The Chinese University of Hong Kong

Department of Systems Engineering & Engineering Management

Ph.D. Qualifying Examination 2012

Area: Operations Research

INSTRUCTIONS

1. Answer ALL questions if your primary area is Operations Research.
2. Answer only those questions that are marked with an asterisk (*) if your secondary area is Operations Research.
3. The score of each question is denoted by $(P : \cdot, S : \cdot)$, where P is the primary area score and S is the secondary area score.

Problem 1* (P:20pts, S:25pts). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ be given. Suppose that A has full column rank. Consider the following problem:

$$\begin{array}{lll} \text{minimize} & \|Ax - \text{diag}(u)b\|_2^2 & \\ \text{subject to} & |u_i| = 1 & \text{for } i = 1, \ldots, m, \\ & x \in \mathbb{R}^n, & \end{array} \tag{1}$$

where $\text{diag}(u)$ is the $m \times m$ diagonal matrix whose diagonal entries are given by u.

(a) **(P:15pts, S:15pts).** Suppose that the vector u is given. Then, Problem (1) becomes

$$\min_{x \in \mathbb{R}^n} \|Ax - \text{diag}(u)b\|_2^2. \tag{2}$$

Write down the KKT conditions for Problem (2) and explain why they are necessary and sufficient for optimality. Hence, or otherwise, express the optimal solution x^* in terms of A, u and b.

(b) **(P:5pts, S:10pts).** Using the result in (a), show that Problem (1) can be reformulated as

$$\begin{array}{lll} \text{minimize} & u^T M u & \\ \text{subject to} & |u_i| = 1 & \text{for } i = 1, \ldots, m \end{array}$$

for some suitable $m \times m$ matrix M.

Problem 2* (P:25pts, S:35pts). Consider the linear program

$$\begin{array}{ll} \text{minimize} & 2x_1 + 9x_2 + 3x_3 \\ \text{subject to} & -2x_1 + 2x_2 + x_3 \geq 1, \\ & x_1 + 4x_2 - x_3 \geq 1, \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{3}$$

(a) **(P:15pts, S:20pts).** Write down the dual of Problem (3) and solve it graphically.

(b) **(P:10pts, S:15pts).** Using the result in (a), or otherwise, determine the optimal solution to Problem (3).

Problem 3* (P:15pts, S:20pts). Let A_i be an $n_{i-1} \times n_i$ matrix, for $i = 1, \ldots, n$. We are interested in the cheapest way to evaluate the product

$$A = A_1 \times A_2 \times \cdots \times A_m,$$

where the cost of multiplying a $p \times q$ matrix by a $q \times r$ matrix is defined to be pqr. Give a dynamic programming algorithm to solve this problem and analyze its runtime.

Problem 4* (P:15pts, S:20pts). If n balls are placed at random into n bins, what is the probability that exactly one bin remains empty? Show your calculations.

Problem 5 (25pts). Consider a Markov chain with state space $\mathcal{S} = \{0, 1, \ldots\}$ and transition probabilities given by

$$p_{0,2} = 1, \quad p_{i,i+2} = p_{i,i-1} = \frac{1}{2} \qquad \text{for } i = 1, 2, \ldots.$$

(a) **(10pts).** Determine the period of state 2.

(b) **(15pts).** Show that every state of the chain is transient.

The Chinese University of Hong Kong

Department of Systems Engineering & Engineering Management
Ph.D. Qualifying Examination 2013
Area: Operations Research

INSTRUCTIONS

1. Answer ALL questions.
2. The score of each question is denoted by $(P:\cdot, S:\cdot)$, where P is the primary area score and S is the secondary area score.

Problem 1 (P:30pts, S:30pts). Let $c \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $A \in \mathbb{R}^{m \times n}$ be given. Suppose that A has full row rank. Consider the following problem:

$$\begin{aligned} v^* \quad = \quad \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = \mathbf{0}, \\ & \|x\|_2^2 \le 1. \end{aligned} \tag{1}$$

(a) **(P:15pts, S:15pts).** Write down the KKT conditions for Problem (1) and explain why they are necessary and sufficient for optimality.

(b) **(P:15pts, S:15pts).** Using the result in (a), or otherwise, express the optimal solution x^* to Problem (1) in terms of A and c.

maximize $\sum_{i=1}^n y_i$
subjectto $A(z+y)=0$
$y_i \le 1, \forall i,$
$z, y \ge 0$

Problem 2 (P:20pts, S:20pts). Let $A \in \mathbb{R}^{m \times n}$ be given. We are interested in finding a vector $x \in \mathbb{R}_+^n$ such that $Ax = \mathbf{0}$ and the number of positive components of x is maximized. Formulate this problem as a linear program. Justify your answer.

Problem 3 (P:25pts, S:25pts). Suppose that we are given n items $\{1, \ldots, n\}$, where the value and size of the i–th item is v_i and w_i, respectively. We are also given a bag of size W, where $W \ge 1$ is an integer. Our goal is to choose a subset $T \subset \{1, \ldots, n\}$ of the items to put into the bag, so that the total size of the selected items, which is defined as $\sum_{i \in T} w_i$, is at most W, and the total value of the selected items, which is defined as $\sum_{i \in T} v_i$, is maximized. Formulate this problem as a dynamic program. Justify your answer.

Problem 4 (P:25pts, S:25pts). Suppose that we have two boxes and $2d$ balls, of which d are black and d are red. Initially, d of the balls are placed in box 1, and the rest of the balls are placed in box 2. At each trial a ball is chosen at random from each of the boxes, and the two balls are then put back in the opposite boxes. Let X_n be the number of black balls in box 1 after n trials, where $n = 0, 1, \ldots$. Then, the process $\mathcal{X} = \{X_n : n \ge 0\}$ forms a Markov chain on the state space $\mathcal{S} = \{0, 1, \ldots, d\}$.

