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Abstract. Add abstract here.

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1. Introduction. Equality-constrained optimization problems arise...

[Lingjun](#): Add a citation to the paper for the unconstrained setting. The unconstrained progressive sampling paper is [1].

1.1. Contributions. Our contributions relate ...

1.2. Notation. We use \mathbb{R} to denote the set of real numbers, $\mathbb{R}_{\geq r}$ (resp., $\mathbb{R}_{>r}$) to denote the set of real numbers greater than or equal to (resp., greater than) $r \in \mathbb{R}$, \mathbb{R}^n to denote the set of n -dimensional real vectors, and $\mathbb{R}^{m \times n}$ to denote the set of m -by- n -dimensional real matrices. We denote the set of nonnegative integers as $\mathbb{N} := \{0, 1, 2, \dots\}$, and, for any integer $N \geq 1$, we use $[N]$ to denote the set $\{1, \dots, N\}$.

For any finite set \mathcal{S} , we use $|\mathcal{S}|$ to denote its cardinality. We consider all vector norms to be Euclidean, i.e., we let $\|\cdot\| := \|\cdot\|_2$, unless otherwise specified. Similarly, we use $\|\cdot\|$ to denote the spectral norm of any matrix input.

For any matrix $A \in \mathbb{R}^{m \times n}$, we use $\sigma_i(A)$ to denote its i th largest singular value. Given any such A , we use $\text{Null}(A)$ to denote its null space, i.e., $\{d \in \mathbb{R}^n : Ad = 0\}$. Assuming $B \in \mathbb{R}^{n \times m}$ has full column rank, we use B^\dagger to denote its pseudoinverse, i.e., $B^\dagger := (B^T B)^{-1} B^T$. For any subspace $\mathcal{X} \subseteq \mathbb{R}^n$ and point $x \in \mathbb{R}^n$, we denote the projection of x onto \mathcal{X} as $\text{Proj}_{\mathcal{X}}(x) := \arg \min_{\bar{x} \in \mathcal{X}} \|\bar{x} - x\|$. Given $B \in \mathbb{R}^{n \times m}$ with full column rank, we use $\mathcal{R}(B) := BB^\dagger$ and $\mathcal{N}(B) = I - \mathcal{R}(B)$ to denote projection matrices onto the span of the columns of B and the null space of B , respectively.

1.3. Organization. In §3, ...

2. Algorithm. Our proposed algorithm is designed to solve a sample average approximation (SAA) of the continuous nonlinear-equality-constrained problem

$$(2.1) \quad \min_{x \in \mathbb{R}^n} f(x) \text{ s.t. } \bar{c}(x) = 0,$$

where the objective and constraint functions, i.e., $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\bar{c} : \mathbb{R}^n \rightarrow \mathbb{R}^m$, respectively, are twice-continuously differentiable, $m \leq n$, and the constraint function c is defined by an expectation. Formally, with respect to a random variable ω defined

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by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, expectation \mathbb{E} defined by \mathbb{P} , and $\bar{C} : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^m$, the constraint function \bar{c} is defined by $\bar{c}(x) = \mathbb{E}[\bar{C}(x, \omega)]$ for all $x \in \mathbb{R}^n$.

The SAA of problem (2.1) that our algorithm is designed to solve is defined with respect to a sample of $N \in \mathbb{N}$ realizations of the random variable ω , say, $\{\omega_i\}_{i \in [N]}$. Defining the SAA constraint function $c : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for all $x \in \mathbb{R}^n$ by

$$c(x) = \frac{1}{N} \sum_{i=1}^N c_i(x), \quad \text{where } c_i(x) \equiv \bar{C}(x, \omega_i) \quad \text{for all } i \in [N],$$

the problem that our algorithm is designed to solve is that given by

$$(2.2) \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } c(x) = 0.$$

Under mild assumptions about c and an assumption that N is sufficiently large, a point that is approximately stationary for problem (2.2) can be shown to be approximately stationary for problem (2.1), at least with high probability. We leave a formal statement and proof of this fact until the end of our analysis. Until that time, we focus on our proposed algorithm and our analysis of it for solving problem (2.2).

The main idea of our proposed algorithm for solving problem (2.2) is to generate a sequence of iterates, each of which is a stationary point (at least approximately) with respect to a subsampled problem involving only a subset $\mathcal{S} \subseteq [N]$ of constraint function terms. For any such \mathcal{S} , we denote the approximate constraint function as $c_{\mathcal{S}} : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and an approximation of problem (2.2) is given by

$$(2.3) \quad \min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t. } c_{\mathcal{S}}(x) = 0, \quad \text{where } c_{\mathcal{S}}(x) = \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} c_i(x).$$

(Observe that, in this manner, the constraint function in (2.2) is $c_{[N]} = c$.) The primary benefit of considering (2.3) for $\mathcal{S} \subseteq [N]$, rather than (2.2) directly, is that any evaluation of a constraint or constraint Jacobian value requires computing a sum of $|\mathcal{S}| \leq N$ terms, as opposed to N terms. Also, under reasonable assumptions about the constraint functions, we show in this paper that, by starting with an approximate stationary point for problem (2.3) and aiming to solve a subsequent instance of (2.3) with respect to a sample set $\bar{\mathcal{S}} \supseteq \mathcal{S}$, our proposed algorithm can obtain an approximate stationary point for the subsequent instance with lower sample complexity than if the problem with the larger sample set were solved directly. Overall, we show that—at least once the sample sets become sufficiently large relative to N —a sufficiently approximate stationary point of problem (2.2) can be obtained more efficiently through progressive sampling than by tackling the problem directly.

For use in our proposed algorithm and our analysis of it, let us introduce stationarity conditions for problem (2.3), which also represent stationarity conditions for problem (2.2) in the particular case when $\mathcal{S} = [N]$. Let the Lagrangian of problem (2.3) be denoted by $L_{\mathcal{S}} : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, defined for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ by

$$L_{\mathcal{S}}(x, y) = f(x) + c_{\mathcal{S}}(x)^T y = f(x) + \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} c_i(x)^T y,$$

where $y \in \mathbb{R}^m$ is a vector of Lagrange multipliers (also known as dual variables). Second-order necessary conditions for optimality for (2.3) can then be stated as

$$(2.4) \quad \begin{bmatrix} \nabla_x L_{\mathcal{S}}(x, y) \\ \nabla_y L_{\mathcal{S}}(x, y) \end{bmatrix} \equiv \begin{bmatrix} \nabla f(x) + \nabla c_{\mathcal{S}}(x) y \\ c_{\mathcal{S}}(x) \end{bmatrix} = 0$$

and, with $[c_S]_j$ denoting the j th component of the constraint function c_S ,

$$(2.5) \quad d^T \nabla_{xx}^2 L_S(x, y) d \equiv d^T \left(\nabla_{xx}^2 f(x) + \sum_{j \in [m]} \nabla_{xx}^2 [c_S]_j(x) y_j \right) d \geq 0$$

for all $d \in \text{Null}(\nabla c_S(x)^T)$.

We refer to any point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying (2.4) as a first-order stationary point with respect to problem (2.3), and we refer to any such point satisfying both (2.4) and (2.5) as a second-order stationary point with respect to problem (2.3). In addition, consistent with the literature on worst-case complexity bounds for nonconvex smooth optimization, we say that a point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ is (ϵ, ε) -stationary with respect to problem (2.3) for some $(\epsilon, \varepsilon) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ if and only if

$$(2.6a) \quad \left\| \begin{bmatrix} \nabla_x L_S(x, y) \\ \nabla_y L_S(x, y) \end{bmatrix} \right\| \leq \epsilon$$

$$(2.6b) \quad \text{and } d^T \nabla_{xx}^2 L_S(x, y) d \geq -\varepsilon \|d\|_2^2 \text{ for all } d \in \text{Null}(\nabla c_S(x)^T).$$

Generally speaking, an algorithm for solving (2.3) can be a *primal* method that might only generate a sequence of primal iterates $\{x_k\}$, or it can be a *primal-dual* method that generates a sequence of primal and dual iterate pairs $\{(x_k, y_k)\}$. For an application of our proposed algorithm, either type of method can be employed, but for certain results in our analysis we refer to properties of *least-square multipliers* corresponding to a given primal point $x \in \mathbb{R}^n$. Assuming that the Jacobian of c_S at x , namely, $\nabla c_S(x)^T$, has full row rank, the least-squares multipliers with respect to x are given by $y_S(x) \in \mathbb{R}^m$ that minimizes $\|\nabla_x L(x, \cdot)\|^2$, which is given by

$$(2.7) \quad y_S(x) = -(\nabla c_S(x)^T \nabla c_S(x))^{-1} \nabla c_S(x)^T \nabla f(x) = -\nabla c_S(x)^\dagger \nabla f(x).$$

Our proposed method is stated as Algorithm 2.1 below.

