1.(a) Farkas Lemma.

(I) $Ax=b,\ x\geq 0$

(II) $A^T y \leq b,\ b^T y<0 \iff \begin{cases} \begin{pmatrix} 3 & -4 \\ -1 & 0 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq 0 \\ (3,2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} > 0 \end{cases} \iff \begin{cases} 3y_1-4y_2\leq 0 \\ -y_1<0 \\ -3y_1+2y_2\leq 0 \\ 3y_1+2y_2>0 \end{cases}$ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution $\Rightarrow$ (I) has no solution $\Rightarrow$ P is empty.

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2017

(Operations Research: Primary Area)

1.(b). (I) $Ax\geq b,\ x\geq 0 \iff (A,-I)\begin{pmatrix} x \\ s \end{pmatrix}=b,\ (x,s)\geq 0.$

Question 1.

(II) $(A,-I)^T y\leq 0,\ b^T y<0. \iff \begin{pmatrix} 3 & -4 \\ -1 & 0 \\ -3 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\leq 0,\ (3,2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}>0.$ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a solution $\Rightarrow$ P is empty.

Let

$$A=\begin{pmatrix} 3 & -1 & -3 \\ -4 & 0 & 2 \end{pmatrix},\quad \text{and}\quad b=\begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

(a) (8pts) Is the polyhedron $P=\{x\in\mathbb{R}^3 \mid Ax=b, x\geq 0\}$ empty or not? Justify your answer.

(b) (8pts) Is the polyhedron $P=\{x\in\mathbb{R}^3 \mid Ax\geq b, x\geq 0\}$ empty or not? Justify your answer.

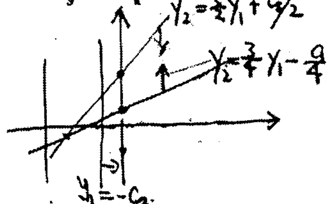
(c) (8pts) Given any vector $c\in\mathbb{R}^3$, what can you say about the linear program

$$\max_y\ b^T y,\ s.t.,\ A^T y\leq c.$$

Justify your answer.

1(c). It is the dual of (P) $\min_x 0^T x$ s.t. $Ax=b,\ x\geq 0$. which is infeasible.

$\forall c\in\mathbb{R}^3$, $A^T y\leq c$ is feasible, $V_d^*=+\infty$. otherwise, (P) is feasible because of strong duality.



Question 2.

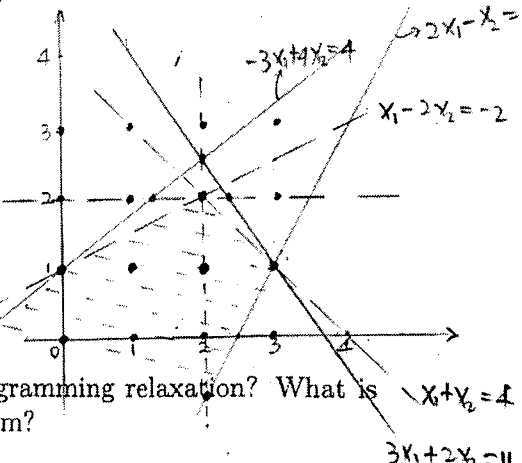
Consider the integer programming problem

$$\begin{aligned} \max\ & x_1+2x_2 \\ s..t\ & -3x_1+4x_2\leq 4 \\ & 3x_1+2x_2\leq 11 \\ & 2x_1-x_2\leq 5 \\ & x_1,x_2 \text{ integer.} \end{aligned}$$

2(a) When $(x_1,x_2)=(2,5/2)$, $V_P^*=7$

When $(x_1,x_2)=(2,2)$, $V_{IP}^*=6$

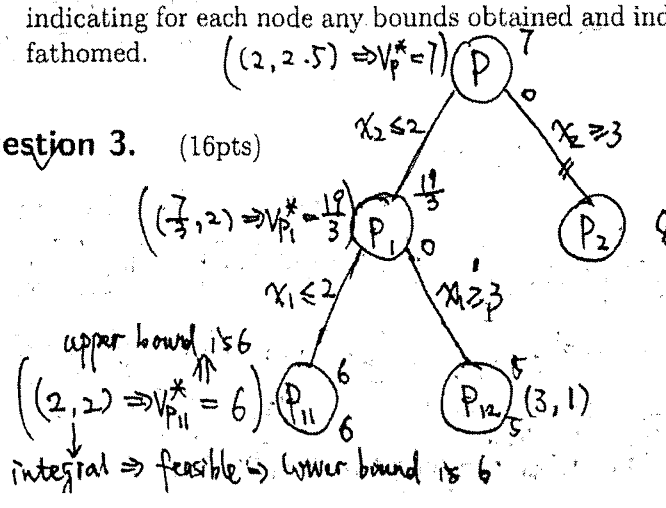
2(b) $\begin{cases} -3x_1+4x_2\leq 4 \\ 2x_1-x_2\leq 5 \\ x_1-2x_2\leq -2 \\ x_1+x_2\leq 4 \end{cases}$



(a) (5pts) What is the optimal value of the linear programming relaxation? What is the optimal cost of the integer programming problem?

(b) (5pts) What is the convex hull of the set of all solutions to the integer programming problem?

(c) (10pts) Solve the problem by branch and bound. Give the branch and bound tree indicating for each node any bounds obtained and indicating when a node has been fathomed.



(P1) $\max\ x_1+2x_2$
s.t. $-3x_1+4x_2\leq 4$
$3x_1+2x_2\leq 11$
$2x_1-x_2\leq 5$
$x_2\leq 2$
$x_1,x_2\in\mathbb{Z}$

(P2) $\max\ x_1+2x_2$
s.t. $-3x_1+4x_2\leq 4$
$3x_1+2x_2\leq 11$
$2x_1-x_2\leq 5$
$x_2\geq 3$
$x_1,x_2\in\mathbb{Z}$

Question 3. (16pts)

2(b) By st

$\therefore 2x_1^*$.

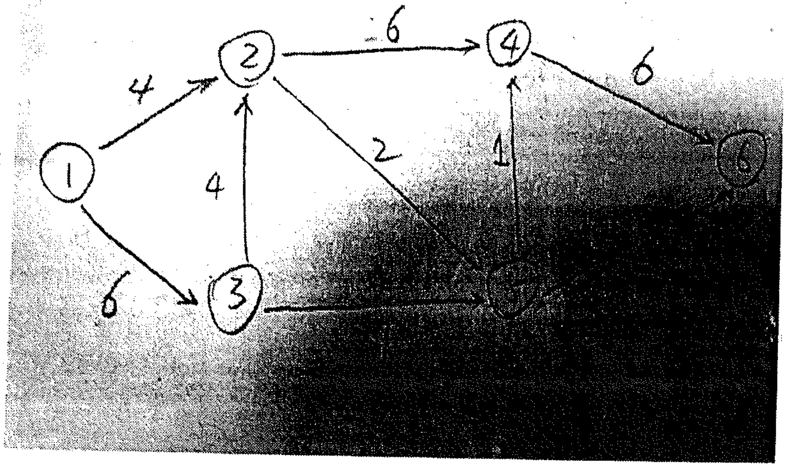
Besides, (x

3 of the 5

$f_1(i) = \begin{cases} 0, & i=1 \\ a_{1i}, & i=2,\dots,6 \end{cases}$ cost from 1 to i containing 1 arc

$f_k(i) = \begin{cases} 0, & i=0 \\ \min\limits_{1\le j\le 6} \{ \max(a_{ij}, f_{k-1}(j)) \} \end{cases}$ cost from 1 to i containing k arcs.

