A+B)(He^{Lt})^3 + \varphi(A+B)L(He^{Lt})^3 + BL(He^{Lt})^2\varphi)s^2 \\
&\quad + (-\frac{4A^2}{B} + (A+B)L(He^{Lt})^3 + 4D)s^3 \\
&\quad + 4A(\varphi' He^{Lt} + \varphi LHe^{Lt})s^4 \\
&\quad + 4AL(He^{Lt})^2 s^5.
\end{aligned}$$

Therefore, if we choose $L = L(n,k,a_0,T_0)$ so large that

$$\begin{aligned}
&\varphi^2 BL - \frac{A^2 + AB - BD}{B} > 0, \\
&-\frac{A}{B\varphi} + \varphi L > 0, \\
&-\frac{4A^2}{B} + (A+B)L(He^{Lt})^3 + 4D > 0,
\end{aligned}$$

for $t \in [0,T_0)$, then we have $Q(M) > 0$. Here notice that $\varphi(t) + He^{Lt}\delta > r(\delta,t)$, so $(He^{Lt}s)^2$ is not small, even it's of form $O(s^2)$. By the maximum principle, we conclude that $r \leq \varphi(t) + He^{Lt}s$. □

Lemma 5.2. *Under the hypotheses of Theorem 1.3 and condition* (1.5)*, we have*

$$|r(s,t) - \varphi(t) - c_1(t)s| \leq Cs^2 \text{ in } [0,1] \times [0,T_0), \tag{5.2}$$

where $C = C(n,k,\Sigma_0,T_0) > 0$.

Proof. It's sufficient to prove that there exists a $\delta > 0$ such that (5.2) holds in $(s,t) \in [0,\delta] \times [0,T_0)$. Consider

$$M = \varphi(t) + c_1(t)s + G(t)s^2$$

such that

$$\varphi(0) + c_1(0)s + G(0)s^2 \geq r(s,0),$$

and

$$\varphi(t) + c_1(t)\delta + G(t)\delta^2 > r(\delta,t) \text{ for } t \in [0,T_0).$$

Here we note that Lemma 5.1 implies $G(t)\delta^2 < C(n,k,\Sigma_0,T_0)\delta$. By a straightforward calculation we get

$$
\begin{aligned}
Q(M) &= (\varphi + c_1 s + Gs^2)(\varphi' + c_1' s + G' s^2) \\
&\quad \times \{4As^3 + As(c_1+2Gs)^2 + B(\varphi + c_1 s + Gs^2)(c_1 + 2Gs) - 2GBs(\varphi + c_1 s + Gs^2)\} \\
&\quad - 2GAs(\varphi + c_1 s + Gs^2) + A(\varphi + c_1 s + Gs^2)(c_1 + 2Gs) + 4Ds^3 + Ds(c_1 + 2Gs)^2 \\
&= \{\varphi\varphi' + (\varphi' c_1 + \varphi c_1')s + (c_1 c_1' + \varphi G' + G\varphi')s^2 + (c_1 G' + Gc_1')s^3 + GG's^4\} \\
&\quad \times \{Bc_1\varphi + (Ac_1^2 + Bc_1^2)s + (4Ac_1 G + BGc_1)s^2 + (4AG^2 + 4A)s^3\} \\
&\quad - 2AG\varphi s - 2AGc_1 s^2 + A(\varphi c_1 + 2G\varphi s + c_1 s^2 + 3Gc_1 s^2) \\
&\quad + 4Ds^3 + Ds(c_1^2 + 4c_1 Gs + 4G^2 s^2).
\end{aligned} \tag{5.3}
$$

We can see that the coefficient of $O(s^2)$ term is:

$$
\begin{aligned}
&(\varphi' c_1 + \varphi c_1')(Ac_1^2 + Bc_1^2) + (c_1 c_1' + \varphi G' + G\varphi')Bc_1\varphi \\
&- 2AGc_1 + 3AGc_1 + 4Dc_1 G + \varphi\varphi'(4Ac_1 G + BGc_1) \\
&= \left(-\frac{A}{B}\right)(4Ac_1 + Bc_1)G + \left(\varphi\Lambda\frac{c_1}{\varphi^2} - \frac{A}{B\varphi}c_1\right)(A+B)c_1^2 \\
&+ \left[\frac{\Lambda c_1^2}{\varphi^2} + \varphi G' + G\left(-\frac{A}{B\varphi}\right)\right]Bc_1\varphi + AGc_1 + 4Dc_1 G.
\end{aligned} \tag{5.4}
$$

Here and in the following of this proof we denote $\Lambda = \frac{A^2+AB-BD}{B^2}$.

The coefficient of $O(s^3)$ term is:

$$
\begin{aligned}
&\varphi\varphi'(4AG^2 + 4A) + (\varphi' c_1 + \varphi c_1')(4Ac_1 + Bc_1)G \\
&+ (c_1 c_1' + \varphi G' + G\varphi')(A+B)c_1^2 + (c_1 G' + Gc_1')Bc_1\varphi + 4D + 4DG^2 \\
&= \left(-\frac{A}{B}\right)4A(G^2+1) + \left(\frac{\Lambda c_1}{\varphi} - \frac{Ac_1}{B\varphi}\right)(4Ac_1 + Bc_1)G \\
&+ \left(\frac{\Lambda c_1^2}{\varphi^2} + \varphi G' - \frac{AG}{B\varphi}\right)(A+B)c_1^2 \\
&+ \left(c_1 G' + \frac{G\Lambda c_1}{\varphi^2}\right)Bc_1\varphi + 4D(1+G^2).
\end{aligned} \tag{5.5}
$$

The coefficient of $O(s^4)$ term is:

$$
\begin{aligned}
&(\varphi' c_1 + \varphi c_1')(4AG^2 + 4A) + (c_1 c_1' + \varphi G' + G\varphi')(4Ac_1 + Bc_1)G \\
&+ (c_1 G' + Gc_1')(A+B)c_1^2 + GG'Bc_1\varphi.
\end{aligned} \tag{5.6}
$$

The coefficient of $O(s^5)$ term is:

$$
(c_1 c_1' + \varphi G' + G\varphi')(4AG^2 + 4A) + (c_1 G' + Gc_1')(4Ac_1 + Bc_1)G + GG'(A+B)c_1^2. \tag{5.7}
$$

The coefficient of $O(s^6)$ term is:

$$
(c_1 G' + Gc_1')4A(G^2+1) + GG'(4Ac_1 + Bc_1)G. \tag{5.8}
$$

The coefficient of $O(s^7)$ term is:

$$GG'4A(G^2+1). \tag{5.9}$$

Combining (5.4)-(5.9), we know that by choosing $G = He^{Lt}$, where $H, L > 0$ large enough, we have $Q(M) > 0$, which yields $M > r$ in $[0,\delta] \times [0, T_0)$. Similarly, by letting $M_- = \varphi + c_1(t)s - He^{Lt}s^2$ we obtain $M_- < r$ in $[0,\delta] \times [0, T_0)$. This completes the proof of Lemma 5.2. $\square$

Remark 5.3. For the higher order case, we use the test function

$$M = \varphi(t) + c_1(t)s + \cdots + c_k(t)s^k + G(t)s^{k+1},$$

where $G = He^{Lt}$ for some large H, L to be determined. It's clear that on the parabolic boundary, we can choose a function G satisfies

$$G(0) \geq C, \ G(t)\delta > C \tag{5.10}$$

for some large positive constant C, such that

$$\begin{aligned} M(s,0) &\geq r(s,0) && \text{for } s \in [0,\delta], \\ M(\delta,t) &> r(\delta,t) && \text{for } t \in [0,T_0). \end{aligned}$$

Furthermore, by induction, we notice that in $[0,\delta] \times [0,T_0)$,

$$|r(t) - (\varphi(t) + c_1(t)s + \cdots + c_{k-1}(t)s^{k-1})| < Cs^k, \tag{5.11}$$

for some constant $C = C(n, k, \Sigma_0, T_0) > 0$. Thus, we know that there exists $C_1 > 0$ independent of δ, such that (5.10) holds and

$$G(t)\delta < C_1. \tag{5.12}$$

From the formal computation, we know that $Q(\varphi(t) + c_1(t)s + \cdots + c_k(t)s^k) = O(s^{k+1})$. The coefficient of $O(s^{k+1})$ term in $Q(M)$ can be expressed as

$$B\varphi^2 c_1 G' + \alpha(t)G + \beta(t),$$

where $\alpha(t), \beta(t)$ are smooth functions uniformly bounded on $[0, T_0)$. It's easy to see that, we can always set H, L large to make this term strictly positive.

For higher order terms, i.e., terms of order $O(s^m)$, $m \geq k+2$, we only worry about terms that are not linear in G and G'. We observe that these terms are equal to Gs^{k+2} times some of the following factors

$$G's^{l-1} \text{ and } Gs^{l-1}, \ 2 \leq l \leq k,$$

which are bounded as $G's^{l-1} = HGs^{l-1} \leq HGs < C_1 H$. This implies,

$$Q(M) \geq [B\varphi^2 c_1 G' + \alpha(t)G + \beta(t)]s^{k+1} + C_2 HGs^{k+2}.$$

Therefore, we can choose $H, L > 0$ large such that (5.10) holds and $Q(M) > 0$. By the maximum principle we have, $M(s,t) > r(s,t)$ in $[0,\delta] \times [0,T_0)$. Similarly, let $M_- = \varphi(t) + c_1(t)s + \cdots + c_k(t)s^k - G(t)s^{k+1}$, we can show $M_-(s,t) < r(s,t)$ in $[0,\delta] \times [0,T_0)$.

6. Improvement on the regularity

Lemma 6.1. *Under the hypotheses of Theorem 1.3 and condition* (1.5)*, we have*

$$\lambda_2 > c(n, k, T_0, \Sigma_0) > 0 \text{ on } (\varphi(t), \varphi_1(t)] \times [0, T_0),$$

where $\varphi(t)$ *is given in* (3.1)*, and* $\varphi_1(t)$ *satisfies* $g(\varphi_1(t), t) = 1$.

Proof. By the smoothness of g up to $\{z = 0\}$ and Lemma 5.2, we know that

$$\lambda_2(\varphi(t)+, t) = \frac{2g_r^2}{\left(\sqrt{1 + 4g^2 g_r^2}\right)^3} = \frac{2}{c_1^2} > C_1 \text{ in } [0, T_0),$$

for some $C_1 = C_1(n, k, \Sigma_0, T_0) > 0$. Moreover, on $\{g = 1\}$, since $f = g^2$ is strictly convex, we have

$$\lambda_2(\varphi_1(t), t) > C_2 \text{ in } [0, T_0),$$

for some $C_2 = C_2(n, k, \Sigma_0, T_0) > 0$. Finally by the assumption on Σ_0 we have

$$\lambda_2(\cdot, 0) > C_3 \text{ on } (\varphi(0), \varphi_1(0)] \times \{0\},$$

for some $C_3 > 0$. We want to show there exists $C_4 > 0$ such that $\lambda_2 > C_4$ on $(\varphi(t), \varphi_1(t)) \times (0, T_0)$.

