Project Report on

Quantum Noise

Submitted by

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November 29, 2019

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Abstract

Understanding Quantum Noise and Quantum-Error Correction Techniques is important in order to understand the behaviour of Open Quantum Systems which interact with the external environment, and recover quantum systems from errors.Therefore,here we present the concept of *mathematical formalism of Quantum Operations* which is used to describe the Quantum Noise.

Chapter 1

Introduction

Chapter 2

Postulates of Quantum Mechanics

In this chapter we introduce the 4 fundamental postulates (**State of a System**, **Evolution of a quantum system**, **Composite Systems**, **Measurement of the system**) of Quantum Mechanics which provide a connection between the physical world and mathematical formalism of Quantum Mechanics/Quantum Computation.We mainly focus more on the postulate of Quantum Measurement.

2.1 State of a System

Postulate 1: Any isolated physical system in a Hilbert space (complex vector space with inner product) C^n known as state space of the system, can be completely described by its ***state vector*** which is a unit vector in it's state space.

Let $[b_1, b_2, b_3, ..., b_n]$ be the any orthonormal basis of C^n, then, any state of the system can described as,

$$\left|\psi\right\rangle = \sum_{i=1}^{i=n} \alpha_i b_i$$

where $\sum_i \left|\alpha_i\right|^2 = 1$ and the **state vector notation of** $\left|\psi\right\rangle = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \end{bmatrix}$

2.2 Evolution of a quantum system

Postulate 2: The evolution of a *closed quantum system* is described by a *unitary (linear) operator* on its state space. $\mathbf{U}$ (i.e) Let the state of the system be $\left|\psi\right\rangle$ at time t_1, $\left|\psi'\right\rangle$ at time t_2, then $|\psi'\rangle$ can be written as

$$|\psi'\rangle = U|\psi\rangle$$

Discussion:

Postulate 2 can be weakened into the following form - The evolution of a closed quantum system can be described by a *linear operator* on its state space.

To see this, we prove that if the evolution can be represented by a linear transformation operator U, then U should be *unitary.*

Let $|\psi\rangle$ be any *arbitrary state* of a quantum system.

Let us assume that a quantum transformation U has been applied to state $|\psi\rangle$, which leaves the system in state $|\psi'\rangle$= U$|\psi\rangle$.

Now, we know that, the state $|\psi'\rangle$ will be valid if and only if $|\psi'\rangle$ is a unit vector (i.e) $\left\| |\psi'\rangle \right\|^2 = 1$.

$$\Rightarrow |\psi'\rangle^* |\psi'\rangle = 1$$

$$\Rightarrow (U|\psi\rangle)^* U|\psi\rangle = 1$$

$$\Rightarrow |\psi\rangle^* U^* U|\psi\rangle = 1$$

Now, let us replace U^*U with A

$$\Rightarrow |\psi\rangle^* A|\psi\rangle = 1 \tag{2.1}$$

Since $|\psi\rangle$ is arbitrary, $\forall_i$ let us substitute e_i= $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$ where 1 lies in i^{th}position, in equation (2.1), in place of $|\psi\rangle$

$$\Rightarrow e_i^* A e_i = 1$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & \cdots & 1 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\Rightarrow \begin{bmatrix} a_{i1} & a_{i2} & \cdots a_{in} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} = 1$$

$$\Rightarrow a_{ii} = 1$$

$$\Rightarrow A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ a_{21} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 1 \end{bmatrix}$$

Now, let us substitute $\vec{v_{ij}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ (where i^{th}, j^{th} entries are 1 and all others 0) in equation (2.1), in place of $\left|\psi\right\rangle$.

$$\Rightarrow \frac{1}{2}\begin{bmatrix}0 & \cdots & 1 & 0 & \cdots & 1 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}1 & a_{12} & \cdots & a_{1n}\\ a_{21} & 1 & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & 1\end{bmatrix}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} = 1$$

$$\Rightarrow \begin{bmatrix}(a_{i1}+a_{j1}) & (a_{i2}+a_{j2}) & \cdots & (a_{in}+a_{jn})\end{bmatrix}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} = 2$$

$$\Rightarrow (a_{ii}+a_{ji}) + (a_{ij}+a_{jj}) = 2$$

Substituting $a_{ii} = a_{jj} = 1$

$$\Rightarrow a_{ji} + a_{ij} = 0 \tag{2.2}$$

Again, let us substitute $\vec{v_{ij}} = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ -j\\ 0\\ \vdots\\ 0\end{bmatrix}$ (where i^{th} entry is $\frac{1}{\sqrt{2}}$, j^{th} entry is $\frac{-j}{\sqrt{2}}$ and all others 0) in equation (2.1), in place of $\left|\psi\right\rangle$.

$$\Rightarrow \frac{1}{2}\begin{bmatrix}0 & \cdots & 1 & 0 & \cdots & +j & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix}1 & a_{12} & \cdots & a_{1n}\\ a_{21} & 1 & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & 1\end{bmatrix}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ -j\\ 0\\ \vdots\\ 0\end{bmatrix} = 1$$

$$\Rightarrow \begin{bmatrix}(a_{i1}+ja_{j1}) & (a_{i2}+ja_{j2}) & \cdots & (a_{in}+ja_{jn})\end{bmatrix}\begin{bmatrix}0\\ \vdots\\ 1\\ 0\\ \vdots\\ -j\\ 0\\ \vdots\\ 0\end{bmatrix} = 2$$

$$\Rightarrow (a_{ii}+ja_{ji}) - j(a_{ij}+ja_{jj}) = 2$$

$$\Rightarrow a_{ii}+ja_{ji} - ja_{ij} + a_{jj} = 2$$

Substituting $a_{ii} = a_{jj} = 1$

$$\Rightarrow ja_{ji} - ja_{ij} = 0$$

$$\Rightarrow a_{ji} - a_{ij} = 0 \tag{2.3}$$

equation (2.2)+(2.3)$\Rightarrow$

$$a_{ji} + a_{ij} + a_{ji} - a_{ij} = 0$$

$$\Rightarrow a_{ji} = 0$$

$$\Rightarrow \forall_{i=1}^{i=n}\forall_{j=1}^{j=n} A_{ij_{(i\neq j)}} = 0$$

$$\Rightarrow A = \begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{bmatrix} = I$$

Result 1: $|\psi\rangle^* A|\psi\rangle = 1$ for any *arbitrary state* $|\psi\rangle \Rightarrow A = I$

$$\Rightarrow A = U^*U = I$$

$$\Rightarrow U^*U = I$$

(i.e) The transformation described by an operator U is a *Unitary Transformation*
Therefore, from now on, we can assume that any operator that describes the evolution of a quantum system, *should be unitary*

2.3 Composite Systems

Postulate 3: If we have n systems numbered from 1 to n (say) and system i is prepared in the state $|\psi_i\rangle$ in state space H_i then, the state space of composite physical systems is $H_1 \otimes H_2 \otimes ... \otimes H_n$ and the joint state of total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$.

