Progressively Sampled Equality-Constrained Optimization

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Abstract. An algorithm is proposed, analyzed, and tested for solving continuous nonlinear-equality-constrained optimization problems where the constraints are defined by an expectation or a large finite-sum of terms. The main idea of the algorithm is to solve a sequence of equality-constrained problems, each involving a finite sample of constraint function terms, over which the sample set grows progressively. Under assumptions about the constraint functions and their first- and second-order derivatives that are reasonable in many real-world settings of interest, it is shown that the sample size is eventually large enough such that, by solving a sequence of problems with progressive sampling, one obtains an improved worst-case sample complexity bound compared to solving a single problem with a full set of samples. The results of numerical experiments with a set of test problems demonstrate that the proposed approach can be effective in practice.

Key words. nonlinear optimization, nonconvex optimization, Newton's method, worst-case iteration complexity, worst-case evaluation complexity, worst-case sample complexity

AMS subject classifications. 49M37, 65K05, 65K10, 65Y20, 68Q25, 90C30, 90C60

1. Introduction.

Equality-constrained optimization problems arise...

Lingjun: Add a citation to the paper for the unconstrained setting. The unconstrained progressive sampling paper is [4].

1.1. Contributions.

Our contributions relate ...

1.2. Notation.

We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}_{\geq r}$ (resp., $\mathbb{R}_{>r}$) to denote the set of real numbers greater than or equal to (resp., greater than) $r \in \mathbb{R}$, $\mathbb{R}^n$ to denote the set of n-dimensional real vectors, and $\mathbb{R}^{m \times n}$ to denote the set of m-by-n-dimensional real matrices. We denote the set of nonnegative integers as $\mathbb{N} := \{0, 1, 2, \dots\}$, and, for any integer $N \geq 1$, we use $[N]$ to denote the set $\{1, \dots, N\}$.

For any finite set $\mathcal{S}$, we use $|\mathcal{S}|$ to denote its cardinality. We consider all vector norms to be Euclidean, i.e., we let $\|\cdot\| := \|\cdot\|_2$, unless otherwise specified. Similarly, we use $\|\cdot\|$ to denote the spectral norm of any matrix input.

For any matrix $A \in \mathbb{R}^{m \times n}$, we use $\sigma_i(A)$ to denote its ith largest singular value. Given any such A, we use $\text{null}(A)$ to denote its null space, i.e., $\{d \in \mathbb{R}^n : Ad = 0\}$. Assuming $B \in \mathbb{R}^{n \times m}$ has full column rank, we use $B^\dagger$ to denote its pseudoinverse, i.e., $B^\dagger := (B^TB)^{-1}B^T$. For any subspace $\mathcal{X} \subseteq \mathbb{R}^n$ and point $x \in \mathbb{R}^n$, we denote the projection of x onto $\mathcal{X}$ as $\text{proj}_{\mathcal{X}}(x) := \arg\min_{\bar{x} \in \mathcal{X}} \|\bar{x} - x\|$. Given $B \in \mathbb{R}^{n \times m}$ with full column rank, we use $\mathcal{R}(B) := BB^\dagger$ and $\mathcal{N}(B) = I - \mathcal{R}(B)$ to denote projection matrices onto the span of the columns of B and the null space of B, respectively.

1.3. Organization.

In §3, ...

2. Algorithm.

Our proposed algorithm is designed to solve a sample average approximation (SAA) of the continuous nonlinear-equality-constrained problem

$$\min_{x \in \mathbb{R}^n} \ f(x) \ \text{ s.t. } \ \bar{c}(x) = 0, \tag{2.1}$$

where the objective and constraint functions, i.e., $f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{c} : \mathbb{R}^n \to \mathbb{R}^m$, respectively, are twice-continuously differentiable, $m \leq n$, and the constraint function $\bar{c}$ is defined by an expectation. Formally, with respect to a random variable ω defined

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by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, expectation $\mathbb{E}$ defined by $\mathbb{P}$, and $\overline{C} : \mathbb{R}^n \times \Omega \to \mathbb{R}^m$, the constraint function $\bar{c}$ is defined by $\bar{c}(x) = \mathbb{E}[\overline{C}(x, \omega)]$ for all $x \in \mathbb{R}^n$.

The SAA of problem (2.1) that our algorithm is designed to solve is defined with respect to a sample of $N \in \mathbb{N}$ realizations of the random variable ω, say, $\{\omega_i\}_{i \in [N]}$. Defining the SAA constraint function $c : \mathbb{R}^n \to \mathbb{R}^m$ for all $x \in \mathbb{R}^n$ by

$$c(x) = \frac{1}{N} \sum_{i \in [N]} c_i(x), \quad \text{where} \quad c_i(x) \equiv \overline{C}(x, \omega_i) \quad \text{for all} \quad i \in [N],$$

the problem that our algorithm is designed to solve is that given by

$$\min_{x \in \mathbb{R}^n} \; f(x) \; \text{ s.t. } \; c(x) = 0. \tag{2.2}$$

Under mild assumptions about c and an assumption that N is sufficiently large, a point that is approximately stationary for problem (2.2) can be shown to the approximately stationary for problem (2.1), at least with high probability. We leave a formal statement and proof of this fact until the end of our analysis. Until that time, we focus on our proposed algorithm and our analysis of it for solving problem (2.2).

The main idea of our proposed algorithm for solving problem (2.2) is to generate a sequence of iterates, each of which is a stationary point (at least approximately) with respect to a subsampled problem involving only a subset $\mathcal{S} \subseteq [N]$ of constraint function terms. For any such $\mathcal{S}$, we denote the approximate constraint function as $c_{\mathcal{S}} : \mathbb{R}^n \to \mathbb{R}^m$ and state the approximation of problem (2.2) as

$$\min_{x \in \mathbb{R}^n} \; f(x) \; \text{ s.t. } \; c_{\mathcal{S}}(x) = 0, \quad \text{where} \quad c_{\mathcal{S}}(x) = \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} c_i(x). \tag{2.3}$$

(Observe that, in this manner, the constraint function in (2.2) is $c_{[N]} = c$.) The primary benefit of considering (2.3) for $\mathcal{S} \subseteq [N]$, rather than (2.2) directly, is that any evaluation of a constraint or constraint Jacobian value requires computing a sum of $|\mathcal{S}| \leq N$ terms, as opposed to N terms. Also, under reasonable assumptions about the constraint functions, we show in this paper that, by starting with an approximate stationary point for problem (2.3) and aiming to solve a subsequent instance of (2.3) with respect to a sample set $\overline{\mathcal{S}} \supseteq \mathcal{S}$, our proposed algorithm can obtain an approximate stationary point for the subsequent instance with lower sample complexity than if the problem with the larger sample set were solved directly. Overall, we show that—at least once the sample sets become sufficiently large relative to N—a sufficiently approximate stationary point of problem (2.2) can be obtained more efficiently through progressive sampling than by tackling the full-sample problem directly.

For use in our proposed algorithm and our analysis of it, let us introduce stationarity conditions for problem (2.3), which also represent stationarity conditions for problem (2.2) in the particular case when $\mathcal{S} = [N]$. Let the Lagrangian of problem (2.3) be denoted by $L_{\mathcal{S}} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, defined for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ by

$$L_{\mathcal{S}}(x, y) = f(x) + c_{\mathcal{S}}(x)^T y = f(x) + \frac{1}{|\mathcal{S}|} \sum_{i \in \mathcal{S}} c_i(x)^T y,$$

where $y \in \mathbb{R}^m$ is a vector of Lagrange multipliers (also known as dual variables). Second-order necessary conditions for optimality for (2.3) can then be stated as

$$\nabla L_{\mathcal{S}}(x, y) \equiv \begin{bmatrix} \nabla_x L_{\mathcal{S}}(x, y) \\ \nabla_y L_{\mathcal{S}}(x, y) \end{bmatrix} \equiv \begin{bmatrix} \nabla f(x) + \nabla c_{\mathcal{S}}(x) y \\ c_{\mathcal{S}}(x) \end{bmatrix} = 0 \tag{2.4}$$

and, with $[c_{\mathcal{S}}]_j$ denoting the jth component of the constraint function $c_{\mathcal{S}}$,

$$d^T \nabla^2_{xx} L_{\mathcal{S}}(x,y) d \equiv d^T \left(\nabla^2 f(x) + \sum_{j \in [m]} \nabla^2 [c_{\mathcal{S}}]_j(x) y_j \right) d \geq 0 \quad \text{for all} \quad d \in \text{null}(\nabla c_{\mathcal{S}}(x)^T). \tag{2.5}$$

We refer to any point $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfying (2.4) as a first-order stationary point (or KKT point) with respect to problem (2.3), and we refer to any such point satisfying both (2.4) and (2.5) as a second-order stationary point with respect to (2.3). In addition, consistent with the literature on worst-case complexity bounds for nonconvex smooth optimization, we say that a point $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ is (ϵ, ε)-stationary with respect to problem (2.3) for some $(\epsilon, \varepsilon) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ if and only if

$$\|\nabla L_{\mathcal{S}}(x,y)\| \leq \epsilon \tag{2.6a}$$

$$\text{and} \quad d^T \nabla^2_{xx} L_{\mathcal{S}}(x,y) d \geq -\varepsilon \|d\|_2^2 \quad \text{for all} \quad d \in \text{null}(\nabla c_{\mathcal{S}}(x)^T). \tag{2.6b}$$

Generally speaking, an algorithm for solving (2.3) can be a *primal* method that might only generate a sequence of primal iterates $\{x_k\}$, or it can be a *primal-dual* method that generates a sequence of primal and dual iterate pairs $\{(x_k, y_k)\}$. For an application of our proposed algorithm, either type of method can be employed, but for certain results in our analysis we refer to properties of *least-square multipliers* corresponding to a given primal point $x \in \mathbb{R}^n$. Assuming that the Jacobian of $c_{\mathcal{S}}$ at x, namely, $\nabla c_{\mathcal{S}}(x)^T$, has full row rank, the least-squares multipliers with respect to x are given by $y_{\mathcal{S}}(x) \in \mathbb{R}^m$ that minimizes $\|\nabla_x L(x, \cdot)\|^2$, which is given by

$$y_{\mathcal{S}}(x) = -(\nabla c_{\mathcal{S}}(x)^T \nabla c_{\mathcal{S}}(x))^{-1} \nabla c_{\mathcal{S}}(x)^T \nabla f(x) = -\nabla c_{\mathcal{S}}(x)^{\dagger} \nabla f(x). \tag{2.7}$$

Our proposed method is stated as Algorithm 2.1 below. In our analysis in the next section, we formalize assumptions under which Algorithm 2.1 is well defined and yields our claimed convergence and worst-case sample complexity guarantees.