Figure 1: Network for problem 3



Consider the network in Figure 1 with distances specified on the arcs. Let the "cost" of a path be the length of the longest edge in that path. For example, the cost of path 1-2-5-6 is 4; cost of path 2-5-4 is 2 etc. Design an ~~efficient algorithm to find a minimum-cost~~ path from node 1 to all other nodes.

Question 4.

Consider the problem of finding a circle of minimum radius that contains r points $y_1, \dots, y_r$ in the plane, i.e., find x and z that minimize z subject to $\|x - y_j\| \le z$ for all $j = 1, \dots, r$, where x is the center of the circle under optimization.

5(

(a) (10pts) Introduce Lagrange multipliers $\mu_j, j = 1, \dots, r$, for the constraints, and show that the dual problem has an optimal solution and there is no duality gap.

(b) (10pts) Show that calculating the dual function at some $\mu \ge 0$ involves the computation of a Weber point of $y_1, \dots, y_r$ with weights $\mu_1, \dots, \mu_r$, i.e., the solution of the problem

$$\min_{x\in\mathbb{R}^2} \sum_{j=1}^{r} \mu_j \|x - y_j\|.$$

Question 5.

$\begin{cases} \pi_1 = 2\pi_0, \\ \pi_2 = 4\pi_0 \\ \vdots \\ \pi_K = 2K\pi_0 \end{cases}$ $\pi_{K+1} = 2(K-1)\pi_0$ $\vdots$ $\pi_{2K} = \pi_0$

$\sum_{i=0}^{2K} \pi_i = 1 \Rightarrow \pi_0 = \frac{1}{2(K^2+1)} = \pi_{2K}$.

$\pi_i = \begin{cases} \frac{i}{K^2+1} & i=1,\dots \\ \frac{2(2K-i)}{2(K^2+1)} & i=K+1,\dots \end{cases}$

Let $\{X_n : n = 0, 1, \dots\}$ be a discrete time Markov chain with state space $\{0, 1, \dots, 2K\}$ for some positive integer K with transition matrix P such that

$$P_{i,i-1} = P_{i,i+1} = 1/2, \quad \text{for } 1 \le i \le 2K-1,$$
$$P_{0,K} = P_{2K,K} = 1.$$

5(a). $\pi = \pi P$

3.

$P = \begin{bmatrix} 0 & \cdots & 0, 1, 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ 0, & \cdots & 0, 1, 0, & \cdots, & 0 \end{bmatrix}$

$\frac{1}{2}\pi_1 = \pi_0$

$\frac{1}{2}\pi_2 = \pi_1$

$\frac{1}{2}(\pi_1 + \pi_3) = \pi_2$

$\frac{1}{2}(\pi_{K-2} + \pi_K) = \pi_{K-1}$

$\frac{1}{2}(\pi_{K-1} + \pi_{K+1}) + \pi_0 + \pi_{2K} = \pi_K$

$\frac{1}{2}(\pi_K + \pi_{K+2}) = \pi_{K+1}$

$\vdots$

$\frac{1}{2}(\pi_{2K-3} + \pi_{2K-1}) = \pi_{2K-2}$

$\frac{1}{2}\pi_{2K-2} = \pi_{2K-1}$

$\frac{1}{2}\pi_{2K-1} = \pi_{2K}$.

b). $P\{Z(s+t)=j \mid Z(s)=i,\ Z(u)=z(u),\ 0\le u<s\}$

$= P(X_{\tilde N(s+t)}=j \mid X_{\tilde N(s)}=i,\ X_{\tilde N(u)}=z(u),\ 0\le u<s)$

$= P(X_{\tilde N(s+t)}=j \mid X_{\tilde N(s)}=i,\ X_{\tilde N(u_0)}=z(u_0), \cdots, X_{\tilde N(u_{\tilde N(s)-1})}=z(u_{\tilde N(s)-1}))$

(a) (10pts) Does $\{X_n : n \ge 1\}$ have a unique stationary distribution π? If yes, find the explicit expressions for $\pi = (\pi_i)$; If not, explain why. $0\le u_0 \le u_1 \le \cdots \le u_{\tilde N(s)-1} \le S$ are transition time points.

(b) (10pts) Let $\{N(t) : t \ge 0\}$ be a Poisson process with rate $\lambda > 0$, independent of the process $\{X_n : n = 0, 1, \ldots\}$. Define for each $t \ge 0$,

$$Z(t) = X_{N(3t)}.$$

Is $\{Z(t) : t \ge 0\}$ a continuous time Markov chain? If yes, find its generator matrix and compute the stationary distribution. If not, explain why.

Denote $\tilde N(t) = N(3t)$. $P(\tilde N(t)=n) = P(N(3t)=n) = e^{-\lambda(3t)}\frac{(\lambda(3t))^n}{n!} = e^{-(3\lambda)t}\frac{((3\lambda)t)^n}{n!}$.

$\therefore$ $\tilde N(t)$ is a Poisson process with rate 3λ.

End of Paper

$\therefore$ v_i of $Z(t)=X_{\tilde N(t)}$ is 3λ $\forall i=0,1,\cdots,2K$.

$\therefore$ $q_{i,i-1} = q_{i,i+1} = v_i P_{i,i-1} = v_i P_{i,i+1} = 3\lambda\cdot\frac{1}{2} = \frac{3}{2}\lambda$ for $1\le i\le 2k-1$

$q_{0,K} = q_{2K,K} = v_0 P_{0,2K} = v_{2K}P_{2K,K} = 3\lambda\cdot 1 = 3\lambda$

$q_{i,i} = -v_i = -3\lambda$

$\therefore$ Generator matrix $Q=[q_{ij}]$ is obtained.

$= P\{X_{\tilde N(s+t)}=j \mid X_{\tilde N(s)}=i,\ X_{\tilde N(s)-1}=z(u_{\tilde N(s)-1}), \cdots, X_0=z(u_0)\}$.

$= P\{X_{\tilde N(s+t)}=j \mid X_{\tilde N(s)}=i\}$ (property of DTMC of X_n)

$\Rightarrow$ $X_{N(t)}$ is CTMC

4(a) $\min_{x,z}\ z$

s.t. $\|x-y_j\|_2 \le z$, $j=1,\cdots,r$

① $z\ge 0 \Rightarrow v_p^* > -\infty$ ② Given x, let $z_j=\|x-y_j\|$ $\exists z$ s.t. $z>\max z_j$ $\Rightarrow$ strictly feasible $\therefore$ Dual attainment

4(b) Lagrangian function : $L(x,z,\mu) = z + \sum_{j=1}^{r}\mu_j(\|x-y_j\|_2 - z) = \sum_{j=1}^{r}\mu_j\|x-y_j\|_2 + (1-\sum_{j=1}^{r}\mu_j)z$.

2k-1 Dual Problem: $\max_{\mu\ge 0}\min_{x,z} L(x,z,\mu)$. Let $\theta(\mu) = \min_{x,z} L(x,z,\mu)$.

$\because$ L is convex with regard to (x,z) $\Rightarrow$ the optimal point x satisfies

$\frac{\partial L}{\partial x}=0 \Rightarrow \sum_{j=1}^{r}\mu_j\frac{x-y_j}{\|x-y_j\|_2}=0$. On the other hand, $f(x)=\sum_{j=1}^{r}\mu_j\|x-y_j\|_2$ is convex

with regard to x $\Rightarrow$ $\min_x \sum_{j=1}^{r}\mu_j\|x-y_j\|_2$ involves $\frac{\partial f}{\partial x} = \sum_{j=1}^{r}\frac{x-y_j}{\|x-y_j\|_2}\mu_j = 0$.