Exercise 2.3.1: Let A,B be the Unitary transformations which transforms two physically isolated systems from $|x\rangle \mapsto A|x\rangle$ and $|y\rangle \mapsto B|y\rangle$ resulting in the state of total system $A|x\rangle \otimes B|y\rangle$, then, **what should be the unitary transformation C on superposed system $|x\rangle \otimes |y\rangle$ such that $C(|x\rangle \otimes |y\rangle) = A|x\rangle \otimes B|y\rangle$?**
Solution:
Properties of Tensor Product:
i) x $\otimes$ y = $Vec(xy^T)$
ii)Vec(AQB)=(A $\otimes B^T$) Vec(Q)
From the above property i), $A|x\rangle \otimes B|y\rangle$ can be written as

$$A|x\rangle \otimes B|y\rangle = Vec(Ax(By)^T)$$

$$\Rightarrow A|x\rangle \otimes B|y\rangle = Vec(Axy^TB^T)$$

$$\Rightarrow A|x\rangle \otimes B|y\rangle = Vec(A(xy^T)B^T)$$

Let $Q = xy^T$

$$\Rightarrow A|x\rangle \otimes B|y\rangle = Vec(AQB^T)$$

applying property ii) to above equation,

$$\Rightarrow A|x\rangle \otimes B|y\rangle = Vec(A(xy^T)B^T) = A \otimes (B^T)^T Vec(Q)$$

$$\Rightarrow A|x\rangle \otimes B|y\rangle = (A \otimes B)Vec(Q)$$

$$\Rightarrow A\big|x\big\rangle \otimes B\big|y\big\rangle = (A \otimes B)Vec(xy^T)$$

$$\Rightarrow A\big|x\big\rangle \otimes B\big|y\big\rangle = (A \otimes B)(\big|x\big\rangle \otimes \big|y\big\rangle)$$

$$\Rightarrow C = A \otimes B$$

2.4 Measurement of a Quantum System

Postulate 4: When an arbitrary state $\big|\psi\big\rangle$, is sent through a quantum measurement system M described by measurement operators $\{M_1, M_2, ...M_k\}$ =$\{M_m\}$ in the state space of the system, we observe the outcome m corresponding to the resultant state $\frac{M_m\big|\psi\big\rangle}{\|M_m\big|\psi\big\rangle\|}$ with probability $\|M_m\big|\psi\big\rangle\|^2$ where $(1 \leq m \leq k)$
Discussion:

the probability of getting the result m is given by

$$p(m) = \|M_m\big|\psi\big\rangle\|^2$$

$$\Rightarrow p(m) = (M_m\big|\psi\big\rangle)^* M_m\big|\psi\big\rangle$$

$$\Rightarrow p(m) = \big|\psi\big\rangle^* M_m^* M_m\big|\psi\big\rangle \tag{2.4}$$

the resultant state corresponding to the result m is given by $\frac{M_m\big|\psi\big\rangle}{\|M_m\big|\psi\big\rangle\|}$

$$= \frac{M_m\big|\psi\big\rangle}{\sqrt{\big|\psi\big\rangle^* M_m^* M_m\big|\psi\big\rangle}} \tag{2.5}$$

since we know that sum of probabilities should always be 1,

$$\Rightarrow \sum_m p(m) = 1$$

$$\Rightarrow \sum_m \big|\psi\big\rangle^* M_m^* M_m\big|\psi\big\rangle = 1$$

$$\Rightarrow \big|\psi\big\rangle^* \left(\sum_m M_m^* M_m\right)\big|\psi\big\rangle = 1$$

In **result 1** in section 2.2, we proved that $\big|\psi\big\rangle^* A\big|\psi\big\rangle = 1$ for any *arbitrary state* $\big|\psi\big\rangle \Rightarrow A = I$

$$\Rightarrow \sum_m M_m^* M_m = I \tag{2.6}$$

Therefore any quantum system described by set of operators $\{M_m\}$ in the state space of system and satisfies the property $\sum_m M_m^* M_m = I$ can be qualified as a General Measurement System

Theorem 1. ***Orthogonal states can be perfectly distinguishable and non-othrogonal states can't be reliably distinguished.***

Proof. Let us assume we have known a set of states S=$\{|\psi_1\rangle, |\psi_2\rangle, ..., |\psi_n\rangle\}$. When we randomly choose a state $|\psi_i\rangle$ from the known set of states and *if we can find out what i is, with 100% certainty, then we say that the states are distinguishable.*

∎

case-i: states $|\psi_i\rangle$ are orthonormal.
Let us design a measurement system M such that $\{M_m\}$ are measurement operators such that $M_i = |\psi_i\rangle|\psi_i\rangle^*$

Since the states are orthonormal, we know that $\forall_{i\neq j}\langle\psi_i|\psi_j\rangle = \langle\psi_j|\psi_i\rangle = 0$ (i.e)

$$\Rightarrow |\psi_i\rangle^*|\psi_j\rangle = |\psi_i\rangle^*|\psi_j\rangle = 0 \tag{2.7}$$

