Calculus Tutorial

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Unit 1: Continuity and Limit

1.1 Limit

Summarized Points

I. The Properties of Limits

1) Scalar multiple: $\lim_{x\to c}[b(f(x))] = b\lim_{x\to c} f(x)$
2) Sum or difference: $\lim_{x\to c}[f(x) + g(\mathrm{x})] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$
3) Product: $\lim_{x\to c}[f(x)g(x)] = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$
4) Quotient: $\lim_{x\to c}[f(x)/g(x)] = [\lim_{x\to c} f(x)]/[\lim_{x\to c} g(x)]$, $\lim_{x\to c} g(x) \neq 0$.

II. One-Sided Limits

1) Right-Sided Limit $\lim_{x\to c+} g(x)$
2) Left-Sided Limit $\lim_{x\to c-} g(x)$

III. Limits at Infinity

1) Limit approaching positive infinity $\lim_{x\to\infty} g(x) = L$ (L is a constant)
2) Limit approaching negative infinity $\lim_{x\to-\infty} g(x) = L$

IV. Limit Nonexistence (Very Important)

1) Left-Sided Limit $\neq$ Right-Sided Limit Example: $f(x) = x/|x|$ at x=0
2) Positive/Negative Infinite Limit Example: $f(x) = 1/x$ at x=0
3) Limit Oscillation Example: $f(x) = \sin(\frac{1}{x})$ at x=0

V. Special Limit

1) $\lim_{x\to 0} \sin(x)/x = 1$
2) $\lim_{x\to 0}(\cos(x) - 1)/x = 0$

Problem-Solving Strategies

In this part, I will show strategy to determine a 0/0 limits.

1) A rational function. (quotients of two polynomials)

   Strategy: If a rational function has 0/0 limit, then the denominator and numerator have one common factor, meaning that this rational function can be simplified. Therefore, just use factorization.

2) A polynomial divided by a trigonometric function, or a trigonometric function divided by a polynomial.

   Strategy: Use the two special limits. Transform the trigonometric function to sin(x) or cos(x) and use quotient or multiplication rule to determine the limit.

Example 1.1.1

Determine the following limits.

a) $\lim_{x\to 0} x/(\cos(x)-1)$

b) $\lim_{x\to 0} \cos(\mathrm{x})/x$

c) $\lim_{x\to 0} \sin(x)$

d) $\lim_{x\to 1} (\mathrm{x}^3-1)/(x-1)$

e) $\lim_{x\to 0} (\sqrt[2]{x+1}-1)/\mathrm{x}$

f) $\lim_{x\to 0} \sin(2x)/x$

g) $\lim_{x\to 0} \tan(x)/x$

Solution:

a) $\lim_{x\to 0} \frac{x}{\sin(\mathrm{x})} = \lim_{x\to 0} (\frac{1}{\frac{\sin(x)}{x}})$ Since $\lim_{x\to 0} \frac{\sin(\mathrm{x})}{x} = 1$, the quotient rule (See Summarized Points I. 4)

works and

$$\lim_{x\to 0} \frac{x}{\sin(\mathrm{x})} = \lim_{x\to 0} \left(\frac{1}{\frac{\sin(x)}{x}}\right) = \frac{\lim_{x\to 0}(1)}{\lim_{x\to 0} \frac{\sin(\mathrm{x})}{x}} = 1$$

b) $\lim_{x\to 0} \frac{\cos(\mathrm{x})}{x} = \lim_{x\to 0} \cos(\mathrm{x}) \lim_{x\to 0} 1/\mathrm{x}$ Since $\lim_{x\to 0} 1/\mathrm{x}$ does Not exist because $\lim_{x\to 0+} 1/\mathrm{x} = \infty \neq$ $\lim_{x\to 0-} 1/\mathrm{x} = -\infty$

c) Does Not exist. See Summarized Point IV.3.

Warning: If this is a multiple choice questions, there may be answers like 'oscillating between -1 and 1'. This type of answer is not correct. A limit can be mathematically described as either a constant or nonexistence.

d) $\lim_{x\to 1} (\mathrm{x}^3-1)/(x-1) = \lim_{x\to 1} (\mathrm{x}-1)(x^2+x+1)/(x-1) = \lim_{x\to 1} x^2+x+1 = 3$

e) $\lim_{x\to 0} (\sqrt[2]{x+1}-1)/\mathrm{x} = \lim_{x\to 0} (\sqrt[2]{x+1}-1)(\sqrt[2]{x+1}+1)/[\mathrm{x}(\sqrt[2]{x+1}+1)] =$

$$\lim_{x\to 0} \frac{x}{[x(\sqrt[2]{x+1}+1)]} = \lim_{x\to 0} \frac{1}{[(\sqrt[2]{x+1}+1)]} = 1/2$$

f) $\lim_{x\to 0} \sin(2x)/x = \lim_{x\to 0} 2\sin(2x)/2x = 2\lim_{x\to 0} \sin(x)/x = 2$

g) $\lim_{x\to 0} \frac{\tan(x)}{x} = \lim_{x\to 0} \left[\frac{\sin(x)}{x} \times \frac{1}{\cos(x)}\right] = \lim_{x\to 0} \frac{\sin(x)}{x} \lim_{x\to 0} \frac{1}{\cos(x)} = 1$

College-Prep Extension

AP Calculus does Not require the precise definition of limit, but a college-equivalent calculus course does. Therefore, I want to introduce to you the precise definition of limit:

$\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta,\ |f(x) - L| < \varepsilon.$

This abstract definition can be explained like this: as long as distance between x and a is limited to a positive number, the difference between f(x) and f(a) will be limited to another positive number. Consider $\lim_{x \to 1} 2x + 1 = 3$. In precise definition,

$\lim_{x \to 1} 2x + 1 = 3$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - 1| < \delta,\ |2x + 1 - 3| = |2x - 2| < \varepsilon$. Suppose $\varepsilon = 0.2$, then δ can be any positive value smaller than 0.1. In fact, as long as $\varepsilon = 2\delta,$ then for every $\varepsilon > 0$, we can deduce that $|2x - 2| = 2|x - 1| < 2\delta = \varepsilon.$

Prove the following limits using the precise definition. (Relate ε and δ)

a) $\lim_{x \to 1} x^2 - 4 = -3$

b) $\lim_{x \to 1} 1/x = 1$

Practice

Determine the following limits.

a) $\lim_{x \to 0} (\sin x)^2/x$

b) $\lim_{x \to 5} (\sqrt{x + 4} - 3)/(x - 5)$

c) $\lim_{x \to 0} \sin(3x)\ /x$

d) $\lim_{x \to 0} \tan(2x)\ /x$

e) $\lim_{x \to \infty} \tan(x)$

Check Time:

A. Know the basic rules of limits

B. Determine limits analytically.

1.2 Continuity

Summarized Points

I. Definition of Continuity

If $\lim_{x \to a} f(x) = f(a)$, then f(x) is continuous at x=a.

A function is said to be continuous on the interval [a, b] if it is continuous at each point in the interval.

II. Intermediate Value Theorem

Suppose that f(x) is continuous on [a, b] and let M be any number between f(a) and f(b). Then there exists a number c such that $a < c < b$ and f(c)=M.

III. Continuity for Each Type of Function

1) Polynomial Function: Continuous at all real number
2) Rational Function: Continuous at points where the denominator does not equal 0.
3) Exponential Function: Continuous at all real number
4) Logarithm Function: Continuous at (0, ∞)
5) Trigonometric Function:

   a) Sine/Cosine Function: Continuous at all real number

   b) Tangent/Cotangent/Cosecant/Secant Function: Continuous at points where denominator does not equal 0.