2016. Q5.(a) $p(Y_n = i_n \mid Y_{n-1} = i_{n-1},\ Y_{n-2} = i_{n-2}, \dots, Y_1 = i_1)$

$$= p(\xi_n \xi_{n-1} = i_n \mid \xi_{n-1}\xi_{n-2} = i_{n-1},\ \xi_{n-2}\xi_{n-3} = i_{n-2}, \dots, \xi_1 \xi_0 = i_1)$$

$$= \frac{p(\xi_n \xi_{n-1} = i_n,\ \xi_{n-1}\xi_{n-2} = i_{n-1}, \dots, \xi_1 \xi_0 = i_1)}{p(\xi_{n-1}\xi_{n-2} = i_{n-1}, \dots, \xi_1\xi_0 = i_1)}$$

$$= \frac{p(\xi_0 = 1,\ \xi_1 = \mathrm{sign}(i_1),\ \xi_2 = \mathrm{sign}(i_2)\,\mathrm{sign}(i_1), \dots, \xi_n = \mathrm{sign}(i_1)\cdots\mathrm{sign}(i_n)) \times 2}{p(\xi_0 = 1,\ \xi_1 = \mathrm{sign}(i_1), \dots, \xi_n = \mathrm{sign}(i_1)\cdots\mathrm{sign}(i_n)) \times 2}$$

$$= \frac{\left(\frac{1}{2}\right)^{n+1} \times 2}{\left(\frac{1}{2}\right)^{n} \times 2} = \frac{1}{2} = \frac{p(\xi_n\xi_{n-1} = i_n,\ \xi_{n-1}\xi_{n-2} = i_{n-1})}{p(\xi_{n-1}\xi_{n-2} = i_{n-1})} = p(\xi_n \xi_{n-1} = i_n \mid \xi_{n-1}$$

$$= p(Y_n = i_n \mid Y_{n-1} = i_{n-1})$$

$(\xi_1 = \mathrm{sign}(i_1)\xi_0,\ \xi_2 = \mathrm{sign}(i_2)\xi_1 = \mathrm{sign}(i_2)\,\mathrm{sign}(i_1)\xi_0, \dots, \xi_n = \mathrm{sign}(i_n)\cdots\mathrm{sign}(i_1)\xi_0.)$

$\max_{x} \sum_{i=1}^{m}\sum_{j=1}^{n} p_j x_{ij} g_{ij}$ 1(b) $\max_{x,y} \sum_{i=1}^{m}\sum_{j=1}^{n} g_{ij} x_{ij}$

s.t. $\sum_{i=1}^{m} g_{ij} x_{ij} \geq b_j$ s.t. $\sum_{j=1}^{n} x_{ij} \leq a_i$, $\sum_{i=1}^{m} g_{ij} x_{ij} = y k_j$, $x_{ij} \geq 0$

$\sum_{j=1}^{n} x_{ij} \leq a_i$

$x_{ij} \geq 0$ (The area of crop j sown in plot i)

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2016

(Operations Research)

INSTRUCTIONS

- You should attempt all questions no matter Operations Research is your primary or secondary area.

Question 1.

Suppose that n different crops (e.g., corn, wheat, etc.) are to be grown on m plots of land with areas of $a_1, a_2, \ldots, a_m$ acres. Further, suppose that the expected yield of the j-th crop when planted on the i-th plot is $g_{i,j}$ tons per acre ($i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$), and that a ton of the j-th crop can be sold for p_j dollars ($j = 1, 2, \ldots, n$).

- (5pts) How much area in each plot should sown in each crop so as to maximize the expected revenue, subject to the additional constraint that at least b_j tons of the j-th crop ($j = 1, 2, \ldots, n$) is produced (in expectation)? Formulate this problem as a linear program.
- (10pts) Suppose that instead of the question asked above, you are asked: how much area in each plot should be sown in each crop so as to maximize the expected yield of all of the crops while ensuring that the ratios of the expected yields of the n crops are $k_1 : k_2 : \cdots : k_n$? Formulate this problem as a linear program.

2(a) ① If $\exists u$ s.t. $A^T u > 0$. $\forall x \in P$, $Ax = b$, $x \geq 0$, then $u^T A x = u^T b$,

Question 2.

$x \geq 0 \Rightarrow \sum_{i=1}^{n} (A^T u)_i x_i = u^T b \Rightarrow \sum_{i=1}^{n} x_i \leq \min_i (A^T u)_i / u^T b$, $x \geq 0$

$\therefore$ P is bounded ② If P is bounded. Consider LP: $\min -\mathbf{1}^T x$ s.t. $Ax = b$, $x \geq 0$

(a) (15pts) Prove that the standard form LP polyhedron $P \equiv \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, is bounded if and only if there exists a vector $u \in \mathbb{R}^m$ such that $A^T u > 0$.

LP has an optimal solution. Dual: $\max b^T y$ s.t. $A^T y \leq -1$ is feasible $\Rightarrow \exists \bar{y}$ s.t. $A^T \bar{y} \leq -1$. Let $u = -\bar{y}$, then $A^T u \geq 1 > 0$

(b) (15pts) Now consider the nonlinear programming problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & c_i(x) \geq 0, \text{ for all } i \in \mathcal{I}, \end{array}$$

where all functions $c_i(x)$, for all $i \in \mathcal{I}$ are continuously differentiable. Suppose that the Mangasarian-Fromovitz constraint qualification (MFCQ) holds at the point $\bar{x}$; i.e., $c_i(\bar{x}) = 0$, for all $i \in \mathcal{A}(\bar{x}) \subseteq \mathcal{I}$ and there exists a vector $p \in \mathbb{R}^n$ such that $\nabla c_i(\bar{x})^T p > 0$, for all $i \in \mathcal{A}(\bar{x})$. Prove that the only nonnegative linear combination of the vectors $\nabla c_i(\bar{x})$, for all $i \in \mathcal{A}(\bar{x})$ equal to zero is the zero linear combination.

2(b). Suppose $\sum_{i \in A(\bar{x})} \alpha_i \nabla c_i(\bar{x}) = 0$. $(\alpha_i \geq 0)$

$\Rightarrow \sum_{i \in A(\bar{x})} \alpha_i \nabla c_i(\bar{x})^T p = 0$. Since $\nabla c_i(\bar{x})^T p > 0$ $\forall i \in A(\bar{x})$

we get $\alpha_i = 0$.

4. (1) If epi(f) is cvx. $\forall x_1, x_2$

Take $(f(x_1), x_1)$, $(f(x_2), x_2) \in$ epi(f)

Then $\lambda(f(x_1), x_1) + (1-\lambda)(f(x_2), x_2) \in$ epi(f) $(0 \le \lambda \le 1)$

$\Leftrightarrow (\lambda f(x_1) + (1-\lambda) f(x_2),\ \lambda x_1 + (1-\lambda)x_2) \in$ epi(f)

$\Leftrightarrow \lambda f(x_1) + (1-\lambda) f(x_2) \ge f(\lambda x_1 + (1-\lambda) x_2)$

$\therefore$ f is cvx

(2) If f is cvx ...

$\in$ epi(f), $t_1 \ge f(x_1)$, ...

Then, $\lambda t_1 + (1-\lambda) t_2 \ge \lambda f(x_1) + (1-\lambda) \dots$

$\ge f(\lambda x_1 + (1-\lambda) \dots$

$\Rightarrow (\lambda t_1, (1-\lambda) t_2,\ \lambda x_1 + (1-\lambda) x_2) \in$ epi(f)

$\Leftrightarrow \lambda(t_1, x_1) + (1-\lambda)(t_2, x_2) \in$ epi(f)

$\therefore$ epi(f) is cvx.

