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Fractionally integrated GARCH model with tempered stable distribution: a simulation study

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ABSTRACT

With the growing availability of high-frequency data, long memory has become a popular topic in finance research. Fractionally Integrated GARCH (FIGARCH) model is a standard approach to study the long memory of financial volatility. The original specification of FIGARCH model is developed using Normal distribution, which cannot accommodate fat-tailed properties commonly existing in financial time series. Traditionally, the Student-*t* distribution and General Error Distribution (GED) are used instead to solve that problem. However, a recent study points out that the Student-*t* lacks stability. Instead, the Stable distribution is introduced. The issue of this distribution is that its second moment does not exist. To overcome this new problem, the tempered stable distribution, which retains most attractive characteristics of the Stable distribution and has defined moments, is a natural candidate. In this paper, we describe the estimation procedure of the FIGARCH model with tempered stable distribution and conduct a series of simulation studies to demonstrate that it consistently outperforms FIGARCH models with the Normal, Student-*t* and GED distributions. An empirical evidence of the S&P 500 hourly return is also provided with robust results. Therefore, we argue that the tempered stable distribution could be a widely useful tool for modelling the high-frequency financial volatility in general contexts with a FIGARCH-type specification.

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1. Introduction

Persistence of a time series describes how fast the effect of current shock will die away. It has been extensively observed and studied in various fields of economics and finance in the past few decades [18,21,24]. The analysis of persistence can help researchers understand how the time series evolves and improve the forecasting quality [19,22,30]. In finance study, the long-memory persistence describes the property that the effects of shocks last far longer than the usual autoregressive moving average process [2,8,16,36]. Although it has different definitions, the one used by Diebold and Inoue [16] is widely accepted as $\mathrm{var}(S_T) = O(T^{2d+1})$, where $S_T = \sum_{t=1}^{T} y_t$, $\{y_t\}$ is a sequence of interested financial time series and T is the number of observations. Then d is the long-memory parameter, and a

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positive value of it suggests the existence of long memory. Among recent finance studies, a growing number of evidence has suggested that the long-memory persistence significantly exists in the volatility of financial return series [3,4,11,22,37].

In particular, the time-varying volatility of financial returns has been a considerable field of research since the introduction of the Autoregressive Conditional Heteroskedasticity (ARCH) model by Engle [17]. Based on this work, the symmetric univariate Generalized ARCH (GARCH) model originally proposed by Bollerslev [6] has been extended further to incorporate long-memory persistence [2,8]. This leads to the Fractional Integrated GARCH (FIGARCH) model, which is based on the application of the fractional differencing operator to the autoregressive structure of the conditional variance by assuming that it follows a hyperbolic rather than exponential decay [2]. The FIGARCH model has thus received considerable interest because of its ability to capture the long-memory persistence in the volatility [3,4,22,37].

Originally, FIGARCH model is constructed based on the assumption that financial time series follow a Normal (Gaussian) distribution. However, significant evidence suggests that the financial time series is rarely Gaussian but typically leptokurtic and exhibits heavy-tail behaviour [7,34,38,39]. Theoretically, GARCH model can accommodate for fat-tailedness through their specification [9], which is also true in the FIGARCH case. In practice, however, there is still excess kurtosis left in the innovation series in most cases [10]. To solve this problem, a common solution is to employ a fat-tailed distribution such as the Student-*t* distribution or General Error Distribution (GED) [13,18,32,42]. Compared with the GARCH model using Normal distribution, estimates of which are believed to be consistent even when the true distribution is fat-tailed [9], that with the true distribution can lead to more efficient results [7]. This result can be straightforwardly extended to the FIGARCH model, since it has similar asymptotic behaviour as the GARCH model [2]. Consequently, for the application in finance study, FIGARCH model should be always fitted with a fat-tailed distribution to provide efficient estimate. This brings in a new question, which fat-tailed distribution should we employ when the true distribution is unknown?

A recent study of Calzolari *et al*. [10] argues that the widely used Student-*t* distribution is problematic. The most outstanding drawback is that this distribution lacks in stability under aggregation, which is of particular importance in portfolio applications and risk management. Since the aim of volatility modelling is to facilitate the decision of risk management, the FIGARCH model with Student-*t* distribution can be problematic in this sense. As a replacement of it, the Stable distribution is recommended by Calzolari *et al*. [10] to overcome this issue. Additionally, they argue that similar to the Student-*t*, Stable distribution can be easily adapted to account for many properties of volatility such as asymmetry in the underlying financial time series. Unfortunately, since the second moment of Stable distribution does not exist in most cases, FIGARCH model with this distribution will lead to ambiguous interpretation. Hence, the sought of an alternative distribution would be of particular interest for the study using FIGARCH model.

The tempered stable distribution is a natural substitute of the Stable for the GARCH-type model [35]. Firstly introduced[1] in [28], tempered stable distribution covers several well-known subclasses like Variance Gamma distributions, bilateral Gamma distributions and CGMY distributions [29]. The advantage of this distribution is that it retains most of the attractive properties of the Stable distribution and has defined moments.

Despite that tempered stable distribution has been adopted within different GARCH-type frameworks (see [27,37]), there is no existing study focussing on its appropriateness to describe the long-memory properties. Additionally, similar studies like [37] suggest that GARCH-type models with tempered stable distribution may outperform those with other fat-tailed distributions, in terms of the estimation consistency and efficiency of the parameters and model fitness. However, it is still unknown if those will hold within the FIGARCH framework and particularly for the long-memory parameter. To the best of our knowledge, this study is the first examining the appropriateness of the tempered stable distribution for the FIGARCH model and discussing its potential superiority to estimate the long-memory parameter over other widely employed fat-tailed distributions.

In this paper, we employ the tempered stable distribution in the FIGARCH model, describe its estimation procedure and argue that it outperforms both the Gaussian and commonly used fat-tailed distributions (Student-*t* and GED). To demonstrate that, we conduct a series of simulation studies to compare the performance of FIGARCH models with those distributions. First, we set the true distribution as Student-*t* and GED, respectively. Via nine combinations of different FIGARCH parameters and sample size, FIGARCH models with Gaussian and three distinct fat-tailed distributions are systematically analysed. It is demonstrated that when the true distribution is Student-*t* or GED, FIGARCH model with tempered stable distribution generates very similar results as true model. More importantly, it outperforms all the other competitors in terms of consistency, efficiency and overall performance. Second, we let the tempered stable be the true distribution. Nine sets of simulations are further constructed, including different combinations of FIGARCH and tempered stable distribution parameters. In this scenario, none of the FIGARCH models with Gaussian, Student-*t* or GED distributions performs as well as the true model. Consequently, we argue that the tempered stable distribution could be a widely useful tool for modelling the long memory of financial volatility in general contexts with a FIGARCH-type specification.

To empirically compare the FIGARCH models with different distributions, we apply them to the hourly return of the S&P 500 index. The results suggest that the FIGARCH model with tempered stable outperforms the others. Additionally, the fitted tempered stable distribution has closer density to that of the standardized data than the other three fitted distributions. Besides, the estimated parameters of FIGARCH models with fat-tailed distributions are close to each other. They are different from those of the FIGARCH model with Normal distribution to some extent. The four fitted conditional volatility series are quite similar.

The remainder of this paper proceeds as follows. Section 2 describes the specification of the FIGARCH model. Section 3 explains how the Student-*t*, GED and tempered stable distributions can be applied to the GARCH model. We conduct three independent simulation studies in Section 4. The empirical results are discussed in Section 5. Section 6 concludes the paper.

2. FIGARCH model

The FIGARCH model is proposed by Baillie *et al.* [2] and is extended from GARCH family models. As concluded by Ho *et al.* [22], GARCH family models have enjoyed popularity because of their ability to capture some of the typical stylized facts of financial return series, such as volatility clustering. French *et al.* [20] and Franses and van Dijk [19] show that GARCH family models take into account the feature of time-varying volatility over a long period, and they provide good in-sample estimates. In addition, the FIGARCH model is particularly designed to model the long-memory characteristic of the financial time series.

