Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
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Dynamical Systems
Lecture 5.02

EEU45C09 / EEP55C09
Self Organising Technological Networks

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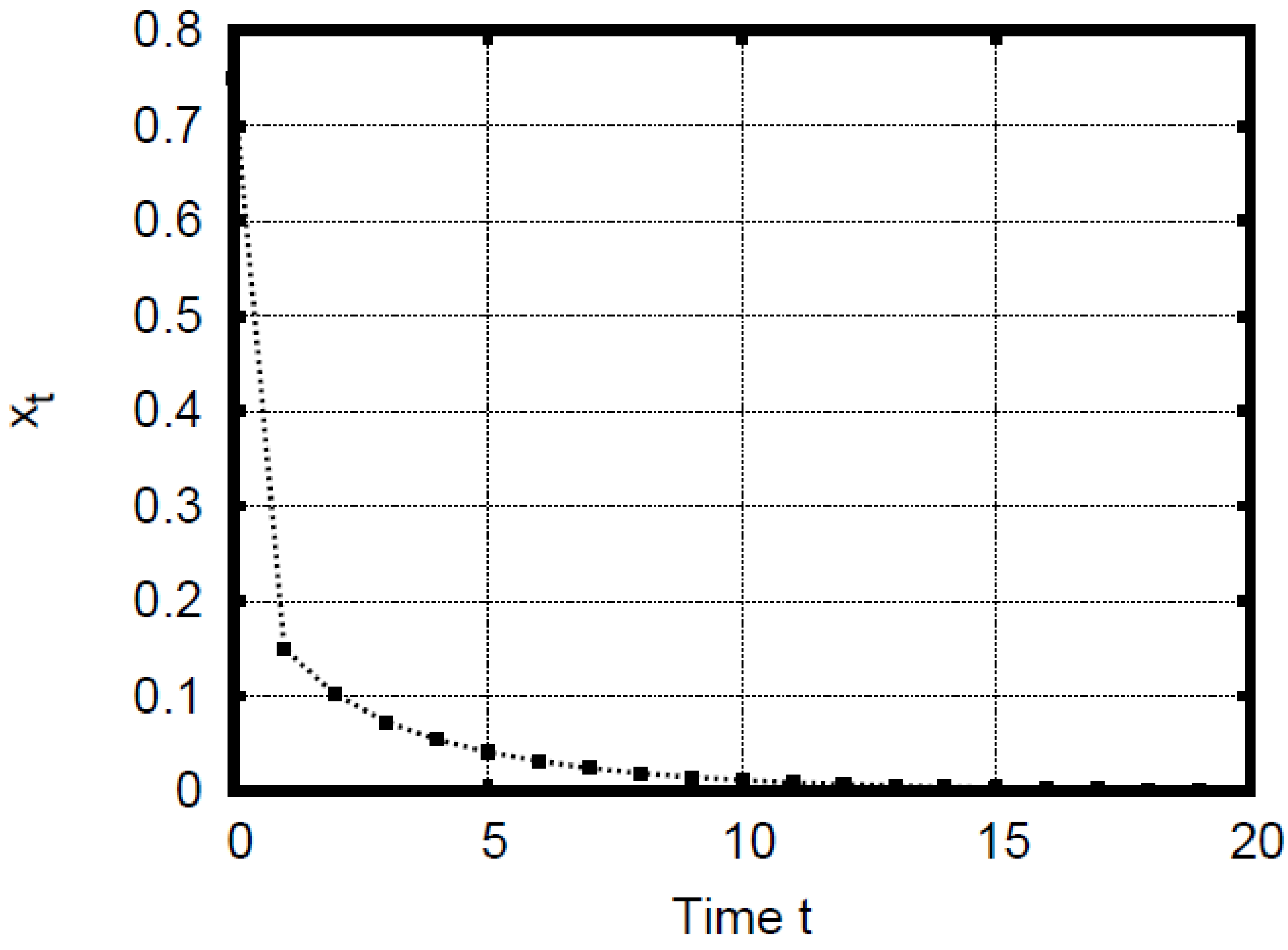
The Logistic Equation

- Logistic equation: $f(x) = rx(1 - x)$.
- A simple model of resource-limited population growth.
- The population x is expressed as a fraction of the carrying capacity. $0 \leq x \leq 1$.
- r is a parameter—the growth rate—that we will vary.
- Let's first see what happens if $r = 0.8$.