Algorithm 2.1 Progressive Constraint-Sampling Method (PCSM) for (2.2)

Require: Initial sample set size $p_1 \in [N]$, initial point $x_0 \in \mathbb{R}^n$, maximum outer iteration index $K = \lceil \log_2 \frac{N}{p_1} \rceil$, and subproblem tolerances $\{(\epsilon_k, \varepsilon_k)\}_{k=1}^K \subset \mathbb{R}_{>0}$

- 1: set $\mathcal{S}_0 \leftarrow \emptyset$
 - 2: **for** $k \in [K]$ **do**
 - 3: choose $\mathcal{S}_k \supseteq \mathcal{S}_{k-1}$ such that $|\mathcal{S}_k| = p_k$
 - 4: using x_{k-1} as a starting point, employ an algorithm to solve (2.3), terminating once a primal iterate x_k has been obtained such that $(x_k, y_{\mathcal{S}_k}(x_k))$ (see (2.7)) is $(\epsilon_k, \varepsilon_k)$ -stationary with respect to problem (2.3) for $\mathcal{S} = \mathcal{S}_k$
 - 5: set $p_{k+1} \leftarrow \min\{2p_k, N\}$
 - 6: **end for**
 - 7: **return** $(x_K, y(x_K))$, which is $(\epsilon_K, \varepsilon_K)$ -stationary with respect to (2.2)
-

3. Analysis. We begin our analysis of Algorithm 2.1 by stating the assumptions under which we prove our convergence guarantees. Our first, Assumption 3.1 below, ensures that any minimizer of each encountered subproblem is a first-order stationary point and that one can expect an algorithm that is employed to solve each subproblem will find a sufficiently approximate first-order stationary point. It would be possible

to prove reasonable convergence guarantees for Assumption 3.1 under looser assumptions. For example, if an algorithm employed to solve (2.3) for some sample set \mathcal{S} were to encounter an (approximate) infeasible stationary point, then it would be reasonable to terminate the subproblem solver and either terminate Algorithm 2.1 in its entirety or move on to solve the next subproblem (with a larger sample set). However, since consideration of such scenarios would distract from the essential properties of our algorithm when each subproblem solve is successful, we make Assumption 3.1.

ASSUMPTION 3.1. *The objective function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and each constraint function $c_i : \mathbb{R}^n \rightarrow \mathbb{R}$ for each $i \in [N]$ are twice-continuously differentiable. In addition, there exists $\sigma_{\min} \in \mathbb{R}_{>0}$ such that, for all $x \in \mathbb{R}^n$ and $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \geq p_1$, the constraint Jacobian has $\sigma_m(\nabla c_{\mathcal{S}}(x)^T) \geq \sigma_{\min}$. Furthermore, for all $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \geq p_1$, the algorithm employed to solve subproblem (2.3) is guaranteed to converge to a second-order stationary point, i.e., one satisfying (2.4) and (2.5).*

The first part of Assumption 3.1 guarantees that the algorithm employed to solve subproblem (2.3) will not, for example, get stuck at an infeasible stationary point. In addition to this assurance, the second part of Assumption 3.1 can be guaranteed if, for example, the algorithm employed to solve subproblem (2.3) is driven by reductions in a merit function that is assumed to be bounded below over the generated iterates.

Our next assumption articulates bounds on derivatives of the objective and constraint functions corresponding to the full-sample problem (2.2).

ASSUMPTION 3.2. *There exists $(\kappa_c, \kappa_{\nabla f}, \kappa_{\nabla c}, \kappa_{\nabla^2 f}, \kappa_{\nabla^2 c}) \in \mathbb{R}_{>0}^5$ such that, for all $x \in \mathbb{R}^n$ and $j \in [m]$, one has $\|c(x)\| \leq \kappa_c$, $\|\nabla f(x)\| \leq \kappa_{\nabla f}$, $\|\nabla c(x)\| \leq \kappa_{\nabla c}$, $\|\nabla^2 f(x)\| \leq \kappa_{\nabla^2 f}$, and $\|\nabla^2[c]_j(x)\| \leq \kappa_{\nabla^2 c}$ for all $j \in [m]$, where $[c]_j$ denotes the j th component of c .*

Our next assumption introduces constants that bound discrepancies between constraint Jacobians corresponding to individual samples and those corresponding to the full set of samples, and introduces constants that similarly bound discrepancies between individual-sample and the full-sample constraint Hessian matrices.

ASSUMPTION 3.3. *There exists $(\gamma_c, \theta_J, \nu_J, \mu_H) \in \mathbb{R}_{>0}^3$ such that the following hold.*

(a) *For all $x \in \mathbb{R}^n$, one has*

$$\frac{1}{N} \sum_{i=1}^N \|c_i(x) - c(x)\|^2 \leq \gamma_c \|c(x)\|^2,$$

$$\frac{1}{N} \sum_{i=1}^N \|\nabla c_i(x)^T \mathcal{R}(\nabla c(x)) - \nabla c(x)^T\|^2 \leq \theta_J \|\nabla c(x)^T\|^2$$

$$\text{and } \frac{1}{N} \sum_{i=1}^N \|\nabla c_i(x)^T \mathcal{N}(\nabla c(x))\|^2 \leq \nu_J \|\nabla c(x)^T\|^2.$$

(b) *For all $x \in \mathbb{R}^n$ and $j \in [m]$, one has*

$$\frac{1}{N} \sum_{i=1}^N \|\nabla^2[c]_j(x) - \nabla^2[c]_j(x)\|^2 \leq \mu_H \|\nabla^2[c]_j(x)\|^2.$$

Finally, we make the following assumption.

ASSUMPTION 3.4. For some $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, problem (2.2) is (α, β) -strongly Morse in the sense that, for any $x \in \mathbb{R}^n$, if $(x, y(x))$ satisfies $\|\nabla L(x, y(x))\| \leq \alpha$, then $|d^T \nabla_{xx}^2 L(x, y(x)) d| \geq \beta \|d\|_2^2$ for all $d \in \text{Null}(\nabla c(x)^T)$.

DEFINITION 3.1. For any full column rank matrices $(A, B) \in \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times m}$ where $n \geq m$, the A and B are acute perturbations to each other, if and only if

$$\text{rank}(AA^\dagger BA^\dagger A) = m.$$

LEMMA 3.2. Under Assumption 3.2 where constant σ_c^{\min} exist. In addition, under Assumption 3.3 where constants $(\gamma_c, \theta_J, \nu_J, \mu_H)$ exist. Then, the following hold

(1). A sample average result holds, that is, for any $(j, x, \mathcal{S}) \in [m] \times \mathbb{R}^n \times [N]$, we have

$$\begin{aligned} \|c_{\mathcal{S}}(x) - c(x)\|_2^2 &\leq N \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) \gamma_c \|c(x)\|_2^2, \\ \|\nabla c_{\mathcal{S}}(x)^T \mathcal{R}(\nabla c(x)) - \nabla c(x)^T\|_2^2 &\leq N \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) \theta_J \|\nabla c(x)^T\|_2^2, \\ \|\nabla c_{\mathcal{S}}(x)^T \mathcal{N}(\nabla c(x))\|_2^2 &\leq N \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) \nu_J \|\nabla c(x)\|_2^2, \\ \|\nabla^2[c_{\mathcal{S}}]_j(x) - \nabla^2[c]_j(x)\|_2^2 &\leq N \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) \mu_H \|\nabla^2[c]_j(x)\|_2^2. \end{aligned}$$

(2). The ∇c^\dagger , $y_{[N]}$ and $\nabla_{xx}^2 L_{[N]}$ are bounded, that is, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned} \|\nabla c(x)^\dagger\|_2 &\leq \frac{1}{\sigma_{\min}} \quad \text{and, } \|y_{[N]}(x)\|_2 \leq \frac{\kappa_{\nabla f}}{\sigma_{\min}}, \text{ moreover} \\ \|\nabla_{xx}^2 L_{[N]}(x, y_{[N]})\|_2 &\leq \kappa_{\nabla^2 f} + \frac{\sqrt{m} \kappa_{\nabla f} \kappa_{\nabla^2 c}}{\sigma_{\min}}. \end{aligned}$$