The original FIGARCH$(1, d, 1)$ model is described as:

$$\begin{aligned} r_t &= \mu + \varepsilon_t \quad \text{and} \quad \varepsilon_t = \eta_t \sqrt{h_t} \\ b(L)h_t &= \omega + [b(L) - \phi(L)(1-L)^d]\varepsilon_t^2 \\ b(L) &= 1 - b_1 L \quad \text{and} \quad \phi(L) = 1 - \phi_1 L, \end{aligned} \tag{1}$$

where ε_t is the error term at time t, h_t is the conditional volatility of ε_t at time t, η_t is an identical and independent innovation sequence following a specific distribution, L is the lag operator and $(1-L)^d$ is the fractional differencing operator as defined by [23]:

$$(1-L)^d = \sum_{k=0}^{\infty} \delta_k(d) L^k, \quad \delta_k(d) = \frac{k-1-d}{k}\delta_{k-1}(d), \quad \delta_0(d) = 1. \tag{2}$$

In particular, at time t, we proxy this infinity process by $(1-L)^d = \sum_{k=0}^{t-1} \delta_k(d) L^k$ as described in [25].[2] Then d is the defined as the long-memory parameter. We have a stationary long-memory process for volatility when $0 < d < 1$. If $d = 1$, the process has a unit root and thus a permanent shock effect, which is equivalent to the IGARCH model. If $d = 0$, the process reduces to an ordinary GARCH process without a long-memory property [2].

In order to estimate parameters of the FIGARCH model, Maximum Likelihood Estimation (MLE) is employed. Therefore, the innovation series η_t needs to follow a specific distribution. Originally, FIGARCH model is developed based on Standard Normal (Gaussian) distribution. In other words, $\eta_t = \varepsilon_t / \sqrt{h_t} \sim N(0, 1)$.[3] Hence, the conditional density of ε_t can be constructed as follows.

$$\begin{aligned} \Omega_{t-1} &= \{\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots, \varepsilon_1,\} \\ \theta &= (\mu, \omega, \phi, \beta, d)' \\ f(\varepsilon_t|\theta, \Omega_{t-1}) &= \frac{1}{\sqrt{2\pi h_t}} \mathrm{e}^{-\varepsilon_t^2/2h_t} \end{aligned} \tag{3}$$

Then, the log-likelihood function corresponding to Equation (3) is:

$$L(\theta|\varepsilon) = \sum_{t=2}^{\mathrm{T}} \log f(\varepsilon_t|\theta, \Omega_{t-1}) \quad \text{where } \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T)', \tag{4}$$

and MLE estimator $\hat{\theta}$ is obtained by maximizing Equation (4). The specific steps of estimation are described in Section 3.2.3. Additionally, standard deviation of $\hat{\theta}$ is obtained by taking square root of diagonal terms of the inverse fisher information (see [2] for details).

However, significant evidence suggests that the financial time series is rarely Gaussian but typically leptokurtic and heavy-tailed (see, e.g. [7]). Therefore, if the true distribution is not Gaussian, MLE standard deviation of $\hat{\theta}$ estimated in the above procedure will be inconsistent. To solve this problem, the Quasi-Maximum Likelihood Estimation (QMLE) based on Gaussian is further derived. The algorithm of QMLE to estimate $\hat{\theta}$ is the same as described by Equations (3) and (4), while the only difference is the way to estimate a robust standard deviation of $\hat{\theta}$ (see [2,9] for details). It is argued that the QMLE standard deviation is asymptotically consistent, even if the true distribution of η_t is not Gaussian.

3. Options of alternative distributions

3.1. Student-t and GED

Although QMLE can lead to consistent estimates, it is argued that QMLE of GARCH model is not efficient. Since FIGARCH model shares the asymptotic properties with the GARCH model, QMLE of FIGARCH can also be inefficient. Among the existing literature, Student-*t* distribution and GED are two widely used alternatives in finance study [13,18,32,42]. Both of them can capture leptokurtic and heavy-tail behaviours. When they are applied to the FIGARCH model, the corresponding density functions of ε_t are described below.

$$\begin{aligned}
\text{Studnet's t:} f(\varepsilon_t|\theta,\Omega_{t-1}) &= \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})\sqrt{\pi(v-2)h_t}}\left[1+\frac{\varepsilon_t^2}{(v-2)h_t}\right]^{(v+1)/2} \\
\text{GED:} f(\varepsilon_t|\theta,\Omega_{t-1}) &= \frac{v\,\mathrm{e}^{-(1/2)|\varepsilon_t/\lambda\sqrt{h_t}|^v}}{\lambda 2^{(v+1)/v}\Gamma(1/v)} \\
\text{where } \lambda &= \left[\frac{2^{(-2/v)}\Gamma(1/v)}{\Gamma(3/v)}\right]^{1/2},
\end{aligned} \tag{5}$$

where v is the degree of freedom. Then, the MLE estimator $\hat{\theta}$ can be obtained in the same way as described in Section 2.

3.2. Tempered stable distribution

Despite their attractive properties to capture excess kurtosis and fat-tails, existing literature argues that the Student-*t* and GED still have unsolved problems. For example, Yang and Brorsen [41] indicate that the tail behaviour of GARCH-type model remains too short even with Student-*t* distributed error terms. Furthermore, Calzolari *et al.* [10] suggests that the Student-*t* distribution lacks the stability-under-addition property. Since stability is desirable in portfolio applications and risk management, a distribution could overcome this issue is of particular importance.

3.2.1. Stable distribution.

As suggested by Calzolari *et al.* [10], the Stable distribution (also known as α-stable distribution) is a replacement of the traditionally used fat-tailed distribution. Its most outstanding characteristic is that the Stable distribution can further overcome the stability problem of the Student-*t*. The Stable distribution constitutes a generalization of the Gaussian distribution by allowing for asymmetry and heavy tails. In general, a random variable x is said to be stably distributed if and only if, for any positive numbers c_1 and c_2, there exists a positive number k and a real number d such that

$$kx + d \stackrel{d}{=} c_1 x_1 + c_2 x_2, \tag{6}$$

where x_1 and x_2 are independent variables and have the same distribution as x. The notation $\stackrel{d}{=}$ indicates equality in distribution. The property described by Equation (6) is also known as the *stability-under-addition* property [40]. In particular, if $d = 0$, x is said to be strictly stable. According to Calzolari *et al.* [10], theoretical foundations of α-stable distribution lay on the generalized central limit theorem, in which the condition of finite variance is replaced by a much less restricting one concerning a regular behaviour of the tails.

Since Stable distribution does not have a close form of density function, the best way to describe it is by means of its characteristic function, which has the following form:

$$\phi(t) = \exp\{i\delta t - \sigma^\alpha |t|^\alpha (1 - i\beta \mathrm{sign}(t)\Phi)\}, \tag{7}$$

where Φ is $-(2/\pi)\log|t|$ when $\alpha = 1$ and is $\tan(\pi\alpha/2)$ when $\alpha \neq 1$. $\alpha \in [0, 2]$ is the index of stability or characteristic exponent that describes the tail-thickness of the distribution (small values correspond to thick tails). $\sigma \in \mathrm{R}^+$ is the scale parameter. $\delta \in \mathrm{R}$ is the location parameter. β describe its skewness. In particular, Calzolari *et al.* [10] only consider the symmetric Stable distribution ($\beta = 0$), which is then characterized by (α, σ, δ) and is denoted as $S(\alpha, \sigma, \delta)$. Therefore, the standardized symmetric version is $S(\alpha, 1, 0)$ with the following characteristic function

$$\phi(t) = \exp\{-|t|^\alpha\}. \tag{8}$$

Despite its attractive properties like stability-under-addition, the second moment of the Stable distribution does not exist in most cases.[4] Consequently, the application of this distribution to the FIGARCH model will cause serious problems. For instance, the interpretation of conditional volatility would fail. Therefore, the seek of a substitute of the Stable distribution, which has similar attractive properties and defined moments, would be of particular interest.

