
mhd-hermes Documentation

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Ondrej Certik

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Contents:

INSTALLATION INSTRUCTIONS

Install `hermes2d`, so that you can `import hermes2d` from Python:

```
In [1]: import hermes2d
```

```
In [2]:
```

Once this works, then just run:

```
cmake .  
make
```

and that's it (cmake will ask the `hermes2d` module where all the `*.h` and `*.pxd` files are).

MHD EQUATIONS

2.1 Introduction

The magnetohydrodynamics (MHD) equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.1)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} \quad (2.2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad (2.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.4)$$

assuming η is constant. See the next section for a derivation. We can now apply the following identities (we use the fact that $\nabla \cdot \mathbf{B} = 0$):

$$\begin{aligned} [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i &= \varepsilon_{ijk} (\nabla \times \mathbf{B})_j B_k = \varepsilon_{ijk} \varepsilon_{jlm} (\partial_l B_m) B_k = (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) (\partial_l B_m) B_k = \\ &= (\partial_k B_i) B_k - (\partial_i B_k) B_k = \left[(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right]_i \\ (\nabla \times \mathbf{B}) \times \mathbf{B} &= (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla) \mathbf{B} + \mathbf{B} (\nabla \cdot \mathbf{B}) - \frac{1}{2} \nabla |\mathbf{B}|^2 = \nabla \cdot (\mathbf{B} \mathbf{B}^T) - \frac{1}{2} \nabla |\mathbf{B}|^2 \\ \nabla \times (\mathbf{v} \times \mathbf{B}) &= (\mathbf{B} \cdot \nabla) \mathbf{v} - \mathbf{B} (\nabla \cdot \mathbf{v}) + \mathbf{v} (\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla) \mathbf{B} = \nabla \cdot (\mathbf{B} \mathbf{v}^T - \mathbf{v} \mathbf{B}^T) \\ \nabla \cdot (\rho \mathbf{v} \mathbf{v}^T) &= (\nabla \cdot (\rho \mathbf{v})) \mathbf{v} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\mathbf{v} \frac{\partial \rho}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} \end{aligned}$$

So the MHD equations can alternatively be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.5)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}^T) = -\nabla p + \frac{1}{\mu} \left(\nabla \cdot (\mathbf{B} \mathbf{B}^T) - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) + \rho \mathbf{g} \quad (2.6)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \cdot (\mathbf{B} \mathbf{v}^T - \mathbf{v} \mathbf{B}^T) + \eta \nabla^2 \mathbf{B} \quad (2.7)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.8)$$

One can also introduce a new variable $p^* = p + \frac{1}{2} \nabla |\mathbf{B}|^2$, that simplifies (2.6) a bit.

2.2 Derivation

The above equations can easily be derived. We have the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Navier-Stokes equations (momentum equation) with the Lorentz force on the right-hand side:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

where the current density \mathbf{j} is given by the Maxwell equation (we neglect the displacement current $\frac{\partial \mathbf{E}}{\partial t}$):

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}$$

and the Lorentz force:

$$\frac{1}{\sigma} \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

from which we eliminate \mathbf{E} :

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma} \mathbf{j} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma \mu} \nabla \times \mathbf{B}$$

and put it into the Maxwell equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

so we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\sigma \mu} \nabla \times \mathbf{B} \right)$$

assuming the magnetic diffusivity $\eta = \frac{1}{\sigma \mu}$ is constant, we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta (\nabla^2 \mathbf{B} - \nabla(\nabla \cdot \mathbf{B})) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

where we used the Maxwell equation:

$$\nabla \cdot \mathbf{B} = 0$$

2.3 Finite Element Formulation

We solve the following ideal MHD equations (we use $p^* = p + \frac{1}{2} \nabla |\mathbf{B}|^2$, but we drop the star):

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - (\mathbf{B} \cdot \nabla) \mathbf{B} + \nabla p = 0 \tag{2.9}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} = 0 \tag{2.10}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.11}$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.12)$$

We discretize in time by introducing a small time step τ and we also linearize the convective terms:

$$\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) \mathbf{u}^n - (\mathbf{B}^{n-1} \cdot \nabla) \mathbf{B}^n + \nabla p = 0 \quad (2.13)$$

$$\frac{\mathbf{B}^n - \mathbf{B}^{n-1}}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) \mathbf{B}^n - (\mathbf{B}^{n-1} \cdot \nabla) \mathbf{u}^n = 0 \quad (2.14)$$

$$\nabla \cdot \mathbf{u}^n = 0 \quad (2.15)$$

$$\nabla \cdot \mathbf{B}^n = 0 \quad (2.16)$$

Testing (2.13) by the velocity test functions (v_1, v_2) , (2.14) by the magnetic field test functions (C_1, C_2) , (2.15) and (2.16) by the pressure test function q , we obtain the following weak formulation:

$$\begin{aligned} \int_{\Omega} \frac{u_1 v_1}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) u_1 v_1 - (\mathbf{B}^{n-1} \cdot \nabla) B_1 v_1 - p \frac{\partial v_1}{\partial x} \, \mathrm{d}\mathbf{x} &= \int_{\Omega} \frac{u_1^{n-1} v_1}{\tau} \, \mathrm{d}\mathbf{x} \\ \int_{\Omega} \frac{u_2 v_2}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) u_2 v_2 - (\mathbf{B}^{n-1} \cdot \nabla) B_2 v_2 - p \frac{\partial v_2}{\partial y} \, \mathrm{d}\mathbf{x} &= \int_{\Omega} \frac{u_2^{n-1} v_2}{\tau} \, \mathrm{d}\mathbf{x} \end{aligned} \quad (2.17)$$

$$\begin{aligned} \int_{\Omega} \frac{B_1 C_1}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) B_1 C_1 - (\mathbf{B}^{n-1} \cdot \nabla) u_1 C_1 \, \mathrm{d}\mathbf{x} &= \int_{\Omega} \frac{B_1^{n-1} C_1}{\tau} \, \mathrm{d}\mathbf{x} \\ \int_{\Omega} \frac{B_2 C_2}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) B_2 C_2 - (\mathbf{B}^{n-1} \cdot \nabla) u_2 C_2 \, \mathrm{d}\mathbf{x} &= \int_{\Omega} \frac{B_2^{n-1} C_2}{\tau} \, \mathrm{d}\mathbf{x} \end{aligned} \quad (2.18)$$

$$\int_{\Omega} \frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \, \mathrm{d}\mathbf{x} = 0 \quad (2.19)$$

$$\int_{\Omega} \frac{\partial B_1}{\partial x} q + \frac{\partial B_2}{\partial y} q \, \mathrm{d}\mathbf{x} = 0 \quad (2.20)$$

Now we write it in the block form:

$$\begin{array}{llllll} a_{11}(u_1, v_1) & + & a_{12}(u_2, v_1) & + & a_{13}(p, v_1) & + & a_{14}(B_1, v_1) & + & a_{15}(B_2, v_1) & = & l_1(v_1) \\ a_{21}(u_1, v_2) & + & a_{22}(u_2, v_2) & + & a_{23}(p, v_2) & + & a_{24}(B_1, v_2) & + & a_{25}(B_2, v_2) & = & l_2(v_2) \\ a_{31}(u_1, q) & + & a_{32}(u_2, q) & + & a_{33}(p, q) & + & a_{34}(B_1, q) & + & a_{35}(B_2, q) & = & l_3(q) \\ a_{41}(u_1, C_1) & + & a_{42}(u_2, C_1) & + & a_{43}(p, C_1) & + & a_{44}(B_1, C_1) & + & a_{45}(B_2, C_1) & = & l_4(C_1) \\ a_{51}(u_1, C_2) & + & a_{52}(u_2, C_2) & + & a_{53}(p, C_2) & + & a_{54}(B_1, C_2) & + & a_{55}(B_2, C_2) & = & l_5(C_2) \end{array}$$

so we get the following nonzero forms:

$$\begin{array}{llllll} a_{11}(u_1, v_1) & + & 0 & + & a_{13}(p, v_1) & + & a_{14}(B_1, v_1) & + & 0 & = & l_1(v_1) \\ 0 & + & a_{22}(u_2, v_2) & + & a_{23}(p, v_2) & + & 0 & + & a_{25}(B_2, v_2) & = & l_2(v_2) \\ a_{31}(u_1, q) & + & a_{32}(u_2, q) & + & 0 & + & a_{34}(B_1, q) & + & a_{35}(B_2, q) & = & 0 \\ a_{41}(u_1, C_1) & + & 0 & + & 0 & + & a_{44}(B_1, C_1) & + & 0 & = & l_4(C_1) \\ 0 & + & a_{52}(u_2, C_2) & + & 0 & + & 0 & + & a_{55}(B_2, C_2) & = & l_5(C_2) \end{array}$$

$$\begin{aligned}
 a_{11}(u, v) &= a_{22}(u, v) = a_{44}(u, v) = a_{55}(u, v) = \int_{\Omega} \frac{uv}{\tau} + (\mathbf{u}^{n-1} \cdot \nabla) uv \, d\mathbf{x} \\
 a_{13}(p, v) &= -a_{31}(v, p) = -a_{34}(v, p) = \int_{\Omega} -p \frac{\partial v}{\partial x} \, d\mathbf{x} \\
 a_{23}(p, v) &= -a_{32}(v, p) = -a_{35}(v, p) = \int_{\Omega} -p \frac{\partial v}{\partial y} \, d\mathbf{x} \\
 a_{14}(B, v) &= a_{25}(B, v) = a_{41}(v, B) = a_{52}(v, B) = - \int_{\Omega} (\mathbf{B}^{n-1} \cdot \nabla) Bv \, d\mathbf{x} \\
 l_1(v) &= \int_{\Omega} \frac{u_1^{n-1} v}{\tau} \, d\mathbf{x} \\
 l_2(v) &= \int_{\Omega} \frac{u_2^{n-1} v}{\tau} \, d\mathbf{x} \\
 l_4(v) &= \int_{\Omega} \frac{B_1^{n-1} v}{\tau} \, d\mathbf{x} \\
 l_5(v) &= \int_{\Omega} \frac{B_2^{n-1} v}{\tau} \, d\mathbf{x}
 \end{aligned}$$

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