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1、 let $g(x) = f(x) - \frac{\mu}{2}\|x\|^2$, $g(x)$ is convex, then

$$\begin{cases} g(y) \geq g(x) + [\nabla g(x)]^T (y-x) \\ g(x) \geq g(y) + [\nabla g(y)]^T (x-y) \end{cases}$$

$$\Leftrightarrow \begin{cases} f(y) - f(x) \geq \frac{\mu}{2}(\|y\|^2 - \|x\|^2) + [\nabla f(x)]^T (y-x) - \mu x^T y + \mu \|x\|^2 & ① \\ f(x) - f(y) \geq \frac{\mu}{2}(\|x\|^2 - \|y\|^2) + [\nabla f(y)]^T (x-y) - \mu y^T x + \mu \|y\|^2 & ② \end{cases}$$

by ①+②, we have $[\nabla f(x) - \nabla f(y)]^T (x-y) \geq \mu(\|x\|^2 + \|y\|^2 - x^T y - y^T x)$
$= \mu \|x-y\|^2$.

Thus, the inequality holds. $\forall x, y \in \mathbb{R}^n$. □

2、(a) $\begin{cases} x_1 - x_2 - x_3 \leq 1 \\ x_2 - x_4 \leq 2 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x_1 \leq 1 + x_2 + x_3 \leq 3 + x_3 + x_4 \\ x_2 \leq 2 + x_4 \\ x_{1,2,3,4} \geq 0 \end{cases}$

So the boundary points are $(3+x_3+x_4, 2+x_4, x_3, x_4)$. $\forall x_3, x_4 \geq 0$.

the extreme/corner point is $(3,2,0,0)$.

$\bar{x} = (3,2,0,0)$ is extreme point because $\bar{A}\bar{x} = \bar{b}$, where $\bar{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

$\bar{A} = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is full-ranked, i.e. there are 4 linearly independent active vectors at $\bar{x}$. ∴ it's a extreme point.

(b) the Linear Relaxation of the problem is

$$(LR) \begin{cases} \max & c^T x \\ \text{s.t.} & Ax \leq b \\ & x \geq 0 \end{cases}$$

, where $c^T = (1,2,3,-4)$, $A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

the dual of (LR) is: $(D) \begin{cases} \min & b^T y \\ \text{s.t.} & A^T y \geq c \\ & y \geq 0 \end{cases}$ i.e. $\begin{cases} \min & y_1 + 2y_2 \\ \text{s.t.} & \begin{cases} y_1 \geq 1 \\ y_2 \geq 2 + y_1 \\ y_1 \leq 3 \\ y_2 \leq 4 \end{cases} \end{cases}$

the optimal solution of (D) is
$\bar{y} = (1,3)$, $v_D^* = 7$

by duality thereom $\max c^T \bar{x} \leq \min b^T \bar{y} = 7$ ①

observe that $\bar{x} = (3,2,0,0)$ is feasible for (LR) as (a) discussed.
and the solution is integer.

Moreover, $c^T \bar{x} = 7$ satisfies ① and is optimal.

∴ an optimal solution is $\bar{x} = (3,2,0,0)$, optimal value is 7. □

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3.

(a) by definition, $F^*(y) = \sup_{\forall u \in \mathbb{R}} \{uy - F(u)\} \geq xy - F(x)$, $\forall x, y \in \mathbb{R}$.

$$\Rightarrow \quad xy \leq F(x) + F^*(y). \quad \forall x, y \in \mathbb{R}.$$

(b) ~~$f(0)=0$, $f(x)$ is increasing $\therefore f(x) \geq 0, \forall x \geq 0$; $f(x) < 0, \forall x < 0$.~~ ✓

let $b = f(a)$, then $g(b) = a$. $b=0$, then $a=0$, as f is $\uparrow$ and $f(0)=0$, $b=y$, $a=g(y)$

$$G(y) = \int_0^y g(b)\, db = \int_0^{g(y)} a \, d f(a)$$

$$= a f(a) \Big|_0^{g(y)} - \int_0^{g(y)} f(a)\, da$$

$$= xy - F(x), \quad \text{where } g(y) \triangleq x. \qquad (*).$$

note that $xy \leq F(x) + F^*(y)$ as (a) proved. ①

and $g = f^{-1}$, f is increasing, $\therefore \forall z$, $F(z) \geq F(x) + y(z-x)$, $y \in \partial f$

$$\Leftrightarrow yx - F(x) \geq yz - F(z) \Leftrightarrow yx - F(x) \geq F^*(y). \quad ②$$

by ①, ②, $xy - F(x) = F^*(y)$

so, $G(y) = F^*(y)$. □

~~6. $\forall x \in \mathbb{R}^n$, $z \in C = \{x \in \mathbb{R}^n : x^T e \geq 0\}$. let $z^* = \Pi_C(x) \in C$ be projection of x onto C.~~

~~then $\forall z \in C$. $(z - z^*)^T (x - z^*) \leq 0$.~~

~~if $x^T e \geq 0$. i.e. $x \in C$. then $z^* = x$.~~

~~if $x^T e < 0$,~~

6. Note that $C = \{x \in \mathbb{R}^n : e^T x \geq 0\}$ is a half-space with hyperplane $H(e, 0) = \{x : e^T x = 0\}$

$\forall x \in \mathbb{R}^n$, if $x^T e > 0$, i.e. $x \in \text{int}(C)$, then the projection is x itself. ①

if $x^T e \leq 0$, then the projection is on the hyperplane $H(e, 0)$,

donated as z^*, $z^* \in H(e, 0) \Rightarrow e^T z^* = 0$ ②

$(x - z^*) \perp H(e, 0) \Rightarrow (x - z^*) = \alpha e$ ③

combine ② ③, $e^T(x - \alpha e) = 0 \Rightarrow \alpha = \frac{1}{n} x^T e$

$\therefore z^* = x - \alpha e = x - \frac{1}{n}(x^T e) e$ ④

with ① ④, we have the projection

$$z^*(x) = \begin{cases} x, & \text{if } x^T e > 0 \\ x - \frac{1}{n}(x^T e) e, & \text{if } x^T e \leq 0. \end{cases}$$

which is the same as $P(x)$. □.

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7. ~~(a)~~

(a). $f(x)$ is convex. $\therefore \forall x \in \mathbb{R}^n$, $f(x) \geq f(\bar{x}) + [\partial f(\bar{x})]^T (x - \bar{x})$

$\because 0 \in \partial f(\bar{x})$ for $\bar{x}$, $\therefore \forall x$, $f(x) \geq f(\bar{x}) + 0^T (x - \bar{x}) = f(\bar{x})$.

$\therefore f(\bar{x}) = \min_{x \in \mathbb{R}^n} f(x)$, $\bar{x}$ is a global optimal solution. $\square$

(b) Let $f(x) = \frac{1}{2} \| x - y \|_2^2 + \lambda \|x\|_1$, f is convex, non-differentiable.

$$\partial f(x) = \begin{cases} x - y + \lambda e, & x > 0 \\ x - y + 0, & x = 0 \\ x - y - \lambda e, & x < 0 \end{cases}$$

as (a) discussed, let $\partial f(\bar{x}) = 0$, then $\bar{x} = \begin{cases} y - \lambda e, & \text{if } y > \lambda e \\ y, & \text{if } y = 0 \\ y + \lambda e, & \text{if } y < -\lambda e. \end{cases}$

is optimal for $\min_{x \in \mathbb{R}^n} \{ \frac{1}{2} \|x - y\|_2^2 + \lambda \| x \|_1 \}$. $\square$