Proof. For the first item, we only show the second inequality in (3.1), and the other two inequalities follow a similar argument. Notice that

$$\begin{aligned} \nabla c_{\mathcal{S}}(x) &= \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} \nabla c_i(x) = \frac{1}{|\mathcal{S}|} \sum_{i \in [N]} \nabla c_i(x) - \frac{1}{|\mathcal{S}|} \sum_{i \in [N] \setminus \mathcal{S}} \nabla c_i(x) \\ &= \frac{N}{|\mathcal{S}|} \nabla c(x) - \frac{1}{|\mathcal{S}|} \sum_{i \in [N] \setminus \mathcal{S}} \nabla c_i(x), \end{aligned}$$

we have

$$\begin{aligned} &\|\nabla c_{\mathcal{S}}(x)^T \mathcal{R}(\nabla c(x)) - \nabla c(x)^T\|_2^2 \\ &= \left\| \frac{N}{|\mathcal{S}|} \nabla c(x)^T \mathcal{R}(\nabla c(x)) - \nabla c(x)^T - \frac{1}{|\mathcal{S}|} \sum_{i \in [N] \setminus \mathcal{S}} \nabla c_i(x)^T \mathcal{R}(\nabla c(x)) \right\|_2^2 \\ &= \underbrace{\left\| \frac{N - |\mathcal{S}|}{|\mathcal{S}|} \nabla c(x)^T - \frac{1}{|\mathcal{S}|} \sum_{i \in [N] \setminus \mathcal{S}} \nabla c_i(x)^T \mathcal{R}(\nabla c(x)^T) \right\|_2^2}_{(i)}. \end{aligned}$$

157 Here, the second line substitutes (3.3) into the equation. For the third line, by the
 158 definition of \mathcal{R} , we have $\nabla c(x)^T \mathcal{R}(\nabla c(x)) = \nabla c(x)^T$, and substitute it to the first
 159 term of the second line gives the result. Further, for (i), we have

$$\begin{aligned}
 (i) &= \frac{1}{|\mathcal{S}|^2} \left\| \sum_{i \in [N] \setminus \mathcal{S}} \{(\nabla c(x)^T - \nabla c_i(x)^T \mathcal{R}(\nabla c(x))) \times I_n\} \right\|_2^2 \\
 &\leq \frac{1}{|\mathcal{S}|^2} \sum_{i \in [N] \setminus \mathcal{S}} \|\nabla c(x)^T - \nabla c_i(x)^T \mathcal{R}(\nabla c(x))\|_2^2 \sum_{i \in [N] \setminus \mathcal{S}} \|I_n\|_2^2 \\
 &= \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) \sum_{i \in [N] \setminus \mathcal{S}} \|\nabla c(x)^T - \nabla c_i(x)^T \mathcal{R}(\nabla c(x))\|_2^2 \\
 &\leq \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) \sum_{i \in [N]} \|\nabla c(x)^T - \nabla c_i(x)^T \mathcal{R}(\nabla c(x))\|_2^2 \\
 &\leq \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) N \theta_J \|\nabla c(x)^T\|_2^2.
 \end{aligned}$$

161 Here, the first line puts the denominator outside the norm and uses a fact that
 162 $(N - |\mathcal{S}|) \nabla c(x)^T = \sum_{i \in [N] \setminus \mathcal{S}} \nabla c_i(x)^T$. The second line uses the Cauchy-Schwarz in-
 163 equality. The third line uses that $\|I_n\|_2 = 1$. The second to last line adds extra $|\mathcal{S}|$
 164 nonnegative terms, and the last line uses the first item of Assumption 3.3.

165 For the second item, the bound on $\|\nabla c(x)^\dagger\|_2$ is obvious since $\nabla c(x)$ has full
 166 column rank, and by the definition $\nabla c(x)^\dagger = (\nabla c_{\mathcal{S}}(x)^T \nabla c_{\mathcal{S}}(x))^{-1} \nabla c_{\mathcal{S}}(x)^T$.

167 For the bound for $\|y_{[N]}(x)\|_2$, by Assumption 3.2, the bound for $\|\nabla c(x)^\dagger\|_2$ and
 168 sub-multiplicity for matrix-vector product, we have

$$\|y_{[N]}(x)\|_2 = \|\nabla c(x)^\dagger \nabla f(x)\|_2 \leq \|\nabla c(x)^\dagger\|_2 \|\nabla f(x)\|_2 \leq \frac{\kappa_{\nabla f}}{\sigma_{\min}}.$$

171 For the last inequality, note that for any vector $y \in \mathbb{R}^m$, we have $\|y\|_1 \leq \sqrt{m} \|y\|_2$.
 172 Combining this result with Assumption 3.2, we have

$$\begin{aligned}
 \|\nabla_{xx}^2 L_{[N]}(x, y_{[N]})\|_2 &= \|\nabla^2 f(x) + \sum_{j=1}^m y_{[N]}^j \nabla^2 [c]_j(x)\|_2 \\
 &\leq \|\nabla^2 f(x)\|_2 + \left\| \sum_{j=1}^m y_{[N]}^j \nabla^2 [c]_j(x) \right\|_2 \\
 &\leq \kappa_{\nabla^2 f} + \max_j \{\|\nabla^2 [c]_j(x)\|_2\} \|y_{[N]}\|_1 \\
 &\leq \kappa_{\nabla^2 f} + \sqrt{m} \max_j \{\|\nabla^2 [c]_j(x)\|_2\} \|y_{[N]}\|_2 \\
 &\leq \kappa_{\nabla^2 f} + \frac{\sqrt{m} \kappa_{\nabla f} \kappa_{\nabla^2 c}}{\sigma_{\min}}.
 \end{aligned}$$

174 Here, the second line uses the triangle inequality. The third line uses multiplicity and
 175 $\|\nabla^2 [c]_j(x)\|_2 \leq \max_j \{\|\nabla^2 [c]_j(x)\|_2\}$. The rest lines use Assumption 3.2 and the norm
 176 relationship. \square

177 With Definition 3.1 and Lemma 3.2, we have the following condition on $(\mathcal{S}, \theta_J, \nu_J)$
 178 to ensure the Jacobian $\nabla c(x)^T$ and $\nabla c_{\mathcal{S}}(x)^T$ are acute perturbations to each other.

LEMMA 3.3. Under Assumption 3.1, Assumption 3.2 where constants $(\sigma_{\min}, \kappa_{\nabla c})$ exist. In addition, under Assumption 3.3 where constants (θ_J, ν_J) exist. Then, if $\mathcal{S} \subseteq [N]$ satisfies

$$|\mathcal{S}| > \frac{2}{1 + \sqrt{1 + \frac{2(\sigma_{\min})^2}{(\theta_J + \nu_J)(\kappa_{\nabla c})^2}}} N,$$

the following hold

- (1). For any $x \in \mathbb{R}^n$ the Jacobian $\nabla c_{\mathcal{S}}(x)^T$ is nondegenerate and the associated least square estimator $y_{\mathcal{S}}$ in (2.7) is well-defined.
- (2). For any $x \in \mathbb{R}^n$, the gradient $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are acute perturbations to each other.

Proof. First, we examine the difference between $\nabla c_{\mathcal{S}}(x)^T$ and $\nabla c(x)^T$. We have

$$\begin{aligned} & \|\nabla c_{\mathcal{S}}(x)^T - \nabla c(x)^T\|_2^2 \\ &= \|\nabla c_{\mathcal{S}}(x)^T (\mathcal{R}(\nabla c(x)) + \mathcal{N}(\nabla c(x))) - \nabla c(x)^T\|_2^2 \\ &= \|\nabla c_{\mathcal{S}}(x)^T \mathcal{R}(\nabla c(x)) - \nabla c(x)^T + \nabla c_{\mathcal{S}}(x)^T \mathcal{N}(\nabla c(x))\|_2^2 \\ &\leq 2 \|\nabla c_{\mathcal{S}}(x)^T \mathcal{R}(\nabla c(x)) - \nabla c(x)^T\|_2^2 + 2 \|\nabla c_{\mathcal{S}}(x)^T \mathcal{N}(\nabla c(x))\|_2^2 \\ &\leq 2N \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right) (\theta_J + \nu_J) \|\nabla c(x)^T\|_2^2. \end{aligned} \tag{3.4}$$

Here, the second line uses $I_n = \mathcal{R}(\nabla c(x)) + \mathcal{N}(\nabla c(x))$. The third line rearranges terms. The second to last line uses the Cauchy-Schwarz inequality, and the last line uses Lemma 3.2.

