AN ISOPERIMETRIC TYPE INEQUALITY IN DE SITTER SPACE

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ABSTRACT. In this paper, we prove an optimal isoperimetric inequality for spacelike, compact, star-shaped, and 2-convex hypersurfaces in de Sitter space.

1. INTRODUCTION

Let $\mathbb{R}^{n+2}_1$ be the $(n+2)$- dimensional Minkowski space, that is, the real vector space $\mathbb{R}^{n+2}$ endowed with the Lorentz metric

$$\langle v, w\rangle = -v^0w^0 + \sum_{i=1}^{n+1} v^iw^i.$$

The one sheeted hyperboloid

$$\mathbb{S}^{n+1}_1 = \{y \in \mathbb{R}^{n+2}_1 : \langle y, y\rangle = 1, y^0 > 0\}$$

consisting of all unit spacelike vectors and equipped with the induced metric is called de Sitter space. It is a geodesically complete simply connected Lorentzian manifold with constant curvature one. We say a hypersurface $M \subset \mathbb{S}^{n+1}_1$ is spacelike if its induced metric is Riemannian.

Let $\mathbb{S}^n$ be the standard round sphere. Then de Sitter space may be parametrised by $Y : \mathbb{S}^n \times \mathbb{R} \to \mathbb{S}^{n+1}_1$ as follows:

$$Y(r, \xi) = \sinh(r)E_1 + \cosh(r)\xi.$$

In this coordinate system, the induced metric is

$$\bar{g} = -dr^2 + \phi^2(r)\sigma$$

where σ is the standard metric on $\mathbb{S}^n$ and $\phi = \cosh$. For a hypersurface $M \subset \mathbb{S}^{n+1}_1$, we define

$$u = -\left\langle \phi\frac{\partial}{\partial r}, \nu\right\rangle$$

to be the support function, where ν is the future directed unit normal to M and $\langle\cdot,\cdot\rangle$ is the inner product with respect to $\bar{g}$.

In this paper, we prove an optimal isoperimetric inequality for spacelike, compact, star-shaped, and 2-convex hypersurfaces in de Sitter space. Before stating our main results, we need the following definition.

Definition 1.1. A C^2 regular hypersurface $M \subset \mathbb{S}^{n+1}_1$ is strictly k-convex, if the principal curvature vector of M at $X \in M$ satisfies $\kappa[X] \in \Gamma_k$ for all $X \in M$, where Γ_k is the Garding's cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k\}$$

and σ_j is the j-th elementary symmetric polynomial. If the principal curvature vector of M at $X \in M$ satisfies $\kappa[X] \in \bar{\Gamma}_k$ for all $X \in M$, then we say M is k-convex.

Theorem 1.2. *Let $M_0 \subset \mathbb{S}^{n+1}_1$ be a spacelike, compact, star-shaped, and strictly 2-convex hypersurface. Then the solution to the following flow equation*

$$\begin{cases} X_t = \left(u - b_{n,2}\phi' \sigma_2^{-1/2}\right)\nu, \\ X_0 = M_0 \end{cases} \tag{1.1}$$

exists for all time, where $b_{n,2} = (\sigma_2(I))^{1/2} = \left[\frac{n(n-1)}{2}\right]^{1/2}$. Moreover, the flow hypersurfaces M_t converge smoothly to a radial coordinate slice as $t \to \infty$.

As a consequence we obtain

Corollary 1.3. *Let $M \subset \mathbb{S}^{n+1}_1$ be a spacelike, compact, star-shaped, and 2-convex hypersurface. Then there holds*

$$\int_M \sigma_2 d\mu_g - (n-1)|M| \leq \xi_{2,0}(|M|) \tag{1.2}$$

with equality is attained if and only if M is a radial coordinate slice. Here, $\xi_{2,0}$ is the associated monotonically increasing function for radial coordinate slices and $|M|$ denotes the surface area of M.

1.1. **Background and motivations.** The classical Minkowski inequality [16] states that: For a convex hypesurface $M \subset \mathbb{R}^{n+1}$ we have

$$\frac{1}{|\mathbb{S}^n|}\int_M \frac{H}{n} d\mu_g \geq \left(\frac{|M|}{|\mathbb{S}^n|}\right)^{\frac{n-1}{n}}, \tag{1.3}$$

with equality holds if and only if M is a sphere. Here, H is the mean curvature of M.

A natural question, raised by several authors (see [6, 11, 21] for example), is whether the Minkowski inequality stays true for larger classes of domains than just for convex ones.

By studying weak solutions of the inverse mean curvature flow in $\mathbb{R}^n$, Huisken-Ilmanen [12, 13] showed that the assumption that M is convex can be replaced by the assumption that M is outward-minimizing. In 2009, by studying a normalized inverse curvature flow, Guan-Li [8] proved (1.3) for the case when M is star-shaped and mean convex. Moreover, they also proved Alexandrov-Fenchel inequalities, which is a general form of Minkowski inequality, for star-shaped and k-convex hypersurfaces in Euclidean space.

There are analogous of Minkowski and Alexandrov-Fenchel inequalities in space form. In hyperbolic space, Wang-Xia [22] proved the Alexandrov-Fenchel type inequalities for horospherically convex hypersurfaces. Since then, many efforts have been carried out to weaken the condition on the convexity. In particular, for star-shaped and mean convex hypersurfaces, a Minkowski type inequality has been proved in [3, 20]; for star-shaped and 2-convex hypersurfaces, an Alexandrov-Fenchel type inequality that involving integral of the scalar curvature has been proved in [3, 15]. Due to technical reasons there are much less such results in sphere, even with the convexity assumption the Alexandrov-Fenchel type inequalities are still open. For difficulties in proving the Alexandrov-Fenchel type inequalities in sphere one may refer to the expository paper [5]. Some variants of Alexandrov-Fenchel type inequalities for convex hypersurfaces in sphere can be found in [10, 23]. In de Sitter space, the Alexandrov-Fenchel type inequalities for convex hypersurfaces were deduced through the well-known duality for strictly convex hypersurfaces of hyperbolic/de Sitter space in [1]; while a Minkowski type inequality for spacelike, compact, star-shaped, and mean-convex hypersurfaces was derived in [18].

1.2. **Outline.** In Section 2, we give basic notations and establish fundamental equations for geometric quantities in de Sitter space that will be used in later sections. In Section 3, we introduce the flow equation and prove the monotonicity properties for the quermassintegrals along the flow. In Section 4, we will establish a priori estimates for the flow equation (1.1) and prove that the flow (1.1) exists for all time. In Section 5, we show the flow converges to a radial coordinate slice. This completes the proof of Theorem 1.2 and Corollary 1.3.

2. Preliminary

In this section, we will collect some formulas and lemmas for k-th symmetric functions as well as hypersurfaces in $\mathbb{S}^{n+1}_1$.

2.1. **Elementary symmetric functions.** For any $k = 1, \cdots, n$, and $\lambda = (\lambda_1, \cdots, \lambda_n)$ the k-th elementary symmetric function is defined by

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k},$$

and we also define $\sigma_0 = 1$. In this paper we will denote $\sigma_k(\lambda|i)$ the symmetric function with $\lambda_i = 0$.

The following properties are well known.

Lemma 2.1. *Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, \cdots, n$, then*

$$\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \ \forall 1 \leq i \leq n,$$
$$\sum_i \lambda_i \sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda),$$

and

$$\sum_i \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda).$$

Lemma 2.2. *Let $\lambda \in \Gamma_k$ with $\lambda_1 \geq \cdots \geq \lambda_k \geq \cdots \geq \lambda_n$, then we have*

$$\sigma_{k-1}(\lambda|n) \geq \sigma_{k-1}(\lambda|n-1) \geq \cdots \geq \sigma_{k-1}(\lambda|1) > 0,$$
$$\lambda_1 \geq \cdots \geq \lambda_k > 0, \ \sigma_k(\lambda) \leq C_n^k \lambda_1 \cdots \lambda_k,$$
$$\sum_i \sigma_{k-1}(\lambda|i)\lambda_i^2 = \sigma_k \sigma_1 - (k+1)\sigma_{k+1},$$

where $C_n^k = \frac{n!}{k!(n-k)!}$.

