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Problem 1. Note that $\cancel{\nabla f}$

$$\nabla f(y) - \nabla f(x) = \int_0^1 \nabla^2 f(x + t(y-x))(y-x)\, dt$$

thus. $\|\nabla f(y) - \nabla f(x) - \nabla^2 f(x)(y-x)\|$

$$= \left\| \int_0^1 \nabla^2 f(x+t(y-x))(y-x)\, dt - \nabla^2 f(x)(y-x) \right\|$$

$$\cancel{= \left\| \left[\nabla^2 f(x+t(y-x))(y-x)t \right]\Big|_0^1 - \int \nabla^2 \right.}$$

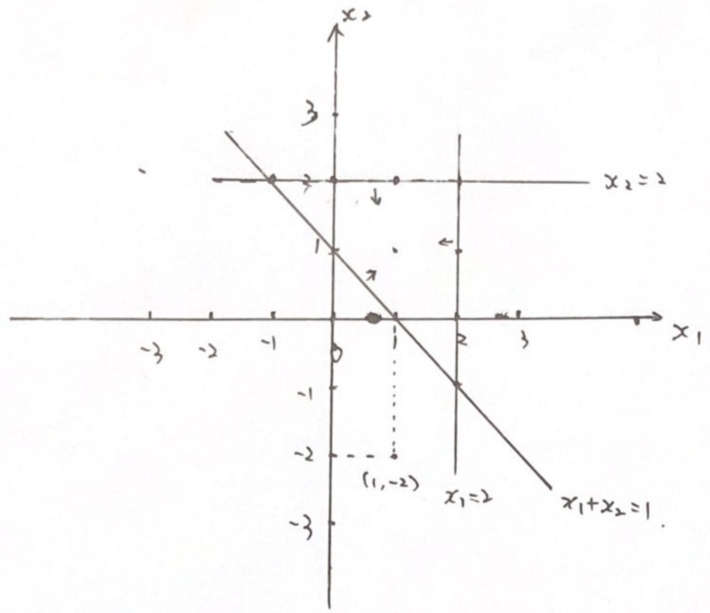
$$\cancel{= \| [}$$

$$\le \left\| \int_0^1 \nabla^2 f(x+t(y-x))\, dt - \nabla^2 f(x) \right\| \, \| y-x \|$$

$$\le \frac{1}{2} \| \nabla^2 f(y) - \nabla^2 f(x) \| \, \| y-x \|$$

$$\le \frac{L}{2} \| y-x \|^2$$

Problem 2 (a) Solution. Let's solve the problem using graph.



Note that all feasible solutions ~~of~~ are $(-1, 2)$, $(0, 2)$. $(0, 1)$. $(1, 2)$ $(1, 1)$ $(1, 0)$. $(2, 2)$, $(2, 1)$. $(2, 0)$. $(2, -1)$.

Thus the optimal solutions are the one that closest ~~the~~ to the point $(1, -2)$, that is $x^* = (2, -1)$, the optimal objective value is $(2-1)^2 + (-1+2)^2 = 2$.

(b) Given $v \in \mathbb{R}_+$ as the multiplier associated with $x_1 + x_2 \geq 1$, the Lagrangian dual is given by

$$\sup_{v \in \mathbb{R}_+} \theta(v),$$

where
$$\theta(v) = \inf_{x \in \mathbb{R}^2} \left\{ (x_1-1)^2 + (x_2+2)^2 + v(1 - x_1 - x_2) \right\}$$

$$= \inf_{x \in \mathbb{R}^2} \left\{ x_1^2 - (2+v)x_1 + x_2^2 + (4-v)x_2 + 5 + v \right\}.$$