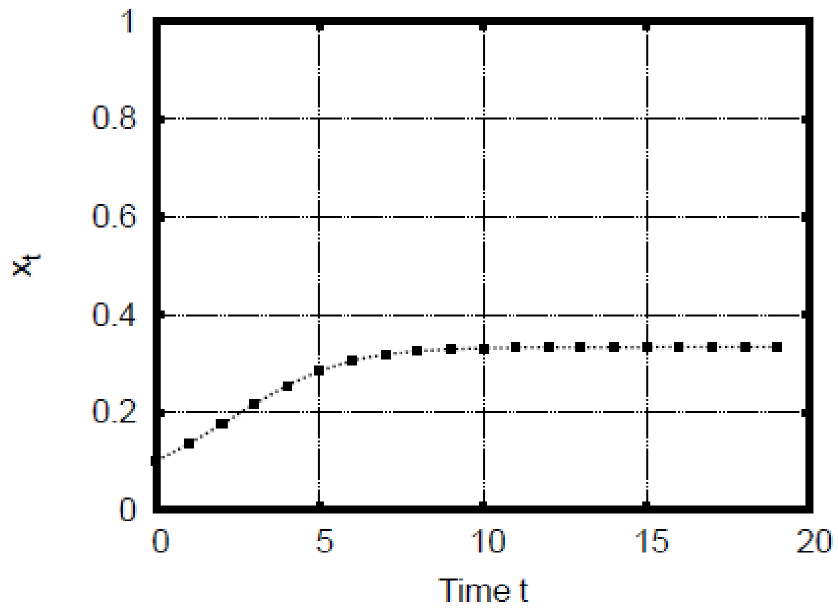
The Logistic Equation

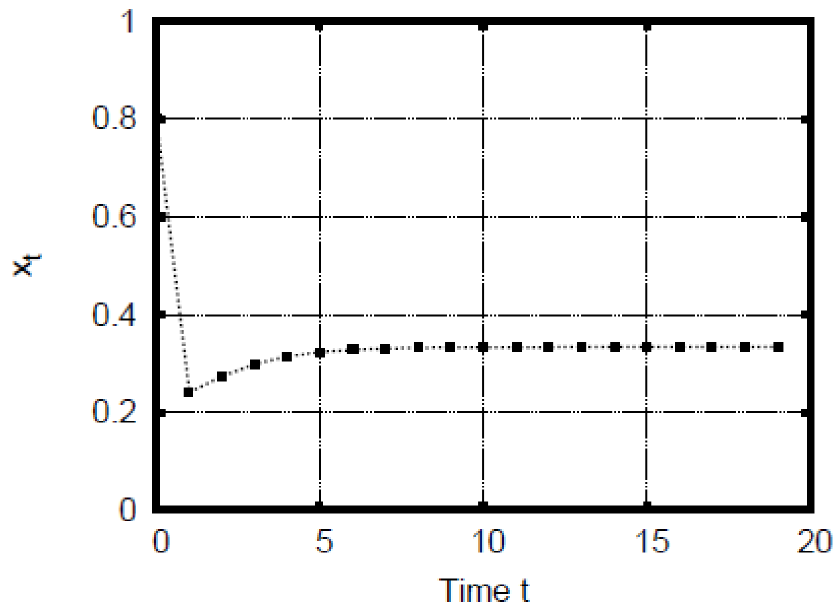


- 0 is an attracting fixed point.