(a) **(P:5pts, S:10pts).** Explain why the transition probabilities are given by

$$p_{i,i-1} = \frac{i^2}{d^2} \quad \text{for } i = 1, 2, \ldots, d; \qquad p_{i,i+1} = \frac{(d-i)^2}{d^2} \quad \text{for } i = 0, 1, \ldots, d-1;$$

$$p_{i,i} = \frac{2i(d-i)}{d^2} \quad \text{for } i = 0, 1, \ldots, d.$$

(b) **(P:20pts, S:15pts).** Using the result in (a), or otherwise, compute the stationary distribution of the Markov chain $\mathcal{X}$. *(Hint: It may be easier to first guess the stationary distribution and then verify that it indeed satisfies the stationarity conditions.)*

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management
PhD Qualifying Examination 2014
(Operations Research)

INSTRUCTIONS

- Answer ALL questions if your primary area is Operations Research.
- If your secondary area is Operations Research, do NOT answer those questions that are marked with an asterisk *
- The score of each question is denoted by $(P : \cdot; S : \cdot)$, where P is the primary area score and S is the secondary area score.

Question 1. (P:20pts; S:20pts)

Consider the following linear program which depends on a parameter θ:

$$\begin{array}{ll} \min & z = x_1 + 2x_2 - \theta(x_1 + x_2) \\ \text{s.t.} & x_1 + x_2 \leq \theta \\ & x_1 \geq \theta \\ & x_2 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

(a) **(P:5pts; S:10pts)** Give the dual of this linear program.

(b) **(P:10pts; S:10pts)** Give the optimal value of the objective function of this linear program, $z = z(\theta)$ and the optimal solution $x(\theta)$ as a function of θ for all nonnegative real values of θ. (Hints: it may be helpful to graph this problem).

(c) * **(P:5pts)** For θ nonnegative, show that or explain why the optimal dual solution is not unique.

Question 2. (P:20pts; S:20pts)

Suppose you have \$3000 to invest. Investments A, B and C have expected rates of return 4%, 7%, and 8%, respectively, and each can be bought in multiples of \$1000. Moreover, there is a fixed commission of \$30, \$70, and \$100 to make purchases of A, B and C, respectively, no matter how many \$1000 multiples of the investment is purchased. Give a Dynamic Programming (DP) formulation for solving the problem of how much you should invest in A, B and C to maximize the net return at the end of one year. Specifically, define (i) the stages in the DP recursion, (ii) the possible states at each stage, (iii) the optimal value function corresponding to each state at each stage, and (iv) give the DP recursion. For the latter, you will need to clearly define net returns for various investments depending on the state.

Question 3. **(P:20pts; S:20pts)**
Consider the following problem

$$\begin{array}{ll} \min_x & c^\top x \\ \text{s.t.} & Ax = b, \\ & \|D^{-1}(x - z)\| \leq \beta, \end{array}$$

where $A \in \Re^{m \times n}$ has full row rank, $z \in \Re^n$ is a given vector with all entries being positive, D is a diagonal matrix with positive diagonal elements $D_{ii} := z_i, i = 1, \ldots, n$, and $\beta \in (0, 1)$.

(a) **(P:10pts; S:20pts)** Give the Karush-Kuhn-Tucker optimality conditions for this problem.

(b) * **(P:10pts)** From the optimality conditions express the optimal solution x^* as $x^* = z + p$; i.e., what is p?

Question 4. **(P:20pts; S:20pts)**
Consider the following integer nonlinear programming problem:

$$\begin{array}{ll} \max & 2x_1^2 - x_1^3 + 5x_2^2 - 3x_2^4 \\ \text{s.t.} & x_1 + x_2 \leq 3, \\ & x_1 \geq 0, x_2 \geq 0, \\ & \text{both } x_1 \text{ and } x_2 \text{ integer} \end{array}$$

Formulate this problem as a Binary Integer Programming problem.

Question 5. **(P:20pts; S:20pts)**
Let $X = \{X_n : n = 0, 1, \ldots\}$ be a Discrete Time Markov Chain on state space $\{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$P = \begin{pmatrix} .25 & .75 & 0 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2 & .25 & .25 & .3 & 0 & 0 \\ .25 & 0 & .4 & .25 & .1 & 0 & 0 \\ .25 & .15 & .1 & .2 & .3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) **(P:10pts; S:20pts)** Find $\mathbb{E}_i(\tau_i)$, where $\tau_i = \inf\{n \geq 1 : X_n = i\}$, for each $i = 1, \ldots, 7$. (You do not need to provide a proof.)

(b) * **(P:10pts)** For each pair (i, j) with $1 \leq i, j \leq 7$, find $\lim_{n \to \infty} P_{i,j}^n$ when the limit exists. Otherwise, state it does not exist.

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2015

(Operations Research)

INSTRUCTIONS

- You should attempt all questions no matter Operations Research is your primary or secondary area.

Question 1. (20pts)

Consider the following linear program with homogeneous linear equality constraints:

$$\begin{array}{ll} \min & c^{\mathrm{T}} x \\ \text{s.t.} & Ax = 0 \\ & x \geq 0. \end{array}$$

(a) (10 points) Prove that the optimal objective value z^* of this linear program is either $-\infty$ or 0.

(b) (10 points) Draw examples that illustrate both cases in part (a) in $\Re^2$ where A is a 1×2 matrix (i.e., a row vector and there is only one equation) and $c \neq 0$ and $A \neq 0$.

Question 2. (20pts)

Consider the following convex optimization problem:

$$(P) \quad \min_{x \in \Re^n} \|Ax - b\|_1,$$

where $A \in \Re^{m \times n}$, $b \in \Re^m$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$.

(a) (5pts) Write down the optimality conditions of problem (P).

(b) (10pts) Write down the dual problem of (P).

(c) (5pts) Prove that the weak duality always holds for problem (P).