Assume by contradiction that λ_2 achieves an interior minimum. All calculations below are done at this point. We can rewrite equation (1.1) as a equation of the graph f:

$$f_t = F(\lambda)W = \frac{D\lambda_1^2 + A\lambda_1\lambda_2}{A\lambda_1 + B\lambda_2} W, \tag{6.1}$$

where $A = k(n - k + 1), B = k(k - 1), D = (n - k + 1)(n - k)$, and $W = \sqrt{1 + f_r^2}$. Since $\lambda_1 = \frac{f_r}{rW}$ and $\lambda_2 = \frac{f_{rr}}{W^3}$, we get

$$\lambda_{2t} = \frac{f_{rrt}}{W^3} - \frac{3\lambda_2}{W} W_t.$$

By a straightforward calculation we obtain

$$\lambda_{2t} = \frac{F_{rr}}{W^2} + \frac{2F_r W_r}{W^3} + \frac{FW_{rr}}{W^3} - \frac{3\lambda_1\lambda_2 r}{W}(F_r W + FW_r). \tag{6.2}$$

Differentiating W and λ_1 with respect to r we get

$$W_r = \frac{f_r f_{rr}}{W} = \lambda_1\lambda_2 r W^3, \tag{6.3}$$

and

$$\begin{aligned} \lambda_{1r} &= \frac{\lambda_2 W^2}{r} - \frac{\lambda_1}{r} - \frac{\lambda_1^2}{W}\lambda_2 r W^3 \\ &= \left(\frac{W^2}{r} - \lambda_1^2 r W^2\right)\lambda_2 - \frac{\lambda_1}{r} := M_1\lambda_2 - \frac{\lambda_1}{r}, \end{aligned} \tag{6.4}$$

where $M_1 = \frac{W^2}{r} - \lambda_1^2 r W^2$. Furthermore,

$$\begin{aligned}\lambda_{1rr} &= \left(\frac{W^2}{r} - \lambda_1^2 r W^2\right)_r \lambda_2 + M_1\lambda_{2r} - \frac{\lambda_{1r}}{r} + \frac{\lambda_1}{r^2} \\ &= -\frac{2\lambda_2 W^2}{r^2} + 2\lambda_1^2\lambda_2 W^2 + \frac{2\lambda_1}{r^2},\end{aligned} \tag{6.5}$$

and

$$W_{rr} = (\lambda_1 r W^3)_r \lambda_2 + \lambda_1 r W^3 \lambda_{2r} = \lambda_2^2 W^5 + 2\lambda_1^2\lambda_2^2 r^2 W^5. \tag{6.6}$$

Next we will differentiate F with respect to r.

$$\begin{aligned}F_r &= \frac{2D\lambda_1\lambda_{1r} + A\lambda_{1r}\lambda_2 + A\lambda_1\lambda_{2r}}{A\lambda_1 + B\lambda_2} - \frac{(D\lambda_1^2 + A\lambda_1\lambda_2)(A\lambda_{1r} + B\lambda_{2r})}{(A\lambda_1 + B\lambda_2)^2} \\ &= \frac{(2D\lambda_1 + A\lambda_2)\lambda_{1r}}{A\lambda_1 + B\lambda_2} - \frac{A(D\lambda_1^2 + A\lambda_1\lambda_2)\lambda_{1r}}{(A\lambda_1 + B\lambda_2)^2} \\ &= \frac{AD\lambda_1^2 + 2BD\lambda_1\lambda_2 + AB\lambda_2^2}{(A\lambda_1 + B\lambda_2)^2}\lambda_{1r} := N_1\lambda_{1r},\end{aligned} \tag{6.7}$$

and

$$\begin{aligned}F_{rr} = &\frac{2D\lambda_{1r}^2 + 2D\lambda_1\lambda_{1rr} + A\lambda_2\lambda_{1rr} + 2A\lambda_{1r}\lambda_{2r} + A\lambda_1\lambda_{2rr}}{A\lambda_1 + B\lambda_2} \\ &- \frac{2(2D\lambda_1\lambda_{1r} + A\lambda_{1r}\lambda_2 + A\lambda_1\lambda_{2r})(A\lambda_{1r} + B\lambda_{2r})}{(A\lambda_1 + B\lambda_2)^2} \\ &- \frac{(D\lambda_1^2 + A\lambda_1\lambda_2)(A\lambda_{1rr} + B\lambda_{2rr})}{(A\lambda_1 + B\lambda_2)^2} + \frac{2(D\lambda_1^2 + A\lambda_1\lambda_2)(A\lambda_{1r} + B\lambda_{2r})^2}{(A\lambda_1 + B\lambda_2)^3} \\ = &\frac{2D}{A\lambda_1 + B\lambda_2}\lambda_{1r}^2 + \frac{A\lambda_2 + 2D\lambda_1}{A\lambda_1 + B\lambda_2}\lambda_{1rr} + \frac{A\lambda_1}{A\lambda_1 + B\lambda_2}\lambda_{2rr} \\ &- \frac{2A(2D\lambda_1 + A\lambda_2)}{(A\lambda_1 + B\lambda_2)^2}\lambda_{1r}^2 - \frac{A(D\lambda_1^2 + A\lambda_1\lambda_2)}{(A\lambda_1 + B\lambda_2)^2}\lambda_{1rr} - \frac{B(D\lambda_1^2 + A\lambda_1\lambda_2)}{(A\lambda_1 + B\lambda_2)^2}\lambda_{2rr} \\ &+ \frac{2A^2(D\lambda_1^2 + A\lambda_1\lambda_2)}{(A\lambda_1 + B\lambda_2)^3}\lambda_{1r}^2.\end{aligned} \tag{6.8}$$

Let

$$N_2 := \frac{A\lambda_1}{A\lambda_1 + B\lambda_2} - \frac{B(D\lambda_1^2 + A\lambda_1\lambda_2)}{(A\lambda_1 + B\lambda_2)^2} = \frac{(A^2 - BD)\lambda_1^2}{(A\lambda_1 + B\lambda_2)^2}, \tag{6.9}$$

and

$$\begin{aligned}N_3 : &= \frac{2D}{A\lambda_1 + B\lambda_2} - \frac{2A(2D\lambda_1 + A\lambda_2)}{(A\lambda_1 + B\lambda_2)^2} + \frac{2A^2(D\lambda_1^2 + A\lambda_1\lambda_2)}{(A\lambda_1 + B\lambda_2)^3} \\ &= \frac{2B(BD - A^2)\lambda_2^2}{(A\lambda_1 + B\lambda_2)^3} < 0.\end{aligned} \tag{6.10}$$

Therefore,

$$F_{rr} = N_1\lambda_{1rr} + N_2\lambda_{2rr} + N_3\lambda_{1r}^2. \tag{6.11}$$

Denote $\mathcal{L} := \frac{\partial}{\partial t} - \frac{N_2}{W^2}\partial_r^2$. Plugging (6.3)-(6.11) into (6.2) we get,

$$
\begin{aligned}
\mathcal{L}\lambda_2 = &\frac{N_1}{W^2}\left(-\frac{2\lambda_2 W^2}{r^2} + 2\lambda_1^2\lambda_2 W^2 + \frac{2\lambda_1}{r^2}\right)\\
&+\frac{N_3}{W^2}\left(M_1\lambda_2 - \frac{\lambda_1}{r}\right)^2 - N_1\lambda_1\lambda_2 r\left(M_1\lambda_2 - \frac{\lambda_1}{r}\right)\\
&+\frac{F}{W^3}(\lambda_2^2 W^5 + 2\lambda_1^2\lambda_2^2 r^2 W^5) - 3\lambda_1^2\lambda_2^2 r^2 W^2 F.
\end{aligned}
\tag{6.12}
$$

Let's look at the terms don't contain λ_2 :

$$
\begin{aligned}
&\left(\frac{2N_1}{W^2r^2} + \frac{\lambda_1 N_3}{W^2 r^2}\right)\lambda_1\\
=&\frac{\lambda_1}{W^2r^2}\left[\frac{2(AD + 2BD\beta + AB\beta^2)}{(A+B\beta)^2} - \frac{2B(A^2 - BD)\beta^2}{(A+B\beta)^3}\right] > 0,
\end{aligned}
\tag{6.13}
$$

where $\beta := \frac{\lambda_2}{\lambda_1}$.

Thus we conclude that at the interior minimum point of λ_2, we have

$$\mathcal{L}\lambda_2 = F_1(\lambda_1, \lambda_2, W, r, n, k)\lambda_2 + C_2,$$

where F_1 is a bounded function and $C_2 > 0$. Now consider $\tilde{\lambda}_2 = e^{C_1 t}\lambda_2$, where $C_1 > |F_1| + 1$. We obtain if $\tilde{\lambda}_2$ achieves its minimum at an interior point, then at this point we have

$$\mathcal{L}\tilde{\lambda}_2 > \tilde{\lambda}_2 + e^{C_1 t}C_2 > 0,$$

which leads to a contradiction.

Therefore $\tilde{\lambda}_2$ doesn't achieve its minimum at interior, this yields Lemma 6.1. $\square$

Lemma 6.2. *Under the assumptions of Theorem 1.3 and condition* (1.5)*, we have*

$$|g_{rr}| < C(n, k, \Sigma_0, T_0) \text{ on } (\varphi(t), \varphi_1(t)] \times [0, T_0),$$

where $\varphi(t)$ is given in (3.1)*, and $\varphi_1(t)$ satisfies $g(\varphi_1(t), t) = 1$.*

Proof. First by (5.11) we know that

$$|g_{rr}(\varphi(t)+, t)| = \left|-\frac{2c_2}{c_1^3}\right| \leq C_1 \text{ in } [0, T_0),$$

for some $C_1 = C_1(n, k, \Sigma_0, T_0) > 0$. Moreover, when $\{g = 1\}$, since

$$C_0 > f_{rr} = 2\left[g g_{rr} + (g_r)^2\right] > 0,$$

by Lemma 4.1 we have

$$|g_{rr}(\varphi_1(t), t)| \leq C_2 \text{ in } [0, T_0),$$

for some $C_2 > 0$. Finally, by our assumption on the initial surface Σ_0, we also have

$$|g_{rr}(\cdot, 0)| \leq C_3 \text{ in } (\varphi(0), \varphi_1(0)],$$

for some $C_3 > 0$.

