ENTIRE SELF-EXPANDERS FOR POWER OF σ_k CURVATURE FLOW IN MINKOWSKI SPACE

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ABSTRACT. In [19], we prove that if an entire, spacelike, convex hypersurface $\mathcal{M}_{u_0}$ has bounded principal curvatures, then the $\sigma_k^{1/\alpha}$ (power of σ_k) curvature flow starting from $\mathcal{M}_{u_0}$ admits a smooth convex solution u for $t>0$. Moreover, after rescaling, the flow converges to a convex self-expander $\tilde{\mathcal{M}}=\{(x,\tilde{u}(x))\mid x\in\mathbb{R}^n\}$ that satisfies $\sigma_k(\kappa[\tilde{\mathcal{M}}])=(-\langle X_0,\nu_0\rangle)^\alpha$. Unfortunately, the existence of self-expander for power of σ_k curvature flow in Minkowski space has not been studied before. In this paper, we fill the gap.

1. INTRODUCTION

Let $\mathbb{R}^{n,1}$ be the Minkowski space with the Lorentzian metric

$$ds^2=\sum_{i=1}^n dx_i^2-dx_{n+1}^2.$$

In this paper, we will devote ourselves to the study of spacelike hypersurfaces $\mathcal{M}$ with prescribed σ_k curvature in Minkowski space $\mathbb{R}^{n,1}$. Here, σ_k is the k-th elementary symmetric polynomial, i.e.,

$$\sigma_k(\kappa)=\sum_{1\leqslant i_1<\cdots<i_k\leqslant n}\kappa_{i_1}\cdots\kappa_{i_k}.$$

Any such hypersurface $\mathcal{M}$ can be written locally as a graph of a function $x_{n+1}=u(x), x\in\mathbb{R}^n$, satisfying the spacelike condition

$$|Du|<1. \tag{1.1}$$

More specifically, we will study self-similar solutions of flow by powers of the σ_k curvature. Namely, we are interested in entire, spacelike, convex hypersurfaces which move under σ_k curvature flows by homothety.

Let $X(\cdot,t)$ be a spacelike, strictly convex solution of

$$\frac{\partial X}{\partial t}(p,t)=\sigma_k^\beta(p,t)\nu(p,t) \tag{1.2}$$

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for some $\beta \in (0, \infty)$. If the hypersurfaces $\mathcal{M}(t)$ given by $X(\cdot, t)$ move homothetically, then $X(\cdot, t) = \phi(t)X_0$ for some positive function ϕ. Since the normal vector field is unchanged by homotheties, by taking the inner product of (1.2) with $\nu_0 = \nu(\cdot, t)$ we obtain

$$\phi' \langle X_0, \nu_0 \rangle = -\sigma_k^\beta(\kappa[X_0])\phi^{-k\beta},$$

where $\kappa[X_0] = (\kappa_1, \cdots, \kappa_n)$ is the principal curvatures of $\mathcal{M}_0$ at X_0. Therefore, we must have

$$\phi'(t)\phi^{k\beta}(t) = \lambda$$

and

$$\sigma_k^\beta = -\lambda \langle X_0, \nu_0 \rangle .$$

In this paper, we will consider the case when $\lambda > 0$, which we call *expanding solutions*. Through rescaling, we may also assume $\lambda = 1$.

Complete noncompact self similar solutions of curvature flows in Euclidean space have been studied intensively (for example [1, 6, 3, 9, 14, 15, 16]). However, in Minkowski space, there is no corresponding known result yet.

It is well-known that the hyperboloid is a self-expander. In [18], we have proved the rescaled convex curvature flows, including Gauss curvature flow, converge to the hyperboloid. Therefore, a natural question to ask is whether there exist self-expanders other than the hyperboloid? Moreover, if such self-expanders exist, can we construct some curvature flows such that their rescaled flows converge to these new self-expanders? In this paper and an upcoming paper [19], we give affirmative answers to both questions.

Now consider $\mathcal{M}_{u_0} = \{(x, u_0(x)) \mid x \in \mathbb{R}^n\}$, an entire, spacelike, convex hypersurface satisfying $u_0(x) - |x| \to \varphi\left(\frac{x}{|x|}\right)$ as $|x| \to \infty$. Here φ is an arbitrary C^2 function defined on $\mathbb{S}^{n-1}$. By translating $\mathcal{M}_{u_0}$ vertically we may also assume $\varphi\left(\frac{x}{|x|}\right) > 0$. In an upcoming paper [19], we prove that, if in addition $\mathcal{M}_{u_0}$ also has bounded principal curvatures, then the equation

$$\begin{cases} \dfrac{\partial X}{\partial t} = \sigma_k^{1/\alpha}\nu \\ X(x, 0) = \mathcal{M}_{u_0}, \end{cases}$$

where $\alpha \in (0, k]$, admits a smooth convex solution u for $t > 0$. Moreover, after rescaling the flow converges to a convex self-expander $\tilde{\mathcal{M}} = \{(x, \tilde{u}(x)) \mid x \in \mathbb{R}^n\}$ that satisfies

$$\sigma_k(\kappa[\tilde{\mathcal{M}}]) = (-\langle X, \nu \rangle)^\alpha \tag{1.3}$$

and

$$\tilde{u} - |x| \to \varphi\left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty. \tag{1.4}$$

Unfortunately, the existence of solutions of equations (1.3) and (1.4) have not been studied before. In this paper, we fill the gap and prove the following theorems.

Theorem 1. *Suppose φ is a positive C^2 function defined on $\mathbb{S}^{n-1}$, i.e., $\varphi \in C^2(\mathbb{S}^{n-1})$. Then there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying*

$$\sigma_n(\kappa[\mathcal{M}_u]) = (-\langle X, \nu\rangle)^\alpha \text{ for any } \alpha \in (0, n], \tag{1.5}$$

and

$$u(x) - |x| \to \varphi\left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty. \tag{1.6}$$

Remark 2. *Note that, unlike previously known results on spacelike hypersurfaces with prescribed Gauss curvature (see* [2, 8, 10, 13]*), the right hand side of* (1.3) *is unbounded. Therefore, the proof of Theorem 1 is different from earlier works, and we need to develop new techniques to prove it.*

Using the solution we obtained in Theorem 1 as a subsolution we can also prove

Theorem 3. *Suppose φ is a positive C^2 function defined on $\mathbb{S}^{n-1}$, i.e., $\varphi \in C^2(\mathbb{S}^{n-1})$. Then there exists a unique, entire, strictly convex, spacelike hypersurface $\mathcal{M}_u = \{(x, u(x)) \mid x \in \mathbb{R}^n\}$ satisfying*

$$\sigma_k(\kappa[\mathcal{M}_u]) = (-\langle X, \nu\rangle)^\alpha \text{ for any } \alpha \in (0, k], \tag{1.7}$$

and

$$u(x) - |x| \to \varphi\left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty, \tag{1.8}$$

where $k \leqslant n-1$.

The paper is organized as follows. In Section 2, we prove Theorem 1. In particular, we develop new techniques to prove the local estimates. In Section 3, combining the result we obtained in Section 2 with our ideas developed in [13] and [17], we prove Theorem 3. The arguments in this section are modifications of our arguments in [13] and [17].

2. GAUSS CURVATURE SELF-EXPANDER

In this section, we want to show there exists an entire, strictly spacelike, convex solution to the following equation

$$\sigma_n(\kappa[\mathcal{M}_u]) = (-\langle X, \nu\rangle)^\alpha, \ 0 < \alpha \leqslant n, \tag{2.1}$$

and

$$u(x) - |x| \to \varphi\left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty, \tag{2.2}$$

where φ is a positive function defined on $\mathbb{S}^{n-1}$. Let u^* be the Legendre transform of u. By Section 3 of [10] and Lemma 14 in [17], we know when u is a solution of (2.1) and (2.2), then u^* satisfies the following PDE,

$$\begin{cases} \sigma_n(D^2u^*) = \dfrac{w^{*\alpha-n-2}}{(-u^*)^\alpha} & \text{in } B_1 \\ u^* = \varphi^*(\xi) & \text{on } \partial B_1, \end{cases} \tag{2.3}$$

where $\varphi^* = -\varphi < 0$ on ∂B_1 and $w^* = \sqrt{1-|\xi|^2}$. Since (2.3) is a degenerate equation, we will study the following approximate problem instead

$$\begin{cases} \sigma_n(D^2u^*) = \dfrac{(1-s|\xi|^2)^{(\alpha-n-2)/2}}{(-u^*)^\alpha} & \text{in } B_1 \\ u^* = \varphi^*(\xi) & \text{on } \partial B_1, \end{cases} \tag{2.4}$$

where $0 < s < 1$.

Remark 4. *In this remark, we want to explain why α needs to be less than or equal to n. Note that here we want to construct an entire solution to equation* (2.1)*. This requires $|Du^*(\xi)| \to \infty$ as $\xi \to \partial B_1$. One can see that if u^* is a solution to* (2.3) *then we have*

$$\begin{aligned} &\int_{B_1} \det D^2u^* = \int_{Du^*(B_1)} 1 \\ &\sim \int_{B_1} (1-|\xi|^2)^{\frac{\alpha-n-2}{2}} \sim \int_{1/2}^1 r^{n-1}(1-r^2)^{\frac{\alpha-n-2}{2}} dr \sim \int_{1/2}^1 (1-r^2)^{\frac{\alpha-n-2}{2}} dr^2. \end{aligned}$$

Since $Du^*(B_1) = \mathbb{R}^n$, we know $\int_{1/2}^1 (1-r^2)^{\frac{\alpha-n-2}{2}} dr^2$ blows up, which implies $\alpha \leqslant n$.

2.1. **Solvability of equation** (2.4)**.** We will show there exists a solution u^{s*} of (2.4) for $0 < s < 1$. For our convenience, in the following, when there is no confusion, we will drop the superscript s and denote u^{s*} by u^*.

Lemma 5. *(**C^0 estimate for** u^*) Let u^* be the solution of* (2.4)*, then*

$$|u^*| < C, \tag{2.5}$$

where $C = C(|\varphi^*|_{C^0})$.