Question 3. (20pts)

A company produces a variety of bathroom accessories, including decorative towel rods and shower curtain rods. Each of the accessories includes a rod made out of stainless steel. However, many different lengths are needed: 12″, 18″, 24″, 40″, and 60″. The company purchases 60″ rods from an outside supplier and then cuts the rods as needed for their products. Each 60″ rod can be used to make a number of smaller rods. For example, a 60″ rod could be used to make a 40″ and an 18″ rod (with 2″ of waste), or

5 12″ rods (with no waste). For the next production period, the company needs 25 12″ rods, 52 18″ rods, 45 24″ rods, 30 40″ rods, and 12 60″ rods. What is the fewest number of 60″ rods that can be purchased to meet their production needs? Formulate an integer programming problem for this problem. (You do not need to solve it).

Question 4. (20pts)

Consider the following continuous knapsack problem. The goal is to maximize the value of N objects packed into a knapsack with limited capacity b. For $i = 1, \ldots, N$, let x_i denote the number of units of object i packed, $f_i(x_i)$ the resulting value and $g_i(x_i)$ the capacity of the knapsack consumed. Suppose that each x_i is a continuous variable, i.e., it can take any value in $[0, \infty)$.

(a) (10pts) Formulate this problem as a dynamic programming problem.

(b) (10pts) Suppose $f_i(x) = f(x)$, $i = 1, \ldots, N$, i.e., the value function is the same for all the N objects, and $g_i(x) = x$, $i = 1, \ldots, N$. In addition, assume that f is differentiable, strictly increasing and concave on $[0, \infty)$, so that $f'(x)$ is decreasing in x. Find the optimal solution of this dynamic programming problem.

Question 5. (20pts; 4pts each)

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables. Suppose

$$P(\xi_1 = 1) = 0.4 \quad \text{and} \quad P(\xi_1 = -1) = 0.6.$$

Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$ for $n \geq 1$. In addition, let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda > 0$, which is independent of the sequences $\xi_1, \xi_2, \ldots$.

(a) Is $\{|S_n| : n \geq 0\}$ a discrete time Markov Chain? Briefly explain why or why not.

(b) Is $\{S_n : n \geq 1\}$ a Martingale with respect to the sequence $\xi_1, \xi_2, \ldots$? Briefly explain why or why not.

(c) Is $\{S_{N(t)} : t \geq 0\}$ a continuous time Markov Chain? Briefly explain why or why not.

(d) For fixed constant $T > 0$, find the optimal solutions for the following problem:

$$\max_{q \geq 0, q \in \mathbb{R}} \left(\mathbb{E}\left[\sum_{i=1}^{N(T)} (\xi_i \cdot 1_{\xi_i \geq q}) \right] - 0.5 \cdot \mathbb{E}\left[\sum_{i=1}^{N(T)} 1_{\xi_i \geq q} \right] \right). \tag{1}$$

Here $1_{\xi_i \geq q}$ is the indicator function. It takes value 1 if $\xi_i \geq q$, and 0 otherwise.

(e) Re-solve the optimization problem (1) if for each i, ξ_i is a positive continuous random variable with an integrable density function $h(x)$, i.e., $\int_0^\infty x h(x) dx < \infty$.

End of Paper

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2016

(Operations Research)

INSTRUCTIONS

- You should attempt all questions no matter Operations Research is your primary or secondary area.

Question 1.

Suppose that n different crops (e.g., corn, wheat, etc.) are to be grown on m plots of land with areas of $a_1, a_2, \ldots, a_m$ acres. Further, suppose that the expected yield of the j-th crop when planted on the i-th plot is $g_{i,j}$ tons per acre ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$), and that a ton of the j-th crop can be sold for p_j dollars ($j = 1, 2, \ldots, n$).

- (5pts) How much area in each plot should sown in each crop so as to maximize the expected revenue, subject to the additional constraint that at least b_j tons of the j-th crop ($j = 1, 2, \ldots, n$) is produced (in expectation)? Formulate this problem as a linear program.

- (10pts) Suppose that instead of the question asked above, you are asked: how much area in each plot should be sown in each crop so as to maximize the expected yield of all of the crops while ensuring that the ratios of the expected yields of the n crops are $k_1 : k_2 : \cdots : k_n$? Formulate this problem as a linear program.

Question 2.

(a) (15pts) Prove that the standard form LP polyhedron $P \equiv \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, is bounded if and only if there exists a vector $u \in \mathbb{R}^m$ such that $A^\top u > 0$.

(b) (15pts) Now consider the nonlinear programming problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & c_i(x) \geq 0, \text{ for all } i \in \mathcal{I}, \end{array}$$

where all functions $c_i(x)$, for all $i \in \mathcal{I}$ are continuously differentiable. Suppose that the Mangasarian-Fromovitz constraint qualification (MFCQ) holds at the point $\bar{x}$; i.e., $c_i(\bar{x}) = 0$, for all $i \in \mathcal{A}(\bar{x}) \subseteq \mathcal{I}$ and there exists a vector $p \in \mathbb{R}^n$ such that $\nabla c_i(\bar{x})^\top p > 0$, for all $i \in \mathcal{A}(\bar{x})$. Prove that the only nonnegative linear combination of the vectors $\nabla c_i(\bar{x})$, for all $i \in \mathcal{A}(\bar{x})$ equal to zero is the zero linear combination.

Question 3. (20pts)

Use Branch-and-Bound method to solve the following Integer Programming problem:

$$\begin{aligned} \max\ z = &\ 5x_1 + 2x_2 \\ \text{s.t.}\quad &\ 3x_1 + x_2 \le 12 \\ &\ x_1 + x_2 \le 5 \\ &\ x_1, x_2 \ge 0; x_1, x_2 \text{ integer} . \end{aligned}$$

Question 4. (15pts)

Prove that a function f is convex if and only if its epigraph is a convex set.

Question 5. (20pts)

(a) (10 points; 5 points each) Let $\xi_0, \xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables. Suppose

$$P(\xi_0 = 1) = 0.5 \quad \text{and} \quad P(\xi_0 = -1) = 0.5.$$

Define $Y_n = \xi_n \cdot \xi_{n-1}$.