The Logistic Equation

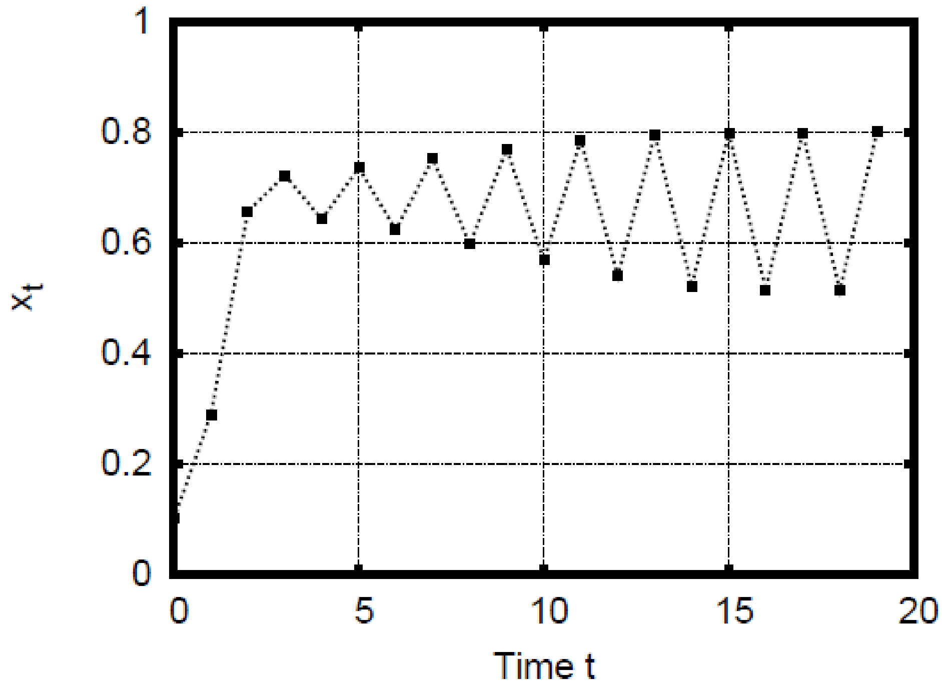
- Logistic equation, $r = 1.5$.





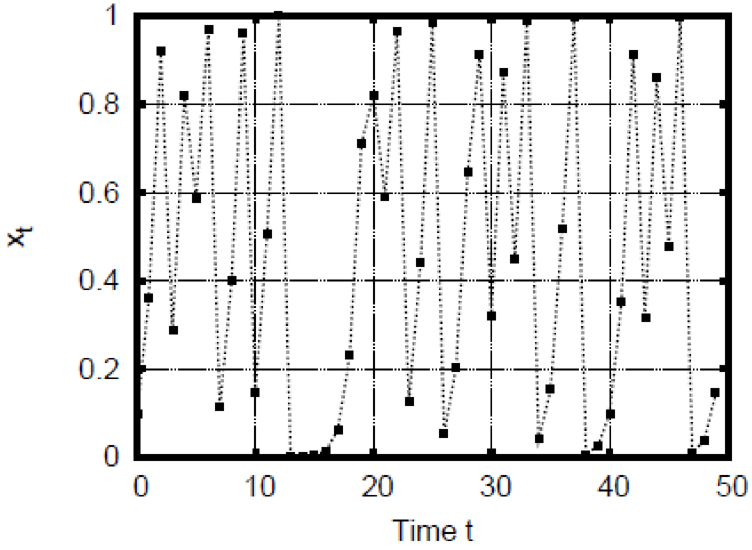
- All initial conditions are pulled toward 0.33.
- 0.33 is an attracting fixed point.