And since every state $|\psi_i\rangle$ is a unit vector,

$$\Rightarrow |\psi_i\rangle^*|\psi_i\rangle = 1$$

Therefore, M when applied to any arbitrary state $|\psi\rangle$, will give output observable output i corresponding to the resultant state $\frac{M_i|\psi\rangle}{\sqrt{|\psi\rangle^* M_i^* M_i|\psi\rangle}}$ with probability $p(i) = |\psi\rangle^* M_i^* M_i|\psi\rangle$

$$\Rightarrow p(i) = |\psi\rangle^*\Big(|\psi_i\rangle|\psi_i\rangle^*\Big)^*\Big(|\psi_i\rangle|\psi_i\rangle^*\Big)|\psi\rangle$$

$$\Rightarrow p(i) = |\psi\rangle^*|\psi_i\rangle|\psi_i\rangle^*|\psi_i\rangle|\psi_i\rangle^*|\psi\rangle$$

$$\Rightarrow p(i) = |\psi\rangle^*|\psi_i\rangle|\psi_i\rangle^*|\psi\rangle$$

If $|\psi\rangle = |\psi_j\rangle$ where $i \neq j$ then probability of observing the outcome i is given by

$$p(i) = |\psi_j\rangle^*|\psi_i\rangle|\psi_i\rangle^*|\psi_j\rangle$$

$\Rightarrow$ **p(i)= 0 when input is $|\psi_j\rangle$, $i \neq j$ ——(a)**

Now, let $|\psi\rangle = |\psi_i\rangle$, then,

$$p(i) = |\psi_i\rangle^*|\psi_i\rangle|\psi_i\rangle^*|\psi_i\rangle$$

$\Rightarrow$ p(i)= 1 when input is $|\psi_i\rangle$ ——(b)

From above results (a)& (b) we can infer that, we can find i with 100% certainty. (i.e) When we see the observable i, then, we can conclude that the state that is chosen is $|\psi_i\rangle$

Therefore, orthogonal states are reliably distinguishable.

case-ii: $\{\ |\psi_i\rangle\}$ are non-orthogonal.

For simplicity let us assume there are only 2 states $\{|\psi_1\rangle, |\psi_2\rangle\}$ which are not orthogonal to each other.

Therefore, $|\psi_2\rangle$ can be expressed in terms of $|\psi_1\rangle$

$$|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\psi\rangle \quad (\alpha \neq 0\&\&\beta < 1) \tag{2.8}$$

where $\alpha|\psi_1\rangle$ is component of $|\psi_2\rangle$ along $|\ \psi_2\rangle$, $\beta|\psi\rangle$ is component of $|\psi_2\rangle$ perpendicular to $|\ \psi_2\rangle$ and $|\alpha|^2 + |\beta|^2 = 1$ since $|\psi_2\rangle$ is a unit vector.

Claim: There doesn't exist any set of Measurement operators {M1,M2} in-order to distinguish $\{|\psi_1\rangle,\ |\psi_2\rangle\}$ with 100% certainty.

Proof by Contradiction:

Let us assume there exists such operators M_1, M_2 in order to distinguish $\{|\psi_1\rangle,\ |\psi_2\rangle\}$

Requirements to be satisfied by M_1, M_2 are :

1) $M_1\ |\psi_1\rangle = |\psi_1\rangle$
2) $M_2\ |\psi_2\rangle = |\psi_2\rangle$
3) $M_1\ |\psi_2\rangle = 0$
4) $M_2\ |\psi_1\rangle = 0$

All above conditions should satisfy because the states $\{|\psi_1\rangle,\ |\psi_2\rangle\}$ should be distinguishable with 100% certainty.

5) $M_1^* M_1 + M_2^* M_2 =$ I

$$1) \Rightarrow |\psi_1\rangle^* M_1^* M_1 |\psi_1\rangle = |\psi_1\rangle^* |\psi_1\rangle = 1 \tag{2.9}$$

$$2) \Rightarrow |\psi_2\rangle^* M_2^* M_2 |\psi_2\rangle = |\psi_2\rangle^* |\psi_2\rangle = 1 \tag{2.10}$$

But we know that, $|\psi_2\rangle = \alpha|\psi_1\rangle + \beta|\psi\rangle$

$$\Rightarrow M_2|\psi_2\rangle = \alpha M_2|\psi_1\rangle + \beta M_2|\psi\rangle$$

$$\Rightarrow M_2|\psi_2\rangle = \beta M_2|\psi\rangle \quad (using\ 4)\ )$$

$$\Rightarrow (M_2|\psi_2\rangle)^* M_2|\psi_2\rangle = (\beta M_2|\psi\rangle)^* \beta M_2|\psi\rangle$$

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle = |\psi\rangle^* M_2^* \beta^* \beta M_2|\psi\rangle$$

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle = |\beta|^2 |\psi\rangle^* M_2^* M_2|\psi\rangle$$

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle = |\beta|^2 \|M_2|\psi\rangle\|^2$$

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle \leq |\beta|^2 \|M_2|\psi\rangle\|^2 + |\beta|^2 \|M_1|\psi\rangle\|^2$$

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle \leq |\beta|^2 (\|M_2|\psi\rangle\|^2 + \|M_1|\psi\rangle\|^2)$$

since sum of probabilities = 1,

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle \leq |\beta|^2$$

since $\beta < 1$,

$$\Rightarrow |\psi_2\rangle^* M_2^* M_2|\psi_2\rangle < 1 \tag{2.11}$$

The above inequality (2.8) is a contradiction for equation (2.7)
Therefore, Non-orthogonal states cannot be distinguished with 100% certainty.

2.4.1 Projective Measurements

Projective Measurements are special type of General Measurements where the measurement system M is a Hermitian operator.

As a consequence of *spectral theorem*, any *hermitian operator* H will have a spectral decomposition such that $H = \sum_i \lambda_i P_i$ where λ_i is the observed spectral value/eigen value and P_i is the projection operator on to the eigen space E_{λ_i}.