Proof. Let $-C_0 = \max_{\xi\in\partial B_1} \varphi^* < 0$, by the convexity of u^* we know that $-C_0 > u^*$ in B_1. On the other hand, [10] proves that there exists a solution $\underline{u}^*$ satisfies

$$\begin{cases} \sigma_n(D^2\underline{u}^*) = \dfrac{1}{K}(1-|\xi|^2)^{-\frac{n+2}{2}} & \text{in } B_1 \\ \underline{u}^* = \varphi^*(\xi) & \text{on } \partial B_1, \end{cases} \tag{2.6}$$

for any $K \in \mathbb{R}_+$.

Now let $K \leqslant C_0^\alpha$ in (2.6), then we have $\sigma_n(D^2\underline{u}^*) > \sigma_n(D^2u^*)$ in B_1, and $\underline{u}^* = u^*$ on ∂B_1. By the maximum principle we obtain

$$-C_0 > u^* > \underline{u}^*. \tag{2.7}$$

□

Following [4], we can obtain the C^1 and C^2 estimates for the solution of (2.4). Applying the method of continuity, we get the solvability of (2.4). Therefore, in the following, we will focus on establishing local estimates for u^{s*}.

2.2. **Local C^1 estimates.** This subsection contains two parts. In the first part, we will prove $(1 - s|\xi|^2)|Du^{s*}(\xi)| < C$, where C is independent of s and ξ. This estimate will be useful for obtaining local C^2 estimates in the next subsection. In the second part, we will show $|Du^{s^*}|(\xi) \to \infty$ as $s, |\xi| \to 1$. This is to illustrate that the Lendrengre transform of u^{s*}, denoted by u^s, converges to an entire solution of (1.3) as $s \to 1$.

2.2.1. *Local C^1 upper bound.* In this part we will show $(1 - s|\xi|^2)|Du^{s*}(\xi)| < C$, where C is a constant independent of s.

First, applying [4] we know that we can solve the following equation

$$\begin{cases} \sigma_n(D^2u^*) = \dfrac{1}{C_1} < \dfrac{1}{(-\min \underline{u}^*)^\alpha} & \text{in } B_1 \\ u^* = \varphi^*(\xi) & \text{on } \partial B_1. \end{cases} \tag{2.8}$$

We will denote the solution to (2.8) by u_0^*. It's clear that $u_0^* > u^{s*} > \underline{u}^*$, where $\underline{u}^*$ is the solution to (2.6).

Next, denote $h^s := 1 - s|\xi|^2$ and $V^s = |Du^{s*}|$, we prove

Lemma 6. *For any $s \in [\frac{1}{2}, 1)$, if $M^s := \max\limits_{\xi \in \bar{B}_1} h^s V^s e^{-u^{s*2}}$ is achieved in B_1, then $M^s \leqslant C$, where $C = C(|u^{s*}|_{C^0})$ is a constant independent of s.*

Proof. For our convenience, in this proof, we drop the superscript s from u^{s*}, h^s, and V^s. Consider $\phi = hVe^{-u^{*2}}$ and assume ϕ achieves its maximum at an interior point $\xi_0 \in B_1$. We may rotate the coordinate such that at ξ_0, we have $V = u_1^*$. Differentiating ϕ we get

$$\frac{2s\xi_i}{h} = \frac{u_{1i}^*}{V} - 2u^*u_i^*, \ 1 \leqslant i \leqslant n.$$

Therefore, when $i = 1$, we obtain

$$\frac{2s\xi_1}{h} = \frac{u_{11}^*}{V} - 2u^*V.$$

By the convexity of u^* we know that $u_{11}^* > 0$, which gives

$$\frac{2s\xi_1}{h} \geqslant 2|u^*|V.$$

This completes the proof of the Lemma 6. $\square$

Finally, we want to show h^sV^s is bounded on ∂B_1.

Lemma 7. *For any $s\in[\frac{1}{2},1)$, if $M^s:=\max\limits_{\xi\in\bar{B}_1}h^sV^se^{-u^{s*2}}$ is achieved in ∂B_1, then $M^s\leqslant C$, where $C>0$ is a constant independent of s.*

Proof. For our convenience, in this proof, we drop the superscript s from u^{s*}, h^s, and V^s. Let us consider the function $\underline{\psi}=-Kh^{1/2}+K(1-s)^{1/2}+u_0^*$. It is clear that, for any $\xi\in B_1$, without loss of generality, we may assume $\xi=(r,0,\cdots,0)$. A direct calculation yields at ξ we have

$$\underline{\psi}_{11}=Ksh^{-3/2}+(u_0^*)_{11},$$
$$\underline{\psi}_{ii}=Ksh^{-1/2}+(u_0^*)_{ii}\text{ for }i\geqslant 2,$$

and

$$\underline{\psi}_{ij}=(u_0^*)_{ij}\text{ for all other cases.}$$

Since u_0^* is strictly convex we get

$$\sigma_n(D^2\underline{\psi})>\sigma_n(D^2(-Kh^{1/2}+K(1-s)^{1/2}))=(Ks)^nh^{-\frac{n+2}{2}}.$$

Choosing $K=\frac{3}{[\min(-\varphi^*)]^{\alpha/n}}$, we can see that $\underline{\psi}$ is a subsolution of (2.4). Thus, on ∂B_1 we have

$$|Du^*|<|D\underline{\psi}|<\frac{C}{\sqrt{1-s}},\tag{2.9}$$

where $C>0$ is a constant independent of s. It is easy to see that (2.9) implies the Lemma. $\square$

Combining Lemma 6 and Lemma 7 we conclude

Lemma 8. *Let u^{s*} be the solution of (2.4) for $s\in[\frac{1}{2},1)$. Then there exists a constant $C>0$, such that $|Du^{s*}(\xi)|(1-s|\xi|^2)\leqslant C$. Here, C is a constant independent of s.*

2.2.2. *Local C^1 lower bound near ∂B_1.* In this part, we will show $|Du^{s^*}|(\xi)\to\infty$ as $s,|\xi|\to 1$.

In order to obtain local C^1 lower bounds, we will construct a supersolution $\bar{u}^{s*}$ to

$$\sigma_n(D^2u^*)=\frac{(1-s|\xi|^2)^{(\alpha-n-2)/2}}{(-u^*)^\alpha}\text{ in }B_1\tag{2.10}$$

for $\alpha\leqslant n$, which satisfies

$$|D\bar{u}^{s*}(\xi)|\to\infty\text{ as }s\to 1\text{ and }|\xi|\to 1.\tag{2.11}$$

In the following, we will restrict ourselves to the case when $s\in[1/2,1)$. Denote $h^s=1-s|\xi|^2$, then $h^s_i=-2s\xi_i$, and $h^s_{ij}=-2s\delta_{ij}$. Consider $g_1(h^s)=-h^s\log|\log h^s|$. By a straightforward calculation we get

$$g_1'=-\log|\log h^s|+\frac{1}{|\log h^s|},\tag{2.12}$$

and

$$g_1'' = \left(1 + \frac{1}{|\log h^s|}\right) \frac{1}{h^s |\log h^s|}. \tag{2.13}$$

Therefore, at any point $\xi \in B_1$ with $|\xi| = r$ we have

$$\begin{aligned} \det(D^2 g_1) = s^n &\left(2\log|\log h^s| - \frac{2}{|\log h^s|}\right)^{n-1} \\ &\times \left[\left(2\log|\log h^s| - \frac{2}{|\log h^s|}\right) + 4sr^2\left(1+\frac{1}{|\log h^s|}\right)\frac{1}{h^s|\log h^s|}\right]. \end{aligned} \tag{2.14}$$

When $h^s < \delta_0$ for some fixed $\delta_0 > 0$ small, we have $\det(D^2 g_1) \leqslant \frac{C}{h^s}$ for some constant $C > 0$. Here, $C = C(\delta_0)$ is independent of s. On the other hand, when $h^s \geqslant \delta_0$, it's easy to see that $g_2 = \frac{|\xi|^2}{2} - \frac{1-\delta_0}{2s} - \delta_0 \log|\log \delta_0|$ satisfying

$$\begin{cases} \det(D^2 g_2) = 1 & \text{in } B_{\sqrt{\frac{1-\delta_0}{s}}} \\ g_2 = -\delta_0 \log|\log\delta_0| & \text{on } \partial B_{\sqrt{\frac{1-\delta_0}{s}}}. \end{cases} \tag{2.15}$$

Define

$$g = \begin{cases} g_1 & \text{for } h^s < \delta_0, \\ g_2 & \text{for } \delta_0 \leqslant h^s \leqslant 1, \end{cases}$$

then g is a continuous and convex function in B_1. By standard smoothing procedure, we can find a convex, rotationally symmetric function $\Phi \in C^2(B_1)$ such that

$$\Phi(g) = \begin{cases} g_1 & \text{for } h^s < \frac{\delta_0}{2}, \\ g_2 & \text{for } 2\delta_0 \leqslant h^s \leqslant 1. \end{cases}$$

We can see that for some suitable choice of $\rho > 0$, $\rho\Phi$ is a supersolution of (2.10) that satisfies (2.11). Here, $\rho > 0$ only depends on $|u^{s*}|_{C^0}$. From now on, ρ is a fixed constant and $\rho\Phi$ is a supersolution of (2.10). For our convenience, we will denote this supersolution by $\bar{u}^{s*}$.