- Is the sequence $\{Y_n : n \ge 1\}$ a Markov chain? Briefly explain why or why not.
- Is the sequence $\{Y_n : n \ge 1\}$ a martingale? Briefly explain why or why not.

(b) (10 points) Consider two independent Poisson processes $\{N_1(t) : t \ge 0\}$ and $\{N_2(t) : t \ge 0\}$, each with rate $\lambda > 0$. Find the probability that the combined process $\{(N_1(t), N_2(t)) : t \ge 0\}$ will hit the point $(2, 2)$. That is, find the probability that at some time t, $N_1(t) = N_2(t) = 2$. (Hint: you can use symmetry to reduce the size of the system of equations.)

End of Paper

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2017

(Operations Research: Primary Area)

Question 1.

Let

$$A = \begin{pmatrix} 3 & -1 & -3 \\ -4 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(a) (8pts) Is the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax = b, x \geq 0\}$ empty or not? Justify your answer.

(b) (8pts) Is the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax \geq b, x \geq 0\}$ empty or not? Justify your answer.

(c) (8pts) Given any vector $c \in \mathbb{R}^3$, what can you say about the linear program

$$\max_y \ b^\top y, \ s.t., \ A^\top y \leq c.$$

Justify your answer.

Question 2.

Consider the integer programming problem

$$\begin{array}{ll} \max & x_1 + 2x_2 \\ s..t & -3x_1 + 4x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 11 \\ & 2x_1 - x_2 \leq 5 \\ & x_1, x_2 \text{ integer.} \end{array}$$

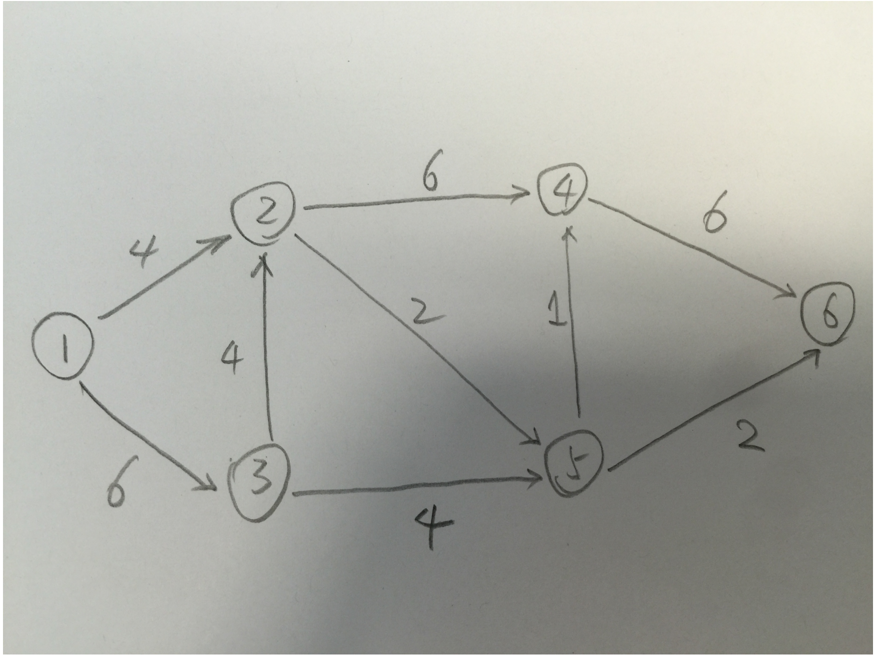
(a) (5pts) What is the optimal value of the linear programming relaxation? What is the optimal cost of the integer programming problem?

(b) (5pts) What is the convex hull of the set of all solutions to the integer programming problem?

(c) (10pts) Solve the problem by branch and bound. Give the branch and bound tree indicating for each node any bounds obtained and indicating when a node has been fathomed.

Question 3. (16pts)

Figure 1: Network for problem 3



Consider the network in Figure 1 with distances specified on the arcs. Let the "cost" of a path be the length of the longest edge in that path. For example, the cost of path 1-2-5-6 is 4; cost of path 2-5-4 is 2 etc. Design an efficient algorithm to find a minimum-cost path from node 1 to all other nodes.

Question 4.

Consider the problem of finding a circle of minimum radius that contains r points $y_1, \ldots, y_r$ in the plane, i.e., find x and z that minimize z subject to $\|x - y_j\| \leq z$ for all $j = 1, \ldots, r$, where x is the center of the circle under optimization.

(a) (10pts) Introduce Lagrange multipliers $\mu_j, j = 1, \ldots, r$, for the constraints, and show that the dual problem has an optimal solution and there is no duality gap.

(b) (10pts) Show that calculating the dual function at some $\mu \geq 0$ involves the computation of a Weber point of $y_1, \ldots, y_r$ with weights $\mu_1, \ldots, \mu_r$, i.e., the solution of the problem

$$\min_{x \in \mathbb{R}^2} \sum_{j=1}^{r} \mu_j \|x - y_j\|.$$

Question 5.

Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete time Markov chain with state space $\{0, 1, ..., 2K\}$ for some positive integer K with transition matrix P such that

$$P_{i,i-1} = P_{i,i+1} = 1/2, \quad \text{for } 1 \leq i \leq 2K - 1,$$
$$P_{0,K} = P_{2K,K} = 1.$$

(a) (10pts) Does $\{X_n : n \geq 1\}$ have a unique stationary distribution π? If yes, find the explicit expressions for $\pi = (\pi_i)$; If not, explain why.

(b) (10pts) Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda > 0$, independent of the process $\{X_n : n = 0, 1, \ldots\}$. Define for each $t \geq 0$,

$$Z(t) = X_{N(3t)}.$$

Is $\{Z(t) : t \geq 0\}$ a continuous time Markov chain? If yes, find its generator matrix and compute the stationary distribution. If not, explain why.

————**End of Paper**————

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2017

(Operations Research: Secondary Area)

Question 1.