Question 3. (20pts)

Use Branch-and-Bound method to solve the following Integer Programming problem:

$$\begin{array}{lll} \max\ z = & 5x_1 + 2x_2 \\ \text{s.t.} & 3x_1 + x_2 \le 12 \\ & x_1 + x_2 \le 5 \\ & x_1, x_2 \ge 0;\ x_1, x_2 \text{ integer}. \end{array}$$

(3.5, 1.5) P 20.5

$x_2 \le 1$ / $x_2 \ge 2$

$(\frac{11}{3}, 1)$ P_1 20.3 ; P_2 20, 20 (4, 0)

$x_1 \le 3$ / $x_1 \ge 4$

(3, 1) P_{11} 17, 17 ; P_{12} ϕ

Question 4. (15pts)

Prove that a function f is convex if and only if its epigraph is a convex set.

Question 5. (20pts)

(a) (10 points; 5 points each) Let $\xi_0, \xi_1, \xi_2, \dots$ be a sequence of independent, identically distributed random variables. Suppose

$$P(\xi_0 = 1) = 0.5 \quad \text{and} \quad P(\xi_0 = -1) = 0.5.$$

Define $Y_n = \xi_n \cdot \xi_{n-1}$.

- Is the sequence $\{Y_n : n \ge 1\}$ a Markov chain? Briefly explain why or why not.
- Is the sequence $\{Y_n : n \ge 1\}$ a martingale? Briefly explain why or why not.

(b) (10 points) Consider two independent Poisson processes $\{N_1(t) : t \ge 0\}$ and $\{N_2(t) : t \ge 0\}$, each with rate $\lambda > 0$. Find the probability that the combined process $\{(N_1(t), N_2(t)) : t \ge 0\}$ will hit the point $(2, 2)$. That is, find the probability that at some time t, $N_1(t) = N_2(t) = 2$. (Hint: you can use symmetry to reduce the size of the system of equations.)

5(a) $P(Y_n \mid Y_{n-1}, Y_{n-2}, \dots, Y_1) = P(Y_n \mid Y_{n-1}) \Rightarrow$ Markov Chain (See 4 pages before)

$E[Y_3 \mid Y_2, Y_1] \ne \cancel{Y_2}$

—End of Paper—

Since $E[Y_3 \mid Y_2 = 1, Y_1 = 1] = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0 \ne 1 = Y_2 \Rightarrow$ Not a martingale.

5(b) $P(N_1(t) = N_2(t) = 2 \text{ at some } t) = P(S_3 > T_2 \text{ and } T_3 > S_2)$

$S_3 = X_1 + X_2 + X_3 \sim \text{Gamma}(3, \lambda)$ with $f_{S_3^{(1)}}(x) = \begin{cases} \frac{1}{2}\lambda e^{-\lambda x}(\lambda x)^2, & x \ge 0 \\ 0, & x < 0 \end{cases}$

$T_2 = Y_1 + Y_2 \sim \text{Gamma}(2, \lambda)$ with $f_{S_2^{(2)}}(y) = \begin{cases} \lambda e^{-\lambda y}(\lambda y), & y \ge 0 \\ 0, & y < 0 \end{cases}$

$P(S_3^{(1)} < S_2^{(2)}) = \int_0^{+\infty} \int_0^{x} f_{S_3^{(1)}}(x) f_{S_2^{(2)}}(y)\, dy\, dx$

∵ If $z^* > -\infty$, (P) min c^Tx s.t. $Ax=0$, $x\ge 0$ has an optimal solution

∴ (D) max 0 s.t. $A^Ty \le c$ has an optimal solution, and $z^* = z_d^* = 0$

1(b). ① $z^* = -\infty$
min $z = -x_1 - x_2$
s.t. $x_1 - x_2 = 0$, $x_1, x_2 \ge 0$.

② $z^* = 0$ min $z = x_1 + x_2$
s.t. $x_1 + x_2 = 0$
$x_1, x_2 \ge 0$

THE CHINESE UNIVERSITY OF HONG KONG

Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2015

(Operations Research)

INSTRUCTIONS

- You should attempt all questions no matter Operations Research is your primary or secondary area.

Question 1. (20pts)

Consider the following linear program with homogeneous linear equality constraints:

$$\begin{aligned} \min \quad & c^\mathsf{T} x \\ \text{s.t.} \quad & Ax = 0 \\ & x \ge 0. \end{aligned}$$

(a) (10 points) Prove that the optimal objective value z^* of this linear program is either $-\infty$ or 0.

(b) (10 points) Draw examples that illustrate both cases in part (a) in $\Re^2$ where A is a 1×2 matrix (i.e., a row vector and there is only one equation) and $c \ne 0$ and $A \ne 0$.

2(a) ($\min_x \|Ax-b\|_1 \Leftrightarrow \min_{x,t} t_1 + \cdots + t_n$ s.t. $t_i = |(Ax-b)_i|$ $(\forall i)$

$\Leftrightarrow \min_{x,t} 1^Tt$ s.t. $t_i \ge |(Ax-b)_i|$ $(\forall i)$

$\Leftrightarrow \min_{x,t} 1^Tt$ s.t. $-t \le Ax-b \le t$) $0 \in A^T \text{sign}(Ax-b)$

Question 2. (20pts)

Consider the following convex optimization problem:

$$(P) \quad \min_{x\in\Re^n} \|Ax - b\|_1,$$

where $A \in \Re^{m\times n}$, $b \in \Re^m$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$.

2(b) $\min_{x,y} \|y\|_1$ s.t. $y = Ax-b$

(D) $\max_{w} \min_{x,y} \|y\|_1 + w^T(Ax-y-b)$

(a) (5pts) Write down the optimality conditions of problem (P).

(b) (10pts) Write down the dual problem of (P).

(c) (5pts) Prove that the weak duality always holds for problem (P).

2(c) $g(w) = \min_{x,y} \|y\|_1 + w^T(Ax-y-b)$

$= \min_{x,y} w^TAx + \|y\|_1 - w^Ty - w^Tb$

$= \begin{cases} -w^Tb, & A^Tw = 0, -1 \le w \le 1 \\ -\infty, & o/w \end{cases}$

Since $\min_y \|y\|_1 - w^Ty = \sum_{i=1}^n |y_i| - \sum_{i=1}^n w_i y_i = \sum_{i=1}^n (|y_i| - w_i \text{sign}(y_i)|y_i|) = \sum_{i=1}^n |y_i|(1 - w_i \text{sign}(y_i)) = \begin{cases} 0, & 1 - \text{sign}(y_i) w_i \ge 0 \\ -\infty, & o/w \end{cases} = \begin{cases} 0, & -1 \le w \le 1 \\ -\infty, & o/w \end{cases}$

Question 3. (20pts)

A company produces a variety of bathroom accessories, including decorative towel rods and shower curtain rods. Each of the accessories includes a rod made out of stainless steel. However, many different lengths are needed: 12″, 18″, 24″, 40″, and 60″. The company purchases 60″ rods from an outside supplier and then cuts the rods as needed for their products. Each 60″ rod can be used to make a number of smaller rods. For example, a 60″ rod could be used to make a 40″ and an 18″ rod (with 2″ of waste), or

∴ (D) max $-w^Tb$
s.t. $A^Tw = 0$
$-1 \le w \le 1$

$\because -b^Tw = -(x^TA^T - y^T)w$
$= -x^TA^Tw - y^Tw$
$= -0^Tw - y^Tw = -\sum_{i=1}^n y_i w_i$
$\le |y_1| + \cdots + |y_n| = \|y\|_1$ (weak duality)

1

Worst Case: $N=164$ 60" rods, with each cut one rod needed

x_{ij}: i^{th} rod cut x_{ij} j^{th}-length rod. $(i=1,\cdots,164,\ j=1,\cdots,5)$; L_j: length of j^{th}-

minimize # of 60" rods is equivalent to maximizing waste.

