mhd-hermes Documentation

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ONE

INSTALLATION INSTRUCTIONS

Install hermes2d, so that you can `import hermes2d` from Python:

```
In [1]: import hermes2d

In [2]:
```

Once this works, then just run:

```
cmake .
make
```

and that's it (cmake will ask the `hermes2d` module where all the `*.h` and `*.pxd` files are).

CHAPTER TWO

MHD EQUATIONS

2.1 Introduction

The magnetohydrodynamics (MHD) equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.1}$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} \tag{2.2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \tag{2.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

assuming η is constant. See the next section for a derivation. We can now apply the following identities (we use the fact that $\nabla \cdot \mathbf{B} = 0$):

$$\begin{aligned}
[(\nabla \times \mathbf{B}) \times \mathbf{B}]_i = \varepsilon_{ijk}(\nabla \times \mathbf{B})_j B_k = \varepsilon_{ijk}\varepsilon_{jlm}(\partial_l B_m) B_k = (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(\partial_l B_m) B_k = \\
= (\partial_k B_i) B_k - (\partial_i B_k) B_k = \left[(\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2 \right]_i
\end{aligned}$$

$$\begin{aligned}
(\nabla \times \mathbf{B}) \times \mathbf{B} &= (\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{1}{2}\nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla)\mathbf{B} + \mathbf{B}(\nabla \cdot \mathbf{B}) - \frac{1}{2}\nabla |\mathbf{B}|^2 = \nabla \cdot (\mathbf{B}\mathbf{B}^T) - \frac{1}{2}\nabla |\mathbf{B}|^2 \\
\nabla \times (\mathbf{v} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}(\nabla \cdot \mathbf{v}) + \mathbf{v}(\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla)\mathbf{B} = \nabla \cdot (\mathbf{B}\mathbf{v}^T - \mathbf{v}\mathbf{B}^T) \\
\nabla \cdot (\rho \mathbf{v}\mathbf{v}^T) &= (\nabla \cdot (\rho \mathbf{v}))\, \mathbf{v} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = -\mathbf{v}\frac{\partial \rho}{\partial t} + \rho(\mathbf{v} \cdot \nabla)\mathbf{v}
\end{aligned}$$

So the MHD equations can alternatively be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{2.5}$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}^T) = -\nabla p + \frac{1}{\mu}\left(\nabla \cdot (\mathbf{B}\mathbf{B}^T) - \frac{1}{2}\nabla |\mathbf{B}|^2 \right) + \rho \mathbf{g} \tag{2.6}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \cdot (\mathbf{B}\mathbf{v}^T - \mathbf{v}\mathbf{B}^T) + \eta \nabla^2 \mathbf{B} \tag{2.7}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.8}$$

One can also introduce a new variable $p^* = p + \frac{1}{2}\nabla |\mathbf{B}|^2$, that simplifies (2.6) a bit.

2.2 Derivation

The above equations can easily be derived. We have the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Navier-Stokes equations (momentum equation) with the Lorentz force on the right-hand side:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

where the current density $\mathbf{j}$ is given by the Maxwell equation (we neglect the displacement current $\frac{\partial \mathbf{E}}{\partial t}$):

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}$$

and the Lorentz force:

$$\frac{1}{\sigma}\mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

from which we eliminate $\mathbf{E}$:

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma}\mathbf{j} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma\mu} \nabla \times \mathbf{B}$$

and put it into the Maxwell equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

so we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\sigma\mu} \nabla \times \mathbf{B} \right)$$

assuming the magnetic diffusivity $\eta = \frac{1}{\sigma\mu}$ is constant, we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \left(\nabla^2 \mathbf{B} - \nabla(\nabla \cdot \mathbf{B}) \right) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

where we used the Maxwell equation:

$$\nabla \cdot \mathbf{B} = 0$$

2.3 Finite Element Formulation

We solve the following ideal MHD equations (we use $p^* = p + \frac{1}{2}\nabla|\mathbf{B}|^2$, but we drop the star):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - (\mathbf{B} \cdot \nabla)\mathbf{B} + \nabla p = 0 \tag{2.9}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} = 0 \tag{2.10}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.11}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.12}$$

We discretize in time by introducing a small time step τ and we also linearize the convective terms:

$$\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^n - (\mathbf{B}^{n-1} \cdot \nabla)\mathbf{B}^n + \nabla p = 0 \tag{2.13}$$

$$\frac{\mathbf{B}^n - \mathbf{B}^{n-1}}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla)\mathbf{B}^n - (\mathbf{B}^{n-1} \cdot \nabla)\mathbf{u}^n = 0 \tag{2.14}$$

$$\nabla \cdot \mathbf{u}^n = 0 \tag{2.15}$$

$$\nabla \cdot \mathbf{B}^n = 0 \tag{2.16}$$

Testing (2.13) by the test functions (v_1, v_2), (2.14) by the functions (C_1, C_2), (2.15) by the test function q and (2.16) by the test function r, we obtain the following weak formulation:

$$\begin{aligned}\int_\Omega \frac{u_1 v_1}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) u_1 v_1 - (\mathbf{B}^{n-1} \cdot \nabla) B_1 v_1 - p\frac{\partial v_1}{\partial x} \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{u_1^{n-1} v_1}{\tau} \,\mathrm{d}\mathbf{x} \\ \int_\Omega \frac{u_2 v_2}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) u_2 v_2 - (\mathbf{B}^{n-1} \cdot \nabla) B_2 v_2 - p\frac{\partial v_2}{\partial y} \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{u_2^{n-1} v_2}{\tau} \,\mathrm{d}\mathbf{x}\end{aligned} \tag{2.17}$$

