1. Let $c \in \mathbb{R}^n \setminus \{0\}$ and $A \in \mathbb{R}^{m \times n}$ be given. Suppose that A has full row rank. Consider the following problem:

$$v^* = \min \ c^T x \quad \text{s.t.} \quad Ax = 0, \quad \|x\|_2^2 \le 1.$$

a) **(P:15pts, S:15pts).** Write down the KKT conditions for Problem (1) and explain why they are necessary and sufficient for optimality.

$$c + 2ux + A^T w = 0 \quad \checkmark$$
$$Ax = 0$$
$$x^T x \le 1 \quad \checkmark$$
$$u(x^T x - 1) = 0. \quad \checkmark$$
$$u \ge 0 \quad \checkmark$$

Since all the rows in A are independent, and we have $Ax=0$. That implies $\frac{d}{dx}(x^Tx-1) = x$ is independent to the rows of A, so it satisfy the linear independence CQ. c^Tx is convex, thus its sufficient too.

b) **(P:15pts, S:15pts).** Using the result in (a), or otherwise, express the optimal solution x^* to Problem (1) in terms of A and c.

If $u=0$, $A^T w = -c$, then all feasible x is optimal.

If $u>0$, $x^T x = 1$

$$w = -(AA^T)^{-1} A c$$

$$c + 2ux - A(AA^T)^{-1} A c = 0.$$