The generalized Newton-Maclaurin inequality is as follows, which will be used all the time (see Lemma 2.10 in [19]).

Proposition 2.3. *For $\lambda \in \Gamma_k$, $k > l \geq 0$, $r > s \geq 0$, $k \geq r$, and $l \geq s$, we have*

$$\left[\frac{\sigma_k(\lambda)/C_n^k}{\sigma_l(\lambda)/C_n^l}\right]^{\frac{1}{k-l}} \leq \left[\frac{\sigma_r(\lambda)/C_n^r}{\sigma_s(\lambda)/C_n^s}\right]^{\frac{1}{r-s}}.$$

Moreover, the equality holds if and only if $\lambda = c(1, \cdots, 1)$ for some $c > 0$.

Let A be a symmetric matrix and $\lambda(A) = (\lambda_1, \cdots, \lambda_n)$ be the eigenvalue vector of A. Let F be the function defined by

$$F(A) = f(\lambda(A))$$

and denote

$$F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}, \ F^{pq,rs} = \frac{\partial^2 F}{\partial a_{pq} \partial a_{rs}}.$$

When A is diagonal, we have

$$F^{ij}(A) = f^i(\lambda(A))\delta_{ij}, \ \text{for } f^i = \tfrac{\partial f}{\partial \lambda_i}.$$

Furthermore, we also have

$$F^{ij}(A)a_{ij} = \sum_i f^i(\lambda(A))\lambda_i, \tag{2.1}$$

$$F^{ij}(A)a_{ik}a_{kj} = \sum_i f^i(\lambda(A))\lambda_i^2. \tag{2.2}$$

In particular, when $F(A) = \sigma_k(\lambda(A))$ and suppose A is diagonalized at p_0, then at p_0, we have

$$F^{ij}(A) = \sigma_k^{ij}(\lambda(A)) = \sigma_{k-1}(\lambda|i)\delta_{ij}, \tag{2.3}$$

$$F^{pq,rs}(A) = \sigma_k^{pq,rs}(\lambda(A)) = \begin{cases} \dfrac{\partial^2 \sigma_k}{\partial \lambda_p \partial \lambda_r}(\lambda) = \sigma_{k-2}(\lambda|pr), & p = q, r = s, p \neq r, \\ -\dfrac{\partial^2 \sigma_k}{\partial \lambda_p \partial \lambda_q}(\lambda) = -\sigma_{k-2}(\lambda|pq), & p = s, q = r, p \neq q, \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

In order to prove the long time existence of the flow (1.1) (see Section 4), we need the following concavity inequality for Hessian operator, which is proved by Siyuan Lu.

Lemma 2.4. *(Lemma 3.1 of* [17]*) Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \Gamma_k$ with $\lambda_1 \geq \cdots \geq \lambda_n$ and let $1 \leq l < k$. For any $\epsilon, \delta, \delta_0 \in (0, 1)$, there exists a constant $\delta' > 0$ depending only on $\epsilon, \delta, \delta_0, n, k$ and l such that if $\lambda_l \geq \delta\lambda_1$ and $\lambda_{l+1} \leq \delta'\lambda_1$, then we have*

$$-\sum_{p \neq q} \frac{\sigma_k^{pp,qq}\xi_p\xi_q}{\sigma_k} + \frac{(\sum_i \sigma_k^{ii}\xi_i)^2}{\sigma_k^2} \geq (1 - \epsilon)\frac{\xi_1^2}{\lambda_1^2} - \delta_0 \sum_{i > l} \frac{\sigma_k^{ii}\xi_i^2}{\lambda_1 \sigma_k},$$

where $\xi = (\xi_1, \cdots, \xi_n)$ is an arbitrary vector in $\mathbb{R}^n$.

Note that from the proof in [17], we can see that for fixed $\delta, \delta_0 \in (0, 1)$, $\delta' = \delta'(\epsilon, \delta, \delta_0, n, k) = O(\epsilon) > 0$ is a small constant.

2.2. **Star-shaped graph in $\mathbb{S}^{n+1}_1$.** Let $M = \{(\rho(\xi), \xi) : \xi \in \mathbb{S}^n\}$, we will use $\tilde{\nabla}$ to denote the standard covariant derivative for the metric σ on $\mathbb{S}^n$. Then the tangent space of the hypersurface at a point $Y \in M$ is spanned by

$$Y_j = \rho_j \partial_r + \partial_j, \text{ where } \partial_j := \tilde{\nabla}_{\xi_j},$$

and the induced metric on M is given by

$$g_{ij} = \langle Y_i, Y_j \rangle = -\rho_i \rho_j + \cosh^2(\rho)\sigma_{ij}.$$

M is spacelike if (g_{ij}) is positive-definite. A unit normal vector ν to M can be obtained by solving the equation $\langle Y_i, \nu \rangle = 0$ for $\forall 1 \leq i \leq n$. Thus we have

$$\nu = \frac{(\cosh\rho, \tilde{\nabla}\rho / \cosh\rho)}{\sqrt{\cosh^2(\rho) - |\tilde{\nabla}\rho|^2}},$$

here $|\tilde{\nabla}\rho|^2 = \sigma^{ij}\rho_i\rho_j$ and (σ^{ij}) is the inverse of (σ_{ij}). In the following, for our convenience we will denote $w := \sqrt{\cosh^2(\rho) - |\tilde{\nabla}\rho|^2}$, then the support function

$$u := -\left\langle \cosh\rho \frac{\partial}{\partial\rho}, \nu \right\rangle = \frac{\cosh^2(\rho)}{w}. \tag{2.5}$$

Moreover, by some routine calculations (for details see [4]) we get

$$g^{ij} = \frac{1}{\cosh^2(\rho)} \left(\sigma^{ij} + \frac{\rho^i \rho^j}{w^2} \right), \tag{2.6}$$

and

$$h_{ij} = \frac{\cosh\rho}{w} \left(\tilde{\nabla}_{ij}\rho - 2\rho_i\rho_j \tanh\rho + \sinh\rho\cosh\rho\sigma_{ij} \right), \tag{2.7}$$

where (g^{ij}) is the inverse of (g_{ij}), $\rho^i = \sigma^{il}\rho_l$, and h_{ij} is the second fundamental form of M.

2.3. **Hypersurfaces in $\mathbb{S}^{n+1}_1$.** In this paper, we will define

$$\Phi = -\int_0^r \cosh s ds = -\sinh r, \tag{2.8}$$

and

$$V = \cosh r \frac{\partial}{\partial r}. \tag{2.9}$$

We note that $\Phi = -\phi'$. Now, let $M \in \mathbb{S}^{n+1}_1$ be a spacelike hypersurface with induced metric g. We will use ∇ to denote the covariant derivative with respect to g. Then the following fundamental equations are well known:

$$\nabla_{\tau_i}\tau_j = h_{ij}\nu \qquad \text{Gauss formula,}$$
$$\nabla_{\tau_i}\nu = h_i^k\tau_k \qquad \text{Weingarten equation,}$$
$$h_{ijk} = h_{ikj} \qquad \text{Codazzi equation.}$$

Following the proof of Lemma 2.2 in [9] (see also equation (2.7) in [18]) we have

Lemma 2.5. *Let $M \subset \mathbb{S}_1^{n+1}$ be a spacelike, compact, connected hypersurfaces with induced metric g. Let Φ be defined as in* (2.8)*. Then $\Phi\mid_M$ satisfies,*

$$\nabla_{ij}\Phi = \phi'(\rho)g_{ij} - h_{ij}u, \tag{2.10}$$

where ∇ is the covariant derivative with respect to g, h_{ij} is the second fundamental form of M, and $u = -\langle V, \nu\rangle$ is the support function of M.