$\min \sum_{i=1}^{164} y_i$ s.t. $\sum_{i=1}^{164} x_{ij} = n_j \ (j=1\sim5)$, $\sum_{j=1}^{5} L_j x_{ij} \le 60$, $x_{ij} \in \mathbb{Z}^+$.

$\sum_{j=1}^{5} x_{ij} \le M y_i$ (M large enough) $y_i \in \{0,1\}$

5 12" rods (with no waste). For the next production period, the company needs 25 12" rods, 52 18" rods, 45 24" rods, 30 40" rods, and 12 60" rods. What is the fewest number of 60" rods that can be purchased to meet their production needs? Formulate an integer programming problem for this problem. (You do not need to solve it).

Question 4. (20pts)

Consider the following continuous knapsack problem. The goal is to maximize the value of N objects packed into a knapsack with limited capacity b. For $i = 1, \ldots, N$, let x_i denote the number of units of object i packed, $f_i(x_i)$ the resulting value and $g_i(x_i)$ the capacity of the knapsack consumed. Suppose that each x_i is a continuous variable, i.e., it can take any value in $[0, \infty)$. $S_1 = b$. $S_{k+1} = S_k - g_k(x_k)$, $F_k(S_k) = \max_{g_k(x_k) \le S_k} \{ f_k(x_k) + F_{k+1}(S_{k+1}) \}$.

(a) (10pts) Formulate this problem as a dynamic programming problem.

(b) (10pts) Suppose $f_i(x) = f(x)$, $i = 1, \ldots, N$, i.e., the value function is the same for all the N objects, and $g_i(x) = x$, $i = 1, \ldots, N$. In addition, assume that f is differentiable, strictly increasing and concave on $[0, \infty)$, so that $f'(x)$ is decreasing in x. Find the optimal solution of this dynamic programming problem.

$F_N(b-\sum_{i=1}^{N-1} x_i) = \max_{x_N \le b-\sum_{i=1}^{N-1} x_i} \{f(x_N)\} \Rightarrow x_N^* = b - \sum_{i=1}^{N-1} x_i$.

$F_{N-1}(b-\sum_{i=1}^{N-2} x_i) = \max_{x_{N-1} \le b-\sum_{i=1}^{N-2} x_i} \{f(x_{N-1}) + F_N(b-\sum_{i=1}^{N-1} x_i)\} \Rightarrow x_{N-1}^* = \frac{1}{2}(b-\sum_{i=1}^{N-2} x_i)$

$x_1^* = \frac{1}{N} b$.

$F_1(b) = f(\frac{b}{N}) + (N-1) f(\frac{b}{N})$

$\therefore F_1(b) = N f(\frac{b}{N})$.

$x_1^* = \cdots = x_N^* = \frac{b}{N}$

Question 5. (20pts; 4pts each)

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent, identically distributed random variables. Suppose

$$P(\xi_1 = 1) = 0.4 \quad \text{and} \quad P(\xi_1 = -1) = 0.6.$$

Define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} \xi_i$ for $n \ge 1$. In addition, let $\{N(t) : t \ge 0\}$ be a Poisson process with rate $\lambda > 0$, which is independent of the sequences $\xi_1, \xi_2, \ldots$.

(a) Is $\{|S_n| : n \ge 0\}$ a discrete time Markov Chain? Briefly explain why or why not.

(b) Is $\{S_n : n \ge 1\}$ a Martingale with respect to the sequence $\xi_1, \xi_2, \ldots$? Briefly explain why or why not.

$E\big[|S_{n+1}| \,\big|\, |S_n|=0, |S_{n-1}|, \cdots, |S_1|\big] = 0.4 \times 1 + 0.6 \times 1 = 1 \ne |S_n| = 0$

(c) Is $\{S_{N(t)} : t \ge 0\}$ a continuous time Markov Chain? Briefly explain why or why not.

$P\{S_{N(t)} = b \mid S_{N(s)} = a, S_{N(u)} = z(u), 0 < u < s\} = P\{\sum_{i=1}^{N(t)} \xi_i = b \mid \sum_{i=1}^{N(s)} \xi_i = a, \cdots\} \Rightarrow$

$= P\{\sum_{i=N(s)+1}^{N(t)} \xi_i = b-a \mid \cdots\}$

$= P\{\sum_{i=N(s)+1}^{N(t)} \xi_i = b-a\}$ (independent increment & independence of ξ_i)

(d) For fixed constant $T > 0$, find the optimal solutions for the following problem:

$$\max_{q \ge 0, q \in \mathbb{R}} \left(\mathbb{E}\left[\sum_{i=1}^{N(T)} (\xi_i \cdot 1_{\xi_i \ge q})\right] - 0.5 \cdot \mathbb{E}\left[\sum_{i=1}^{N(T)} 1_{\xi_i \ge q}\right] \right). \quad (1)$$

Here $1_{\xi_i \ge q}$ is the indicator function. It takes value 1 if $\xi_i \ge q$, and 0 otherwise.

(e) Re-solve the optimization problem (1) if for each i, ξ_i is a positive continuous random variable with an integrable density function $h(x)$, i.e., $\int_0^\infty x h(x) dx < \infty$.

End of Paper

5(d). If $q \in [0,1]$

$E\left[\sum_{i=1}^{N(T)} \xi_i \cdot 1_{\xi_i \ge q}\right] = \sum_{k=0}^{\infty} E\left[\sum_{i=1}^{k} \xi_i \cdot 1_{\xi_i \ge q} \,\middle|\, N(T)=k\right] P(N(T)=k) = \sum_{k=0}^{\infty} k\, E[\xi_1 \cdot 1_{\xi_1 \ge q}]\, P(N(T)=k)$

$= \sum_{k=0}^{\infty} 0.4k \cdot \frac{e^{-\lambda T}(\lambda T)^k}{k!}$ ~~$= 0.4 e^{-\lambda T} \sum_{k=0}^{\infty} \frac{(\lambda T)^k}{(k-1)!} = 0.4 e^{-\lambda T}\left[\sum_{k=1}^{\infty} \frac{(\lambda T)^{k-1}}{(k-1)!} \lambda T + 1\right] = 0.4 e^{-\lambda T}[\lambda T e^{\lambda T} + 1]$~~

$=$ ~~$0.4 e^{-\lambda T}$~~ $+ 0.4 \lambda T$. $E\left[\sum_{i=1}^{N(T)} 1_{\xi_i \ge q}\right] = \sum_{k=0}^{\infty} E\left[\sum_{i=1}^{N(T)} 1_{\xi_i \ge q} \,\middle|\, N(T)=k\right] P(N(T)=k) = 0.4$

$\therefore \max = 0.4($~~$e^{-\lambda T}$~~$+ \lambda T + 1)$ $(0 \le q \le 1)$. (If $q > 1$, objective $= 0$.)