3.2.2. Tempered stable distribution.

A natural substitute of the Stable is the tempered stable distribution for the GARCH-type model [35]. A general case of the tempered stable distribution is characterized by six parameters and denoted as $\mathrm{TS}(\alpha^+, C^+, \lambda^+; \alpha^-, C^-, \lambda^-)$. The levy measure of such random variable x is

$$\nu(x) = \frac{c^-}{|x|^{1+\alpha^-}} e^{-\lambda^-|x|} 1_{x<0} + \frac{c^+}{|x|^{1+\alpha^+}} e^{-\lambda^+|x|} 1_{x>0}. \tag{9}$$

Therefore, a tempered stable distribution with zero mean has the following characteristic function [15].

$$\begin{aligned}\phi(t) = \exp\Bigg\{&\Gamma(-\alpha^+)(\lambda^+)^{\alpha^+} C^+ \left[\left(1 - \frac{it}{\lambda^+}\right)^{\alpha^+} - 1 + \frac{it\alpha^+}{\lambda^+}\right] \\ &+\Gamma(-\alpha^-)(\lambda^-)^{\alpha^-} C^- \left[\left(1 - \frac{it}{\lambda^-}\right)^{\alpha^-} - 1 + \frac{it\alpha^-}{\lambda^-}\right]\Bigg\},\end{aligned} \tag{10}$$

where $\alpha^+, \alpha^- < 2$ and $C^+, C^-, \lambda^+, \lambda^- > 0$. Its first four cumulants are hereby defined as:

$$
\begin{aligned}
\kappa_1 &= 0 \\
\kappa_2 &= \Gamma(2-\alpha^+)C^+(\lambda^+)^{\alpha^+-2} + \Gamma(2-\alpha^-)C^-(\lambda^-)^{\alpha^--2} \\
\kappa_3 &= \Gamma(3-\alpha^+)C^+(\lambda^+)^{\alpha^+-3} + \Gamma(3-\alpha^-)C^-(\lambda^-)^{\alpha^--3} \\
\kappa_4 &= \Gamma(4-\alpha^+)C^+(\lambda^+)^{\alpha^+-4} + \Gamma(4-\alpha^-)C^-(\lambda^-)^{\alpha^--4}.
\end{aligned}
\tag{11}
$$

Hence, the first four moments of x can be found via the following relations:

$$
\begin{aligned}
m_1 &= \kappa_1 \\
m_2 &= \kappa_2 + \kappa_1^2 \\
m_3 &= \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3 \\
m_4 &= \kappa_4 + 4\kappa_3\kappa_1 + 3\kappa_2^2 + 6\kappa_2\kappa_1^2 + \kappa_1^4.
\end{aligned}
\tag{12}
$$

Therefore, it is clear that the tempered stable distribution has defined moments, so that it can be further applied to describe the innovation of GARCH-type model.

Küchler and Tappe [29] suggest that sample paths of x can be classified as some well-known processes, which mainly depend on values of α^+ and α^-. In particular, if $\alpha^+ = \alpha^-$, x follows a classical tempered stable distribution. Additionally, if we further require $C^+ = C^-$, then x follows a CGMY distribution [12].

- For $\alpha^+, \alpha^- < 0$ we have $\nu(\mathrm{R}) < \infty$, and thus, x is a compound Poisson process.
- For $\alpha^+, \alpha^- \in [0, 1)$ we have $\nu(\mathrm{R}) = \infty$, but $\int_{-1}^{1} |x|\nu(\mathrm{d}x) < \infty$. Therefore, x is a finite-variation process making infinitely many jumps in each interval of positive length, which we can express as $x_t = \sum_{s \le t} \Delta x_s$.
- For $\alpha^+, \alpha^- \in (1, 2)$ we have $\int_{-1}^{1} |x|\nu(\mathrm{d}x) = \infty$. Thus, x has sample paths of infinite variation.

To apply this distribution into the FIGARCH model, we require x to be standardized. Bianchi *et al.* [5] argues that one way to achieve the standardization is letting

$$
C^+ = \frac{p(\lambda^+)^{2-\alpha^+}}{\Gamma(2-\alpha^+)} \quad \text{and} \quad C^- = \frac{(1-p)(\lambda^-)^{2-\alpha^-}}{\Gamma(2-\alpha^-)},
\tag{13}
$$

where $p \in (0, 1)$, then $x \sim \mathrm{TS}(\alpha^+, \lambda^+, \alpha^-, \lambda^-, p)$ has zero mean and unit variance.

Combining Equations (10) and (13), we now have a standardized tempered stable distribution. This distribution is expected to retain all the attractive properties similar to those of Stable distribution and can be further employed to describe the innovation of the FIGARCH model.

3.2.3. FIGARCH model with tempered stable distribution.

Since the tempered stable distribution has defined fourth moment, the sufficient condition to ensure the asymptotic properties of the FIGARCH model will not be affected.[5] Hence, we expect the MLE of FIGARCH model with tempered stable distribution would provide consistent estimators.

In terms of estimation, we can still use the similar MLE approach as discussed in Section 2. Since the tempered stable distribution also does not have a closed form of density function, a certain numerical algorithm is needed to compute it [27]. As argued by Mittnik *et al.* [33], compared with other approximation methods, discrete Fourier transform is accurate and efficient to estimate parameters of stable family distributions, especially when $N = 2^{13}$ or above. Therefore, we employ the discrete Fourier transform method to obtain the estimated $f(\varepsilon_t|\theta, \Omega_{t-1})$ and perform the MLE in the following iteration steps:

(1) Give initial estimates for the parameters θ_0.
(2) Follow Equations (1) and (2) to generate h_t and ε_t with the value of θ_0, where initial values (at $t=0$) of h_t and ε_t^2 are set to sample variance of r_t.
(3) Obtain the minimum and maximum of $\eta_t = \varepsilon_t/\sqrt{h_t}$ as η_1 and η_2, respectively;
(4) Calculate the values of $\phi(s)$ for the tempered stable distribution determined by the estimates of α^+, C^+, λ^+, α^-, C^- and λ^- via Equation (10), where s evenly ranges from η_1-0.1 to η_2+0.1 with size $N = 2^{15}$;[6]
(5) Use discrete Fourier transform to find the values of the corresponding density function for s;
(6) Employ the linear interpolation to find the density values of η_t that fall between the prespecified equally-spaced values of s;
(7) Sum up the logarithms of the density values of η_t, which is the corresponding log-likelihood of θ_0; and
(8) Employ the BFGS algorithm[7] with numeric derivatives to search the θ_0^* that increases the log-likelihood of θ_0.

By repeating steps (1)–(8) with the new estimates of θ_0^*, the MLE estimates of the FIGARCH model with tempered stable distribution can be generated when the convergence is reached. [8]

4. Comparisons between distributions: simulation studies

In this Section, we will conduct three simulation studies to compare the performance of FIGARCH models with Normal, Student-t, GED and tempered stable distributions. The data generation process is FIGARCH(1,d,1) in all cases.[9] True distributions are therefore Student-t, GED and tempered stable, respectively. Regarding the simulation method, the exact inverse transform based on the uniform distribution between 0 and 1 is adopted to simulate the Student-t and GED random variates, for which density functions are well defined. We adopt an approximative acceptance-rejection sampling technique proposed by Baeumer and Meerschaert [1] to simulate the tempered stable random variates. A brief overview on the related simulation techniques is provided in Section 4.3.

4.1. Student-t distribution

First, we set the true distribution as Student-t with 3 degrees of freedom.[10] Altogether, nine sets of simulations of the FIGARCH(1,d,1) process with different ϕ, β, d and T (sample

Table 1. Simulation results: Student-*t* distribution.

