Question 1.

Let

$$A = \begin{pmatrix} 3 & -1 & -3 \\ -4 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(a) (8pts) Is the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax = b, x \geq 0\}$ empty or not? Justify your answer.

(b) (8pts) Is the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax \geq b, x \geq 0\}$ empty or not? Justify your answer.

(c) (8pts) Given any vector $c \in \mathbb{R}^3$, what can you say about the linear program

$$\max_y \; b^\top y, \; s.t., \; A^\top y \leq c.$$

Justify your answer.

(a) using Farkas' Lemm

(i) $Ax = b$, $x \geq 0$

(ii) $A^T y \leq 0$, $b^T y > 0$ $\longrightarrow$ (ii) $\begin{cases} 3y_1 - 4y_2 \leq 0 \\ -y_1 \leq 0 \\ -3y_1 + 2y_2 \leq 0 \\ 3y_1 + 2y_2 > 0 \end{cases}$ $\Rightarrow$ easily to find a feasible solution $y = (1, 1)$

$\therefore$ (ii) is solvable, (i) is not solvable $\therefore$ P is empty

(b)

using Farkas' Lemm

$Ax \geq b$ $x \geq 0$ $\Rightarrow$ (i) $(A \; -I)\begin{pmatrix} x \\ s \end{pmatrix} = b$ $\begin{pmatrix} x \\ s \end{pmatrix} \geq 0$

(ii) $(A \; -I)^T y \leq 0$, $b^T y > 0$

$\therefore$ $\begin{cases} 3y_1 - 4y_2 \leq 0 \\ -y_1 \leq 0 \\ -3y_1 + 2y_2 \leq 0 \\ -y_1 - y_2 \leq 0 \\ 3y_1 + 2y_2 > 0 \end{cases}$ easily to find a feasible solution

$y = (1, 1)$

$\therefore$ (ii) is solvable, (i) is not solvable

$\therefore$ P is empty

(c) Dual: $\max \; b^T y$

s.t. $A^T y \leq C$

Primal: $\min \; C^T x$

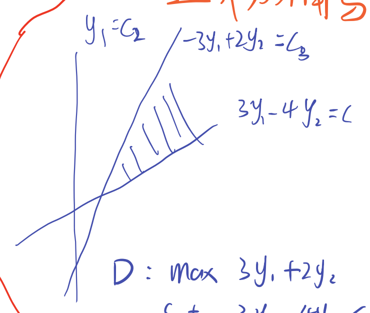
s.t. $Ax = b$

$x \geq 0$

$\because$ P is not feasible (proved in (a))

$\therefore$ D's optimal value is $+\infty$

这部分不用写



D: $\max \; 3y_1 + 2y_2$

s.t. $3y_1 - 4y_2 \leq C_1$

$-y_1 \leq C_2$

$-3y_1 + 2y_2 \leq C_3$

Question 2.

Consider the integer programming problem

$$
\begin{array}{ll}
\max & x_1 + 2x_2 \\
s..t & -3x_1 + 4x_2 \le 4 \quad ① \\
 & 3x_1 + 2x_2 \le 11 \quad ② \\
 & 2x_1 - x_2 \le 5 \quad ③ \\
 & x_1, x_2 \text{ integer.}
\end{array}
$$

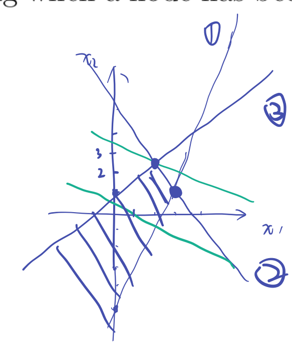
(a) (5pts) What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem?

(b) (5pts) What is the convex hull of the set of all solutions to the integer programming problem?

(c) (10pts) Solve the problem by branch and bound. Give the branch and bound tree indicating for each node any bounds obtained and indicating when a node has been fathomed.

(a) The fesible area is upper-bounded, There must exist a BFS optimal sol

$$
\begin{cases} -3x_1 + 4x_2 = 4 & ① \\ 3x_1 + 2x_2 = 11 & ② \\ 2x_1 - x_2 = 5 & ③ \end{cases}
$$



We have 2 BFS, solve {①,②} or {②③},

$$
\begin{cases} x = (2, \tfrac{5}{2}) & f = 7 \leftarrow \text{OPT for LP} \\ x = (3, 1) & f = 5 \end{cases}
$$

For ILP, It's $(2,2)$, $z = 6$

(b) ~~Shaded area~~ $\begin{cases} ① \\ ② \\ ③ \end{cases} + x_2 \le 2$

(c)

$\boxed{LP0: x = (2, \tfrac{5}{2}),\ z = 7}$

- $x_2 \ge 3$: $\boxed{LP_1: \text{infeasible, fathomed}}$
- $x_2 \le 2$: $\boxed{LP2: x = (\tfrac{7}{3}, 2),\ z = \tfrac{19}{3}}$
  - $x_1 \ge 3$: $\boxed{LP3: (3,1),\ z = 5,\ \text{fathomed}}$
  - $x_1 \le 2$: $\boxed{LP4: (2,2),\ z = 6,\ \text{fathomed}}$