Let

$$A = \begin{pmatrix} 3 & -1 & -3 \\ -4 & 0 & 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(a) (8pts) Is the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax = b, x \geq 0\}$ empty or not? Justify your answer.

(b) (8pts) Is the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax \geq b, x \geq 0\}$ empty or not? Justify your answer.

(c) (8pts) Given any vector $c \in \mathbb{R}^3$, what can you say about the linear program

$$\max_y \; b^\top y, \; s.t., \; A^\top y \leq c.$$

Justify your answer.

Question 2.

Consider the integer programming problem

$$\begin{array}{ll} \max & x_1 + 2x_2 \\ s..t & -3x_1 + 4x_2 \leq 4 \\ & 3x_1 + 2x_2 \leq 11 \\ & 2x_1 - x_2 \leq 5 \\ & x_1, x_2 \text{ integer.} \end{array}$$

(a) (5pts) What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem?

(b) (5pts) What is the convex hull of the set of all solutions to the integer programming problem?

(c) (10pts) Solve the problem by branch and bound. Give the branch and bound tree indicating for each node any bounds obtained and indicating when a node has been fathomed.

Question 3. (16pts)

The minimum number of employees needed and the rate of pay for each of 6 periods during a 24-hour day is given in the table below. Each employee works during a 3-period shift with the middle period off. For example an employee that starts work at 4am works periods 2 and 4. note that the period 1 follows immediately after period 6, i.e., an employee that start work at 8pm(period 6) has a break during period 1 and continues working during period 2.

Formulate the problem of finding a daily schedule of employees that minimizes cost while providing a sufficient number of employees for each period as a linear program, ignoring integrality requirements. Clearly define all variables.

Period	Time of Day	Minimal number of employees required	Hourly rate of pay
1	midnight - 4am	20	13
2	4am-8am	30	12
3	8am-noon	100	9
4	noon-4pm	120	9
5	4pm-8pm	80	10
6	8pm-midnight	50	12

Question 4.

Consider the following problem

$$\begin{array}{ll} \max & f(x) = \log(x_1+1) - x_2^2 \\ s.t. & x_1 + 2x_2 \leq 3 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

(a) (10pts) Is the problem convex or not? Justify your answer.

(b) (10pts) Use the KKT conditions to find an optimal solution.

Question 5. (20pts)

Let $\{X_n : n = 0, 1, \ldots\}$ be a discrete time Markov chain with state space $\{0, 1, ..., 2K\}$ for some positive integer K with transition matrix P such that

$$\begin{array}{ll} P_{i,i-1} = P_{i,i+1} = 1/2, & \text{for } 1 \leq i \leq 2K-1, \\ P_{0,K} = P_{2K,K} = 1. & \end{array}$$

Question: Does $\{X_n : n \geq 1\}$ have a unique stationary distribution π? If yes, find the explicit expressions for $\pi = (\pi_i)$; If not, explain why.

End of Paper

The Chinese University of Hong Kong

Department of Systems Engineering & Engineering Management

Ph.D. Qualifying Examination 2018

Area: Operations Research

INSTRUCTIONS

1. Answer ALL questions.
2. The score of each question is denoted by $(P : \cdot, S : \cdot)$, where P is the primary area score and S is the secondary area score.

Problem 1 (P:10pts, S:10pts). Let P be the polyhedron defined as follows:

$$P = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{array}{rcl} x_1 + x_2 & \geq & 1, \\ x_1 + x_3 & \geq & 1, \\ x_1 + x_4 & \geq & 1, \\ x_2 + x_4 & \geq & 1, \\ x_3 + x_4 & \geq & 1, \\ x & \geq & \mathbf{0}. \end{array} \right\}.$$

Identify an extreme point of P and justify why the point you identified is an extreme point of P.

Problem 2 (P:15pts, S:20pts). Consider the following LP:

$$\begin{array}{ll} \text{minimize} & -3x_1 + x_2 + 3x_3 - x_4 \\ \text{subject to} & x_1 + 2x_2 - x_3 + x_4 = 0, \\ & 2x_1 - 2x_2 + 3x_3 + 3x_4 = 9, \\ & x_1 - x_2 + 2x_3 - x_4 = 6, \\ & x \geq \mathbf{0}. \end{array} \tag{1}$$

Let $\bar{x} = (1, 1, 3, 0)$. Determine whether $\bar{x}$ is optimal for Problem (1) by considering the dual of Problem (1).

Problem 3 (P:35pts, S:30pts). Let $a_1, \ldots, a_n > 0$ be given. Consider the following problem:

$$\begin{array}{ll} \text{maximize} & x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} x_i = 1, \\ & x \geq \mathbf{0}. \end{array} \tag{2}$$

(a) **(P:15pts, S:15pts).** Write down the KKT conditions for Problem (2) and explain why they are necessary for optimality.

(b) **(P:20pts, S:15pts).** Using the result in (a), or otherwise, find a globally optimal solution to Problem (2) and show that it is unique.

Problem 4 (P:25pts, S:20pts). Let $P \in \mathbb{R}^{n \times n}$ be the transition probability matrix of a certain Markov chain on n states. We assume that P is symmetric. Thus, the matrix P satisfies the following properties:

$$P \geq \mathbf{0}, \quad Pe = e, \quad P = P^T.$$

Here, as usual, e is the vector of all ones.

(a) **(P:5pts, S:5pts).** Find a stationary distribution associated with P. Justify your answer.

It is known that the asymptotic rate of convergence of the Markov chain to the stationary distribution is determined by the second largest eigenvalue modulus of P, which is defined by

$$\mu(P) = \max_{i \in \{2,\ldots,n\}} |\lambda_i(P)|.$$

Here, $\lambda_i(P)$ is the i-th largest eigenvalue of P.