We will denote the angular derivative $\xi_k \frac{\partial}{\partial \xi_l} - \xi_l \frac{\partial}{\partial \xi_k}$ on $\mathbb{S}^{n-1}(r)$ by $\partial_{k,l}$, or simply by ∂, when no confusion arises. Then following [10] we can prove following Lemmas

Lemma 9. *Let u^{s*} be the solution of* (2.4)*, then $|\partial u^{s*}|$ is bounded above by a constant $C_1 = C_1(|\varphi^*|_{C^1})$.*

Proof. For our convenience, in this proof, we drop the superscript s from u^{s*}. We take the logarithms of both sides of (2.4) and differentiate it with respect to ξ_k, then find

$$u^{*ij} u^*_{kij} = \frac{\partial}{\partial \xi_k}\left[\log(1 - s|\xi|^2) \cdot \frac{\alpha - n - 2}{2}\right] - \alpha \frac{\partial \log(-u^*)}{\partial \xi_k}.$$

This implies

$$\sum u^{*ij} (\partial u^*)_{ij} = -\alpha \frac{\partial(-u^*)}{-u^*} = \alpha \frac{\partial u^*}{-u^*}. \tag{2.16}$$

If ∂u^* achieves interior positive maximum, we would have $0 \geqslant \alpha\frac{\partial u^*}{-u^*} > 0$. This leads to a contradiction. Similarly, if ∂u^* achieves interior negative minimum, we would have $0 \leqslant \alpha\frac{\partial u^*}{-u^*} < 0$. This also leads to a contradiction. Therefore we conclude

$$|\partial u^*| \leqslant \max_{\partial B_1} |\partial \varphi^*|.$$

□

Lemma 10. *Let u^{s*} be the solution of* (2.4)*, then $\partial^2 u^{s*}$ is bounded above by a constant $C_2 = C_2(|\varphi^*|_{C^2})$.*

Proof. For our convenience, in this proof, we drop the superscript s from u^{s*}. We have shown

$$\sum u^{*ij}(\partial u^*)_{ij} = -\alpha\frac{\partial(-u^*)}{-u^*} = \alpha\frac{\partial u^*}{-u^*}.$$

Differentiating this equation once again we obtain

$$\sum u^{*ij}\partial[(\partial u^*)_{ij}] + \partial(u^{*ij})(\partial u^*)_{ij} = \frac{\alpha}{(-u^*)^2}(\partial u^*)^2 + \frac{\alpha}{(-u^*)}\partial^2 u^*. \tag{2.17}$$

Following the argument of Lemma 5.2 in [10], we get

$$\sum u^{*ij}[(\partial^2 u^*)_{ij}] \geqslant \frac{\alpha}{(-u^*)}(\partial^2 u^*). \tag{2.18}$$

Therefore, $\partial^2 u^*$ does not achieve positive maximum at interior points and we conclude

$$\partial^2 u^* \leqslant \max_{\partial B_1} |\partial^2 \varphi^*|.$$

□

Lemma 11. *Let $s \in [1/2, 1)$, $\sqrt{\frac{2-\delta_0}{2s}} < r < 1$, and $\mathbb{S}^{n-1}(r) = \{\xi \in \mathbb{R}^n \mid \sum \xi_i^2 = r^2\}$. For any point $\hat{\xi} \in \mathbb{S}^{n-1}(r)$ there exists a function*

$$\bar{u}^*_s = \bar{u}^{s*} + b_1\xi_1 + \cdots + b_n\xi_n + d$$

such that $\bar{u}^*_s(\hat{\xi}) = u^{s*}(\hat{\xi})$ and $\bar{u}^*_s(\xi) > u^{s*}(\xi)$ for any $\xi \in \mathbb{S}^{n-1}(r) \setminus \{\hat{\xi}\}$. Here, u^{s*} is a solution of (2.4)*, $\bar{u}^{s*}$ is the supersolution, $b_1, \cdots, b_n$ are bounded constants depending on $\hat{\xi}$, and d is a positive constant independent of $\hat{\xi}$ and r.*

Proof. By rotating the coordinate we may assume $\hat{\xi} = (r, 0, \cdots, 0)$. We choose $b_k = \frac{\partial u^{s*}}{\partial \xi_k}(r, 0, \cdots, 0)$, $k = 2, 3, \cdots, n$, and choose b_1 such that $u^{s*}(r, 0, \cdots, 0) = \bar{u}^{s*}(r, 0, \cdots, 0) + b_1 r + d$. To choose d we consider an arbitrary great circle $c(t)$ on $\mathbb{S}^{n-1}(r)$ passing through $\hat{\xi}$, for example the circle

$$\xi_1 = r\cos t,\ \xi_2 = r\sin t,\ -\pi \leqslant t \leqslant \pi,\ \xi_3 = \xi_4 = \cdots = \xi_n = 0.$$

Let

$$\begin{aligned} F(t) = (\bar{u}^*_s - u^{s*})|_{c(t)} &= \bar{u}^{s*}|_{c(t)} + b_1\xi_1 + \cdots + b_n\xi_n + d - u^{s*}|_{c(t)} \\ &= \bar{u}^{s*}|_{c(t)} + b_1 r\cos t + b_2 r\sin t + d - u^{s*}|_{c(t)}. \end{aligned}$$

Note that by our construction of $\bar{u}^{s*}$, when $\frac{2-\delta_0}{2s}<r<1$ we have

$$\bar{u}^{s*}|_{c(t)}=-\rho h^s\log|\log h^s||_{\{|\xi|=r\}}:=\bar{u}^{s*}(r).$$

Therefore, we get

$$F(t)=\bar{u}^{s*}(r)+[u^{s*}(r,0,\cdots,0)-\bar{u}^{s*}(r)-d]\cos t+b_2r\sin t+d-u^{s*}(t),$$

here and in the rest of this proof we denote $u^{s*}|_{c(t)}$ by $u^{s*}(t)$. It's clear that $F(0)=0$ and $\frac{dF}{dt}(0)=0$. We will look at the second derivative of F. Since

$$\frac{d^2F(t)}{dt^2}=[d+\bar{u}^{s*}(r)-u^{s*}(r,0,\cdots,0)]\cos t-b_2r\sin t-\frac{d^2u^{s*}}{dt^2},$$

when $-\frac{\pi}{4}\leqslant t\leqslant\frac{\pi}{4}$ we choose $d>u^{s*}(r,0,\cdots,0)-\bar{u}^{s*}(r)$ then we get

$$\begin{aligned}\frac{d^2F(t)}{dt^2}&\geqslant\frac{1}{\sqrt{2}}[d+\bar{u}^{s*}(r)-u^{s*}(r,0,\cdots,0)]-\left|\frac{du^{s*}}{dt}(0)\right|-\frac{d^2u^{s*}}{dt^2}\\&\geqslant\frac{1}{\sqrt{2}}[d+\bar{u}^{s*}(r)-u^{s*}(r,0,\cdots,0)]-C_3\end{aligned}$$

for some $C_3>0$ determined by Lemma 9 and 10. When $t\in[-\pi,-\frac{\pi}{4}]\cup[\frac{\pi}{4},\pi]$ we have

$$\begin{aligned}F(t)&=d(1-\cos t)+[u^{s*}(r,0\cdots,0)-\bar{u}^{s*}(r)]\cos t+b_2r\sin t-u^{s*}(t)+\bar{u}^{s*}(r)\\&\geqslant d\left(1-\frac{\sqrt{2}}{2}\right)-C_4\end{aligned}$$

By choosing $d>0$ sufficiently large we prove this lemma. □

Finally, we can prove

Lemma 12. *Let u^{s*} be the solution of* (2.4) *for $s\in[1/2,1)$. Then there exists $C=C(|\varphi^*|_{C^2})>0$, such that when $\sqrt{\frac{2-\delta_0}{2s}}\leqslant\sqrt{\frac{2-\delta_1}{2s}}<|\xi|<1$, we have*

$$\frac{|Du^{s*}(\xi)|}{\log|\log h^s|}\geqslant C.$$

Here, $\delta_1>0$ is a small constant.

Proof. When $\sqrt{\frac{2-\delta_0}{2s}}\leqslant\sqrt{\frac{2-\delta_1}{2s}}<r<1$, for any $\hat{\xi}\in\mathbb{S}^{n-1}(r)$ we assume $\hat{\xi}=(r,0,\cdots,0)$. By Lemma 11, there exists a supersolution of (2.4) $\bar{u}^*_s$, such that $\bar{u}^*_s(\hat{\xi})=u^{s*}(\hat{\xi})$ and $\bar{u}^*_s(\xi)>u^{s*}(\xi)$ for any $\xi\in\mathbb{S}^{n-1}(r)\setminus\{\hat{\xi}\}$. By the maximum principle we get $\bar{u}^*_s(\xi)>u^{s*}(\xi)$ in B_r. Hence at $\hat{\xi}$ we obtain

$$\frac{\partial u^{s*}}{\partial\xi_1}>\frac{\partial\bar{u}^*_s}{\partial\xi_1}=\frac{\partial\bar{u}^{s*}}{\partial\xi_1}+b_1.$$

Therefore, when $\delta_1>0$ is chosen to be small we complete the proof of this Lemma. □

2.3. **Local C^2 estimates.** Lemma 8 gives us local C^1 estimates for u^{s*}. In the following we will establish local C^2 estimates for the solution u^{s*} of equation (2.4). Comparing with usual local C^2 estimates, the complication here is as $s\to 1$ and $|\xi|\to 1$, by Lemma 12 we know that $|Du^{s*}(\xi)|\to\infty$. In other words, we don't have uniform C^1 estimates. Therefore, we need to introduce some new techniques to overcome this difficulty.

Let u_0^* be the solution of (2.8), denote $\eta^s := u_0^* - u^{s*}$ and $f^s = (-u^{s*})^{-\alpha}(1-s|\xi|^2)^{(\alpha-n-2)/2}$, we prove

Lemma 13. *Let u^{s*} be a solution of* (2.4) *for* $s\in[1/2,1)$. *Then we have*

$$\eta^s < C(h^s)^{m_\alpha}, \tag{2.19}$$

where $m_\alpha := \frac{n-2+\alpha}{2n}$, $h^s = 1-s|\xi|^2$, *and* $C = C(n,\alpha,\inf(-\varphi^*)) > 0$ *is a constant independent of* s.