IV. Type of Discontinuity (Very Important)

	Left-Sided and Right-Sided Limits Existence?	Why Discontinuous?
Removable Discontinuity	Yes	$\lim_{x \to a} f(x) \neq f(a)$
Jump Discontinuity	Yes	Left-Sided Limit Not Equal to Right-Sided Limit
Infinite Discontinuity	No	Both sided-limits not exist (Infinite or negative infinite)

Example 1.2.1

Determine whether the following functions are continuous at given points. If they are discontinuous, identify the types of discontinuity.

h) $f(x) = \frac{1}{x}, x = 0$

i) $f(x) = \frac{(x-1)(x+1)}{(2x-3)(x+1)}, x = -1, x = 1.5$

j) $f(x) = \log x, x = 0$

k) $\mathrm{f(x)} = \frac{x}{|x|}, x = 0$

Solution:

h) $\lim_{x\to 0+} \frac{1}{\mathrm{x}} = \infty$ $\lim_{x\to 0-} \frac{1}{\mathrm{x}} = -\infty$ $\lim_{x\to 0+} \frac{1}{\mathrm{x}} \neq \lim_{x\to 0-} \frac{1}{\mathrm{x}}$ Type of Discontinuity: Infinite Discontinuity; Jump Discontinuity

i) x=-1: $\lim_{x\to -1} f(x) = 0.4$, but x cannot be -1. So this is a removable discontinuity.

x=1.5: $\lim_{x\to 1.5+} f(x) = \infty$ $\lim_{x\to 1.5-} f(x) = -\infty$ $\lim_{x\to 1.5+} \neq \lim_{x\to 1.5-}$ Type of Discontinuity: Infinite Discontinuity; Jump Discontinuity

j) $\lim_{x\to 0-} \log x$ does Not exist. $\lim_{x\to 0+} \log x = -\infty$. Type of Discontinuity: Infinite Discontinuity

k) $\mathrm{x} > 0$, $\mathrm{f(x)} = \frac{x}{|x|} = 1$; $\mathrm{x} < 0$, $\mathrm{f(x)} = \frac{x}{|x|} = -1$. $\lim_{x\to 0+} f(x) = 1 \neq \lim_{x\to 0-} f(x) = -1$

Example 1.2.2

Given a function $f(x) = \begin{cases} \frac{\sin(x)}{x}, x \neq 0 \\ k, \ x = 0 \end{cases}$, for what value of k will f(x) be continuous at x=0?

Solution:

$$\lim_{x\to 0} \frac{\sin x}{\mathrm{x}} = 1 = k$$

Example 1.2.3

Show that $f(x) = x^3 + 3x - 1$ has at least one root in the interval [-1, 1].

Solution:

According to intermediate value theorem, there exists at least one value c between -1 and 1 that has a value 0, which is between -5 and 3.

Practice

1. Determine whether the following functions are continuous at given points. If they are discontinuous, identify the types of discontinuity.

a) $f(x) = \frac{(x-1)}{(2x-3)(x+1)}, x = -1, x = 1.5$

b) $f(x) = \tan x\,, x = \pi/2$

c) $f(x) = \frac{\sin x}{x}, x = 0$

2. Show that $f(x) = ax^3 - 3x - 1$ (a is a constant) has at least one root.

Check Time:

C. Identify the types of discontinuity for different functions

D. Apply intermediate value theorem

Unit 2: Differentiation

2.1 Rate of Change

Summarized Points

I. Rate of Change

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

Rate of change is essentially how sensitive is the change of y related to change of x

II. Instantaneous Rate of Change (Derivative)

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example 2.1.1

Determine the derivative of following functions at the given points.

l) $f(x) = 2x^3 + 3x - 1, x = 1$

m) $f(x) = \frac{x-1}{2x-3}, x = 0$

n) $f(x) = \sqrt[2]{1 - x^2}, x = 1/2$

Solution:

l) $\frac{dy}{dx} = \lim_{h \to 0} \frac{2(1+h)^3 + 3(1+h) - 1 - [2+3-1]}{\mathrm{h}} = \lim_{h \to 0} \frac{2h^3 + 6h^2 + 6h + 3h}{h} = \lim_{h \to 0} 9 = 9$

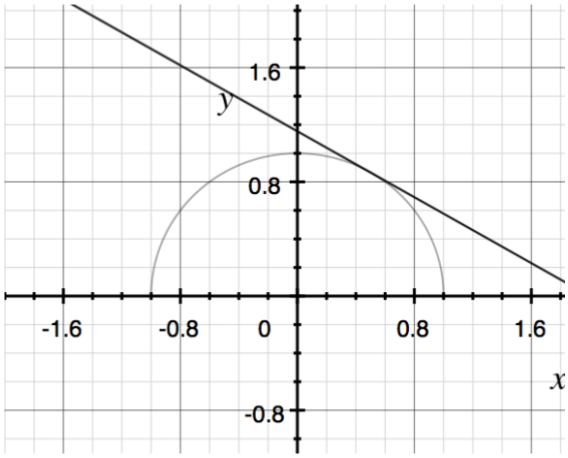
m) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\frac{h-1}{2h-3} - 1/3}{h} = \lim_{h \to 0} \frac{\frac{3h-3}{6h-9} - \frac{2h-3}{6h-9}}{h} = \lim_{h \to 0} \frac{1}{6h-9} = -1/9$

n) $\frac{dy}{dx} = \lim_{h \to 0} \frac{\sqrt[2]{1-(1/2+h)^2} - \sqrt[2]{1-(\frac{1}{2})^2}}{h} = \lim_{h \to 0} \frac{(\sqrt[2]{1-\left(\frac{1}{2}+h\right)^2} - \sqrt[2]{1-\left(\frac{1}{2}\right)^2})(\sqrt[2]{1-\left(\frac{1}{2}+h\right)^2} + \sqrt[2]{1-\left(\frac{1}{2}\right)^2})}{h(\sqrt[2]{1-\left(\frac{1}{2}+h\right)^2} + \sqrt[2]{1-\left(\frac{1}{2}\right)^2})} =$

$\lim_{h \to 0} \frac{-h-h^2}{h(\sqrt[2]{1-\left(\frac{1}{2}+h\right)^2} + \sqrt[2]{1-\left(\frac{1}{2}\right)^2})} = \lim_{h \to 0} \frac{-1-h}{(\sqrt[2]{1-\left(\frac{1}{2}+h\right)^2} + \sqrt[2]{1-\left(\frac{1}{2}\right)^2})} = \frac{-1}{(\sqrt[2]{1-\left(\frac{1}{2}+0\right)^2} + \sqrt[2]{1-\left(\frac{1}{2}\right)^2})} = -1/\sqrt[2]{3}$

Note: Another way to solve problem c is to evaluate the slope of the tangent to the circle.

$f(x) = \sqrt[2]{1 - x^2}$ *is essentially a semicircle, as the following graph shows. Based on what we learn in plane geometry,* $PO \perp$ *the tangent line, where p has coordinate (1/2,* $\sqrt[2]{3}/2$*). And the acute angle formed by PO and positive x-axis is* $\tan^{-1} \sqrt[2]{3} = \pi/3$*. Therefore, the obtuse angle between x-axis and tangent line is* $\pi - (\pi - \frac{\pi}{3} - \frac{\pi}{2}) = \frac{5\pi}{6}$ *and the slope of the tangent (derivative) is* $\tan \frac{5\pi}{6} = -1/\sqrt[2]{3}$



Practice

1. Determine the derivative of following functions at given points.

d) $f(x) = \frac{(x-1)}{(x+1)}, x = 1$

e) $f(x) = \sqrt[2]{1 + x^2}, x = 1/2$

Check Time:

E. Know definition of (instantaneous) rate of change

F. Know how to find derivatives

2.2 Fundamental Derivative

Summarized Points

I. Rules of Differentiation

a. $\frac{d}{dx}\left(f(x)+g(x)\right)=\frac{df}{dx}+\frac{dg}{dx}$ (Sum rule)

b. $\frac{d}{dx}\left(f(x)g(x)\right)=\frac{df}{dx}g+\frac{dg}{dx}f$ (Product rule)

c. $\frac{d}{dx}\left(f(x)/g(x)\right)=\frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$ (Quotient rule)

d. $\frac{d}{dx}f\left(g(x)\right)=g'(x)f'(g(x))$ (Chain rule)

II. Derivative of Different Functions

1) Polynomial Function: $f(x)=ax^n$ $\quad f'(x)=anx^{n-1}$ (n$\neq$ 0)

2) Exponential Function: $f(x)=a^x$ $\quad f'(x)=a^x\cdot\ln a$

3) Trigonometric Function:

a) Sine/Cosine Function:

$f(x)=sin(x)$ $\quad f'(x)=\cos(x)$

$f(x)=cos(x)$ $\quad f'(x)=-\sin(x)$

b) Tangent/Cotangent/Cosecant/Secant Function:

$f(x)=tan(x)$ $\quad f'(x)=(\sec x)^2$

$f(x)=cot(x)$ $\quad f'(x)=-(\csc x)^2$

$f(x)=csc(x)$ $\quad f'(x)=-(\csc x)\,cot(x)$

$f(x)=sec(x)$ $\quad f'(x)=(\sec x)\,tan(x)$

Problem-Solving Strategy

In many cases product rules, quotient rules and chain rules all work. However, based on my experience, it is usually better to consider chain rules first, then product rules, and eventually quotient rules. You need to make the denominator a numerator.

For instance, when you differentiate $\frac{1}{\sqrt[2]{x^2+1}}$ and $\frac{x}{\cos x}$, you need to turn it to

$(x^2+1)^{-1/2}$ and $x\sec x$. Example 2.2.1 will show why chain rule and product rule are more convenient.

