OR20

Problem 1. **[P: 15pts, S: 20pts]** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. For some $\mu \in \mathbb{R}$, suppose that $f(x) - \frac{\mu}{2}\|x\|^2$ is a convex function. Show that the following inequality holds:

$$\langle \nabla f(x) - \nabla f(y) \,|\, x - y \rangle \geq \mu \|x - y\|^2, \quad \forall\, x, y \in \mathbb{R}^n.$$

Remark: For $x, y \in \mathbb{R}^n$, $\langle x \,|\, y \rangle$ denotes the inner product between the two vectors, i.e., $\langle x \,|\, y \rangle = x^\top y$.

$\because$ $g(x) = f(x) - \frac{\mu}{2}\|x\|^2$ is convex

$\therefore$ $g(x) \geq g(y) + \nabla g(y)^T (x-y) \quad \forall\ x, y \in \mathbb{R}$

$f(x) - \frac{\mu}{2}\|x\|^2 \geq f(y) - \frac{\mu}{2}\|y\|^2 + [\nabla f(y) - \mu y]^T (x-y)$ ①

$f(y) - \frac{\mu}{2}\|y\|^2 \geq f(x) - \frac{\mu}{2}\|x\|^2 + [\nabla f(x) - \mu x]^T (y-x)$ ②

$\therefore$ ① + ②: $\cancel{f(x) + f(y)} - \cancel{\frac{\mu}{2}(\|x\|^2 + \|y\|^2)} \geq \cancel{f(y) + f(x)} - \cancel{\frac{\mu}{2}(\|y\|^2 + \|x\|^2)}$

$+ [\nabla f(x) - \mu x - \nabla f(y) + \mu y]^T (y-x)$

$0 \geq + [\nabla f(x) - \nabla f(y)]^T (y-x) + (\mu y - \mu x)^T (y-x)$

$\therefore$ $[\nabla f(x) - \nabla f(y)]^T (x-y) \geq \mu \|x-y\|^2$

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Problem 2. [P: 20pts, S: 30pts] Consider the following polyhedral set:

$$X = \left\{ x \in \mathbb{R}^4 : \begin{array}{rl} x_1 - x_2 - x_3 & \leq 1 \\ x_2 - x_4 & \leq 2 \\ x_1, x_2, x_3, x_4 & \geq 0 \end{array} \right\}$$

(a) Find **all** the extreme/corner points of X, and justify why they are extreme/corner points.

(b) Consider the following **linear integer program**:

$$\begin{array}{rl} \max\limits_{x_1,x_2,x_3,x_4} & x_1 + 2x_2 + 3x_3 + 4x_4 \\ & x_1 - x_2 - x_3 \leq 1 \\ \text{s.t.} & x_2 - x_4 \leq 2 \\ & x_1, x_2, x_3, x_4 \geq 0 \\ & x_1, x_2, x_3, x_4 \in \mathbb{Z} \end{array}$$

Find an **optimal solution** to the above integer program. Justify your answer.

(a) Matrix of the constraints

$$\begin{array}{cccc|c|c} x_1 & x_2 & x_3 & x_4 & b & \\ 1 & -1 & -1 & 0 & 1 & ① \\ 0 & 1 & 0 & -1 & 2 & ② \\ 1 & 0 & 0 & 0 & 0 & ③ \\ 0 & 1 & 0 & 0 & 0 & ④ \\ 0 & 0 & 1 & 0 & 0 & ⑤ \\ 0 & 0 & 0 & 1 & 0 & ⑥ \end{array}$$

Take ①②③④, can be simplified to

$$\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array} \rightarrow x = (0, 0, -1, -2)$$

not satisfy ⑤, ⑥

Take ①②③⑤,

$$\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \rightarrow x = (0, -1, 0, -3)$$

not satisfy

Take ①②③⑥

$$\begin{array}{cccc|c} 0 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}$$

$\rightarrow x = (0, 2, -3, 0)$

not satisfy ⑤

Take ①②④⑤

$$\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}$$

$\rightarrow x = (1, 0, 0, -2)$

not satisfy

Take ①②④⑥

not linear independent

Take ①②⑤⑥

$$\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \rightarrow x = (3, 2, 0, 0)$$

✓

Take ③④⑤⑥

$(0, 0, 0, 0)$ ✓

①④⑤⑥ : $x = (1, 0, 0, 0)$ ✓

②④⑤⑥ : $x = (0, 2, 0, 0)$ ✓

另一个思路是，

x_1, x_2, x_3, x_4 至少 2 个为 0，

代入看是否为合法解

(b) ∵ The LP have BFS (vertex),

∴ either optimal value is $+\infty$ or exists a BFS optimal sol

We can specify $x_3 \rightarrow +\infty$, $x_4 \rightarrow +\infty$ ∴ The optimal value is $+\infty$

IR20 **Problem 3. [P: 30pts, S: 15pts]** Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable, increasing function with $f(0) = 0$. Consider

$$F(x) = \int_0^x f(a)\, da.$$

(a) Let $F^\star(y) = \sup_u\{uy - F(u)\}$ be the conjugate function of $F(x)$. Show the following inequality:

$$xy \le F(x) + F^\star(y), \quad \forall\, x, y \in \mathbb{R}.$$

(b) Let $g = f^{-1}$ such that g is an inverse function and suppose that it is differentiable. Show that the conjugate function of $F(x)$ can be written as

$$G(y) = F^\star(y) = \int_0^y g(b)\, db.$$

Remark: You may prove either using a graphical method by drawing the integration region; or apply the change of variable and integration-by-part, respectively, whose formulas are provided as follows for your convenience:

$$\int_a^b h(\phi(x))\phi'(x)\, dx = \int_{\phi(a)}^{\phi(b)} h(u)\, du, \qquad \int_a^b u(x)v'(x)\, dx = \big[u(x)v(x)\big]_a^b - \int_a^b u'(x)v(x)\, dx.$$