Therefore projective measurement system M is described by set of othrogonal projection operators $\{P_i\}$ where $P_i P_{i'} = \delta_{i,i'} P_i$(orthogonal projections) and $P_i^* = P_i$(hermitian) and $P_i^2 = P_i$(projection onto eigen space λ_i twice, is same as projecting once)

Therefore when an arbitrary state $|\psi\rangle$, is sent through the projective measurement system M, we observe the outcome λ_i corresponding to the resultant state $\frac{P_i|\psi\rangle}{\|P_i|\psi\rangle\|}$ with probability $\|P_i|\psi\rangle\|^2$
Discussion:
probability of observing the result λ_i is given by

$$p(\lambda_i) = \|P_i|\psi\rangle\|^2$$

$$\Rightarrow p(\lambda_i) = \big|\psi\big\rangle^* P_i^* P_i \big|\psi\big\rangle$$

$$\Rightarrow p(\lambda_i) = \big|\psi\big\rangle^* P_i \big|\psi\big\rangle \tag{2.12}$$

the resultant state corresponding to λ_i is given by $\frac{P_i\big|\psi\big\rangle}{\|P_i\big|\psi\big\rangle\|}$

$$= \frac{P_i\big|\psi\big\rangle}{\sqrt{\big|\psi\big\rangle^* P_i \big|\psi\big\rangle}} \tag{2.13}$$

since the sum of probabilities should be 1, $\sum_i P_i^* P_i = I$

$$\Rightarrow \sum_i P_i = I \tag{2.14}$$

2.4.2 Realization of any General Quantum Measurement as some Unitary transformation followed by a Projective Measurement

Theorem 2. ***Any General Quantum Measurement can be realized as some Unitary transformation followed by a Projective Measurement.***

Proof. Consider a General Quantum Measurement system $\{M_m\} = \{M_1, M_2....., M_k\}$ with state space $Q = \mathbf{C}^{2^n}$, Let the measurement outcomes be 1,2,3....k corresponding to the resultant states $\frac{M_1\big|\psi\big\rangle}{\|M_1\big|\psi\big\rangle\|}, \frac{M_2\big|\psi\big\rangle}{\|M_2\big|\psi\big\rangle\|}...., \frac{M_k\big|\psi\big\rangle}{\|M_k\big|\psi\big\rangle\|}$ for any arbitrary state $\big|\psi\big\rangle$ as shown in figure 2.1.

Let us extend the system with state space Q to $V = Q \otimes S$ where S=$\mathbf{C}^{2^k}$ and $V = \mathbf{C}^{2^n} \otimes \mathbf{C}^{2^k} = \mathbf{C}^{2^{(n+k)}}$ as shown in the figure 2.2.

Let $\{b_1, b_2, b_3, ...b_k\}$ be any orthonormal basis of state space S of sub system of the extended system with state space $V = Q \otimes S$.

Let us define a linear operator U on the subspace W of V to V such that, W = $\mathbf{C}^{2^n} \otimes \big|0^k\big\rangle$ and $U : W \rightarrow V$ defined by,

$$U\big(\big|\psi\big\rangle\big|0^k\big\rangle\big) = \sum_i \big(M_i\big|\psi\big\rangle\big) \otimes \big|b_i\big\rangle \tag{2.15}$$

Lemma 2.1. ***The above Linear Operator U defined by*** $U\big(\big|\psi\big\rangle\otimes\big|0^k\big\rangle\big) = \sum_i \big(M_i\big|\psi\big\rangle\big)\otimes \big|b_i\big\rangle$ ***preserves the inner product.***

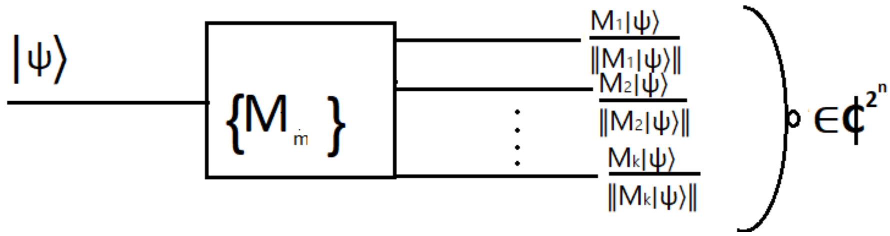

Figure 2.1: General Quantum Measurement system

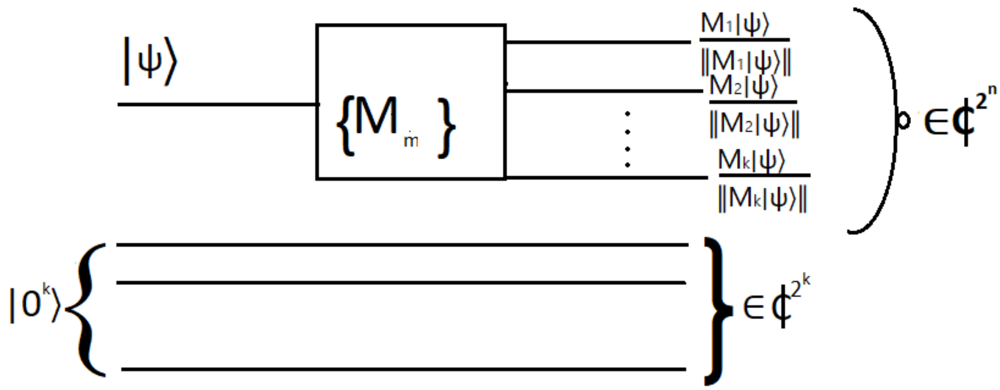

Figure 2.2: Extended composite system

Proof. Prove that $\Big(U\big(|\psi_1\rangle \otimes |0^k\rangle\big), U\big(|\psi_2\rangle \otimes |0^k\rangle\big)\Big) = \Big(|\psi_1\rangle \otimes |0^k\rangle, |\psi_2\rangle \otimes |0^k\rangle\Big)$

Consider $\Big(U\big(|\psi_1\rangle \otimes |0^k\rangle\big), U\big(|\psi_2\rangle \otimes |0^k\rangle\big)\Big)$

$$= \Big(\sum_{i=1}^{i=k} \big(M_i|\psi_1\rangle\big) \otimes |b_i\rangle, \sum_{j=1}^{j=k} \big(M_j|\psi_2\rangle\big) \otimes |b_j\rangle\Big)$$

$$= \sum_{i=1}^{i=k}\sum_{j=1}^{j=k} \Big(\big(M_i|\psi_1\rangle\big) \otimes |b_i\rangle, \big(M_j|\psi_2\rangle\big) \otimes |b_j\rangle\Big)$$