Example 2.2.1

Determine the derivative of following functions at the given points.

o) $y = x^2/e^x, x = 1$

p) $y = \frac{x-1}{2x-3}, x = 0$

q) $y = 1/\sqrt[2]{1-x^2}, x = 0$

r) $y = (\cos x)^2, x = 0$

Solution:

o) Method 1: $u = x^2,\ v = e^x.\ \frac{dy}{dx} = \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} = \frac{e^x\times 2x - x^2e^x}{e^{2x}} = \frac{2x-x^2}{e^x} = 1/e$

Method 2: $u = x^2,\ v = e^{-x}.\ \frac{dy}{dx} = \frac{d(uv)}{dx} = \frac{du}{dx}v + \frac{dv}{dx}u = 2xe^{-x} - x^2e^{-x} = 1/e$

p) Method 1: $v = 2x-3, u = x-1.\ \frac{dy}{dx} = \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} = \frac{(2x-3)-2(x-1)}{(2x-3)^2} = \frac{-1}{(2x-3)^2} = -1/9$

Method 2: $f(x) = \frac{x-1}{2x-3} = \frac{x-1.5+0.5}{2x-3} = \frac{1}{2} + 0.5(2x-3)^{-1}$

A constant's derivative is 0. Therefore, we only need to differentiate $0.5(2x-3)^{-1}$.

Now we use chain rule. $2x-3 = u.\ 0.5(2x-3)^{-1} = 0.5(u)^{-1} = -0.5(u)^{-2}\times(2) = -1/9$

q) Method 1: $u = 1, v = \sqrt[2]{1-x^2},\ t = 1-x^2. \frac{dy}{dx} = \frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} = \frac{-\frac{dv}{dx}}{1-x^2} = \frac{-\frac{dv}{dt}\times\frac{dt}{dx}}{1-x^2} =$

$\frac{-\frac{1}{2}(1-x^2)^{-0.5}\times(-2x)}{1-x^2} = x(1-x^2)^{-1.5} = 0$

Method 2: $u = 1-x^2.\ \frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = -\frac{1}{2}u^{-1.5}\times(-2x) = \frac{1}{2}(1-x^2)^{-1.5}\times(-2x) =$

$x(1-x^2)^{-1.5} = 0$

r) Method 1: Use product rule. $u = v = \cos x. \frac{dy}{dx} = \frac{dv}{dx}u + \frac{du}{dx}v = \cos x\times(-\sin x) + \cos x\times(-\sin x) = -2\sin x\cos x = 0$

Method 2: Use chain rule. $v = \cos x.\ \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = 2\cos x\,(-\sin x) = -2\sin x\cos x = 0$

Note: on the above four examples, method 2 is always more convenient than method 1, which shows that chain rule is more convenient than product rule and product rule is more convenient than quotient rule.

Example 2.2.2

Use product rule and chain rule to prove quotient rule.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x) \times \frac{1}{g(x)}\right) = \frac{d\big(f(x)\big)}{dx} \times \frac{1}{g(x)} + \frac{d}{dx}\left(\frac{1}{g(x)}\right) \times f(x)$$

$$= \frac{f'}{g} + f\left(\frac{1}{g(x)}\right)'$$

$$u = \frac{1}{g(x)} = g^{-1} \quad \frac{d(g^{-1})}{dx} = \frac{du}{dg}\frac{dg}{dx} = -g^{-2} \times g'$$

$$\frac{f'}{g} + f\left(\frac{1}{g(x)}\right)' = \frac{f'}{g} + f \times (-g^{-2} \times g') = \frac{f' \times g - f \times g'}{g^2}$$

Note: The deduction is actually a good way to memorize quotient rule.

Practice

1. Determine the derivative of following functions at given points.

f) $f(x) = \frac{(x-1)}{(3x+1)}, x = 1$

g) $f(x) = \frac{\cos x}{\sqrt[2]{1+x^2}}, x = 1/2$

h) $f(x) = (\cos x)^3, x = 1$

i) $f(x) = \frac{\cos x}{e^{x^2}}, x = 0$

Check Time:

G. Know how to find fundamental derivatives

H. Know what rules should be used for different derivatives problems.

2.3 Implicit Derivative

Summarized Points

I. Derivative of Different Functions

1) Logarithm Function: $f(x) = \log_a x \quad f'(x) = \frac{1}{(x \cdot \ln a)}$

2) Inverse Trigonometric Function:

a) Inverse Sine/Cosine Function:

$$f(x) = \sin^{-1} x \quad f'(x) = \frac{1}{\sqrt[2]{1-x^2}}$$

$$f(x) = \cos^{-1} x \quad f'(x) = \frac{-1}{\sqrt[2]{1-x^2}}$$

b) Inverse Tangent/Cotangent/Cosecant/Secant Function:

$$f(x) = \tan^{-1} x \quad f'(x) = \frac{1}{1+x^2}$$

$$f(x) = \cot^{-1} x \quad f'(x) = \frac{-1}{1+x^2}$$

$$f(x) = \sec^{-1} x \quad f'(x) = \frac{1}{|x|\sqrt[2]{x^2-1}}$$

$$f(x) = \csc^{-1} x \quad f'(x) = \frac{-1}{|x|\sqrt[2]{x^2-1}}$$

II. Derivative of an inverse function

1) Given a differentiable function f and its inverse function g. If the derivative of f at point (a, b) is f', then the derivative of g at point (b, a) is 1/f'.

Problem-Solving Strategy

1. One challenge in this section is that you need to differentiate a function of y (or some other variables than x) with respect to x. The strategy is simple: let $f(y) = u$, then $\frac{d(f(y))}{dx} = \frac{d(f(y))}{dy}\frac{dy}{dx}$. (Essentially chain rule) Then simplify the equation and you will get $\frac{dy}{dx}$.

2. As for determining a derivative of an inverse function g at point (a, b) (suppose f is original function), you just need to determine f'(b) and the answer is $\frac{1}{f'(b)}$. Sometimes b will not be stated directly and you need to determine the value of b through either function g or f.

3. If the exponent of an expression is not a constant, then you need to consider add a ln at both sides. See question (c) of example 2.3.1.

Example 2.3.1

Determine $\frac{dy}{dx}$ of following equations at the given points.

a) $xy + x^2y^2 = 1$

b) $y^2 = \cos x$

c) $y = e^{\sin x}$

d) $y^{\sin x} + \sin x = 1$

e) $y = \log_2[\tan^{-1}(x^x)]$

Solution:

a) $\frac{d}{dx}[xy + x^2y^2] = \frac{d}{dx}(1)$

To differentiate $\frac{d}{dx}(xy)$, let x=u, y=v. Then use product rule so that $\frac{d}{dx}(xy) = \frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u = \frac{dx}{dx}y + \frac{dy}{dx}x = y + x\frac{dy}{dx}$.

Similarly, let x^2=u, y^2=v. $\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u = \frac{d(x^2)}{dx}y^2 + \frac{d(y^2)}{dx}x^2 =$

$2xy^2 + \frac{d(y^2)}{dx}x^2$

To simplify, $\frac{d(y^2)}{dx}x^2 = \frac{d(y^2)}{dy}x^2\frac{dy}{dx} = 2x^2y\frac{dy}{dx}$

Then the equation becomes $y + x\frac{dy}{dx} + 2x^2y\frac{dy}{dx} = 0$. $(x + 2x^2y)\frac{dy}{dx} = -y$

$\frac{dy}{dx} = \frac{-y}{x+2x^2y}$

Warning: $\frac{d(y^2)}{dx}x^2 \neq \frac{d(x^2)}{dx}y^2$ *you cannot switch the function inside and outside d( ).*

b) $\frac{d}{dx}(y^2) = \frac{d}{dx}(\cos x)$

$\frac{d(y^2)}{dy}\frac{dy}{dx} = -\sin x$ $\quad 2y\frac{dy}{dx} = -\sin x$ $\quad \frac{dy}{dx} = \frac{-\sin x}{2y}$

c) Add a ln at both sides, the equation becomes $\ln y = \sin x$

Differentiate with respect to x, the equation becomes $\frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x)$

$$\frac{d}{dx}(\ln y) = \frac{d}{dy}(\ln y)\frac{dy}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \quad \frac{dy}{dx} = y\cos x = e^{\sin x}\cos x$$

d) $\frac{d}{dx}(y^{\sin x} + \sin x) = \frac{d}{dx}(1)$

$$\frac{d}{dx}(y^{\sin x}) + \cos x = 0$$

To differentiate $y^{\sin x}$, let $u = y^{\sin x}$. Then add a ln on both sides so that the equation becomes: $\ln u = \sin x \ln y \quad \frac{d(\ln u)}{dx} = \frac{d(\sin x \ln y)}{dx}$