$$\begin{aligned}\int_\Omega \frac{B_1 C_1}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) B_1 C_1 - (\mathbf{B}^{n-1} \cdot \nabla) u_1 C_1 \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{B_1^{n-1} C_1}{\tau} \,\mathrm{d}\mathbf{x} \\ \int_\Omega \frac{B_2 C_2}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) B_2 C_2 - (\mathbf{B}^{n-1} \cdot \nabla) u_2 C_2 \,\mathrm{d}\mathbf{x} &= \int_\Omega \frac{B_2^{n-1} C_2}{\tau} \,\mathrm{d}\mathbf{x}\end{aligned} \tag{2.18}$$

$$\int_\Omega \frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \,\mathrm{d}\mathbf{x} = 0 \tag{2.19}$$

$$\int_\Omega \frac{\partial B_1}{\partial x} r + \frac{\partial B_2}{\partial y} r \,\mathrm{d}\mathbf{x} = 0 \tag{2.20}$$

Now we write it in the block form:

$$\begin{array}{lclclclclclcl} a_{11}(u_1, v_1) & + & a_{12}(u_2, v_1) & + & a_{13}(p, v_1) & + & a_{14}(B_1, v_1) & + & a_{15}(B_2, v_1) & + & a_{16}(r, v_1) & = & l_1(v_1) \\ a_{21}(u_1, v_2) & + & a_{22}(u_2, v_2) & + & a_{23}(p, v_2) & + & a_{24}(B_1, v_2) & + & a_{25}(B_2, v_2) & + & a_{26}(r, v_2) & = & l_2(v_2) \\ a_{31}(u_1, q) & + & a_{32}(u_2, q) & + & a_{33}(p, q) & + & a_{34}(B_1, q) & + & a_{35}(B_2, q) & + & a_{36}(r, q) & = & l_3(q) \\ a_{41}(u_1, C_1) & + & a_{42}(u_2, C_1) & + & a_{43}(p, C_1) & + & a_{44}(B_1, C_1) & + & a_{45}(B_2, C_1) & + & a_{46}(r, C_1) & = & l_4(C_1) \\ a_{51}(u_1, C_2) & + & a_{52}(u_2, C_2) & + & a_{53}(p, C_2) & + & a_{54}(B_1, C_2) & + & a_{55}(B_2, C_2) & + & a_{56}(r, C_2) & = & l_5(C_2) \\ a_{61}(u_1, r) & + & a_{62}(u_2, r) & + & a_{63}(p, r) & + & a_{64}(B_1, r) & + & a_{65}(B_2, r) & + & a_{66}(r, r) & = & l_6(r) \end{array}$$

so we get the following nonzero forms:

$$\begin{array}{lclclclclcl} a_{11}(u_1, v_1) & + & 0 & + & a_{13}(p, v_1) & + & a_{14}(B_1, v_1) & + & 0 & = & l_1(v_1) \\ 0 & + & a_{22}(u_2, v_2) & + & a_{23}(p, v_2) & + & 0 & + & a_{25}(B_2, v_2) & = & l_2(v_2) \\ a_{31}(u_1, q) & + & a_{32}(u_2, q) & + & 0 & + & 0 & + & 0 & = & 0 \\ a_{41}(u_1, C_1) & + & 0 & + & 0 & + & a_{44}(B_1, C_1) & + & 0 & = & l_4(C_1) \\ 0 & + & a_{52}(u_2, C_2) & + & 0 & + & 0 & + & a_{55}(B_2, C_2) & = & l_5(C_2) \\ 0 & + & 0 & + & 0 & + & a_{64}(B_1, r) & + & a_{65}(B_2, r) & = & 0 \end{array}$$

$$
\begin{aligned}
a_{11}(u,v) = a_{22}(u,v) = a_{44}(u,v) = a_{55}(u,v) &= \int_\Omega \frac{uv}{\tau} + (\mathbf{u}^{n-1}\cdot\nabla)uv\,\mathrm{d}\mathbf{x} \\
a_{13}(p,v) = -a_{31}(v,p) = -a_{64}(v,p) &= \int_\Omega -p\frac{\partial v}{\partial x}\,\mathrm{d}\mathbf{x} \\
a_{23}(p,v) = -a_{32}(v,p) = -a_{65}(v,p) &= \int_\Omega -p\frac{\partial v}{\partial y}\,\mathrm{d}\mathbf{x} \\
a_{14}(B,v) = a_{25}(B,v) = a_{41}(v,B) = a_{52}(v,B) &= -\int_\Omega (\mathbf{B}^{n-1}\cdot\nabla)Bv\,\mathrm{d}\mathbf{x} \\
l_1(v) &= \int_\Omega \frac{u_1^{n-1}v}{\tau}\,\mathrm{d}\mathbf{x} \\
l_2(v) &= \int_\Omega \frac{u_2^{n-1}v}{\tau}\,\mathrm{d}\mathbf{x} \\
l_4(v) &= \int_\Omega \frac{B_1^{n-1}v}{\tau}\,\mathrm{d}\mathbf{x} \\
l_5(v) &= \int_\Omega \frac{B_2^{n-1}v}{\tau}\,\mathrm{d}\mathbf{x}
\end{aligned}
$$

CHAPTER

THREE

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