(b) **(P:20pts, S:15pts).** Since the largest eigenvalue of P is 1 and the corresponding eigenvector is e, by the Courant-Fischer theorem, we have

$$\lambda_2(P) = \sup \left\{ u^T P u \mid \|u\|_2 \leq 1, \, e^T u = 0 \right\}.$$

Using this result, show that $P \mapsto \mu(P)$ is convex. Also, show that

$$\mu(P) = \left\| (I - (1/n)ee^T) P (I - (1/n)ee^T) \right\| = \left\| P - (1/n)ee^T \right\|,$$

where $\| \cdot \|$ denotes the spectral norm (i.e., the largest singular value).

Problem 5 (P:15pts, S:20pts). Suppose that there are two types of cars on the road — red and blue. The red cars will pass a point on the road according to a Poisson process of rate λ, while the blue cars will pass that point according to a Poisson process of rate μ. The two processes are independent. Determine the probability that the first car that passes the point is red. Show your calculations.

The Chinese University of Hong Kong

Department of Systems Engineering & Engineering Management

Ph.D. Qualifying Examination 2019

Area: Operations Research

INSTRUCTIONS

1. There are SIX questions in total. Answer ALL of them.
2. The score of each question is denoted by $(P:\cdot, S:\cdot)$, where P is the primary area score and S is the secondary area score.

Problem 1 (P:15pts, S:20pts). Consider the following optimization problem:

$$\min_{x,y,z} \left\{ 2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9 \right\}. \tag{QP}$$

(a) **(P:5pts, S:5pts).** Write down the first-order optimality condition of Problem (QP).

(b) **(P:10pts, S:15pts).** Using the result in (a), determine the optimal solution(s) to Problem (QP). Justify your answer.

Problem 2 (P:15pts, S:20pts). Consider the following integer linear programming problem:

$$\begin{array}{lll} \text{minimize} & -2x_1 - x_2 & \\ \text{subject to} & 4x_1 + 5x_2 \leq 10, & \\ & 0 \leq x_j \leq 3 & \text{for } j = 1,2, \\ & x_j \text{ integer} & \text{for } j = 1,2. \end{array} \tag{ILP}$$

(a) **(P:5pts, S:5pts).** Determine the optimal value of and optimal solution to Problem (ILP).

(b) **(P:10pts, S:15pts).** Let v be the Lagrange multiplier associated with the constraint $4x_1 + 5x_2 \leq 10$. Write down the Lagrangian dual of Problem (ILP) that dualizes only the constraint $4x_1 + 5x_2 \leq 10$. Hence, determine the optimal value of and optimal solution to the Lagrangian dual you derived.

Problem 3 (P:25pts, S:20pts). Let $E \in \mathcal{S}^n$ be the $n \times n$ matrix of all ones and $\Omega \subseteq \{(i,j) : 1 \leq i < j \leq n\}$. Consider the following optimization problem:

$$\begin{array}{lll} \inf & \lambda_{\max}(X + E) & \\ \text{subject to} & X_{ij} = 0 & \text{for } (i,j) \in \Omega, \\ & X_{ii} = 0 & \text{for } i = 1,2,\dots,n. \end{array} \tag{Q}$$

Here, $\lambda_{\max}(A)$ denotes the largest eigenvalue of A.

(a) **(P:15pts, S:10pts).** Show that Problem (Q) can be formulated as an SDP. Justify your answer.

(b) **(P:10pts, S:10pts).** Derive the dual of the SDP found in (a).

Problem 4 (P:20pts, S:15pts). Consider the following standard primal-dual pair of LPs:

$$(P): \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq \mathbf{0}. \end{array} \qquad (D): \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{array}$$

Here, as usual, we have $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, where $m \leq n$. Suppose that (P) has a unique optimal solution x^* that is non-degenerate (i.e., there are exactly n linearly independent active constraints at x^*). Show that (D) also has a unique optimal solution y^* that is non-degenerate. (*Hint: Consider the complementary slackness condition.*)

Problem 5 (P:10pts, S:10pts). Let $\{(X_n, \mathscr{F}_n) : n \geq 0\}$ be a martingale and $f : \mathbb{R} \to \mathbb{R}$ be a convex function. Suppose that $\mathbb{E}\left[|f(X_n)|\right] < \infty$ for all $n \geq 0$. Show that $\{f(X_n), \mathscr{F}_n) : n \geq 0\}$ is a submartingale; i.e.,

$$\mathbb{E}\left[f(X_{n+1}) \,|\, \mathscr{F}_n\right] \geq f(X_n) \quad \text{for } n \geq 0.$$

Problem 6 (P:15pts, S:15pts). Suppose that passengers arriving at a bus station follow a Poisson process with rate λ. Moreover, suppose that the bus' arrival time T follows an exponential distribution with parameter μ, independent of the passengers' arrival process. Let W be the combined waiting time of all passengers up to time T. Compute $\mathbb{E}\left[W\right]$. State clearly the probability result(s) that you use without proof in your derivation.

The Chinese University of Hong Kong
Department of Systems Engineering & Engineering Management
PhD Qualifying Exam 2019/2020: Operation Research Area

INSTRUCTIONS:

For students who is taking OR as the primary area:

- Attempt Problem 1 to 5. The marks distribution are indicated next to the question. There is no need to attempt Problem 6 to 7.

For students who is taking OR as the secondary area:

- Attempt Problem 1 to 3, then Problem 4 to 5 **OR** Problem 6 to 7. The marks distribution are indicated next to the question.

Problem 1. [P: 15pts, S: 20pts] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. For some $\mu \in \mathbb{R}$, suppose that $f(x) - \frac{\mu}{2}\|x\|^2$ is a convex function. Show that the following inequality holds:

$$\langle \nabla f(x) - \nabla f(y) \,|\, x - y \rangle \geq \mu \|x - y\|^2, \quad \forall\, x, y \in \mathbb{R}^n.$$

Remark: For $x, y \in \mathbb{R}^n$, $\langle x \,|\, y \rangle$ denotes the inner product between the two vectors, i.e., $\langle x \,|\, y \rangle = x^\top y$.