Further, [?, Theorem 1.1] gives us a bound on the difference of the smallest singular values, i.e. $|\sigma_m(\nabla c_{\mathcal{S}}(x)^T) - \sigma_m(\nabla c(x)^T)| \leq \|\nabla c_{\mathcal{S}}(x) - \nabla c(x)^T\|_2$. Combining it with (3.4) we have

$$|\sigma_m(\nabla c_{\mathcal{S}}(x)^T) - \sigma_m(\nabla c(x)^T)| \leq \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} \|\nabla c(x)^T\|_2. \tag{3.5}$$

By the choice of \mathcal{S} , we have $\sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} < \frac{\sigma_{\min}}{\kappa_{\nabla c}}$, which gives a bound for the smallest singular value of $\nabla c_{\mathcal{S}}(x)^T$,

$$\begin{aligned} \sigma_m(\nabla c_{\mathcal{S}}(x)^T) &= \sigma_m(\nabla c(x)^T) + \sigma_m(\nabla c_{\mathcal{S}}(x)^T) - \sigma_m(\nabla c(x)^T) \\ &\geq \sigma_m(\nabla c(x)^T) - |\sigma_m(\nabla c_{\mathcal{S}}(x)^T) - \sigma_m(\nabla c(x)^T)| \\ &\geq \sigma_m(\nabla c(x)^T) - \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} \|\nabla c(x)\|_2 \\ &> \sigma_{\min} - \sigma_{\min} = 0. \end{aligned}$$

Here, the first line adds and subtracts a term. The third line plugs in (3.5). The above result indicates that the smallest singular value of $\nabla c_{\mathcal{S}}(x)^T$ is positive, and we can conclude that $\nabla c_{\mathcal{S}}(x)$ is of full column rank and the dual variable $y(x)$ in (2.7) is well defined.

For the second item, by Assumption 3.2, for any $x \in \mathbb{R}^n$, we have the Jacobian $\nabla c(x)^T$ is of full row rank. Further, we have

$$\nabla c(x) \nabla c(x)^\dagger \nabla c_{\mathcal{S}}(x) \nabla c(x)^\dagger \nabla c(x)$$

$$\begin{aligned}
&= \nabla c(x) \nabla c(x)^\dagger \nabla c_{\mathcal{S}}(x) \\
&= \nabla c(x) \nabla c(x)^\dagger (\nabla c(x) + \nabla c_{\mathcal{S}}(x) - \nabla c(x)) \\
&= \nabla c(x) \left(I_m + \underbrace{\nabla c(x)^\dagger (\nabla c_{\mathcal{S}}(x) - \nabla c(x))}_{(ii)} \right).
\end{aligned}$$

Here, the second line uses the definition of pseudo-inverse that $\nabla c(x)^\dagger \nabla c(x) = I_m$. The second to last line adds and subtracts a term, and the last line combines the product of the last two terms from the previous equality.

Combining the sub-multiplicity of the matrix product, the first item of Lemma 3.2, inequality (3.4) and the choice of \mathcal{S} , we have:

$$\begin{aligned}
\|(ii)\|_2 &= \|\nabla c(x)^\dagger (\nabla c_{\mathcal{S}}(x) - \nabla c(x))\|_2 \\
&\leq \|\nabla c(x)^\dagger\|_2 \|(\nabla c_{\mathcal{S}}(x) - \nabla c(x))\|_2 \\
&\leq \frac{1}{\sigma_{\min}} \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} \|\nabla c(x)\|_2 \\
&< \frac{1}{\sigma_{\min}} \sigma_{\min} = 1.
\end{aligned}$$

Again, by [?, Theorem 1.1], we have

$$\sigma_m(I_m) - \sigma_m(I_m + (ii)) \leq |\sigma_m(I_m) - \sigma_m(I_m + (ii))| \leq \|(ii)\|_2 < 1,$$

and the most left and right terms of the above inequality give us

$$\sigma_m(I_m + (ii)) > \sigma_m(I_m) - 1 = 1 - 1 = 0,$$

which is positive. Hence, the matrix $I_m + (ii)$ is of full rank. Combining with the fact that $\nabla c_{[N]}(x)$ has full column rank, we have that $\nabla c(x)^T(I_m + (ii))$ has full row rank, that gives us

$$\text{rank}(\nabla c(x) \nabla c(x)^\dagger \nabla c_{\mathcal{S}}(x) \nabla c(x)^\dagger \nabla c(x)) = m.$$

By Definition 3.1, the $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are acute perturbations to each other. \square

Now, we can present the first type of bounds.

LEMMA 3.4. Under Assumption 3.1 and Assumption 3.2 where $(\kappa_{\nabla f}, \sigma_{\min}, \kappa_{\nabla c})$ exist. In addition, under Assumption 3.3 where constants (θ_J, ν_J) exist. Then, for any $x \in \mathbb{R}^n$, if the sample set $\mathcal{S} \subseteq [N]$ satisfies

$$|\mathcal{S}| \geq \frac{2}{1 + \sqrt{1 + \frac{2\sigma_{\min}^2}{9(\theta_J + \nu_J)\kappa_{\nabla c}^2}}} N$$

and let $y_{\mathcal{S}}(x) = -\nabla c_{\mathcal{S}}(x)^\dagger \nabla f(x)$, we have

$$\|y_{[N]}(x) - y_{\mathcal{S}}(x)\|_2 \leq \frac{3\kappa_{\nabla f}\kappa_{\nabla c}}{2\sigma_{\min}^2} \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}}.$$

8

241 *Proof.* By the choice of \mathcal{S} , we have $\sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} \leq \frac{\sigma_{\min}}{3\kappa_{\nabla c}} < \frac{\sigma_{\min}}{\kappa_{\nabla c}}$ since
 242 $\sigma_{\min} > 0$, which satisfies requirements of Lemma 3.3. Hence, we have that $\nabla c(x)$ and
 243 $\nabla c_{\mathcal{S}}(x)$ are of full column rank and are acute perturbations to each other. By [2,
 244 Theorem 5.2], we have the following:

$$245 \quad (3.6) \quad \|y_{[N]}(x) - y_{\mathcal{S}}(x)\|_2 \leq \underbrace{\frac{\|\nabla c(x)^\dagger\|_2 \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2}{1 - \|\nabla c(x)^\dagger\|_2 \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2}}_{(iii)} \|y_{[N]}(x)\|_2.$$

246 By inequality (3.4) and the choice of \mathcal{S} , we have

$$247 \quad \|\nabla c_{\mathcal{S}}(x) - \nabla c(x)\|_2 \leq \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}} \|\nabla c(x)\|_2 \leq \frac{1}{3}\sigma_{\min},$$

248 which further gives us

$$249 \quad (3.7) \quad 1 - \|\nabla c(x)^\dagger\|_2 \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2 \geq 1 - \frac{1}{\sigma_{\min}} \frac{\sigma_{\min}}{3} = 2/3.$$

250 Hence, we have

$$251 \quad (3.8) \quad \begin{aligned} (iii) &\leq \frac{3}{2} \|\nabla c(x)^\dagger\|_2 \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2 \\ &\leq \frac{3\kappa_{\nabla c}}{2\sigma_{\min}} \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}}. \end{aligned}$$

252 Here, the first line uses (3.7) at the denominator, and the last line uses the bound for
 253 $\|\nabla c(x)^\dagger\|_2$. Combining with the bound $\|y_{[N]}(x)\|_2 \leq \frac{\kappa_{\nabla c}}{\sigma_{\min}}$, we have

$$254 \quad \|y_{[N]}(x) - y_{\mathcal{S}}(x)\|_2 \leq \frac{3\kappa_{\nabla c}}{2\sigma_{\min}^2} \sqrt{\frac{2(\theta_J + \nu_J)N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}}. \quad \square$$

255 The difference in the Hessian conditions is more complicated compared to the
 256 gradient condition. Recall the Hessian condition in Definition 3.4 that

$$257 \quad |d^T \nabla_{xx}^2 L(x, y) d| \geq \beta \|d\|_2^2, \quad \forall d \in \text{Null}(\nabla c(x)^T).$$

258 When we consider the empirical system (2.3), not only did the Lagrangian function
 259 L change, but also the null space $\text{Null}(\nabla c(x)^T)$ change. We start by giving a general
 260 result for two perturbed null spaces by examining the difference between vectors in
 261 one null space and their projections onto the other null space.