Next, following the proof of Lemma 2.6 in [9], we derive the gradient and hessian of the support function u under the induced metric g on M.

Lemma 2.6. *The support function u satisfies*

$$\nabla_i u = -h_i^k\nabla_k\Phi, \tag{2.11}$$

$$\nabla_{ij}u = -g^{kl}\nabla_k h_{ij}\nabla_l\Phi - \phi' h_{ij} + u h_i^k h_{kj}, \tag{2.12}$$

where $h_i^k = g^{kl}h_{li}$.

3. Curvature flow and monotonicity formula

In this paper, we consider hypersurface flows related to the quermassintegrals. Similar to [5], let $M = \partial\Omega$, set

$$\begin{aligned} \mathcal{A}_{-1} &= \mathrm{Vol}(\Omega),\ \mathcal{A}_0 = \int_M d\mu_g \\ \mathcal{A}_1 &= \int_M \sigma_1 d\mu_g - n\mathrm{Vol}(\Omega) \\ \mathcal{A}_m &= \int_M \sigma_m d\mu_g - \frac{n-m+1}{m-1}\mathcal{A}_{m-2}, \end{aligned} \tag{3.1}$$

where $2 \le m \le n$, g is the induced metric on M, and $d\mu_g$ is the associated volume element. Let M_t be a smooth family of spacelike, compact, connected hypersurfaces in $\mathbb{S}_1^{n+1}$ evolving along the flow

$$X_t = S\nu. \tag{3.2}$$

From [7] we get

$$\partial_t g_{ij} = 2Sh_{ij} \tag{3.3}$$

and

$$\partial_t \nu = \nabla S. \tag{3.4}$$

Moreover, by Lemma 3.1 of [18] we have

$$\partial_t h_i^j = \nabla^j \nabla_i S - S h_i^k h_k^j + S\delta_i^j. \tag{3.5}$$

By Lemma 3.2 of [18] we also have

$$\begin{aligned} \partial_t \mathcal{A}_{-1} &= \int_{M_t} S d\mu_g, \\ \partial_t \mathcal{A}_0 &= \int_{M_t} \sigma_1 S d\mu_g. \end{aligned} \tag{3.6}$$

Combining (3.5), (3.6) with an induction argument, we derive for $0 \leq l \leq n-1$

$$\partial_t \mathcal{A}_l = (l+1) \int_{M_t} S\sigma_{l+1} d\mu_g. \tag{3.7}$$

3.1. **A specific flow equation.** In order to obtain an isoperimetric type inequality, we will consider the following curvature flow

$$X_t = \left(u - b_{n,k}\phi' \sigma_k^{-1/k}\right) \nu, \tag{3.8}$$

where $b_{n,k} = \left(C_n^k\right)^{1/k}$ and $1 \leq k \leq n$. Then from now on, our normal velocity $S = u - b_{n,k}\phi'\sigma_k^{-1/k}$.

Recall the Hsiung-Minkowski identities (see (2.8) of [18] or (1.4) of [5])

$$\int_M u\sigma_{m+1} d\mu_g = C_{n,m} \int_M \phi' \sigma_m d\mu_g, \; 0 \leq m \leq n-1 \tag{3.9}$$

for $C_{n,m} = \frac{n-m}{m+1} = \frac{C_n^{m+1}}{C_n^m}$ we obtain the following lemma.

Lemma 3.1. *Let M_t be a smooth family of spacelike, compact, connected, strictly k-convex hypersurfaces in $\mathbb{S}_1^{n+1}$ evolving along the flow* (3.8)*. Then the surface area $\mathcal{A}_0$ is non-increasing and the quantity*

$$\mathcal{A}_k(M_t) = \begin{cases} \displaystyle\int_{M_t} \sigma_1 d\mu_g - n\mathit{Vol}(\Omega_t), \; k = 1 \\ \displaystyle\int_{M_t} \sigma_k d\mu_g - \frac{n-k+1}{k-1} \mathcal{A}_{k-2}(M_t), \; 2 \leq k \leq n-1 \end{cases}$$

is non-decreasing. Moreover, $\mathcal{A}_k$ is strictly increasing unless M_t is totally umbilic.

Proof. In view of (3.7) and (3.9) we get, along the flow (3.8)

$$
\begin{aligned}
\partial_t \mathcal{A}_0 &= \int_{M_t} u\sigma_1 - b_{n,k}\phi' \sigma_k^{-1/k}\sigma_1 d\mu_g \\
&= \int_{M_t} n\phi' - b_{n,k}\phi' \sigma_k^{-1/k}\sigma_1 d\mu_g.
\end{aligned} \tag{3.10}
$$

By the Newton-Maclaurin inequality (see Proposition 2.3) we know that when M_t is strictly k-convex,

$$\frac{\sigma_1}{n} \geq \frac{\sigma_k^{1/k}}{b_{n,k}}.$$

Thus,we conclude that $\partial_t \mathcal{A}_0 \leq 0$.

Similarly, we can also obtain for $1 \leq k \leq n-1$

$$
\begin{aligned}
\partial_t \mathcal{A}_k &= (k+1)\int_{M_t} \sigma_{k+1}(u - b_{n,k}\phi' \sigma_k^{-1/k}) d\mu_g \\
&= (k+1)\int_{M_t} C_{n,k}\phi'\sigma_k - b_{n,k}\phi' \sigma_k^{-1/k}\sigma_{k+1} d\mu_g.
\end{aligned} \tag{3.11}
$$

It's easy to see that at the point where $\sigma_{k+1} \leq 0$ we have $C_{n,k}\phi'\sigma_k - b_{n,k}\phi'\sigma_k^{-1/k}\sigma_{k+1} > 0$; while at the point where $\sigma_{k+1} > 0$, applying Newton-Maclaurin inequality we still have $C_{n,k}\phi'\sigma_k - b_{n,k}\phi'\sigma_k^{-1/k}\sigma_{k+1} \geq 0$. Moreover, the equality holds if and only if at this point the principal curvature vector of M_t is $\kappa = c(1, \cdots, 1)$ for some $c > 0$. Therefore, the lemma is proved. ☐

We want to point out that a straightforward calculation yields $\partial_t \mathcal{A}_n \equiv 0$. Hence, if one can prove the flow (3.8) moves an arbitrary k-convex hypersurface to a round sphere, then the following conjecture would turn into a theorem:

Conjecture 3.2. *Let $M \subset \mathbb{S}_1^{n+1}$ be a spacelike, compact, star-shaped, and k-convex hypersurface. Then there holds*

$$\mathcal{A}_k \leq \xi_{k,0}(\mathcal{A}_0),\ 1 \leq k \leq n \tag{3.12}$$

with equality holds if and only if M is a radial coordinate slice. Here, $\xi_{k,0}$ is the associated monotonically increasing function for radial coordinate slices.

Note that when $k = 1$ the above conjecture has been proved in [18]. In this paper, we solve the case when $k = 2$.

4. LONG TIME EXISTENCE OF (1.1)

In this section, we will establish a priori estimates for the flow equation (1.1) and prove the long time existence theory. For greater generality, we will start with the study of (3.8) instead. In the rest of this section, we will assume the initial hypersurface M_0 is spacelike, compact, star-shaped, and strictly k-convex. Then by the short time existence theorem we know there exists $T^* > 0$ such that the flow (3.8) has a unique solution M_t for $t \in [0, T^*)$. Moreover, the flow hypersurface M_t is also spacelike, compact, star-shaped, and strictly k-convex.

4.1. **Estimates up to first order.** In this subsection, we will derive the C^0 and C^1 estimates for the solution of (3.8).

Lemma 4.1. *Along the flow* (3.8) *there holds for all* $(\xi, t) \in \mathbb{S}^n \times (0, T^*)$ *we have*

$$\min_{\mathbb{S}^n} \rho(\cdot, 0) \leq \rho(\xi, t) \leq \max_{\mathbb{S}^n} \rho(\cdot, 0).$$

Proof. The proof is the same as the one in [18], for completeness, we include it here.