Let $g(x) = x_1^2 - (2+v)x_1 + x_2^2 + (4-v)x_2 + 5 + v$

Note that $g(x)$ is twicely differentiable on $\mathbb{R}^2$. 3

$$\nabla g(x) = \begin{bmatrix} 2x_1 - (v+2) \\ 2x_2 + 4 - v \end{bmatrix}$$

$$\nabla^2 g(x) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \succ 0.$$

~~Thus the~~

let $\nabla g(x) = 0$, we get

$$\begin{cases} x_1 = \frac{v+2}{2} \\ x_2 = \frac{v-4}{2} \end{cases}$$

thus $\inf_{x \in \mathbb{R}^2} \{g(x)\} = g\left(\begin{bmatrix} \frac{v+2}{2} \\ \frac{v-4}{2} \end{bmatrix}\right)$

$$= \frac{(v+2)^2}{4} - \frac{(v+2)^2}{2} + \frac{(v-4)^2}{4} - \frac{(v-4)^2}{2} + 5 + v$$

$$= -\frac{(v+2)^2}{4} - \frac{(v-4)^2}{4} + 5 + v$$

$$= -\frac{1}{2}v^2 + v - 5 + 5 + v$$

$$= -\frac{1}{2}v^2 + 2v$$

thus, the ~~Lagrangian~~ Lagrangian dual can be further written as

$$\sup_{v \in \mathbb{R}^+} \left\{-\frac{1}{2}v^2 + 2v\right\}$$

The optimal solution to the dual problem is $v = 2$, with the optimal objective value as $-\frac{1}{2} \times 2^2 + 2 \times 2 = 2$.

Problem 3. (a). Note that ~~x^2~~ $\|x\|^2 \le yz \Rightarrow yz \ge 0$

thus $\|x\|^2 \le yz$

$$\Leftrightarrow \quad \|x\|^2 \le \frac{1}{4}\left[(y+z)^2 - (y-z)^2\right]$$

$$\Leftrightarrow \quad 4\|x\|^2 \le (y+z)^2 - (y-z)^2$$

$$\Leftrightarrow \quad 4\|x\|^2 + (y-z)^2 \le (y+z)^2$$

$$\Leftrightarrow \quad \sqrt{4\|x\|^2 + (y-z)^2} \le |y+z|$$

$$\Leftrightarrow \quad \left\| \begin{matrix} 2x \\ y-z \end{matrix} \right\| \le |y+z|.$$

Since y and z are nonnegative, then we have

$$\left\| \begin{matrix} 2x \\ y-z \end{matrix} \right\| \le y+z.$$

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problem 3.(b)

Note that the problem can be formulated as follows.

$$\frac{1}{\min\limits_{x \in \mathbb{R}^n} \left(\sum_{i=1}^{m} \frac{1}{a_i^T x - b_i} \right)}$$

Let $y_i = \dfrac{1}{a_i^T x - b_i}$, ~~$t = 1, \ldots$~~ $i = 1, \ldots, m$

Problem 4. 1(a).

Suppose T is bounded, ~~let's prove that~~

i.e. $\exists\ \varepsilon > 0$. s.t. $\forall\ y \in T$. we have

$$\|y\|^2 \leq \varepsilon.$$

~~Let's then prove that S is unbounded, suppose not.~~

Let's the prove that S is unbounded,

~~then~~.

Note that $\forall\ x \in S$. we have

$$Ax \geq b,\ x \geq 0.$$

①

conversely. if S is unbounded. then $\exists\ d \in \mathbb{R}^n$. s.t. $Ad \geq b.\ d \geq 0$

$d \neq 0$..

4(b). Yes, they can.

let $A = [0 \ 0] \in \mathbb{R}^{1\times 2}$.

$b = 0$.

$C = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \mathbb{R}^2$.

then $\forall\, x \in \mathbb{R}^2_+$, we have $Ax = 0 \geq b = 0$

$\forall\, y \in \mathbb{R}^1_+$, we have $A^T y = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \leq C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

both S and T ~~there~~ in this case are unbounded.

Problem 5. (a) Solution.