$\frac{d(\ln u)}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \cos x \ln y + \sin x \frac{d(\ln y)}{dy}\frac{dy}{dx} = \cos x \ln y + \sin x \frac{1}{y}\frac{dy}{dx}$

$\frac{1}{u}\frac{du}{dx} = \cos x \ln y + \sin x \frac{1}{y}\frac{dy}{dx} \quad \frac{du}{dx} = u\left(\cos x \ln y + \sin x \frac{1}{y}\frac{dy}{dx}\right) =$

$y^{\sin x}\left(\cos x \ln y + \sin x \frac{1}{y}\frac{dy}{dx}\right)$

Substitute to the original equation,

$$y^{\sin x}\cos x \ln y + y^{\sin x}\sin x \frac{1}{y}\frac{dy}{dx} + \cos x = 0$$

$$\frac{dy}{dx} = \frac{(-\cos x - y^{\sin x}\cos x \ln y)y}{y^{\sin x}\sin x}$$

e) $2^y = \tan^{-1}(x^x)$

let $x^x = u$, then the equation becomes

$$2^y = \tan^{-1} u$$

Differentiate with respect to x, we get

$$2^y \ln 2 \frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

To solve $\frac{du}{dx}$, add ln on both sides of equation $x^x = u$, then the equation becomes

$$x \ln x = \ln u$$

$$\frac{d}{dx}(x\ln x) = \ln x + 1$$

$$\frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$\ln x + 1 = \frac{1}{u}\frac{du}{dx} \quad \frac{du}{dx} = x^x(\ln x + 1)$$

$$2^y \ln 2\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx} = \frac{x^x(\ln x + 1)}{1 + x^{2x}}$$

$$\frac{dy}{dx} = \frac{x^x(\ln x + 1)}{2^y(1 + x^{2x})\ln 2} = \frac{x^x(\ln x + 1)}{\tan^{-1}(x^x)\,(1 + x^{2x})\ln 2}$$

Example 2.3.2

g(x) is an inverse function of f(x)=$x^2 - 4x + 1$. Find g'(1).

Method 1: Let f(x)=1, then x=0 or 4. f'(0)=2(0)-4=-4 f'(4)=2(4)-4=4

For point (0, 1), f'(0)×g'(1)=1 g'(1)=-1/4

For point (4, 1), f'(4)×g'(1)=1 g'(1)=1/4

Therefore, g'(1)=1/4 or -1/4

Method 2: the inverse function is $x = y^2 - 4y + 1$.

Then differentiate with respect to x, we get $1 = (2y-4)\frac{dy}{dx} \quad \frac{dy}{dx} = \frac{1}{2y-4}$

Let $x = 1, y = 0\ and\ 4$. $\frac{dy}{dx} = \frac{1}{2y-4} = \frac{1}{4}\ and -\frac{1}{4}$.

Practice

1. Determine $\frac{dy}{dx}$ of following equations.

j) $(xy)^2 = \cot x$

k) $xy = \cos(x+y)$

l) $y = x^{\cos x}$

m) $y^x = x^y$

n) $y = \sin^{-1}(e^{\tan^{-1}x})$

2. Determine the derivative of inverse function of following function at given point.

a) $y - x = \cos y$ (-1, 0)

b) $y = \frac{1}{x} + 3x + 1$ (1, 5)

Check Time:

I. Know how to evaluate implicit derivatives

J. Know how to evaluate derivatives of inverse functions

2.4 Application of Differentiation

Summarized Points

I. Determine Maximum/Minimum Value

1) Define Local Maximum and Minimum

   a) f is a local maximum value at c if and only if $f(x) \leq f(c)$ for all x in some open interval containing c.

   b) f is a local minimum value at c if and only if $f(x) \geq f(c)$ for all x in some open interval containing c.

2) Define Absolute Maximum and Minimum

   a) f is an absolute minimum value at c if and only if $f(x) \geq f(c)$ for all x in the domain.

   b) f is an absolute maximum value at c if and only if $f(x) \leq f(c)$ for all x in the domain.

3) Extreme Value Theorem

   If f is continuous on a closed interval [a, b], then f has both a maximum value and a minimum value on the interval.

4) Local Extreme Value Theorem

   If a function f has a local maximum value or a local minimum value at an interior point c of its domain, and if f' exists at c, then f'(c)=0.

5) First Derivative Test of Local Extreme Value (Apply to a continuous function)

At a critical point x = c	Local Maxima	Local Minima
$x < c$	$f'(x) > 0$	$f'(x) < 0$
$x > c$	$f'(x) < 0$	$f'(x) > 0$
Left Endpoint x=a	$f'(a) < 0$	$f'(a) > 0$
Right Endpoint x=b	$f'(b) > 0$	$f'(b) < 0$

6) Second Derivative Test of Local Extreme Value (Apply to a continuous function)

At a critical point x = c	Local Maxima	Local Minima
$f''(x)$	$f''(x) < 0$	$f''(x) > 0$

7) Compare f' and f''

	$f''(x) < 0$	$f''(x) > 0$

$f'(x) < 0$	Decrease, Concave down	Decrease, Concave up
$f'(x) > 0$	Increase, Concave down	Increase Concave up

II. Linear Approximation

1) $f(x) = f'(a)(x-a) + f(a)$

III. Related Rate

1) $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Problem-Solving Strategy

1. As you determine absolute extreme value, don't forget to compare local extrema and determine whether the values of two endpoints are extrema.

2. Procedure for determining related rate problem:

a) Decide what the two variables are.

b) Find an equation relating them.

c) Differentiate with respect to a variable.

d) Plug all know values to a related rate equation.

e) Solve for the unknown rate.

Example 2.4.1

Determine all extreme values of following functions at given interval. Find the absolute maximum and minimum.

a) $y = \frac{1}{3}x^3 + 2x^2 \ \ [-\infty, \infty]$

b) $y = \cos x - \sin x \ \ [0, 2\pi]$

c) $y = \frac{1}{3}x^3 + 2x^2 \ \ [-2,1]$

Solution:

a) $\frac{d}{dx}\left[\frac{1}{3}x^3 + 2x^2\right] = x^2 + 4x = 0 \quad x = 0, -4$

$y'' = 2x + 4$ Substitute x=0 and -4 respectively we get $y'' = 4$ and -4. Therefore (0, 0) is a relative minimum and (-4, 32/3) is a relative maximum.

Boundary value: $\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{3}x^3 + 2x^2 = \infty \quad \lim_{x\to-\infty} y = \lim_{x\to-\infty} \frac{1}{3}x^3 + 2x^2 = -\infty$

Due to infinity of boundary value, there is no absolute maximum and minimum.

b) $\frac{d}{dx}[\cos x - \sin x] = -\sin x - \cos x = 0$

Square both sides, we get $(\sin x)^2 + (\cos x)^2 + 2\sin x \cos x = 1 + \sin 2x = 0$

$2x = \frac{3\pi}{2}$ and $\frac{7\pi}{2}$ $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ $y'' = \sin x - \cos x$ Substitute $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$,

$y''\left(\frac{3\pi}{4}\right) = \sqrt{2} > 0$ and $y''\left(\frac{7\pi}{4}\right) = -\sqrt{2} > 0$

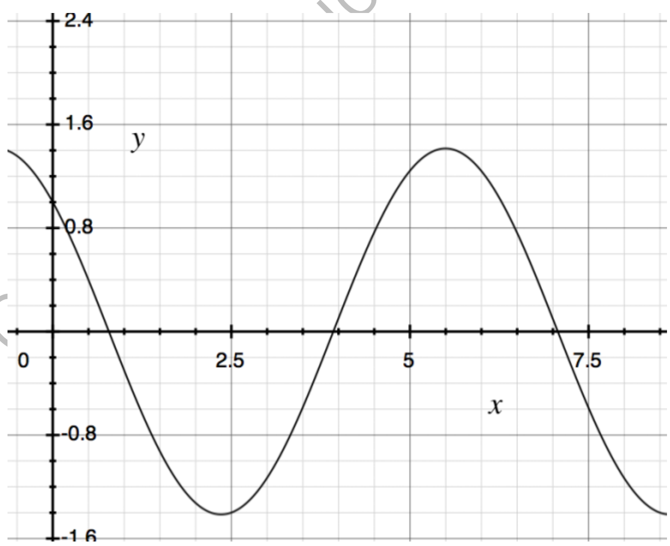
So $(\frac{3\pi}{4}, -\sqrt{2})$ is a relative minimum and $(\frac{7\pi}{4}, \sqrt{2})$ is a relative maximum

Boundary Value: Substitute $x = 0$ and 2π, $y = 1$; $y'(0) = y'(2\pi) = -1$.