The radial function ρ satisfies

$$\rho_t = \left(u - b_{n,k}\phi' \sigma_k^{-1/k}\right) \frac{\cosh \rho}{w}.$$

At the critical point of ρ, we get

$$\tilde{\nabla}\rho = 0 \text{ and } w = \cosh \rho = u.$$

In view of (2.6) and (2.7) we can see that at the the critical point,

$$h^i_j = g^{ik} h_{kj} = \frac{1}{\cosh^2(\rho)} \left(\tilde{\nabla}_{ij}\rho + \sinh \rho \cosh \rho \delta_{ij}\right).$$

Therefore, we obtain that at the critical point

$$\rho_t = \cosh \rho - \frac{b_{n,k} \sinh \rho}{\sigma_k^{1/k}\left(\frac{\tilde{\nabla}_{ij}\rho}{\cosh^2(\rho)} + \frac{\sinh \rho}{\cosh \rho}\delta_{ij}\right)}.$$

Note that at the spacial maximal points of ρ we have $(\tilde{\nabla}_{ij}\rho) \leq 0$, which implies that

$$\sigma_k^{1/k}\left(\frac{\tilde{\nabla}_{ij}\rho}{\cosh^2(\rho)} + \tanh \rho \delta_{ij}\right) \leq b_{n,k} \tanh \rho.$$

Thus we have $\max \rho$ is non-increasing. Similarly, we can show that $\min \rho$ is non-decreasing. □

Define $L := \partial_t - b_{n,k}\phi' F^{-2} F^{ij}\nabla^j\nabla_i$ for $F = \sigma_k^{1/k}$ and $F^{ij} = \frac{\partial F}{\partial h_j^i}$, denote

$$\hat{L} := L + \langle V, \nabla \cdot \rangle,$$

we will use Lemma 2.5 and Lemma 2.6 to derive the evolution equations for Φ and u.

Lemma 4.2. *Along the flow* (3.8), Φ *and* u *evolve as follows*

$$\hat{L}\Phi = 2b_{n,k}F^{-1}\phi' u - b_{n,k}(\phi')^2 F^{-2}\sum f^i - \cosh^2(\rho), \tag{4.1}$$

and

$$\hat{L}u = \phi' u \left(1 - b_{n,k}F^{-2}\sum f^i\kappa_i^2\right) - b_{n,k}F^{-1}|\nabla\Phi|^2, \tag{4.2}$$

where $F(A) = f(\kappa[A])$ *and* $\sum f^i = \sum F^{ii}$.

Proof. Since the values of $\hat{L}\Phi$ and $\hat{L}u$ are independent of the choice of coordinates, we may always choose an orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that (h_{ij}) is diagonalized. Then at the point of consideration, we have $F^{ij} = f^i\delta_{ij}$ and $h_{ij} = \kappa_i\delta_{ij}$. In view of Lemma 2.5 we get

$$\begin{aligned}
\hat{L}\Phi &= \partial_t\Phi - b_{n,k}\phi' F^{-2}F^{ii}\nabla_i\nabla_i\Phi + \langle V, \nabla\Phi\rangle \\
&= \langle V, S\nu\rangle - b_{n,k}\phi' F^{-2}F^{ii}(\phi'\delta_{ii} - h_{ii}u) + |\nabla\Phi|^2 \\
&= -(u - b_{n,k}F^{-1}\phi')u - b_{n,k}(\phi')^2F^{-2}\sum f^i + b_{n,k}\phi' F^{-1}u + |\nabla\Phi|^2 \\
&= 2b_{n,k}F^{-1}\phi' u - b_{n,k}(\phi')^2F^{-2}\sum f^i - \cosh^2(\rho).
\end{aligned}$$

Here, we have used $\sum f^i\kappa_i = f$ and

$$\langle V, \nabla\Phi\rangle = \sum \langle V, \langle V, \tau_i\rangle \tau_i\rangle = |V|^2 + u^2 = u^2 - \cosh^2(\rho) = |\nabla\Phi|^2.$$

Similarly, by Lemma 2.6 and (3.4) we have

$$\begin{aligned}
\hat{L}u &= \partial_t u - b_{n,k}\phi' F^{-2}F^{ii}\nabla_i\nabla_i u + \langle V, \nabla u\rangle \\
&= \phi' S - \langle V, \nabla S\rangle + \langle V, \nabla u\rangle - b_{n,k}\phi' F^{-2}F^{ii}(-h_{iik}\nabla_k\Phi - \phi' h_{ii} + uh_i^l h_{li}) \\
&= \phi' S + b_{n,k}\left\langle V, (\nabla\phi')F^{-1}\right\rangle + b_{n,k}\left\langle V, \phi'\nabla F^{-1}\right\rangle + b_{n,k}\phi' F^{-2}\langle \nabla F, V\rangle \\
&+ b_{n,k}(\phi')^2F^{-1} - ub_{n,k}\phi' F^{-2}F^{ii}h_i^l h_{li} \\
&= \phi' u - b_{n,k}F^{-1}|\nabla\Phi|^2 - ub_{n,k}\phi' F^{-2}\sum f^i\kappa_i^2 \\
&= \phi' u\left(1 - b_{n,k}F^{-2}\sum f^i\kappa_i^2\right) - b_{n,k}F^{-1}|\nabla\Phi|^2.
\end{aligned}$$

$\square$

From Lemma 4.2 we obtain the C^1 estimate.

Lemma 4.3. *Along the flow* (3.8) *there holds for all* $(\xi,t)\in\mathbb{S}^n\times(0,T^*)$ *we have*

$$u(\xi,t)\le\max_{\mathbb{S}^n}u(\cdot,0).$$

Proof. Let $\kappa=(\kappa_1,\cdots,\kappa_n)$ be the principal curvature vector of M_t, then $\kappa\in\Gamma_k$. In view of Lemma 2.2 and Proposition 2.3 we get

$$\begin{aligned}\sum f^i\kappa_i^2&=\frac{1}{k}\sigma_k^{1/k-1}\sigma_{k-1}(\kappa|i)\kappa_i^2\\&=\frac{1}{k}\sigma_k^{1/k-1}[\sigma_k\sigma_1-(k+1)\sigma_{k+1}]\\&\ge\frac{1}{k}\sigma_k^{1/k}\left[\frac{n}{b_{n,k}}\sigma_k^{1/k}-(k+1)C_{n,k}\frac{\sigma_k^{1/k}}{b_{n,k}}\right].\end{aligned}$$

Therefore, we have

$$\sum f^i\kappa_i^2\ge\frac{f^2}{b_{n,k}}. \tag{4.3}$$

Combining with equation (4.2) we obtain, along the flow (3.8) $\hat{L}u\le 0$. Then the lemma follows from the standard maximum principle. □

Recall equation (2.5), we can see that Lemma 4.3 implies that along the flow (3.8), $\frac{|\tilde{\nabla}\rho|^2}{\cosh^2(\rho)}$ is uniformly bounded away from 1.

4.2. **Uniform bounds of F.** In this subsection, we will prove that F is uniformly bounded from below along the flow (3.8). However, due to technical reasons, to obtain the uniform upper bound of F we have to restrict ourselves to the case when $k=2$, that is, the flow (1.1).