Since we know that $\Big(|a\rangle \otimes |b\rangle, |x\rangle \otimes |y\rangle\Big) = \Big(|a\rangle, |x\rangle\Big)\Big(|y\rangle, |b\rangle\Big)$

$$= \sum_{i=1}^{i=k}\sum_{j=1}^{j=k} \Big(\big(M_i|\psi_1\rangle\big), \big(M_j|\psi_2\rangle\big)\Big)\Big(|b_j\rangle, |b_i\rangle\Big)$$

$$= \sum_{i=1}^{i=k}\sum_{j=1}^{j=k} \Big(\big(M_i|\psi_1\rangle\big), \big(M_j|\psi_2\rangle\big)\Big)\delta_{i,j}$$

$$= \sum_{i=1}^{i=k} \Big(\big(M_i|\psi_1\rangle\big), \big(M_i|\psi_2\rangle\big)\Big)$$

$$= \sum_{i=1}^{i=k} \left(M_i|\psi_1\rangle\right)^* \left(M_i|\psi_2\rangle\right)$$

$$= \sum_{i=1}^{i=k} |\psi_1\rangle^* M_i^* M_i |\psi_2\rangle$$

$$= |\psi_1\rangle^* \left(\sum_{i=1}^{i=k} M_i^* M_i\right) |\psi_2\rangle$$

Since we know that $\sum_i M_i^* M_i = I$ for any general measurement system,

$$= |\psi_1\rangle^* |\psi_2\rangle$$

$$= |\psi_1\rangle^* |\psi_2\rangle |0^k\rangle^* |0^k\rangle$$

$$= \left(|\psi_1\rangle, |\psi_2\rangle\right)\left(|0^k\rangle, |0^k\rangle\right)$$

$$= \left(|\psi_1\rangle \otimes |0^k\rangle, |\psi_2\rangle \otimes |0^k\rangle\right)$$

Hence proved. ■

Lemma 2.2. ***Any linear operator*** $U : W \mapsto V$ ***that preserves inner product can extended to a Unitary Operator*** $U' : V \mapsto V$ ***such that, the properties of U are unchanged.***

Proof. **Find a map** U' **such that** $U'|w\rangle = U|w\rangle\ \forall |w\rangle \in W$

Since $U : W \mapsto V$ where W is the subspace of V, preserves inner product, for any $|w_1\rangle$, $|w_2\rangle \in W$

$$\Rightarrow \left(U|w_1\rangle, U|w_2\rangle\right) = \left(|w_1\rangle, |w_2\rangle\right)$$

Let $\{|b_1\rangle, |b_2\rangle, ...|b_n\rangle\}$ be the orthonormal basis of subspace W.

Note: *Here the dimension of subspace* $W = \boldsymbol{C}^{2^n} \otimes |0^k\rangle$*, of the vector space* $V = \boldsymbol{C}^{2^n} \otimes \boldsymbol{C}^{2^k}$*, is n.*

Since U is inner product preserving, (i.e) $\left(U|b_i\rangle, U|b_j\rangle\right) = \left(|b_i\rangle, |b_j\rangle\right)$, $\{U|b_1\rangle, U|b_2\rangle, ..., U|b_n\rangle\}$ form an orthonormal basis of the subspace formed by vectors $\left(U|w\rangle\right)$ of V, where $|w\rangle \in W$.

Therefore the map $f : |w\rangle \in W \mapsto U|w\rangle \in W'$ is a Unitary map. (Since the vector spaces W, W' are of equal dimension(=n) and have a one-one map between their orthonormal basis $|b_i\rangle \mapsto U|b_i\rangle$)

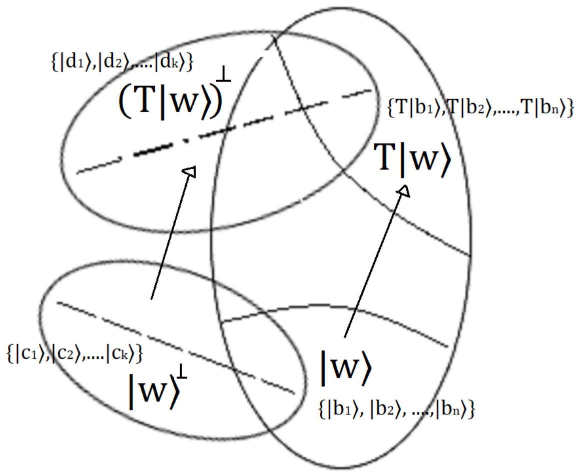

Figure 2.3: Extension of Inner product preserving Linear operator T to Unitary operator U'

Similarly we can design a map $f' : |w\rangle^{\perp} \in W^{\perp} \mapsto (U|w\rangle)^{\perp} \in (W')^{\perp}$ such that some one-one mapping $|c_i\rangle \mapsto |d_i\rangle$ of orthonormal basis $\{|c_1\rangle, |c_2\rangle, ...|c_k\rangle\}$ of $W^{\perp}$, $\{|d_1\rangle, |d_2\rangle, ...|d_k\rangle\}$ of $(W')^{\perp}$ is considered.

Therefore the above map f' is Unitary since the spaces $W^{\perp}$, $W'^{\perp}$ have same dimension $(=(n+k)-n=n)$ and we designed f' such that there is a one-one map $|c_i\rangle \mapsto |d_i\rangle$ between their orthonormal basis.

Therefore $U' = f \cup f'$ is a Unitary map from $V \mapsto V$ and U' preserves the property $U'|w\rangle = U|w\rangle$.

Therefore, any Linear Operator $U : W \mapsto V$ that preserves inner product, can extended to a Unitary Operator $U' : V \mapsto V$ such that, $U'|w\rangle = U|w\rangle \ \forall |w\rangle \in W$.