Proof. For our convenience, in this proof, we will drop the superscript s on η^s, h^s, f^s, and u^{s*}. Let $\gamma = \frac{1}{m_\alpha}$, since $\alpha\in(0,n]$, it's clear that $\gamma>1$. Assume $\max\limits_{\xi\in B_1}\eta^\gamma h^{-1}$ is achieved at an interior point ξ_0. We may rotate the coordinate such that at this point $u^*_{ij} = u^*_{ii}\delta_{ij}$. Moreover, at ξ_0 we have

$$0 = \gamma\frac{\eta_i}{\eta} - \frac{h_i}{h},$$

and

$$\begin{aligned}0 &\geqslant \gamma\sigma_n^{ii}\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) - \frac{\sigma_n^{ii}h_{ii}}{h} + \sigma_n^{ii}\frac{h_i^2}{h^2}\\ &= \gamma\sigma_n^{ii}\frac{\eta_{ii}}{\eta} + (\gamma^2-\gamma)\sigma_n^{ii}\frac{\eta_i^2}{\eta^2} + 2s\frac{\sum\sigma_n^{ii}}{h}.\end{aligned}$$

Since u_0^* is convex, we get $\sigma_n^{ii}(u_0^*)_{ii} > 0$ and the above inequality becomes

$$0 \geqslant \frac{-n\gamma\sigma_n}{\eta} + 2s\frac{\sum\sigma_n^{ii}}{h}.$$

Recall that $\sum\sigma_n^{ii} = \sigma_{n-1} \geqslant c(n)\sigma_n^{\frac{n-1}{n}}$ and $s\in[1/2,1)$, we conclude

$$n\gamma \geqslant \frac{c(n)\eta}{h\sigma_n^{1/n}} \geqslant c_0\frac{\eta}{h^{1-\frac{n+2-\alpha}{2n}}} = c_0\frac{\eta}{h^{m_\alpha}},$$

where $c(n)>0$ is a constant depending on n and c_0 is a constant depending on $\inf(-u^*)$, n, and α. It's easy to see that $\inf(-u^*) = \inf(-\varphi^*)$. Therefore, we conclude that at ξ_0

$$\eta^\gamma h^{-1} \leqslant C,$$

where $C = C(n,\alpha,\inf(-\varphi^*))$ is independent of s. $\square$

Lemma 14. *Let u^{s*} be a solution of (2.4) for $s \in [1/2, 1)$. Then we have*

$$\max_{\xi\in B_1,\ \zeta\in\mathbb{S}^n} \eta^\beta u^{s*}_{\zeta\zeta} \leqslant C.$$

Here, $\beta = \frac{8}{m_\alpha}$ and C only depends on the C^0 estimates of u^{s*} and the local C^1 estimates we obtained in Lemma 8.

Proof. In this proof, for our convenience, we will drop the superscript s. We denote $h = 1 - s|\xi|^2$, then $h_i = -2s\xi_i$ and $h_{ij} = -2s\delta_{ij}$. We also note, differentiating $f = (-u^*)^{-\alpha} h^{\frac{\alpha-n-2}{2}}$ twice we get

$$f_i = f\left[\frac{\alpha u^*_i}{-u^*} + \frac{(\alpha-n-2)}{2}h^{-1}h_i\right]$$

and

$$\begin{aligned} f_{ii} = & f\left[\frac{\alpha u^*_i}{-u^*} + \frac{(\alpha-n-2)}{2}h^{-1}h_i\right]^2 \\ & + f\left[\frac{\alpha u^*_{ii}}{-u^*} + \frac{\alpha u^{*2}_i}{u^{*2}} - \frac{(\alpha-n-2)}{2}h^{-2}h_i^2 + \frac{(\alpha-n-2)}{2}h^{-1}h_{ii}\right]. \end{aligned}$$

Moreover, applying Lemma 8 we may assume

$$h^2|Du^*|^2 < \mathfrak{m}_0 \text{ and } h^4|Du^*|^2 < \mathfrak{m}_0,$$

for some positive constant $\mathfrak{m}_0 > 1$. Let $g = h^4|Du^*|^2$ and differentiate g twice, we get

$$g_i = 4h^3h_i|Du^*|^2 + 2h^4\sum_k u^*_k u^*_{ki}, \tag{2.20}$$

and

$$\begin{aligned} g_{ii} = & 12h^2h_i^2|Du^*|^2 + 4h^3|Du^*|^2h_{ii} + 16h^3\sum_k h_i u^*_k u^*_{ki} \\ & + 2h^4\sum_k u^{*2}_{ki} + 2h^4\sum_k u^*_k u^*_{kii}. \end{aligned} \tag{2.21}$$

Now we consider $\phi = \frac{\eta^\beta u^*_{\zeta\zeta}}{1-\frac{g}{M}}$, where $\beta > 0$, $M > 2\mathfrak{m}_0$ are some constants to be determined, and $\zeta \in \mathbb{S}^n$ is some direction. Suppose

$$\hat{M} := \max_{\xi\in B_1,\ \zeta\in\mathbb{S}^n} \phi$$

is achieved at an interior point $\xi_0 \in B_1$ in the direction of $\zeta_0 \in \mathbb{S}^n$. We may choose a local orthonormal frame $\{e_1, \cdots, e_n\}$ at ξ_0, such that $u^*_{ij}(\xi_0)$ is diagonal and we also assume $\zeta_0 = e_1$. Then at ξ_0 we have

$$\log\phi = \beta\log\eta - \log\left(1 - \frac{g}{M}\right) + \log u^*_{11}.$$

Differentiating $\log\phi$ twice we get

$$0 = \frac{\phi_i}{\phi} = \frac{\beta\eta_i}{\eta} + \frac{g_i}{M-g} + \frac{u^*_{11i}}{u^*_{11}} \tag{2.22}$$

and

$$0 \geqslant \sigma_n^{ii}\left[\frac{\beta\eta_{ii}}{\eta} - \frac{\beta\eta_i^2}{\eta^2} + \frac{g_{ii}}{M-g} + \frac{g_i^2}{(M-g)^2} + \frac{u^*_{11ii}}{u^*_{11}} - \left(\frac{u^*_{11i}}{u^*_{11}}\right)^2\right]. \tag{2.23}$$

By (2.22) we can see that when $i=1$ we have

$$\left(\frac{u^*_{111}}{u^*_{11}}\right)^2 = \left(\frac{\beta\eta_1}{\eta} + \frac{g_1}{M-g}\right)^2 \leqslant \frac{2\beta^2\eta_1^2}{\eta^2} + \frac{2g_1^2}{(M-g)^2}. \tag{2.24}$$

When $i \geqslant 2$

$$\beta\left(\frac{\eta_i}{\eta}\right)^2 = \frac{1}{\beta}\left(\frac{g_i}{M-g} + \frac{u^*_{11i}}{u^*_{11}}\right)^2 \leqslant \frac{2g_i^2}{\beta(M-g)^2} + \frac{2}{\beta}\left(\frac{u^*_{11i}}{u^*_{11}}\right)^2. \tag{2.25}$$

Note also that

$$\sum_k \sigma_n^{ii}u_k^*u^*_{kii} = \sum_k u_k^* f_k = f\left(\frac{\alpha|Du^*|^2}{-u^*} + \frac{(\alpha-n-2)}{2}h^{-1}\sum_k h_k u_k^*\right).$$

Therefore,

$$\begin{aligned}\sigma_n^{ii}g_{ii} \geqslant\ & 12h^2|Du^*|^2\sigma_n^{ii}\xi_i^2 - 8h\mathfrak{m}_0\sum\sigma_n^{ii} - 32h^2\sqrt{\mathfrak{m}_0}n\sigma_n \\ &+ 2h^4\sigma_n\sigma_1 + 2h^4\sigma_n\left(\frac{\alpha|Du^*|^2}{-u^*} + \frac{(\alpha-n-2)}{2}h^{-1}\sum_k h_k u_k^*\right),\end{aligned} \tag{2.26}$$

and

$$\sigma_n^{11}g_1^2 \leqslant \sigma_n^{11}\left(32h^6h_1^2|Du^*|^4 + 8h^8u_1^{*2}u_{11}^{*2}\right) < 128\mathfrak{m}_0h^4|Du^*|^2\sigma_n^{11}\xi_1^2 + 8h^6\mathfrak{m}_0\sigma_n u^*_{11}. \tag{2.27}$$

Combining (2.26) and (2.27) we obtain

$$\begin{aligned}&\frac{\sigma_n^{ii}g_{ii}}{M-g} - \frac{\sigma_n^{11}g_1^2}{(M-g)^2} \\ &\geqslant \frac{1}{(M-g)^2}\big\{(M-g)[12h^2|Du^*|^2\sigma_n^{ii}\xi_i^2 - 36h^2\sqrt{\mathfrak{m}_0}n\sigma_n - 8h\mathfrak{m}_0\sum\sigma_n^{ii} + 2h^4\sigma_n\sigma_1] \\ &- 128\mathfrak{m}_0h^4|Du^*|^2\sigma_n^{11}\xi_1^2 - 8h^6\mathfrak{m}_0\sigma_n u^*_{11}\big\}\end{aligned} \tag{2.28}$$

Choose $M = 13\mathfrak{m}_0 + N$ such that $M - g \geqslant 12\mathfrak{m}_0 + N$ then

$$\begin{aligned}&\frac{\sigma_n^{ii}g_{ii}}{M-g} - \frac{\sigma_n^{11}g_1^2}{(M-g)^2} \\ &\geqslant \frac{1}{(M-g)^2}[-(M-g)36h^2\sqrt{\mathfrak{m}_0}n\sigma_n - 8(M-g)h\mathfrak{m}_0\sum\sigma_n^{ii} + 2Nh^4\sigma_n\sigma_1],\end{aligned} \tag{2.29}$$

where we have used $\sigma_1 > u^*_{11}$. Differentiating $\sigma_n = f$ twice we get

$$\sigma_n^{ii}u^*_{11ii} + \sigma_n^{pq,rs}u^*_{pq1}u^*_{rs1} = f_{11}.$$

Thus,

$$\begin{aligned}\sigma_n^{ii}u_{11ii}^* &= f_{11} + \sum_{p\neq q}\sigma_n^{pp,qq}u_{pq1}^{*2} - \sum_{p\neq q}\sigma_n^{pp,qq}u_{pp1}^*u_{qq1}^* \\ &\geqslant f_{11} + 2\sum_{p=2}^{n}\frac{\sigma_n}{u_{pp}^*u_{11}^*}u_{11p}^{*2} - \frac{f_1^2}{f}.\end{aligned} \tag{2.30}$$

