OR-18:

1. Extreme point = BFS = Vertex.

$(1, 0, 0, 1) \Rightarrow \begin{cases} x_1+x_2=1 \\ x_1+x_3=1 \\ x_2+x_4=1 \\ x_3+x_4=1 \end{cases}$

or $(0, 1, 1, 1)$

$\Downarrow \begin{cases} x_1+x_2=1 \\ x_1+x_3=1 \\ x_1+x_4=1 \\ x_1=0 \end{cases}$

2.

$$\text{(P)}\quad \min\ (-3,1,3,-1)^T(x_1,x_2,x_3,x_4)$$
$$\text{s.t.}\quad \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & -2 & 3 & 3 \\ 1 & -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 6 \end{bmatrix}$$
$$x \geq 0$$

$\Rightarrow$

$$\text{(D)}\quad \max\ (0,9,6)^T(y_1,y_2,y_3)$$
$$\text{s.t.}\quad \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -1 \\ -1 & 3 & 2 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} -3 \\ 1 \\ 3 \\ -1 \end{bmatrix}$$

$\bar{x} = (1, 1, 3, 0)$. $\quad c^T\bar{x} = -3+1+9=7$.

$x_1 > 0 \Rightarrow y_1+2y_2+y_3 = -3$

$x_2 > 0 \Rightarrow 2y_1-2y_2-y_3 = 1$

$x_3 > 0 \Rightarrow -y_1+3y_2+2y_3 = 3$

$x_4 = 0 \Rightarrow y_1+3y_2-y_3 < -1$

$\Rightarrow y_1 = -\frac{2}{3}$, $\quad y_3 = \frac{35}{3}$, $\quad y_2 = -7$

$(-\frac{2}{3}, -7, \frac{35}{3})$ is feasible for (D) so $\bar{x}$ is optimal.

3. a.

$$\text{maximize}\ x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} \quad \text{s.t.}\ \sum_{i=1}^n x_i = 1,\ x \geq 0$$

(P) $\Leftrightarrow$

$$-\text{minimize}\ -x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$$
$$\text{s.t.}\quad e^Tx=1 \qquad e=[1,1,\cdots 1]$$
$$e_i^Tx \geq 0 \qquad i=1,2,\cdots n$$

$\Rightarrow -e_i^Tx \leq 0$.

KKT condition:

$$\nabla f(\bar{x}) + \sum_{i=1}^n v_i \nabla g_i(\bar{x}) + w\nabla h(\bar{x})$$
$$= \begin{bmatrix} -a_1\bar{x}_1^{a_1-1}\bar{x}_2^{a_2}\cdots\bar{x}_n^{a_n} \\ -a_2\bar{x}_1^{a_1}\bar{x}_2^{a_2-1}\cdots\bar{x}_n^{a_n} \\ \vdots \\ -a_n\bar{x}_1^{a_1}\bar{x}_2^{a_2}\cdots\bar{x}_n^{a_n-1} \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + w\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0 \qquad v_i \geq 0 \ \text{(dual feasibility)}$$

$v_i g_i(\bar{x}) = v_i(-\bar{x}_i) = v_i\bar{x}_i = 0$ (Complementery Slackness)

$\bar{x} \geq 0 \quad \sum_{i=1}^{n} \bar{x}_i = 1$ (primal feasibility)

Since this is a linear constrained optimization problem, Necessary for optimal.

6. $\because v_i\bar{x}_i = 0 \quad v_i \geq 0$

$\therefore$ if $v_i > 0$, $\bar{x}_i = 0$

the optimal value for P is 0.

However, it's easily to find $\bar{x} = (\frac{1}{n}, \frac{1}{n}, \cdots \frac{1}{n})$. $\bar{x}_i = \frac{1}{n} > 0$

to make the optimal value > 0. In this way, $v_i = 0$.

$\because v_i = 0 \quad \therefore \nabla f(\bar{x}) + w\nabla h(\bar{x}) = 0$

$$\begin{bmatrix} -a_1 x_1^{a_1-1} x_2^{a_2} \cdots x_n^{a_n} \\ -a_2 x_1^{a_1} x_2^{a_2-1} \cdots x_n^{a_n} \\ \vdots \\ -a_n x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n-1} \end{bmatrix} + w \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow \quad a_1 x_1^{a_1-1} x_2^{a_2} \cdots x_n^{a_n} = a_2 x_1^{a_1} x_2^{a_2-1} \cdots x_n^{a_n} = \cdots = a_n x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n-1}$$

$$\Rightarrow \quad \begin{aligned} a_1 x_2 &= a_2 x_1 \\ a_1 x_3 &= a_3 x_1 \\ &\vdots \\ a_1 x_n &= a_n x_1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_2 &= \frac{a_2}{a_1} x_1 \\ x_3 &= \frac{a_3}{a_1} x_1 \\ &\vdots \\ x_n &= \frac{a_n}{a_1} x_1 \\ \sum_{i=1}^{n} x_i &= 1 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 + \frac{a_2}{a_1}x_1 + \cdots + \frac{a_n}{a_1}x_1 &= 1 \\ \frac{a_1+a_2+\cdots+a_n}{a_1} x_1 &= 1 \\ x_1 &= \frac{a_1}{a_1+a_2+\cdots+a_n} \\ x_2 &= \frac{a_2}{a_1+a_2+\cdots+a_n} \\ &\vdots \end{aligned}$$