ϕ	β	d	T	Mean_{ll}	Bias_ϕ	SE_ϕ	RMSE_ϕ	Bias_β	SE_β	RMSE_β	Bias_d	SE_d	RMSE_d
Panel A: Normal													
0.60	0.20	0.25	3000	−2659	−0.1056	0.2568	0.2777	−0.0397	0.2257	0.2292	0.0126	0.2363	0.2366
			4000	−3553	−0.0782	0.2523	0.2641	−0.0323	0.2025	0.2051	0.0019	0.2102	0.2102
			5000	−4486	−0.0707	0.2390	0.2492	−0.0146	0.2024	0.2029	0.0101	0.2030	0.2033
0.30	0.30	0.50	3000	−3202	−0.0602	0.3682	0.3731	−0.0195	0.3364	0.3370	0.0215	0.2703	0.2712
			4000	−4299	−0.0822	0.3627	0.3719	−0.0448	0.3710	0.3737	0.0196	0.2473	0.2481
			5000	−5392	−0.1112	0.3841	0.3999	−0.0941	0.3637	0.3757	−0.0012	0.2182	0.2182
0.20	0.60	0.75	3000	−4018	−0.0861	0.2693	0.2827	−0.0716	0.2963	0.3048	0.0026	0.2141	0.2141
			4000	−5431	−0.0674	0.3002	0.3077	−0.0566	0.2759	0.2816	−0.0054	0.2173	0.2174
			5000	−6778	−0.0681	0.2665	0.2751	−0.0515	0.2558	0.2609	0.0086	0.1841	0.1843
Panel B: Student-*t*													
0.60	0.20	0.25	3000	−2138	−0.0318	0.0814	0.0874	−0.0014	0.0886	0.0886	0.0211	0.1304	0.1321
			4000	−2867	−0.0204	0.0757	0.0784	0.0067	0.0839	0.0842	0.0251	0.1160	0.1187
			5000	−3593	−0.0206	0.0630	0.0663	0.0014	0.0632	0.0632	0.0146	0.1003	0.1014
0.30	0.30	0.50	3000	−2669	−0.0547	0.3624	0.3665	−0.0433	0.3677	0.3702	0.0054	0.1361	0.1362
			4000	−3599	−0.0343	0.3290	0.3308	−0.0349	0.3318	0.3336	−0.0042	0.1048	0.1049
			5000	−4511	−0.0288	0.3343	0.3355	−0.0211	0.3404	0.3411	0.0049	0.1054	0.1055
0.20	0.60	0.75	3000	−3510	−0.0238	0.0950	0.0979	−0.0212	0.1288	0.1305	0.0024	0.1080	0.1080
			4000	−4708	−0.0117	0.0836	0.0844	0.0003	0.0997	0.0997	0.0097	0.0902	0.0907
			5000	−5882	−0.0115	0.0763	0.0772	−0.0087	0.0911	0.0915	0.0020	0.0759	0.0759
Panel C: GED													
0.60	0.20	0.25	3000	−2185	−0.0510	0.0868	0.1007	−0.0536	0.0774	0.0941	−0.0955	0.1221	0.1550
			4000	−2929	−0.0368	0.0729	0.0817	−0.0531	0.0628	0.0822	−0.1083	0.0989	0.1467
			5000	−3673	−0.0331	0.0643	0.0723	−0.0527	0.0555	0.0765	−0.1187	0.0754	0.1406
0.30	0.30	0.50	3000	−2719	0.0711	0.3039	0.3121	0.0074	0.2597	0.2598	−0.0991	0.2006	0.2237
			4000	−3664	0.0857	0.2824	0.2951	0.0156	0.2531	0.2536	−0.0991	0.1885	0.2130
			5000	−4594	0.1210	0.2459	0.2741	0.0512	0.2138	0.2198	−0.0966	0.1758	0.2006
0.20	0.60	0.75	3000	−3560	−0.0142	0.1971	0.1976	−0.0559	0.2038	0.2113	−0.0416	0.1758	0.1807
			4000	−4775	−0.0045	0.1469	0.1470	−0.0317	0.1366	0.1402	−0.0271	0.1380	0.1406
			5000	−5967	−0.0069	0.1287	0.1289	−0.0358	0.1190	0.1243	−0.0263	0.1250	0.1277

size) are generated, where $\mu = 0$ and $\omega = 0.1$. To avoid the starting bias, 10000 points are generated for each simulation, and then only the last 3000, 4000 or 5000 points are used. For each set, we generate 300 Monte Carlo replicates.

The simulated data are fitted into FIGARCH model with Normal (FIGARCH-N), Student-*t* (FIGARCH-t), GED (FIGARCH-G) and tempered stable (FIGARCH-S) distributions, respectively. In Table 1 and panel A of Table 4, the log-likelihood (LL), bias, standard error (SE) and root-mean-square-error (RMSE) of ϕ, β and d are reported. Bias is the mean difference between the true parameter and its estimate, SE is the standard error of the estimates, and RMSE is the square root of the mean of squared difference between the true parameter and its estimate.

Although tempered stable distribution has more parameters than Normal, Student-*t* and GED, the log-likelihood is still a preliminary indicator of the model performance. Not surprisingly, FIGARCH-N model has the smallest values in all cases. Results of FIGARCH-G are generally smaller than those of FIGARCH-t. It is worth noticing that FIGARCH-S can yield slightly better log-likelihood than the true model in all cases.

For the long-memory parameter d, the biases obtained from the FIGARCH-N, FIGARCH-t and FIGARCH-S are quite similar and fairly small. Most of them are less than 0.02. However, the results of FIGARCH-G are comparatively worse, some of which can be above 0.1. Hence FIGARCH-N and FIGARCH-S can lead to consistent estimates of d, while those of FIGARCH-G are less preferred. In terms of the efficiency, as expected,

Table 2. Simulation results: GED.

ϕ	β	d	T	Mean_{ll}	Bias_{ϕ}	SE_{ϕ}	RMSE_{ϕ}	Bias_{β}	SE_{β}	RMSE_{β}	Bias_{d}	SE_{d}	RMSE_{d}
Panel A: Normal													
0.60	0.20	0.25	3000	−3338	−0.0163	0.0776	0.0793	0.0039	0.0780	0.0781	0.0201	0.1277	0.1293
			4000	−4471	−0.0108	0.0883	0.0890	0.0003	0.0721	0.0721	0.0085	0.1020	0.1024
			5000	−5610	−0.0164	0.0615	0.0636	0.0012	0.0582	0.0582	0.0151	0.0919	0.0931
0.30	0.30	0.50	3000	−4250	−0.0876	0.3847	0.3945	−0.0713	0.3925	0.3989	0.0133	0.1255	0.1262
			4000	−5666	−0.0635	0.3678	0.3732	−0.0523	0.3745	0.3781	0.0059	0.1086	0.1088
			5000	−7037	−0.0327	0.3224	0.3241	−0.0327	0.3127	0.3144	−0.0020	0.1027	0.1027
0.20	0.60	0.75	3000	−5022	−0.0275	0.1294	0.1323	−0.0135	0.1550	0.1556	0.0123	0.1103	0.1110
			4000	−6722	−0.0143	0.0925	0.0936	0.0048	0.1200	0.1201	0.0193	0.0916	0.0936
			5000	−8331	−0.0065	0.0763	0.0766	0.0043	0.0915	0.0916	0.0148	0.0793	0.0807
Panel B: Student-*t*													
0.60	0.20	0.25	3000	−3150	−0.0371	0.1153	0.1211	0.0487	0.1089	0.1193	0.1312	0.1183	0.1767
			4000	−4217	−0.0257	0.0923	0.0958	0.0559	0.0945	0.1098	0.1193	0.1021	0.1570
			5000	−5287	−0.0309	0.0669	0.0737	0.0485	0.0804	0.0939	0.1171	0.0871	0.1459
0.30	0.30	0.50	3000	−4057	−0.1386	0.3422	0.3692	−0.0890	0.3763	0.3867	0.0518	0.1037	0.1159
			4000	−5410	−0.1487	0.3744	0.4028	−0.1162	0.3992	0.4158	0.0303	0.0842	0.0895
			5000	−6708	−0.0942	0.3288	0.3420	−0.0636	0.3453	0.3511	0.0300	0.0732	0.0791
0.20	0.60	0.75	3000	−4826	−0.0239	0.0933	0.0963	0.0055	0.1281	0.1282	0.0256	0.1105	0.1134
			4000	−6463	−0.0133	0.0690	0.0703	0.0139	0.0872	0.0883	0.0231	0.0809	0.0841
			5000	−7977	−0.0124	0.0591	0.0604	0.0051	0.0777	0.0779	0.0137	0.0672	0.0686
Panel C: GED													
0.60	0.20	0.25	3000	−3122	−0.0162	0.0704	0.0722	0.0022	0.0684	0.0684	0.0182	0.1164	0.1178
			4000	−4180	−0.0061	0.0617	0.0620	0.0063	0.0573	0.0576	0.0084	0.0951	0.0955
			5000	−5241	−0.0122	0.0528	0.0542	0.0027	0.0546	0.0547	0.0138	0.0867	0.0878
0.30	0.30	0.50	3000	−4028	−0.0170	0.3067	0.3072	−0.0086	0.3134	0.3135	0.0072	0.1160	0.1162
			4000	−5372	−0.0096	0.2824	0.2826	−0.0043	0.2866	0.2866	0.0005	0.0967	0.0967
			5000	−6664	0.0059	0.2740	0.2741	0.0031	0.2668	0.2668	−0.0040	0.0908	0.0909
0.20	0.60	0.75	3000	−4800	−0.0188	0.0962	0.0980	−0.0099	0.1267	0.1271	0.0080	0.1040	0.1043
			4000	−6428	−0.0081	0.0693	0.0698	0.0044	0.0881	0.0882	0.0128	0.0790	0.0800
			5000	−7936	−0.0080	0.0619	0.0624	−0.0021	0.0761	0.0761	0.0073	0.0644	0.0648