Lemma 4.4. *Along the flow* (3.8) *there holds for all* $(\xi,t)\in\mathbb{S}^n\times(0,T^*)$ *we have*

$$F(\xi,t)\ge\min_{\mathbb{S}^n}F(\cdot,0).$$

Proof. At the critical point of F we may choose an orthonormal frame $\{\tau_1,\cdots,\tau_n\}$ such that $h_{ij}=\kappa_i\delta_{ij}$ is diagonalized. By virtue of (3.5) we have

$$\begin{aligned}\partial_tF=&F^{ii}\left\{u_{ii}+b_{n,k}F^{-1}\Phi_{ii}+2b_{n,k}F^{-1}F_i\left(\frac{\phi'}{F}\right)_i+b_{n,k}\phi'F^{-2}\nabla_{ii}F\right\}\\&-(u-b_{n,k}\phi'F^{-1})\sum f^i\kappa_i^2+(u-b_{n,k}\phi'F^{-1})\sum f^i.\end{aligned}$$

Applying Lemma 2.5 and Lemma 2.6 we obtain

(4.4)
$$\begin{aligned}\partial_t F = {}& b_{n,k}\phi' F^{-2}F^{ii}\nabla_{ii}F + F^{ii}(-h_{iil}\nabla_l\Phi - \phi' h_{ii} + uh_i^l h_{li}) + b_{n,k}F^{-1}F^{ii}(\phi'\delta_{ii} - h_{ii}u)\\ &+ 2b_{n,k}F^{-1}F^{ii}F_i\left(\frac{\phi'}{F}\right)_i - (u - b_{n,k}\phi' F^{-1})\sum f^i\kappa_i^2\\ &+ (u - b_{n,k}\phi' F^{-1})\sum f^i.\end{aligned}$$

Let $F_{\min}(t) = \min\limits_{\xi\in\mathbb{S}^n} F(\xi, t)$, then $F_{\min}$ satisfies

$$\begin{aligned}\frac{d}{dt}F_{\min} \geq {}& -\phi' F + u\sum f^i\kappa_i^2 + b_{n,k}F^{-1}\phi'\sum f^i - b_{n,k}u\\ &- u\sum f^i\kappa_i^2 + b_{n,k}\phi' F^{-1}\sum f^i\kappa_i^2 + u\sum f^i - b_{n,k}\phi' F^{-1}\sum f^i\\ = {}& u(\sum f^i - b_{n,k}) + \phi' F^{-1}\left(b_{n,k}\sum f^i\kappa_i^2 - F^2\right)\end{aligned}$$

By the concavity of F we get

$$\sum f^i \geq f(1,\cdots,1) = b_{n,k}. \tag{4.5}$$

In conjunction with inequality (4.3) we conclude

$$\frac{d}{dt}F_{\min} \geq 0,$$

which yields this lemma. □

In the proofs of Lemma 4.5 and Lemma 4.6 below, we will explicitly use the fact that $k = 2$. Therefore, from now on, we will restrict ourselves to the locally constrained inverse scalar curvature flow

$$\begin{cases} X_t = \left(u - b_{n,2}\phi'\sigma_2^{-1/2}\right)\nu,\\ X_0 = M_0.\end{cases} \tag{4.6}$$

Lemma 4.5. *Along the flow* (4.6) *there holds for all* $(\xi, t) \in \mathbb{S}^n \times (0, T^*)$ *we have*

$$F(\xi, t) \leq C,$$

where $C > 0$ *is a constant depending on* $M_0, n, \rho,$ *and* u.

Proof. Recall the flow equation (4.1) of Φ we have

$$\begin{aligned}\hat{L}\Phi &= 2b_{n,2}F^{-1}\phi' u - b_{n,2}(\phi')^2F^{-2}\sum f^i - \cosh^2(\rho)\\ &\leq -\left(u - \frac{b_{n,2}\phi'}{F}\right)^2 + u^2 - \cosh^2(\rho),\end{aligned}$$

where we have used $\sum f^i \geq b_{n,2}$. Moreover, by Lemma 4.3 we get

$$u^2 - \cosh^2(\rho) = \frac{\cosh^2(\rho)}{1 - |\tilde{\nabla}\rho|^2/\cosh^2(\rho)} - \cosh^2(\rho) \leq \beta_0 u^2,$$

for some $0 < \beta_0 = \beta_0(|\tilde{\nabla}\rho|/\cosh\rho) < 1$. Therefore, we can see that there exists some uniform constant $0 < \beta_1 = \beta_1(n, \beta_0) < 1$ such that whenever $F \geq C_0(n, \rho, u)$

$$\hat{L}\Phi \leq -\beta_1 u^2. \tag{4.7}$$

Here, $C_0(n, \rho, u)$ is a large constant that only depends on n, ρ, and u.

Now, consider $\Psi = \log F + \lambda u + \alpha\Phi$, where $\lambda, \alpha > 0$ to be determined. Assume Ψ achieves its maximum at an interior point $X_0 \in M_{t_0}$ for some $t_0 \in (0, T^*)$. Then at X_0 we can choose an orthonormal frame $\{\tau_1, \cdots, \tau_n\}$ such that $h_{ij} = \kappa_i\delta_{ij}$. We have, at X_0

$$\Psi_i = \frac{F_i}{F} + \lambda u_i + \alpha\Phi_i = 0 \tag{4.8}$$

and

$$\begin{aligned}
0 \leq \hat{L}\Psi &= \frac{\hat{L}F}{F} + b_{n,2}\phi' F^{-2} F^{ii}\left(\frac{F_i}{F}\right)^2 + \lambda\hat{L}u + \alpha\hat{L}\Phi \\
&\leq \frac{1}{F}\Big[u(\sum f^i - b_{n,2}) + \phi' F^{-1}(b_{n,2}\sum f^i\kappa_i^2 - F^2) \\
&\quad + 2b_{n,2}F^{-1}F^{ii}\left(\frac{\phi_i' F_i}{F} - \frac{\phi' F_i^2}{F^2}\right)\Big] + b_{n,2}\phi' F^{-4}F^{ii}F_i^2 \\
&\quad + \lambda\left(\phi' u - b_{n,2}\phi' u F^{-2}\sum f^i\kappa_i^2 - b_{n,2}F^{-1}|\nabla\Phi|^2\right) + \alpha\hat{L}\Phi,
\end{aligned} \tag{4.9}$$

where we have used equations (4.2) and (4.4). We can see that (4.9) implies

$$\begin{aligned}
0 \leq &\frac{u}{F}\sum f^i + \phi' F^{-2} b_{n,2}\sum f^i\kappa_i^2\,(1 - \lambda u) \\
&+ 2b_{n,2}F^{-3}F^{ii}\phi_i' F_i - b_{n,2}\phi' F^{-4}F^{ii}F_i^2 + \lambda\phi' u + \alpha\hat{L}\Phi.
\end{aligned} \tag{4.10}$$

Since

$$\begin{aligned}
2b_{n,2}F^{-3}F^{ii}\phi_i' F_i &\leq b_{n,2}\phi' F^{-4}F^{ii}F_i^2 + \frac{b_{n,2}}{\phi'}F^{-2}F^{ii}(\phi_i')^2 \\
&\leq b_{n,2}\phi' F^{-4}F^{ii}F_i^2 + \frac{b_{n,2}}{\phi'}F^{-2}|\nabla\Phi|^2\sum f^i,
\end{aligned}$$

if at the critical point $F \geq C_1 = C_1(u, n, \rho)$ very large, applying (4.7) we can see that (4.10) becomes

$$\begin{aligned}
0 \leq \hat{L}\Psi \leq &\sum f^i + \phi' F^{-2} b_{n,2}\sum f^i\kappa_i^2\,(1 - \lambda u) \\
&+ \lambda\phi' u - \alpha\beta_1 u^2.
\end{aligned} \tag{4.11}$$

From equation (2.3), Lemma 2.1, Lemma 2.2, and Proposition 2.3 we obtain

$$\sum f^i = \frac{1}{2}F^{-1}(n-1)\sigma_1 \tag{4.12}$$

and

$$\sum f^i\kappa_i^2 = \frac{1}{2}F\sigma_1 - \frac{3}{2}F^{-1}\sigma_3 \geq \frac{1}{n}\sigma_1 F. \tag{4.13}$$

Thus we may choose $\lambda = \lambda(\rho, n, u) > 0$ such that

$$\sum f^i + \phi' F^{-2} b_{n,2} \sum f^i \kappa_i^2 \left(1 - \lambda u\right) \leq -\sum f^i.$$