Notice that

$$\begin{aligned}&f_{11} - \frac{f_1^2}{f} \\ &= f\left[\frac{\alpha u_{11}^*}{-u^*} + \frac{\alpha u_1^{*2}}{u^{*2}} + \frac{(n+2-\alpha)}{2}h^{-2}h_1^2 + (n+2-\alpha)sh^{-1}\right] \\ &\geqslant C_5 u_{11}^* f,\end{aligned}$$

we conclude

$$\frac{\sigma_n^{ii}u_{11ii}^*}{u_{11}^*} \geqslant C_5 f + 2\sum_{p=2}^{n}\frac{\sigma_n^{pp}}{u_{11}^{*2}}u_{11p}^{*2}. \tag{2.31}$$

By a straightforward calculation we can see

$$\sigma_n^{ii}\eta_{ii} = \sigma_n^{ii}((u_0^*)_{ii} - u_{ii}^*) \geqslant C_6\sum\sigma_n^{ii} - n\sigma_n.$$

Combining (2.24), (2.25) with (2.23) we obtain

$$\begin{aligned}0 \geqslant\ &\sigma_n^{ii}\frac{\beta\eta_{ii}}{\eta} - \frac{(\beta+2\beta^2)\sigma_n^{11}\eta_1^2}{\eta^2} + \frac{\sigma_n^{ii}g_{ii}}{M-g} - \frac{\sigma_n^{11}g_1^2}{(M-g)^2} \\ &- \left(1+\frac{2}{\beta}\right)\sum_{i\geqslant 2}\sigma_n^{ii}\left(\frac{u_{11i}^*}{u_{11}^*}\right)^2 + \frac{\sigma_n^{ii}u_{11ii}^*}{u_{11}^*} + \left(1-\frac{2}{\beta}\right)\sum_{i\geqslant 2}\frac{\sigma_n^{ii}g_i^2}{(M-g)^2}.\end{aligned}$$

When $\beta \geqslant 2$, applying (2.29) and(2.31) we get

$$\begin{aligned}0 \geqslant\ &\frac{\beta}{\eta}(C_6\sigma_n^{ii} - n\sigma_n) - \frac{(\beta+2\beta^2)\sigma_n((u_0^*)_1 - u_1^*)^2}{\eta^2} \\ &- \frac{36h^2 n\sqrt{\mathfrak{m}_0}\sigma_n}{N} - \frac{8\mathfrak{m}_0 h\sum\sigma_n^{ii}}{N} + \frac{2Nh^4\sigma_n\sigma_1}{M^2} + C_5\sigma_n.\end{aligned} \tag{2.32}$$

Here we have used the fact that when $u_{11}^* \geqslant 1$ then $\sigma_n^{11} \leqslant \sigma_n$. We will choose N large such that $\frac{C_6\beta}{\eta} > \frac{8\mathfrak{m}_0}{N}$. Therefore, (2.32) becomes

$$0 \geqslant -\frac{n\beta}{\eta}\sigma_n - \frac{(\beta+2\beta^2)\sigma_n(C_6+|Du^*|)^2}{\eta^2} - C_7\sigma_n + \frac{2Nh^4\sigma_n u_{11}^*}{M^2}. \tag{2.33}$$

By Lemma 13 we know $\eta^{\frac{1}{m\alpha}} < Ch$, which gives $h > C\eta^{\frac{1}{m\alpha}}$. Thus, we have

$$\eta^{\frac{1}{m\alpha}}|Du^*| < C\mathfrak{m}_0.$$

Now, let $\beta = \frac{8}{m_\alpha} > 8$ and multiplying (2.33) by $\eta^{\frac{\beta}{2}}$, we obtain

$$0 \geqslant -n\beta\eta^{\frac{\beta}{2}-1} - (\beta + 2\beta^2)(C_6 + |Du^*|)^2\eta^{\frac{\beta}{2}-2} - C_7\eta^{\frac{\beta}{2}} + C_8 N\frac{\eta^\beta u^*_{11}}{M^2}.$$

Therefore, we conclude that $\eta^\beta u^*_{11} < C_9$ at its interior maximum point, which implies $\phi < 2C_9$. $\square$

Proof of Theorem 1. By subsection 2.1 we know there exists a solution u^{s*} of (2.4) for any $s \in (0, 1)$. Combining Lemma 5, 8, 14 with the classic regularity theorem, we know that there exists a subsequence of u^{s*} denoted by $\{u^{s_j*}\}_{j=1}^\infty$, converging locally smoothly to a convex function u^*, which satisfies (2.3). Here, $s_j \to 1$ as $j \to \infty$. Moreover, applying Lemma 12 and Lemma 14 of [17] we conclude, the Legendre transform of u^*, denoted by u, is the desired entire solution of (2.1) satisfying the asymptotic condition (2.2). This completes the proof of Theorem 1. $\square$

3. σ_k CURVATURE SELF-EXPANDER

In this section we will show that there exists an entire, strictly spacelike solution to the following equation

$$\sigma_k(\kappa[\mathcal{M}_u]) = (-\langle X, \nu\rangle)^\alpha, \tag{3.1}$$

and

$$u(x) - |x| \to \varphi\left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty, \tag{3.2}$$

where $0 < \alpha \leqslant k$ are constants. If u is a strictly convex solution satisfying (3.1) and (3.2), then subsection 2.3 and Lemma 14 of [17] imply its Legendre transform u^* satisfies

$$\begin{cases} \dfrac{\sigma_n}{\sigma_{n-k}}(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \left(\dfrac{\sqrt{1-|\xi|^2}}{-u^*}\right)^\alpha & \text{in } B_1 \\ u^* = \varphi^* & \text{on } \partial B_1, \end{cases} \tag{3.3}$$

where $\varphi^*(\xi) = -\varphi(\xi)$, $w^* = \sqrt{1-|\xi|^2}$, and $\gamma^*_{ij} = \delta_{ij} - \frac{\xi_i\xi_j}{1+w^*}$. By Section 2 we know there exists $\underline{u}$ such that

$$\sigma_n(\kappa[\mathcal{M}_{\underline{u}}]) = \frac{1}{\binom{n}{k}^{\frac{n}{k}}}(-\langle X, \nu\rangle)^{\frac{\alpha n}{k}},$$

and $\underline{u}(x) - |x| \to \varphi\left(\frac{x}{|x|}\right)$ as $|x| \to \infty$. Applying Maclaurin's inequality we obtain

$$\sigma_k(\kappa[\mathcal{M}_{\underline{u}}]) \geqslant (-\langle X, \nu\rangle)^\alpha.$$

We will denote the Lengendre transform of $\underline{u}$ by $\underline{u}^*$, then $\underline{u}^*$ satisfies

$$\begin{cases} \dfrac{\sigma_n}{\sigma_{n-k}}(w^*\gamma^*_{ik}\underline{u}^*_{kl}\gamma^*_{lj}) \leqslant \left(\dfrac{\sqrt{1-|\xi|^2}}{-\underline{u}^*}\right)^\alpha & \text{in } B_1, \\ \underline{u}^* = \varphi^* & \text{on } \partial B_1. \end{cases} \tag{3.4}$$

We will study the following approximate equation

$$\begin{cases} \dfrac{\sigma_n}{\sigma_{n-k}}(w^*\gamma^*_{ik}u^*_{kl}\gamma^*_{lj}) = \left(\dfrac{\sqrt{1-|\xi|^2}}{-u^*}\right)^\alpha & \text{in } B_r, \\ u^* = \underline{u}^* & \text{on } \partial B_r, \end{cases} \tag{3.5}$$

where $0 < r < 1$. In the following we denote $\Psi^* := \left(\frac{\sqrt{1-|\xi|^2}}{-\underline{u}^*}\right)^\alpha$, and we can see that as long as $-u^* > 0$ we have

$$\frac{\partial \Psi^*}{\partial u^*} = \alpha(\sqrt{1-|\xi|^2})^\alpha(-u^*)^{-\alpha-1} > 0.$$

This guarantees that the maximum principle holds for (3.5). Let $\bar{u}$ be a constant σ_k curvature hypersurface satisfying $\sigma_k(\kappa[\mathcal{M}_{\bar{u}}]) = C_0^\alpha$, where $0 > -C_0 = \max_{\xi\in\partial B_1} \varphi^*(\xi)$, $\bar{u}$ is strictly convex, and $\bar{u}(x) - |x| \to \varphi\left(\frac{x}{|x|}\right)$ as $|x| \to \infty$. We denote the Legendre transform of $\bar{u}$ by $\bar{u}^*$, then $\bar{u}^*$ satisfies

$$\begin{cases} \dfrac{\sigma_n}{\sigma_{n-k}}(w^*\gamma^*_{ik}\bar{u}^*_{kl}\gamma^*_{lj}) = \dfrac{1}{C_0^\alpha} \geqslant \left(\dfrac{\sqrt{1-|\xi|^2}}{-\underline{u}^*}\right)^\alpha & \text{in } B_1 \\ \bar{u}^* = \varphi^* & \text{on } \partial B_1. \end{cases} \tag{3.6}$$

By the maximal principle we know $\bar{u}^* < \underline{u}^*$ in B_1. Moreover, for any solution u^{r*} of (3.5), it is easy to see that

$$\bar{u}^* < u^{r*} < \underline{u}^* \text{ in } B_r.$$

Therefore, we conclude

Lemma 15. *Let u^{r*} be a solution of* (3.5) *and $\underline{u}^*, \bar{u}^*$ are constructed above. Then we have*

$$\bar{u}^* < u^{r*} < \underline{u}^* \textit{ in } B_r.$$

3.1. **Global a priori estimates.** In the subsection, we will prove a priori estimates that needed for the solvability of (3.5).