According to first derivative test, $(0, -1)$ is relative maximum and $(2\pi, -1)$ relative minimum. Pay attention to whether each endpoint is a left boundary one or right one.

Compare the boundary value and critical point (y'=0), the absolute maximum is $(\frac{7\pi}{4}, \sqrt{2})$. The absolute minimum is $(\frac{3\pi}{4}, -\sqrt{2})$.

A graph of the function is given below.



Picture 2.4.1.1 Graph of $y = \cos x - \sin x$

c) From part a we know (0, 0) is a relative minimum. The only difference is the boundary value. Substitute x=-2 and 1 respectively, we get y=16/3 and 7/3 respectively. Notice that y'(-2)=-4; y'(1)=5. Therefore (-2, 16/3) and (1, 7/3) are relative maximums. Compare all the endpoints and critical points, we conclude that (-2, 16/3) is the absolute maximum and (0, 0) is the absolute minimum.

Example 2.4.2

Define $f(x) = x^{1/3}$. Determine the approximate value of $f(1.4)$ using linear approximation at x=1.

$f'(1) = \frac{1}{3}x^{-2/3} = \frac{1}{3}$ $f(1) = 1$ $f(1.4) = f'(1)(1.4 - 1) + f(1) = \frac{17}{15}$

Note: recall the definition of derivative f'(a)=$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ *It can be transformed to* h[f'(a)]+$f(a)$=$\lim_{h \to 0} f(a+h)$. *Linear approximation is just equivalent to the transformed formula when h does not approach 0. Substitute h with* Δx, *the transformed equation becomes* Δx[f'(a)]+$f(a)$=$f(a+\Delta x)$. *Connecting with this example, you can tell that* Δx *is (1.4-1); f'(a) is f'(1); f(1) is f(a);f(1.4)=f(1+0.4)is* $f(a+\Delta x)$.

Example 2.4.3

If the height, length and width of a rectangular solid $(V = xyz)$ all change constantly, find the relation between $\frac{dv}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ when x=y=2, z=1.

$$\frac{dv}{dt} = \frac{d(xyz)}{dt} = yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt} + 4\frac{dz}{dt}$$

Practice

1. (a) Determine all the extrema of each of the following function. (b) Determine whether each of the extreme value is a relative maximum or minimum. (c) Determine the absolute maximum and minimum.

o) $y = \cot x$ $[\frac{\pi}{4}, \frac{\pi}{3}]$

p) $y = x^3 - 3x - 1$ $[0, 2]$

q) $y = \sin x + \cos x$ $[0, 2\pi]$

r) $y = \frac{x}{1+x^2}$ $[-2, 2]$

s) $y = ax^3 + bx^2 + cx + d$ $(a \neq 0, a, b, c, d \text{ are all constants})$

2. Use linear approximation to estimate the following value. Note that the result should be in decimal.

c) Estimate $4.1^{1.5}$ using $f(x) = x^{1.5}$ at x=4

d) Estimate $33^{1/5}$ using a proper function to approximate linearly

3. What is the largest possible area of a rectangle that is inscribed in the region under $y = \sqrt[2]{1-x^2}$ and above x-axis? (Inscribed here means that there are two points of the rectangle on the function y and other two on the x-axis.)

4. Suppose $x^2 + y^2 + y = 3$. Find the absolute maximum value of the function $z = yx^2{+}y^3$.

Warning: Pay attention to the domain of the function z when making substitutions.

5. A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at 0.1 meters per second. How fast is the foot of the latter approaching the wall when the area enclosed by the horizontal ground, ladder and vertical wall is 30?

Check Time:

K. Know how to determine maximum and minimum of functions

L. Know how to approximate value linearly

M. Know how to build equations with related rates

Unit 3: Integration and Differential Equation

3.1 Computation of Integration

Summarized Points

I. Geometric View

a) If the integration value is positive, then it is equivalent to the area between x-axis and the function

b) If the integration value is negative, then it is equivalent to the negative area between x-axis and the function

II. Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

III. Integration by Part

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

IV. Integral of tangent/cotangent/secant/cosecant function

a) $\int \tan x\, dx = -\ln|\cos x|$

b) $\int \cot x\, dx = \ln|\sin x|$

c) $\int \sec x\, dx = \ln|\sec x + \tan x|$

d) $\int \csc x\, dx = -\ln|\csc x + \cot x|$

Problem-Solving Strategy

For each of the following type of function, common integration methods are shown:

a) A rational function (a quotient of two polynomials): Simple substitution or Partial fraction or Inverse tangent function

Example: $f(x) = \frac{1+x}{x^2+1}$

b) $f(x) = g(x)h(x)$, where g(x) is a polynomial function and h(x) is a sine or cosine function or exponential function: Integration by part. Make sure $u = g(x)$ *and* $dv = h(x)dx$

Example: $f(x) = x\cos x$

c) $f(x) = g(x)h(x)$, where g(x) is a polynomial function and h(x) is an inverse trigonometric function or a log function. Integration by part. Make sure $u = h(x)$ *and* $dv = g(x)dx$

Example: $f(x) = x\tan^{-1} x$

d) Remember the four trigonometric formulae:

1) $(\sin x)^2 + (\cos x)^2$=1

2) $\sin(2x) = 2\sin x \cos x$

3) $1 + (\tan x)^2 = (\sec x)^2$

4) $\cos(2x) = (\cos x)^2 - (\sin x)^2$

Example 3.1.1 (Substitution)

Compute the following integral:

(e) $\int \sin(2x + \pi)\, dx$

(f) $\int x\sqrt{1 - x^2}\, dx$

Solutions

e) Let $u = 2x + \pi$, then $\int \sin(2x + \pi)\, dx = \int \sin(u)\, dx$, $dx = du/2$. $\int \sin(u)\, dx = 0.5 \int \sin(u)\, du = -0.5\cos u = -0.5\cos(2x + \pi) + C$

f) Let $u = 1 - x^2$, then $\int x\sqrt{1 - x^2}\, dx = \int x\sqrt{u}\, dx$, $du = -2xdx$. $\int x\sqrt{u}\, dx = -0.5 \int u^{0.5}\, du = -\frac{1}{3}u^{1.5} + C$

Example 3.1.2(Integration by Part)

Compute the following integral:

a) $\int x \cos x\, dx$

b) $\int (\sec x)^3\, dx$

Solutions:

a) Let $u = x, dv = \cos x\, dx$. Then Let $du = dx, v = sinx$. $\int x\cos x\, dx = xsinx - \int vdu = xsinx + cosx + C$

b) $\int (\sec x)^2\, dx = dv, u = \sec x\, .\, v = \tan x\, , du = \sec x \tan x\, dx.$

$$\int (\sec x)^3\, dx = \sec x \tan x - \int \sec x\, (\tan x)^2 dx$$

$$= \sec x \tan x - \int \sec x\, [(\sec x)^2 - 1] dx$$

$$= \sec x \tan x - \int [(\sec x)^3 - \sec x] dx$$

$$= \sec x \tan x - \int [(\sec x)^3 dx] + \int \sec x\, dx$$

$$=\frac{\sec x\tan x+\int \sec x\, dx}{2}$$

$$=\frac{\sec x\tan x+\ln|\sec x+\tan x|}{2}+C$$

Example 3.1.3(Partial Fraction)

Compute the following integrals:

a) $\int \frac{dx}{x^2-3x-4}$

b) $\int \frac{xdx}{x^2-2x+3}$

Solutions:

a) Solve $x^2-3x-4=0, x=4\ or-1$ $\frac{1}{x^2-3x-4}=\frac{1}{(x-4)(x+1)}=\frac{A}{x-4}+\frac{B}{x+1}$

$$\mathrm{A}\ (\ x+1)+B(x-4)=1$$

$$\mathrm{A}=0.2\ \mathrm{B}=-0.2$$

b) There is no real solution to $x^2-2x+3=0$. Simple substitution is not possible since $d(x^2-2x+3)=(2x-2)dx\ which\ cannot\ substitute\ 2xdx$. But transforming denominator to the form of $(x-1)^2+2$ is necessary since $\int \frac{dx}{x^2+1}=\tan^{-1}x+C$ Therefore when the denominator is in the form $(x-A)^2+B$, where B is greater than 0, we can consider inverse tangent function.

$$\int \frac{xdx}{x^2-2x+3}=\int \frac{(x-1)dx}{(x-1)^2+2}+\int \frac{dx}{(x-1)^2+2}$$

Using substitution that $u=(x-1)^2+2$, we can get that $\int \frac{(x-1)dx}{(x-1)^2+2}=\frac{1}{2}\ln|(x-1)^2+2|$

