OR19 **Problem 1 (P:15pts, S:20pts).** Consider the following optimization problem:

$$\min_{x,y,z} \left\{2x^2 + xy + y^2 + yz + z^2 - 6x - 7y - 8z + 9\right\}. \qquad \text{(QP)}$$

(a) **(P:5pts, S:5pts).** Write down the first-order optimality condition of Problem (QP).

(b) **(P:10pts, S:15pts).** Using the result in (a), determine the optimal solution(s) to Problem (QP). Justify your answer.

(a) First order optimality condition : $\nabla f(\bar{x}, \bar{y}, \bar{z}) = 0$

$$\nabla f(\bar{x}, \bar{y}, \bar{z}) = \begin{cases} 4x + y - 6 = 0 \\ x + 2y + z - 7 = 0 \\ y + 2z - 8 = 0 \end{cases}$$

(b)

$$\begin{array}{ccc|c} x & y & z & b \\ 4 & 1 & 0 & 6 \\ 1 & 2 & 1 & 7 \\ 0 & 1 & 2 & 8 \end{array} \Rightarrow \begin{array}{ccc|c} 0 & 1 & 0 & \frac{6}{5} \\ 1 & 0 & 0 & \frac{6}{5} \\ 0 & 0 & 1 & \frac{17}{5} \end{array} \Rightarrow \begin{cases} \bar{x} = \frac{6}{5}, \\ \bar{y} = \frac{6}{5}. \\ \bar{z} = \frac{17}{5} \end{cases}$$

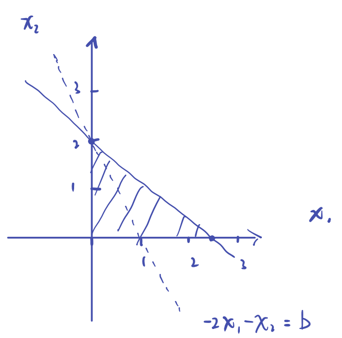
OR19 **Problem 2 (P:15pts, S:20pts).** Consider the following integer linear programming problem:

$$\begin{array}{lll} \text{minimize} & -2x_1 - x_2 & \\ \text{subject to} & 4x_1 + 5x_2 \le 10, & \\ & 0 \le x_j \le 3 & \text{for } j = 1, 2, \\ & x_j \text{ integer} & \text{for } j = 1, 2. \end{array} \qquad \text{(ILP)}$$

(a) **(P:5pts, S:5pts).** Determine the optimal value of and optimal solution to Problem (ILP).

(b) **(P:10pts, S:15pts).** Let v be the Lagrange multiplier associated with the constraint $4x_1 + 5x_2 \le 10$. Write down the Lagrangian dual of Problem (ILP) that dualizes only the constraint $4x_1 + 5x_2 \le 10$. Hence, determine the optimal value of and optimal solution to the Lagrangian dual you derived.

(a)



the feasible field is boundes

all possible solution (0,0) (0,1) (0,2) (1,0) (1,1) (2,0)

we can verify that,

it's (2,0), and opt value is -4

(b) (P) Lagrangian Function: 注意第二题是 Partial Lagrangian Dual, 只是部分对偶!

$$L = -2x_1 - x_2 + v(4x_1 + 5x_2 - 10) = (4v-2)x_1 + (5v-1)x_2 - 10v$$

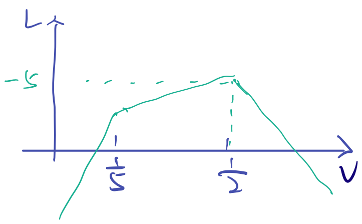
$$\therefore (P) \rightarrow \min_{x_1, x_2} \max_{v \ge 0} L \qquad \therefore (D) \rightarrow \max_{v \ge 0} \min_{x_1, x_2} L$$

s.t. $x_1, x_2 \in [0,1,2,3]$ for (P) and (D)

$$\min_{x_1 \in [0,1,2,3]} (4v-2)x_1 = \begin{cases} 0, & \text{if } v \geq \frac{1}{2} \\ 12v-6, & \text{if } v < \frac{1}{2} \end{cases}$$

$$\min_{x_2 \in [0,1,2,3]} (5v-1)x_2 = \begin{cases} 0, & \text{if } v \geq \frac{1}{5} \\ 15v-3, & \text{if } v < \frac{1}{5} \end{cases}$$

$$\min L = \begin{cases} -10v, & \text{if } v \geq \frac{1}{2} \\ 2v-6, & \text{if } v \in [\frac{1}{5}, \frac{1}{2}] \\ 17v-9 & \text{if } v < \frac{1}{5} \end{cases}$$



$\therefore \max_{v \geq 0} \min_{x_1, x_2} L = -5$,

$v^* = \frac{1}{2}$

OR19

Problem 3 (P:25pts, S:20pts). Let $E \in \mathcal{S}^n$ be the $n \times n$ matrix of all ones and $\Omega \subseteq \{(i,j) : 1 \leq i < j \leq n\}$. Consider the following optimization problem:

$$\begin{array}{lll} \inf & \lambda_{\max}(X+E) & \\ \text{subject to} & X_{ij} = 0 & \text{for } (i,j) \in \Omega, \\ & X_{ii} = 0 & \text{for } i = 1, 2, \ldots, n. \end{array} \tag{Q}$$

Here, $\lambda_{\max}(A)$ denotes the largest eigenvalue of A.

(a) **(P:15pts, S:10pts).** Show that Problem (Q) can be formulated as an SDP. Justify your answer.

(b) **(P:10pts, S:10pts).** Derive the dual of the SDP found in (a).

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ORI9 **Problem 4 (P:20pts, S:15pts).** Consider the following standard primal-dual pair of LPs:

$$(P): \quad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \\ & x \geq \mathbf{0}. \end{array} \qquad (D): \quad \begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c. \end{array}$$

Here, as usual, we have $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$, where $m \leq n$. Suppose that (P) has a unique optimal solution x^* that is non-degenerate (i.e., there are exactly n linearly independent active constraints at x^*). Show that (D) also has a unique optimal solution y^* that is non-degenerate. (*Hint: Consider the complementary slackness condition.*)

$\because$ $Ax = b \in \mathbb{R}^m$ $\therefore$ m constraints are active ,

i.e. m of n x's elements > 0

(why ? suppose we have one row of Ax=b like this:

- $x_1 + x_2 = 1$, Then x_1, x_2 can not both be 0
- $x_1 + x_2 = 0$, if $x_1 = 0$, then x_2 must be 0

However $\begin{cases} x_1 + x_2 = 0 \\ x_1 = 0,\ x_2 = 0 \end{cases}$ is not linear independent

$\therefore$ There must be m of x's elem > 0

Complementary slackness: $\underbrace{\bar{x}_i}_{\geq 0} \underbrace{(c - A^T \bar{y})_i}_{\geq 0} = 0$

$\therefore$ if $\bar{x}_i \neq 0$, $(c - A^T \bar{y})_i = 0$, m of $c - A^T \bar{y}$ satisfy this

$\therefore$ m linearly independent active constraints

$\therefore$ y^* is a unique optimal sol that is non-degenerate.

(BFS)