Question 1.

Suppose that n different crops (e.g., corn, wheat, etc.) are to be grown on m plots of land with areas of $a_1, a_2, \ldots, a_m$ acres. Further, suppose that the expected yield of the j-th crop when planted on the i-th plot is $g_{i,j}$ tons per acre ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$), and that a ton of the j-th crop can be sold for p_j dollars ($j = 1, 2, \ldots, n$).

- (5pts) How much area in each plot should sown in each crop so as to maximize the expected revenue, subject to the additional constraint that at least b_j tons of the j-th crop ($j = 1, 2, \ldots, n$) is produced (in expectation)? Formulate this problem as a linear program.
- (10pts) Suppose that instead of the question asked above, you are asked: how much area in each plot should be sown in each crop so as to maximize the expected yield of all of the crops while ensuring that the ratios of the expected yields of the n crops are $k_1 : k_2 : \cdots : k_n$? Formulate this problem as a linear program.

(1) Define x_{ij} as the acres of crop j planted on plot i

$$\max \sum_{j=1}^{n} \sum_{i=1}^{m} p_j \cdot g_{ij} \cdot x_{ij}$$

s.t. $\sum_{i=1}^{m} x_{ij} \le a_i$

$\sum_{j=1}^{n} g_{ij} x_{ij} \ge b_j$

$x_{ij} \ge 0$

(2) $$\max \sum_{j=1}^{n} \sum_{i=1}^{m} g_{ij} x_{ij}$$

s.t. $\sum_{j=1}^{n} x_{ij} \le a_i$

$$\frac{\sum_{j=1}^{n} g_{ij} x_{ij}}{\sum_{l=1}^{n} g_{il} x_{il}} = \frac{k_j}{k_l} \quad \text{for } \forall i, l \in \{0, 1, \ldots, m\}$$

$x_{ij} \ge 0$

Question 2.

(a) (15pts) Prove that the standard form LP polyhedron $P \equiv \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, is bounded if and only if there exists a vector $u \in \mathbb{R}^m$ such that $A^\top u > 0$.

(b) (15pts) Now consider the nonlinear programming problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & c_i(x) \geq 0, \ \text{for all } i \in \mathcal{I}, \end{array}$$

where all functions $c_i(x)$, for all $i \in \mathcal{I}$ are continuously differentiable. Suppose that the Mangasarian-Fromovitz constraint qualification (MFCQ) holds at the point $\bar{x}$; i.e., $c_i(\bar{x}) = 0$, for all $i \in \mathcal{A}(\bar{x}) \subseteq \mathcal{I}$ and there exists a vector $p \in \mathbb{R}^n$ such that $\nabla c_i(\bar{x})^\top p > 0$, for all $i \in \mathcal{A}(\bar{x})$. Prove that the only nonnegative linear combination of the vectors $\nabla c_i(\bar{x})$, for all $i \in \mathcal{A}(\bar{x})$ equal to zero is the zero linear combination.

(a) ($\Rightarrow$) $u^T A x = u^T b$

$\sum_{i=1}^{n} (A^T u)_i x_i = u^T b \qquad \because A^T u > 0 \quad \therefore \min_i (A^T u)_i > 0$

$\therefore \quad \min_i (A^T u)_i \cdot \sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} (A^T u)_i x_i = u^T b$

$\therefore \quad \sum_{i=1}^{n} x_i \leq \dfrac{u^T b}{\min_i (A^T u)_i}$

$\because \quad x \geq 0 \qquad \therefore \quad 0 \leq \sum_{i=1}^{n} x_i \leq \dfrac{u^T b}{\min_i (A^T u)_i} \qquad \therefore P$ is bounded

($\Leftarrow$) Suppose a LP Problem

(P) $\min -e^T x$ s.t. $Ax = b$, $x \geq 0$ and its (D) $\max b^T y$ s.t $A^T y \leq -e$

$\because$ $Ax=b$, $x \geq 0$ bounded, Strong Duality

$\therefore$ (D) has a optimal solution

$\therefore$ $A^T y \leq -e$ is feasible, set $u = -y$, $A^T u \geq e > 0$

(b) Suppose $\sum_{i \in A(\bar{x})} a_i \nabla C_i(\bar{x}) = 0$

We have $\left(\sum_{i \in A(\bar{x})} a_i \nabla C_i(\bar{x})\right)^T P = \sum_{i \in A(\bar{x})} a_i \nabla C_i(\bar{x})^T P = 0$