Lemma 16. *Let u^{r*} be a solution of* (3.5)*, then there exists $C > 0$ such that*

$$|Du^{r*}| < C.$$

Proof. By Section 2 of [5], we know that for any $0<r<1$, we can construct a subsolution $\underline{u}^{r*}$ such that

$$\frac{\sigma_n}{\sigma_{n-k}}(w^*\gamma^*_{ik}\underline{u}^{r*}_{kl}\gamma^*_{lj}) \geqslant \frac{1}{C_0^\alpha} \qquad \text{in } B_r$$
$$\underline{u}^{r*} = \underline{u}^* \qquad \text{on } \partial B_r.$$

Then by the convexity of u^{r*} we have

$$|Du^{r*}| \leqslant \max_{\partial B_r}|D\underline{u}^{r*}|.$$

□

Let $v=\langle X,\nu\rangle=\frac{x\cdot Du-u}{\sqrt{1-|Du|^2}}=\frac{u^*}{\sqrt{1-|Du|^2}}$. We will consider the hyperbolic model of (3.5) (seeing [17] for detail).

$$\begin{cases} F(v_{ij}-v\delta_{ij}) = (-v)^{-\alpha} & \text{in } U_r, \\ v = \dfrac{\underline{u}^*}{\sqrt{1-r^2}} & \text{on } \partial U_r, \end{cases} \tag{3.7}$$

where v_{ij} denotes the covariant derivative with respect to the hyperbolic metric, $U_r=P^{-1}(B_r)\subset\mathbb{H}^n(-1)$, $F(v_{ij}-v\delta_{ij})=\frac{\sigma_n}{\sigma_{n-k}}(\lambda[v_{ij}-v\delta_{ij}])$, and $\lambda[v_{ij}-v\delta_{ij}]=(\lambda_1,\cdots,\lambda_n)$ denotes the eigenvalues of $(v_{ij}-v\delta_{ij})$. Equations of the type (3.7) have been studied by Bo Guan in [7]. However, our function F is slightly different from the functions in [7]. More precisely, our function F doesn't satisfy the assumption (1.7) in [7]. We want to point out that the only place that assumption (1.7) is needed in [7] is in the proof of Lemma 6.2. Therefore, in order to apply [7] and obtain the C^2 boundary estimate we only need to find a replacement of Lemma 6.2 in [7] as follows.

Lemma 17. *There exist some uniformly positive constants $B,\delta,\epsilon>0$ such that*

$$h=(v-\underline{v})+B\left(\frac{1}{\sqrt{1-r^2}}-x_{n+1}\right)$$

satisfying $\mathfrak{L}h\leqslant -a(1+\sum_i F^{ii})$ in $U_{r\delta}$ and $h\geqslant 0$ on $\partial U_{r\delta}$. Here $a>0$ is some positive constant, $\underline{v}=\frac{\underline{u}^{r*}}{\sqrt{1-|\xi|^2}}$ is a subsolution, $\mathfrak{L}f:=F^{ij}\nabla_{ij}f-f\sum_i F^{ii}$, and $U_{r\delta}:=\left\{x\in U_r \mid \frac{1}{\sqrt{1-r^2}}-x_{n+1}<\delta\right\}$.

The above Lemma is the Lemma 27 of [12] and its proof can be found there. Now following the argument in [7], we obtain a C^2 boundary estimate for u^{r*}. So far, we have obtained the C^0, C^1, and C^2 boundary estimates for the solution of (3.5). To prove the solvability of (3.5), we only need to obtain the C^2 global estimates. We consider

$$\hat{F}=\left(\frac{\sigma_n}{\sigma_{n-k}}\right)^{\frac{1}{k}}(\Lambda_{ij})=(-v)^{-\frac{\alpha}{k}}:=\tilde{\Psi}, \tag{3.8}$$

where $\Lambda_{ij}=v_{ij}-v\delta_{ij}$.

Lemma 18. *Let v be the solution of (3.8) in a bounded domain $U \subset \mathbb{H}^n$. Denote the eigenvalues of $(v_{ij} - v\delta_{ij})$ by $\lambda[v_{ij} - v\delta_{ij}] = (\lambda_1, \cdots, \lambda_n)$. Then*

$$\lambda_{\max} \leqslant \max\{C, \lambda|_{\partial U}\},$$

and C is a positive constant only depending on U and $\tilde{\Psi}$.

Proof. The proof of this Lemma is a modification of the proof of Lemma 18 in [17]. Details on calculations in hyperbolic space can be found in subsection 2.2 of [17].

Set $M = \max\limits_{p \in \bar{U}} \max\limits_{|\xi|=1, \xi \in T_p\mathbb{H}^n} (\log \Lambda_{\xi\xi}) + N x_{n+1}$, where x_{n+1} is the coordinate function. Without loss of generality, we may assume M is achieved at an interior point $p_0 \in U$ for some direction ξ_0. Choose an orthonormal frame $\{e_1, \cdots, e_n\}$ around p_0 such that $e_1(p_0) = \xi_0$ and $\Lambda_{ij}(p_0) = \lambda_i \delta_{ij}$. Now, lets consider the test function

$$\phi = \log \Lambda_{11} + N x_{n+1}.$$

At its maximum point p_0, we have

$$0 = \phi_i = \frac{\Lambda_{11i}}{\Lambda_{11}} + N(x_{n+1})_i, \tag{3.9}$$

and

$$0 \geqslant \hat{F}^{ii}\phi_{ii} = \frac{\hat{F}^{ii}\Lambda_{11ii}}{\Lambda_{11}} - \hat{F}^{ii}\left(\frac{\Lambda_{11i}}{\Lambda_{11}}\right)^2 + N(x_{n+1})\sum_i \hat{F}^{ii} \tag{3.10}$$

Since $\Lambda_{11ii} = \Lambda_{ii11} + \Lambda_{ii} - \Lambda_{11}$ and

$$\hat{F}_{11} = \hat{F}^{ii}\Lambda_{ii11} + \hat{F}^{pq,rs}\Lambda_{pq1}\Lambda_{rs1} = \tilde{\Psi}_{11},$$

we get

$$\begin{aligned} \hat{F}^{ii}\Lambda_{11ii} &= \hat{F}^{ii}\Lambda_{ii11} + \tilde{\Psi} - \Lambda_{11}\sum_i \hat{F}^{ii} \\ &= \tilde{\Psi}_{11} - \hat{F}^{pp,qq}\Lambda_{pp1}\Lambda_{qq1} - \sum_{p\neq q}\frac{\hat{F}^{pp} - \hat{F}^{qq}}{\lambda_p - \lambda_q}\Lambda_{pq1}^2 + \tilde{\Psi} - \Lambda_{11}\sum_i \hat{F}^{ii}. \end{aligned} \tag{3.11}$$

Since $\hat{F}$ is concave, combining (3.11) and (3.10) we have

$$\begin{aligned} 0 \geqslant{}& \frac{1}{\Lambda_{11}}\left\{\tilde{\Psi}_{11} + 2\sum_{i\geqslant 2}\frac{\hat{F}^{ii} - \hat{F}^{11}}{\lambda_1 - \lambda_i}\Lambda_{11i}^2 + \tilde{\Psi} - \Lambda_{11}\sum_i \hat{F}^{ii}\right\} \\ &- \frac{\hat{F}^{ii}\Lambda_{11i}^2}{\Lambda_{11}^2} + N x_{n+1}\sum_i \hat{F}^{ii}. \end{aligned} \tag{3.12}$$

We need an explicit expression of $\hat{F}^{ii}$. A straightforward calculation gives

$$k\hat{F}^{k-1}\hat{F}^{ii} = \frac{\sigma_{n-1}(\lambda|i)}{\sigma_{n-k}} - \frac{\sigma_n}{\sigma_{n-k}^2}\sigma_{n-k-1}(\lambda|i).$$

Since

$$\begin{aligned}&\sigma_{n-1}(\lambda|i)\sigma_{n-k}-\sigma_n\sigma_{n-k-1}(\lambda|i)\\&=\sigma_{n-1}(\lambda|i)[\lambda_i\sigma_{n-k-1}(\lambda|i)+\sigma_{n-k}(\lambda|i)]-\sigma_n\sigma_{n-k-1}(\lambda|i)\\&=\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i),\end{aligned}$$

we get

$$k\hat{F}^{k-1}\hat{F}^{ii}=\frac{\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\kappa|i)}{\sigma_{n-k}^2}.$$

Therefore, we have

$$\begin{aligned}&k\hat{F}^{k-1}(\hat{F}^{ii}-\hat{F}^{11})\\&=\frac{1}{\sigma_{n-k}^2}[\sigma_{n-1}(\lambda|i)\sigma_{n-k}(\lambda|i)-\sigma_{n-1}(\lambda|1)\sigma_{n-k}(\lambda|1)]\\&=\frac{1}{\sigma_{n-k}^2}[\sigma_{n-2}(\lambda|i1)\lambda_1\sigma_{n-k}(\lambda|i)-\sigma_{n-2}(\lambda|1i)\lambda_i\sigma_{n-k}(\lambda|1)]\\&=\frac{\sigma_{n-2}(\lambda|1i)}{\sigma_{n-k}^2}[\lambda_1\sigma_{n-k}(\lambda|i)-\lambda_i\sigma_{n-k}(\lambda|1)]\\&=\frac{\sigma_{n-2}(\lambda|1i)(\lambda_1-\lambda_i)}{\sigma_{n-k}^2}[(\lambda_1+\lambda_i)\sigma_{n-k-1}(\lambda|1i)+\sigma_{n-k}(\lambda|1i)]\end{aligned}$$

When $i\geqslant 2$ we can see that

$$\begin{aligned}&k\hat{F}^{k-1}\left(\frac{\hat{F}^{ii}-\hat{F}^{11}}{\lambda_1-\lambda_i}-\frac{\hat{F}^{ii}}{\lambda_1}\right)\\&=\frac{\sigma_{n-2}(\lambda|1i)}{\sigma_{n-k}^2}[(\lambda_1+\lambda_i)\sigma_{n-k-1}(\lambda|1i)+\sigma_{n-k}(\lambda|1i)-\sigma_{n-k}(\lambda|i)]\\&=\frac{\sigma_{n-1}(\lambda|1)}{\sigma_{n-k}^2}\sigma_{n-k-1}(\lambda|1i)>0\end{aligned}$$

Thus, (3.12) can be reduced to

$$\begin{aligned}0&\geqslant\frac{1}{\Lambda_{11}}\tilde{\Psi}_{11}+(Nx_{n+1}-1)\sum_i\hat{F}^{ii}-\frac{\hat{F}^{11}\Lambda_{111}^2}{\Lambda_{11}^2}\\&=\frac{\tilde{\Psi}_{11}}{\Lambda_{11}}+(Nx_{n+1}-1)\sum_i\hat{F}^{ii}-\hat{F}^{11}N^2(x_{n+1})_1^2.\end{aligned}\tag{3.13}$$