LP1: infeasible

LP2: $\begin{cases} x_2 = 2 \\ 3x_1 + 2x_2 = 11 \end{cases}$

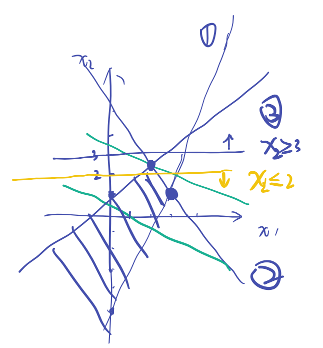
$\therefore x = (\tfrac{7}{3}, 2)$

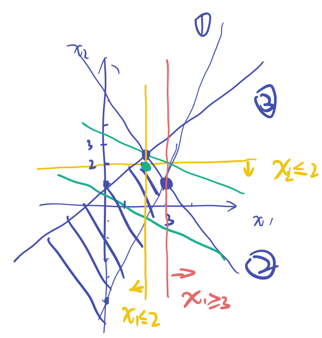
LP3: $\begin{cases} x_1 = 3 \\ ②, ③ \end{cases}$

$x = (3,1)$

LP4: $\begin{cases} x_1 = 2 \\ x_2 = 2 \end{cases}$

$x = (2,2)$ $z = 6$





Question 3. (16pts)

The minimum number of employees needed and the rate of pay for each of 6 periods during a 24-hour day is given in the table below. Each employee works during a 3-period shift with the middle period off. For example an employee that starts work at 4am works periods 2 and 4. note that the period 1 follows immediately after period 6, i.e., an employee that start work at 8pm(period 6) has a break during period 1 and continues working during period 2.

Formulate the problem of finding a daily schedule of employees that minimizes cost while providing a sufficient number of employees for each period as a linear program, ignoring integrality requirements. Clearly define all variables.

Period	Time of Day	Minimal number of employees required	Hourly rate of pay
1	midnight - 4am	20	13
2	4am-8am	30	12
3	8am-noon	100	9
4	noon-4pm	120	9
5	4pm-8pm	80	10
6	8pm-midnight	50	12

Define $x_1, x_2, x_3, x_4, x_5, x_6$ are employees numbers who start working at period 1, 2, 3, 4, 5, 6, respectively.

$$\min \quad 22x_1 + 21x_2 + 19x_3 + 21x_4 + 23x_5 + 24x_6$$

s.t.

$$x_5 + x_1 \geq 20$$
$$x_6 + x_2 \geq 30$$
$$x_1 + x_3 \geq 100$$
$$x_2 + x_4 \geq 120$$
$$x_3 + x_5 \geq 80$$
$$x_4 + x_6 \geq 50$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

x_i is integer.

Question 4.

Consider the following problem

$$(P) \qquad \begin{array}{ll} \max & f(x) = \log(x_1+1) - x_2^2 \\ s.t. & x_1 + 2x_2 \le 3 \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

(a) (10pts) Is the problem convex or not? Justify your answer.

(b) (10pts) Use the KKT conditions to find an optimal solution.

(a) Rewrite (P):

$$\begin{array}{lll} \min & x_2^2 - \log(x_1+1) & (f(x)) \\ \text{s.t.} & x_1 + 2x_2 - 3 \le 0 & (g_1(x)) \\ & -x_1 \le 0 & (g_2(x)) \\ & -x_2 \le 0 & (g_3(x)) \end{array}$$

$\because$ x_2^2 is convex, $-\log(x_1+1)$ is convex

$\therefore$ $f(x)$ is convex

$\because$ g_1, g_2, g_3 are linear constraints, thus convex.

$\therefore$ it's a convex problem

(b) KKT Conditions:

- (Dual feasibility)

$$\nabla f(\bar{x}) + \sum_{i=1}^{3} v_i \nabla g_i(\bar{x}) = \begin{bmatrix} \frac{-1}{\bar{x}_1+1} \\ 2\bar{x}_2 \end{bmatrix} + v_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + v_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\bar{x}_1+1} + v_1 - v_2 \\ 2\bar{x}_2 + 2v_1 - v_3 \end{bmatrix} = 0$$

- (Complementary slackness)

$$\begin{cases} v_1(\bar{x}_1 + 2\bar{x}_2 - 3) = 0 \\ -v_2 \bar{x}_1 = 0 \\ -v_3 \bar{x}_2 = 0 \end{cases}$$

- (Primal feasibility)

$$\begin{cases} x_1 + 2x_2 - 3 \le 0 \\ -x_1 \le 0 \\ -x_2 \le 0 \end{cases}$$

if $v_1 = 0$, $v_2 = \frac{-1}{x_1+1}$, infeasible $\times$

Therefore $x_1 + 2x_2 - 3 = 0$

Further, if $v_2 = 0$

$$\begin{cases} v_1 = \frac{1}{x_1+1} \\ 2x_2 + 2v_1 - v_3 = 0 \\ v_3 x_2 = 0 \end{cases} \Rightarrow v_3 = 2x_2 + \frac{2}{x_1+1}$$

$\therefore v_3 x_2 = 2x_2^2 + \frac{2x_2}{x_1+1} = 0$

$\therefore \underbrace{x_2^2(x_1+1)}_{\geq 0} + \underbrace{x_2}_{\geq 0} = 0$

$\therefore x_2 = 0$, $x_1 = 3$

$v_1 = \frac{1}{4}$ $v_2 = 0$ $v_3 = \frac{1}{2}$

Verify it satisfy KKT condition

Prove necessary:

(P) is linearly constrained

prove sufficiency:

(P) is a convex problem

$\therefore$ the solution $(3, 0)$ is the optimal solution