Algorithm 2.1 Progressive Constraint-Sampling Method (PCSM) for (2.2)

Require: Initial sample size $p_1 \in [N]$, initial point $x_0 \in \mathbb{R}^n$, maximum iteration index $K = \lceil \log_2 \frac{N}{p_1} \rceil$, and subproblem tolerances $\{(\epsilon_k, \varepsilon_k)\}_{k=1}^K \subset \mathbb{R}_{>0} \times \mathbb{R}_{>0}$
1: set $\mathcal{S}_0 \leftarrow \emptyset$
2: **for** $k \in [K]$ **do**
3: choose $\mathcal{S}_k \supseteq \mathcal{S}_{k-1}$ such that $|\mathcal{S}_k| = p_k$
4: using x_{k-1} as a starting point, employ an algorithm to solve (2.3), terminating once a primal iterate x_k has been obtained such that $(x_k, y_{\mathcal{S}_k}(x_k))$ (see (2.7)) is $(\epsilon_k, \varepsilon_k)$-stationary with respect to problem (2.3) for $\mathcal{S} = \mathcal{S}_k$
5: set $p_{k+1} \leftarrow \min\{2p_k, N\}$
6: **end for**
7: **return** $(x_K, y(x_K))$, which is $(\epsilon_K, \varepsilon_K)$-stationary with respect to (2.2)

3. Analysis. We begin our analysis of Algorithm 2.1 by stating the assumptions under which we prove our theoretical guarantees. Our first, Assumption 3.1 below, ensures that any minimizer of each encountered subproblem is a second-order stationary point and that one can expect an algorithm that is employed to solve each subproblem will find a sufficiently approximate second-order stationary point. It would be possible to prove reasonable theoretical guarantees for Assumption 3.1 under looser assumptions. For example, if an algorithm employed to solve (2.3) for some sample set $\mathcal{S}$ were to encounter an (approximate) infeasible stationary point, then it would be reasonable to terminate the subproblem solver and either terminate Algorithm 2.1 in its entirety or move on to solve the next subproblem (with a larger sample set). However, since consideration of such scenarios would distract from the essential properties of our algorithm when each subproblem solve is successful, we make Assumption 3.1. Our remarks after the assumption justify it further.

ASSUMPTION 3.1. *The functions $f : \mathbb{R}^n \to \mathbb{R}$ and $c_i : \mathbb{R}^n \to \mathbb{R}$ for each $i \in [N]$ are twice-continuously differentiable. In addition, the following hold for problem (2.2), instances of subproblem (2.3), and the algorithm employed to solve them.*

(a) There exists $\sigma_{\min} \in \mathbb{R}_{>0}$ such that, for all $x \in \mathbb{R}^n$ and $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \geq p_1$, the constraint Jacobian has $\sigma_m(\nabla c_{\mathcal{S}}(x)^T) \geq \sigma_{\min}$.

(b) For all $\mathcal{S} \subset [N]$ with $|\mathcal{S}| \geq p_1$, the algorithm employed to solve (2.3) is guaranteed to converge from any initial point to a second-order stationary point, i.e., one satisfying (2.4) and (2.5).

(c) For some $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, problem (2.2) is (α, β)-strongly Morse in the sense that, for any $x \in \mathbb{R}^n$, if $(x, y(x))$ satisfies $\|\nabla L(x, y(x))\| \leq \alpha$, then $|d^T \nabla^2_{xx} L(x, y(x)) d| \geq \beta \|d\|_2^2$ for all $d \in \operatorname{null}(\nabla c(x)^T)$.

The first part of Assumption 3.1 guarantees that the algorithm employed to solve subproblem (2.3) will not, for example, get stuck at an infeasible stationary point. In addition to this assurance, the second part of Assumption 3.1 implicitly requires one of various types of conditions in the literature on equality-constrained optimization that guarantee convergence of an algorithm to a second-order stationary point. See, for example, [3], where for certain $(\epsilon, \varepsilon) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ it is shown that an algorithm can produce an (ϵ, ε)-stationary point in $\mathcal{O}(\max\{\epsilon^{-2}, \varepsilon^{-3}\})$ iterations.

The third part of Assumption 3.1 is an essential element for proving our desired worst-case sample complexity properties of Algorithm 2.1. Related to this, the following comment is important, so we emphasize it as a remark.

REMARK 3.1. *If one were to change the requirement for the subproblem solver in Algorithm 2.1 and only require that the solver produce an approximate first-order stationary point, rather than an approximate second-order stationary point, then under Assumption 3.1 (with "second-order" replaced by "first-order" in part (b) and part (c) of the assumption removed) the algorithm would be well defined and, by its construction, would guarantee convergence to an approximate first-order stationary point of problem (2.2). Therefore, in practice, Algorithm 2.1 might be run with only approximate first-order stationary requirements, as we explore in our numerical experiments in §4. However, such a set-up would not allow us to prove strong sample complexity guarantees. Therefore, for the purposes of our analysis, we state Algorithm 2.1 and Assumption 3.1 as they are given, and note that our analysis leverages approximate second-order stationarity in order to ensure good sample complexity.*

Our next assumption articulates bounds on derivatives of the objective and constraint functions corresponding to the full-sample problem (2.2).

ASSUMPTION 3.2. *There exists $(\kappa_{\nabla f}, \kappa_{\nabla c}, \kappa_{\nabla^2 f}, \kappa_{\nabla^2 c}) \in \mathbb{R}^5_{>0}$ such that, for all $x \in \mathbb{R}^n$ and $j \in [m]$, one has $\|\nabla f(x)\| \leq \kappa_{\nabla f}$, $\|\nabla c(x)\| \leq \kappa_{\nabla c}$, $\|\nabla^2 f(x)\| \leq \kappa_{\nabla^2 f}$, and $\|\nabla^2 [c]_j(x)\| \leq \kappa_{\nabla^2 c}$ for all $j \in [m]$, where $[c]_j$ denotes the jth component of c.*

Our final assumption introduces constants that bound discrepancies between constraint Jacobians corresponding to individual samples and those corresponding to the full set of samples, and introduces constants that similarly bound discrepancies between individual-sample and the full-sample constraint Hessian matrices.

Assumption 3.3. *There exists* $(\gamma_c, \gamma_{\nabla c}, \gamma_{\nabla^2 c}) \in \mathbb{R}^3_{>0}$ *such that, for all* $x \in \mathbb{R}^n$ *and* $j \in [m]$*, one has*

$$\frac{1}{N}\sum_{i\in[N]} \|c_i(x) - c(x)\|^2 \le \gamma_c,$$

$$\frac{1}{N}\sum_{i\in[N]} \|\nabla c_i(x) - \nabla c(x)\|^2 \le \gamma_{\nabla c}\|\nabla c(x)\|^2,$$

$$\textit{and}\quad \frac{1}{N}\sum_{i\in[N]} \|\nabla^2[c_i]_j(x) - \nabla^2[c]_j(x)\|^2 \le \gamma_{\nabla^2 c}\|\nabla^2[c]_j(x)\|^2.$$

Our first lemma leverages Assumption 3.3 in order to provide bounds that are similar to those in the assumption, except that they are with respect to $c_{\mathcal{S}}$ and its derivatives for given $\mathcal{S} \subseteq [N]$. The resulting bounds depend on the sample size $|\mathcal{S}|$.

Lemma 3.1. *For all* $x \in \mathbb{R}^n$*,* $\mathcal{S} \subseteq [N]$*, and* $j \in [m]$*, one has*

$$\|c_{\mathcal{S}}(x) - c(x)\|^2 \le N\left(\frac{N-|\mathcal{S}|}{|\mathcal{S}|^2}\right)\gamma_c, \tag{3.1a}$$

$$\|\nabla c_{\mathcal{S}}(x) - \nabla c(x)\|^2 \le N\left(\frac{N-|\mathcal{S}|}{|\mathcal{S}|^2}\right)\gamma_{\nabla c}\|\nabla c(x)\|^2, \tag{3.1b}$$

$$\textit{and}\quad \|\nabla^2[c_{\mathcal{S}}]_j(x) - \nabla^2[c]_j(x)\|^2 \le N\left(\frac{N-|\mathcal{S}|}{|\mathcal{S}|^2}\right)\gamma_{\nabla^2 c}\|\nabla^2[c]_j(x)\|^2. \tag{3.1c}$$

Proof. Each of the desired bounds can be proved in a similar manner. We prove the second bound, namely, (3.1b), with respect to the constraint Jacobians. The other two bounds follow in a similar manner. First, observe that

$$\begin{aligned}\nabla c_{\mathcal{S}}(x) = \frac{1}{|\mathcal{S}|}\sum_{i\in\mathcal{S}}\nabla c_i(x) &= \frac{1}{|\mathcal{S}|}\sum_{i\in[N]}\nabla c_i(x) - \frac{1}{|\mathcal{S}|}\sum_{i\in[N]\setminus\mathcal{S}}\nabla c_i(x)\\ &= \frac{N}{|\mathcal{S}|}\nabla c(x) - \frac{1}{|\mathcal{S}|}\sum_{i\in[N]\setminus\mathcal{S}}\nabla c_i(x).\end{aligned}$$