Since $\tilde{\Psi}=(-v)^{-\frac{\alpha}{k}}$ and $-v=\frac{|u^*|}{\sqrt{1-|\xi|^2}}>\min_{\xi\in\bar{B}_r}|u^*|>\left|\max_{\xi\in\partial B_1}\varphi^*\right|>0$, a direct calculation yields

$$\begin{aligned}\tilde{\Psi}_{11}&=\frac{\alpha}{k}\left(\frac{\alpha}{k}+1\right)(-v)^{-\frac{\alpha}{k}-2}v_1^2+\frac{\alpha}{k}(-v)^{-\frac{\alpha}{k}-1}v_{11}\\&\geqslant\frac{\alpha}{k}(-v)^{-\frac{\alpha}{k}-1}(\lambda_1+v)\\&\geqslant C_1\lambda_1-C_2.\end{aligned}$$

Here, C_1 depends on U, since $-v \leqslant \frac{C}{\sqrt{1-|\xi|^2}}$. Plugging the above inequality into (3.13) we obtain

$$0 \geqslant C_1 - \frac{C_2}{\lambda_1} + (Nx_{n+1} - 1)\sum_i \hat{F}^{ii} - N^2(x_{n+1})_1^2 \frac{C_3}{\lambda_1}.$$

Here we have used

$$k\hat{F}^{k-1}\hat{F}^{11} = \frac{\sigma_n \sigma_{n-k}(\lambda|1)}{\lambda_1 \sigma_{n-k}^2} < \frac{1}{\lambda_1}\hat{F}^k \leqslant \frac{C_3}{\lambda_1},$$

where C_3 depends on U. Let $N = 2$ we can see that when λ_1 is large, we get an contradiction. This completes the proof of Lemma 18. $\square$

Therefore, we conclude that the approximate problem (3.5) is solvable.

3.2. **Local a priori estimates.** Let u^{r*} be the solution of (3.5), u_r be the Legendre transform of u^{r*}. In this section, we will study interior estimates of u_r, which will enable us to show there exists a subsequence of $\{u_r\}$ that converges to the desired entire solution u of (3.1).

Lemma 19. *(Lemma 5.1 of* [2]*) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $u, \bar{u}, \Psi : \Omega \to \mathbb{R}^n$ be strictly spacelike. Assume that u is strictly convex and $u < \bar{u}$ in Ω. Also assume that near $\partial\Omega$, we have $\Psi > \bar{u}$. Consider the set where $u > \Psi$. For every x in this set, we have the following gradient estimate for u:*

$$\frac{1}{\sqrt{1-|Du|^2}} \leqslant \frac{1}{u(x) - \Psi(x)} \cdot \sup_{\{u>\Psi\}} \frac{\bar{u} - \Psi}{\sqrt{1-|D\Psi|^2}}.$$

3.2.1. *Construction of* Ψ. In order to obtain the local C^1 estimate, we introduce a new subsolution $\underline{u}_1$ of (3.1), where $\underline{u}_1$ satisfies

$$\sigma_n(\kappa[\mathcal{M}_{\underline{u}_1}]) = 100\left(-\langle X, \nu\rangle\right)^{\frac{\alpha n}{k}},$$

and

$$\underline{u}_1(x) - |x| \to \varphi\left(\frac{x}{|x|}\right) \text{ as } |x| \to \infty.$$

Lemma 20. *Let $\underline{u}$ be a solution of*

$$\sigma_n(\kappa[\mathcal{M}_u]) = \frac{1}{\binom{n}{k}^{\frac{n}{k}}}\left(-\langle X, \nu\rangle\right)^{\frac{\alpha n}{k}}$$

satisfying $\underline{u}(x) - |x| \to \varphi\left(\frac{x}{|x|}\right)$ as $|x| \to \infty$, then $\underline{u}_1 < \underline{u}$.

Proof. We look at the Legendre transform of $\underline{u}_1$, denoted by $\underline{u}_1^*$. Then $\underline{u}_1^*$ satisfies

$$\sigma_n(w^* \gamma_{ik}^* (\underline{u}_1^*)_{kl} \gamma_{lj}^*) = \frac{1}{100}\left(\frac{\sqrt{1-|\xi|^2}}{-\underline{u}_1^*}\right)^{\frac{\alpha n}{k}};$$

while $\underline{u}^*$ satisfies

$$\sigma_n(w^*\gamma^*_{ik}(\underline{u}^*)_{kl}\gamma^*_{lj}) = \binom{n}{k}^{\frac{n}{k}} \left(\frac{\sqrt{1-|\xi|^2}}{-\underline{u}^*}\right)^{\frac{\alpha n}{k}}.$$

Moreover, $\underline{u}_1^* = \underline{u}^* = \varphi^*(\xi)$ on ∂B_1. Applying the maximal principle we conclude $\underline{u}_1^* > \underline{u}^*$ in B_1. Following the proof of Lemma 13 of [17] we get $\underline{u}_1 < \underline{u}$ in $\mathbb{R}^n$. □

Now, for any compact domain $K \subset \mathbb{R}^n$, let $2\delta = \min_K(\underline{u} - \underline{u}_1)$. We define $\Psi = \underline{u}_1 + \delta$. Denote $K' = \{x \in \mathbb{R}^n \mid \Psi \leqslant \bar{u}\}$, notice that as $|x| \to \infty$, we have $\underline{u}_1 - \bar{u} \to 0$, this implies K' is compact. Applying Lemma 19, for any (Ω_r, u^r), if $K' \subset \Omega_r$ we have

$$\sup_K \frac{1}{\sqrt{1-|Du^r|^2}} \leqslant \frac{1}{\delta} \sup_{K'} \frac{\bar{u} - \Psi}{\sqrt{1-|D\Psi|^2}}.$$

3.2.2. *Local C^2 estimates.* We will follow the proof of Lemma 24 in [17].

Lemma 21. *Let u^{r*} be the solution of* (3.5)*, u_r be the Legendre transform of u^{r*}, and $\Omega_r = Du^{r*}(B_r)$. For any giving $s > 1$, let $r_s > 0$ be a positive number such that when $r > r_s$, $u_r|_{\partial\Omega_r} > s$. Let $\kappa_{\max}(x)$ be the largest principal curvature of $\mathcal{M}_{u_r}$ at x, where $\mathcal{M}_{u_r} = \{(x, u_r(x))|x \in \Omega_r\}$. Then, for $r > r_s$ we have*

$$\max_{\{x\in\Omega_r|u_r\leqslant s\}} (s - u_r)\kappa_{\max} \leqslant C.$$

Here, C only depends on the C^0 and local C^1 estimates of u_r.

Proof. Consider the test function

$$\phi = m\log(s-u) + \log P_m - mN\langle \nu, E\rangle, \tag{3.14}$$

where $P_m = \sum_j \kappa_j^m$, $E = (0, \cdots, 0, 1)$, and $N, m > 0$ are some undetermined constants. Assume that ϕ achieves its maximum value on $\mathcal{M}$ at some point x_0. We may choose a local orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that at x_0, $h_{ij} = \kappa_i\delta_{ij}$ and $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$. Differentiating ϕ twice at x_0 we have

$$\frac{\sum_j \kappa_j^{m-1}h_{jji}}{P_m} - Nh_{ii}\langle \tau_i, E\rangle + \frac{\langle \tau_i, E\rangle}{s-u} = 0, \tag{3.15}$$

and

$$\begin{aligned} 0 \geqslant{}& \frac{1}{P_m}\left[\sum_j \kappa_j^{m-1}h_{jjii} + (m-1)\sum_j \kappa_j^{m-2}h_{jji}^2 + \sum_{p\neq q}\frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q}h_{pqi}^2\right] \\ &- \frac{m}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 - Nh_{iil}\langle \tau_l, E\rangle - Nh_{ii}^2\langle \nu, E\rangle + \frac{h_{ii}\langle \nu, E\rangle}{s-u} - \frac{u_i^2}{(s-u)^2}. \end{aligned} \tag{3.16}$$

Denote $\hat{v} = -\langle X, \nu \rangle$ then

$$\hat{v}_j = -h_{jk}\langle X, \tau_k\rangle = -h_{jj}\langle X, \tau_j\rangle,$$

and

$$\begin{aligned}\hat{v}_{jj} &= -h_{jjk}\langle X, \tau_k\rangle - h_{jk}\langle \tau_j, \tau_k\rangle - h_{jk}^2\langle X, \nu\rangle\\ &= -h_{jjk}\langle X, \tau_k\rangle - h_{jj} - h_{jj}^2\langle X, \nu\rangle.\end{aligned}$$

Since $\sigma_k = \hat{v}^\alpha := G$, we can see that $\sigma_k^{ii}h_{iij} = G_j$ and $\sigma_k^{ii}h_{iijj} + \sigma_k^{pq,rs}h_{pqj}h_{rsj} = G_{jj}$. Recall also that in Minkowski space we have

$$h_{jjii} = h_{iijj} + h_{ii}^2 h_{jj} - h_{ii}h_{jj}^2,$$

thus (3.16) becomes

$$\begin{aligned}0 \geqslant\ & \frac{1}{P_m}\Bigg[\sum_j \kappa_j^{m-1}\sigma_k^{ii}(h_{iijj} + h_{ii}^2h_{jj} - h_{ii}h_{jj}^2)\\ & + (m-1)\sigma_k^{ii}\sum_j \kappa_j^{m-2}h_{jji}^2 + \sum_{p\neq q}\frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q}\sigma_k^{ii}h_{pqi}^2\Bigg]\\ & - \frac{m}{P_m^2}\sigma_k^{ii}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 - N\langle \nabla G, E\rangle - N\sigma_k^{ii}\kappa_i^2\langle \nu, E\rangle + \frac{kG\langle \nu, E\rangle}{s-u} - \frac{\sigma_k^{ii}u_i^2}{(s-u)^2}.\end{aligned} \tag{3.17}$$

