1. Consider the following linear program with homogeneous linear equality constraints:

$$\begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = 0 \\ & x \geq 0. \end{array}$$

(a) (10 points) Prove that the optimal objective value z^* of this linear program is either $-\infty$ or 0.

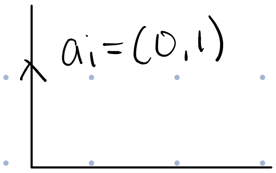
If $z^* > -\infty$, then consider the dual

$$\begin{array}{ll} \max & 0 \\ \text{s.t.} & A^T y \leq c \end{array}$$

By Strong Duality Theorem, $z^* = 0$.

b) (1) Let $c = (-1, 0)$

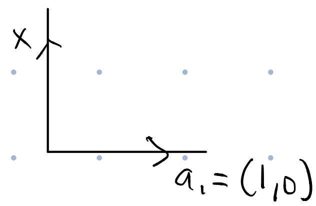
$a_1 = (0, 1)$



$x = (d, 0)$ is a feasible solution for all $d > 0$

(2) Let $c = (-1, 0)$

$a_1 = (1, 0)$



$x = (0, d)$ is feasible for all $d > 0$,

but $c^T x = 0$.

2. Consider the following convex optimization problem:

$$(P) \quad \min_{x \in \Re^n} \|Ax - b\|_1,$$

where $A \in \Re^{m \times n}$, $b \in \Re^m$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$.

a) Optimal Conditions:

By handout 1, (P) is equaivalent to

$$\begin{array}{ll} \min & \sum_{i=1}^m z_i = e^T z \\ \text{s.t.} & Ax - b \leq z \\ & b - Ax \leq z \\ & x \in \mathbb{R}^n \end{array}$$

The KKT condition is.

$$\begin{array}{c} x \\ z \end{array} \quad \underbrace{\begin{bmatrix} 0 \\ e \end{bmatrix}}_{\nabla f} + \begin{bmatrix} A^T \\ -I \end{bmatrix} v + \begin{bmatrix} -A^T \\ -I \end{bmatrix} w = \begin{bmatrix} A^T(v-w) \\ e-v-w \end{bmatrix} = 0$$

$$v \geq 0, \quad w \geq 0$$

$$v_i((Ax)_i - b_i - z_i) = 0$$

$$w_i(b_i - (Ax)_i - z_i) = 0$$

$$Ax - b \leq z$$

$$b - Ax \leq z$$

b) Find the dual.

$$(P') \quad \begin{array}{ll} \min & (0,e)^T(x,z) \\ \text{s.t.} & Ax - z \leq b \\ & -Ax - z \leq -b \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \max & b^Tv - b^Tw \\ \text{s.t.} & A^T(v-w) = 0 \\ & v \leq 0 \\ & w \leq 0 \end{array}$$

(directly from rule).

By Langrange function,

$$L(x,z,v,w) = e^Tz + v^T(Ax-b-z) + w^T(b-Ax-z)$$

$$= (e-v-w)^Tz + (A^Tv - A^Tw)^Tx + w^Tb - v^Tb$$

$$g(v,w) = \inf_{\substack{x \in \mathbb{R}^n \\ z \in \mathbb{R}^m}} L(x,z,v,w) = \begin{cases} w^Tb - v^Tb & \text{if } \begin{array}{l} Av - Aw = 0 \\ e - v - w = 0 \end{array} \\ -\infty & \text{otherwise.} \end{cases}$$

So, the dual is

$$(D) \quad \begin{array}{ll} \max & b^Tw - b^Tv \\ \text{s.t.} & A^Tv - A^Tw = 0 \\ & e - v - w = 0 \\ & v \geq 0 \\ & w \geq 0 \end{array}$$

(same as above).

c). Prove that weak duality always holds for problem (P).

For (D), consider the solution $(\bar{v}, \bar{w})$ with $\bar{v}=\bar{w}$.

Then, $e=\bar{v}+\bar{w}=2\bar{v} \Rightarrow \bar{v}=\frac{1}{2}e$. The objective value is 0.

Thus, the weak duality holds as.

$$0 \leq \|Ax-b\|_1 .$$