Second, observe that for any vector in $v \in \mathbb{R}^{N-|\mathcal{S}|}$ one has from the Cauchy-Schwarz inequality that $(\mathbb{1}^T v)^2 \le \|\mathbb{1}\|^2\|v\|_2^2 = (N-|\mathcal{S}|)\|v\|_2^2$. Consequently, with the triangle inequality and Assumption 3.3, one finds that

$$\begin{aligned}\|\nabla c_{\mathcal{S}}(x) - \nabla c(x)\|^2 &= \left\|\frac{N}{|\mathcal{S}|}\nabla c(x) - \nabla c(x) - \frac{1}{|\mathcal{S}|}\sum_{i\in[N]\setminus\mathcal{S}}\nabla c_i(x)\right\|^2\\ &= \left\|\frac{N-|\mathcal{S}|}{|\mathcal{S}|}\nabla c(x) - \frac{1}{|\mathcal{S}|}\sum_{i\in[N]\setminus\mathcal{S}}\nabla c_i(x)\right\|^2\\ &= \frac{1}{|\mathcal{S}|^2}\left\|\sum_{i\in[N]\setminus\mathcal{S}}(\nabla c(x) - \nabla c_i(x))\right\|^2\end{aligned}$$

$$\leq \frac{1}{|\mathcal{S}|^2}\left(\sum_{i\in[N]\setminus\mathcal{S}}\|\nabla c(x)-\nabla c_i(x)\|\right)^2$$
$$\leq \left(\frac{N-|\mathcal{S}|}{|\mathcal{S}|^2}\right)\sum_{i\in[N]\setminus\mathcal{S}}\|\nabla c(x)-\nabla c_i(x)\|^2$$
$$\leq \left(\frac{N-|\mathcal{S}|}{|\mathcal{S}|^2}\right)\sum_{i\in[N]}\|\nabla c(x)-\nabla c_i(x)\|^2$$
$$\leq \left(\frac{N-|\mathcal{S}|}{|\mathcal{S}|^2}\right)N\gamma_{\nabla c}\|\nabla c(x)\|^2,$$

which gives the desired conclusion. ∎

LEMMA 3.2. *The values of $\|\nabla c^\dagger\|_2$, $\|y\|_2$ and $\|\nabla^2_{xx}L\|_2$ are bounded, that is, for any $x\in\mathbb{R}^n$ we have*

$$\|\nabla c(x)^\dagger\| \leq \frac{1}{\sigma_{\min}} \text{ and,} \tag{3.2a}$$
$$\|y(x)\| \leq \frac{\kappa_{\nabla f}}{\sigma_{\min}}, \text{ moreover} \tag{3.2b}$$
$$\|\nabla^2_{xx}L(x,y)\| \leq \kappa_{\nabla^2 f} + \frac{\sqrt{m}\kappa_{\nabla f}\kappa_{\nabla^2 c}}{\sigma_{\min}}. \tag{3.2c}$$

Proof. For (3.2a), [1, Chapter 21] tells us that the spectral norm of the pseudo inverse defined in our paper for a full column rank matrix, equals the inverse of the smallest singular value of that matrix. In our case of $\nabla c(x)$, by Assumption 3.1 $\nabla c(x)$ has full column rank and the singular value is bounded below by $\sigma_{\min}$, we have that the spectral norm of $\nabla c(x)^\dagger$ is bounded above by $\frac{1}{\sigma_{\min}}$.

For (3.2b), by sub-multiplicity for matrix-vector product, Assumption 3.2 that $\|\nabla f(x)\| \leq \kappa_{\nabla f}$, and (3.2a), we have

$$\|y(x)\| = \|-\nabla c(x)^\dagger\nabla f(x)\| \leq \|\nabla c(x)^\dagger\|\|\nabla f(x)\| \leq \frac{\kappa_{\nabla f}}{\sigma_{\min}}.$$

For (3.2c), we have

$$\begin{aligned}
\|\nabla^2_{xx}L(x,y)\| &= \|\nabla^2 f(x) + \sum_{j=1}^m \nabla^2[c]_j(x)y_j\| \\
&\leq \|\nabla^2 f(x)\| + \sum_{j=1}^m \|\nabla^2[c]_j(x)\|\cdot|y| \\
&\leq \kappa_{\nabla^2 f} + \kappa_{\nabla^2 c}\|y\|_1 \\
&\leq \kappa_{\nabla^2 f} + \sqrt{m}\kappa_{\nabla^2 c}\|y\| \\
&\leq \kappa_{\nabla^2 f} + \frac{\sqrt{m}\kappa_{\nabla f}\kappa_{\nabla^2 c}}{\sigma_{\min}}.
\end{aligned}$$

Here, the second line uses the triangle inequality, and the absolute homogeneity of any norm. The third line uses Assumption 3.2 that $\|\nabla^2 f(x)\| \leq \kappa_{\nabla^2 f}$ and $\|\nabla^2[c]_j(x)\|_2 \leq \kappa_{\nabla^2 c}$ for all $j\in[m]$. The second from last line uses again the Cauchy-Schwarz inequality that for any vector $y\in\mathbb{R}^m$, it holds $\left(\sum_{j=1}^m y_j\cdot 1\right)^2 \leq \left(\sum_{j=1}^m 1^2\right)\|y\|^2 = m\|y\|^2$, and taking square roots on both sides. The last line uses (3.2b). ∎

Definition 3.3. *For any full column rank matrices $(A, B) \in \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m}$ where $n \geq m$, the A and B are acute perturbations to each other, if and only if*

$$\operatorname{rank}(AA^{\dagger}BA^{\dagger}A) = m.$$

With Definition 3.3 and Lemma 3.1, we have the following condition on $(\mathcal{S}, \gamma_{\nabla c})$ to ensure $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are acute perturbations to each other.

Lemma 3.4. *Under Assumption 3.1, Assumption 3.2 and Assumption 3.3 and suppose constants $(\sigma_{\min}, \kappa_{\nabla c}, \gamma_{\nabla c})$ notates the same quantity. If $\mathcal{S} \subseteq [N]$ satisfies*

$$|\mathcal{S}| > \frac{2}{1 + \sqrt{1 + \frac{4\sigma_{\min}^2}{\gamma_{\nabla c}\kappa_{\nabla c}^2}}} N, \tag{3.3}$$

we have that for any $x \in \mathbb{R}^n$, the gradient $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are acute perturbations to each other.

Proof. By Assumption 3.1 item a, for any $x \in \mathbb{R}^n$, we have the Jacobian $\nabla c_{\mathcal{S}}(x)^T$ and $\nabla c(x)^T$ are of full row rank since their smallest singular values are bounded away from zero. Hence we have that $\nabla c_{\mathcal{S}}(x)$ and $\nabla c(x)$ are n by m matrices with full column rank. And according to Definition 3.3, we only need to show the equality (3.3) holds for $\nabla c_{\mathcal{S}}(x)$ and $\nabla c(x)$. We have

$$\begin{aligned}
&\nabla c(x)\nabla c(x)^{\dagger}\nabla c_{\mathcal{S}}(x)\nabla c(x)^{\dagger}\nabla c(x)\\
&= \nabla c(x)\nabla c(x)^{\dagger}\nabla c_{\mathcal{S}}(x)\\
&= \nabla c(x)\nabla c(x)^{\dagger}\left(\nabla c(x) + \nabla c_{\mathcal{S}}(x) - \nabla c(x)\right)\\
&= \nabla c(x)\left(I_m + \underbrace{\nabla c(x)^{\dagger}\left(\nabla c_{\mathcal{S}}(x) - \nabla c(x)\right)}_{(3.4.1)}\right).
\end{aligned} \tag{3.4}$$

Here, the second line uses the definition of pseudo inverse that $\nabla c(x)^{\dagger}\nabla c(x) = \left(\left(\nabla c(x)^T\nabla c(x)\right)^{-1}\nabla c(x)^T\right)\nabla c(x) = I_m$. The second from last line adds and subtracts a term, and the last line combines the product of the last two terms from the previous equality, and uses $\nabla c(x)^{\dagger}\nabla c(x) = I_m$ again.

Further, for the value of $\|(3.4.1)\|$, we have

$$\begin{aligned}
\|(3.4.1)\| &= \left\|\nabla c(x)^{\dagger}\left(\nabla c_{\mathcal{S}}(x) - \nabla c(x)\right)\right\|\\
&\leq \left\|\nabla c(x)^{\dagger}\right\| \left\|\nabla c_{\mathcal{S}}(x) - \nabla c(x)\right\|\\
&\leq \frac{1}{\sigma_{\min}}\sqrt{\frac{\gamma_{\nabla c}N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}}\|\nabla c(x)\|\\
&< \frac{1}{\sigma_{\min}}\sigma_{\min} = 1.
\end{aligned} \tag{3.5}$$

Here, the second line uses the sub-multiplicity of the matrix product. The third line uses inequalities (3.1b,3.2a) and the last line follows that the inequality (3.3) gives $\sqrt{\frac{\gamma_{\nabla c}N(N-|\mathcal{S}|)}{|\mathcal{S}|^2}}\|\nabla c(x)\| \leq \sigma_{\min}$. Moreover, [5, bound (3)] gives a bound on the differences of singular values for two matrices of the same size, which gives us

$$|\sigma_m(I_m) - \sigma_m(I_m + (3.4.1))| \leq \|(3.4.1)\|. \tag{3.6}$$

Using the definition of the absolute value and combining (3.6) with (3.5), we have

$$\sigma_m(I_m) - \sigma_m(I_m + (3.4.1)) \leq |\sigma_m(I_m) - \sigma_m(I_m + (3.4.1))| < 1,$$

and the left-most and right-most terms of the above inequality give us

$$\sigma_m(I_m + (3.4.1)) > \sigma_m(I_m) - 1 = 1 - 1 = 0.$$

Hence, the square matrix $I_m + (3.4.1)$ is of full rank. Combining with the fact that $\nabla c(x)$ has full column rank, we have that $\nabla c(x)(I_m + (3.4.1))$ also has full column rank,which gives us

$$\operatorname{rank}\left(\nabla c(x)\nabla c(x)^\dagger \nabla c_{\mathcal{S}}(x)\nabla c(x)^\dagger \nabla c(x)\right) = \operatorname{rank}\left(I_m + (3.4.1)\right) = m. \qquad \square$$

Now, we can present the first type of bounds.

LEMMA 3.5. *Under Assumption 3.1, Assumption 3.2 and Assumption 3.3 where constants* $(\kappa_{\nabla f}, \sigma_{\min}, \kappa_{\nabla c}, \gamma_{\nabla c})$ *notated the same quantity. Then, for any* $x \in \mathbb{R}^n$*, if the sample set* $\mathcal{S} \subseteq [N]$ *satisfies*

$$|\mathcal{S}| \geq \frac{2}{1 + \sqrt{1 + \frac{4\sigma_{\min}^2}{9\gamma_{\nabla c}\kappa_{\nabla c}^2}}} N \tag{3.7}$$

and let $y_{\mathcal{S}}(x)$ *be defined in (2.7), we have*

$$\|y(x) - y_{\mathcal{S}}(x)\| \leq \frac{3\kappa_{\nabla f}\kappa_{\nabla c}}{2\sigma_{\min}^2}\sqrt{\frac{\gamma_{\nabla c}N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}}.$$

Proof. First note that when $|\mathcal{S}|$ satisfies (3.7), it satisfies (3.3) of Lemma 3.4. Hence by Lemma 3.4 we have that $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are acute perturbations to each other. In addition, [6, Theorem 5.2] gives a perturbation bound for the least square solutions of the systems $A^T x = b$ and $\tilde{A}^T x = b$ where A and $\tilde{A}$ are acute perturbations to each other. Combing their result with our case where $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are of full column rank by Assumption 3.1, and relaxing the constant $\bar{\kappa}$ in their paper as in [6, Corollary 3.9], we have

$$\|y(x) - y_{\mathcal{S}}(x)\| \leq \underbrace{\left(\frac{\|\nabla c(x)^\dagger\|\|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|}{1 - \|\nabla c(x)^\dagger\|\|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|}\right)}_{(3.8.1)}\|y(x)\|. \tag{3.8}$$