After we fix the value of λ, we can choose $\alpha = \alpha(\rho, u, \lambda, \beta_1) > 0$ such that

$$\lambda\phi' u - \alpha\beta_1 u^2 < 0.$$

We conclude that if $F > C_1$ large at X_0 we would have the right hand side of (4.11) is negative. This leads to a contradiction. Therefore, Ψ is uniformly bounded on $[0, T^*)$. $\square$

4.3. **Uniform bounds for principal curvatures.** In this subsection, we will show that the principal curvatures of M_t remain bounded along the flow (4.6). More precisely, we will prove the following lemma

Lemma 4.6. *Let M_t be the solution of* (4.6) *on* $[0, T^*)$, *then there exists a constant C depending on $M_0, n, \rho, u,$ and F such that*

$$|\kappa[M_t]| \leq C. \tag{4.14}$$

Proof. Let us consider the test function

$$G = \log \kappa_1 + \lambda\Phi,$$

where κ_1 is the largest principal curvature and $\lambda > 0$ is a large constant to be determined. Assume G achieves its maximum at an interior point $P_0 \in M_{t_0}$ for some $t_0 \in (0, T^*)$. Then at this point, we can choose an orthonormal frame such that $h_{ij} = \kappa_i\delta_{ij}$ is diagonalized. Without loss of generality, we may assume κ_1 has multiplicity m, i.e.,

$$\kappa_1 = \cdots = \kappa_m > \kappa_{m+1} \geq \cdots \geq \kappa_n.$$

By Lemma 5 in [2] (see also [17]) we have at P_0,

$$\delta_{kl}\kappa_{1i} = h_{kli}, \text{for } 1 \leq k, l \leq m, \tag{4.15}$$

and

$$\kappa_{1ii} \geq h_{11ii} + 2\sum_{p>m} \frac{h_{1pi}^2}{\kappa_1 - \kappa_p} \tag{4.16}$$

in the viscosity sense. We want to point out that (4.15) yields that when $m \geq 2$, $h_{11i} = 0$ for $1 < i \leq m$.

We will start with computing the evolution equation of κ_1 at P_0. Recall (3.5) we have

$$\begin{aligned}\partial_t h_i^i &= \nabla_{ii} S - S\kappa_i^2 + S \\ &= u_{ii} - b_{n,2}\phi'_{ii}F^{-1} + 2b_{n,2}\phi'_i F^{-2}F_i - 2b_{n,2}\phi' F^{-3}F_i^2 \\ &\quad + b_{n,2}\phi' F^{-2}\left(F^{kk}h_{kkii} + F^{pq,rs}h_{pqi}h_{rsi}\right) - S\kappa_i^2 + S.\end{aligned}$$

Combining with the following commuting formula in de Sitter space (for details see page 10 of [4])

$$h_{kkii} = h_{iikk} + \kappa_i^2\kappa_k - \kappa_i\delta_{kk} - \kappa_i\kappa_k^2 + \kappa_k\delta_{ii},$$

we deduce

$$\begin{aligned}\partial_t h_i^i = &(-h_{iik}\nabla_k\Phi - \phi'\kappa_i + u\kappa_i^2) - b_{n,2}F^{-1}(-\phi'\delta_{ii} + \kappa_i u) \\ &+ 2b_{n,2}F^{-2}\phi'_i F_i - 2b_{n,2}\phi' F^{-3}F_i^2 \\ &+ b_{n,2}\phi' F^{-2}F^{kk}(h_{iikk} + \kappa_i^2\kappa_k - \kappa_i\delta_{kk} - \kappa_i\kappa_k^2 + \kappa_k\delta_{ii}) \\ &- (u - b_{n,2}\phi' F^{-1})\kappa_i^2 + (u - b_{n,2}\phi' F^{-1}) \\ &+ b_{n,2}\phi' F^{-2}F^{pq,rs}h_{pqi}h_{rsi}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\hat{L}h_i^i = &\kappa_i\left(-\phi' - b_{n,2}F^{-1}u - b_{n,2}\phi' F^{-2}\sum f^k - b_{n,2}\phi' F^{-2}\sum f^k\kappa_k^2\right) \\ &+ \left(b_{n,2}\phi' F^{-1} + u\right) + 2b_{n,2}\phi' F^{-1}\kappa_i^2 \\ &+ \left(2b_{n,2}F^{-2}\phi'_i F_i - 2b_{n,2}\phi' F^{-3}F_i^2 + b_{n,2}\phi' F^{-2}F^{pq,rs}h_{pqi}h_{rsi}\right).\end{aligned} \tag{4.17}$$

From (4.16) we get, at P_0

$$\hat{L}\kappa_1 \leq \hat{L}h_1^1 - 2b_{n,2}\phi' F^{-2}\sum_i\sum_{p>m} F^{ii}\frac{h_{1pi}^2}{\kappa_1 - \kappa_p}.$$

By our assumption we have, at P_0

$$G_i = \frac{\kappa_{1i}}{\kappa_1} + \lambda\Phi_i = 0. \tag{4.18}$$

Moreover, in view of Lemma 4.2 and equation (4.17) we derive

$$
\begin{aligned}
0 \le \hat{L}G =& \frac{\hat{L}\kappa_1}{\kappa_1} + \frac{b_{n,2}\phi' F^{-2}F^{ii}}{\kappa_1^2} h_{11i}^2 + \lambda \hat{L}\Phi \\
\le& \frac{1}{\kappa_1}\Big[\kappa_1\Big(-\phi' - b_{n,2}F^{-1}u - b_{n,2}\phi' F^{-2}\sum f^k - b_{n,2}\phi' F^{-2}\sum f^k\kappa_k^2\Big) \\
&+ (b_{n,2}\phi' F^{-1} + u) + 2b_{n,2}\phi' F^{-1}\kappa_1^2 + \big(2b_{n,2}F^{-2}\phi'_1 F_1 - 2b_{n,2}\phi' F^{-3}F_1^2 \\
&+ b_{n,2}\phi' F^{-2}F^{pq,rs}h_{pq1}h_{rs1}\big) - 2b_{n,2}\phi' F^{-2}\sum_i\sum_{p>m}F^{ii}\frac{h_{1pi}^2}{\kappa_1-\kappa_p}\Big] \\
&+ \frac{b_{n,2}\phi' F^{-2}}{\kappa_1^2}F^{ii}h_{11i}^2 + \lambda\Big(2b_{n,2}F^{-1}\phi' u - b_{n,2}(\phi')^2F^{-2}\sum f^k - \cosh^2(\rho)\Big).
\end{aligned}
\tag{4.19}
$$

Since

$$
2b_{n,2}F^{-2}\phi'_1 F_1 \le \frac{1}{2}b_{n,2}\phi' F^{-3}F_1^2 + 2\frac{b_{n,2}F^{-1}(\phi'_1)^2}{\phi'}
$$

(4.19) becomes

$$
\begin{aligned}
0 \le \hat{L}G \le& \lambda C_1 + \Big(-b_{n,2}\phi' F^{-2}\sum f^k - b_{n,2}\phi' F^{-2}\sum f^k\kappa_k^2\Big) \\
&- \frac{3}{2\kappa_1}b_{n,2}\phi' F^{-3}F_1^2 + \frac{b_{n,2}}{\kappa_1}\phi' F^{-2}F^{pq,rs}h_{pq1}h_{rs1} \\
&+ 2b_{n,2}\phi' F^{-1}\kappa_1 - \frac{2b_{n,2}}{\kappa_1}\phi' F^{-2}\sum_i\sum_{p>m}F^{ii}\frac{h_{1pi}^2}{\kappa_1-\kappa_p} \\
&+ \frac{b_{n,2}\phi' F^{-2}}{\kappa_1^2}F^{ii}h_{11i}^2 - \lambda b_{n,2}(\phi')^2F^{-2}\sum f^k.
\end{aligned}
\tag{4.20}
$$