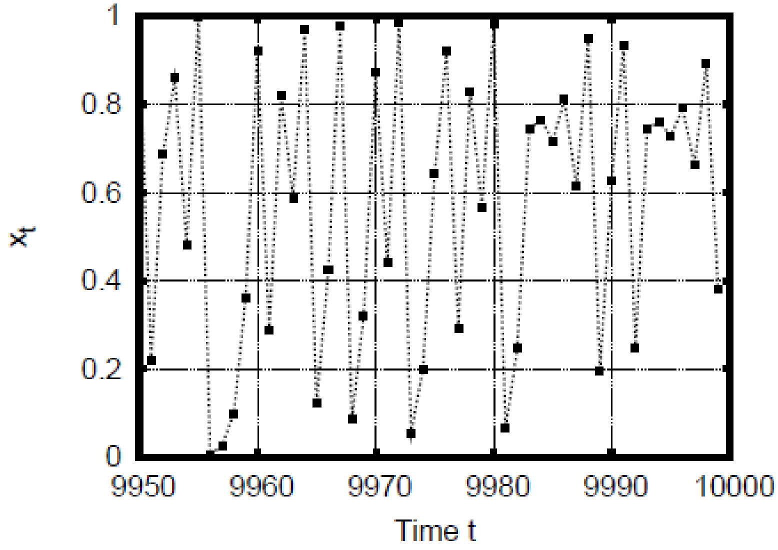
The Logistic Equation



- Logistic equation, $r = 3.2$.
- Initial conditions are pulled toward a **cycle** of period 2.
- The orbit oscillates between 0.513045 and 0.799455.
- This cycle is an attractor. Many different initial conditions get pulled to it.

The Logistic Equation

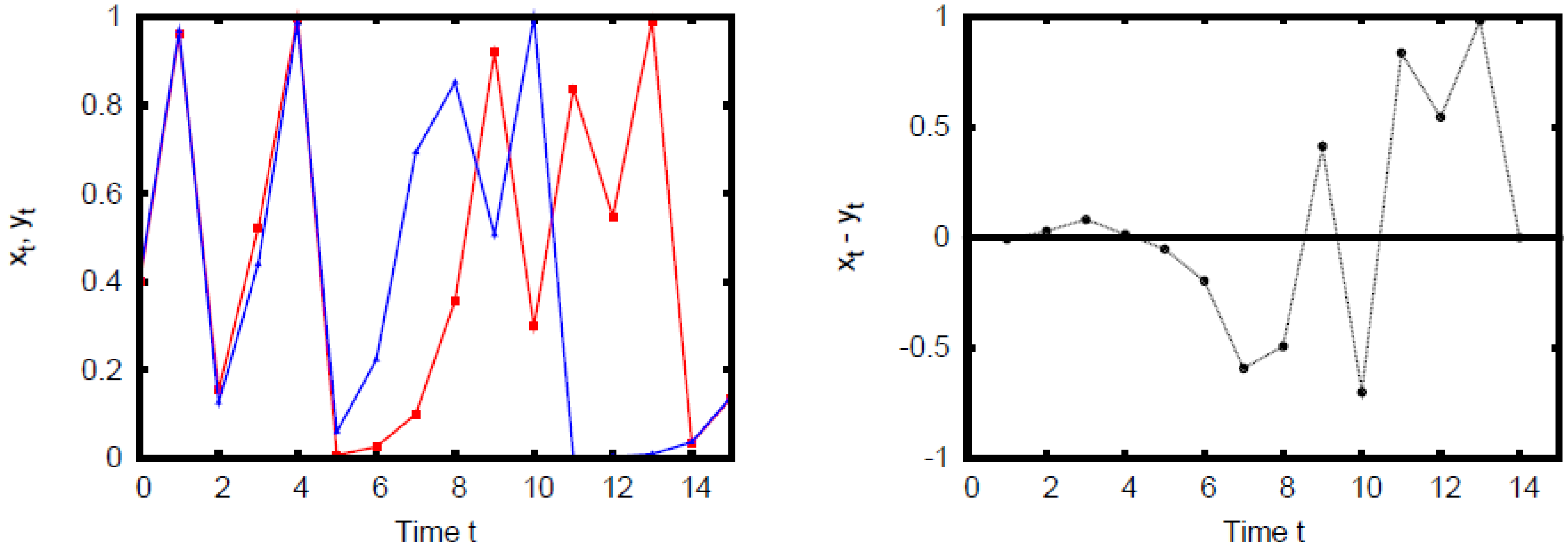




- Logistic equation, $r = 4.0$.
- What's going on here?!
- The orbit is not periodic. In fact, it never repeats.
- This is a rigorous result; it doesn't rely on computers.
- What happens if we try different initial conditions?