Let $f(x) = c^T x$

$g_i(x) = d_i^T x - l_i$, $i = 1, \cdots, m$. (1) u

$h_j(x) = f_j^T x - g_j$, $j = 1, \cdots, p$. (2) w

$m_i(x) = -x_i$, $i = 1, \cdots, n$. (3) v

Let $u \in \mathbb{R}_+^m$, $v \in \mathbb{R}_+^n$, $w \in \mathbb{R}^p$ be the multiplier associated with constraint (1), (3) and (2), respectively.

~~Note that~~

The KKT conditions are as follows.

$$\begin{cases} \nabla f(x) + \sum_{i=1}^{m} u_i \nabla g_i(x) + \sum_{j=1}^{p} w_j \nabla h_j(x) + \sum_{k=1}^{n} v_k \nabla m_k(x) = 0 \\ u \in \mathbb{R}_+^m, \quad v \in \mathbb{R}_+^n. \\ u_i g_i(x) = 0, \quad i = 1, \cdots, m. \\ v_k x_k = 0, \quad k = 1, \cdots, n. \end{cases}$$

$$\Rightarrow \begin{cases} c + \sum_{i=1}^{m} u_i d_i + \sum_{j=1}^{p} w_j f_j + \sum_{k=1}^{n} v_k(-e_k) = 0. \\ u \in \mathbb{R}_+^m, \quad v \in \mathbb{R}_+^n. \\ u_i g_i(x) = 0, \quad i = 1, \cdots, m \\ v_k x_k = 0, \quad k = 1, \cdots, n. \end{cases}$$

$$\Rightarrow \begin{cases} c + \sum_{i=1}^{m} u_i d_i + \sum_{j=1}^{p} w_j f_j - v = 0. \\ u \in \mathbb{R}_+^m, \quad v \in \mathbb{R}_+^n \\ u_i g_i(x) = 0, \quad i = 1, \cdots, m \\ v_k x_k = 0, \quad k = 1, \cdots, n. \end{cases}$$

Problem 5. (b) proof.

~~Since $g_i(x)$, $i=1,\dots,m$,~~

Note that $g_i(x), \forall i$, $m_k(x), \forall k$ are concave functions. $h_j(x)$ $\forall i$ are affine functions, then we know that KKT conditions are necessary for the optimality of the original problem.

Since ~~the~~ $f(x)$ is convex, the problem (3) is a convex optimization problem, thus KKT conditions are also sufficient for the optimality.

Note that if x^* is optimal for problem (3), then KKT conditions must hold for x^*. thus we have

$$\begin{cases} c + \sum_{i=1}^{m} u_i d_i + \sum_{j=1}^{p} w_j f_j - v = 0. \\ u \in \mathbb{R}_+^m, \ v \in \mathbb{R}_+^n, \\ u_i g_i(x^*) = 0. \ i=1,\dots,m \\ v_k x_k^* = 0, \ k=1,\dots,n. \end{cases}$$

if $\sum_{j=1}^{p} \|f_j\|_0 \le \bar{p} \le n.$

then $\| \sum_{j=1}^{p} w_j f_j \|_0 \le \sum_{j=1}^{p} \| w_j f_j \|_0 \le \sum_{j=1}^{p} \|f_j\|_0 \le \bar{p} \le n.$

~~Note that Σ~~

By the KKT condition, we know that

$$\sum_{j=1}^{p} w_i f_j = v - \sum_{i=1}^{m} u_i d_i - c.$$

then we have $\| v - \sum_{i=1}^{m} u_i d_i - c \|_0 \leq \bar{p} \leq n.$

since $v \in \mathbb{R}_+^n$, $d_i \geq 0$, $\forall i$, $c > 0$.

then we have $\| v - \sum_{i=1}^{m} u_i d_i \|_0 \geq n - \bar{p}$.

Then by the KKT condition that $v_k x_k^* = 0$, $k = 1, \ldots, n$.

we know that

$$\| x^* \|_0 \leq \bar{p}.$$