This gives

$$\begin{aligned}0 \geqslant\ & \frac{1}{P_m}\Bigg\{\sum_j \kappa_j^{m-1}[G_{jj} - \sigma_k^{pq,rs}h_{pqj}h_{rsj} - kGh_{jj}^2]\\ & + (m-1)\sigma_k^{ii}\sum_j \kappa_j^{m-2}h_{jji}^2 + \sum_{p\neq q}\frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q}\sigma_k^{ii}h_{pqi}^2\Bigg\}\\ & - \frac{m}{P_m^2}\sigma_k^{ii}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 - N\langle \nabla G, E\rangle - N\sigma_k^{ii}\kappa_i^2\langle \nu, E\rangle + \frac{kG\langle \nu, E\rangle}{s-u} - \frac{\sigma_k^{ii}u_i^2}{(s-u)^2}.\end{aligned} \tag{3.18}$$

We denote $A_i = \frac{\kappa_i^{m-1}}{P_m}\left[K(\sigma_k)_i^2 - \sum_{p,q}\sigma_k^{pp,qq}h_{ppi}h_{qqi}\right]$, $B_i = 2\frac{\kappa_j^{m-1}}{P_m}\sum_j \sigma_k^{jj,ii}h_{jji}^2$, $C_i = \frac{m-1}{P_m}\sigma_k^{ii}\sum_j \kappa_j^{m-2}h_{jji}^2$, $D_i = \frac{2\sigma_k^{jj}}{P_m}\sum_{j\neq i}\frac{\kappa_j^{m-1}-\kappa_i^{m-1}}{\kappa_j - \kappa_i}h_{jji}^2$, and $E_i = \frac{m}{P_m^2}\sigma_k^{ii}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2$.

Here K is a sufficiently large positive constant. Then (3.18) can be reduced to

$$\begin{aligned}0 \geqslant & \sum_i (A_i + B_i + C_i + D_i - E_i) - \sum_i \frac{K\kappa_i^{m-1}(G_i)^2}{P_m} \\ & + \frac{\sum_j \kappa_j^{m-1} G_{jj}}{P_m} - N\langle \nabla G, E\rangle - N\sigma_k^{ii}\kappa_i^2 \langle \nu, E\rangle \\ & - \frac{\sum_j \kappa_j^{m+1}}{P_m} kG + \frac{kG\langle \nu, E\rangle}{s-u} - \frac{\sigma_k^{ii} u_i^2}{(s-u)^2}.\end{aligned} \tag{3.19}$$

A straightforward calculation shows

$$\begin{aligned}&\frac{\sum_j \kappa_j^{m-1} G_{jj}}{P_m} = \frac{\sum_j \kappa_j^{m-1}[\alpha(\alpha-1)\hat{v}^{\alpha-2}\hat{v}_j^2 + \alpha\hat{v}^{\alpha-1}\hat{v}_{jj}]}{P_m} \\ &= \alpha(\alpha-1)\hat{v}^{\alpha-2}\frac{\sum_j \kappa_j^{m-1}\hat{v}_j^2}{P_m} + \alpha\hat{v}^{\alpha-1}\frac{\sum_j \kappa_j^{m-1}\left(h_{jjl}\langle -X, \tau_l\rangle - \kappa_j + \kappa_j^2\hat{v}\right)}{P_m} \\ &= \alpha(\alpha-1)\hat{v}^{\alpha-2}\frac{\sum_j \kappa_j^{m-1}\hat{v}_j^2}{P_m} + \alpha\hat{v}^{\alpha-1}\frac{\sum_j \kappa_j^{m-1} h_{jjl}\langle -X, \tau_l\rangle}{P_m} \\ &- \alpha\hat{v}^{\alpha-1} + \alpha\hat{v}^{\alpha}\frac{\sum_j \kappa_j^{m+1}}{P_m}.\end{aligned} \tag{3.20}$$

Moreover, we have

$$\begin{aligned}&\frac{\alpha\hat{v}^{\alpha-1}\sum \kappa_j^{m-1} h_{jjl}\langle -X, \tau_l\rangle}{P_m} - N\langle \nabla G, E\rangle \\ &= \frac{\alpha\hat{v}^{\alpha-1}\sum \kappa_j^{m-1} h_{jjl}\langle -X, \tau_l\rangle}{P_m} - N\alpha\hat{v}^{\alpha-1}\hat{v}_l\langle \tau_l, E\rangle \\ &= \alpha\hat{v}^{\alpha-1}\left(\frac{\sum \kappa_j^{m-1} h_{jjl}\langle -X, \tau_l\rangle}{P_m} - N\kappa_l\langle X, \tau_l\rangle u_l\right) \\ &= \alpha\hat{v}^{\alpha-1}\sum \langle X, \tau_l\rangle\left(N\kappa_l u_l - \frac{u_l}{s-u} - N\kappa_l u_l\right) \\ &= -\frac{\alpha\hat{v}^{\alpha-1}\sum \langle X, \tau_l\rangle u_l}{s-u},\end{aligned} \tag{3.21}$$

where we have used (3.15). Combing (3.20), (3.21) with (3.19) we obtain

$$
\begin{aligned}
0 \geqslant & \sum_i (A_i + B_i + C_i + D_i - E_i) - \sum_i \frac{K\kappa_i^{m-1}(G_i)^2}{P_m} \\
& + \alpha(\alpha-1)\hat{v}^{\alpha-2}\frac{\sum_j \kappa_j^{m-1}\hat{v}_j^2}{P_m} - \alpha\hat{v}^{\alpha-1} + \alpha\hat{v}^{\alpha}\frac{\sum_j \kappa_j^{m+1}}{P_m} \\
& - \frac{\alpha\hat{v}^{\alpha-1}\sum_l \langle X, \tau_l\rangle u_l}{s-u} - N\sigma_k^{ii}\kappa_i^2\langle \nu, E\rangle \\
& - kG\frac{\sum_j \kappa_j^{m+1}}{P_m} + \frac{kG\langle \nu, E\rangle}{s-u} - \frac{\sigma_k^{ii}u_i^2}{(s-u)^2}.
\end{aligned} \tag{3.22}
$$

Recall that

$$
\langle X, X\rangle + \langle \nu, X\rangle^2 = \sum_i \langle X, \tau_i\rangle^2,
$$

we know $|\langle X, \tau_i\rangle|$ can be controlled by some constants depending on s and local C^1 estimates. Therefore, applying Lemma 8 and 9 of [11] we may assume

$$
\begin{aligned}
0 \geqslant & -C\kappa_1 + \sum_{i=2}^n \frac{\sigma_k^{ii}}{P_m^2}\left(\sum_j \kappa_j^{m-1}h_{jji}\right)^2 - \frac{C}{s-u} \\
& - N\sigma_k^{ii}\kappa_i^2\langle \nu, E\rangle + \frac{kG\langle \nu, E\rangle}{s-u} - \frac{\sigma_k^{ii}u_i^2}{(s-u)^2}.
\end{aligned} \tag{3.23}
$$

Now, for any fixed $i \geqslant 2$ by (3.15) we have

$$
\begin{aligned}
\frac{\sigma_k^{ii}u_i^2}{(s-u)^2} &= \sigma_k^{ii}\left[\frac{\sum \kappa_j^{m-1}h_{jji}}{P_m} + N\kappa_i u_i\right]^2 \\
&= \sigma_k^{ii}\left(\frac{\sum \kappa_j^{m-1}h_{jji}}{P_m}\right)^2 + 2N\sigma_k^{ii}\kappa_i u_i\left(-N\kappa_i u_i + \frac{u_i}{s-u}\right) + N^2\sigma_k^{ii}\kappa_i^2 u_i^2 \\
&= \sigma_k^{ii}\left(\frac{\sum \kappa_j^{m-1}h_{jji}}{P_m}\right)^2 - N^2\sigma_k^{ii}\kappa_i^2 u_i^2 + 2N\frac{\sigma_k^{ii}\kappa_i u_i^2}{s-u}
\end{aligned} \tag{3.24}
$$

Plugging (3.24) into (3.23) we get,

$$
\begin{aligned}
0 \geqslant & -C\kappa_1 - \frac{C}{s-u} - N\sigma_k^{ii}\kappa_i^2\langle \nu, E\rangle + \frac{kG\langle \nu, E\rangle}{s-u} \\
& - \frac{\sigma_k^{11}u_1^2}{(s-u)^2} + \sum_{i=2}^n N^2\sigma_k^{ii}\kappa_i^2 u_i^2 - 2N\sum_{i=2}^n \frac{\sigma_k^{ii}\kappa_i u_i^2}{s-u}
\end{aligned}
$$

Since there is some constant c_0 such that $\sigma_k^{11}\kappa_1 \geqslant c_0 > 0$, we have

$$0 \geqslant \left(-\frac{c_0 N \langle \nu, E\rangle}{2} - C\right)\kappa_1 - \frac{N}{2}\sigma_k^{11}\kappa_1^2 \langle \nu, E\rangle$$
$$-\sum_{i=2}^{n} \frac{2N\sigma_k u_i^2}{s-u} + \frac{kG\langle \nu, E\rangle - C}{s-u} - \frac{\sigma_k^{11} u_1^2}{(s-u)^2},$$

where we have used for any $1 \leqslant i \leqslant n$ (no summation), $\sigma_k = \sigma_k^{ii}\kappa_i + \sigma_k(\kappa|i) \geqslant \sigma_k^{ii}\kappa_i$. Moreover, it's clear that

$$\sum_{i=2}^{n} u_i^2 = \sum_{i=2}^{n} \langle \tau_i, E\rangle^2 < \frac{1}{1-|Du|^2} = \langle \nu, E\rangle^2 .$$

We conclude

$$\left(\frac{2NC}{s-u} + \frac{\sigma_k^{11}}{(s-u)^2}\right)\langle \nu, E\rangle^2 \geqslant \frac{Nc_0\kappa_1}{4}\langle -\nu, E\rangle + \frac{N}{2}\sigma_k^{11}\kappa_1^2 \langle -\nu, E\rangle .$$

This implies $(s-u)\kappa_1 \leqslant C$, where C depends on s and local C^1 estimates. Therefore, we obtain the desired Pogorelov type C^2 local estimates. □

Following the argument in subsection 6.4 of [17], we prove Theorem 3.

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