FIGARCH-N generate much greater SEs than the others. They are mostly above 0.2, while SEs of the other fat-tailed models are at around 0.1 and quite close to each other. Therefore, it suggests that the QMLE of FIGARCH model is consistent but not efficient. Finally, the overall performance indicator RMSE shows that FIGARCH-S outperforms FIGARCH-N and FIGARCH-G. Additionally, the RMSEs of FIGARCH-S are very close to those of the true model. To quantitatively compare the similarities between estimates of the true model and those of the alternative models, we calculate the absolute differences of their RMSEs in the nine sets. The average differences of FIGARCH-N, FIGARCH-G and FIGARCH-S for d are 0.1144, 0.0617 and 0.0084, respectively. Clearly, the overall performance of FIGARCH-S is much closer to that of the true model.

Parameters ϕ and β measure the short-memory persistence, and they are not as important as d. For both measures, FIGARCH-N generates slightly greater absolute biases than the others. Some of them exceed 0.1. The results of the fat-tailed models are relatively similar and are mostly below 0.05. The SEs of FIGARCH-N are also the largest in most cases. In particular, its SEs for the sets $d = 0.25$ and $d = 0.75$ are at around 0.25, while SEs of the fat-tailed models are mostly below 0.1. Therefore, RMSE still suggests that FIGARCH-N is the least preferred model in the estimation of short-memory persistence. The results of the fat-tailed models are fairly close to each other. Also, FIGARCH-S slightly outperforms FIGARCH-G in most cases. More specifically, the average absolute differences between RMSEs of FIGARCH-N, FIGARCH-G and FIGARCH-S and those of the true model for

Table 3. Simulation results: tempered stable distribution.

ϕ	β	d	α^+	α^-	λ^+	λ^-	Mean$_{ll}$	Bias$_\phi$	SE$_\phi$	RMSE$_\phi$	Bias$_\beta$	SE$_\beta$	RMSE$_\beta$	Bias$_d$	SE$_d$	RMSE$_d$
Panel A: Normal																
0.60	0.20	0.25	0.5	0.5	1.0	1.0	−5546	−0.0060	0.0601	0.0604	0.0080	0.0671	0.0676	0.0205	0.0973	0.0994
			0.2	1.2	1.2	0.2	−4733	−0.0283	0.1100	0.1136	−0.0139	0.0896	0.0907	0.0034	0.1446	0.1446
			1.2	0.2	0.2	1.2	−4705	−0.0199	0.0984	0.1004	−0.0125	0.0908	0.0917	−0.0147	0.1279	0.1287
0.30	0.30	0.50	0.5	0.5	1.0	1.0	−6962	−0.0754	0.3645	0.3722	−0.0579	0.3778	0.3822	0.0123	0.0963	0.0971
			0.2	1.2	1.2	0.2	−5767	−0.0662	0.3500	0.3562	−0.0577	0.3471	0.3519	0.0027	0.1496	0.1496
			1.2	0.2	0.2	1.2	−5776	−0.0614	0.3755	0.3805	−0.0593	0.3693	0.3740	−0.0032	0.1503	0.1503
0.20	0.60	0.75	0.5	0.5	1.0	1.0	−8391	−0.0131	0.0714	0.0726	−0.0075	0.0861	0.0864	0.0024	0.0778	0.0778
			0.2	1.2	1.2	0.2	−7150	−0.0205	0.1128	0.1146	−0.0081	0.1407	0.1409	0.0198	0.1045	0.1064
			1.2	0.2	0.2	1.2	−7179	−0.0435	0.1689	0.1744	−0.0387	0.2164	0.2198	0.0032	0.1320	0.1320
Panel B: Student-*t*																
0.60	0.20	0.25	0.5	0.5	1.0	1.0	−5187	−0.0255	0.0587	0.0640	0.0419	0.0681	0.0800	0.1017	0.0803	0.1296
			0.2	1.2	1.2	0.2	−3978	−0.0385	0.0812	0.0899	0.0556	0.0949	0.1100	0.1397	0.0972	0.1702
			1.2	0.2	0.2	1.2	−3951	−0.0396	0.0993	0.1069	0.0531	0.0865	0.1015	0.1354	0.0847	0.1597
0.30	0.30	0.50	0.5	0.5	1.0	1.0	−6610	−0.1074	0.3287	0.3458	−0.0647	0.3576	0.3634	0.0352	0.0691	0.0775
			0.2	1.2	1.2	0.2	−5028	−0.0331	0.3222	0.3239	0.0160	0.3430	0.3434	0.0459	0.0812	0.0933
			1.2	0.2	0.2	1.2	−5031	0.0053	0.3011	0.3011	0.0501	0.3200	0.3239	0.0430	0.0912	0.1008
0.20	0.60	0.75	0.5	0.5	1.0	1.0	−8028	−0.0070	0.0589	0.0593	0.0004	0.0705	0.0705	0.0029	0.0625	0.0626
			0.2	1.2	1.2	0.2	−6408	−0.0111	0.0643	0.0653	0.0113	0.0774	0.0782	0.0188	0.0704	0.0729
			1.2	0.2	0.2	1.2	−6436	−0.0145	0.0631	0.0647	0.0021	0.0831	0.0831	0.0104	0.0759	0.0766
Panel C: GED																
0.60	0.20	0.25	0.5	0.5	1.0	1.0	−5172	−0.0046	0.0460	0.0462	−0.0008	0.0548	0.0548	−0.0002	0.0820	0.0820
			0.2	1.2	1.2	0.2	−4002	−0.0199	0.0615	0.0646	−0.0362	0.0596	0.0697	−0.0705	0.0958	0.1189
			1.2	0.2	0.2	1.2	−3973	−0.0182	0.0600	0.0627	−0.0367	0.0520	0.0636	−0.0810	0.0810	0.1146
0.30	0.30	0.50	0.5	0.5	1.0	1.0	−6592	−0.0402	0.3186	0.3211	−0.0283	0.3298	0.3310	0.0069	0.0812	0.0815
			0.2	1.2	1.2	0.2	−5049	0.0809	0.2508	0.2635	0.0494	0.2366	0.2417	−0.0444	0.1297	0.1371
			1.2	0.2	0.2	1.2	−5054	0.0890	0.2538	0.2690	0.0471	0.2292	0.2340	−0.0553	0.1358	0.1466
0.20	0.60	0.75	0.5	0.5	1.0	1.0	−8013	−0.0070	0.0606	0.0610	−0.0086	0.0715	0.0720	−0.0023	0.0632	0.0632
			0.2	1.2	1.2	0.2	−6430	−0.0075	0.0766	0.0770	−0.0170	0.0991	0.1005	−0.0036	0.0772	0.0773
			1.2	0.2	0.2	1.2	−6458	−0.0131	0.0805	0.0816	−0.0275	0.1075	0.1110	−0.0132	0.0904	0.0914

Table 4. Summary of the FIGARCH model with the tempered stable distribution.