By inequality (3.2a,3.1b), Assumption 3.2 that gives $\|\nabla c(x)\| \leq \kappa_{\nabla c}$ and (3.7), we have

$$\|\nabla c(x)^\dagger\|\|\nabla c_{\mathcal{S}}(x) - \nabla c(x)\| \leq \sqrt{\frac{\gamma_{\nabla c}N(N - |\mathcal{S}|)}{|\mathcal{S}|^2}}\|\nabla c(x)^\dagger\|\|\nabla c(x)\| \leq \frac{1}{3},$$

which further gives us

$$1 - \|\nabla c(x)^\dagger\|\|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\| \geq 2/3. \tag{3.9}$$

Hence, we have

$$
\begin{aligned}
(3.8.1) &\le \frac{3}{2}\|\nabla c(x)^\dagger\|\|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\| \\
&\le \frac{3\kappa_{\nabla c}}{2\sigma_{\min}}\sqrt{\frac{\gamma_{\nabla c} N(N-|\mathcal{S}|)}{|\mathcal{S}|^2}}.
\end{aligned}
$$

Here, the first line uses (3.9) at the denominator. The last line uses again (3.1b, 3.2a), and $\|\nabla c(x)\|_2 \le \kappa_{\nabla c}$ in Assumption 3.2. Combining with the bound for $\|y(x)\|$ in (3.2b), we have

$$
\|y(x) - y_{\mathcal{S}}(x)\| \le \frac{3\kappa_{\nabla f}\kappa_{\nabla c}}{2\sigma_{\min}^2}\sqrt{\frac{\gamma_{\nabla c} N(N-|\mathcal{S}|)}{|\mathcal{S}|^2}}. \qquad \square
$$

The difference in the Hessian conditions is more complicated compared to the gradient condition. Recall the Hessian condition in item c of Assumption 3.1 that

$$
|d^T \nabla_{xx}^2 L(x,y) d| \ge \beta \|d\|_2^2, \ \forall \ d \in \mathrm{null}(\nabla c(x)^T).
$$

When we consider the empirical system (2.3), not only did the Lagrangian function L change, but also the null space $\mathrm{null}(\nabla c(x)^T)$ change. We start by giving a general result for two perturbed null spaces by examining the difference between vectors in one null space and their projections onto the other null space.

LEMMA 3.6. *Under Assumption.3.1, Assumption.3.2 and Assumption 3.3. In addition, let constants $(\sigma_{\min}, \kappa_{\nabla c}, \gamma_{\nabla c})$ notate the same quantities. If $\mathcal{S} \subseteq [N]$ satisfies (3.3), then for any $x \in \mathbb{R}^n$ and any $d_{\mathcal{S}} \in \mathrm{null}(\nabla c_{\mathcal{S}}(x)^T)$, we have*

$$
\frac{\|\mathcal{R}(\nabla c(x))(d_{\mathcal{S}})\|}{\|d_{\mathcal{S}}\|} \le \frac{\kappa_{\nabla c}}{\sigma_{\min}}\sqrt{\frac{N\left(N-|\mathcal{S}|\right)\gamma_{\nabla c}}{|\mathcal{S}|^2}} < 1.
$$

Proof. Since $d_{\mathcal{S}} \in \mathrm{null}(\nabla c_{\mathcal{S}}(x)^T)$, and $\mathcal{N}(\nabla c_{\mathcal{S}}(x))$ is a projection matrix to $\mathrm{null}(\nabla c_{\mathcal{S}}(x)^T)$. [2, Lemma 1.17] tells us that a projection matrix maps any vector in its image space to the vector itself. We have $\mathcal{N}(\nabla c_{\mathcal{S}}(x)) d_{\mathcal{S}} = d_{\mathcal{S}}$. Combining with sub-multiplicity, we have

$$
\|\mathcal{R}(\nabla c(x)) d_{\mathcal{S}}\| = \|\mathcal{R}(\nabla c(x))\mathcal{N}(\nabla c_{\mathcal{S}}(x)) d_{\mathcal{S}}\| \le \|\mathcal{R}(\nabla c(x))\mathcal{N}(\nabla c_{\mathcal{S}}(x))\|\|d_{\mathcal{S}}\|.
$$

In addition, Assumption 3.1 tells us that $\nabla c_{\mathcal{S}}(x)$ and $\nabla c(x)$ are of full column rank. Moreover, [6, Theorem 2.4] gives us a bound for the product of two projection matrices, one projects to the column space of an n by m matrix, and the other projects to the left null space of another n by m matrix who has the same rank as the one whose column space is projected. We further have

$$
\begin{aligned}
\|\mathcal{R}(\nabla c(x)) d_{\mathcal{S}}\| &\le \|\mathcal{R}(\nabla c(x))\mathcal{N}(\nabla c_{\mathcal{S}}(x))\|\|d_{\mathcal{S}}\| \\
&\le \|\nabla c(x)^\dagger\|\|\nabla c(x) - \nabla c_{\mathcal{S}}(x)\|\|d_{\mathcal{S}}\|.
\end{aligned} \tag{3.10}
$$

Dividing $\|d_{\mathcal{S}}\|$ on both sides and combining Lemma 3.1 item b, Assumption 3.2 that $\|\nabla c(x)\| \le \kappa_{\nabla c}$ and Lemma 3.2 item a, we have

$$
\frac{\|\mathcal{R}(\nabla c(x))(d_{\mathcal{S}})\|}{\|d_{\mathcal{S}}\|} \le \frac{\kappa_{\nabla c}}{\sigma_{\min}}\sqrt{\frac{N\left(N-|\mathcal{S}|\right)\gamma_{\nabla c}}{|\mathcal{S}|^2}}.
$$

Moreover, the choice of $|\mathcal{S}|$ in (3.3) gives us

$$\frac{\kappa_{\nabla c}}{\sigma_{\min}}\sqrt{\frac{N\left(N-|\mathcal{S}|\right)\gamma_{\nabla c}}{|\mathcal{S}|^2}}<1. \qquad \square$$

With this result, we can now look into the Hessian condition for empirical constraint Morse problem. In particular, let $d\in \operatorname{null}(\nabla c_{\mathcal{S}}(x)^T)$, vector $\tilde{d}=\mathcal{N}\left(\nabla c(x)\right)d$ and $r=d-\tilde{d}$, we tend to look at the difference

$$\left|d^T\nabla^2_{xx}L_{\mathcal{S}}(x,y_{\mathcal{S}})d-\tilde{d}^T\nabla^2_{xx}L_{[N]}(x,y_{[N]})\tilde{d}\right|,$$

and the following lemma gives us a general bound for the above term.

In summary, we have the result for the Morse property of the empirical problem. We define the following four parameters

$$\begin{cases}\eta_1:=\sqrt{\gamma_{\nabla c}}\frac{\kappa_{\nabla c}}{\sigma_{\min}},\\ \eta_2:=\frac{\kappa_{\nabla f}\kappa_{\nabla^2 c}}{\sigma_{\min}}\sqrt{m\gamma_{\nabla^2 c}},\\ \eta_3:=\eta_2+3\eta_1\kappa_{\nabla^2 f}+\frac{3\eta_1\eta_2}{2\sqrt{\gamma_{\nabla^2 c}}}.\end{cases}$$

THEOREM 3.7. *Under Assumption 3.1, Assumption 3.2 and Assumption 3.3 and the constants $(\sigma_{\min},\kappa_{\nabla c},\kappa_{\nabla f},\kappa_{\nabla^2 c},\kappa_{\nabla^2 f},\gamma_c,\gamma_{\nabla c},\gamma_{\nabla^2 c})$ exist. In addition, the dual variable y and $y_{\mathcal{S}}$ are chosen as in (2.7). Then, for any $\mathcal{S}\subseteq[N]$ when satisfies:*

$$g_{\mathcal{S}}:=\sqrt{\frac{N(N-|\mathcal{S}|)}{|\mathcal{S}|^2}}\le\min\left\{\frac{1}{3\eta_1},\frac{\alpha}{2\sqrt{\gamma_c+\kappa^2_{\nabla f}\eta_1^2}},\frac{\beta}{2\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}}\right\}, \tag{3.11}$$

the problem (2.3) is $(\alpha_{\mathcal{S}},\beta_{\mathcal{S}})$-morse, where

$$\begin{cases}\alpha_{\mathcal{S}}=\alpha-\sqrt{\gamma_c+\kappa^2_{\nabla f}\eta_1^2}\,g_{\mathcal{S}}\ge\frac{1}{2}\alpha>0 \text{ and}\\ \beta_{\mathcal{S}}=\beta-\eta_3 g_{\mathcal{S}}-\left(\beta\eta_1+2\eta_3\right)\eta_1 g_{\mathcal{S}}^2\ge\frac{1}{2}\beta>0.\end{cases}$$

Proof. We start by investigating what can be implied about $\|\nabla L(x,y)\|$ when we have $\|\nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})\|\le\alpha_{\mathcal{S}}$. To do so, we first look into the difference between $\nabla_x L(x,y)$ and $\nabla_x L_{\mathcal{S}}(x,y_{\mathcal{S}})$. We have

$$\begin{aligned}&\|\nabla_x L(x,y)-\nabla_x L_{\mathcal{S}}(x,y_{\mathcal{S}})\|\\ &=\|\nabla c(x)y-\nabla c_{\mathcal{S}}(x)y_{\mathcal{S}}\|\\ &=\|-\nabla c(x)\nabla c(x)^\dagger\nabla f(x)+\nabla c_{\mathcal{S}}(x)\nabla c_{\mathcal{S}}(x)^\dagger\nabla f(x)\|\\ &\le\|-\mathcal{R}\left(\nabla c_{\mathcal{S}}(x)\right)+\mathcal{R}\left(\nabla c(x)\right)\|\|\nabla f(x)\|\\ &\le\|-\mathcal{R}\left(\nabla c_{\mathcal{S}}(x)\right)+\mathcal{R}\left(\nabla c(x)\right)\|\kappa_{\nabla f}.\end{aligned} \tag{3.12}$$