The Logistic Equation

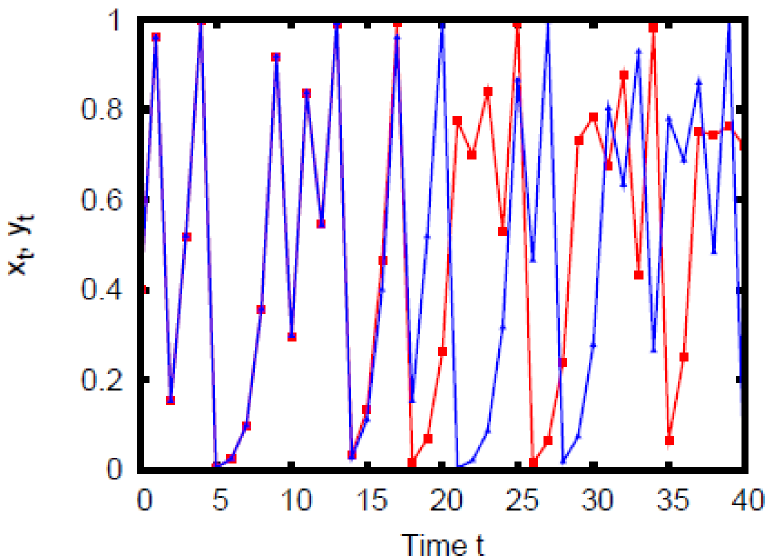
- Two slightly different initial conditions, $x_0 = 0.4$ and $x_0 = 0.41$.

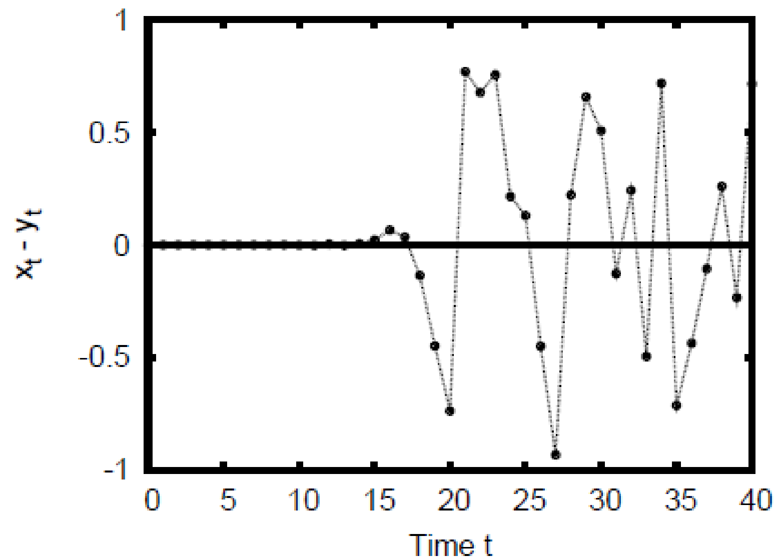


- The right graph plots the difference between the two orbits on the left
- Note that the difference between the two orbits grows.
- Can think of the blue as the true values, and the red as the predicted values.
- The plot on the right can be viewed as prediction error over time.
- How can we improve our predictions?

Sensitive Dependence on Initial Conditions

- Two different initial conditions, $x_0 = 0.4$ and $x_0 = 0.4000001$.