In the following, we will study the evolution of g_{rr}. Recall that $f_t = F(\lambda)W := \lambda_1\hat{F}(\lambda)W$, where $\hat{F}(\lambda) = \frac{D\lambda_1+A\lambda_2}{A\lambda_1+B\lambda_2}$, we get

$$g_t = \frac{g_r}{r}\hat{F}(\lambda). \tag{6.14}$$

Differentiating g_t with respect to r twice we obtain

$$g_{rt} = \left(\frac{g_{rr}}{r} - \frac{g_r}{r^2}\right)\hat{F} + \frac{g_r}{r}\hat{F}_r, \tag{6.15}$$

and

$$\begin{aligned} g_{rrt} = &\left(\frac{g_{rrr}}{r} - 2\frac{g_{rr}}{r^2} + \frac{2g_r}{r^3}\right)\hat{F}(\lambda) \\ &+ 2\left(\frac{g_{rr}}{r} - \frac{g_r}{r^2}\right)\hat{F}_r + \frac{g_r}{r}\hat{F}_{rr}. \end{aligned} \tag{6.16}$$

At the point where $G := g_{rr}$ achieves its interior extreme value we have

$$G_t = \frac{-2G}{r^2}\hat{F} + \frac{2g_r}{r^3}\hat{F} + \frac{2G}{r}\hat{F}_r - \frac{2g_r}{r^2}\hat{F}_r + \frac{g_r}{r}\hat{F}_{rr}. \tag{6.17}$$

Recall that

$$\lambda_{1r} = M_1\lambda_2 - \frac{\lambda_1}{r} = \frac{W^2}{r}\lambda_2 - \frac{\lambda_1}{r} - \lambda_1^2\lambda_2 rW^2, \tag{6.18}$$

$$\lambda_{1rr} = M_1\lambda_{2r} - \frac{2\lambda_2 W^2}{r^2} + 2\lambda_1^2\lambda_2 W^2 + \frac{2\lambda_1}{r^2}, \tag{6.19}$$

and

$$\lambda_2 = \frac{f_{rr}}{W^3} = \frac{2g_r^2 + 2gg_{rr}}{W^3}. \tag{6.20}$$

Differentiating λ_2 with respect to r we get

$$\begin{aligned} \lambda_{2r} &= \frac{6g_rG}{W^3} + \frac{2gG_r}{W^3} - \frac{3\lambda_2}{W}W_r \\ &= \frac{6g_rG}{W^3} + \frac{2gG_r}{W^3} - 3\lambda_1\lambda_2^2 rW^2. \end{aligned} \tag{6.21}$$

Substituting (6.21) into (6.19) we obtain

$$\begin{aligned} \lambda_{1rr} &= M_1\left(\frac{6g_rG}{W^3} - 3\lambda_1\lambda_2^2 rW^2\right) - \frac{2\lambda_2 W^2}{r^2} + O(\lambda_1) \\ &= \frac{6g_rG}{rW} - \frac{2\lambda_2 W^2}{r^2} + O(\lambda_1). \end{aligned} \tag{6.22}$$

Here, we want to point out that by Lemma 4.1 and Theorem 2.2 in [4] we have

$$|\lambda_1 G| = \left|\frac{2gg_rg_{rr}}{rW}\right| < C_4.$$

Next, let's compute the second derivative of λ_2 at the extreme point of G.

$$\begin{aligned}
\lambda_{2rr} &= \frac{6G^2}{W^3} + \frac{6g_rG_r}{W^3} - \frac{18g_rG}{W^4}W_r + \frac{2g_rG_r}{W^3} + \frac{2gG_{rr}}{W^3} \\
&\quad - \frac{6gG_rW_r}{W^4} - \frac{3\lambda_{2r}}{W}W_r - \frac{3\lambda_2}{W}W_{rr} + \frac{3\lambda_2}{W^2}W_r^2 \\
&= \frac{2gG_{rr}}{W^3} + \frac{6G^2}{W^3} - \frac{54\lambda_1\lambda_2 r}{W}g_rG - 3\lambda_2^3W^4 + O(\lambda_1).
\end{aligned} \tag{6.23}$$

Finally, we will compute the derivative of $\hat{F}$ with respect to r.

$$\begin{aligned}
\hat{F}_r &= \frac{D\lambda_{1r} + A\lambda_{2r}}{A\lambda_1 + B\lambda_2} - \frac{(D\lambda_1 + A\lambda_2)(A\lambda_{1r} + B\lambda_{2r})}{(A\lambda_1 + B\lambda_2)^2} \\
&= \frac{(A^2 - BD)(\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r})}{(A\lambda_1 + B\lambda_2)^2}.
\end{aligned} \tag{6.24}$$

By equation (6.18) and (6.21) we get,

$$\begin{aligned}
\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r} &= \frac{6\lambda_1 g_rG}{W^3} - \frac{\lambda_2^2W^2}{r} + O(\lambda_1) \\
&= \frac{12gg_r^2G}{rW^4} - \frac{W^2}{r}\left(\frac{2g_r^2 + 2gG}{W^3}\right)^2 + O(\lambda_1) \\
&= \frac{1}{rW^4}\left[-3g_r^4 - (g_r^2 - 2gG)^2\right] + O(\lambda_1).
\end{aligned} \tag{6.25}$$

$$\begin{aligned}
\hat{F}_{rr} &= \frac{A^2 - BD}{(A\lambda_1 + B\lambda_2)^2}(\lambda_1\lambda_{2rr} - \lambda_2\lambda_{1rr}) - 2\frac{A^2 - BD}{(A\lambda_1 + B\lambda_2)^3}(\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r})(A\lambda_{1r} + B\lambda_{2r}) \\
&= \frac{A^2 - BD}{(A\lambda_1 + B\lambda_2)^3}\left[(\lambda_1\lambda_{2rr} - \lambda_2\lambda_{1rr})(A\lambda_1 + B\lambda_2) - 2(\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r})(A\lambda_{1r} + B\lambda_{2r})\right] \\
&:= \frac{A^2 - BD}{(A\lambda_1 + B\lambda_2)^3}(I - 2II).
\end{aligned} \tag{6.26}$$

By equations (6.22) and (6.23) we have

$$\begin{aligned}
I &= \left[\lambda_1\left(\frac{2gG_{rr}}{W^3} + \frac{6G^2}{W^3} - \frac{54\lambda_1\lambda_2 r}{W}g_rG - 3\lambda_2^3W^4\right)\right. \\
&\quad \left. -\lambda_2\left(\frac{6g_rG}{rW} - \frac{2\lambda_2W^2}{r^2}\right)\right](A\lambda_1 + B\lambda_2) + O(1) \\
&= \left(\frac{2g\lambda_1G_{rr}}{W^3} + \frac{6\lambda_1G^2}{W^3} - \frac{6\lambda_2g_rG}{rW}\right)(A\lambda_1 + B\lambda_2) + O(1) \\
&= \left(\frac{2g\lambda_1G_{rr}}{W^3} - \frac{12g_r^3}{rW^4}G\right)(A\lambda_1 + B\lambda_2) + O(1).
\end{aligned} \tag{6.27}$$

While

$$
\begin{aligned}
II &= (\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r})(A\lambda_{1r} + B\lambda_{2r}) \\
&= \left\{\frac{1}{rW^4}[-3g_r^4 - (g_r^2 - 2gG)^2] + O(\lambda_1)\right\} A\lambda_{1r} \\
&+ \left\{\frac{1}{rW^4}[-3g_r^4 - (g_r^2 - 2gG)^2] + O(\lambda_1)\right\} B\left(\frac{6g_rG}{W^3} - 3\lambda_1\lambda_2^2 rW^2\right) \\
&= \frac{B}{rW^4}[-3g_r^4 - (g_r^2 - 2gG)^2]\frac{6g_rG}{W^3} + O(1).
\end{aligned}
\tag{6.28}
$$

Let's denote $\hat{\mathcal{L}} := \frac{\partial}{\partial t} - \frac{2gg_r\lambda_1(A^2-BD)}{rW^3(A\lambda_1+B\lambda_2)^2}\partial_r^2$. Then at the point where G achieves its interior extreme value we have

$$\hat{\mathcal{L}}G = F_2(gG, r, g_r, \lambda_1, \lambda_2, W)G + F_3(gG, r, g_r, \lambda_1, \lambda_2, W).$$

Here, using Lemma 6.1 we know that F_2 and F_3 are bounded.

Now consider the function $\tilde{G} = e^{-C_5 t}G$, where $C_5 > |F_2| + 1$. By the maximum principle we can see that either $|\tilde{G}|$ is bounded at its interior extreme point or $\tilde{G}$ doesn't achieve its negative minimum or positive maximum at an interior point. Therefore, we finish the proof of Lemma 6.2. □

7. Regularity Estimates of the remainder term

In previous sections, we have showed on $\{g \le 1, 0 \le t < T_0\}$, $|g(\cdot,t)|_{C^2} < C(n,k,\Sigma_0,T_0)$. In this section, by studying the regularity of the remainder term, we will show that for any $\alpha \in (0,1)$, $\|r\|_{C^{2+\alpha}_{w,s}} < C(n,k,\Sigma_0,T_0)$ (see Definition 3.4 in [4]). This yields that $\Sigma_{T_0} \in \mathfrak{S}$ and satisfies the non-degeneracy condition (1.3). Then, we can apply Theorem 1.2 and extend T_0 to $T_0 + \tau$, which contradicts to the maximality assumption of T_0. Therefore, we conclude that $T_0 = T$.