∎

Therefore, from Lemma (2.1) and Lemma (2.2), linear operator defined by $U : W \to V$ such that $U(|\psi\rangle|0^k\rangle) = \sum_i (M_i|\psi\rangle) \otimes |b_i\rangle$ can be extended to a Unitary Operator $U' : V \to V$ such that

$$U'(|\psi\rangle|0^k\rangle) = U(|\psi\rangle|0^k\rangle) = \sum_i (M_i|\psi\rangle) \otimes |b_i\rangle$$

Let $\{P_i\}$ be the set of Projective Measurement operators defined on state space S the sub-system of the extended system with space $Q \otimes S$

Lemma 2.3. ***The set of operators*** $\{I_Q \otimes P_i\}$ ***act as Projective Measurement operators to extended composite system, where*** I_Q ***is identity operator on acting on state space*** Q***.***

Proof. 1. $(I_Q \otimes P_i)^* = I_Q^* \otimes P_i^* = I_Q \otimes P_i$

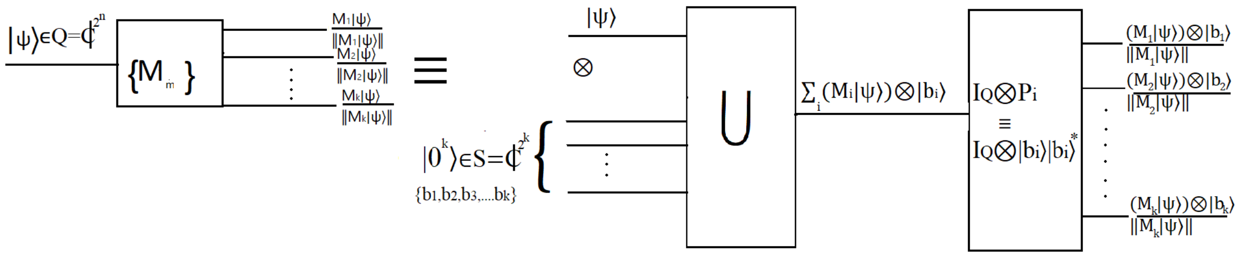

Figure 2.4: Realization of General Quantum Measurement system $\{M_m\}$ as a Unitary Transformation U followed by Projective Measurement system $\{I_Q \otimes P_i\}$

2. $\big(I_Q \otimes P_i\big).\big(I_Q \otimes P_j\big) = \big(I_Q.I_Q\big)\big(P_i.P_j\big) = 0$
3. $\big(I_Q \otimes P_i\big).\big(I_Q \otimes P_i\big) = \big(I_Q.I_Q\big)\big(P_i.P_i\big) = I_Q \otimes P_i$
4. $\sum_i \big(I_Q \otimes P_i\big) = I_Q \otimes \big(\sum_i P_i\big) = I_Q \otimes I_S = I$

■

Therefore, the set of operators $\{I_Q \otimes P_i\}$ qualify as Projective Measurement operators to extended composite system.

Now let us perform Projective Measurement described by set of operators $\{I_Q \otimes P_i\}$ on the resultant state $\sum_i \big(M_i\big|\psi\big\rangle\big) \otimes \big|b_i\big\rangle$ obtained after unitary transformation U on the extended composite system, as shown in figure (2.4).

Consider, $(I_Q \otimes P_i)(\sum_j \big(M_j\big|\psi\big\rangle\big) \otimes \big|b_j\big\rangle)$

$$= \sum_j (I_Q \otimes P_i)\big((M_j\big|\psi\big\rangle) \otimes \big|b_j\big\rangle\big)$$

$$= \sum_j \big(I_Q.(M_j\big|\psi\big\rangle)\big) \otimes (P_i\big|b_j\big\rangle)$$

$$= \sum_j (M_j\big|\psi\big\rangle) \otimes (\big|b_i\big\rangle\big|b_i\big\rangle^*\big|b_j\big\rangle)$$

$$= \sum_j (M_j\big|\psi\big\rangle) \otimes (\delta_{i,j}\big|b_i\big\rangle)$$

$$= (M_i\big|\psi\big\rangle) \otimes \big|b_i\big\rangle$$

Therefore the measurement outcomes of the composite system are some observables 1',2',3'...k' corresponding to the resultant state $\frac{(M_i\big|\psi\big\rangle)\otimes\big|b_i\big\rangle}{\|(M_i\big|\psi\big\rangle)\otimes\big|b_i\big\rangle\|}$ with probability $\|(M_i\big|\psi\big\rangle) \otimes \big|b_i\big\rangle\|^2$

. (i.e) the probability of observing an outcome i is given by,

$$p(i) = \|(M_i|\psi\rangle) \otimes |b_i\rangle\|^2$$

$$\Rightarrow p(i) = \big((M_i|\psi\rangle) \otimes |b_i\rangle\big)^*\big((M_i|\psi\rangle) \otimes |b_i\rangle\big)$$

$$\Rightarrow p(i) = \big((M_i|\psi\rangle)^* \otimes |b_i\rangle^*\big)\big((M_i|\psi\rangle) \otimes |b_i\rangle\big)$$

$$\Rightarrow p(i) = \big((M_i|\psi\rangle)^*(M_i|\psi\rangle)\big) \otimes \big(|b_i\rangle^*|b_i\rangle\big)$$

$$\Rightarrow p(i) = \big((M_i|\psi\rangle)^*(M_i|\psi\rangle)\big(|b_i\rangle^*|b_i\rangle\big)^*$$

$$\Rightarrow p(i) = \big((M_i|\psi\rangle)^*(M_i|\psi\rangle)\big(|b_i\rangle^*|b_i\rangle\big)$$

$$\Rightarrow p(i) = \big((M_i|\psi\rangle)^*(M_i|\psi\rangle)\big)$$

$$\Rightarrow p(i) = \|M_i|\psi\rangle\|^2 \tag{2.16}$$

the resultant state of the system corresponding to the outcome i is given by,

$$= \frac{(M_i|\psi\rangle) \otimes |b_i\rangle}{\|(M_i|\psi\rangle) \otimes |b_i\rangle\|}$$

$$= \frac{(M_i|\psi\rangle) \otimes |b_i\rangle}{\|M_i|\psi\rangle\|}$$

$$= \left(\frac{M_i|\psi\rangle}{\|M_i|\psi\rangle\|}\right) \otimes |b_i\rangle \tag{2.17}$$

Comparing equation (2.16) with (2.4), (2.17) with (2.5) we can say that the composite system which we have designed is equivalent to general measurement system as shown in figure (2.4)

Therefore, any General Quantum Measurement can be realized by some Unitary transformation followed by a Projective Measurement. ■

In the light of the above result, from now on, we focus on projective measurements.

2.4.3 Expected value of observable of a projective measurement:

Let a Projective Measurement is described by Hermitian operator M, with a spectral decomposition $M = \sum_i \lambda_i P_i$.