Here, the third line uses the definition for y and $y_{\mathcal{S}}$ in (2.7). The second from last line uses the definition of the column space projector $\mathcal{R}$ and sub-multiplicity of matrix-vector product. The last line uses Assumption 3.2 that $\|\nabla f(x)\|_2\le\kappa_{\nabla f}$. By Assumption 3.1 that $\nabla c(x)$ and $\nabla c_{\mathcal{S}}(x)$ are of the same column rank, which satisfies the requirement of [6, Theorem 2.3, 2.4], which tells that for (3.12), we have

$$\begin{aligned}\|-\mathcal{R}\left(\nabla c_{\mathcal{S}}(x)\right)+\mathcal{R}\left(\nabla c(x)\right)\|\kappa_{\nabla f}&=\|\mathcal{R}\left(\nabla c_{\mathcal{S}}(x)\right)\mathcal{N}\left(\nabla c(x)\right)\|\kappa_{\nabla f}\\ &\le\|\nabla c(x)^\dagger\|\|\nabla c_{\mathcal{S}}(x)-\nabla c(x)\|\kappa_{\nabla f}.\end{aligned}$$

Combining the above result with Lemma 3.2 item a and Lemma 3.1 item b, we have

$$\begin{aligned}\|\nabla_x L(x,y) - \nabla_x L_{\mathcal{S}}(x,y_{\mathcal{S}})\| &\le \| - \mathcal{R}\left(\nabla c_{\mathcal{S}}(x)\right) + \mathcal{R}\left(\nabla c(x)\right)\|_2 \kappa_{\nabla f} \\ &\le \frac{\kappa_{\nabla c}\kappa_{\nabla f}}{\sigma_{\min}}\sqrt{\gamma_{\nabla c}}g_{\mathcal{S}}.\end{aligned} \tag{3.13}$$

Second, we look into the difference between $\nabla_y L(x,y)$ and $\nabla_y L_{\mathcal{S}}(x,y_{\mathcal{S}})$. By Assumption 3.3 item a we have

$$\|\nabla_y L(x,y) - \nabla_y L_{\mathcal{S}}(x,y_{\mathcal{S}})\| = \|c(x) - c_{\mathcal{S}}(x)\| \le \sqrt{\gamma_c} g_{\mathcal{S}}. \tag{3.14}$$

As a result, the difference between $\nabla L(x,y)$ and $\nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})$ can be bounded by

$$\begin{aligned}&\|\nabla L(x,y) - \nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})\| \\ &= \sqrt{\|\nabla_x L(x,y) - \nabla_x L_{\mathcal{S}}(x,y_{\mathcal{S}})\|^2 + \|\nabla_y L(x,y) - \nabla_y L_{\mathcal{S}}(x,y_{\mathcal{S}})\|^2} \\ &\le \sqrt{\gamma_c + \frac{\gamma_{\nabla c}\kappa_{\nabla c}^2\kappa_{\nabla f}^2}{\sigma_{\min}^2}} g_{\mathcal{S}}.\end{aligned} \tag{3.15}$$

Here, the second line uses $\nabla L(x,y) - \nabla L_{\mathcal{S}}(x,y_{\mathcal{S}}) = \begin{bmatrix}\nabla_x L(x,y) - \nabla_x L_{\mathcal{S}}(x,y_{\mathcal{S}} \\ \nabla_y L(x,y) - \nabla_y L_{\mathcal{S}}(x,y_{\mathcal{S}}\end{bmatrix}$, and the last line combines (3.13, 3.14). Further, for $\|\nabla L(x,y)\|$ we have

$$\begin{aligned}\|\nabla L(x,y)\| &\le \|\nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})\| + \|\nabla L(x,y) - \nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})\| \\ &= \|\nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})\|_2 + \sqrt{\gamma_c + \kappa_{\nabla f}^2\eta_1^2} g_{\mathcal{S}}.\end{aligned}$$

Here, the first line adds, subtracts a term and uses the triangle inequality. The last line uses (3.15). As a result, for any $x \in \mathbb{R}^n$ satisfying $\|\nabla L_{\mathcal{S}}(x,y_{\mathcal{S}})\| \le \alpha_{\mathcal{S}} = \alpha - \sqrt{\gamma_c + \kappa_{\nabla f}^2\eta_1^2} g_{\mathcal{S}}$, we have $\|\nabla L(x,y)\| \le \alpha$. In addition, the choice of $\mathcal{S}$ gives $\sqrt{\gamma_c + \kappa_{\nabla f}^2\eta_1^2} g_{\mathcal{S}} \le \frac{1}{2}\alpha$, we have $\alpha_{\mathcal{S}} \ge \frac{1}{2}\alpha > 0$.

By Assumption 3.1 item c, the problem (2.2) is (α,β)-strongly Morse. Combining with $\|\nabla L(x,y)\| \le \alpha$, we have

$$|d^T \nabla_{xx}^2 L(x,y) d| \ge \beta \|d\|^2 \text{ for all } d \in \operatorname{null}(\nabla c(x)^T).$$

Now, we investigate what can be implied for the value $|d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x,y_{\mathcal{S}}) d_{\mathcal{S}}|$ for any $d_{\mathcal{S}} \in \operatorname{null}(\nabla c_{\mathcal{S}}(x)^T)$, when the above condition holds. For such $d_{\mathcal{S}}$, we have

$$\begin{aligned}&\left|d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x,y_{\mathcal{S}}) d_{\mathcal{S}}\right| \\ &= \left|d_{\mathcal{S}}^T \nabla_{xx}^2 L(x,y) d_{\mathcal{S}} + d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x,y_{\mathcal{S}}) d_{\mathcal{S}} - d_{\mathcal{S}}^T \nabla_{xx}^2 L(x,y) d_{\mathcal{S}}\right| \\ &\ge \underbrace{\left|d_{\mathcal{S}}^T \nabla_{xx}^2 L(x,y) d_{\mathcal{S}}\right|}_{(3.16.1)} - \underbrace{\left|d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x,y_{\mathcal{S}}) d_{\mathcal{S}} - d_{\mathcal{S}}^T \nabla_{xx}^2 L(x,y) d_{\mathcal{S}}\right|}_{(3.16.2)}.\end{aligned} \tag{3.16}$$

Here, the last line adds, subtracts a term and uses the triangle inequality.

Let $\bar{d}_{\mathcal{S}} := \mathcal{N}(\nabla c(x)) d_{\mathcal{S}}$ and $r_{\mathcal{S}} := d_{\mathcal{S}} - \bar{d}_{\mathcal{S}}$. Note that

$$\begin{aligned}\bar{d}_{\mathcal{S}}^T r_{\mathcal{S}} &= \bar{d}_{\mathcal{S}}^T \left(d_{\mathcal{S}} - \bar{d}_{\mathcal{S}}\right) = \bar{d}_{\mathcal{S}}^T \left(d_{\mathcal{S}} - \mathcal{N}(\nabla c(x)) d_{\mathcal{S}}\right) \\ &= d_{\mathcal{S}}^T \mathcal{N}(\nabla c(x)) \left(I_n - \mathcal{N}(\nabla c(x))\right) d_{\mathcal{S}} = 0.\end{aligned}$$

Here, the last equality uses the fact that $\mathcal{N}$ is a projection matrix, which satisfies $\mathcal{N} = \mathcal{N}^2$. As a result, we have $\|d_\mathcal{S}\|^2 = \|\bar{d}_\mathcal{S}\|^2 + \|r_\mathcal{S}\|^2$ which gives $\max\{\|\bar{d}_\mathcal{S}\|, \|r_\mathcal{S}\|\} \leq \|d_\mathcal{S}\|$. For term (3.16.1) we have

$$\begin{aligned}
(3.16.1) &= \left|\bar{d}_\mathcal{S}^T \nabla_{xx}^2 L(x,y)\bar{d}_\mathcal{S} + 2\bar{d}_\mathcal{S}^T \nabla_{xx}^2 L(x,y) r_\mathcal{S} + r_\mathcal{S}^T \nabla_{xx}^2 L(x,y) r_\mathcal{S}\right| \\
&\geq \left|\bar{d}_\mathcal{S}^T \nabla_{xx}^2 L(x,y)\bar{d}_\mathcal{S}\right| - 2\|\nabla_{xx}^2 L(x,y)\|\|\bar{d}_\mathcal{S}\|\|r_\mathcal{S}\| \\
&\quad - \left\|\nabla_{xx}^2 L(x,y)\right\|\|r_\mathcal{S}\|^2 \\
&\geq \beta\|\bar{d}_\mathcal{S}\|^2 - 3\|\nabla_{xx}^2 L(x,y)\|\|d_\mathcal{S}\|\|r_\mathcal{S}\|.
\end{aligned}$$

Here, the first line substitutes $d_\mathcal{S}$ by $\bar{d}_\mathcal{S} + r_\mathcal{S}$. The first inequality uses the triangle inequality. The last line uses that since (2.2) is (α,β)-strongly Morse and $\bar{d}_\mathcal{S} \in \operatorname{null}(\nabla c(x)^T)$, which gives $\left|\bar{d}_\mathcal{S}^T \nabla_{xx}^2 L(x,y)\bar{d}_\mathcal{S}\right| \geq \beta\|\bar{d}_\mathcal{S}\|^2$, and the fact that since $\max\{\|\bar{d}_\mathcal{S}\|, \|r_\mathcal{S}\|\} \leq \|d_\mathcal{S}\|$, we have $\|\bar{d}_\mathcal{S}\|\|r_\mathcal{S}\| \leq \|d_\mathcal{S}\|\|r_\mathcal{S}\|$ and $\|r_\mathcal{S}\|^2 \leq \|d_\mathcal{S}\|\|r_\mathcal{S}\|$. Further, we have

$$\begin{aligned}
(3.16.1) &\geq \beta\|\bar{d}_\mathcal{S}\|^2 - 3\left(\kappa_{\nabla^2 f} + \frac{\sqrt{m}\kappa_{\nabla f}\kappa_{\nabla^2 c}}{\sigma_{\min}} g_{\mathcal{S}_k})\right)\|d_\mathcal{S}\|\|r_\mathcal{S}\| \\
&\geq \beta(1-\eta_1^2 g_\mathcal{S}^2)\|d_\mathcal{S}\|^2 - 3\left(\kappa_{\nabla^2 f} + \frac{\eta_2}{\sqrt{\gamma_{\nabla^2 c}}} g_\mathcal{S}\right)\eta_1 g_\mathcal{S}\|d_\mathcal{S}\|^2 \qquad (3.17)\\
&= \left(\beta - 3\eta_1\kappa_{\nabla^2 f} g_\mathcal{S} - \left(\beta\eta_1^2 + 3\frac{\eta_1\eta_2}{\sqrt{\gamma_{\nabla^2 c}}}\right) g_\mathcal{S}^2\right)\|d_\mathcal{S}\|^2.
\end{aligned}$$

Here, the first line uses Lemma 3.2 item c on $\|\nabla_{xx}^2 L(x,y)\|$. The second line uses the fact that $\|\bar{d}_\mathcal{S}\|^2 = \|d_\mathcal{S}\|^2 - \|r_\mathcal{S}\|^2$, and the fact that our requirement $g_\mathcal{S} \leq \frac{1}{3\eta_1}$ satisfies (3.3), hene Lemma 3.6 can be applied for $\|r_\mathcal{S}\|$. The last line simply rearranges terms.