5(e) $E\left[\sum_{i=1}^{N(T)} \xi_i \cdot 1_{\xi_i \ge q}\right] = \sum_{k=0}^{\infty} k \cdot \int_{-\infty}^{\infty} x \cdot 1_{x \ge q}\, h(x) dx \cdot P(N(T)=k) = \sum_{k=0}^{\infty} k \cdot \int_q^{\infty} x h(x) dx \cdot P(N(T)=k)$

$\min\ (1-\theta)x_1+(2-\theta)x_2$

s.t. $\begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \begin{pmatrix} -\theta \\ \theta \\ -4 \end{pmatrix}$

$(x_1, x_2) \geq 0$

Dual: $\max\ -\theta y_1+\theta y_2-4y_3$

s.t. $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \leq \begin{pmatrix} 1-\theta \\ 2-\theta \end{pmatrix}$

$(y_1, y_2, y_3) \geq 0$

i.e. s.t. $-y_1+y_2 \leq 1-\theta$

$-y_1-y_3 \leq 2-\theta$, $(y_1, y_2, y_3) \geq 0$

THE CHINESE UNIVERSITY OF HONG KONG

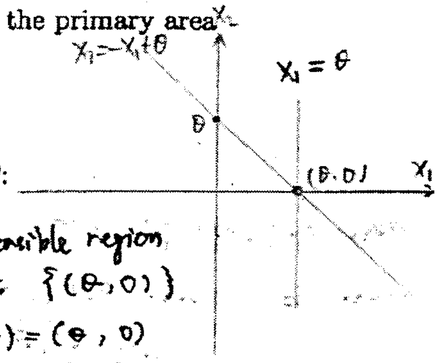
Department of Systems Engineering and Engineering Management

PhD Qualifying Examination 2014

(Operations Research)

INSTRUCTIONS

- Answer ALL questions if your primary area is Operations Research.
- If your secondary area is Operations Research, do NOT answer those questions that are marked with an asterisk *
- The score of each question is denoted by $(P:\cdot;S:\cdot)$, where P is the primary area score and S is the secondary area score.



Question 1. (P:20pts; S:20pts)

Consider the following linear program which depends on a parameter θ:

$$\begin{aligned} \min\quad & z = x_1 + 2x_2 - \theta(x_1 + x_2) \\ \text{s.t.}\quad & x_1 + x_2 \leq \theta \\ & x_1 \geq \theta \\ & x_2 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

2(b) Feasible region is $\{(\theta, 0)\}$

$\therefore x(\theta) = (\theta, 0)$

$z(\theta) = (1-\theta)\theta$.

(a) (P:5pts; S:10pts) Give the dual of this linear program.

(b) (P:10pts; S:10pts) Give the optimal value of the objective function of this linear program, $z = z(\theta)$ and the optimal solution $x(\theta)$ as a function of θ for all nonnegative real values of θ. (Hints: it may be helpful to graph this problem).

(c) * (P:5pts) For θ nonnegative, show that or explain why the optimal dual solution is not unique. 2(c) By strong duality, $V_d^* = (1-\theta)\theta$.

Let $\begin{cases} -\theta y_1+\theta y_2-4y_3 = (1-\theta)\theta \\ -y_1+y_2 \leq 1-\theta \\ -y_1-y_3 \leq 2-\theta \\ y_1, y_2, y_3 \geq 0 \end{cases}$ We just need $\begin{cases} -y_1+y_2 = 1-\theta \\ y_3 = 0 \\ -y_1-y_3 \leq 2-\theta \\ y_1, y_2, y_3 \geq 0 \end{cases} \Leftrightarrow \begin{cases} y_2 = y_1-\theta+1 \\ y_1 \geq \theta-2 \\ y_3 = 0 \\ y_1, y_2 \geq 0 \end{cases}$

Solutions to this system is not unique.

Question 2. (P:20pts; S:20pts)

Suppose you have $3000 to invest. Investments A, B and C have expected rates of return 4%, 7%, and 8%, respectively, and each can be bought in multiples of $1000. Moreover, there is a fixed commission of $30, $70, and $100 to make purchases of A, B and C, respectively, no matter how many $1000 multiples of the investment is purchased. Give a Dynamic Programming (DP) formulation for solving the problem of how much you should invest in A, B and C to maximize the net return at the end of one year. Specifically, define (i) the stages in the DP recursion, (ii) the possible states at each stage, (iii) the optimal value function corresponding to each state at each stage, and (iv) give the DP recursion. For the latter, you will need to clearly define net returns for various investments depending on the state.

$\max\ 40x_1-30y_1+70x_2-70y_2+80x_3-100y_3$

s.t. $x_1+x_2+x_3 \leq 3$

$x_i \leq 3y_i$

$x_i \geq 0$

$y_i \in \{0,1\}$.

$\max_{\xi \geq 0} \int_{\xi}^{\infty} (x-\tfrac{1}{2})h(x)dx = \int_{\frac{1}{2}}^{\infty} x\,h(x)\,dx \triangleq M$.

$\Rightarrow \max_{\xi \geq 0} E\left[\sum_{t=1}^{N(T)} \xi_i \cdot \mathbb{1}_{\xi_i \geq \xi}\right] = \sum_{k=0}^{\infty} M \cdot k \cdot \frac{e^{-\lambda T}(\lambda T)^k}{k!}$

and $\xi^* = \frac{1}{2}$

$f_k(S_k) = \max_{0 \leq x_k \leq S_k} \{r_i x_i - c_i \operatorname{sign}(x_i) + f_{k+1}(S_{k+1})\}$

3(a). $L(x, v, w) = c^T x + v(\|D^{-1}(x-z)\|_2^2 - \beta^2) + w^T(Ax-b)$

KKT:
$$\begin{cases} c + 2v D^{-2}(x-z) + A^T w = 0 \\ v \geq 0 \\ v(\|D^{-1}(x-z)\|_2^2 - \beta^2) = 0 \\ Ax = b \\ \|D^{-1}(x-z)\|_2^2 \leq \beta^2 \end{cases}$$

3(b) If $v = 0$. …

If $v > 0 \Rightarrow$
$$\begin{cases} c + 2v D^{-2}(x-z) + A^T w = 0 & ① \\ \|D^{-1}(x-z)\|_2^2 = \beta^2 & ② \\ Ax = b & ③ \end{cases}$$

Question 3. (P:20pts; S:20pts)

Consider the following problem

(since, $c^T x$ is affine, $Ax=b$ affine and $\|D^{-1}(x-z)\|_2 - \beta$ is convex, slater's holds, KKT conditions hold)

$$\begin{aligned} \min_x \quad & c^T x \\ \text{s.t.} \quad & Ax = b, \\ & \|D^{-1}(x-z)\| \leq \beta, \end{aligned}$$

where $A \in \Re^{m \times n}$ has full row rank, $z \in \Re^n$ is a given vector with all entries being positive, D is a diagonal matrix with positive diagonal elements $D_{ii} := z_i, i = 1, \ldots, n$, and $\beta \in (0, 1)$.

(a) **(P:10pts; S:20pts)** Give the Karush-Kuhn-Tucker optimality conditions for this problem.

(b) * **(P:10pts)** From the optimality conditions express the optimal solution x^* as $x^* = z + p$; i.e., what is p?

Question 4. (P:20pts; S:20pts)

Consider the following integer nonlinear programming problem:

$$\begin{aligned} \max \quad & 2x_1^2 - x_1^3 + 5x_2^2 - 3x_2^4 \\ \text{s.t.} \quad & x_1 + x_2 \leq 3, \\ & x_1 \geq 0, x_2 \geq 0, \\ & \text{both } x_1 \text{ and } x_2 \text{ integer} \end{aligned}$$

Let $x_1 = y_1 + 2y_2$

$x_2 = y_3 + 2y_4$

$y_1, y_2, y_3, y_4 \in \{0, 1\}$

max $y_1 y_2$ is reformulated as

max z s.t. $y_1 + y_2 - 1 \leq z$

$y_1 \geq z$

$y_2 \geq z$, $z \in \{0, 1\}$

or max $1 - z$

s.t. $2 - (y_1 + y_2) \leq Mz$

(M large enough)

Formulate this problem as a Binary Integer Programming problem.