The expected value of the observable outcome λ given that the state of the system is $|\psi\rangle$

just before the measurement(i.e state of the system is $\big|\psi\big\rangle$ with probability 1)is given by,

$$E\left(\lambda\Big|\big|\psi\big\rangle\right) = \sum_i \lambda_i p(\lambda_i)$$

$$\Rightarrow E\left(\lambda\Big|\big|\psi\big\rangle\right) = \sum_i \lambda_i \big|\psi\big\rangle^* P_i \big|\psi\big\rangle$$

$$\Rightarrow E\left(\lambda\Big|\big|\psi\big\rangle\right) = \sum_i \big|\psi\big\rangle^* \lambda_i P_i \big|\psi\big\rangle$$

$$\Rightarrow E\left(\lambda\Big|\big|\psi\big\rangle\right) = \big|\psi\big\rangle^* (\sum_i \lambda_i P_i) \big|\psi\big\rangle$$

$$\Rightarrow E\left(\lambda\Big|\big|\psi\big\rangle\right) = \big|\psi\big\rangle^* M \big|\psi\big\rangle \tag{2.18}$$

The expected value of the observable outcome λ given that the state of the system is $\big|\phi\big\rangle = \{p_i, \big|\psi_i\big\rangle\}$ just before the measurement(i.e state of the system is $\big|\psi_i\big\rangle$ with probability p_i) is given by,

$$E\left(\lambda\Big|\big|\phi\big\rangle\right) = \sum_i p_i E\left(\lambda\Big|\big|\psi_i\big\rangle\right)$$

$$\Rightarrow E\left(\lambda\Big|\big|\phi\big\rangle\right) = \sum_i p_i \sum_j \lambda_j \big|\psi_i\big\rangle^* P_j \big|\psi_i\big\rangle$$

$$\Rightarrow E\left(\lambda\Big|\big|\phi\big\rangle\right) = \sum_i p_i \sum_j \big|\psi_i\big\rangle^* \lambda_j P_j \big|\psi_i\big\rangle$$

$$\Rightarrow E\left(\lambda\Big|\big|\phi\big\rangle\right) = \sum_i p_i \big|\psi_i\big\rangle^* \left(\sum_j \lambda_j P_j\right) \big|\psi_i\big\rangle$$

$$\Rightarrow E\left(\lambda\Big|\big|\phi\big\rangle\right) = \sum_i p_i \big|\psi_i\big\rangle^* M \big|\psi_i\big\rangle \tag{2.19}$$

Consider an operator such that $\rho = \sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*$, we claim that $E\left(\lambda\Big|\big|\phi\big\rangle\right) = \left(\rho, M\right)$

Consider $E\left(\lambda\Big|\big|\phi\big\rangle\right)$

$$= \sum_i p_i \big|\psi_i\big\rangle^* M \big|\psi_i\big\rangle$$

$$= \sum_i p_i \big|\psi_i\big\rangle^* \left(\sum_j \lambda_j P_j\right) \big|\psi_i\big\rangle$$

$$= \sum_i \sum_j p_i \big|\psi_i\big\rangle^* \lambda_j P_j \big|\psi_i\big\rangle$$

$$= \sum_i \sum_j ((p_i \big|\psi_i\big\rangle)^* \lambda_j P_j)(I^* \big|\psi_i\big\rangle)$$

$$= \sum_i \sum_j \Big((p_i \big|\psi_i\big\rangle), (\lambda_j P_j) \Big) \Big(I, \big|\psi_i\big\rangle \Big)$$

$$= \sum_i \sum_j \Big((p_i \big|\psi_i\big\rangle) \otimes \big|\psi_i\big\rangle, (\lambda_j P_j) \otimes I \Big)$$

$$= \sum_i \sum_j \Big(p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*, \lambda_j P_j I \Big)$$

$$= \sum_i \sum_j \Big(p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*, \lambda_j P_j \Big)$$

$$= \Big(\sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*, \sum_j \lambda_j P_j \Big)$$

$$= \Big(\rho, M \Big)$$

$$\Rightarrow E\Big(\lambda \Big| \big|\phi\big\rangle \Big) = \Big(\rho, M \Big) \tag{2.20}$$

where $\rho = \sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*$ and $\big|\phi\big\rangle = \{p_i, \big|\psi_i\big\rangle\}$ (i.e) $\big|\phi\big\rangle$ is a mixture of states $\big|\psi_i\big\rangle$ with probability p_i

Since the operator defined by $\rho = \sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*$ plays a role probability density matrix as shown above, we call the ρ ***as the Density Operator***

As a consequence of Quantum Measurement postulate, the resultant state of the system after Performing a measurement described by $\{M_m\}$, on an arbitrary state $\big|\psi\big\rangle$ can be one of the states $\frac{M_i \big|\psi\big\rangle}{\|M_i \big|\psi\big\rangle\|}$ with corresponding probabilities $\|M_i \big|\psi\big\rangle\|^2$

We can simply consider the above situation as, at a particular time t a quantum system can be in one of the $\big|\psi_i\big\rangle$ with probability p_i (i.e) $\{p_i, \big|\psi_i\big\rangle\}$

2.4.4 Density Operator

If a state of a Quantum System is described by $\{p_i, \big|\psi_i\big\rangle\}$, then the Density Operator of the Quantum System is defined by

$$\rho = \sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^* \tag{2.21}$$

2.4.5 Reformulation of Quantum Mechanics Postulates using Density operator notation

1)State of a System:

Any isolated physical system in a Hilbert space (complex vector space with inner product) C^n known as state space of the system, can be completely described by its ***Density Operator*** acting on the state space of the system.