Here and in the rest of this proof, we will use $C_1, C_2, \cdots$ and $c_0, c_1, c_2, \cdots$ to denote universal positive constants that only depend on n, ρ, u, and F. Equation (4.20) yields

$$
\begin{aligned}
0 \le \hat{L}G \le& \lambda C_1 - \big(b_{n,2}\phi' F^{-2} + \lambda b_{n,2}(\phi')^2F^{-2}\big)\sum f^k \\
&- b_{n,2}\phi' F^{-2}\sum f^k\kappa_k^2 + 2b_{n,2}\phi' F^{-1}\kappa_1 \\
&+ \frac{b_{n,2}\phi'}{\kappa_1 F^2}\left(-\frac{3}{2}F^{-1}F_1^2 + F^{pq,rs}h_{pq1}h_{rs1} - 2\sum_i\sum_{p>m}\frac{F^{ii}h_{1pi}^2}{\kappa_1-\kappa_p} + \sum_i\frac{F^{ii}h_{11i}^2}{\kappa_1}\right).
\end{aligned}
\tag{4.21}
$$

By virtue of (2.4) we can see that

$$\begin{aligned}\sigma_2^{pq,rs}h_{pq1}h_{rs1} &= \sum_{p\neq q}\sigma_2^{pp,qq}h_{pp1}h_{qq1} + 2\sum_{p>q}\sigma_2^{pq,qp}h_{pq1}^2\\ &= \sum_{p\neq q}h_{pp1}h_{qq1} - 2\sum_{p>q}h_{pq1}^2\\ &\leq \sum_{p\neq q}h_{pp1}h_{qq1} - 2\sum_{p>m}h_{11p}^2\end{aligned}$$

Note that $F^2 = \sigma_2$, thus we have

$$2FF^{pq,rs}h_{pq1}h_{rs1} + 2(F_1)^2 \leq \sum_{p\neq q}h_{pp1}h_{qq1} - 2\sum_{p>m}h_{11p}^2.$$

This gives

$$F^{pq,rs}h_{pq1}h_{rs1} \leq \frac{1}{2}F^{-1}\left(\sum_{p\neq q}h_{pp1}h_{qq1} - 2\sum_{p>m}h_{11p}^2 - 2(F_1)^2\right). \tag{4.22}$$

Therefore, we get

$$\begin{aligned}&-\frac{3}{2}F^{-1}F_1^2 + F^{pq,rs}h_{pq1}h_{rs1} - 2\sum_i\sum_{p>m}\frac{F^{ii}h_{1pi}^2}{\kappa_1-\kappa_p} + \sum_i\frac{F^{ii}h_{11i}^2}{\kappa_1}\\ &\leq -\frac{5}{2}F^{-1}F_1^2 + \frac{1}{2}F^{-1}\sum_{p\neq q}h_{pp1}h_{qq1} - F^{-1}\sum_{p>m}h_{11p}^2 - 2\sum_{p>m}\frac{F^{11}h_{11p}^2}{\kappa_1-\kappa_p}\\ &-2\sum_{p>m}\frac{F^{pp}h_{pp1}^2}{\kappa_1-\kappa_p} + \sum_{p>m}\frac{F^{pp}h_{11p}^2}{\kappa_1} + \frac{F^{11}h_{111}^2}{\kappa_1},\end{aligned}$$

where we have used $h_{11i} = 0$ for $1 < i \leq m$. By a straightforward calculation we can see that for each fixed $p > m$,

$$\begin{aligned}&\left(-F^{-1} - 2\frac{F^{11}}{\kappa_1-\kappa_p} + \frac{F^{pp}}{\kappa_1}\right)h_{11p}^2\\ &= F^{-1}\left(-1 - \frac{\sigma_1-\kappa_1}{\kappa_1-\kappa_p} + \frac{\sigma_1-\kappa_p}{2\kappa_1}\right)h_{11p}^2\\ &= F^{-1}\frac{(\kappa_1+\kappa_p)(\kappa_p-\sigma_1)}{2(\kappa_1-\kappa_p)\kappa_1}h_{11p}^2 \leq 0,\end{aligned}$$

here we have used equality (2.3) and the first inequality in Lemma 2.2. Thus we conclude

$$\begin{aligned}
&-\frac{3}{2}F^{-1}F_1^2+F^{pq,rs}h_{pq1}h_{rs1}-2\sum_i\sum_{p>m}\frac{F^{ii}h_{1pi}^2}{\kappa_1-\kappa_p}+\sum_i\frac{F^{ii}h_{11i}^2}{\kappa_1}\\
&\le\frac{1}{2}F^{-1}\sum_{p\neq q}h_{pp1}h_{qq1}-\frac{5}{2}F^{-1}F_1^2-2\sum_{p>m}\frac{F^{pp}h_{pp1}^2}{\kappa_1-\kappa_p}+\frac{F^{11}h_{111}^2}{\kappa_1}.
\end{aligned}\tag{4.23}$$

Now let $\epsilon>0$ be a small constant that will be determined later, $\delta=\delta_0=\frac{1}{2}$, and let $\delta'=\delta'(\epsilon,\delta,\delta_0)=O(\epsilon)>0$ be a constant determined by Lemma 2.4. We will divide the rest of this proof into two cases.

Case 1. When $\kappa_2\le\delta'\kappa_1$ at P_0, by Lemma 2.4 we get

$$\sum_{p\neq q}\frac{h_{pp1}h_{qq1}}{\sigma_2}-\frac{(\sigma_2^{ii}h_{ii1})^2}{\sigma_2^2}\le(\epsilon-1)\frac{h_{111}^2}{\kappa_1^2}+\frac{1}{2}\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{\kappa_1\sigma_2}.\tag{4.24}$$

Moreover, since $\kappa\in\Gamma_2$ we have $\sigma_2^{11}=\sum_{i=2}^{n-1}\kappa_i+\kappa_n>0$, which implies

$$\kappa_n>-(n-2)\delta'\kappa_1.\tag{4.25}$$

When $\delta'>0$ small we obtain

$$2\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{\kappa_1-\kappa_p}>2\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{(1+(n-2)\delta')\kappa_1}>\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{\kappa_1}$$

Therefore, in this case (4.23) becomes

$$\begin{aligned}
&-\frac{3}{2}F^{-1}F_1^2+F^{pq,rs}h_{pq1}h_{rs1}-2\sum_i\sum_{p>1}\frac{F^{ii}h_{1pi}^2}{\kappa_1-\kappa_p}+\sum_i\frac{F^{ii}h_{11i}^2}{\kappa_1}\\
&\le\frac{1}{2}F^{-1}\left\{\sum_{p\neq q}h_{pp1}h_{qq1}-\frac{5}{4}\sigma_2^{-1}(\sigma_2^{ii}h_{ii1})^2-2\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{\kappa_1-\kappa_p}+\frac{\sigma_2^{11}h_{111}^2}{\kappa_1}\right\}\\
&\le\frac{1}{2}F^{-1}\left\{(\epsilon-1)\sigma_2\frac{h_{111}^2}{\kappa_1^2}+\frac{1}{2}\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{\kappa_1}-\frac{1}{4}\sigma_2^{-1}(\sigma_2)_1^2-2\sum_{p>1}\frac{\sigma_2^{pp}h_{pp1}^2}{\kappa_1-\kappa_p}+\frac{\sigma_2^{11}h_{111}^2}{\kappa_1}\right\}\\
&\le\frac{1}{2}F^{-1}\left[(\epsilon-1)\sigma_2+\sigma_2^{11}\kappa_1\right]\frac{h_{111}^2}{\kappa_1^2}
\end{aligned}\tag{4.26}$$

Plugging (4.26) into (4.21) and applying the first equality in Lemma 2.1 we obtain

$$
\begin{aligned}
0 \leq \hat{L}G \leq \lambda C_1 - &\left(b_{n,2}\phi' F^{-2} + \lambda b_{n,2}(\phi')^2 F^{-2}\right)\sum f^k \\
- b_{n,2}\phi' F^{-2}\sum f^k\kappa_k^2 &+ 2b_{n,2}\phi' F^{-1}\kappa_1 + \frac{b_{n,2}\phi'}{2F^3\kappa_1}[\epsilon\sigma_2 - \sigma_2(\kappa|1)]\frac{h_{111}^2}{\kappa_1^2}.
\end{aligned} \tag{4.27}
$$