7.1. **Improved gradient estimates.**

Lemma 7.1. *Under the assumptions of Theorem 1.3 and condition* (1.5)*, we have*

$$g_r < C(n,k,\Sigma_0)r \text{ on } (\varphi(t), \varphi_1(t)] \times [0, T_0),$$

where $\varphi(t)$ is given in (3.1)*, $\varphi_1(t)$ satisfies $g(\varphi_1(t),t) = 1$.*

Proof. By equation (3.3) and Lemma 5.2, we have

$$g_r(\varphi(t)+,t) = \left(\frac{\partial r}{\partial s}\right)^{-1} = \frac{\varphi^{\frac{BM_{k,1}}{A}}}{a_0^{\frac{BM_{k,1}}{A}c_1(0)}} < C\varphi = Cr, \text{for } t \in [0, T_0).$$

Moreover, by Lemma 4.1 we have

$$g_r(\varphi_1(t),t) \le C\varphi_1 = Cr, \text{for } t \in [0, T_0),$$

where C is independent of T_0. Now let's consider $G = g_r r^{-1}$. If G achieves its global maximum at an interior point (r^*, t^*), then at this point, we have

$$G_r = g_{rr}r^{-1} - r^{-2}g_r = 0, \tag{7.1}$$

and

$$G_{rr} = g_{rrr}r^{-1} - 2r^{-2}g_{rr} + 2r^{-3}g_r \leq 0. \tag{7.2}$$

Moreover, since we know at this point $0 \leq G_t$, combining with equations (7.1) and (7.2) we get

$$0 \leq \lambda_1\lambda_{2r} - \lambda_2\lambda_{1r}. \tag{7.3}$$

Plugging (6.18) and (6.21) into (7.3) then applying (7.1) and (7.2), we obtain at this point

$$r^2G_0^2 \leq 2gG_0, \tag{7.4}$$

where $G_0 = \sup\limits_{(0,1)\times[0,T_0)} \frac{g_r}{r}$. Note that $\frac{g_r}{r} \leq G_0$ implies that $g \leq \frac{G_0r^2}{2} - \frac{G_0r_0^2}{2}$, here $r_0 = \varphi(t^*)$. Therefore we have

$$r^2G_0^2 \leq G_0r^2 - G_0r_0^2,$$

which leads to a contradiction. Thus, we conclude that G achieves its global maximum at its parabolic boundary point. This completes the proof of Lemma 7.1. ☐

Lemma 7.2. *Under the hypotheses of Theorem 1.3 and condition* (1.5)*, we have*

$$g_r > C(n, k, \Sigma_0)r^{N_1} \text{ on } (\varphi(t), \varphi_1(t)] \times [0, T_0),$$

where $\varphi(t)$ *is given in* (3.1)*,* $g(\varphi_1(t), t) = 1$*, and* $N_1 \geq \frac{B}{A}M_{k,1}$.

Proof. By the smoothness of g up to $\{z = 0\}$, Lemma 5.2, and equation (3.3) we know there exists $C > 0$ such that

$$g_r(r, 0) > Cr^{N_1} \text{ when } r \in (\varphi(0), \varphi_1(0)], \tag{7.5}$$

$$g_r(\varphi(t)+, t) = c_1^{-1} > C\varphi(t)^{N_1} \text{ when } t \in [0, T_0), \tag{7.6}$$

and

$$g_r(\varphi_1(t), t) > C\varphi_1(t)^{N_1} \text{ when } t \in [0, T_0). \tag{7.7}$$

We will prove by contradiction. Let's assume $r^{-N_1}g_r$ achieves an interior minimum at (r_0, t_0). Then at this point we have,

$$G_r = -N_1r^{-N_1-1}g_r + r^{-N_1}g_{rr} = 0,$$

which implies

$$g_{rr} = \frac{N_1g_r}{r}; \tag{7.8}$$

and

$$G_{rr} = N_1(N_1+1)r^{-N_1-2}g_r - 2N_1r^{-N_1-1}g_{rr} + r^{-N_1}g_{rrr} \geq 0,$$

which implies

$$g_{rrr} \geq \frac{N_1^2 - N_1}{r^2} g_r. \tag{7.9}$$

Moreover, by (6.15) we can see that

$$G_t = r^{-N_1-2} g_r \left[(N_1 - 1)\hat{F} + r\hat{F}_r \right]. \tag{7.10}$$

A straightforward calculation gives

$$\begin{aligned} r\hat{F}_r &= r \cdot \frac{A^2 - BD}{(A\lambda_1 + B\lambda_2)^2} \left[\lambda_1 \left(\frac{6 g_r g_{rr}}{W^3} + \frac{2 g g_{rrr}}{W^3} \right) \right. \\ &\quad \left. - \lambda_2 \left(\frac{W^2}{r} \lambda_2 - \frac{\lambda_1}{r} \right) - 2\lambda_1^2 \lambda_2^2 r W^2 \right] \\ &\geq \frac{r(A^2 - BD)}{(A\lambda_1 + B\lambda_2)^2} \left[\lambda_1 \left(\frac{6 N_1 g_r^2}{r W^3} + \frac{2g(N_1^2 - N_1) g_r}{r^2 W^3} \right) \right. \\ &\quad \left. - \frac{W^2}{r} \lambda_2^2 + \frac{\lambda_1 \lambda_2}{r} - 2\lambda_1^2 \lambda_2^2 r W^2 \right] \\ &\geq -C_2. \end{aligned} \tag{7.11}$$

Thus, we have at (r_0, t_0)

$$G_t \geq r^{-N_1-2} g_r [(N_1 - 1)C_1 - C_2],$$

where $C_i = C_i(n, k, \Sigma_0) > 0,\ i = 1, 2$. It's easy to see that when $N_1 > 0$ large we have a contradiction. This completes the proof of Lemma 7.2. □

Remark 7.3. When g is small, $0 < \lambda_1 = \frac{2 g g_r}{rW} < Cg$ is small. Then in (7.11),

$$\begin{aligned} -\frac{W^2}{r}\lambda_2^2 + \frac{\lambda_1 \lambda_2}{r} - 2\lambda_1^2 \lambda_2^2 r W^2 &= -\frac{1}{r}\lambda_2^2 + \frac{\lambda_1\lambda_2}{r} - \lambda_1 \lambda_2 \cdot O(\lambda_1) \\ &> -\frac{1}{r}\lambda_2^2. \end{aligned}$$

Thus we have at (r_0, t_0)

$$G_t \geq r^{-N_1-2} g_r \left[(N_1 - 1)\hat{F} - \frac{A^2 - BD}{(A\lambda_1 + B\lambda_2)^2} \lambda_2^2 \right]$$

which is a contradiction if we set

$$N_1 = \frac{B}{A} M_{k,1}.$$

So as in Lemma 7.2, we can prove

$$g_r > C(n, k, \Sigma_0) r^{\frac{B}{A} M_{k,1}} \text{ on } (\varphi(t), \varphi_\epsilon(t)] \times [0, T_0),$$

where ϵ is small and $g(\varphi_\epsilon(t), t) = \epsilon$. Notice here C is independent of T_0 and ϵ. As $g_r > 0$, it implies Lemma 7.2 holds on $(\varphi(t), \varphi_1(t)] \times [0, T_1]$ with $N_1 = \frac{B}{A} M_{k,1}$.

7.2. **Proof of Theorem 1.3.** Denote the Euclidean coordinates $\{x_1, x_2, \cdots, x_n\}$ such that

$$\frac{\partial}{\partial x_n} = \frac{\partial}{\partial r}$$

at P_0. Then $|Dg(P_0)| = g_{x_n}(P_0)$. By

$$x_n = r(z, x_1, \cdots, x_{n-1}),$$

We derive that, as $z = g^2$,

$$1 = r_z \frac{\partial z}{\partial x_n} = r_z \cdot 2g g_{x_n} = r_z \cdot 2\sqrt{z} g_{x_n}.$$

Equation (2.1) can be rewritten as follows: for $1 \leq i \leq n-1$,

$$\lambda_1 = \kappa_i = \frac{1}{r\sqrt{1+r_z^2}} = \frac{2\sqrt{z}}{r\sqrt{4z + g_{x_n}^{-2}}},$$

and

$$\lambda_2 = \frac{g_{x_n}^{-1} + g_{x_n}^{-3} g_{x_n x_n} z^{\frac{1}{2}}}{4\left(z + \frac{1}{4} g_{x_n}^{-2}\right)^{\frac{3}{2}}}.$$

Applying Lemma 5.2, Lemma 6.2, and condition (1.5) we can see that $\frac{\lambda_1}{\sqrt{z}}$ is bounded in $(z, t) \in [0, 1] \times [0, T_0)$.

Consider $v = r - u_2 = O(s^3)$, where $u_2 = \varphi + c_1(t)s + c_2(t)s^2$.

Let

$$\bar{Q}(r_{zz}, r_z, r) = r_t + \frac{S_k^n(\kappa)}{S_{k-1}^n(\kappa)} w = r_t + \frac{D\lambda_1^2 + A\lambda_1\lambda_2}{A\lambda_1 + B\lambda_2} w,$$

where $w = \sqrt{1 + r_z^2}$. Then by the construction of u_2 we get $\bar{Q}(r) - \bar{Q}(u_2) = F(\varphi, s) = O(s^3)$. Hence v satisfies

$$v_t - \bar{Q}^{zz} v_{zz} - \bar{Q}^z v_z - \bar{Q}^u v = F(\varphi, s),$$

where

$$\begin{aligned}
\bar{Q}^{zz} &= -\int_0^1 \frac{\partial \bar{Q}}{\partial r_{zz}} (D^2(u_2 + \xi v), D(u_2 + \xi v), u_2 + \xi v) d\xi, \\
\bar{Q}^{z} &= -\int_0^1 \frac{\partial \bar{Q}}{\partial r_{z}} (D^2(u_2 + \xi v), D(u_2 + \xi v), u_2 + \xi v) d\xi, \\
\bar{Q}^{u} &= -\int_0^1 \frac{\partial \bar{Q}}{\partial r} (D^2(u_2 + \xi v), D(u_2 + \xi v), u_2 + \xi v) d\xi.
\end{aligned}$$

By a straightforward calculation, we get

$$\begin{aligned}\bar{Q}^{zz}(r_{zz}, r_z, r) &= \frac{(A^2 - BD)\lambda_1^2}{w^2(A\lambda_1 + B\lambda_2)^2} \\ &= \frac{z^2(A^2 - BD)}{(A\lambda_1 + B\lambda_2)^2 \left(z + \frac{1}{4g_r^2}\right)^2},\end{aligned}$$

$$\begin{aligned}\bar{Q}^{z}(r_{zz}, r_z, r) &= \frac{2(BD - A^2)\lambda_1^2\lambda_2}{(A\lambda_1 + B\lambda_2)^2} \\ &= \frac{2z(BD - A^2)\lambda_2}{r^4(A\lambda_1 + B\lambda_2)^2 \left(z + \frac{1}{4g_r^2}\right)},\end{aligned}$$

and

$$\bar{Q}^{u}(r_{zz}, r_z, r) = -\frac{1}{r^2}\left[\frac{2D\lambda_1 + A\lambda_2}{A\lambda_1 + B\lambda_2} - \frac{AD\lambda_1^2 + A^2\lambda_1\lambda_2}{(A\lambda_1 + B\lambda_2)^2}\right].$$

Note that we have $A^2 - BD > 0$.

Then v satisfies

$$v_t - (z^2 a_{zz} v_{zz} + z b_z v_z + cv) = F(\varphi, s)$$

where $c(n,k,\Sigma_0,T_0) \le a_{zz} \le C(n,k,\Sigma_0,T_0)$, $|b_2| \le C(n,k,\Sigma_0,T_0)$, and $|c| \le C(n,k,\Sigma_0,T_0)$. Note that by [4] we have uniform upper bound on λ_1, λ_2.

Now take any $(z_0, t_0) \in [0,1) \times (0, T_0)$, set $\lambda = \frac{z_0}{2}$.