262 **LEMMA 3.5.** *Under Assumption 3.2 where constants $(\sigma_{\min}, \kappa_{\nabla c})$ exist. In addi-*
 263 *tion, under Assumption 3.3 where constants (θ_J, ν_J) exist. For any $x \in \mathbb{R}^n$ and any*
 264 *$\mathcal{S} \subseteq [N]$ such that*

$$265 \quad (3.9) \quad |\mathcal{S}| > \frac{2}{1 + \sqrt{1 + \frac{2\sigma_{\min}^2}{(\theta_J + \nu_J)\kappa_{\nabla c}^2}}} N.$$

266 Then, for any $d_{\mathcal{S}} \in \text{Null}(\nabla c_{\mathcal{S}}(x)^T)$, we have

$$267 \quad \frac{\|\mathcal{R}(\nabla c(x))(d_{\mathcal{S}})\|_2}{\|d_{\mathcal{S}}\|_2} \leq \frac{\kappa_{\nabla c}}{\sigma_{\min}} \sqrt{\frac{2N(N - |\mathcal{S}|)(\theta_J + \nu_J)}{|\mathcal{S}|^2}} < 1.$$

9

269 *Proof.* Since $d_S \in \text{Null}(\nabla c_S(x)^T)$ and $\mathcal{N}(\nabla c_S(x))$ is a projection matrix to the
 270 null space $\text{Null}(\nabla c_S(x)^T)$, we have $\mathcal{N}(\nabla c_S(x))d_S = d_S$. In addition, by Lemma 3.3,
 271 (3.9) ensures $\nabla c_S(x)$ and $\nabla c(x)$ are of full column rank. Combining with [2, Theorem
 272 2.4] we have

$$\begin{aligned} 273 \quad (3.10) \quad \|\mathcal{R}(\nabla c(x))d_S\|_2 &= \|\mathcal{R}(\nabla c(x))\mathcal{N}(\nabla c_S(x))d\|_2 \\ &\leq \|\nabla c(x)^\dagger\|_2 \|\nabla c(x) - \nabla c_S(x)\|_2 \|d_S\|_2. \end{aligned}$$

274 Then, combined with Lemma 3.2 gives the desired result. Moreover, the choice of $|\mathcal{S}|$
 275 (3.9) gives us

$$276 \quad \frac{\kappa_{\nabla c}}{\sigma_{\min}} \sqrt{\frac{2N(N-|\mathcal{S}|)(\theta_J + \nu_J)}{|\mathcal{S}|^2}} < 1. \quad \square$$

277 With this result, we can now look into the Hessian condition for empirical con-
 278 straint Morse problem. In particular, let $d \in \text{Null}(\nabla c_S(x)^T)$, vector $\tilde{d} = \mathcal{N}(\nabla c(x))d$
 279 and $r = d - \tilde{d}$, we tend to look at the difference

$$280 \quad \left| d^T \nabla_{xx}^2 L_S(x, y_S) d - \tilde{d}^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) \tilde{d} \right|,$$

281 and the following lemma gives us a general bound for the above term.

282 In summary, we have the result for the Morse property of the empirical problem.
 283 We define the following four parameters

$$284 \quad \begin{cases} \eta_0 := \sqrt{\gamma_c \kappa_c^2}, \\ \eta_1 := \sqrt{\frac{2(\theta_J + \nu_J) \kappa_{\nabla c}^2}{\sigma_{\min}^2}}, \\ \eta_2 := \frac{\kappa_{\nabla f} \kappa_{\nabla^2 c}}{\sigma_{\min}} \sqrt{m \mu_H}, \\ \eta_3 := \eta_2 + 3\eta_1 \lambda_f^{\max} + \frac{9\eta_1 \eta_2}{2\sqrt{\mu_H}}. \end{cases}$$

285 **THEOREM 3.6.** *Under Assumption 3.1, Assumption 3.2, Assumption 3.3 and As-*
 286 *sumption 3.4 where the constants $(\sigma_{\min}, \kappa_c, \kappa_{\nabla c}, \kappa_{\nabla f}, \kappa_{\nabla^2 c}, \kappa_{\nabla^2 f}, \gamma_c, \theta_J, \nu_J, \mu_H)$ exist,*
 287 *and in addition, the dual variable $y_{[N]}$ and y_S are chosen as in (2.7). Then, for any*
 288 *$\mathcal{S} \subseteq [N]$ when satisfies:*

$$289 \quad (3.11) \quad g_S := \sqrt{\frac{N(N-|\mathcal{S}|)}{|\mathcal{S}|^2}} \leq \min \left\{ \frac{1}{3\eta_1}, \frac{\alpha}{2\sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2}}, \frac{\beta}{2\sqrt{(\eta_1 \beta + \eta_3)^2 + 3\eta_1 \eta_2 \beta}} \right\},$$

290 the problem (2.3) is (α_S, β_S) -morse, where

$$291 \quad \begin{cases} \alpha_S = \alpha - \sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_S > 0 \text{ and} \\ \beta_S = \beta - (\eta_1 \beta + \eta_3) g_S - \frac{3}{2} \eta_1 \eta_2 g_S^2 > 0. \end{cases}$$

292 *Proof.* By inequality (3.4) and triangle inequality, we have that for any $x \in \mathbb{R}^n$

$$293 \quad \|\nabla c_S(x)\|_2 \leq \|\nabla c(x)\|_2 + \|\nabla c_S(x) - \nabla c(x)\|_2 \leq \left(1 + \sqrt{2(\theta_J + \nu_J)} g_S\right) \|\nabla c(x)\|_2.$$

294 Next, we look into the difference between $\nabla_x L_{[N]}(x, y_{[N]})$ and $\nabla_x L_S(x, y_S)$. We

295 have

$$\begin{aligned}
& \|\nabla_x L_{[N]}(x, y_{[N]}) - \nabla_x L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 \\
&= \|\nabla c(x) y_{[N]} - \nabla c_{\mathcal{S}}(x) y_{\mathcal{S}}\|_2 \\
296 \quad (3.12) \quad &= \|\nabla c(x) \nabla c(x)^\dagger \nabla f(x) + \nabla c_{\mathcal{S}}(x) \nabla c_{\mathcal{S}}(x)^\dagger \nabla f(x)\|_2 \\
&\leq \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2 \|\nabla f(x)\|_2 \\
&\leq \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2 \kappa_{\nabla f}.
\end{aligned}$$

297 Here, the third line uses the definition for $y_{[N]}$ and $y_{\mathcal{S}}$. The second to last line uses
298 the definition of \mathcal{R} , and the last line uses the bound for $\|\nabla f(x)\|_2$. By choice of \mathcal{S}
299 (3.11), the requirement of Lemma 3.3 is satisfied, and both $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are of
300 full column rank. By [2, Theorem 2.4] and previous bounds, we have

$$\begin{aligned}
& \|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|_2 \kappa_{\nabla f} \\
301 \quad (3.13) \quad &= \|\nabla c(x) (I_n - \mathcal{R}(\nabla c(x)))\|_2 \kappa_{\nabla f} \\
&= \|\mathcal{R}(\nabla c_{\mathcal{S}}(x)) \mathcal{N}(\nabla c(x))\|_2 \kappa_{\nabla f} \leq \frac{\kappa_{\nabla c} \kappa_{\nabla f}}{\sigma_{\min}} \sqrt{2(\theta_J + \nu_J)} g_{\mathcal{S}}.
\end{aligned}$$

302 For the difference between $\nabla_y L_{[N]}(x, y_{[N]})$ and $\nabla_y L_{\mathcal{S}}(x, y_{\mathcal{S}})$, we have

$$\begin{aligned}
& \|\nabla_y L_{[N]}(x, y_{[N]}) - \nabla_y L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 = \|c(x) - c_{\mathcal{S}}(x)\|_2 \\
303 \quad (3.14) \quad &\leq \sqrt{\gamma_c N \left(\frac{N - |\mathcal{S}|}{|\mathcal{S}|^2} \right)} \|c(x)\|_2 \leq \sqrt{\gamma_c} \kappa_c g_{\mathcal{S}}.
\end{aligned}$$

304 Combining (3.12, 3.14) and note that

$$305 \quad \|\nabla L_{[N]}(x, y_{[N]})\|_2^2 = \|\nabla_x L_{[N]}(x, y_{[N]})\|_2^2 + \|\nabla_y L_{[N]}(x, y_{[N]})\|_2^2,$$

306 we have

$$\begin{aligned}
307 \quad \|\nabla L_{[N]}(x, y_{[N]})\|_2 &\leq \|\nabla L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 + \|\nabla L_{[N]}(x, y_{[N]}) - \nabla L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 \\
308 \quad &\leq \|\nabla L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 + \sqrt{\frac{2(\theta_J + \nu_J) \kappa_{\nabla c}^2 \kappa_{\nabla f}^2}{\sigma_{\min}^2}} + \gamma_c \kappa_c^2 g_{\mathcal{S}} \\
309 \quad &= \|\nabla L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 + \sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}}. \\
310
\end{aligned}$$

311 Hence for any $x \in \mathbb{R}^n$ satisfies $\|\nabla_x L_{\mathcal{S}}(x, y_{\mathcal{S}})\|_2 \leq \alpha - \sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}} = \alpha_{\mathcal{S}}$, we
312 have $\|\nabla_x L_{[N]}(x, y_{[N]})\|_2 \leq \alpha$. In addition, the choice of \mathcal{S} gives $\sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}} \leq \frac{1}{2}\alpha$,
313 we have

$$314 \quad \alpha_{\mathcal{S}} \geq \frac{1}{2}\alpha > 0.$$

315 Since the problem (2.2) is (α, β) -morse, by the definition of morse we have

$$316 \quad |d^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) d| \geq \beta \|d\|_2^2 \text{ for all } d \in \text{Null}(\nabla c(x)^T).$$

317 Now, for any $d_{\mathcal{S}} \in \text{Null}(\nabla c_{\mathcal{S}}(x)^T)$, we look into the value $|d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x, y_{\mathcal{S}}) d_{\mathcal{S}}|$. We
318 have

$$319 \quad |d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x, y_{\mathcal{S}}) d_{\mathcal{S}}|$$

$$\begin{aligned}
&= |d_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) d_S + d_S^T \nabla_{xx}^2 L_S(x, y_S) d_S - d_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) d_S| \\
&\geq \underbrace{|d_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) d_S|}_{(v.1)} - \underbrace{|d_S^T \nabla_{xx}^2 L_S(x, y_S) d_S - d_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) d_S|}_{(v.2)}.
\end{aligned}$$

Here, we get the third line by adding and subtracting a term and using the triangle inequality.