In view of our assumption that $\kappa_2 \leq \delta'\kappa_1$, we know

$$|\sigma_2(\kappa|1)| \leq c_0(\delta'\kappa_1)^2.$$

Therefore, we have

$$
\begin{aligned}
0 \leq \hat{L}G \leq \lambda C_1 - &\left(b_{n,2}\phi' F^{-2} + \lambda b_{n,2}(\phi')^2 F^{-2}\right)\sum f^k \\
- b_{n,2}\phi' F^{-2}\sum f^k\kappa_k^2 &+ 2b_{n,2}\phi' F^{-1}\kappa_1 + \frac{b_{n,2}\phi'}{2F^3\kappa_1}[c_1\epsilon + c_0(\delta'\kappa_1)^2]\frac{h_{111}^2}{\kappa_1^2}.
\end{aligned} \tag{4.28}
$$

By (4.18) we get at P_0

$$\frac{h_{111}}{\kappa_1} = -\lambda\Phi_1 = -\lambda\langle V, \tau_1\rangle.$$

Combining with (4.28) and (4.12) yields

$$0 \leq \hat{L}G \leq \lambda C_1 + C_2(\delta')^2\lambda^2\kappa_1 + C_3\kappa_1 - \frac{n-1}{2}\lambda b_{n,2}(\phi')^2F^{-3}\sigma_1. \tag{4.29}$$

Without loss of generality, we will always assume $\delta' \leq \frac{1}{2n^2}$, then by (4.25) we have,

$$\sigma_1 > \kappa_1 + (n-1)\kappa_n > \frac{\kappa_1}{2}.$$

Therefore, (4.29) implies

$$0 \leq \hat{L}G \leq \lambda C_1 + C_2(\delta')^2\lambda^2\kappa_1 + C_3\kappa_1 - \frac{n-1}{4}\lambda b_{n,2}(\phi')^2F^{-3}\kappa_1. \tag{4.30}$$

We can choose $\lambda = \lambda(n, \rho, F, C_3) > 0$ large such that $\frac{n-1}{4}\lambda b_{n,2}(\phi')^2F^{-3} > 2C_3 + 1$, then choose $\delta' = \delta'(\lambda, C_2) > 0$ small (this can be achieved by choosing a small ϵ) such that $C_2(\delta')^2\lambda^2 < \frac{1}{2}$. Then if at P_0 we have $\kappa_1 > N_0 = N_0(\lambda, C_1) > 0$ large, the right hand side of (4.30) would be strictly negative. This leads to a contradiction.

Case 2. When $\kappa_2 \geq \delta'\kappa_1$ at P_0. By Lemma 4.4 and Lemma 4.5 we know $f\left(1, \frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_1}, \cdots, \frac{\kappa_n}{\kappa_1}\right) = O\left(\frac{1}{\kappa_1}\right)$. In view of our assumption that $\frac{\kappa_2}{\kappa_1} \geq \delta'$, we have

$$\frac{C_4}{\kappa_1} > f\left(1, \frac{\kappa_2}{\kappa_1}, \frac{\kappa_3}{\kappa_1}, \cdots, \frac{\kappa_n}{\kappa_1}\right) > \delta' + c_2\frac{\kappa_n}{\kappa_1}$$

for some $c_2 = c_2(n) > 0$. Therefore, if $\kappa_1 > N_1 = N_1(1/\delta', C_4)$ very large at P_0 , then there exists $\eta_0 = \eta_0(\delta', c_2) > 0$ such that $\frac{\kappa_n}{\kappa_1} \leq -\eta_0 < 0$. This implies

$$\sum f^k \kappa_k^2 \geq f^n \kappa_n^2 \geq \frac{1}{n}\left(\sum f^i\right) \eta_0^2 \kappa_1^2 \geq c_3 \kappa_1^2,$$

where $c_3 = c_3(n, \eta_0) > 0$. Combining with (4.21) we obtain

$$0 \leq \hat{L}G \leq \lambda C_1 - b_{n,2}\phi' F^{-2} c_3 \kappa_1^2 + 2b_{n,2}\phi' F^{-1}\kappa_1 + \frac{b_{n,2}\phi'}{F^2}\sum F^{ii}\lambda^2 \langle V, \tau_i\rangle^2, \tag{4.31}$$

where we have used (4.18) and the fact that $F = \sigma_2^{1/2}$ is concave. We can see that when $\kappa_1 > N_2 = N_2(N_1, n, \rho, u, F, \lambda, c_3) > 0$ very large

$$\lambda C_1 - b_{n,2}\phi' F^{-2} c_3 \kappa_1^2 + 2b_{n,2}\phi' F^{-1}\kappa_1 + \frac{b_{n,2}\phi'}{F^2}\sum F^{ii}\lambda^2 \langle V, \tau_i\rangle^2 < 0.$$

This leads to a contradiction. It follows that if G achieves its maximum at an interior point, then G is uniformly bounded. Therefore, the lemma is proved. □

So far we have obtained uniform C^2 bounds for the flow hypersurfaces M_t. This implies the uniform parabolicity of the operator L. Due to the concavity of the operator, we can apply the Krylov and Safonov regularity theorem (see [14]) to deduce $C^{2,\alpha}$ bounds. The C^∞ bounds follow from the Schauder theory. We conclude:

Proposition 4.7. *Let $M_0 \subset \mathbb{S}_1^{n+1}$ be a spacelike, compact, star-shaped, and strictly* 2*-convex hypersurface. Then the flow* (4.6) *exists for all time with uniform C^∞-estimates.*

5. Convergence and inequality

In this section, we complete the proof of the geometric inequality, that is, Corollary 1.3.

Proof of Corollary 1.3. Recall Lemma 3.1 we know that $\mathcal{A}_0$ is decreasing. By the C^2 estimates obtained in the subsection 4.3, we have $\mathcal{A}_2$ is uniformly bounded from above. Moreover, Lemma 3.1 also tells us that $\mathcal{A}_2$ is increasing. Therefore, we have

$$\int_0^\infty \partial_t \mathcal{A}_2 < \infty,$$

which implies that

$$\partial_t \mathcal{A}_2 = 3\int_{M_t} C_{n,2}\phi'\sigma_2 - b_{n,2}\phi'\sigma_2^{-1/2}\sigma_3 d\mu_g \to 0 \tag{5.1}$$

as $t \to \infty$. By Lemma 4.1 we know $\mathcal{A}_0 = \int_{M_t} d\mu_g$ is bounded away from 0. Thus by (5.1) we have $C_{n,2}\phi'\sigma_2 - b_{n,2}\phi'\sigma_2^{-1/2}\sigma_3 \to 0$ as $t \to \infty$. In view of the Newton-Maclaurin inequality we conclude that any convergent subsequence $\{M_{t_i}\}$ must converge to a totally umbilical hypersurface, that is, a radial coordinate slice as $t_i \to \infty$. Note that by the proof of Lemma 4.1 we can see that both $\rho_{\min}$ and $\rho_{\max}$ are monotonic. Therefore, we conclude that the limiting radial coordinate slice is unique.

In sum, we have the flow hypersurfaces M_t converge to a radial coordinate slice smoothly as $t \to \infty$, and the inequality

$$\int_M \sigma_2 d\mu_g - (n-1)\mathcal{A}_0 \leq \xi_{2,0}(\mathcal{A}_0) \tag{5.2}$$

follows from Lemma 3.1 easily. Moreover, the equality holds if and only if M is a radial coordinate slice. Following the argument in [8], when M is 2-convex instead of strictly 2-convex, we may approximate it by strictly 2-convex star-shaped hypersurfaces. The inequality (5.2) follows from the approximation. Therefore, we complete the proof of Corollary 1.3. □

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