1. **Problem 1*** **(P:20pts, S:25pts).** Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be given. Suppose that A has full column rank. Consider the following problem:

$$\begin{array}{lll} \text{minimize} & \|Ax - \text{diag}(u)b\|_2^2 & \\ \text{subject to} & |u_i| = 1 & \text{for } i = 1,\dots,m, \\ & x \in \mathbb{R}^n, & \end{array} \quad (1)$$

where $\text{diag}(u)$ is the $m \times m$ diagonal matrix whose diagonal entries are given by u.

a) (a) **(P:15pts, S:15pts).** Suppose that the vector u is given. Then, Problem (1) becomes

$$\min_{x\in\mathbb{R}^n} \|Ax - \text{diag}(u)b\|_2^2. \quad (2)$$

Write down the KKT conditions for Problem (2) and explain why they are necessary and sufficient for optimality. Hence, or otherwise, express the optimal solution x^* in terms of A, u and b.

KKT: $A^T(Ax - \text{diag}(u)b) = 0$

Let $f(x) = \|Ax - \text{diag}(u)b\|_2^2$

$f''(x) = A^TA \succeq 0$

so f is convex

$x = (A^TA)^{-1}A^T \text{diag}(u)b$.

b) **(P:5pts, S:10pts).** Using the result in (a), show that Problem (1) can be reformulated as

$$\begin{array}{lll} \text{minimize} & u^TMu & \\ \text{subject to} & |u_i| = 1 & \text{for } i = 1,\dots,m \end{array}$$

for some suitable $m \times m$ matrix M.

$$\|Ax - \text{diag}(u)b\|_2^2 \Rightarrow \|A(A^TA)^{-1}A^T\text{diag}(u)b - \text{diag}(u)b\|_2^2$$

$$= \|\underbrace{(A(A^TA)^{-1}A^T - I)\text{diag}(b)}_{C}u\|_2^2$$

$$= (Cu)^T(Cu)$$

$$= u^TC^TCu$$

$$= u^TMu$$

So, (1) becomes $\min\ u^TMu$

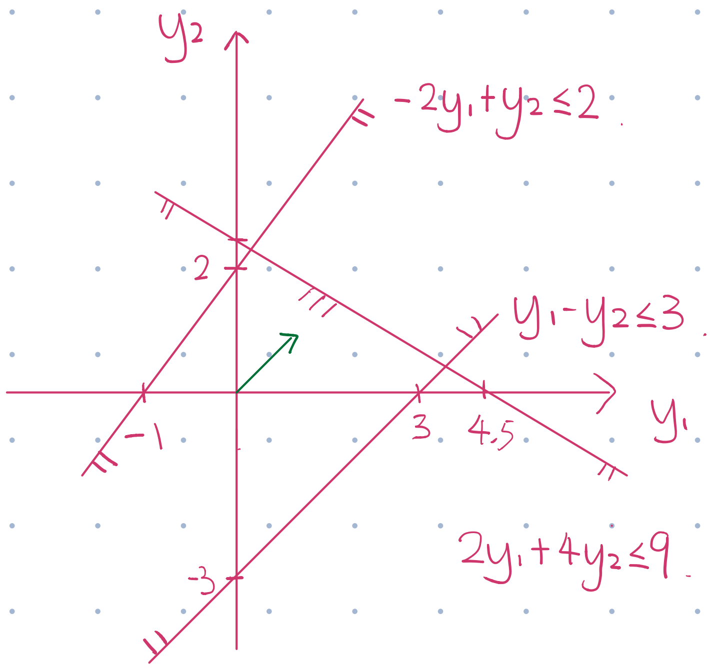
s.t. $|u_i| = 1$ for $i = 1,\dots,m$.

2. Consider the linear program

$$\min \quad 2x_1 + 9x_2 + 3x_3$$
$$\text{s.t.} \quad -2x_1 + 2x_2 + x_3 \geq 1 \qquad (P)$$
$$x_1 + 4x_2 - x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0.$$

a) Write down the dual and solve it graphically.

$$\max \quad y_1 + y_2$$
$$\text{s.t.} \quad -2y_1 + y_2 \leq 2$$
$$2y_1 + 4y_2 \leq 9.$$
$$y_1 - y_2 \leq 3.$$
$$y_1 \geq 0, \; y_2 \geq 0$$



$$y_1 - y_2 = 3.$$
$$2y_1 + 4y_2 = 9.$$

$$y_2 = \frac{1}{2} = 0.5$$
$$y_1 = \frac{7}{2} = 3.5 \qquad \Rightarrow v_d^* = 4$$

b). **(P:10pts, S:15pts).** Using the result in (a), or otherwise, determine the optimal solution to Problem ~~(3)~~. (P)

By Strong Duality Theorem, the optimal value is 4.

$$2x_1 + 9x_2 + 3x_3 = 4.$$
$$-2x_1 + 2x_2 + x_3 = 1$$
$$x_1 + 4x_2 - x_3 = 1.$$

$$x_1 = 0.$$
$$x_2 = 1/3. \quad \Rightarrow V_P^* = 4.$$
$$x_3 = 1/3$$