$\int \frac{dx}{(x-1)^2+2}=\frac{1}{2}\int \frac{dx}{[(x-1)^2+2]/2}=\frac{1}{2}\int \frac{dx}{[(x-1)/\sqrt{2}]^2+1}=\frac{1}{2}\tan^{-1}(\frac{x}{\sqrt{2}}-\frac{1}{\sqrt{2}})$

$$\int \frac{xdx}{x^2-2x+3}=\frac{1}{2}\ln|(x-1)^2+2|+\frac{1}{2}\tan^{-1}(\frac{x}{\sqrt{2}}-\frac{1}{\sqrt{2}})+C$$

Example 3.1.4(Algebraic Transformation of Variables)

Compute the integral: $\int \frac{xdx}{(x+1)^{100}}$

Solution:

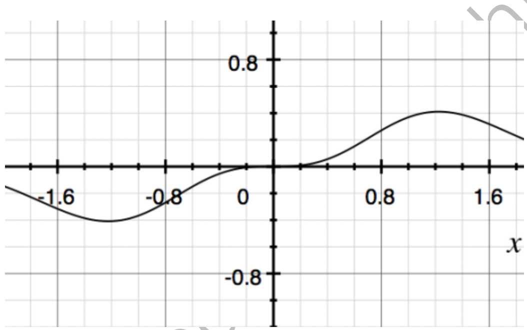
Let $x+1=u$. In this case however we don't use du to substitute xdx. Instead, we let $x=u-1$ and $dx=du$. Then the integral becomes $\int \frac{xdx}{(x+1)^{100}} = \int \frac{udu}{u^{100}} - \int \frac{du}{u^{100}} = \int \frac{du}{u^{99}} - \int \frac{du}{u^{100}} = \frac{u^{-98}}{-98} + \frac{u^{-99}}{99} = \frac{(x+1)^{-98}}{-98} + \frac{(x+1)^{-99}}{99} + C$

Example 3.1.4(Symmetric Method)

Compute the integral: $\int_{-1}^{1} x^3 e^{-x^2}\, dx$

Solution:

Look at the graph. Observe that $f(x) = x^3 e^{-x^2}$ is symmetric with origin. Then the area between curve and x-axis on interval 0 to 1 is equal to that on interval -1 to 0. Therefore, this integral is equal 0.



Note: if a function is symmetric with origin, then f(a)=-f(-a) for all constants a. If a function is symmetric with a point (x, y) then f(x+a)-y=y-f(x-a) for all constants a.

Practice

1. Compute the following integrals:

a) $\int (\sec x)^2\, dx$

b) $\int x \sin x\, d\mathrm{x}$

c) $\int x \ln(x^3)\, dx$

d) $\int \frac{1}{x^2+4x+3} dx$

e) $\int \frac{1}{x^2+4x+5} dx$

f) $\int \frac{dx}{\sqrt{x^2+1}}$ (hint: consider $1+(\tan x)^2 = (\sec x)^2$)

g) $\int \frac{dx}{\sqrt{x^2-1}}$

h) $\int \frac{xdx}{(x^2+1)^{\frac{3}{2}}}$

i) $\int \frac{x^2dx}{(x^2+1)^{\frac{3}{2}}}$

j) $\int x \cos x \sin x \, dx$

Note: Essentially question f, g, h, i are required to use certain algebraic (trigonometric) transformation/substitution.

2. Consider the following function:

$$F = \int (\sin x)^n dx$$

a) Let $\sin x \, dx = dv$ $(\sin x)^{n-1} = u$, build an algebraic relationship between $\int (\sin x)^n dx$ and $\int (\sin x)^{n-2} dx$

b) Compute $\int (\sin x)^2 dx$ and $\int (\sin x)^3 dx$

c) Compute $\int (\cos x)^4 dx$

Check Time:

N. Use proper ways to solve integrals

3.2 Differential Equation

Summarized Points

I. Definition of Differential Equation

a) An equation that related functions and their derivatives.

b) The typical differential equation we need to solve is in form of $\frac{dy}{dx} = f(x)g(y) or\ f(x)/g(y)$

II. Euler's Method

Euler's method is used mainly when the first derivative of a function is given. The step is similar to multiple-times linear approximations: Given a Δx, then use $\frac{dy}{dx}$ to predict the change of dependent variable.

III. Slope field

Slope field is used to show the slope at every point of domain.

IV. Logistic Growth

$$\frac{dp}{dt} = KP(M - P)$$

$$P = \frac{M}{1 + Ae^{-MKt}}$$

V. Exponential Growth

$$y = y_0 e^{kt}$$

Problem-Solving Strategy

How to choose the right slope field graph: Let x be a constant, then think about what the slope should behave as y changes. For instance, for the differential equation $\frac{dy}{dx} = x + y$, we let x=0, then we think about the slope at y-axis. At x=0, $\frac{dy}{dx} = y$. We can tell that when y is getting smaller, $\frac{dy}{dx}$ is getting smaller, meaning that the slope should point downward gradually. Similarly, when y is getting larger, $\frac{dy}{dx}$ is getting greater, meaning that the slope should point upward gradually. We can also observe the pattern of the slope by making y constant. For instance, if we let y=0 here, then the slope will also get smaller when x decreases and bigger when x increases.

Example 3.2.1

Solve the differential equations

(a) $\frac{dy}{dx} = x^2 y$

(b) $\frac{dy}{dx} = \cos x / e^y$

Solutions

a) $\frac{dy}{y} = x^2 dx \quad \ln y = \frac{x^3}{3} + C \quad y = e^{\frac{x^3}{3}+C}$

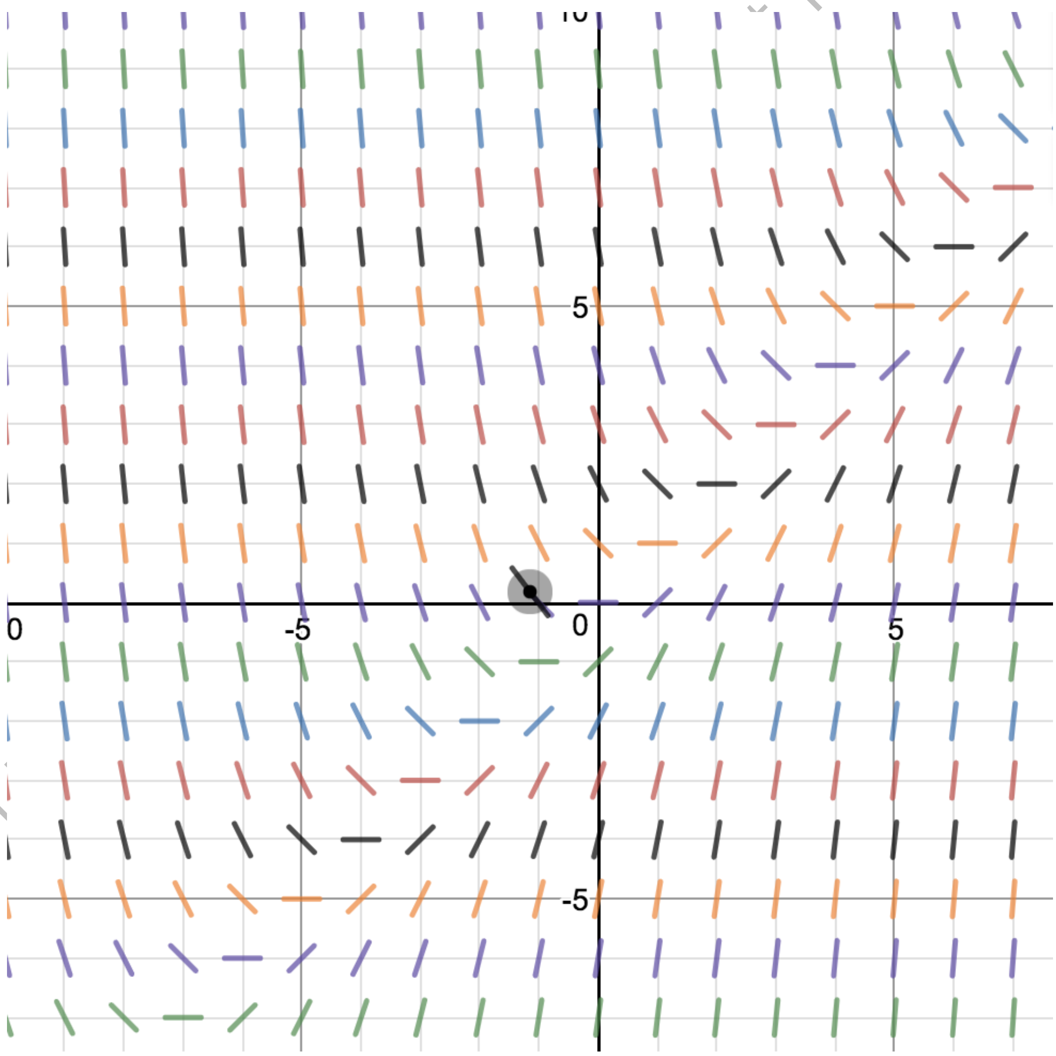
b) $e^y dy = \cos x\, dx \quad e^y = \sin x + C \quad y = \ln(\sin x + C)$

Example 3.2.2

Given a differential equation $\frac{dy}{dx} = x - y$, sketch a graph of its slope field.