For the term (3.16.2), we have

$$\begin{aligned}
(3.16.2) &= \left|d_\mathcal{S}^T\left(\nabla_{xx}^2 L_\mathcal{S}(x,y_\mathcal{S}) - \nabla_{xx}^2 L(x,y)\right) d_\mathcal{S}\right| \\
&= \left|d_\mathcal{S}^T\left(\sum_{j=1}^m \nabla^2[c_\mathcal{S}]_j(x)[y_\mathcal{S}]_j - \sum_{j=1}^m \nabla^2[c]_j(x) y_j\right) d_\mathcal{S}\right| \\
&\leq \left\|\sum_{j=1}^m \left(\nabla^2[c_\mathcal{S}]_j(x)[y_\mathcal{S}]_j - \nabla^2[c]_j(x) y_j\right)\right\| \|d_\mathcal{S}\|^2.
\end{aligned}$$

Here, the last line uses the sub-multiplicity for matrix-vector product. For the term $\left\|\sum_{j=1}^m \left(\nabla^2[c_\mathcal{S}]_j(x)[y_\mathcal{S}]_j - \nabla^2[c]_j(x) y_j\right)\right\|$, we have

$$\begin{aligned}
&\left\|\sum_{j=1}^m \left(\nabla^2[c_\mathcal{S}]_j(x)[y_\mathcal{S}]_j - \nabla^2[c]_j(x) y_j\right)\right\| \\
&= \left\|\sum_{j=1}^m \left(\left(\nabla^2[c_\mathcal{S}]_j(x) - \nabla^2[c]_j(x)\right)\right)[y_\mathcal{S}]_j + \sum_{j=1}^m \left(\left([y_\mathcal{S}]_j - y_j\right)\nabla^2[c]_j(x)\right)\right\| \\
&\leq \sum_{j=1}^m \left\|\nabla^2[c_\mathcal{S}]_j(x) - \nabla^2[c]_j(x)\right\| |[y_\mathcal{S}]_j| + \sum_{j=1}^m \left\|\nabla^2[c]_j(x)\right\| |[y_\mathcal{S}]_j - y_j|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{m} \sqrt{\gamma_{\nabla^2 c}} g_{\mathcal{S}} \|\nabla^2 [c]_j(x)\| \, |[y_{\mathcal{S}}]_j| + \sum_{j=1}^{m} \left\|\nabla^2 [c]_j(x)\right\| |[y_{\mathcal{S}}]_j - y_j| \\
&\le \sqrt{\gamma_{\nabla^2 c}} g_{\mathcal{S}} \kappa_{\nabla^2 c} \sum_{j=1}^{m} |[y_{\mathcal{S}}]_j| + \kappa_{\nabla^2 c} \sum_{j=1}^{m} |[y_{\mathcal{S}}]_j - y_j|.
\end{aligned}
$$

Here, the second line adds, subtracts, and rearranges terms. The third line uses triangle inequality and absolute homogeneity of matrix norm. The second from last line uses Lemma 3.4 item c, and the last line uses Assumption 3.1 that $\|\nabla^2 [c]_j(x)\| \le \kappa_{\nabla^2 c}$ for all $j \in [m]$. Further, from the last line of the above inequality, we have

$$
\begin{aligned}
&\sqrt{\gamma_{\nabla^2 c}} g_{\mathcal{S}} \kappa_{\nabla^2 c} \sum_{j=1}^{m} |[y_{\mathcal{S}}]_j| + \kappa_{\nabla^2 c} \sum_{j=1}^{m} |[y_{\mathcal{S}}]_j - y_j| \\
&\le \sqrt{m\gamma_{\nabla^2 c}} \kappa_{\nabla^2 c} g_{\mathcal{S}} \|y_{\mathcal{S}}\| + \sqrt{m} \kappa_{\nabla^2 c} \|y_{\mathcal{S}} - y\| \\
&\le \sqrt{m\gamma_{\nabla^2 c}} \kappa_{\nabla^2 c} g_{\mathcal{S}} \|y\| + (1 + \sqrt{\gamma_{\nabla^2 c}} g_{\mathcal{S}}) \sqrt{m} \kappa_{\nabla^2 c} \|y_{\mathcal{S}} - y\| \\
&\le \sqrt{m\gamma_{\nabla^2 c}} \frac{\kappa_{\nabla^2 c} \kappa_{\nabla f}}{\sigma_{\min}} g_{\mathcal{S}} + (1 + \sqrt{\gamma_{\nabla^2 c}} g_{\mathcal{S}}) \sqrt{m} \frac{3\kappa_{\nabla f} \kappa_{\nabla^2 c} \kappa_{\nabla c}}{2\sigma_{\min}^2} \sqrt{\gamma_{\nabla c}} g_{\mathcal{S}} \\
&= \eta_2 g_{\mathcal{S}} + \frac{3\eta_1 \eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} g_{\mathcal{S}} + \frac{3\eta_1 \eta_2}{2} g_{\mathcal{S}}^2.
\end{aligned}
$$

Here, the second line uses again the Cauchy-Schwarz inequality that $\left(\sum_{j=1}^m |y_j|\right)^2 \le m\|y\|^2$. The third line uses a triangle inequality that $\|y_{\mathcal{S}}\| \le \|y\| + \|y - y_{\mathcal{S}}\|$ and rearragens terms. The second from last line uses Lemma 3.2 item a on $\|y\|$, and the fact that our requirement $g_{\mathcal{S}} \le \frac{1}{3\eta_1}$ satisfies (3.7) hence Lemma 3.5 is applied on $\|y_{\mathcal{S}} - y\|$. The last line rearranges terms and uses the definition of (η_1, η_2).

As a result, for (3.16.2) we have

$$
(3.16.2) \le \left(\eta_2 g_{\mathcal{S}} + \frac{3\eta_1 \eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} g_{\mathcal{S}} + \frac{3\eta_1 \eta_2}{2} g_{\mathcal{S}}^2\right) \|d_{\mathcal{S}}\|^2. \tag{3.18}
$$

Combining (3.17, 3.18) and rearranging terms, we have

$$
\begin{aligned}
&\left|d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x, y_{\mathcal{S}}) d_{\mathcal{S}}\right| \\
&\ge \left(\beta - \left(\eta_2 + 3\eta_1 \kappa_{\nabla^2 f} + \frac{3\eta_1 \eta_2}{2\sqrt{\gamma_{\nabla^2 c}}}\right) g_{\mathcal{S}} - \left(\beta\eta_1^2 + \frac{3}{2}\eta_1 \eta_2 + 3\frac{\eta_1 \eta_2}{\sqrt{\gamma_{\nabla^2 c}}}\right) g_{\mathcal{S}}^2\right) \|d_{\mathcal{S}}\|^2 \\
&= \left(\beta - \eta_3 g_{\mathcal{S}} - (\beta\eta_1 + 2\eta_3)\, \eta_1 g_{\mathcal{S}}^2\right) \|d_{\mathcal{S}}\|^2.
\end{aligned}
$$

Since $\beta_{\mathcal{S}}$ is defined as $\beta_{\mathcal{S}} = \beta - \eta_3 g_{\mathcal{S}} - (\beta\eta_1 + 2\eta_3)\, \eta_1 g_{\mathcal{S}}^2$, we now have that when $\|\nabla L_{\mathcal{S}}(x, y_{\mathcal{S}})\| \le \alpha_{\mathcal{S}}$, it holds $\left|d_{\mathcal{S}}^T \nabla_{xx}^2 L_{\mathcal{S}}(x, y_{\mathcal{S}}) d_{\mathcal{S}}\right| \ge \beta_{\mathcal{S}} \|d_{\mathcal{S}}\|$, $\forall d_{\mathcal{S}} \in \mathrm{null}(\nabla c_{\mathcal{S}}(x)^T)$, which is equalvently to say that (2.2) is $(\alpha_{\mathcal{S}}, \beta_{\mathcal{S}})$-strongly Morse.

In addition, to ensure $\beta_{\mathcal{S}} \ge \frac{1}{2}\beta$, we need

$$
\left(\eta_3 g_{\mathcal{S}} + (\beta\eta_1 + 2\eta_3)\, \eta_1 g_{\mathcal{S}}^2\right) \le \frac{1}{2}\beta,
$$

where the nonnegative solutions for $g_{\mathcal{S}}$ are

$$
0 \le g_{\mathcal{S}} \le \frac{-\eta_3 + \sqrt{\eta_3^2 + 2\beta\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)}}{2\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)}.
$$

The right-hand side can be bounded below by

$$
\begin{aligned}
&\frac{-\eta_3+\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}}{2\left(\beta\eta_1^2+2\eta_1\eta_3\right)}\\
&=\frac{2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}{2\left(\beta\eta_1^2+2\eta_1\eta_3\right)\left(\eta_3+\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}\right)}\\
&\geq\frac{\beta}{2\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}}.
\end{aligned}
$$

Here, the second line multiplies a $\left(\eta_3+\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}\right)$ at both the numerator and denominator. The last line uses that $\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}\geq\eta_3$. Hence when $|\mathcal{S}|\leq\frac{\beta}{2\sqrt{\eta_3^2+2\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}}$, we have that $\beta_{\mathcal{S}}\geq\frac{1}{2}\beta$. ∎

THEOREM 3.8. *Under Assumption 3.1, Assumption 3.2 and Assumption 3.3 such that the constants* $(\sigma_{\min},\kappa_{\nabla c},\kappa_{\nabla f},\kappa_{\nabla^2 c},\kappa_{\nabla^2 f},\gamma_c,\gamma_{\nabla c},\gamma_{\nabla^2 c})$ *exist. Define tolerances*

$$
\epsilon_k:=\sqrt{\gamma_c+\kappa_{\nabla f}^2\eta_1^2}\sqrt{\frac{N(N-|\mathcal{S}_k|)}{|\mathcal{S}_k|^2}}\ \textit{and}\ \varepsilon_k:=\kappa_{\nabla^2 f}\eta_1\sqrt{\frac{N(N-|\mathcal{S}_k|)}{|\mathcal{S}_k|^2}},\ \forall\ k\in[K].
$$