Question 5. (P:20pts; S:20pts)

Let $X = \{X_n : n = 0, 1, \ldots\}$ be a Discrete Time Markov Chain on state space $\{1, 2, 3, 4, 5, 6, 7\}$ with transition matrix

$$P = \begin{pmatrix} 0.25 & 0.75 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0.25 & 0.25 & 0.3 & 0 & 0 \\ 0.25 & 0 & 0.4 & 0.25 & 0.1 & 0 & 0 \\ 0.25 & 0.15 & 0.1 & 0.2 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

5(a).
Consider 3 irreducible DTMC

① $P_1 = \begin{bmatrix} 0.25 & 0.75 \\ 0.5 & 0.5 \end{bmatrix}$

$\Rightarrow \pi_1 = 0.4, \quad \pi_2 = 0.6 . \quad d_1 = d_2 = 1$

$E_1(\tau_1) = \frac{1}{0.4} = 2.5, \; E_2(\tau_2) = \frac{1}{0.6} = \frac{5}{3}$

(a) **(P:10pts; S:20pts)** Find $\mathbf{E}_i(\tau_i)$, where $\tau_i = \inf\{n \geq 1 : X_n = i\}$, for each $i = 1, \ldots, 7$. (You do not need to provide a proof.)

(b) * **(P:10pts)** For each pair (i, j) with $1 \leq i, j \leq 7$, find $\lim_{n \to \infty} P_{ij}^n$ when the limit exists. Otherwise, state it does not exist.

②
state 3, 4, 5 are transient, then

$E_3(\tau_3) = E_4(\tau_4) = E_5(\tau_5) = \infty$

③ $P_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$(\pi_6, \pi_7) = (\frac{1}{2}, \frac{1}{2})$

$d_6 = d_7 = 2$

$\lim_{n \to \infty} P_{jj}^{nd} = 1, \; j = 6, 7$

$E_6(\tau_6) = E_7(\tau_7) = \frac{d}{\lim_{n \to \infty} P_{jj}^{nd}} = 2$

5(b) $\lim_{n \to \infty} P_{i,1}^n = \pi_1 = 0.4, \; i = 1, 2, 3, 4, 5$

$\lim_{n \to \infty} P_{i,2}^n = \pi_2 = 0.6, \; i = 1, 2, 3, 4, 5$

$\lim_{n \to \infty} P_{i,j}^n = \frac{1}{E(\tau_j)} = 0, \; i = 1, 2, 3, 4, 5, 6, 7, \; j = 3, 4, 5$

$\lim_{n \to \infty} P_{ij}^n$ doesn't exist $i, j = 6, 7$

since they aren't aperiodic

$L(x,v,w) = c^T x + v(\|x\|_2^2 - 1) + w^T A x$

KT: $c + 2vx + A^T w = 0$ ①

$v \geq 0$

$v(\|x\|_2^2 - 1) = 0$

$Ax = 0$

$\|x\|_2^2 \leq 1$

1(b) ① ⇒ $Ac + 2vAx + AA^T w = 0$

⇒ $AA^T w = -Ac$ ⇒ $w = -(AA^T)^{-1} Ac$ (A full col rank)

If $v > 0$, then $\|x\|_2^2 - 1 = 0 \Leftrightarrow x^T x = 1$

① ⇒ $x = \frac{1}{2v}[A^T(AA^T)^{-1}Ac - c]$ } ⇒ $v = \frac{1}{2}[c^Tc - c^TA^T(AA^T)^{-1}Ac]^{\frac{1}{2}}$

Then $x^* = [A^T(AA^T)^{-1}Ac - c] / [c^Tc - c^TA^T(AA^T)^{-1}Ac]^{1/2}$.

If $v = 0$, $c^T x = -w^T A x = 0$ $\forall x \in$ feasible region, x is optimal

The Chinese University of Hong Kong

Department of Systems Engineering & Engineering Management

Ph.D. Qualifying Examination 2013

Area: Operations Research

INSTRUCTIONS

1. Answer ALL questions.
2. The score of each question is denoted by $(P:\cdot, S:\cdot)$, where P is the primary area score and S is the secondary area score.

Problem 1 (P:30pts, S:30pts). Let $c \in \mathbb{R}^n \setminus \{0\}$ and $A \in \mathbb{R}^{m\times n}$ be given. Suppose that A has full row rank. Consider the following problem:

$$\begin{aligned} v^* = \ & \text{minimize} && c^T x \\ & \text{subject to} && Ax = 0, \\ & && \|x\|_2^2 \leq 1. \end{aligned} \tag{1}$$

2. max $1^T z$

s.t. $A(y+z) = 0$

$y \geq 0$, $0 \leq z \leq 1$

(a) **(P:15pts, S:15pts).** Write down the KKT conditions for Problem (1) and explain why they are necessary and sufficient for optimality.

(b) **(P:15pts, S:15pts).** Using the result in (a), or otherwise, express the optimal solution x^* to Problem (1) in terms of A and c.

Problem 2 (P:20pts, S:20pts). Let $A \in \mathbb{R}^{m\times n}$ be given. We are interested in finding a vector $x \in \mathbb{R}^n_+$ such that $Ax = 0$ and the number of positive components of x is maximized. Formulate this problem as a linear program. Justify your answer.

Problem 3 (P:25pts, S:25pts). Suppose that we are given n items $\{1,\ldots,n\}$, where the value and size of the i-th item is v_i and w_i, respectively. We are also given a bag of size W, where $W \geq 1$ is an integer. Our goal is to choose a subset $T \subset \{1,\ldots,n\}$ of the items to put into the bag, so that the total size of the selected items, which is defined as $\sum_{i\in T} w_i$, is at most W, and the total value of the selected items, which is defined as $\sum_{i\in T} v_i$, is maximized. Formulate this problem as a dynamic program. Justify your answer.

Problem 4 (P:25pts, S:25pts). Suppose that we have two boxes and $2d$ balls, of which d are black and d are red. Initially, d of the balls are placed in box 1, and the rest of the balls are placed in box 2. At each trial a ball is chosen at random from each of the boxes, and the two balls are then put back in the opposite boxes. Let X_n be the number of black balls in box 1 after n trials, where $n = 0,1,\ldots$. Then, the process $\mathcal{X} = \{X_n : n \geq 0\}$ forms a Markov chain on the state space $S = \{0,1,\ldots,d\}$.

(a) **(P:5pts, S:10pts).** Explain why the transition probabilities are given by

$$p_{i,i-1} = \frac{i^2}{d^2} \quad \text{for } i = 1,2,\ldots,d; \qquad p_{i,i+1} = \frac{(d-i)^2}{d^2} \quad \text{for } i = 0,1,\ldots,d-1;$$

$$p_{i,i} = \frac{2i(d-i)}{d^2} \quad \text{for } i = 0,1,\ldots,d.$$

$s_1 = W$, $S_{k+1} = S_k - w_k x_k$

$\max_x \sum_{i=1}^{n} v_i x_i$

s.t. $\sum_{i=1}^{n} w_i x_i \leq W$

$x_i \in W$

$F_k(S_k) = \max_{w_k x_k \leq S_k} \{ v_k x_k + F_{k+1}(S_{k+1}) \}$.