Let $t = t_0 + t', z = \lambda(h+1)$ and $v^\lambda(t', h) = v(t, z)$. Then

$$v^\lambda_{t'} - [(h+1)^2 a_{zz} v^\lambda_{hh} + (h+1) b_z v^\lambda_h + cv^\lambda] = F(\varphi, s).$$

By Lemma 6.2 we know that, when $(z,t) \in [0,1] \times [0,T_0)$ a_{zz}, b_z, and c are in C^α for any $\alpha \in (0,1)$. Applying standard Schauder estimates gives us that $v^\lambda \in C^{2+\alpha, 1+\frac{\alpha}{2}}$ in $(-\tau', \tau') \times [\frac{1}{2}, \frac{3}{2}]$. We evaluate v^λ at $(0,1)$ and get

$$|v_t| + z_0|Dv(t_0, z_0)| + z_0^2|D^2 v(t_0, z_0)| \le C(n,k,\Sigma_0,T_0)s^3.$$

Then

$$\frac{|v_s|}{s^2} + \frac{|v_{ss}|}{s} + \frac{|v_t|}{s^3} \le C(n,k,\Sigma_0,T_0). \tag{7.12}$$

Then we derive that $\|r\|_{C^{2+\alpha}_{w,s}}$ (see Definition 3.4 in [4]) is uniformly bounded on $[0,1] \times [0,T_0)$ for any $\alpha \in (0,1)$ as long as $\varphi(T_0) > 0$. Thus, we have $\Sigma_{T_0} \in C^{2+\alpha}_{w,\bar{s}}$. One can prove Σ_{T_0} is smooth up to the boundary by repeated differentiation. We conclude that Σ_{T_0} belongs to the class $\mathfrak{S}$, then applying Theorem 1.2 leads to a contradiction. So far, we have finished the proof of Theorem 1.3.

8. $C^{1,\alpha}$ estimates for g

Lemma 8.1. *Under the assumptions of Theorem 1.3 and condition* (1.5), *we have*

$$gg_{rr} + c_0 g_r^2 \geq 0. \tag{8.1}$$

holds for some constant $c_0 \in (0,1)$ *on* $\{g \leq \delta_0\} \times [0,T]$. *Here* c_0 *depends on* Σ_0 *and* δ_0 *is a small constant depending on the upper bounds of* g_r *and* λ_2.

Proof. By the nondegeneracy condition (1.3), we know that on Σ_0 there exists $c_0 \in (0,1)$ such that

$$gg_{rr} + c_0 g_r^2 \geq 0 \text{ on } \{g \leq 1\}.$$

Let $\{\Sigma_0^\epsilon\}$ be a sequence of smooth strictly convex hypersurfaces approaching Σ_0, and Σ_0^ϵ satisfies

$$g^\epsilon g^\epsilon_{rr} + c_0 \left(g^\epsilon_r\right)^2 \geq \epsilon. \tag{8.2}$$

We will show there exists $\tilde{c}_0 \in (0,1)$ such that

$$g^\epsilon g^\epsilon_{rr} + \tilde{c}_0 \left(g^\epsilon_r\right)^2 > 0 \text{ for } \{g^\epsilon \leq \delta_0\} \times [0,T].$$

Here $\tilde{c}_0$ is chosen as follows:
When $\{g^\epsilon = \delta_0\}$, by our assumptions and earlier results we have

$$g^\epsilon_r < a_0$$

and

$$g^\epsilon g^\epsilon_{rr} + \left(g^\epsilon_r\right)^2 > a_1 > 0 \text{ for } t \in [0,T].$$

Therefore, there exists $c_1 < 1$ such that $g^\epsilon g^\epsilon_{rr} + c_1 \left(g^\epsilon_r\right)^2 \geq \epsilon$ on $\{g^\epsilon = \delta_0\} \times [0,T]$. We will let $\tilde{c}_0 := \max\{c_0, c_1\}$. In the following, for our convenience, we will denote g^ϵ by g and $\tilde{c}_0$ by c_0.

Now, consider $M := gg_{rr} + c_0 g_r^2$, by our assumption we know $M \geq \epsilon$ on $\{g \leq \delta_0\} \times \{t = 0\} \cup \{g = \delta_0\} \times (0,T]$. We will prove by contradiction. If $M = 0$ at an interior point (r_0, t_0) for the first time, where $g(r_0, t_0) < \delta_0$ and $t_0 \in (0,T]$. Then at this point we have

$$gg_{rr} + c_0 g_r^2 = 0, \tag{8.3}$$

and

$$g_r g_{rr} + gg_{rrr} + 2c_0 g_r g_{rr} = 0. \tag{8.4}$$

These yields

$$g_{rr} = -\frac{c_0}{g} g_r^2, \tag{8.5}$$

and

$$g_{rrr} = \frac{c_0(2c_0+1)}{g^2} g_r^3. \tag{8.6}$$

Moreover, since at this point we have $M_{rr} \geq 0$ which implies

$$gg_{rrrr} \geq -(1+2c_0)\frac{c_0^2}{g^2} g_r^4 - \frac{(2+2c_0)c_0(2c_0+1)}{g^2} g_r^4. \tag{8.7}$$

On the other hand, at (r_0, t_0) we have

$$
\begin{aligned}
0 \geq M_t &= g_t g_{rr} + g g_{rrt} + 2c_0 g_r g_{rt} \\
&= \frac{g_r}{r}\hat{F} g_{rr} + g\left(\frac{g_{rrr}}{r} - 2\frac{g_{rr}}{r^2} + 2\frac{g_r}{r^3}\right)\hat{F} \\
&+ 2g\left(\frac{g_{rr}}{r} - \frac{g_r}{r^2}\right)\hat{F}_r + \frac{g g_r}{r}\hat{F}_{rr} \\
&+ 2c_0 g_r\left(\frac{g_{rr}}{r} - \frac{g_r}{r^2}\right)\hat{F} + 2c_0\frac{g_r^2}{r}\hat{F}_r,
\end{aligned} \tag{8.8}
$$

where we have used equations (6.14), (6.15), and (6.16). Substituting (8.5) and (8.6) into (8.8) we get

$$
\frac{2g g_r}{r^3}\hat{F} - \frac{2g g_r}{r^2}\hat{F}_r + \frac{g g_r}{r}\hat{F}_{rr} \leq 0.
$$

Since $g g_r \geq 0$ we have

$$
2\hat{F} - 2r\hat{F}_r + r^2\hat{F}_{rr} \leq 0. \tag{8.9}
$$

Next, we will compute λ_2 and the derivatives of λ_2 at (r_0, t_0). First, by (6.20) and (8.5) we obtain

$$
\lambda_2 = \frac{2(g_r^2 + g g_{rr})}{W^3} = \frac{2(1-c_0)g_r^2}{W^3}. \tag{8.10}
$$

Then, plugging (8.6) and (8.5) into (6.21) yields

$$
\begin{aligned}
\lambda_{2r} &= \frac{6g_r g_{rr}}{W^3} + \frac{2g g_{rrr}}{W^3} - 3\lambda_1\lambda_2^2 r W^2 \\
&= \frac{6g_r}{W^3}\left(-\frac{c_0}{g}g_r^2\right) + \frac{2g}{W^3}\frac{c_0(2c_0+1)}{g^2}g_r^3 - 3\lambda_1\lambda_2^2 r W^2 \\
&= -\frac{\sqrt{2}c_0\lambda_2^{\frac{3}{2}}W^{\frac{3}{2}}}{\sqrt{1-c_0}g} - 3\lambda_1\lambda_2^2 r W^2 \\
&:= -\frac{A_1}{g}W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 r W^2,
\end{aligned} \tag{8.11}
$$

where $A_1 = \frac{\sqrt{2}c_0}{\sqrt{1-c_0}}$. Finally, differentiating (8.11) then applying (6.18) and (8.7) gives

$$
\begin{aligned}
\lambda_{2rr} \geq & -A_1 \left\{ \frac{\frac{3}{2}W^{\frac{1}{2}}W_r\lambda_2^{\frac{3}{2}} + \frac{3}{2}W^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}\lambda_{2r}}{g} - \frac{W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}}{g^2}g_r \right\} \\
& - 3\lambda_{1r}\lambda_2^2 rW^2 - 6\lambda_1\lambda_2\lambda_{2r}rW^2 - 3\lambda_1\lambda_2^2W^2 - 6\lambda_1\lambda_2^2 rWW_r \\
= & -A_1 \left\{ \frac{\frac{3}{2}W^{\frac{1}{2}}\lambda_2^{\frac{3}{2}}\lambda_1\lambda_2 rW^3}{g} + \frac{3}{2}\frac{W^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}}{g}\left(-\frac{A_1}{g}W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 rW^2\right) \right. \\
& \left. - \frac{W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}}{g^2} \cdot \frac{W^{\frac{3}{2}}\lambda_2^{\frac{1}{2}}}{\sqrt{2}\sqrt{1-c_0}} \right\} - 3\lambda_2^2 rW^2\left(\frac{W^2}{r}\lambda_2 - \frac{\lambda_1}{r} - \lambda_1^2\lambda_2 rW^2\right) \\
& - 6\lambda_1\lambda_2 rW^2\left(-\frac{A_1}{g}W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 rW^2\right) \\
& - 3\lambda_1\lambda_2^2W^2 - 6\lambda_1\lambda_2^2 rW\lambda_1\lambda_2 rW^3.
\end{aligned}
\tag{8.12}
$$

Note that

$$
\frac{\lambda_1}{g} = \frac{2g_r}{rW} = \frac{\sqrt{2}}{\sqrt{1-c_0}} \cdot \frac{W^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}}{r} = \frac{A_1}{c_0}\frac{W^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}}{r}.
$$

Therefore, (8.12) becomes

$$
\begin{aligned}
\lambda_{2rr} \geq & -A_1 \left\{ \frac{3}{2}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}r \cdot \frac{A_1}{c_0}\frac{W^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}}{r} - \frac{3}{2}\frac{A_1}{g^2}W^3\lambda_2^2 \right. \\
& \left. - \frac{9}{2}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}r \cdot \frac{A_1}{c_0}\frac{W^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}}{r} - \frac{W^3\lambda_2^2 A_1}{g^2 2c_0} \right\} \\
& - 3W^4\lambda_2^3 + 3\lambda_1\lambda_2^2W^2 + 3\lambda_1^2\lambda_2^3r^2W^4 \\
& + 6A_1W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}r \cdot \frac{A_1}{c_0}\frac{W^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}}{r} + 18\lambda_1^2\lambda_2^3r^2W^4 \\
& - 3\lambda_1\lambda_2^2W^2 - 6\lambda_1^2\lambda_2^3r^2W^4 \\
= & \frac{3A_1^2}{c_0}W^4\lambda_2^3 + \left(\frac{3}{2}A_1^2 + \frac{A_1^2}{2c_0}\right)\frac{W^3\lambda_2^2}{g^2} \\
& + \frac{6A_1^2}{c_0}W^4\lambda_2^3 + 15\lambda_1^2\lambda_2^3r^2W^4 - 3W^4\lambda_2^3 \\
= & \left(\frac{9A_1^2}{c_0} - 3\right)W^4\lambda_2^3 + \left(\frac{3}{2}A_1^2 + \frac{A_1^2}{2c_0}\right)\frac{W^3\lambda_2^2}{g^2} + 15\lambda_1^2\lambda_2^3r^2W^4.
\end{aligned}
\tag{8.13}
$$

