WANG Disong 1155113946

第 1 題
(答題不得寫在紅綫外) 第 頁

let $g(x) = f(x) - \frac{\mu}{2}\|x\|^2$. we can see $g(x)$ is convex and differentiable function.

$\Rightarrow \forall x, y \in R^n$. $g(x) \geq g(y) + \nabla g^T(y)(x-y)$ ①

$g(y) \geq g(x) + \nabla g^T(x)(y-x)$ ②

① + ② $\Rightarrow (\nabla g^T(x) - \nabla g^T(y))(x-y) \geq 0$

$\Downarrow$

$(\nabla f^T(x) - \mu x^T - \nabla f^T(y) + \mu y^T)(x-y) \geq 0$

$\Downarrow$

$(\nabla f^T(x) - \nabla f^T(y))(x-y) \geq \mu (x-y)^T(x-y)$

$\| \quad \mu\|x-y\|^2$.

$\Rightarrow \langle \nabla f(x) - \nabla f(y) \mid x-y \rangle \geq \mu\|x-y\|^2 \quad \forall x, y \in R^n$.

WANG Disong 1155113946

第 2 題
（答題不得寫在紅綫外） 第 頁

(a) To find extreme points, we firstly need to find 4 linearly independent constraints, the extreme points are those that are active at these constraints and belong to X. X can be reformulated as:

$$X = \{ x \in \mathbb{R}^4 \mid Ax \le b \}$$

where $A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Through verification, we can find the rows of following matrices $\{A_i\}$ are linearly independent and derive feasible solutions:

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_1} x = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}}_{b_1} \qquad \underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_2} x = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_2}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_3} x = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_3} \qquad \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{A_4} x = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{b_4}$$

$\Rightarrow$ extreme points: $(3,2,0,0)$
$(1,0,0,0)$
$(0,2,0,0)$
$(0,0,0,0)$

WANG Disong 1155113946

第 2 題
（答題不得寫在紅綫外）　　第 頁

(b) The LIP is equivalent to

$$\min \ -x_1-2x_2+3x_3+4x_4$$
$$\text{s.t } x \in X$$
$$x \in \mathbb{Z}$$

its LP relaxation is $\min \ -x_1-2x_2+3x_3+4x_4$ s.t $x \in X$

From (a), we know X has 4 extreme points.

∴ LP relaxation problem either is unbounded below, or there exists an extreme point that is optimal.

From the constraints of X, we observe $-x_1+x_2+x_3 \geq -1$ ①

$-3x_2+3x_4 \geq -6$ ②

①+② $\Rightarrow -x_1-2x_2+x_3+3x_4 \geq -7$. ∵ $x_1, x_2, x_3, x_4 \geq 0$

$\Rightarrow -x_1-2x_2+3x_3+4x_4 \geq -7$.

which means $-x_1-2x_2+3x_3+4x_4$ can not be unbounded below.

∴ there exists an extreme point that is optimal.

From (a), it can be easily verified that (3,2,0,0) is the minimizer.

∵ (3,2,0,0) is also the integer solution.

(3,2,0,0) is also optimal to the original LIP.

∴ one optimal solution to LIP: (3,2,0,0).

WANG Disong 1155113946

第 3 题 第 页
（答題不得寫在紅綫外）

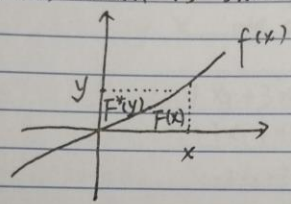
(a) From the definition of $F^*(y)$.

we can see for $\forall x, y \in R$.

$$xy - F(x) \le \sup_u \{uy - F(u)\} = F^*(y)$$

$$\Rightarrow \quad xy \le F(x) + F^*(y)$$

(b) The graphical illustration is shown below



we can see that $F(x)$ is the area under $f(x)$ from $0 \to x$

$\because$ $xy \le F(x) + F^*(y)$. xy is the area of rectangle by $\{(0,0), (x,0), (0,y), (x,y)\}$

$\therefore$ $F^*(y)$ can be the area above $f(x)$ from $0 \to y$.

and $F(x) + F^*(y) = xy$ iff ~~when~~ $y = f(x)$.

i.e. $G(y) = \int_0^y g(b)\, db = F^*(y)$.

WANG Pisong 1155113946

第 6 题
(答題不得寫在紅綫外)

第 頁

For $\forall x \in R^n$.

① we can see when $x^Te > 0$. x is in C.

and $P(x) = x$.

② when $x^Te \leq 0$. $P(x)^Te = x^Te - \frac{1}{n}(x^Te)e^Te$

$$= x^Te - \frac{1}{n} \cdot n \cdot (x^Te)$$

$$= 0$$

in either cases. $P(x)^Te \geq 0$. $\Rightarrow P(x) \in C$.

$\therefore$ $P(x)$ is the projection of x into C. for $\forall x \in R^n$

WANG Disong 1155113946

(a) $\because 0 \in \partial f(\bar{x})$

$\Rightarrow \forall y \in R^n. \quad f(y) \geq f(\bar{x}) + 0^T(y-x)$

$\Rightarrow f(y) \geq f(\bar{x})$.

$\Rightarrow \bar{x}$ is a global minimizer of $\min_{x \in R^n} f(x)$.

(b). let $f_1(x) = \frac{1}{2}\|x-y\|_2^2 \quad f_2(x) = \lambda\|x\|_1. \quad f(x) = f_1(x) + f_2(x)$

it can be easily verified that $f_1(x)$ and $f_2(x)$ are convex functions. and $f_1(x)$. $f_2(x)$ are not identically $+\infty$. Besides, $f_1(x)$ is a continuous function at any $x \in R^n$.

$\Rightarrow \quad \partial f(x) = \partial f_1(x) + \partial f_2(x)$.

where $\partial f_1(x) = \{\nabla f_1(x)\} = \{x-y\}$

$\partial f_2(x) = \{g \mid g_i \in [-\lambda, \lambda] \text{ if } x_i = 0,\ g_i = \lambda \text{ if } x_i > 0,\ g_i = -\lambda \text{ if } x_i < 0\}$

From (a), we need to find $\bar{x}$. s.t. $0 \in \partial f(\bar{x})$

let $\bar{x} - y + g = 0$

$$\Rightarrow \bar{x}_i = \begin{cases} y_i - \lambda, & \text{if } y_i > \lambda \\ 0, & \text{if } -\lambda \leq y_i \leq \lambda \\ y_i + \lambda, & \text{if } y < -\lambda. \end{cases}$$

$i = 1, 2, \cdots, n$. $\bar{x}_i$ is ith element of $\bar{x}$.