Let (2.2) proceed with Algorithm 2.1. Then, for all sample set $\mathcal{S}_k\subseteq[N]$ *that satisfies*

$$
\sqrt{\frac{N(N-|\mathcal{S}_k|)}{|\mathcal{S}_k|^2}}\leq\min\left\{\frac{1}{3\eta_1},\frac{\alpha}{4\sqrt{\gamma_c+\kappa_{\nabla f}^2\eta_1^2}},\frac{\beta}{4\sqrt{\eta_3^2+\beta\left(\beta\eta_1^2+2\eta_1\eta_3\right)}}\right\}, \tag{3.19}
$$

if $x_{\mathcal{S}_k}\in\mathbb{R}^n$ *is a* $(\epsilon_k,\varepsilon_k)$ *stationary solution, the* $x_{\mathcal{S}_k}$ *must satisfy the following for problem (2.3) with* $\mathcal{S}=\mathcal{S}_{k+1}$

$$
\begin{aligned}
&\|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k},y_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}))\|\leq 3\epsilon_k\leq\alpha_{\mathcal{S}_{k+1}},\ \textit{and}\\
&d^T\nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k},y_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}))d\geq\beta_{\mathcal{S}_{k+1}}\|d\|^2,\forall\ d\in\operatorname{null}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})^T).
\end{aligned}
$$

Proof. Let the dual variables y and $y_{\mathcal{S}_k}$ be defined as in (2.7). In addition, define $z_{\mathcal{S}_{k+1}}=-\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})^\dagger\nabla f(x_{\mathcal{S}_k})$. Similarly, we have

$$
\begin{aligned}
&\|\nabla_x L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k},z_{\mathcal{S}_{k+1}})-\nabla_x L_{\mathcal{S}_k}(x_{\mathcal{S}_k},y_{\mathcal{S}_k})\|\\
&\leq\|\mathcal{R}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}))-\mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))\|\kappa_{\nabla f}\\
&\leq\big(\|\mathcal{R}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}))-\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))\|\\
&\qquad+\|\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))-\mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))\|\big)\|\kappa_{\nabla f}.
\end{aligned}
$$

Here, the second line follows similar arguments as (3.12). The last inequality adds, subtracts a term, and uses the triangle inequality. Recall in (3.13), under the same assumptions as here, we showed that for all $x\in\mathbb{R}^n$ and for all $\mathcal{S}\subseteq[N]$ we have

$$
\|\mathcal{R}\left(\nabla c_{\mathcal{S}}(x)\right)-\mathcal{R}\left(\nabla c(x)\right)\|\|\nabla f(x)\|\leq\kappa_{\nabla f}\eta_1 g_{\mathcal{S}}.
$$

Note the right-hand side is a function depending on $|\mathcal{S}|$, which decreases when $|\mathcal{S}|$ increases. Hence we have

$$
\big(\|\mathcal{R}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}))-\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))\|+\|\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))-\mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))\|\big)\|\kappa_{\nabla f}
$$

$$\leq \kappa_{\nabla f}\eta_1 g_{\mathcal{S}_{k+1}} + \kappa_{\nabla f}\eta_1 g_{\mathcal{S}_k} \leq 2\kappa_{\nabla f}\eta_1 g_{\mathcal{S}},$$

which gives us that

$$\|\nabla_x L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla_x L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\| \leq 2\kappa_{\nabla f}\eta_1 g_{\mathcal{S}_k}. \tag{3.20}$$

On the other hand, similar to (3.14) we have

$$\begin{aligned}
&\|\nabla_y L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla_y L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\| \\
&= \|c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}) - c_{\mathcal{S}_k}(x_{\mathcal{S}_k})\| \\
&\leq \|c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}) - c(x_{\mathcal{S}_k})\|_2 + \|c(x_{\mathcal{S}_k}) - c_{\mathcal{S}_k}(x_{\mathcal{S}_k})\| \\
&\leq 2\eta_0 g_{\mathcal{S}_k}.
\end{aligned} \tag{3.21}$$

Here, the third line adds and subtracts a term, and uses the triangle inequality. The last line follows Lemma 3.2 item a, and the nonincreasing property of $g_{\mathcal{S}}$. Combining (3.20, 3.21) and following a similar argument as (3.15), we have

$$\|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\| \leq 2\sqrt{\gamma_c + \kappa_{\nabla f}^2\eta_1^2}\, g_{\mathcal{S}_k}, \tag{3.22}$$

which further gives us

$$\begin{aligned}
\|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})\| &\leq \|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) - \nabla L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\| \\
&\qquad + \|\nabla L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\| \\
&\leq 2\sqrt{\gamma_c + \kappa_{\nabla f}^2\eta_1^2}\, g_{\mathcal{S}_k} + \epsilon_k \\
&= 3\epsilon_k \leq \frac{3}{4}\alpha.
\end{aligned}$$

Here, the first inequality adds and subtracts a term, and uses the triangle inequality. The second inequality combines (3.22) with the fact that $x_{\mathcal{S}_k}$ is a $(\epsilon_k, \varepsilon_k)$ stationary point, and the last line comes from the definition of ϵ_k, and that (3.19) ensures $\epsilon_k \leq \frac{1}{4}\alpha$.

Combining $\epsilon_k \leq \frac{1}{4}\alpha$ with the definition of $\alpha_{\mathcal{S}}$, we have

$$\alpha_{\mathcal{S}_k} = \alpha - \sqrt{\gamma_c + \kappa_{\nabla f}^2\eta_1^2}\, g_{\mathcal{S}_k} = \alpha - \epsilon_k \geq \frac{3}{4}\alpha.$$

As a result, we have

$$\|\nabla L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})\|_2 \leq \alpha_{\mathcal{S}_k} \leq \alpha_{\mathcal{S}_{k+1}}. \tag{3.23}$$

Here, the last inequality utilizes the fact that $\alpha_{\mathcal{S}}$ is nondecreasing with respect to $|\mathcal{S}|$.

One can verify that under the three assumptions and inequality (3.19) of this theorem, i.e. Theorem 3.8, the assumptions and inequality (3.11) in Theorem 3.7 are satisfied, hence the subproblem (2.2) for $\mathcal{S}_{k+1}$ is $(\alpha_{\mathcal{S}_{k+1}}, \beta_{\mathcal{S}_{k+1}})$-strongly Morse. Combining with that $x_{\mathcal{S}_k}$ satisfies (3.23), we have

$$\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \right| \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2,\ \forall d_{\mathcal{S}_{k+1}} \in \mathrm{null}(\nabla c_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k})^T).$$

Define $\bar{d}_{\mathcal{S}_{k+1}} := \mathcal{N}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}$, we have

$$\begin{aligned}
&d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})d_{\mathcal{S}_{k+1}} \\
&\geq \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\bar{d}_{\mathcal{S}_{k+1}} \\
&\quad - \underbrace{\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\bar{d}_{\mathcal{S}_{k+1}} \right|}_{(3.24.1)} \\
&\geq -\varepsilon_k \|d_{\mathcal{S}_{k+1}}\|^2 - (3.24.1).
\end{aligned} \tag{3.24}$$

Here, the first inequality adds and subtracts a term, and uses the triangle inequality. The last line uses the termination condition (2.6b) and the fact that $\|\bar{d}_{\mathcal{S}_{k+1}}\| \leq \left\|d_{\mathcal{S}_{k+1}}\right\|$ which follows a similar argument as in (3.16.1).

Now, to give a bound for (3.16.1), we add and subtract two terms. Define the variable $z := -\nabla c(x_{\mathcal{S}_k})^\dagger \nabla f(x_{\mathcal{S}_k})$, we have

$$\begin{aligned}
(3.24.1) = &\Big| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})d_{\mathcal{S}_{k+1}} - d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)d_{\mathcal{S}_{k+1}} \\
&+ d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)\bar{d}_{\mathcal{S}_{k+1}} \\
&+ \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)\bar{d}_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\bar{d}_{\mathcal{S}_{k+1}} \Big| \\
\leq &\underbrace{\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})d_{\mathcal{S}_{k+1}} - d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)d_{\mathcal{S}_{k+1}} \right|}_{(3.25.1)} \\
&+ \underbrace{\left| d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)\bar{d}_{\mathcal{S}_{k+1}} \right|}_{(3.25.2)} \\
&+ \underbrace{\left| \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L(x_{\mathcal{S}_k}, z)\bar{d}_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_k}(x_{\mathcal{S}_k}, y_{\mathcal{S}_k})\bar{d}_{\mathcal{S}_{k+1}} \right|}_{(3.25.3)}.
\end{aligned} \tag{3.25}$$

Recall that the assumptions and inequality (3.19) of this theorem, i.e. Theorem 3.8, satisfy the assumptions and inequality (3.11) of Theorem 3.7. Hence, the intermediate results of analysis of Theorem 3.7 can be applied here. Thanks to the inequality (3.18) that gives a bound for the difference between the projected sub-sample Hessian value and full-sample Hessian value in Theorem 3.7, for term (3.24.1, 3.24.3) we have

$$\begin{aligned}
(3.24.1) &\leq \left(\eta_2 g_{\mathcal{S}_{k+1}} + \frac{3\eta_1\eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} g_{\mathcal{S}_{k+1}} + \frac{3\eta_1\eta_2}{2} g_{\mathcal{S}_{k+1}}^2 \right) \|d_{\mathcal{S}_{k+1}}\|^2 \\
&\leq \left(\eta_2 g_{\mathcal{S}_k} + \frac{3\eta_1\eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} g_{\mathcal{S}_k} + \frac{3\eta_1\eta_2}{2} g_{\mathcal{S}_k}^2 \right) \|d_{\mathcal{S}_{k+1}}\|^2.
\end{aligned} \tag{3.26}$$

Here, the second line uses the fact that $g_{\mathcal{S}}$ is non-increasing with respect to $|\mathcal{S}|$, which gives $g_{\mathcal{S}_{k+1}} \leq g_{\mathcal{S}_k}$. And

$$\begin{aligned}
(3.24.3) &\leq \left(\eta_2 g_{\mathcal{S}_k} + \frac{3\eta_1\eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} g_{\mathcal{S}_k} + \frac{3\eta_1\eta_2}{2} g_{\mathcal{S}_k}^2 \right) \|\bar{d}_{\mathcal{S}_{k+1}}\|^2 \\
&\leq \left(\eta_2 g_{\mathcal{S}_k} + \frac{3\eta_1\eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} g_{\mathcal{S}_k} + \frac{3\eta_1\eta_2}{2} g_{\mathcal{S}_k}^2 \right) \|d_{\mathcal{S}_{k+1}}\|^2,
\end{aligned} \tag{3.27}$$

where the second line uses $\|\bar{d}_{\mathcal{S}_{k+1}}\| \leq \left\|d_{\mathcal{S}_{k+1}}\right\|$ again.