Substituting equation (8.11) into equation (6.19) we get

$$
\begin{aligned}
\lambda_{1rr} &= \left(\frac{W^2}{r} - \lambda_1^2 r W^2\right)\left(-\frac{A_1}{g} W^{\frac{3}{2}} \lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 r W^2\right) \\
&\quad - \frac{2\lambda_2 W^2}{r^2} + \frac{2\lambda_1}{r^2} + 2\lambda_1^2\lambda_2 W^2 \\
&= -\frac{A_1}{gr} W^{\frac{7}{2}} \lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 W^4 + \frac{A_1}{g} W^{\frac{7}{2}} \lambda_2^{\frac{3}{2}} r \lambda_1^2 \\
&\quad + 3\lambda_1^3\lambda_2^2 r^2 W^4 - \frac{2\lambda_2 W^2}{r^2} + \frac{2\lambda_1}{r^2} + 2\lambda_1^2\lambda_2 W^2 \\
&= -\frac{A_1}{gr} W^{\frac{7}{2}} \lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 W^4 + \frac{A_1^2}{c_0} W^4 \lambda_1 \lambda_2^2 \\
&\quad + 3\lambda_1^3\lambda_2^2 r^2 W^4 - \frac{2\lambda_2 W^2}{r^2} + \frac{2\lambda_1}{r^2} + 2\lambda_1^2\lambda_2 W^2.
\end{aligned} \tag{8.14}
$$

Now combining equation (8.9) with equations (6.24) and (6.26) we have

$$
\begin{aligned}
0 &\geq 2\hat{F} - 2r\hat{F}_r + r^2\hat{F}_{rr} \\
&= 2\frac{D\lambda_1 + A\lambda_2}{A\lambda_1 + B\lambda_2} - \frac{2r(A^2 - BD)}{(A\lambda_1 + B\lambda_2)^2}(\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r}) \\
&\quad + \frac{r^2(A^2 - BD)}{(A\lambda_1 + B\lambda_2)^3}\left[(\lambda_1\lambda_{2rr} - \lambda_2\lambda_{1rr})(A\lambda_1 + B\lambda_2)\right. \\
&\quad \left. 2(\lambda_2\lambda_{1r} - \lambda_1\lambda_{2r})(A\lambda_{1r} + B\lambda_{2r})\right]
\end{aligned} \tag{8.15}
$$

By a straightforward calculation we obtain

$$
\begin{aligned}
\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r} &= \lambda_1\left(-\frac{A_1}{g} W^{\frac{3}{2}} \lambda_2^{\frac{3}{2}} - 3\lambda_1\lambda_2^2 r W^2\right) \\
&\quad - \lambda_2\left(\frac{W^2}{r}\lambda_2 - \frac{\lambda_1}{r} - \lambda_1^2\lambda_2 r W^2\right) \\
&= -A_1 W^{\frac{3}{2}} \lambda_2^{\frac{3}{2}} \cdot \frac{A_1}{c_0} \frac{W^{\frac{1}{2}} \lambda_2^{\frac{1}{2}}}{r} - 3\lambda_1^2\lambda_2^2 r W^2 - \frac{W^2}{r}\lambda_2^2 \\
&\quad + \frac{\lambda_1\lambda_2}{r} + \lambda_1^2\lambda_2^2 r W^2 \\
&= -\left(\frac{1 + c_0}{1 - c_0}\right)\frac{W^2\lambda_2^2}{r} - 2\lambda_1^2\lambda_2^2 r W^2 + \frac{\lambda_1\lambda_2}{r}.
\end{aligned} \tag{8.16}
$$

Plugging (8.16) into (8.15) yields

$$
\begin{aligned}
0 &\geq 2AD\lambda_1^2 + 2AB\lambda_2^2 + 4BD\lambda_1\lambda_2 + 2(A^2 - BD)\left(\frac{1 + c_0}{1 - c_0}\right) W^2\lambda_2^2 \\
&\quad + 4(A^2 - BD)\lambda_1^2\lambda_2^2 r^2 W^2 + \frac{r^2(A^2 - BD)}{(A\lambda_1 + B\lambda_2)}\left[(\lambda_1\lambda_{2rr} - \lambda_2\lambda_{1rr})(A\lambda_1 + B\lambda_2)\right. \\
&\quad \left. +2(\lambda_2\lambda_{1r} - \lambda_1\lambda_{2r})(A\lambda_{1r} + B\lambda_{2r})\right].
\end{aligned} \tag{8.17}
$$

Equations (8.13) and (8.14) implies

$$
\begin{aligned}
&\lambda_1\lambda_{2rr}-\lambda_2\lambda_{1rr}\\
&\geq \lambda_1\left\{\left(\frac{9A_1^2}{c_0}-3\right)W^4\lambda_2^3+\left(\frac{3}{2}A_1^2+\frac{A_1^2}{2c_0}\right)\frac{W^3\lambda_2^2}{g^2}+15\lambda_1 62\lambda_2^3r^2W^4\right\}\\
&-\lambda_2\left\{-\frac{A_1}{gr}W^{\frac{7}{2}}\lambda_2^{\frac{3}{2}}-3\lambda_1\lambda_2^2W^4+\frac{A_1^2}{c_0}W^4\lambda_1\lambda_2^2\right.\\
&\left.+3\lambda_1^3\lambda_2^2r^2W^4-\frac{2\lambda_2W^2}{r^2}+\frac{2\lambda_1}{r^2}+2\lambda_1^2\lambda_2W^2\right\}\\
&=\frac{8A_1^2}{c_0}\lambda_2^3\lambda_1W^4+\left(\frac{3}{2}A_1^2+\frac{A_1^2}{2c_0}\right)\frac{W^3\lambda_2^2}{g}\cdot\frac{A_1W^{\frac{1}{2}}\lambda_2^{\frac{1}{2}}}{c_0r}\\
&+12\lambda_1^3\lambda_2^3r^2W^4+\frac{A_1}{gr}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}+\frac{2\lambda_2^2W^2}{r^2}-\frac{2\lambda_1\lambda_2}{r^2}-2\lambda_1^2\lambda_2^2W^2\\
&=\frac{8A_1^2}{c_0}\lambda_2^3\lambda_1W^4+\frac{2+2c_0}{1-c_0}\cdot\frac{A_1}{gr}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}\\
&+12\lambda_1^3\lambda_2^3r^2W^4+\frac{2\lambda_2^2W^2}{r^2}-\frac{2\lambda_1\lambda_2}{r^2}-2\lambda_1^2\lambda_2^2W^2.
\end{aligned}
\tag{8.18}
$$

Equation (6.18) and equation (8.16) gives

$$
\begin{aligned}
&2(\lambda_2\lambda_{1r}-\lambda_1\lambda_{2r})A\lambda_{1r}\\
&=2A\left[\left(\frac{1+c_0}{1-c_0}\right)\frac{W^2\lambda_2^2}{r}+2\lambda_1^2\lambda_2^2rW^2-\frac{\lambda_1\lambda_2}{r}\right]\left(\frac{W^2}{r}\lambda_2-\frac{\lambda_1}{r}-\lambda_1^2\lambda_2rW^2\right)\\
&=2A\left[\left(\frac{1+c_0}{1-c_0}\right)\frac{W^4\lambda_2^3}{r^2}-\frac{2}{1-c_0}\frac{W^2\lambda_1\lambda_2^2}{r^2}-\left(\frac{3c_0-1}{1-c_0}\right)\lambda_1^2\lambda_2^3W^4\right.\\
&\left.-\lambda_1^3\lambda_2^2W^2-2\lambda_1^4\lambda_2^3r^2W^4+\frac{\lambda_1^2\lambda_2}{r^2}\right].
\end{aligned}
\tag{8.19}
$$

Moreover, equation (8.11) and equation (8.16) gives

$$
\begin{aligned}
&2(\lambda_2\lambda_{1r}-\lambda_1\lambda_{2r})B\lambda_{2r}\\
&=2B\left[\left(\frac{1+c_0}{1-c_0}\right)\frac{W^2\lambda_2^2}{r}+2\lambda_1^2\lambda_2^2rW^2-\frac{\lambda_1\lambda_2}{r}\right]\left(-\frac{A_1}{g}W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}-3\lambda_1\lambda_2^2rW^2\right)\\
&=2B\left[-\frac{A_1}{g}\left(\frac{1+c_0}{1-c_0}\right)\frac{W^{\frac{7}{2}}\lambda_2^{\frac{7}{2}}}{r}-3\left(\frac{1+c_0}{1-c_0}\right)\lambda_1\lambda_2^4W^4\right.\\
&\left.-\frac{2A_1^2}{c_0}W^4\lambda_2^4\lambda_1-6\lambda_1^3\lambda_2^4r^2W^4+\frac{A_1^2}{c_0}\frac{W^2\lambda_2^3}{r^2}+3\lambda_1^2\lambda_2^3W^2\right]
\end{aligned}
\tag{8.20}
$$

Combining (8.18), (8.19), and (8.20) with (8.17) we get

$$
\begin{aligned}
0 \geq & \Bigg[2AD\lambda_1^2 + 2AB\lambda_2^2 + 4BD\lambda_1\lambda_2 + 2(A^2 - BD)\left(\frac{1+c_0}{1-c_0}\right)W^2\lambda_2^2 \\
& + 4(A^2 - BD)\lambda_1^2\lambda_2^2 r^2 W^2\Bigg](A\lambda_1 + B\lambda_2) \\
& + r^2(A^2 - BD)\left[\frac{8A_1^2}{c_0}\lambda_2^3\lambda_1 W^4 + \frac{2+2c_0}{1-c_0}\frac{A_1}{gr}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}\right. \\
& \left. + 12\lambda_1^3\lambda_2^3 r^2 W^4 + \frac{2\lambda_2^2 W^2}{r^2} - 2\frac{\lambda_1\lambda_2}{r^2} - 2\lambda_1^2\lambda_2^2 W^2\right](A\lambda_1 + B\lambda_2) \\
& + 2Ar^2(A^2 - BD)\left[\left(\frac{1+c_0}{1-c_0}\right)\frac{W^4\lambda_2^3}{r^2} - \frac{2}{1-c_0}\frac{W^2\lambda_1\lambda_2^2}{r^2}\right. \\
& \left. - \left(\frac{3c_0 - 1}{1-c_0}\right)\lambda_1^2\lambda_2^3 W^4 - \lambda_1^3\lambda_2^2 W^2 - 2\lambda_1^4\lambda_2^3 r^2 W^4 + \frac{\lambda_1^2\lambda_2}{r^2}\right] \\
& + 2Br^2(A^2 - BD)\left[-\frac{A_1}{g}\left(\frac{1+c_0}{1-c_0}\right)\frac{W^{\frac{7}{2}}\lambda_2^{\frac{7}{2}}}{r} - 3\left(\frac{1+c_0}{1-c_0}\right)\lambda_1\lambda_2^4 W^4\right. \\
& \left. - \frac{2A_1^2}{c_0}\lambda_1\lambda_2^4 W^4 - 6\lambda_1^3\lambda_2^4 r^2 W^4 + \frac{A_1^2}{c_0}\frac{W^2\lambda_2^3}{r^2} + 3\lambda_1^2\lambda_2^3 W^2\right]
\end{aligned} \tag{8.21}
$$