$\because$ $\nabla C_i(\bar{x})^T P > 0 \quad \therefore \quad a_i = 0$ for $i \in A(\bar{x})$

Question 3. (20pts)

Use Branch-and-Bound method to solve the following Integer Programming problem:

$$
\begin{aligned}
\max\ z = \ & 5x_1 + 2x_2 \\
\text{s.t.}\ \ & 3x_1 + x_2 \le 12 \quad ① \\
& x_1 + x_2 \le 5 \quad ② \\
& x_1, x_2 \ge 0; x_1, x_2 \text{ integer} .
\end{aligned}
$$

LP0: $x = (\frac{7}{2}, \frac{3}{2})$

$V = 20.5$

$x_1 \le 3$ | $x_1 \ge 4$

LP1: $x = (3,2)$

$z^* = 19$

fathomed

LP2: $x = (4,0)$

$z^* = 20$

fathomed

$\therefore \max z = 20$

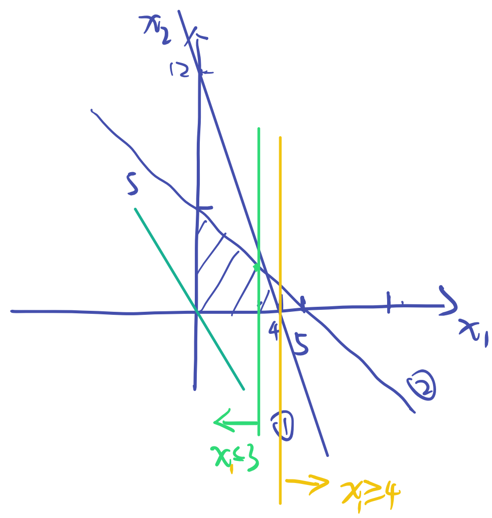
For LP0:

$$\begin{cases} 3x_1 + x_2 = 12 \\ x_1 + x_2 = 5 \end{cases}$$

$x = (\frac{7}{2}, \frac{3}{2})$

For LP1:

$$\begin{cases} x_1 = 3 \\ x_1 + x_2 = 5 \end{cases}$$



Question 4. (15pts)
Prove that a function f is convex if and only if its epigraph is a convex set.

$epi(f) \triangleq \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq t\} \subseteq \mathbb{R}^{n+1}$

$(\Rightarrow)$ Suppose $(x_1, t_1), (x_2, t_2) \in epi(f)$, for $\forall x_1, x_2 \in \mathbb{R}^n$

i.e. $f(x_1) \leq t_1$, $f(x_2) \leq t_2$

for $\forall \alpha \in [0,1]$, $\alpha f(x_1) \leq \alpha t_1$, $(1-\alpha) f(x_2) \leq (1-\alpha) t_2$

$\alpha f(x_1) + (1-\alpha) f(x_2) \leq \alpha t_1 + (1-\alpha) t_2$

$\because$ f is convex

$\therefore f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \leq \alpha t_1 + (1-\alpha) t_2$

i.e. $\alpha (x_1, t_1) + (1-\alpha)(x_2, t_2)$

$= (\alpha x_1 + (1-\alpha) x_2, \alpha t_1 + (1-\alpha) t_2) \in epi(f)$

i.e. $epi(f)$ is convex

$\Leftarrow$ $\because$ $epi(f)$ is convex

$\therefore$ for $(x_1, t_1), (x_2, t_2) \in epi(f)$, $\forall x_1, x_2 \in \mathbb{R}^n$

$(\alpha x_1 + (1-\alpha) x_2, \alpha t_1 + (1-\alpha) t_2) \in epi(f)$ for $\alpha \in [0,1]$

Set $t_1 = f(x_1)$ $t_2 = f(x_2)$

$(\alpha x_1 + (1-\alpha) x_2, \alpha f(x_1) + (1-\alpha) f(x_2)) \in epi(f)$

i.e. $f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

$\therefore$ f is convex