Let $\tilde{d}_S := \mathcal{N}(\nabla c(x)) d_S$ and $r_S := d_S - \tilde{d}_S$, and substitute $d_S = \tilde{d}_S + r_S$ for term (v.1) we have

$$\begin{aligned}
(v.1) &= \left| \tilde{d}_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) \tilde{d}_S + 2\tilde{d}_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) r_S + r_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) r_S \right| \\
&\geq \left| \tilde{d}_S^T \nabla_{xx}^2 L_{[N]}(x, y_{[N]}) \tilde{d}_S \right| - 2\|\nabla_{xx}^2 L_{[N]}(x, y_{[N]})\|_2 \|\tilde{d}_S\|_2 \|r_S\|_2 \\
&\quad - \|\nabla_{xx}^2 L_{[N]}(x, y_{[N]})\|_2^2 \|r_S\|_2^2 \\
&\geq \beta \|\tilde{d}_S\|_2^2 - 3\|\nabla_{xx}^2 L_{[N]}(x, y_{[N]})\|_2 \|d_S\|_2 \|r_S\|_2.
\end{aligned}$$

Here, the first equality and inequality follow by adding, subtracting a term, and using the triangle inequality. The second inequality uses the fact that $\|d_S\|_2 = \|\tilde{d}_S\|_2^2 + \|r_S\|_2^2$ which gives $\|\tilde{d}_S\|_2 \leq \|d_S\|_2$, and substitute this result with the last two terms. Further we have

$$\begin{aligned}
(v.1) &\geq \beta \|\tilde{d}_S\|_2^2 - 3 \left(\kappa_{\nabla^2 f} + \frac{\sqrt{m} \kappa_{\nabla f} \kappa_{\nabla^2 c}}{\sigma_{\min}} g_{S_k} \right) \|d_S\|_2 \|r_S\|_2 \\
&\geq \beta (1 - \eta_1 g_S) \|d_S\|_2^2 - 3 \left(\kappa_{\nabla^2 f} + \frac{\eta_2}{\sqrt{\mu_H}} g_S \right) \eta_1 g_S \|d_S\|_2^2 \\
&= \left(\beta - \left(\eta_1 \beta + 3\eta_1 \kappa_{\nabla^2 f} + 3 \frac{\eta_1 \eta_2}{\sqrt{\mu_H}} \right) g_S \right) \|d_S\|_2^2.
\end{aligned}$$

Here, the first line uses Lemma 3.2, the second line uses Lemma 3.5, and the last line rearranges terms.

For the term (v.2), we have

$$\begin{aligned}
(v.2) &= |d_S^T (\nabla_{xx}^2 L_S(x, y_S) - \nabla_{xx}^2 L_{[N]}(x, y_{[N]})) d_S| \\
&= \left| d_S^T \left(\sum_{j=1}^m y_S^j \nabla^2 [c_S]_j(x) - \sum_{j=1}^m y_{[N]}^j \nabla^2 [c]_j(x) \right) d_S \right| \\
&\leq \left\| \sum_{j=1}^m (y_S^j \nabla^2 [c_S]_j(x) - y_{[N]}^j \nabla^2 [c]_j(x)) \right\|_2 \|d_S\|_2^2,
\end{aligned}$$

where for the term of Hessian, we have

$$\begin{aligned}
&\left\| \sum_{j=1}^m (y_S^j \nabla^2 [c_S]_j(x) - y_{[N]}^j \nabla^2 [c]_j(x)) \right\|_2 \\
&= \left\| \sum_{j=1}^m (y_S^j (\nabla^2 [c_S]_j(x) - \nabla^2 [c]_j(x))) + \sum_{j=1}^m ((y_S^j - y_{[N]}^j) \nabla^2 [c]_j(x)) \right\|_2
\end{aligned}$$

$$\begin{aligned}
& \leq \sum_{j=1}^m \|y_S^j\|_2 \|\nabla^2[c_S]_j(x) - \nabla^2[c]_j(x)\|_2 + \sum_{j=1}^m \|y_S^j - y_{[N]}^j\|_2 \|\nabla^2[c]_j(x)\|_2 \\
& \leq \sqrt{m\mu_H\kappa_{\nabla^2 c}g_S} \|y_S\|_2 + \sqrt{m\kappa_{\nabla^2 c}} \|y_S - y_{[N]}\|_2 \\
& \leq \sqrt{m\mu_H\kappa_{\nabla^2 c}g_S} \|y_{[N]}\|_2 + (1 + \sqrt{\mu_H g_S}) \sqrt{m\kappa_{\nabla^2 c}} \|y_S - y_{[N]}\|_2 \\
& \leq \sqrt{m\mu_H} \frac{\kappa_{\nabla^2 c} \kappa_{\nabla f}}{\sigma_{\min}} g_S + (1 + \sqrt{\mu_H g_S}) \sqrt{m} \frac{3\kappa_{\nabla f} \kappa_{\nabla^2 c} \kappa_{\nabla c}}{2(\sigma_{\min})^2} \sqrt{2(\theta_J + \nu_J)g_S} \\
& = \eta_2 g_S + \frac{3\eta_1 \eta_2}{2\sqrt{\mu_H}} g_S + \frac{3\eta_1 \eta_2}{2} g_S^2.
\end{aligned}$$

Here, the second is to add, subtract, and rearrange terms. The third line uses triangle inequality and submultiplicity. The fourth line uses similar arguments as in (v.1). The fifth line uses the fact that $\|y_S\|_2 \leq \|y_{[N]}\|_2 + \|y_S - y_{[N]}\|_2$ and rearranges terms. The last two lines use Lemma 3.4 since $g_S \leq \frac{1}{3\eta_1}$, and the definition of (η_1, η_2) .

Combining the above results for (v.1, v.2), we have

$$\begin{aligned}
& |d_S^T \nabla_{xx}^2 L_S(x, y_S) d_S| \\
& \geq \left(\beta - \left(\eta_1 \beta + \eta_2 + 3\eta_1 \kappa_{\nabla^2 c} + \frac{9\eta_1 \eta_2}{2\sqrt{\mu_H}} \right) g_S - \frac{3}{2} \eta_1 \eta_2 g_S^2 \right) \|d_S\|_2^2 \\
& = \left(\beta - (\eta_1 \beta + \eta_3) g_S - \frac{3}{2} \eta_1 \eta_2 g_S^2 \right) \|d_S\|_2^2.
\end{aligned}$$

By the requirement of $|\mathcal{S}|$ that , we have

$$\left((\eta_1 \beta + \eta_3) g_S + \frac{3}{2} \eta_1 \eta_2 g_S^2 \right) \leq \frac{1}{2} \beta,$$

where the nonnegative solution for g_S is

$$0 \leq g_S \leq \frac{-(\eta_1 \beta + \eta_3) + \sqrt{(\eta_1 \beta + \eta_3)^2 + 3\eta_1 \eta_2 \beta}}{3\eta_1 \eta_2},$$

where the right-hand side can be bounded below by

$$\begin{aligned}
& \frac{-(\eta_1 \beta + \eta_3) + \sqrt{(\eta_1 \beta + \eta_3)^2 + 3\eta_1 \eta_2 \beta}}{3\eta_1 \eta_2} \\
& = \frac{3\eta_1 \eta_2 \beta}{9\eta_1 \eta_2 \left((\eta_1 \beta + \eta_3) + \sqrt{(\eta_1 \beta + \eta_3)^2 + 3\eta_1 \eta_2 \beta} \right)} \\
& \geq \frac{\beta}{2\sqrt{(\eta_1 \beta + \eta_3)^2 + 3\eta_1 \eta_2 \beta}}.
\end{aligned}$$

Here, the second line multiplies a $\left((\eta_1 \beta + \eta_3) + \sqrt{(\eta_1 \beta + \eta_3)^2 + 3\eta_1 \eta_2 \beta} \right)$ at both the numerator and denominator. Hence the last requirement for \mathcal{S} ensures that. \square