Now, we observe that

$$
\begin{aligned}
& r^2(A^2 - BD)\frac{2+2c_0}{1-c_0}\frac{A_1}{gr}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}(A\lambda_1 + B\lambda_2) \\
& - 2Br^2(A^2 - BD)\frac{A_1}{g}\left(\frac{1+c_0}{1-c_0}\right)\frac{W^{\frac{7}{2}}\lambda_2^{\frac{7}{2}}}{r} \\
& = r^2(A^2 - BD)\frac{2+2c_0}{1-c_0}\frac{A_1}{gr}W^{\frac{7}{2}}\lambda_2^{\frac{5}{2}}A\lambda_1,
\end{aligned} \tag{8.22}
$$

$$
\begin{aligned}
& 2(A^2 - BD)\left(\frac{1+c_0}{1-c_0}\right)W^2\lambda_2^2(A\lambda_1 + B\lambda_2) \\
& + r^2(A^2 - BD)\frac{2\lambda_2^2 W^2}{r^2}(A\lambda_1 + B\lambda_2) \\
& - 2Ar^2(A^2 - BD)\frac{2}{1-c_0}\frac{W^2\lambda_1\lambda_2^2}{r^2} \\
& = 2(A^2 - BD)\left(\frac{2}{1-c_0}\right)BW^2\lambda_2^3,
\end{aligned} \tag{8.23}
$$

$$\begin{aligned}
&r^2(A^2-BD)\frac{8A_1^2}{c_0}\lambda_2^3\lambda_1W^4(A\lambda_1+B\lambda_2)\\
&-2Ar^2(A^2-BD)\left(\frac{3c_0-1}{1-c_0}\right)\lambda_1^2\lambda_2^3W^4\\
&-2Br^2(A^2-BD)\frac{2A_1^2}{c_0}\lambda_1\lambda_2^4W^4\\
&=\frac{r^2(A^2-BD)}{1-c_0}\lambda_2^3\lambda_1W^4(10c_0A\lambda_1+8Bc_0\lambda_2+2A\lambda_1),
\end{aligned}\tag{8.24}$$

$$\begin{aligned}
&r^2(A^2-BD)12\lambda_1^3\lambda_2^3r^2W^4(A\lambda_1+B\lambda_2)\\
&-2Ar^2(A^2-BD)\cdot 2\lambda_1^4\lambda_2^3r^2W^4-2Br^2(A^2-BD)\cdot 6\lambda_1^3\lambda_2^4r^2W^4\\
&=8Ar^4(A^2-BD)\lambda_1^4\lambda_2^3W^4,
\end{aligned}\tag{8.25}$$

and

$$\begin{aligned}
&4(A^2-BD)\lambda_1^2\lambda_2^2r^2W^2(A\lambda_1+B\lambda_2)-2r^2(A^2-BD)\lambda_1^2\lambda_2^2W^2(A\lambda_1+B\lambda_2)\\
&=2r^2(A^2-BD)\lambda_1^2\lambda_2^2W^2(A\lambda_1+B\lambda_2).
\end{aligned}\tag{8.26}$$

Combining (8.21)–(8.26) we get

$$\begin{aligned}
0\geq{}&[2AD\lambda_1^2+2AB\lambda_2^2+4BD\lambda_1\lambda_2+2(A^2-BD)\lambda_1^2\lambda_2^2r^2W^2](A\lambda_1+B\lambda_2)\\
&-2(A^2-BD)\lambda_1\lambda_2(A\lambda_1+B\lambda_2)+2A(A^2-BD)\lambda_1^2\lambda_2\\
&-2Ar^2(A^2-BD)\lambda_1^3\lambda_2^2W^2-6Br^2(A^2-BD)\left(\frac{1+c_0}{1-c_0}\right)\lambda_1\lambda_2^4W^4\\
&+2B(A^2-BD)\left(\frac{1+c_0}{1-c_0}\right)W^2\lambda_2^3.
\end{aligned}\tag{8.27}$$

Since when $\delta_0>0$ small we have

$$r^2\lambda_1\lambda_2^4=2g\frac{g_rr}{W}\lambda_2^4<2\delta_0\frac{g_rr}{W}\lambda_2^4<\frac{\lambda_2^3}{3W^2}.$$

It's easy to see that in this case the right hand side of (8.27) is positive, which leads to a contradiction. This completes the proof of Lemma 8.1. □

Remark 8.2. Since g is smooth and λ_2 is bounded away from 0 when $g\geq\delta_0$, we know that (8.1) holds for $\{g\leq 1\}\times[0,T]$, possibly for a different constant $c_0\in(0,1)$.

Lemma 8.3. *Under the same assumptions as Lemma 7.2, we have there exists a* $N\geq\max\{N_1+1,\frac{B}{A}(M_{k,2}-3M_{k,1})\}$ *depends on* n,k, *and* Σ_0, *such that*

$$|r^Ng_{rr}|<C(n,k,\Sigma_0)\ \text{on}\ (\varphi(t),\varphi_1(t)]\times[0,T_1],$$

for any $0<T_1<T$.

Proof. Let $G=r^Ng_{rr}$, by equations (3.3), (3.5), Remark 5.3, and our assumptions we can see that $|G|$ is bounded on $(\varphi(0),\varphi_1(0)]\times\{t=0\}\cup\{r\to\varphi(t)+\}\times(0,T_1]\cup\{r=$

$\varphi_1(t)\} \times (0, T_1]$. Now assume G achieves its negative minimum at an interior point (r_0, t_0). Then at this point, we have

$$G_r = Nr^{N-1} g_{rr} + r^N g_{rrr} = 0,$$

which implies

$$g_{rrr} = -\frac{N}{r} g_{rr}; \tag{8.28}$$

and

$$0 \le G_{rr} = N(N-1) r^{N-2} g_{rr} + 2N r^{N-1} g_{rrr} + r^N g_{rrrr},$$

which implies

$$g_{rrrr} \ge -\frac{N(N-1)}{r^2} g_{rr} - \frac{2N}{r} g_{rrr} = \frac{N^2+N}{r^2} g_{rr}. \tag{8.29}$$

Moreover, by (6.16) we know that at (r_0, t_0) the following equality holds

$$\begin{aligned} G_t &= r^N g_{rrt} \\ &= r^N \left[-\frac{(N+2)}{r^2} g_{rr} + \frac{2g_r}{r^3} \right] \hat{F} \\ &+ 2r^N \left(\frac{g_{rr}}{r} - \frac{g_r}{r^2} \right) \hat{F}_r + r^N \frac{g_r}{r} \hat{F}_{rr}. \end{aligned} \tag{8.30}$$

Since

$$\hat{F}_r = \frac{(A^2 - BD)}{(A\lambda_1 + B\lambda_2)^2} (\lambda_1 \lambda_{2r} - \lambda_2 \lambda_{1r}),$$

using equations (6.18) and (6.21) we compute

$$\begin{aligned} &\lambda_1 \lambda_{2r} - \lambda_2 \lambda_{1r} \\ &= \lambda_1 \left[\frac{6 g_r g_{rr}}{W^3} + \frac{2g}{W^3} \left(-\frac{N}{r} g_{rr} \right) - 3\lambda_1 \lambda_2^2 r W^2 \right] \\ &- \lambda_2 \left(\frac{W^2 \lambda_2}{r} - \frac{\lambda_1}{r} - \lambda_1^2 \lambda_2 r W^2 \right) \\ &= \frac{\lambda_1}{W^3} \left(6 g_r - \frac{2Ng}{r} \right) g_{rr} - \frac{W^2 \lambda_2^2}{r} + \frac{\lambda_1 \lambda_2}{r} - 2\lambda_1^2 \lambda_2^2 r W^2 \\ &:= \frac{\lambda_1}{W^3} M_2 g_{rr} - \frac{W^2 \lambda_2^2}{r} + \frac{\lambda_1 \lambda_2}{r} - 2\lambda_1^2 \lambda_2^2 r W^2. \end{aligned} \tag{8.31}$$

Here and in the following we denote $M_2 = 6g_r - \frac{2Ng}{r}$, then it's easy to see that at (r_0, t_0)

$$\lambda_{2r} = \frac{M_2}{W^3} g_{rr} - 3\lambda_1 \lambda_2^2 r W^2.$$

Note that if at (r_0, t_0) $M_2 \le 0$, then by Lemma 7.2 we get

$$g \ge \frac{3 g_r r}{N} \ge \frac{C r^{N_1+1}}{N}.$$

Thus, by condition (1.5) and Remark 8.2 we obtain at this point

$$C \geq |gg_{rr}| \geq \left|\frac{Cr^{N_1+1}}{N} g_{rr}\right|.$$

Since $N \geq N_1+1$ we know that at this point G is bounded from below, then Lemma 8.3 follows directly. Therefore, in the following, we assume $M_2 > 0$. Then equation (8.31) implies

$$\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r} < -\frac{W^2}{r}\lambda_2 + \frac{\lambda_1\lambda_2}{r} - 2\lambda_1^2\lambda_2^2 rW^2.$$

Plugging this into (8.30) yields

$$\begin{aligned}G_t > r^{-2}\Big\{ &\left[-(N+2)G + 2g_r r^{N-1}\right]\hat{F} \\ &+2\frac{A^2-BD}{(A\lambda_1+B\lambda_2)^2}(G - g_r r^{N-1})(-W^2\lambda_2^2 + \lambda_1\lambda_2 - 2\lambda_1^2\lambda_2^2 r^2W^2) + g_r r^{N+1}\hat{F}_{rr}\Big\}.\end{aligned} \tag{8.32}$$