If a Quantum System is in state $\rho_i = \sum_j p_{ij}|\psi_{ij}\rangle|\psi_{ij}\rangle^*$ with probability p_i, then the density operator of the system is

$$\rho = \sum_i \sum_j p_i p_{ij}|\psi_{ij}\rangle|\psi_{ij}\rangle^*$$

$$\Rightarrow \rho = \sum_i p_i \sum_j p_{ij}|\psi_{ij}\rangle|\psi_{ij}\rangle^*$$

$$\Rightarrow \rho = \sum_i p_i \rho_i \tag{2.22}$$

2)Evolution of Quantum System

Suppose the evolution of a *closed quantum system* is described by a *unitary operator* **U** on its state space. If the initial state of the system be $\{p_i, |\psi_i\rangle\}$ then, the state of the system after evolution will be $\{p_i, U|\psi_i\rangle\}$

Therefore the density operator after evolution of the quantum system is given by,

$$\rho' = \sum_i p_i(U|\psi_i\rangle)(U|\psi_i\rangle)^*$$

$$\Rightarrow \rho' = \sum_i p_i U|\psi_i\rangle\psi_i|\rangle^* U^*$$

$$\Rightarrow \rho' = \sum_i U p_i|\psi_i\rangle\psi_i|\rangle^* U^*$$

$$\Rightarrow \rho' = U\big(\sum_i p_i|\psi_i\rangle|\psi_i\rangle^*\big)U^*$$

$$\Rightarrow \rho' = U\rho U^* \tag{2.23}$$

3)Composite Systems:

If we have n systems numbered from 1 to n (say) and system i is prepared in the state ρ_i in state space H_i then, the state space of composite physical systems is $H_1 \otimes H_2 \otimes ... \otimes H_n$

and the joint state of total system is $\rho_1 \otimes \rho_2 \otimes ... \otimes \rho_n$.

Discussion:

Consider two isolated physical systems 1,2 with their initial states $\{p_i, |\psi_i\rangle\}$, $\{q_j, |\phi_j\rangle\}$ and density operators $\rho_1 = \sum_i p_i |\psi_i\rangle |\psi_i\rangle^*$, $\rho_2 = \sum_j q_j |\phi_j\rangle |\phi_j\rangle^*$ respectively.

The resultant state of the composite system is $\{p_i q_j, |\psi_i\rangle \otimes |\phi_j\rangle\}$

Therefore the density operator of the quantum system is given by

$$\rho' = \sum_{ij} p_i q_j \big(|\psi_i\rangle \otimes |\phi_j\rangle\big)\big(|\psi_i\rangle \otimes |\phi_j\rangle\big)^*$$

$$\Rightarrow \rho' = \sum_{ij} p_i q_j \big(|\psi_i\rangle \otimes |\phi_j\rangle\big)\big(|\psi_i\rangle^* \otimes |\phi_j\rangle^*\big)$$

$$\Rightarrow \rho' = \sum_{ij} p_i q_j \big(|\psi_i\rangle |\psi_i\rangle^*\big) \otimes \big(|\phi_j\rangle |\phi_j\rangle^*\big)$$

$$\Rightarrow \rho' = \sum_{ij} \big(p_i |\psi_i\rangle |\psi_i\rangle^*\big) \otimes \big(q_j |\phi_j\rangle |\phi_j\rangle^*\big)$$

$$\Rightarrow \rho' = \sum_i \sum_j \big(p_i |\psi_i\rangle |\psi_i\rangle^*\big) \otimes \big(q_j |\phi_j\rangle |\phi_j\rangle^*\big)$$

$$\Rightarrow \rho' = \sum_i \big(p_i |\psi_i\rangle |\psi_i\rangle^*\big) \otimes \sum_j \big(q_j |\phi_j\rangle |\phi_j\rangle^*\big)$$

$$\Rightarrow \rho' = \rho_1 \otimes \rho_2 \tag{2.24}$$

4)Measurement of a Quantum System:

When an arbitrary state of the system described by $\{p_i, |\psi_i\rangle\}$, is sent through a quantum measurement system M described by measurement operators $\{M_1, M_2, ...M_k\}$ $=\{M_m\}$ in the state space of the system, we observe the outcome m corresponding to the resultant state $\left\{p_i \|M_m|\psi_i\rangle\|^2, \frac{M_m|\psi_i\rangle}{\|M_m|\psi_i\rangle\|}\right\}$ with probability $\sum_i p_i \|M_m|\psi_i\rangle\|^2$ where $(1 \leq m \leq k)$

Discussion:

the probability of observing the result m is given by,

$$p(m) = \sum_i p_i \|M_m|\psi_i\rangle\|^2$$

$$\Rightarrow p(m) = \sum_i p_i |\psi_i\rangle^* (M_m^* M_m) |\psi_i\rangle$$

$$\Rightarrow p(m) = \sum_i p_i tr\big((M_m^* M_m) |\psi_i\rangle |\psi_i\rangle^*\big)$$

$$\Rightarrow p(m) = \sum_i tr\Big((M_m^* M_m) p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*\Big)$$

$$\Rightarrow p(m) = tr\Big((M_m^* M_m) \sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*\Big)$$

$$\Rightarrow p(m) = tr\Big(M_m^* M_m \rho\Big) \tag{2.25}$$

the density operator of the system when the result observed is m is given by,

$$\rho_m = \sum_i \left(\frac{p_i \|M_m \big|\psi_i\big\rangle\|^2}{p(m)}\right) \left(\frac{M_m \big|\psi_i\big\rangle}{\|M_m \big|\psi_i\big\rangle\|}\right) \left(\frac{M_m \big|\psi_i\big\rangle}{\|M_m \big|\psi_i\big\rangle\|}\right)^*$$

$$\Rightarrow \rho_m = \sum_i \left(\frac{p_i}{p(m)}\right) (M_m \big|\psi_i\big\rangle)(M_m \big|\psi_i\big\rangle)^*$$

$$\Rightarrow \rho_m = \sum_i \left(\frac{p_i}{p(m)}\right) M_m \big|\psi_i\big\rangle \big|\psi_i\big\rangle^* M_m^*$$

$$\Rightarrow \rho_m = \sum_i \left(\frac{1}{p(m)}\right) M_m p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^* M_m^*$$

$$\Rightarrow \rho_m = \left(\frac{1}{p(m)}\right) M_m \left(\sum_i p_i \big|\psi_i\big\rangle \big|\psi_i\big\rangle^*\right) M_m^*$$

$$\Rightarrow \rho_m = \left(\frac{1}{p(m)}\right) M_m \rho M_m^*$$

$$\Rightarrow \rho_m = \frac{M_m \rho M_m^*}{tr\Big(M_m^* M_m \rho\Big)} \tag{2.26}$$

the density operator of the system when the observed value m is lost is given by,

$$\rho' = \sum_m p(m) \rho_m$$

$$\Rightarrow \rho' = \sum_m p(m) \left(\frac{1}{p(m)}\right) M_m \rho M_m^*$$

$$\Rightarrow \rho' = \sum_m M_m \rho M_m^* \tag{2.27}$$

Bibliography

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