(a) $F^*(y) = \sup_u\{uy - F(u)\}$

take $x = u$, $\therefore F^*(y) = \sup_x\{xy - F(x)\}$

$\therefore F^*(y) \ge xy - F(x) \to xy \le F^*(y) + F(x)$

(b) $\because g = f^{-1}$ $\therefore$ take $g(y) = u$ $\quad y = f(u)$

$G(y) = \int_0^y g(b)\,db = \int_{f(0)}^{f(u)} g(b)\,db$

$= \int_0^u g(f(x))f'(x)\,dx = \int_0^u x f'(x)\,dx$

$= xf(x)\Big|_0^u - \int_0^u f(x)\,dx$

$= uf(u) - \int_0^u f(x)\,dx$

$= uy - \int_0^u f(x)\,dx = F^*(y)$

OR20 **Problem 6.** **[S: 20pts]** Consider the constraint set $\mathcal{C} = \{x \in \mathbb{R}^n : x^\top e \geq 0\}$, where $e \in \mathbb{R}^n$ is an all-one n-dimensional vector. Define the operator:

$$\mathcal{P}(x) = \begin{cases} y - \frac{1}{n}(y^\top e)e, & \text{if } y^\top e \leq 0 \\ y, & \text{if } y^\top e > 0. \end{cases}$$

Show that for any $x \in \mathbb{R}^n$, $\mathcal{P}(x)$ is the projection of x onto $\mathcal{C}$.

① verify $P(x) \in C$

if $x^T e > 0$, obviously $P(x) = x \in C$

if $x^T e \leq 0$, $P(x) = x - \frac{1}{n}(x^T e)e$

$$P(x)^T e = x^T e - \frac{1}{n}(x^T e)\underbrace{e^T e}_{n} = 0$$

$\therefore P(x) \in C$

② for $\forall z \in C$ verify $(z - P(x))^T(x - P(x)) \leq x$

$z \in C,\ x \in \mathbb{R}^n$

if $x^T e > 0$,

$$(z - P(x))^T(x - P(x))$$
$$= (z - x)^T(x - x) = 0,$$

if $x^T e \leq 0$

$$(z - P(x))^T(x - P(x))$$
$$= (z - x + \tfrac{1}{n}(x^T e)e)^T(\tfrac{1}{n}(x^T e)e)$$
$$= z^T(\tfrac{1}{n}(x^T e)e) - P(x)^T(\tfrac{1}{n}(x^T e)e)$$
$$= \frac{1}{n}\underbrace{(x^T e)}_{\leq 0}\underbrace{z^T e}_{\geq 0} - \frac{1}{n}(x^T e)\underbrace{P(x)^T e}_{=0} \leq 0$$

$\therefore (z - P(x))^T(x - P(x)) \leq 0$, i.e. $P(x) = \Pi_C(x)$

OR20 **Problem 7. [S: 15pts]** Consider a non-differentiable, convex function $f : \mathbb{R}^n \to \mathbb{R}$.

(a) Consider a point $\overline{x} \in \mathbb{R}^n$ such that $0 \in \partial f(\overline{x})$. Show that $\overline{x}$ is a global optimal solution to the optimization problem $\min_{x\in\mathbb{R}^n} f(x)$.

(b) Let $y \in \mathbb{R}^n$ and $\lambda > 0$ be fixed constant vector/scalar. Consider the optimization problem:

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\|x-y\|_2^2 + \lambda\|x\|_1 \qquad \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Using the result from (a), find an optimal solution $\overline{x} \in \mathbb{R}^n$ to the above problem. Express your solution in terms of y and λ.

(a) $\partial f(\bar{x}) = \{ s \in \mathbb{R}^n : f(x) \geq f(\bar{x}) + s^T(x-\bar{x}),\ \forall x \in \mathbb{R}^n \}$

$\because 0 \in \partial f(\bar{x}) \quad \therefore \quad f(x) \geq f(\bar{x}) + 0^T(x-\bar{x})$, for $\forall x \in \mathbb{R}^n$

i.e. $f(x) \geq f(\bar{x})$ for $\forall x \in \mathbb{R}^n$

(b) Take $f_1(x) = \frac{1}{2}\|x-y\|_2^2$, f_1 is convex and differentiable

$f_2(x) = \lambda \|x\|_1$, f_2 is convex but non-differentable

$\therefore \partial f(x) = \partial f_1(x) + \partial f_2(x)$

where, $\partial f_1(x) = \nabla f_1(x) = x - y$

$$\partial f_2(x) = \left\{ \frac{\partial f_2}{\partial x_i} \right\} \qquad \frac{\partial f_2}{\partial x_i} = \begin{cases} \lambda & \text{if } x_i > 0 \\ [-\lambda, \lambda] & \text{if } x_i = 0 \\ -\lambda & \text{if } x_i < 0 \end{cases}$$

Suppose $\bar{x}$ is an optimal solution, we have $0 \in \partial f(\bar{x})$

i.e. we can find $\bar{x} - y + \partial f_2(\bar{x}) = 0$

$\bar{x} = y - \partial f_2(\bar{x})$

where $$\bar{x}_i = \begin{cases} y_i - \lambda & \text{if } y_i > \lambda \\ 0 & \text{if } y_i \in [-\lambda, \lambda] \\ y_i + \lambda & \text{if } y_i < -\lambda \end{cases}$$