Next, we will compute $\hat{F}_{rr}$ carefully. Recall that

$$\begin{aligned}\hat{F}_{rr} = &\frac{A^2-BD}{(A\lambda_1+B\lambda_2)^3}[(\lambda_1\lambda_{2rr} - \lambda_2\lambda_{1rr})(A\lambda_1+B\lambda_2) \\ &- 2(\lambda_1\lambda_{2r} - \lambda_2\lambda_{1r})(A\lambda_{1r} + B\lambda_{2r})].\end{aligned}$$

At (r_0, t_0), by a straightforward calculation we get

$$\begin{aligned}\lambda_{2rr} &\geq \frac{M_2}{W^3}g_{rrr} + \frac{M_{2r}g_{rr}}{W^3} - \frac{3M_2}{W^4}g_{rr}W_r - 3(\lambda_1\lambda_2^2 rW^2)_r \\ &= \frac{M_2}{W^3}\left(-\frac{N}{r}g_{rr}\right) + \frac{1}{W^3}\left(6g_{rr}^2 - \frac{2Ng_r}{r}g_{rr} + \frac{2Ng}{r^2}g_{rr}\right) \\ &\quad - \frac{3M_2}{W^4}W_r g_{rr} - 6\lambda_1\lambda_2 rW^2\left(\frac{M_2}{W^3}g_{rr} - 3\lambda_1\lambda_2^2 rW^2\right) - 3\lambda_2^2(\lambda_1 rW^2)_r.\end{aligned} \tag{8.33}$$

Note that by Remark 8.2 we have

$$\lambda_2 = \frac{2g_r^2 + 2gg_{rr}}{W^3} \geq \frac{2(1-c_0)g_r^2}{W^3},$$

which implies

$$g_r \leq \frac{\lambda_2^{\frac{1}{2}}W^{\frac{3}{2}}}{\sqrt{2(1-c_0)}}, \tag{8.34}$$

$$\lambda_1 g_{rr} = \frac{2gg_r}{rW^3}g_{rr} \geq \frac{2g_r}{rW^3}(-c_0g_r^2) \geq \frac{-c_0W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}}{r\sqrt{2}(\sqrt{1-c_0})^3}, \tag{8.35}$$

and

$$M_2 = 6g_r - \frac{2Ng}{r} \leq \frac{6\lambda_2^{\frac{1}{2}}W^{\frac{3}{2}}}{\sqrt{2(1-c_0)}}. \tag{8.36}$$

Combining (8.34), (8.35), (8.36) with (8.33) we obtain

$$
\begin{aligned}
\lambda_1\lambda_{2rr} &\geq \frac{2Ng\lambda_1}{r^2W^3}g_{rr} - 3\lambda_1\lambda_2^2(\lambda_1 rW^2)_r \\
&\geq -\frac{2Ng}{r^2W^3}\frac{c_0W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}}{r\sqrt{2}(\sqrt{1-c_0})^3} + O(\lambda_1\lambda_2^2),
\end{aligned}
\tag{8.37}
$$

and

$$
\begin{aligned}
-\lambda_2\lambda_{1rr} &= \lambda_2\left[\left(\frac{W^2}{r} - \lambda_1^2 rW^2\right)\left(-\frac{M_2}{W^3}g_{rr} + 3\lambda_1\lambda_2^2 rW^2\right)\right. \\
&\quad \left. + \frac{2\lambda_2W^2}{r^2} - 2\lambda_2^2\lambda_2W^2 - \frac{2\lambda_1}{r^2}\right] \\
&\geq -\lambda_1^2\lambda_2 rW^2\left(-\frac{M_2}{W^3}g_{rr} + 3\lambda_1\lambda_2^2 rW^2\right) - 2\lambda_1^2\lambda_2^3W^2 - \frac{2\lambda_1\lambda_2}{r^2} \\
&\geq O\left(\frac{(\lambda_1+\lambda_2)^2}{r^2}\right).
\end{aligned}
\tag{8.38}
$$

Here we have used the assumption that at (r_0,t_0), $g_{rr}<0$. From (8.37) and (8.38) we conclude that

$$
\begin{aligned}
&\lambda_1\lambda_{2rr} - \lambda_2\lambda_{1rr} \\
&\geq -\frac{2Ng}{r^2W^3}\frac{c_0W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}}{r\sqrt{2}(\sqrt{1-c_0})^3} + O\left(\frac{(\lambda_1+\lambda_2)^2}{r^2}\right).
\end{aligned}
\tag{8.39}
$$

Moreover, by equation (8.31) we have

$$
\begin{aligned}
&2(\lambda_2\lambda_{1r} - \lambda_1\lambda_{2r})(A\lambda_{1r} + B\lambda_{2r}) \\
&= 2A\left(-\frac{\lambda_1}{W^3}M_2g_{rr} + \frac{W^2}{r}\lambda_2^2 - \frac{\lambda_1\lambda_2}{r} + 2\lambda_1^2\lambda_2^2 rW^2\right)\left(\frac{W^2}{r}\lambda_2 - \frac{\lambda_1}{r} - \lambda_1^2\lambda_2 rW^2\right) \\
&+ 2B\left(-\frac{\lambda_1}{W^3}M_2g_{rr} + \frac{W^2}{r}\lambda_2^2 - \frac{\lambda_1\lambda_2}{r} + 2\lambda_1^2\lambda_2^2 rW^2\right)\left(\frac{M_2}{W^3}g_{rr} - 3\lambda_1\lambda_2^2 rW^2\right) \\
&\geq 2B\left(\frac{-\lambda_1}{W^3}M_2g_{rr}\right)\frac{M_2}{W^3}g_{rr} + 2B\frac{W^2}{r}\lambda_2^2\frac{M_2}{W^3}g_{rr} + O\left(\frac{(\lambda_1+\lambda_2)^{\frac{5}{2}}}{r^2}\right) \\
&\geq \left(\frac{18\sqrt{2}Bc_0\lambda_2^{\frac{5}{2}}}{W^{\frac{3}{2}}(\sqrt{1-c_0})^5 r} + \frac{6\sqrt{2}BW^{\frac{1}{2}}\lambda_2^{\frac{5}{2}}}{r\sqrt{1-c_0}}\right)\frac{G}{r^N} + O\left(\frac{(\lambda_1+\lambda_2)^{\frac{5}{2}}}{r^2}\right).
\end{aligned}
\tag{8.40}
$$

Therefore,

$$
\begin{aligned}
&[(\lambda_1\lambda_{2rr}-\lambda_2\lambda_{1rr})(A\lambda_1+B\lambda_2)+2(\lambda_2\lambda_{1r}-\lambda_1\lambda_{2r})(A\lambda_{1r}+B\lambda_{2r})]g_r r^{N+1}\\
&\geq\left[-\frac{2Ng}{r^2W^3}\frac{c_0W^{\frac{3}{2}}\lambda_2^{\frac{3}{2}}}{r\sqrt{2}(\sqrt{1-c_0})^3}+O\left(\frac{(\lambda_1+\lambda_2)^2}{r^2}\right)\right](A\lambda_1+B\lambda_2)\frac{\lambda_2^{\frac{1}{2}}W^{\frac{3}{2}}}{\sqrt{2(1-c_0)}}r^{N+1}\\
&+\left[\left(\frac{18\sqrt{2}Bc_0\lambda_2^{\frac{5}{2}}}{W^{\frac{3}{2}}(\sqrt{1-c_0})^5r}+\frac{6\sqrt{2}BW^{\frac{1}{2}}\lambda_2^{\frac{5}{2}}}{r\sqrt{1-c_0}}\right)\frac{G}{r^N}+O\left(\frac{(\lambda_1+\lambda_2)^{\frac{5}{2}}}{r^2}\right)\right]\frac{\lambda_2^{\frac{1}{2}}W^{\frac{3}{2}}}{\sqrt{2(1-c_0)}}r^{N+1}.
\end{aligned}
\tag{8.41}
$$

This implies

$$
g_r r^{N+1}\hat{F}_{rr}\geq C_1(n,k,\Sigma_0)G+C_2(n,k,\Sigma_0,N). \tag{8.42}
$$

Recalling (8.32) we conclude that at (r_0,t_0) we have

$$
G_t>r^{-2}\left\{\left[-(N+2)G+2g_rr^{N-1}\right]C_3+2\left(G-g_rr^{N-1}\right)C_4+C_1G+C_2\right\}, \tag{8.43}
$$

where C_3, C_4 only depends on n,k. Therefore, if we choose N large such that $(N+2)C_3>2C_4+C_1+1$, then we can see that G is bounded from below at this point. By a similar argument, we can also show that if the positive maximal value of G is achieved at an interior point, then G is bounded from above at this point. This completes the proof of Lemma 8.3. □

Remark 8.4. Similar to Remark 7.3, we point out that N can be selected to be $\max\{N_1+1,\frac{B}{A}(M_{k,2}-3M_{k,1})\}$. The terms in $C_1, C_2, -C_4$ are either positive or $O(\lambda_1)$. If g is small, $(N+2)C_3-2C_4-C_1>C_3$ and (8.43) implies G has a lower bound.

The following lemma is well known.

Lemma 8.5. *Assume on $[0,\delta]$, a function f satisfies $|f|<Cr$ and $r^N|f_r|<C$. Then $f\in C^{\frac{1}{N+1}}[0,\delta]$.*

Proof. It's easy to see that f is $C^{0,1}$ at points away from $r=0$. Now, for $0\leq r_1<r_2<\delta$, if $|r_2-r_1|>\frac{1}{2}r_2^{N+1}$, then

$$
\frac{|f(r_2)-f(r_1)|}{|r_2-r_1|^{\frac{1}{N+1}}}\leq C\frac{|f(r_1)|+|f(r_2)|}{r_2}\leq\frac{Cr_1+Cr_2}{r_2}\leq C.
$$

If $|r_2-r_1|<\frac{1}{2}r_2^{N+1}$, then $r_1>\frac{1}{2}r_2$ and so $|r_2-r_1|<Cr_1^{N+1}$. We have

$$
\frac{|f(r_2)-f(r_1)|}{|r_2-r_1|^{\frac{1}{N+1}}}\leq\frac{\int_{r_1}^{r_2}|f_r(r)|dr}{|r_2-r_1|^{\frac{1}{N+1}}}\leq\frac{|r_2-r_1|\cdot C|r_1|^{-N}}{|r_2-r_1|^{\frac{1}{N+1}}}\leq C\frac{|r_2-r_1|^{\frac{N}{N+1}}}{|r_1|^N}\leq C.
$$

Therefore Lemma 8.5 is proved. □

Theorem 1.4 follows from Lemma 7.1, 7.2, 8.3, 8.5, and Remark 7.3, 8.4 immediately.

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Department of Mathematics, Fordham University, New York, NY 10023
E-mail address: xjiang77@fordham.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06268
E-mail address: ling.2.xiao@uconn.edu