Problem 2. [P: 20pts, S: 30pts] Consider the following polyhedral set:

$$X = \left\{ x \in \mathbb{R}^4 : \begin{array}{rl} x_1 - x_2 - x_3 & \leq 1 \\ x_2 - x_4 & \leq 2 \\ x_1, x_2, x_3, x_4 & \geq 0 \end{array} \right\}$$

(a) Find **all** the extreme/corner points of X, and justify why they are extreme/corner points.

(b) Consider the following **linear integer program**:

$$\begin{array}{rrl} \max\limits_{x_1,x_2,x_3,x_4} & x_1 + 2x_2 + 3x_3 + 4x_4 & \\ & x_1 - x_2 - x_3 & \leq 1 \\ \text{s.t.} & x_2 - x_4 & \leq 2 \\ & x_1, x_2, x_3, x_4 & \geq 0 \\ & x_1, x_2, x_3, x_4 & \in \mathbb{Z} \end{array}$$

Find an **optimal solution** to the above integer program. Justify your answer.

Problem 3. [P: 30pts, S: 15pts] Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable, increasing function with $f(0) = 0$. Consider

$$F(x) = \int_0^x f(a)\, da.$$

(a) Let $F^\star(y) = \sup_u \{uy - F(u)\}$ be the conjugate function of $F(x)$. Show the following inequality:

$$xy \leq F(x) + F^\star(y), \quad \forall\, x, y \in \mathbb{R}.$$

(b) Let $g = f^{-1}$ such that g is an inverse function and suppose that it is differentiable. Show that the conjugate function of $F(x)$ can be written as

$$G(y) = F^{\star}(y) = \int_0^y g(b)\, db.$$

Remark: You may prove either using a graphical method by drawing the integration region; or apply the change of variable and integration-by-part, respectively, whose formulas are provided as follows for your convenience:

$$\int_a^b h(\phi(x))\phi'(x)\, dx = \int_{\phi(a)}^{\phi(b)} h(u)\, du, \quad \int_a^b u(x)v'(x)\, dx = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\, dx.$$

Problem 4. [P: 15pts, S: 20pts] Let $\alpha \in (0,1)$ be fixed, and n be a finite integer. Consider a Markov chain with n states $\mathsf{S} = \{1, ..., n\}$ and the following transition property:

$$S_{t+1} = \begin{cases} \min\{n, S_t + 1\}, & \text{with probability } \alpha, \\ \max\{1, S_t - 1\}, & \text{with probability } 1 - \alpha. \end{cases}$$

(a) Write down the transition probability matrix $\mathbf{P}$ for the Markov chain whose (i,j)th entry is such that $P_{ij} = P(S_{t+1} = j | S_t = i)$, where $i, j = 1, ..., n$. Is the Markov chain irreducible? Is the Markov chain aperiodic?

(b) Let $n = 3$. Find the stationary distribution of the Markov chain, i.e., what is $\mathbb{P}(S_\infty = i)$ for $i = 1, 2, 3$?

(c) Let $\alpha = 0.5$ and $n = 3$. Define the function $f : \mathsf{S} \to \mathbb{R}$ such that $f(s) = s^2$. Evaluate the following quantities:

$$\text{(i) } \mathbb{E}\big[\{f(S_1) - \mathbb{E}[f(S_\infty)]\}^2 | S_0 = 1\big], \quad \text{(ii) } \mathbb{E}\big[\{f(S_2) - \mathbb{E}[f(S_\infty)]\}^2 | S_0 = 1\big]$$

Problem 5. [P: 20pts, S: 15pts] Suppose that a marathon race begins at time $t = 0$ for n people. For $1 \le i \le n$, the time to complete the marathon for person i is given by a random variable (r.v.) $X_i \in \mathbb{N}$, which is exponentially distributed with the density $f_i(x) = \lambda_i e^{-\lambda_i x}$, $x \in \mathbb{N}$; note that the CDF is given by $\mathbb{P}(X_i \le t) = 1 - e^{-\lambda_i x}$. Moreover, these r.v.s are independent.

(a) Let $Z \in \mathbb{N}$ be the r.v. of the time when the **last person** finishes the marathon. Find the cumulative distribution of Z, i.e., $\mathbb{P}(Z \le x)$.

(b) Let T_1 be the r.v. of the time when the **first person** finishes the marathon; and T_2 be the r.v. of the time difference between when the **first person** and **second person** finishes the marathon.

Find the cumulative distributions of T_1 and T_2, i.e., $\mathbb{P}(T_1 \le x)$ and $\mathbb{P}(T_2 \le x)$. Determine if each of them is an exponential r.v..

Remark: You may find the following result useful when evaluating the distribution of T_2:

$$\mathbb{P}(\text{person } i \text{ is the first to finish}) = \mathbb{P}(X_i = \min\{X_1, ..., X_n\}) = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \cdots + \lambda_n}$$

Problem 6. [**S: 20pts**] Consider the constraint set $\mathcal{C} = \{x \in \mathbb{R}^n : x^\top e \geq 0\}$, where $e \in \mathbb{R}^n$ is an all-one n-dimensional vector. Define the operator:

$$\mathcal{P}(x) = \begin{cases} y - \frac{1}{n}(y^\top e)e, & \text{if } y^\top e \leq 0 \\ y, & \text{if } y^\top e > 0. \end{cases}$$

Show that for any $x \in \mathbb{R}^n$, $\mathcal{P}(x)$ is the projection of x onto $\mathcal{C}$.

Problem 7. [**S: 15pts**] Consider a non-differentiable, convex function $f : \mathbb{R}^n \to \mathbb{R}$.

(a) Consider a point $\overline{x} \in \mathbb{R}^n$ such that $0 \in \partial f(\overline{x})$. Show that $\overline{x}$ is a global optimal solution to the optimization problem $\min_{x \in \mathbb{R}^n} f(x)$.

