
mhd-hermes Documentation

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Contents:

INSTALLATION INSTRUCTIONS

Install `hermes2d`, so that you can `import hermes2d` from Python:

```
In [1]: import hermes2d
```

```
In [2]:
```

Once this works, then just run:

```
cmake .  
make
```

and that's it (cmake will ask the `hermes2d` module where all the `*.h` and `*.pxd` files are).

MHD EQUATIONS

2.1 Introduction

The magnetohydrodynamics (MHD) equations are:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.1)$$

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \rho \mathbf{g} \quad (2.2)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B} \quad (2.3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.4)$$

assuming η is constant. See the next section for a derivation. We can now apply the following identities (we use the fact that $\nabla \cdot \mathbf{B} = 0$):

$$\begin{aligned} [(\nabla \times \mathbf{B}) \times \mathbf{B}]_i &= \varepsilon_{ijk} (\nabla \times \mathbf{B})_j B_k = \varepsilon_{ijk} \varepsilon_{jlm} (\partial_l B_m) B_k = (\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}) (\partial_l B_m) B_k = (\partial_k B_i) B_k - (\partial_i B_k) B_k = \\ &= \left[(\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 \right]_i \end{aligned}$$

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \frac{1}{2} \nabla |\mathbf{B}|^2 = (\mathbf{B} \cdot \nabla) \mathbf{B} + \mathbf{B} (\nabla \cdot \mathbf{B}) - \frac{1}{2} \nabla |\mathbf{B}|^2 = \nabla \cdot (\mathbf{B} \mathbf{B}^T) - \frac{1}{2} \nabla |\mathbf{B}|^2$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla) \mathbf{v} - \mathbf{B} (\nabla \cdot \mathbf{v}) + \mathbf{v} (\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla) \mathbf{B} = \nabla \cdot (\mathbf{B} \mathbf{v}^T - \mathbf{v} \mathbf{B}^T)$$

$$\nabla \cdot (\rho \mathbf{v} \mathbf{v}^T) = (\nabla \cdot (\rho \mathbf{v})) \mathbf{v} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = -\mathbf{v} \frac{\partial \rho}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v}$$

So the MHD equations can alternatively be written as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \quad (2.5)$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}^T) = -\nabla p + \frac{1}{\mu} \left(\nabla \cdot (\mathbf{B} \mathbf{B}^T) - \frac{1}{2} \nabla |\mathbf{B}|^2 \right) + \rho \mathbf{g} \quad (2.6)$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \cdot (\mathbf{B} \mathbf{v}^T - \mathbf{v} \mathbf{B}^T) + \eta \nabla^2 \mathbf{B} \quad (2.7)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (2.8)$$

2.2 Derivation

The above equations can easily be derived. We have the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Navier-Stokes equations (momentum equation) with the Lorentz force on the right-hand side:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g}$$

where the current density \mathbf{j} is given by the Maxwell equation (we neglect the displacement current $\frac{\partial \mathbf{E}}{\partial t}$):

$$\mathbf{j} = \frac{1}{\mu} \nabla \times \mathbf{B}$$

and the Lorentz force:

$$\frac{1}{\sigma} \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}$$

from which we eliminate \mathbf{E} :

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma} \mathbf{j} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\sigma \mu} \nabla \times \mathbf{B}$$

and put it into the Maxwell equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

so we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \left(\frac{1}{\sigma \mu} \nabla \times \mathbf{B} \right)$$

assuming the magnetic diffusivity $\eta = \frac{1}{\sigma \mu}$ is constant, we get:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \nabla \times (\nabla \times \mathbf{B}) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta (\nabla^2 \mathbf{B} - \nabla(\nabla \cdot \mathbf{B})) = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

where we used the Maxwell equation:

$$\nabla \cdot \mathbf{B} = 0$$

INDICES AND TABLES

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