1(a) $\min_{x} \|Ax - \mathrm{Diag}(u)b\|_2^2$

$\Leftrightarrow \min_{x,y} \|y\|_2^2$

s.t. $y = Ax - \mathrm{Diag}(u)b$

KKT: $\frac{\partial L}{\partial x} = A^T w = 0$. $\frac{\partial L}{\partial y} = 2y - w = 0$

$\Rightarrow x^* = (A^TA)^{-1}A^T \mathrm{Diag}(u)b$.

1(b) $\min_{u} \|A(A^TA)^{-1}A^T \mathrm{Diag}(u)b - \mathrm{Diag}(u)b\|_2^2$

$= \min_{u} \|(A(A^TA)^{-1}A^T - I)\,\mathrm{Diag}(b)\,u\|_2^2$

$= \min_{u} \|Cu\|_2^2 = \min_{u} u^TC^TCu \triangleq \min_{u} u^T M u$

s.t. $|u_i| = 1$

The Chinese University of Hong Kong

Department of Systems Engineering & Engineering Management

Ph.D. Qualifying Examination 2012

Area: Operations Research

INSTRUCTIONS

1. Answer ALL questions if your primary area is Operations Research.
2. Answer only those questions that are marked with an asterisk (*) if your secondary area is Operations Research.
3. The score of each question is denoted by $(P:\cdot, S:\cdot)$, where P is the primary area score and S is the secondary area score.

Problem 1* (P:20pts, S:25pts). Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ be given. Suppose that A has full column rank. Consider the following problem:

$$\begin{array}{lll} \text{minimize} & \|Ax - \mathrm{diag}(u)b\|_2^2 & \\ \text{subject to} & |u_i| = 1 & \text{for } i = 1,\ldots,m, \\ & x \in \mathbb{R}^n, & \end{array} \tag{1}$$

where $\mathrm{diag}(u)$ is the $m \times m$ diagonal matrix whose diagonal entries are given by u.

(a) **(P:15pts, S:15pts).** Suppose that the vector u is given. Then, Problem (1) becomes

$$\min_{x\in\mathbb{R}^n} \|Ax - \mathrm{diag}(u)b\|_2^2. \tag{2}$$

Write down the KKT conditions for Problem (2) and explain why they are necessary and sufficient for optimality. Hence, or otherwise, express the optimal solution x^* in terms of A, u and b.

(b) **(P:5pts, S:10pts).** Using the result in (a), show that Problem (1) can be reformulated as

$$\begin{array}{lll} \text{minimize} & u^TMu & \\ \text{subject to} & |u_i| = 1 & \text{for } i = 1,\ldots,m \end{array}$$

for some suitable $m \times m$ matrix M.

Problem 2* (P:25pts, S:35pts). Consider the linear program

$$\begin{array}{ll} \text{minimize} & 2x_1 + 9x_2 + 3x_3 \\ \text{subject to} & -2x_1 + 2x_2 + x_3 \geq 1, \\ & x_1 + 4x_2 - x_3 \geq 1, \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{3}$$

2(a)

(P) $\min\ (2, 9, 3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

s.t. $\begin{bmatrix} -2 & 2 & 1 \\ 1 & 4 & -1 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$x_1, x_2, x_3 \geq 0$

(D) $\max\ y_1 + y_2$

s.t. $\begin{bmatrix} -2 & 1 \\ 2 & 4 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 9 \\ 3 \end{pmatrix}$

$y_1, y_2, y_3 \geq 0$

i.e. $\max\ y_1 + y_2$

s.t. $-2y_1 + y_2 \leq 2$

$2y_1 + 4y_2 \leq 9$

$y_2 - y_2 \leq 3$

$y_1, y_2, y_3 \geq 0$

$\Rightarrow v_d^* = 4$

by strong duality $V_p^* = 4$

$\therefore 2x_1^* + 9x_2^* + 3x_3^* = 4$

Besides, (x_1^*, x_2^*, x_3^*) should satisfy 3 of the 5 equations
$$\begin{cases} -2x_1 + 2x_2 + x_3 = 1 \\ x_1 + 4x_2 - x_3 = 1 \\ x_1 = 0 \\ x_2 = 0, \quad x_3 = 0 \end{cases}$$

If $x_1 = 0$, $x_2 \neq 0$, $x_3 \neq 0$

$\Rightarrow x_2 = \frac{1}{3}$, $x_3 = \frac{1}{3}$

$(0, \frac{1}{3}, \frac{1}{3})$ satisfies $2x_1 + 9x_2 + 3x_3 = 4 = V_d^*$

$\Rightarrow$ is an opt. sol.

The other cases are not true.

(a) **(P:15pts, S:20pts).** Write down the dual of Problem (3) and solve it graphically.

(b) **(P:10pts, S:15pts).** Using the result in (a), or otherwise, determine the optimal solution to Problem (3).

Problem 3* (P:15pts, S:20pts). Let A_i be an $n_{i-1} \times n_i$ matrix, for $i = 1, \dots, n$. We are interested in the cheapest way to evaluate the product

$$A = A_1 \times A_2 \times \cdots \times A_m,$$

where the cost of multiplying a $p \times q$ matrix by a $q \times r$ matrix is defined to be pqr. Give a dynamic programming algorithm to solve this problem and analyze its runtime.

Problem 4* (P:15pts, S:20pts). If n balls are placed at random into n bins, what is the probability that exactly one bin remains empty? Show your calculations.

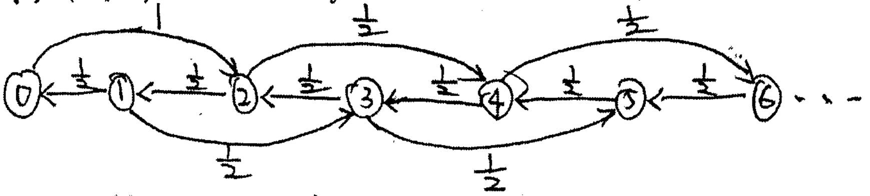
$$p = C_n^1 C_{n-1}^1 C_n^2 A_{n-2}^{n-2} / n^n = \frac{C_n^2 \cdot n!}{n^n}$$

Problem 5 (25pts). Consider a Markov chain with state space $S = \{0, 1, \dots\}$ and transition probabilities given by

$$p_{0,2} = 1, \quad p_{i,i+2} = p_{i,i-1} = \frac{1}{2} \quad \text{for } i = 1, 2, \dots.$$

(a) **(10pts).** Determine the period of state 2. 3

(b) **(15pts).** Show that every state of the chain is transient.



5(b) For state 0 coming back, there will be n steps going right, $2n$ steps going left.

$P_{02} = 1$, $P_{ij} = \frac{1}{2}$ for other (i,j) pair $\quad \therefore P_{00}^{3n} = C_{3n-1}^{n-1} \left(\frac{1}{2}\right)^{3n-1} = \frac{(3n-1)!}{(2n)!\,(n-1)!} \left(\frac{1}{2}\right)^{3n-1}$

$$\approx \frac{\sqrt{2\pi(3n-1)} \left(\frac{3n-1}{e}\right)^{3n-1}}{\sqrt{2\pi(2n)} \left(\frac{2n}{e}\right)^{2n} \cdot \sqrt{2\pi(n-1)} \left(\frac{2(n-1)}{e}\right)^{2n-1}} \left(\frac{1}{2}\right)^{3n-1}$$

3. $S_1 = \{n_1, n_2, \dots, n_{m-1}\}$ ordered set, $S_{k+1} = S_k \setminus \{a_k\}$. a_k is an integer chosen from S_k

Optimal cost from state k to $m-1$ $\quad C_k(S_k) = \max_{a_k \in S_k} \{ a_{k-1} a_k a_{k+1} + C_{k+1}(S_{k+1}) \}$

where a_{k-1}, a_k, a_{k+1} are successive in S_k.