Solution:

Below is a graph of slope field. You can easily check the graph by checking special cases. For instance, when $\frac{dy}{dx} = x - y = 0$, the slope field should be horizontal. If you don't know how to draw the approximate graph, look back at the problem-solving strategy.



Picture 3.2.2.1

Practice

1. Solve following differential equations with initial conditions:

a) $\frac{dy}{dx} = \frac{x}{y}\ y(0) = 1$

b) $\frac{dy}{y} = \frac{dx}{-x^2}\ y(1) = 1$

c) $\frac{dy}{dx} = x \csc y\ \ y(1) = \frac{\pi}{2}$

2. Draw slope field of $\frac{dy}{dx} = x^2 + y^2$.

3. Suppose a population of butterflies is growing according to the logistic equation. If the carrying capacity is 500 butterflies and r = 0.1 individuals/(individuals*month), what is the maximum possible growth rate for the population? How would the maximum possible growth rate change when r increases from 0.1 to ∞?

4. Suppose a population of butterflies is growing at a rate proportional to its population. If the growth rate is 10% and the current population is 100, what will the population be after 5 years? If the growth rate is proportional to time, will the population after 5 years be greater or smaller?

Check Time:

O. Solve separable differential equations

P. Write differential equations in different problems and solve them.

3.3 Application of Integration

Summarized Points

I. Area

$$A = \int_a^b [f(x) - g(x)]\, dx, where\ f(x) \geq g(x)$$

$$= \int_c^d [g(y) - h(y)]\, dy, where\ g(y) \geq h(y)$$

II. Motion

$$\frac{d^2\mathrm{x}}{\mathrm{dt}^2} = \frac{dv}{dt} = a$$

III. Volume

$$V = \int_a^b s\, dh$$

IV. Arc Length

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}\, dx$$

Problem-Solving Strategy

1. Make differential be easily defined upper and lower bound. For instance, when you are calculating the work done by a force on a straight line, it is easy to give the integral $W = \int_a^b F\, dr$, where b-a represents the passing distance. In fact, you can also write $W = \int r\, dF$. But dF's upper and lower bound is really hard to determine and we don't use F to express r most of the time. Therefore, $W = \int_a^b F\, dr$ is a better form.

2. When you are calculating the base area of a volume, distinguish $[f(x) - g(x)]^2$ and $[f(x)]^2 - [g(x)]^2$. See example 3.3.1

Example 3.3.1

Find the volume obtained when the region bounded by $y = x^2$ and $x = 2$ and $y = 0$ is rotated about:

a) x=0

b) x=2

Solutions

a) When rotating about y-axis, the base is a disk with a hole. $V = \pi \int_0^4 [2^2 - x^2] dy =$

$\pi \int_0^4 [2^2 - y] dy$=$8\pi$

b) When rotating about x=2, the base is a disk without a hole. $V = \int_0^4 (2-x)^2 \, dy =$

$\int_0^4 (2-\sqrt{y})^2 \, dy = \frac{8\pi}{3}$

Example 3.3.2

A car moves along a straight line with a constant acceleration a. Given that the car starts from x=x_0 and the initial velocity is v=v_0, find a formula relating position and time.

$$\frac{dv}{dt} = a$$

$$v = at + C$$

Substitute (0, v_0) into v, C= v_0. $$v = at + v_0$$

$$\frac{dx}{dt} = v = at + v_0$$

$$x = \frac{a}{2}t^2 + v_0 t + K$$

Substitute (0, x_0) into x, C= x_0. $$x = \frac{a}{2}t^2 + v_0 t + x_0$$

Practice

1. Find the area of the followings:

a) The area between x-axis and $y = \tan^{-1} x$.

b) The area between $x = y^2$ and y-axis.

2. Find the volume of the followings:

a) The volume of a sphere with radius 1.

b) Suppose region S is bounded by $y = 1 - x^2$ and x-axis. Then rotate region S with respect to axis x=1 to form a new region R. Find the volume of R.

3. Find the perimeter of region S in the last question. (Use a calculator to approximate the integral)

4. Find the work done by a force $F = \mathrm{e}^{-x}$ from x=1 to m. What happened when m approaches infinity?

Check Time:

Q. Apply integration to geometry, physics and other math-intensive fields.

3.4 Parameter and Polar Coordinate Calculus

Summarized Points

I. Polar Coordinate

$$x = r\cos\theta \;\; y = r\sin\theta \;\; \theta = \tan^{-1}\frac{y}{x} \;\; r = x^2 + y^2$$

II. Slope in Polar Coordinate

$$\frac{dy}{dx} = \frac{dy/\,d\theta}{dx/\,d\theta}$$

III. Area in Polar Coordinate

$$A = \int \frac{1}{2} r^2\, d\theta$$

IV. Parametric Equation Calculus

$$x = g(t) \;\; y = h(t)$$

$$\frac{dy}{dx} = \frac{dy/\,dt}{dx/\,dt}$$

$$A = \int y\,dx = \int h(t)\,d[g(t)] = \int h(t)\,g'(t)dt$$

$$L = \int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$$

Problem-Solving Strategy

1. When you deal with the type of function that has an arc, use polar coordinate.

2. When you are calculating the second derivative of a parametric or polar function, you cannot differentiate the first derivative with respect to dt. Specific procedures are shown in example 3.4.1.

Example 3.4.1

Given a polar function $f\colon \begin{cases} x = \cos t \\ y = 2\sin t \end{cases}$ $(2\pi \geq t \geq 0)$, determine the followings:

a) The shape of the function

b) The slope of the function

c) $\frac{d^2y}{dx^2}$

Solutions

a) Based on the identity $\sin^2 t + \cos^2 t = 1$, we can transform the polar function to $x^2 + \frac{y^2}{4} = 1$. This is a standard form of an eclipse. Therefore, the shape is elliptical.

b) The shape is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-\sin t} = -2\cot t$

c) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2\cot t) = \frac{d}{dt}(-2\cot t)\frac{dt}{dx} = 2\csc^2 t \times \frac{1}{\frac{dx}{dt}} = 2\csc^2 t \times \frac{1}{-\sin t} =$

$-2\csc^3 t$

Example 3.4.2

Given a circle with radius 1, deduce the circumference and area of it.

We can write the circle in an equation $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $(2\pi \geq t \geq 0)$.

Then,

$$A = \int \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} 0.5 \, d\theta = \pi$$

$$C = \int \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 dt} = \int_0^{2\pi} 1 \, dt = 2\pi$$

Practice

1. Calculate the area between the following functions and given axis:

a) $f{:}\begin{cases} x = t^2 \\ y = \cos t \end{cases}, 1 \geq t \geq 0$ x-axis

b) $g{:}\begin{cases} x = e^{-t} \\ y = te^{-t} \end{cases}, 1 \geq t \geq 0$ y-axis

2. Calculate the length of the following functions: (You may a calculator for evaluating integral in question d)

a) $h{:}\begin{cases} x = \sin t \cos t \\ y = \cos t \end{cases}, 2\pi \geq t \geq 0$

b) $k{:}\begin{cases} x = t + \sin t \\ y = \cos t \end{cases}, 1 \geq t \geq 0$

3. Determine $\frac{d^2y}{dx^2}$ for functions h and k of last questions. (Express results in terms of t)

Check Time:

R. Transform parameter functions and polar coordinate functions to rectangular coordinate functions and the opposite procedure.

S. Calculate length, area and derivative of functions in non-rectangular coordinate form.

Unit 4: Sequence and Series

4.1 Testifying Convergence and Divergence

Summarized Points

I. Sequence

A sequence is a function with domain the natural number 1, 2, 3, etc.

II. Series

A series is a sum of all value of a sequence.

III. Geometric Series

A geometric series has $\frac{a_{n+1}}{a_n} = r$, which is a constant for all positive integers n. (Assume a_1 is the first term)

Sum of a geometric series with infinite terms

Ratio r	Sum
0	0
$\|r\| < 1$	$\frac{a_1}{1-r}$
$\|r\| \geq 1$	Infinite

IV. Divergence and Convergence

Convergence: $\lim_{n\to\infty}(a\ series\ or\ a\ sequence) = L$, which is a constant.