For (3.24.2), we have

$$
(3.28)\qquad
\begin{aligned}
(3.24.2) &= \left|\left(d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\right)^T \nabla^2_{xx} L(x_{\mathcal{S}_k}, z)\left(d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\right)\right| \\
&\leq \left\|\nabla^2_{xx} L(x_{\mathcal{S}_k}, z)\right\| \left\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\right\|^2 \\
&\leq \left(\kappa_{\nabla^2 f} + \frac{\eta_2}{\sqrt{\gamma_{\nabla^2 c}}}\right)\left\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\right\|^2 .
\end{aligned}
$$

Here, the second line uses the sub-multiplicity of matrix-vector product, and the last line is given by Lemma 3.2 item c. In addition, for $\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\|$, we have

$$
(3.29)\qquad
\begin{aligned}
&\|d_{\mathcal{S}_{k+1}} - \bar{d}_{\mathcal{S}_{k+1}}\| \\
&= \|\mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}\| \\
&\leq \|\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}\| + \|\left(\mathcal{R}(\nabla c(x_{\mathcal{S}_k})) - \mathcal{R}(\nabla c_{\mathcal{S}_k}(x_{\mathcal{S}_k}))\right) d_{\mathcal{S}_{k+1}}\| \\
&\leq 2\eta_1 g_{\mathcal{S}_k}\|d_{\mathcal{S}_{k+1}}\|.
\end{aligned}
$$

Here, the second line uses the fact that $\mathcal{R}(\nabla c_{\mathcal{S}_k}) = I_n - \mathcal{N}(\nabla c_{\mathcal{S}_k})$. The third line adds and subtracts a term, and uses the triangle inequality. The last line uses Lemma 3.6 for $\|\mathcal{R}(\nabla c(x_{\mathcal{S}_k}))d_{\mathcal{S}_{k+1}}\|$, and inequality (3.13).

Combining (3.24, 3.25, 3.26, 3.27, 3.28, 3.29), we have

$$
\begin{aligned}
&d^T_{\mathcal{S}_{k+1}} \nabla^2_{xx} L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}})d_{\mathcal{S}_{k+1}} \\
&\geq -\eta_1\kappa_{\nabla^2 f} g_{\mathcal{S}_k}\left\|d_{\mathcal{S}_{k+1}}\right\|^2 - 2\left(\eta_2 + \frac{3\eta_1\eta_2}{2\sqrt{\gamma_{\nabla^2 c}}} + \frac{3\eta_1\eta_2}{2} g_{\mathcal{S}_k}\right) g_{\mathcal{S}_k}\|d_{\mathcal{S}_{k+1}}\|^2 \\
&\qquad - \left(\kappa_{\nabla^2 f} + \frac{\eta_2}{\sqrt{\gamma_{\nabla^2 c}}}\right) 4\eta_1^2 g^2_{\mathcal{S}_k}\|d_{\mathcal{S}_{k+1}}\|^2 \\
&= -\left(\left(\eta_1\kappa_{\nabla^2 f} + 2\eta_2 + \frac{3\eta_1\eta_2}{\sqrt{\gamma_{\nabla^2 c}}}\right) g_{\mathcal{S}_k} + \left(3\eta_1\eta_2 + 4\eta_1^2\kappa_{\nabla^2 f} + \frac{4\eta_1^2\eta_2}{\sqrt{\gamma_{\nabla^2 c}}}\right) g^2_{\mathcal{S}_k}\right)\left\|d_{\mathcal{S}_{k+1}}\right\|^2 \\
&\geq -\left(2\eta_3 g_{\mathcal{S}_k} + 3\eta_1\eta_3 g^2_{\mathcal{S}_k}\right)\left\|d_{\mathcal{S}_{k+1}}\right\|^2 .
\end{aligned}
$$

Here, the second line also uses that $\varepsilon_k = \eta_1\kappa_{\nabla^2 f} g_{\mathcal{S}_k}$. The second from last line rearranges terms, and the last line uses the definition of (η_1, η_2, η_3).

Now, the nonnegative solutions to $\left(\eta_3 g_{\mathcal{S}} + (\beta\eta_1 + 2\eta_3)\,\eta_1 g^2_{\mathcal{S}}\right) \leq \frac{1}{4}\beta$ are

$$
(3.30)\qquad g_{\mathcal{S}_k} \leq \frac{-\eta_3 + \sqrt{\eta_3^2 + \beta\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)}}{2\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)}.
$$

By multiplying $\left(\eta_3 + \sqrt{\eta_3^2 + \beta\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)}\right)$ at both the numerator and denominator of the above inequality, and using $\sqrt{\eta_3^2 + \beta\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)} \geq \eta_3$, we have that when $g_{\mathcal{S}_k}$ satisfies

$$
g_{\mathcal{S}_k} \leq \frac{\beta}{4\sqrt{\eta_3^2 + \beta\left(\beta\eta_1^2 + 2\eta_1\eta_3\right)}},
$$

it will satisfy (3.30). Further, we have

$$
-\left(2\eta_3 g_{\mathcal{S}_k} + 3\eta_1\eta_3 g^2_{\mathcal{S}_k}\right) \geq -2\left(\eta_3 g_{\mathcal{S}} + (\beta\eta_1 + 2\eta_3)\,\eta_1 g^2_{\mathcal{S}}\right) \geq -\frac{1}{2}\beta,
$$

which gives us

$$d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \geq -\frac{1}{2}\beta \left\|d_{\mathcal{S}_{k+1}}\right\|^2. \tag{3.31}$$

Recall that $\beta_{\mathcal{S}_k} = \beta - \left((\eta_1\beta + \eta_3)\, g_{\mathcal{S}_k} + 2\eta_1\eta_3 g_{\mathcal{S}_k}^2\right)$, and (3.30) gives us that $\beta_{\mathcal{S}_k} \geq \frac{3}{4}\beta$. Note that $\beta_{\mathcal{S}}$ is a non-decreasing function with respect to $|\mathcal{S}|$, we have $\beta_{\mathcal{S}_{k+1}} \geq \beta_{\mathcal{S}_k} \geq \frac{3}{4}\beta$. Moreover, since subproblem (2.3) for $\mathcal{S} = \mathcal{S}_{k+1}$ is $(\alpha_{k+1}, \beta_{k+1})$-strongly morse, we have $|d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}}| \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2 \geq \frac{3}{4}\beta \|d_{\mathcal{S}_{k+1}}\|^2$. Combining wirh (3.31), it must hold

$$d_{\mathcal{S}_{k+1}}^T \nabla_{xx}^2 L_{\mathcal{S}_{k+1}}(x_{\mathcal{S}_k}, z_{\mathcal{S}_{k+1}}) d_{\mathcal{S}_{k+1}} \geq \beta_{\mathcal{S}_{k+1}} \|d_{\mathcal{S}_{k+1}}\|^2,$$

which completes the proof. □

Assumption 3.4. *We make the following assumptions for each element of the expected constraint function, ci. There exists a $(r, \tau) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ such that for all $(x, i) \in \mathbb{B}^n(r) \times \{1, \cdots, m\}$,*

(1). the gradient of $c^i(x)$ is τ^2-sub-Gaussian. Namely, for any $a \in \mathbb{R}^n$,

$$\mathbb{E}_\xi \left[\exp\left(a^T \left(\nabla c^i(x;\xi) - \mathbb{E}_\xi\left[\nabla c^i(x;\xi)\right]\right)\right)\right] \leq \exp\left(\frac{\tau^2 \|x\|_2^2}{2}\right).$$

z_{ξ_j} finite sample distribution

(2). the Hessian of $c^i(x)$, evaluated on a unit vector, is τ^2-sub-exponential. Namely, for any $a \in \mathbb{B}^n(1)$, let $z_{a,x,\xi} := a^T \nabla^2 c^i(x;\xi) a$, then

$$\mathbb{E}_\xi \left[\exp\left(\frac{1}{\tau^2}\left|z_{a,x,\xi} - \mathbb{E}[z_{a,x,\xi}]\right|\right)\right] \leq 2.$$

(3). within $\mathbb{B}^n(r)$, the Hessian of c^i is L-Lipschitz continuous, and the gradient of c^i is $\lambda_c^{\max}$-Lipschitz continuous. Moreover, there exists a constant $h > 0$ such that

$$L \leq \tau^3 n^h, \text{ and } \lambda_c^{\max} \leq \tau^2 n^h.$$

Theorem 3.9. *Under Assumption.3.4 and let $(n, r, \tau, h) \in \mathbb{N} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ be defined in the same way. There exists a universal constant C_0 and for any $\delta \in [0, 1]$ let $C := C_0 \max\{h, \log\frac{r\tau}{\delta}, 1\}$. Then, for any sample size $p \geq Cn\log n$, the following holds with probability at least $(1 - \delta)$:*

$$\begin{aligned} &\sup_{\forall x \in \mathbb{B}^n(r)} \|\nabla c(x) - \nabla c_p(x)\|_2 \leq g(p) := \tau\sqrt{\frac{Cn\log p}{p}} \text{ and} \\ &\sup_{i \in \{1,\cdots,m\}} \left\{\sup_{\forall x \in \mathbb{B}^n(r)} \|\nabla^2 c_p^i(x) - \nabla^2 c^i(x)\|_2\right\} \leq G(p) := \tau^2\sqrt{\frac{Cn\log p}{p}}. \end{aligned} \tag{3.32}$$

Lemma 3.10. *Under Assumption.3.4 and let $(n, r, \tau, h) \in \mathbb{N} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ be defined in the same way. Let (C, p) be defined in the same way as Theorem.3.9,* *then the following holds with probability at least* $(1-\delta)$*:*

(3.33)

$$\sup_{\forall x \in \mathbb{B}^n(r)} \left\| \frac{1}{p} \sum_{i \in \mathcal{S}_p} \nabla c(x, \xi_i) - \frac{1}{2p} \sum_{i \in \mathcal{S}_{2p}} \nabla c(x, \xi_i) \right\|_2 \leq \tau \sqrt{\frac{Cn \log p}{p}} \text{ and}$$

$$\sup_{i \in \{1, \cdots, m\}} \left\{ \sup_{\forall x \in \mathbb{B}^n(r)} \left\| \frac{1}{p} \sum_{i \in \mathcal{S}_p} \nabla^2 c^i(x, \xi_i) - \frac{1}{2p} \sum_{i \in \mathcal{S}_{2p}} \nabla^2 c^i(x, \xi_i) \right\|_{op} \right\} \leq \tau^2 \sqrt{\frac{Cn \log p}{p}}.$$

Proof. ...... ∎

4. Numerical Results.

5. Conclusion.

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