ϕ	β	d	T	α^-	α^-	λ^+	λ^-	Mean_{ll}	Bias_ϕ	SE_ϕ	RMSE_ϕ	Bias_β	SE_β	RMSE_β	Bias_d	SE_d	RMSE_d
Panel A: Student-*t* distribution																	
0.60	0.20	0.25	3000					−2132	−0.0335	0.0793	0.0861	−0.0087	0.0935	0.0939	0.0000	0.1388	0.1388
			4000					−2862	−0.0261	0.0754	0.0798	−0.0115	0.0748	0.0757	−0.0108	0.1265	0.1270
			5000					−3588	−0.0204	0.0625	0.0657	−0.0176	0.0609	0.0634	−0.0304	0.1002	0.1047
0.30	0.30	0.50	3000					−2664	−0.0321	0.3897	0.3910	−0.0284	0.3903	0.3913	−0.0056	0.1565	0.1566
			4000					−3595	−0.0316	0.3772	0.3785	−0.0412	0.3747	0.3770	−0.0168	0.1205	0.1217
			5000					−4506	−0.0230	0.3714	0.3721	−0.0249	0.3734	0.3742	−0.0056	0.1176	0.1177
0.20	0.60	0.75	3000					−3504	−0.0259	0.0984	0.1018	−0.0206	0.1328	0.1344	0.0050	0.1113	0.1114
			4000					−4703	−0.0140	0.0834	0.0846	−0.0023	0.1022	0.1022	0.0096	0.0930	0.0935
			5000					−5878	−0.0122	0.0768	0.0778	−0.0103	0.0937	0.0943	0.0019	0.0780	0.0780
Panel B: GED																	
0.60	0.20	0.25	3000					−3116	−0.0162	0.0999	0.1012	0.0015	0.0830	0.0830	0.0193	0.1073	0.1090
			4000					−4174	−0.0037	0.0602	0.0603	0.0087	0.0522	0.0529	0.0111	0.0840	0.0847
			5000					−5236	−0.0105	0.0501	0.0512	0.0057	0.0521	0.0524	0.0178	0.0840	0.0859
0.30	0.30	0.50	3000					−4023	−0.0358	0.3095	0.3116	−0.0181	0.3132	0.3137	0.0177	0.1034	0.1049
			4000					−5366	−0.0025	0.2989	0.2989	0.0017	0.3067	0.3067	0.0028	0.0858	0.0858
			5000					−6656	−0.0112	0.2879	0.2881	−0.0106	0.2819	0.2821	0.0007	0.0803	0.0803
0.20	0.60	0.75	3000					−4793	−0.0024	0.1073	0.1073	0.0041	0.1065	0.1066	0.0088	0.1031	0.1035
			4000					−6422	−0.0058	0.0707	0.0709	0.0078	0.0792	0.0796	0.0146	0.0704	0.0719
			5000					−7928	−0.0056	0.0729	0.0731	−0.0021	0.0720	0.0720	0.0041	0.0673	0.0674
Panel C: Tempered stable distribution																	
0.60	0.20	0.25	5000	0.5	0.5	1.0	1.0	−5159	−0.0088	0.0506	0.0514	0.0060	0.0562	0.0565	0.0195	0.0839	0.0861
			5000	0.2	1.2	1.2	0.2	−3909	−0.0207	0.0650	0.0682	0.0061	0.0639	0.0642	0.0344	0.1080	0.1133
			5000	1.2	0.2	0.2	1.2	−3882	−0.0166	0.0665	0.0685	0.0080	0.0681	0.0686	0.0291	0.0975	0.1017
0.30	0.30	0.50	5000	0.5	0.5	1.0	1.0	−6580	−0.0234	0.3474	0.3482	−0.0053	0.3590	0.3590	0.0126	0.0827	0.0837
			5000	0.2	1.2	1.2	0.2	−4956	−0.0491	0.3645	0.3678	−0.0367	0.3742	0.3760	0.0106	0.0943	0.0949
			5000	1.2	0.2	0.2	1.2	−4961	−0.0212	0.3347	0.3354	−0.0135	0.3391	0.3394	0.0070	0.0957	0.0960
0.20	0.60	0.75	5000	0.5	0.5	1.0	1.0	−8000	−0.0092	0.0634	0.0641	−0.0085	0.0750	0.0755	−0.0012	0.0636	0.0636
			5000	0.2	1.2	1.2	0.2	−6337	−0.0065	0.0641	0.0644	−0.0017	0.0767	0.0767	0.0071	0.0662	0.0666
			5000	1.2	0.2	0.2	1.2	−6365	−0.0133	0.0645	0.0659	−0.0116	0.0838	0.0846	−0.0006	0.0764	0.0764

ϕ (β) are 0.1419, 0.0431 and 0.0130 (0.1150, 0.0541 and 0.0134), respectively. Hence, the estimates of the short-memory parameters in FIGARCH-S are much closer to those in the true model.

In conclusion, when the true model is FIGARCH-t, FIGARCH-S model outperforms FIGARCH-N and FIGARCH-G models in the estimation of the long- and short-memory persistence. Its results are overall very close to those of the true model.

4.2. GED

Next, we set the true distribution as GED with 1.1 degree of freedom.[11] Nine sets of simulations with the same combinations of parameters as those in Section 4.1 are constructed. We also generate 300 Monte Carlo replicates for each set. The same truncation strategy described in Section 4.1 is adopted to avoid the simulation bias for each replicate.

Simulation results are reported in Table 2 and panel B of Table 4. As a preliminary evidence of model performance, log-likelihood results are consistent with those presented in Section 4.1. More specifically, FIGARCH-t leads to much larger values than FIGARCH-N, but its log-likelihood are still smaller than those of the true model. FIGARCH-S model generates larger log-likelihood than FIGARCH-G in all cases.

In terms of the long-memory persistence, the absolute biases of FIGARCH-N and FIGARCH-S are similar and fairly close to those of the true model. The results of FIGARCH-t, however, are relatively larger than the others. For instance, its biases of the set $d = 0.25$ are above 0.1, while those of the others are at around 0.01. The SEs of FIGARCH-N are mostly above 0.1, which are slightly greater than those of the others. It is worth noticing that FIGARCH-S can produce smaller SEs than the true model in eight out of nine cases. Hence, RMSE indicates that FIGARCH-S model outperforms all the other three in almost all sets. QMLE of FIGARCH is still consistent but not efficient. Moreover, the average absolute differences between RMSEs of FIGARCH-N, FIGARCH-t and FIGARCH-S and those of the true model for d are 0.0104, 0.0239 and 0.0073, respectively. Clearly, the overall performance of FIGARCH-S is much closer to that of the true model.

For the short-memory persistence, the four models generate similar absolute biases in most cases. However, FIGARCH-t leads to the largest values for the sets $d = 0.5$. Some of them even exceed 0.1, while those of FIGARCH-N and FIGARCH-S are mostly below 0.05. The SEs of FIGARCH-t are even larger than those of FIGARCH-N in the cases of $d = 0.25$ and $d = 0.5$. FIGARCH-S is more efficient than FIGARCH-N and FIGARCH-t in almost all sets. Indicated by RMSE, FIGARCH-S is overall preferred to FIGARCH-N and FIGARCH-t. Its performance is quite similar as FIGARCH-G. More specifically, the average absolute differences between RMSEs of FIGARCH-N, FIGARCH-t and FIGARCH-S and those of the true model for ϕ (β) are 0.0382, 0.0396 and 0.0099 (0.0777, 0.0480 and 0.0100), respectively. Hence, the estimates of the short-memory parameters in FIGARCH-S are much closer to those in the true model.

In conclusion, when the true model is FIGARCH-G, FIGARCH-S model outperforms FIGARCH-N and FIGARCH-t models in the estimation of the long- and short-memory persistence. Its results are overall very close to those of the true model.

4.3. Tempered stable distribution

Next, we set the true distribution as tempered stable with three sets of different parameters (all the p are set to 0.5), including one case of CGMY distribution (when $\alpha^+ = \alpha^- = 0.5$ and $\lambda^+ = \lambda^- = 1.0$) and two general cases. Altogether, nine sets of simulations are constructed, where the combination of ϕ, β and d are the same as those in Section 4.1 and 4.2. Sample size is set to 5000 in all cases. We also generate 300 Monte Carlo replicates for each set. The same truncation strategy described in Section 4.1 is adopted to avoid the simulation bias for each replicate.