Convergence: $\lim_{n\to\infty}(a\ series\ or\ a\ sequence)$ doesn't exist or approach infinity or negative infinity.

V. Testing Divergence or Convergence

Test Name	Specific Content	Usually applied in
Integral Test	Suppose $f(x) > 0$ and is decreasing on the infinite interval $[k, \infty]$ for some $k \geq 1$ and that $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.	p-series ($\sum_{n=1}^{\infty}\frac{1}{n^p}$), where p is a constant
Alternative Series Test	Suppose that $\{a_n\}_1^{\infty}$ is a non-increasing sequence of positive numbers and $\lim_{n\to\infty} a_n = 0$. Then	Alternative series described in the specific content, typically having a

	the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.	$(-1)^{n-1}$ or $(-1)^n$ in each term.
Comparison Test	Suppose a_n and b_n are non-negative for all n and that $a_n \leq b_n$ when $n \geq N$, for some N. Then, If $\sum_{n=0}^{\infty} b_n$ converges, so does $\sum_{n=0}^{\infty} a_n$. If $\sum_{n=0}^{\infty} a_n$ diverges, so does $\sum_{n=0}^{\infty} b_n$.	Series whose terms consist of log functions or (inverse) trigonometric functions.
Limit Comparison Test	If $\lim_{n\to\infty} b_n/a_n = L$, which is a constant, then both $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} a_n$ diverge or converge.	Terms expressed as ratio of two polynomials.
Absolute Convergence Test	If $\sum_{n=0}^{\infty}\lvert a_n\rvert$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.	Series typically having $(-1)^{n-1}$ or $(-1)^n$ in each term.
The Ratio Test	Suppose that $\lim_{n\to\infty}\lvert a_{n+1}/a_n\rvert = L$. If $L < 1$, the series $\sum a_n$ converges absolutely, if $L > 1$ the series diverges, and $L = 1$ gives no information.	Series typically having $n!\ or\ a^n$ in each term.
The Root Test	Suppose that $\lim_{n\to\infty}\lvert a_n\rvert^{1/n} = L$. If $L < 1$ the series $\sum_{n=0}^{\infty}\lvert a_n\rvert$ converges absolutely, if $L > 1$ the series diverges, and if $L = 1$ this test gives no information.	Series typically having $[f(n)]^n$, where f(n) is a polynomial expression.

VI. Taylor Series

Definition of Taylor Series: $\sum_{n=0}^{\infty}\frac{f^{(n)}(0)(x-a)^n}{n!}$

VII. Taylor's Inequality

If $\left|f^{(n+1)}(x)\right| \leq M$, then

$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$, where $R_n(x)$ is the error(remainder) of the predictive value.

(Note that the Taylor's inequality is derived from the fact that any function can be written as $f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(z)}{(N+1)!}(x-a)^{N+1}$ where z is between center a and x.)

Example 4.1.1

A geometric series with r=0.2 and $a_0 = 1$, determine the followings:

a) a_{10}

b) the sum of the series

Solutions

g) Based on the given conditions, we can deduce that $a_{10} = a_0 \times r^{10} = 0.2^{10}$

h) Sum$= \frac{a_0}{1-r} = 5/4$

Example 4.1.2

Test whether the following series converge:

1. $\sum_1^\infty \frac{1}{x^3}$
2. $\sum_1^\infty e^{-n}$
3. $\sum_1^\infty \frac{1}{2^n+1}$
4. $\sum_1^\infty \frac{\ln k}{k}$
5. $\sum_1^\infty \frac{n}{\sqrt[3]{n^4+3n-2}}$
6. $\sum_1^\infty \frac{1}{3^n+4^n}$
7. $\sum_1^\infty \frac{1}{n!}$
8. $\sum_1^\infty \frac{1}{n!} \times (-1)^n$
9. $\sum_1^\infty (\frac{2n-1}{3n-3})^n$

Solutions:

1. Using integral test (you can check that the conditions are satisfied), we get to know that $\sum_1^\infty \frac{1}{x^3} < \int_1^\infty \frac{1}{x^3} = \lim_{b\to\infty} -\frac{x^{-2}}{2}\Big|_1^b = 1/2$ Therefore, the series converge.

2. Using ratio test, we have $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{-(n+1)}}{e^{-n}} = e^{-1} < 1$. Therefore, we have the series converge.

3. Using comparison test, we get $\sum_1^\infty \frac{1}{2^n+1} < \sum_1^\infty \frac{1}{2^n} =1$. Therefore, we have the series converge.

4. Using comparison test, we get $\sum_3^\infty \frac{\ln k}{k} > \sum_3^\infty \frac{1}{k}$. Since by integral test we know that $\sum_3^\infty \frac{1}{k}$, the sum $\sum_3^\infty \frac{\ln k}{k}$ diverge. And so does $\sum_1^\infty \frac{\ln k}{k}$.

5. Using limit comparison test, we get that $\lim_{n\to\infty} \frac{n}{\sqrt[3]{n^4+3n-2}} / \frac{n}{\sqrt[3]{n^4}} = 1$. Since $\sum_1^\infty \frac{n}{\sqrt[3]{n^4}}$ diverge by integral test, $\sum_1^\infty \frac{n}{\sqrt[3]{n^4+3n-2}}$ also diverge.

6. Using comparison test, $\frac{1}{3^n+4^n} < \frac{1}{3^n}$. Since $\sum_1^\infty \frac{1}{3^n}$ converges, the series $\sum_1^\infty \frac{1}{3^n+4^n}$ also converges.

7. Using ratio test, we get $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = 0$. Therefore, the series converge.

8. By absolute convergence test, since $\sum_1^\infty \left|\frac{1}{n!} \times (-1)^n\right| = \sum_1^\infty \frac{1}{n!}$ converges, $\sum_1^\infty \frac{1}{n!} \times (-1)^n$ also converges.

9. Using root test, since $\lim_{n\to\infty} \left|[(\frac{2n-1}{3n-3})^n]^{1/n}\right| = \frac{2}{3} < 1$. The series $\sum_1^\infty (\frac{2n-1}{3n-3})^n$ converges.

Example 4.1.3

Determine the Taylor series for the following functions at given center.

1. $y = e^{-x}$ $x = 2$
2. $y = \frac{1}{x^3+1}$ $x = 0$
3. $y = \sin(x-1)$ $x = 1$

Solutions:

1. $y^{(n)} = (-1)^n e^{-x}$ When x = 2, $y^{(n)}(2) = (-1)^n e^{-2}$

Substitute this to the standard formula, we get $y = e^{-x} = \sum_0^\infty \frac{(-1)^n e^{-2} (x-2)^n}{n!}$

2. Recall the series of $\frac{1}{x+1}$ at x=0 is $\sum_0^\infty (-1)^n x^n$. Then we replace x with x^3, so the series for $\frac{1}{x^3+1}$ becomes $\sum_0^\infty (-1)^n x^{3n}$. (Note that in this situation doing differentiation is not practical. However, Through the series expression we are able to find the derivative of any order quickly now.)

3. Following a periodic order, $y^{(4n)}(1) = 0$; $y^{(4n+1)}(1) = 1$; $y^{(4n+2)}(1) = 0$; $y^{(4n+3)}(1) = -1$ for any non-negative integer n. Summarizing this, we can conclude that $y^{(2n+1)}(1) = (-1)^n$ Then $\sin(x-1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)(x-a)^n}{n!} = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)(x-1)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^{2n+1}}{(2n+1)!}$ (We don't need to consider derivatives of even order since they are all 0)

Practice

1. Determine whether the following series diverge or converge:

a) $\sum_1^{\infty} 2^{1/n}$

b) $\sum_1^{\infty} \cos n$

c) $\sum_1^{\infty} \frac{1}{n^2-1}$

d) $\sum_1^{\infty} \frac{2^n}{n!}$

e) $\sum_1^{\infty} (\frac{2n}{3n-1})^n$

f) $\sum_1^{\infty} \frac{2^n}{n!}$

g) $\sum_1^{\infty} (-1)^n$

2. Find the Taylor series for the following functions and indicate the convergence interval:

a) $f(x) = \tan^{-1}(x^2)$ at x=0

b) $g(x) = cos(2x)$ at x=1

c) $k(x) = xe^{-x}$ at x=1.

3. Find the sixth derivatives of all functions at center point in question 2.

Check Time:

T. Use correct tests to test convergence or divergence

U. Determine Taylor series of given functions.

V. Approximate errors