(b) Let $y \in \mathbb{R}^n$ and $\lambda > 0$ be fixed constant vector/scalar. Consider the optimization problem:

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|x - y\|_2^2 + \lambda \|x\|_1$$

Using the result from (a), find an optimal solution $\overline{x} \in \mathbb{R}^n$ to the above problem. Express your solution in terms of y and λ.

The Chinese University of Hong Kong
Department of Systems Engineering & Engineering Management
PhD Qualifying Exam 2020/2021: Operation Research Area

INSTRUCTIONS:

For students who is taking OR as the primary area:

- Attempt Problem 1 to 5. The marks distribution are indicated next to the question.

For students who is taking OR as the secondary area:

- Attempt Problem 1 to 5. The marks distribution are indicated next to the question.

Problem 1. [P: 10pts, S: 20pts] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Suppose that there exists a constant $L > 0$ such that

$$\|\nabla^2 f(x) - \nabla^2 f(y)\| \leq L\|x - y\|, \ \forall\ x, y \in \mathbb{R}^n.$$

Show that the following inequality holds for any $x, y \in \mathbb{R}^n$,

$$\|\nabla f(y) - \nabla f(x) - \nabla^2 f(x)(y - x)\| \leq \frac{L}{2}\|x - y\|^2. \tag{1}$$

For any $x \in \mathbb{R}^n$, $\|x\| := \sqrt{x^\top x}$ denotes the Euclidean norm; while for any $A \in \mathbb{R}^{n \times n}$, $\|A\|$ denotes the matrix norm induced by the Euclidean norm, i.e., $\|A\| := \sup_{x \in \mathbb{R}^n, x \neq 0} \|Ax\|/\|x\|$.

Hint: from calculus, we know that $\nabla f(y) = \nabla f(x) + \int_0^1 \nabla^2 f(x + t(y - x))(y - x)dt$.

Problem 2. [P: 20pts, S: 30pts] Consider the integer programming problem:

$$\begin{array}{rl} \min\limits_{x_1, x_2 \in \mathbb{R}} & (x_1 - 1)^2 + (x_2 + 2)^2 \\ \text{s.t.} & x_1 + x_2 \geq 1, \\ & -2 \leq x_1 \leq 2, -2 \leq x_2 \leq 2 \\ & x_1, x_2 \in \mathbb{Z}. \end{array} \tag{2}$$

(a) Find all the optimal solutions to (2). Also, write down the optimal objective value.

(b) Let v be the dual variable associated with the constraint $x_1 + x_2 \geq 1$. Write down the Lagrangian dual problem to (2) which dualizes only the constraint $x_1 + x_2 \geq 1$. You must express the objective function of the dual problem in terms of v.

Then, determine the optimal solution and optimal objective value to the dual problem.

Problem 3. [P: 20pts, S: 20pts] Answer the following questions regarding second order cone programs (SOCPs):

(a) Verify the following identity

$$\left\{x \in \mathbb{R}^n, y, z \in \mathbb{R} : \|x\|^2 \leq yz\right\} = \left\{x \in \mathbb{R}^n, y, z \in \mathbb{R} : \left\|\begin{pmatrix} 2x \\ y - z \end{pmatrix}\right\| \leq y + z\right\},$$

where for any $x \in \mathbb{R}^n$, $\|x\| := \sqrt{x^\top x}$ denotes the Euclidean norm.

(b) Show that the following problem can be written as an SOCP:

$$\max_{x \in \mathbb{R}^n} \left(\sum_{i=1}^{m} \frac{1}{a_i^\top x - b_i} \right)^{-1} \quad \text{s.t.} \quad a_i^\top x - b_i \geq \delta,$$

where $\delta > 0$ is a fixed constant. You may use the result from part (a) to help.

Problem 4. [P: 25pts, S: 15pts] Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, consider the following two sets:

$$\mathcal{S} = \{x \in \mathbb{R}^n : Ax \geq b,\ x \geq 0\}, \quad \mathcal{T} = \{y \in \mathbb{R}^m : A^\top y \leq c,\ y \geq 0\}$$

(a) Suppose that $\mathcal{T} \neq \emptyset$, $\mathcal{S} \neq \emptyset$. Prove that *at least* one of the above two sets is unbounded.

Hint: notice that if there exists a direction $d \in \mathbb{R}^n$ such that $Ad \geq b, d \geq 0, d \neq 0$, then $\mathcal{S}$ must be unbounded.

(b) Could both of the two sets be unbounded simultaneously? Justify your answer with an example $\mathcal{T}, \mathcal{S}$ by specifying such A, b, c.

Remark: You may find the following variant of the Farka's lemma useful:

Lemma 1: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Exactly one of the following holds:

1. There exists $x \in \mathbb{R}^n$ such that $Ax > 0$ and $x \geq 0$.
2. There exists $y \in \mathbb{R}^m$ such that $y^\top A \leq 0$, $y \geq 0$, $y \neq 0$.

Notice that $Ax > 0$ means that the vector Ax is strictly positive.

Problem 5. [P: 25pts, S: 15pts] Consider the following linear program (LP):

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^\top x \\ \text{s.t.} \quad & d_i^\top x \leq \ell_i,\ i = 1, ..., m, \\ & f_j^\top x = g_j,\ j = 1, ..., p, \\ & x_i \geq 0,\ i = 1, ..., n. \end{aligned} \tag{3}$$

We assume that $c \in \mathbb{R}^n$ is a positive vector such that $c_i > 0$ for $i = 1, ..., n$. Furthermore, $d_i \in \mathbb{R}^n$ and $f_j \in \mathbb{R}^n$ are non-negative vectors.

(a) Write down the KKT conditions for the above problem.

(b) Suppose that

$$\sum_{j=1}^{p} ||f_j||_0 \leq \bar{p} \leq n,$$

where $\|f_j\|_0$ counts the number of non-zero elements in the vector f_j. Show that there must exist an optimal solution $x^\star$ to (3) such that

$$\|x^\star\|_0 \leq \bar{p}.$$

Hint: consider checking the slackness condition in the KKT conditions.