There are three different approaches to simulate the tempered stable random variates: acceptance-rejection sampling, a Gaussian approximation of a small jump component and infinite shot noise series representations (see [26] for details of those methods). With a given computing budget, Kawai and Masuda [26] suggest that an approximative acceptance-rejection sampling technique proposed by Baeumer and Meerschaert [1] is both most efficient and handiest. Moreover, it is argued that any desired level of accuracy may be attained with a small amount of additional computing effort via this method. Therefore, we adopt this acceptance-rejection sampling technique to simulate the tempered stable random variates used in this paper.

Simulation results are reported in Table 3 and panel C of Table 4. Not surprisingly, log-likelihood values of all the FIGARCH-N, FIGARCH-t and FIGARCH-G are all smaller than those of the true model. The fat-tailed models are preferred to the FIGARCH-N. Also, FIGARCH-G outperforms FIGARCH-t only in the CGMY distribution cases, while FIGARCH-t generates lager log-likelihood in the other general cases.

In terms of the long-memory parameter, the absolute biases of FIGARCH-N, FIGARCH-G and FIGARCH-S are quite small and fairly close to each other. Most of them are below 0.05. The results of FIGARCH-t, however, are comparatively larger. For instance, when $d = 0.25$, its absolute biases can be greater than 0.1. The SEs of the fat-tailed models are similar and are mostly below 0.1. Those of FIGARCH-N, however, are almost all greater than 0.1. Therefore, the QMLE of FIGARCH model for data of tempered stable distribution is also consistent but not efficient. Finally, the RMSE suggests that the overall performance of the fat-tailed models are at the same scale, while FIGARCH-S is preferred to FIGARCH-t or FIGARCH-G in most cases. The FIGARCH-N still cannot perform as well as the fat-tailed models. Moreover, the average absolute differences between RMSEs of FIGARCH-N, FIGARCH-t and FIGARCH-G and those of the true model for d are 0.0337, 0.0198 and 0.0160, respectively.

For the short-memory persistence, the absolute biases across the four models are relatively similar. Most of them are smaller than 0.05, indicating a consistent estimate. With respect to the efficiency, the SEs of FIGARCH-N are still overall greater than those of the fat-tailed models. FIGARCH-t and FIGARCH-G have similar SEs, which are not far from those of FIGARCH-S. Therefore, in the estimation of the short-memory persistence, the overall performance of the fat-tailed models are on the same scale and slightly better than that of the FIGARCH-N. Moreover, the average absolute differences between RMSEs of FIGARCH-N, FIGARCH-t and FIGARCH-G and those of the true model for ϕ (β) ar e0.0371 0.0178 and 0.0271 (0.0392, 0.0181 and 0.0371), respectively.

To summarize, when the true model is FIGARCH-S, none of the FIGARCH-N, FIGARCH-t and FIGARCH-G models can perform as well as it to estimate d. In terms of ϕ and β, the three fat-tailed models lead to similar result, while FIGARCH-N is less preferred.

5. Empirical results

To empirically compare the FIGARCH models with Normal, Student-t, GED and tempered stable distributions, we work on the data set of the hourly S&P 500 Index (SP500). The hourly closing prices for SP500 over the period from 1 January 2004 to 31 December 2013 are obtained from the Thomson Reuters Tick History (TRTH) database, which contains microsecond-time-stamped tick data dating back to January 1996. The database covers 35 million OTC and exchange-traded instruments worldwide, which are provided by the Securities Industry Research Centre of Australasia (SIRCA). The corresponding return in the percentage series is defined as the logarithm of the daily closing price differences times 100; that is, $r_t = 100 \times \log(S_t/S_{t-1})$. Altogether, there are 20,414 observations in our sample.

The level and return of SP500 are plotted in Figure 1. In the level plot, SP500 level clearly decreases between 2008 and 2009, which is the global financial crisis (GFC) period. Demonstrated in the return plot, from 2008 to 2010, the return becomes much more volatile. After 2010, the volatility tends to be smaller, with some turbulence around the end of 2010 and the beginning of 2012. In addition, the mean and standard error of the SP500 return are 0.0025 and 0.4179, respectively. The skewness is 0.2728, indicating that the SP500 return is slightly positively skewed. The kurtosis is 25.8934, suggesting a non-Gaussian distribution. Thus, we firstly fit the FIGARCH-N model and perform the Kolmogorov–Smirnov and Jarque–Bera normality tests (not presented) for its estimated standardized residuals. The null hypotheses indicating normality are rejected in both cases (p-values are 0.0000). As a result, FIGARCH models with fat-tailed distributions are expected to outperform the FIGARCH-N model.

The estimates are presented in Table 5. Overall, all estimates of parameters are significant at 5% level in all models. Additionally, estimates of individual parameters from the fat-tailed models are quite close, but differ from those of FIGARCH-N. Specifically, the estimated d of the fat-tailed models are at around 0.90, indicating that the long memory of the conditional volatility is remarkably persistent over our sample period. FIGARCH-N, in contrast, only leads to an estimate of 0.4054. The ϕ and β obtained from the fat-tailed

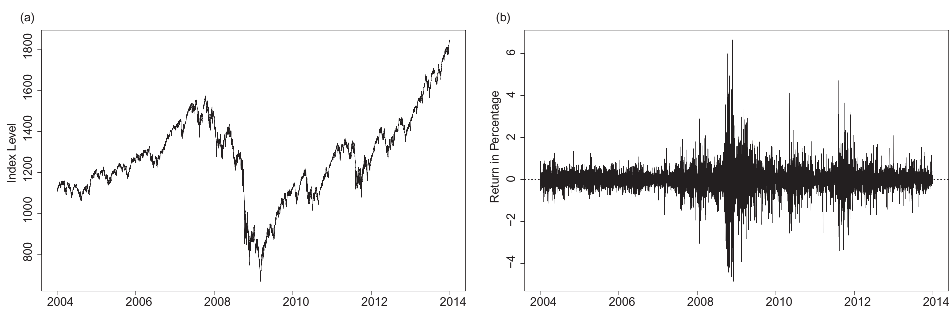

Figure 1. S&P 500 Index and Return.

Table 5. Empirical results: S& P 500 Index.

	Norm	Std-t	GED	T-Stb
ϕ	0.3632	0.0865	0.0684	0.0794
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
β	0.7516	0.9493	0.9557	0.9565
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
d	0.4054	0.9071	0.9236	0.9014
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
υ		2.8216	0.7335	
		(0.0000)	(0.0000)	
α^+				−0.2534
				(0.0000)
α^-				0.1488
				(0.0000)
λ^+				1.6367
				(0.0000)
λ^-				0.8680
				(0.0000)
p				0.4271
				(0.0000)
Log.lik	−4880	−2449	−1797	−1732
AIC	9770	4909	3605	3483
BIC	9810	4957	3613	3562

Note: The numbers in the parentheses are the corresponding p-values.

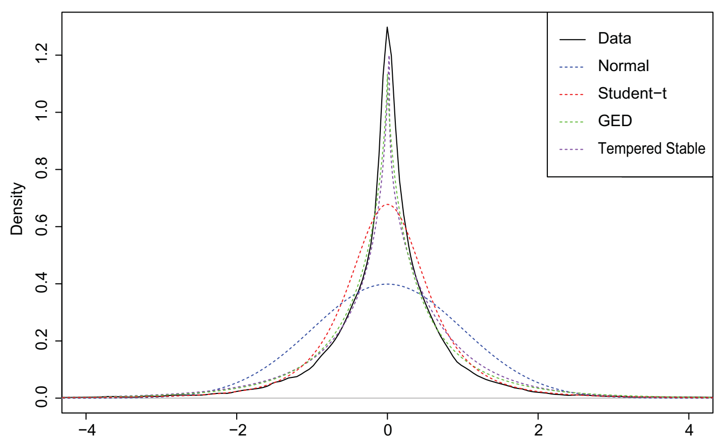

Figure 2. Empirical results: comparison of estimated densities.

models are at around 0.08 and 0.95, respectively. Those of the FIGARCH-N are different and are 0.3632 and 0.7516, respectively. To compare the model performance, the log-likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) are presented. It can be seen that fat-tailed models are all preferred to the FIGARCH-N model. More specially, FIGARCH-G outperforms FIGARCH-t, while FIGARCH-S leads to the best results.