THEOREM 3.7. *Under Assumptino 3.1, Assumption 3.2 Assumption 3.3 and Assumption 3.4 where the constants $(\sigma_{\min}, \kappa_c, \kappa_{\nabla c}, \kappa_{\nabla f}, \kappa_{\nabla^2 c}, \kappa_{\nabla^2 f}, \gamma_c, \theta_J, \nu_J, \mu_H)$ exist. Define tolerances*

$$\epsilon_k := \sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} \sqrt{\frac{N(N - |\mathcal{S}_k|)}{|\mathcal{S}_k|^2}} \text{ and } \varepsilon_k := \kappa_{\nabla^2 f} \eta_1 \sqrt{\frac{N(N - |\mathcal{S}_k|)}{|\mathcal{S}_k|^2}}, \forall k \in [K].$$

Let (2.2) proceed with Algorithm 2.1. Then, for all sample set $\mathcal{S}_k \subseteq [N]$ that satisfies

$$(3.17) \quad \sqrt{\frac{N(N - |\mathcal{S}_k|)}{|\mathcal{S}_k|^2}} \leq \min \left\{ \frac{1}{3\eta_1}, \frac{\alpha}{4\sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2}}, \frac{\beta}{4\sqrt{(\eta_1\beta + \eta_3)^2 + \frac{3}{2}\eta_1\eta_2\beta}} \right\},$$

if $x_{\mathcal{S}_k} \in \mathbb{R}^n$ is a $(\epsilon_k, \varepsilon_k)$ stationary solution, the $x_{\mathcal{S}_k}$ must satisfy the following for problem (2.3) with $\mathcal{S} = \mathcal{S}_{k+1}$

$$\begin{aligned} & \|\nabla_x L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, y_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}))\|_2 \leq \alpha_{\mathcal{S}_{k+1}}, \text{ and} \\ & d^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, y_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})) d \geq \beta_{\mathcal{S}_{k+1}} \|d\|_2^2, \forall d \in \text{Null}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})^T). \end{aligned}$$

Proof. Let the dual variables $y_{[N]}$ and $y_{\mathcal{S}_k}$ be defined as in (2.7). In addition, define $z_{\mathcal{S}_{k+1}} = -\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})^\dagger \nabla f(x_{\mathcal{S}_k})$. Similar to (3.12) we have

$$\begin{aligned} & \|\nabla_x L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla_x L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\|_2 \\ & \leq \|\mathcal{R}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})) - \mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))\|_2 \|\nabla f(x_{\mathcal{S}_k})\|_2 \\ & \leq (\|\mathcal{R}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})) - \mathcal{R}(\nabla c(x_{\mathcal{S}_k}))\|_2 \\ & \quad + \|\mathcal{R}(\nabla c(x_{\mathcal{S}_k})) - \mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))\|_2) \|\nabla f(x_{\mathcal{S}_k})\|_2. \end{aligned}$$

Here, the last inequality uses the triangle inequality. In (3.13) we already have

$$\|\mathcal{R}(\nabla c_{\mathcal{S}}(x)) - \mathcal{R}(\nabla c(x))\|_2 \|\nabla f(x_{\mathcal{S}_k})\|_2 \leq \kappa_{\nabla f} \eta_1 g_{\mathcal{S}}.$$

Note the right-hand side depends on $g_{\mathcal{S}}$, which by definition decreases when $|\mathcal{S}|$ increases. Hence we have

$$\|\nabla_x L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla_x L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\|_2 \leq 2\kappa_{\nabla f} \eta_1 g_{\mathcal{S}}.$$

On the other hand, similar to (3.14) we have

$$\begin{aligned} & \|\nabla_y L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla_y L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\|_2 \\ & = \|c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}) - c_{\mathcal{S}_k}(x_{\mathcal{S}_k})\|_2 \\ & \leq \|c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}) - c_{[N]}(x_{\mathcal{S}_k})\|_2 + \|c_{[N]}(x_{\mathcal{S}_k}) - c_{\mathcal{S}_k}(x_{\mathcal{S}_k})\|_2 \\ & \leq 2\eta_0 g_{\mathcal{S}_k}. \end{aligned}$$

Combining the above two results we have

$$\begin{aligned} \|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})\|_2 & \leq \|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\|_2 \\ & \quad + \|\nabla L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\|_2 \\ & \leq 2\sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}_k} + \sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}_k} \\ & = 3\sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}_k} g_{\mathcal{S}_k} \leq \frac{3}{4}\alpha. \end{aligned}$$

Here, the first inequality uses the triangle inequality. The second inequality combines with the fact that $x_{\mathcal{S}_k}$ is a $(\epsilon_k, \varepsilon_k)$ stationary point, and the last inequality comes from the first requirement for $|\mathcal{S}|$ that $\sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}_k} \leq \frac{1}{4}\alpha$.

Moreover, the same requirement for $|\mathcal{S}|$ gives us

$$\alpha_{\mathcal{S}_k} = \alpha - \sqrt{\eta_0^2 + \kappa_{\nabla f}^2 \eta_1^2} g_{\mathcal{S}_k} g_{\mathcal{S}_k} \geq \frac{3}{4}\alpha,$$

404 and combining with the fact that $\alpha_{\mathcal{S}}$ decreases when $|\mathcal{S}|$ increases, we have

$$405 \quad (3.18) \quad \|\nabla_x L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})\|_2 \leq \alpha_{\mathcal{S}_k} \leq \alpha_{\mathcal{S}_{k+1}}.$$

406 Now, we turn to the condition for hessian. Since the subproblem for \mathcal{S}_{k+1} is
407 $(\alpha_{\mathcal{S}_{k+1}}, \beta_{\mathcal{S}_{k+1}})$ -morse and with (3.18), we have

$$408 \quad \left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \right| \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|_2^2, \quad \forall d_{\mathcal{S}_{k+1}} \in \text{Null}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})^T).$$

409 Similar to the analysis for Theorem (3.6), define $\bar{d}_{\mathcal{S}_{k+1}} := \mathcal{N}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k})) d_{\mathcal{S}_{k+1}}$,
410 by triangle inequality we have

$$\begin{aligned} 411 \quad & d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \\ 412 \quad & \geq \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k}) \bar{d}_{\mathcal{S}_{k+1}} \\ 413 \quad & - \underbrace{\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k}) \bar{d}_{\mathcal{S}_{k+1}} \right|}_{(vi)} \\ 414 \quad & \geq -\varepsilon_k \|d_{\mathcal{S}_{k+1}}\|_2^2 - (vi). \end{aligned}$$

416 Here, the last line uses the termination condition (2.6b) and the fact that $\|\bar{d}_{\mathcal{S}_{k+1}}\|_2^2 \leq$
417 $\|d_{\mathcal{S}_{k+1}}\|_2^2$.

418 To give a bound for (vi), we add and subtract four terms. Define the variable
419 $z_{[N]} := -\nabla c(x_{\mathcal{S}_k})^\dagger \nabla f(x_{\mathcal{S}_k})$, following the triangle inequality, we have

$$\begin{aligned} (vi) &= \left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} - d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) d_{\mathcal{S}_{k+1}} \right. \\ &\quad + d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) \bar{d}_{\mathcal{S}_{k+1}} \\ &\quad \left. + \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) \bar{d}_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k}) \bar{d}_{\mathcal{S}_{k+1}} \right| \\ 420 \quad &\leq \underbrace{\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} - d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) d_{\mathcal{S}_{k+1}} \right|}_{(vi.1)} \\ &\quad + \underbrace{\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) \bar{d}_{\mathcal{S}_{k+1}} \right|}_{(vi.2)} \\ &\quad + \underbrace{\left| \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]}) \bar{d}_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k}) \bar{d}_{\mathcal{S}_{k+1}} \right|}_{(vi.3)}. \end{aligned}$$

421 Thanks to the previous result in Theorem 3.6 on the term (v.2), we have

$$\begin{aligned} 422 \quad (3.19) \quad (vi.1) &\leq \left(\eta_2 g_{\mathcal{S}_{k+1}} + \frac{3\eta_1 \eta_2}{2\sqrt{\mu_H}} g_{\mathcal{S}_{k+1}} + \frac{3\eta_1 \eta_2}{2} g_{\mathcal{S}_{k+1}}^2 \right) \|d_{\mathcal{S}_{k+1}}\|_2^2, \text{ and} \\ (vi.3) &\leq \left(\eta_2 g_{\mathcal{S}_k} + \frac{3\eta_1 \eta_2}{2\sqrt{\mu_H}} g_{\mathcal{S}_k} + \frac{3\eta_1 \eta_2}{2} g_{\mathcal{S}_k}^2 \right) \|\bar{d}_{\mathcal{S}_{k+1}}\|_2^2. \end{aligned}$$

423 For (vi.2), by Lemma 3.2 we have $\|\nabla_{xx}^2 L_{[N]}(x_{\mathcal{S}_k}, z_{[N]})\|_2 \leq \kappa_{\nabla^2 f} + \frac{\eta_2}{\sqrt{\mu_H}}$, and that

$$\begin{aligned}
& \|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\|_2 = \|d_{\mathcal{S}_{k+1}} - \mathcal{N}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}\|_2 \\
& = \|\mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}\|_2 \\
& \leq \|\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}\|_2 + \|(\mathcal{R}(\nabla c(x_{\mathcal{S}_k})) - \mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k})))d_{\mathcal{S}_{k+1}}\|_2 \\
& \leq 2\eta_1 g_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|_2.
\end{aligned}$$

424

425 Here, the first line uses the definition of $\bar{d}_{\mathcal{S}_{k+1}}$. The second line uses the definition
426 of \mathcal{R} . The third line uses the triangle inequality. The last line uses Lemma 3.5,
427 and inequality (3.13). In addition, the second line also gives us $\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\|_2 \leq$
428 $\|d_{\mathcal{S}_{k+1}}\|_2$.