To further explore the models, we report the smoothed density plot of the standardized SP500 return in Figure 2. To compare the fitness of the distributions, we also plot the densities of Student-t and GED distributions with 2.8216 and 0.7335 degrees of freedom, respectively. Those values are the estimates presented in Table 5. The densities of the standard Normal distribution and tempered stable distribution with parameters as estimated

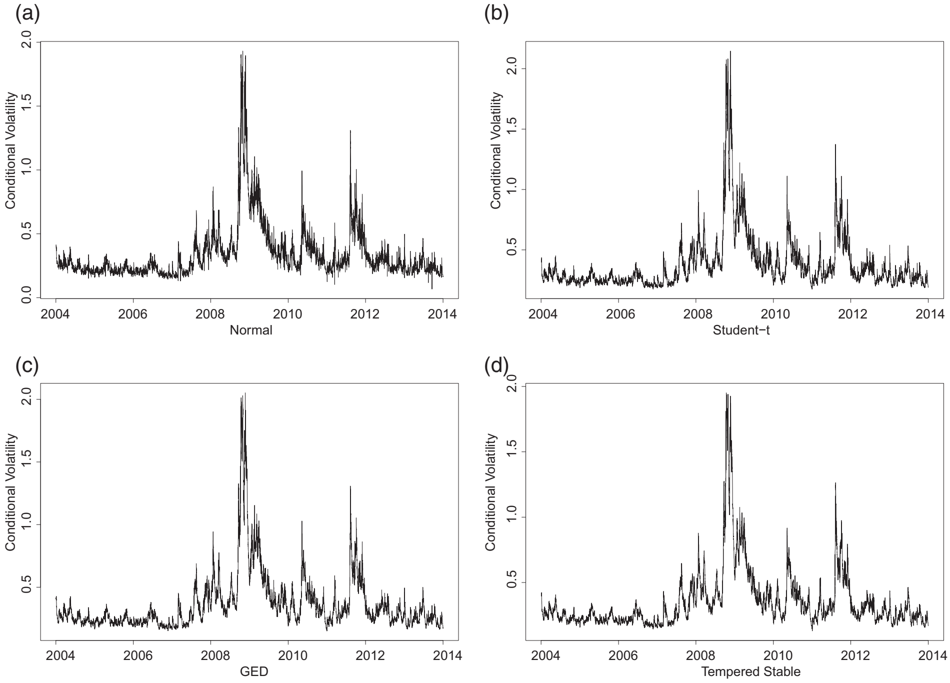

Figure 3. Empirical results: comparison of estimated conditional volatility.

in Table 5 are also reported. It is clear that the shapes of tempered stable and GED density are much closer to that of the data than the others. Relatively speaking, the density of the fitted tempered stable distribution is slightly closer to that of the data than the fitted GED. This provides further evidence that the FIGARCH-S outperforms the other models when applied to fit the SP500 data.

Finally, all the four estimated conditional volatility series[12] are plotted in Figure 3. Despite their different model performances, the four estimated conditional volatility series have similar shapes. The trends of those series are consistent with our observation of the return series, since the conditional volatility is comparatively larger from 2008 to 2010. This is concordant with the real macro-economic situation: the 2008 GFC with an effect lasting for around 3 years. Despite their similarities, the difference between the fitted conditional volatilities can be described as the different in-sample forecasting performance of the fitted models. To quantitatively compare this performance, we measure the prediction error for each model by $|\sqrt{\hat{h}_t} - |r_t||$, where $\hat{h}_t$ is the fitted h_t for each model, and $|r_t|$ is employed to proxy the true conditional volatility. The results are reported in Panel A of Table 6. Moreover, to consider the out-of-sample forecasts, we use the last 100 observations as the prediction sample and the others as the training sample. Then, we fit each model for the training sample and calculate the one-step ahead forecast of h_t. After that, we include the first observation in the training sample and generate another one-step ahead forecast. We repeat this rolling-window approach until the 100 one-step ahead forecasts are produced. Then, we calculate the absolute difference between it and the corresponding $|r_t|$, which measures the out-of-sample forecasting errors. The results are reported in

Table 6. Forecasting results: S& P 500 Index.

	Mean	SD	Median	Q_1	Q_3
Panel A: In-sample forecasts					
Norm	0.1412	0.3192	0.1769	0.0495	0.2636
Std-t	0.1275	0.3200	0.1610	0.0336	0.2505
GED	0.1226	0.3203	0.1554	0.0279	0.2465
T-Stb	0.1128	0.3162	0.1520	0.0235	0.2383
Panel B: Out-of-sample forecasts					
Norm	0.2826	0.1719	0.3240	0.2476	0.3764
Std-t	0.2706	0.1724	0.3129	0.2363	0.3686
GED	0.2638	0.1751	0.3050	0.2269	0.3622
T-Stb	0.2155	0.1707	0.2561	0.1841	0.3033

Note: SD, $Q1$ and $Q3$ stand for the standard deviation, first quartile and third quartile, respectively.

Panel B of Table 6. It is observed that FIGARCH-S leads to the smallest average forecasting errors in both the in-sample and out-of-sample cases. It also has the smallest variations of forecasting errors, as measured by the standard deviation. Additionally, in terms of the average forecasting errors, FIGARCH-G outperforms the FIGARCH-t, and FIGARCH-N is the least preferred model in both cases.

6. Conclusion

FIGARCH model has enjoyed particular popularity in the study of long memory for financial volatility. Despite its well-established properties, in practice, FIGARCH model can lead to more efficient estimates when an appropriate fat-tailed distribution is employed. As widely used alternatives, the Student-*t* distribution and the GED are common choices in most existing literature. A recent study by Calzolari *et al.* [10] points out that due to the instability under aggregation, the Student-*t* is not an optimal option. To overcome that, the Stable distribution is introduced. However, the undefined second moment of it brings in even more serious problems to be applied in the FIGARCH model.

To solve this issue, this paper argues that the tempered stable distribution should be used. Via three different sets of simulation studies on the FIGARCH process, we systematically demonstrate the appropriateness of the tempered stable distribution applied in the FIGARCH model. The first two studies assume that the true distributions are the Student-*t* and GED, respectively. In those cases, results of FIGARCH-S are close to those of the true models. Additionally, FIGARCH-S generally outperforms its competitor (the other models except the true model) in terms of consistency, efficiency and overall performance. We construct different combinations of the underlying tempered stable distribution in the third study. Our results suggest that none of the FIGARCH-N, FIGARCH-t and FIGARCH-S can perform as well as the FIGARCH-S model.

Finally, empirical evidence is further provided as the robustness check of our simulation results. We fit the hourly return of the S&P 500 index into the four FIGARCH models, respectively. Our results suggest that FIGARCH-S is still preferred to the others, according to AIC, BIC and forecasting performance. Also, the density of fitted tempered stable distribution is closer to that of the standardized data than the others. Hence, we argue that the tempered stable distribution could be a widely useful tool for modelling the long memory of financial volatility in general contexts with a FIGARCH-type specification.

Notes

1. In this case, the associated Levy processes are called 'truncated Levy flights', the appropriateness of which to be applied in the original GARCH model is also discussed in [14].
2. The calculation of the summation is conducted via fast Fourier transformation algorithm [25]. We thank Jensen and Nielsen for making the computation codes available.
3. Since the mean of ε_t is 0, η_t is sometimes named as standardized residual.
4. The second moment of the Stable distribution only exists when $\alpha = 2$. In this case, the symmetric Stable distribution collapses to a Gaussian distribution and cannot describe the fat fails.
5. Specific conditions can be found in [2].
6. As discussed in [33], discrete Fourier transform works most efficiently for N being expressed in terms of a power of 2.
7. The details of which can be found in [31].
8. For distributions with defined density function (like Normal, Student-t and GED), steps (2)–(6) are replaced by generating the density values of η_t with the estimates of the distribution parameters.
9. The FIGARCH(0,d,0), FIGARCH(1,d,0) and FIGARCH(0,d,1) processes can lead to similar results, which are available upon request.
10. We also considered the cases of 4, 5 and 6 degrees of freedom. The results are robust and available upon request.
11. We also considered the cases of 1.3, 1.5 and 1.8 degrees of freedom. The results are robust and available upon request.
12. We report conditional volatility here as the square root of h_t, so that it has the same scale as r_t.

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