429 Combining all the above results, noticing that $g_{\mathcal{S}}$ is nondecreasing with respect
430 to $|\mathcal{S}|$, which gives $g_{\mathcal{S}_{k+1}} \leq g_{\mathcal{S}_k}$. And remember that $\|\bar{d}_{\mathcal{S}_{k+1}}\|_2 \leq \|d_{\mathcal{S}_{k+1}}\|_2$ and
431 $\varepsilon_k = \eta_1 \kappa_{\nabla^2 f} g_{\mathcal{S}_k}$, we have

$$\begin{aligned}
& d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \\
& \geq -\eta_1 \kappa_{\nabla^2 f} g_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|_2^2 - 2 \left(\eta_2 + \frac{3\eta_1 \eta_2}{2\sqrt{\mu_H}} + \frac{3\eta_1 \eta_2}{2} g_{\mathcal{S}_k} \right) g_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|_2^2 \\
& \quad - \left(\kappa_{\nabla^2 f} + \frac{\eta_2}{\sqrt{\mu_H}} \right) 2\eta_1 g_{\mathcal{S}_k} \|d_{\mathcal{S}_{k+1}}\|_2^2 \\
& = - \left(\left(3\eta_1 \kappa_{\nabla^2 f} + 2\eta_2 + \frac{5\eta_1 \eta_2}{\sqrt{\mu_H}} \right) g_{\mathcal{S}_k} + 3\eta_1 \eta_2 g_{\mathcal{S}_k}^2 \right) \|d_{\mathcal{S}_{k+1}}\|_2^2 \\
& \geq - (2\eta_3 g_{\mathcal{S}_k} + 3\eta_1 \eta_2 g_{\mathcal{S}_k}^2) \|d_{\mathcal{S}_{k+1}}\|_2^2.
\end{aligned}$$

432

433 Similar to the analysis for Theorem 3.6. Recall $\beta_{\mathcal{S}} = \beta - ((\eta_1 \beta + \eta_3) g_{\mathcal{S}_k} + \frac{3}{2} \eta_1 \eta_2 g_{\mathcal{S}_k}^2)$
434 and $\beta_{\mathcal{S}_{k+1}} \geq \beta_{\mathcal{S}_k}$. To ensure $\beta_{\mathcal{S}_{k+1}} \geq \frac{3}{4} \beta$, we need $\frac{1}{4} \beta \geq (\eta_1 \beta + \eta_3) g_{\mathcal{S}_k} + \frac{3}{2} \eta_1 \eta_2 g_{\mathcal{S}_k}^2$,
435 whose nonnegative solution is

$$g_{\mathcal{S}_k} \leq \frac{-(\eta_1 \beta + \eta_3) + \sqrt{(\eta_1 \beta + \eta_3)^2 + \frac{3}{2} \eta_1 \eta_2 \beta}}{3\eta_1 \eta_2},$$

436

437 and it is ensured by

$$g_{\mathcal{S}_k} \leq \frac{\beta}{4\sqrt{(\eta_1 \beta + \eta_3)^2 + \frac{3}{2} \eta_1 \eta_2 \beta}}.$$

438

439 Moreover, we have

$$- (2\eta_3 g_{\mathcal{S}_k} + 3\eta_1 \eta_2 g_{\mathcal{S}_k}^2) \geq -2 \left((\eta_3 + \eta_1 \beta) g_{\mathcal{S}_k} - \frac{3}{2} \eta_1 \eta_2 g_{\mathcal{S}_k}^2 \right) \geq -\frac{1}{2} \beta > -\beta_{\mathcal{S}_{k+1}}.$$

440
441

442 The above analysis gives us that $d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} > -\beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|_2^2$.
443 However, since subproblem (2.3) for $\mathcal{S} = \mathcal{S}_{k+1}$ is $(\alpha_{k+1}, \beta_{k+1})$ -morse, which says that
444 $|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}}| \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|_2^2$. Combining these two, we must
445 have

$$d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|_2^2,$$

446

447 which completes proof. \square

448

ASSUMPTION 3.5. We make the following assumptions for each element of the expected constraint function, c^i . There exists a $(r, \tau) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that for all $(x, i) \in \mathbb{B}^n(r) \times \{1, \dots, m\}$,

(1). the gradient of $c^i(x)$ is τ^2 -sub-Gaussian. Namely, for any $a \in \mathbb{R}^n$,

$$\mathbb{E}_\xi \left[\exp \left(a^T (\nabla c^i(x; \xi) - \mathbb{E}_\xi [\nabla c^i(x; \xi)]) \right) \right] \leq \exp \left(\frac{\tau^2 \|a\|_2^2}{2} \right).$$

z_{ξ_j} finite sample distribution

(2). the Hessian of $c^i(x)$, evaluated on a unit vector, is τ^2 -sub-exponential. Namely, for any $a \in \mathbb{B}^n(1)$, let $z_{a,x,\xi} := a^T \nabla^2 c^i(x; \xi) a$, then

$$\mathbb{E}_\xi \left[\exp \left(\frac{1}{\tau^2} |z_{a,x,\xi} - \mathbb{E}[z_{a,x,\xi}]| \right) \right] \leq 2.$$

(3). within $\mathbb{B}^n(r)$, the Hessian of c^i is L -Lipschitz continuous, and the gradient of c^i is λ_c^{\max} -Lipschitz continuous. Moreover, there exists a constant $h > 0$ such that

$$L \leq \tau^3 n^h, \text{ and } \lambda_c^{\max} \leq \tau^2 n^h.$$

THEOREM 3.8. Under Assumption.3.5 and let $(n, r, \tau, h) \in \mathbb{N} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ be defined in the same way. There exists a universal constant C_0 and for any $\delta \in [0, 1]$ let $C := C_0 \max\{h, \log \frac{\tau\tau}{\delta}, 1\}$. Then, for any sample size $p \geq Cn \log n$, the following holds with probability at least $(1 - \delta)$:

$$(3.20) \quad \sup_{\forall x \in \mathbb{B}^n(r)} \|\nabla c(x) - \nabla c_p(x)\|_2 \leq g(p) := \tau \sqrt{\frac{Cn \log p}{p}} \text{ and} \\ \sup_{i \in \{1, \dots, m\}} \left\{ \sup_{\forall x \in \mathbb{B}^n(r)} \|\nabla^2 c_p^i(x) - \nabla^2 c^i(x)\|_2 \right\} \leq G(p) := \tau^2 \sqrt{\frac{Cn \log p}{p}}.$$

LEMMA 3.9. Under Assumption.3.5 and let $(n, r, \tau, h) \in \mathbb{N} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ be defined in the same way. Let (C, p) be defined in the same way as Theorem.3.8, then the following holds with probability at least $(1 - \delta)$:

$$(3.21) \quad \sup_{\forall x \in \mathbb{B}^n(r)} \left\| \frac{1}{p} \sum_{i \in \mathcal{S}_p} \nabla c(x, \xi_i) - \frac{1}{2p} \sum_{i \in \mathcal{S}_{2p}} \nabla c(x, \xi_i) \right\|_2 \leq \tau \sqrt{\frac{Cn \log p}{p}} \text{ and} \\ \sup_{i \in \{1, \dots, m\}} \left\{ \sup_{\forall x \in \mathbb{B}^n(r)} \left\| \frac{1}{p} \sum_{i \in \mathcal{S}_p} \nabla^2 c^i(x, \xi_i) - \frac{1}{2p} \sum_{i \in \mathcal{S}_{2p}} \nabla^2 c^i(x, \xi_i) \right\|_{op} \right\} \leq \tau^2 \sqrt{\frac{Cn \log p}{p}}.$$

Proof. \square

4. Numerical Results.

5. Conclusion.

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