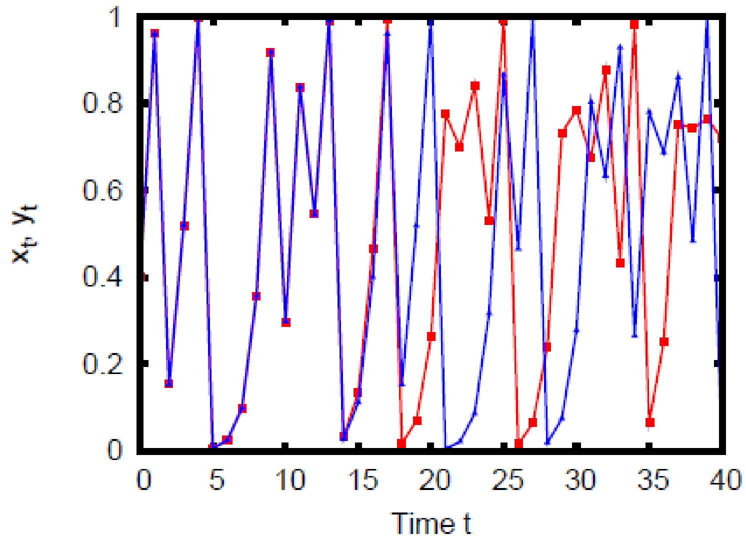


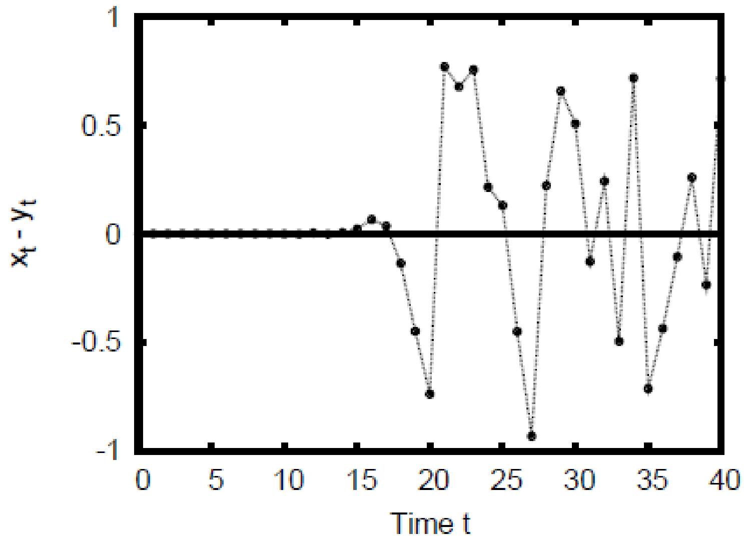


- The two initial conditions differ by one part in one million
- The orbits differ significantly after around 20 iterations, whereas before they differed after around 4 iterations.
- Increasing the accuracy of the initial condition by a factor of 10^5 allow us to predict the outcome 5 times further.

Sensitive Dependence on Initial Conditions

- Two different initial conditions, $x_0 = 0.4$ and $x_0 = 0.4000001$.





- Thus, for all practical purposes, this system is unpredictable, even though it is deterministic.
- This phenomena is known as **Sensitive Dependence on Initial Conditions**, or, more colloquially, **The Butterfly Effect**.
- Arbitrarily small differences in initial conditions grow to become arbitrarily large.

Formal Definition of SDIC

- A dynamical system has sensitive dependence on initial conditions (SDIC) if arbitrarily small differences in initial conditions eventually lead to arbitrarily large differences in the orbits.

More formally

- Let X be a metric space, and let f be a function that maps X to itself: $f : X \mapsto X$.
- The function f has SDIC if there exists a $\delta > 0$ such that $\forall x_1 \in X$ and $\forall \epsilon > 0$, there is an $x_2 \in X$ and a natural number $n \in N$ such that $d[x_1, x_2] < \epsilon$ and $d[f^n(x_1), f^n(x_2)] > \delta$.
- In other words, two initial conditions that start ϵ apart will, after n iterations, be separated by a distance δ.

Formal Definition of Chaos

There is not a 100% standard definition of chaos. But here is one of the most commonly used ones:

An iterated function is **chaotic** if:

1. The function is **deterministic**.
2. The system's orbits are **bounded**.
3. The system's orbits are **aperiodic**; i.e., they never repeat.
4. The system has **sensitive dependence on initial conditions**.

Other properties of a chaotic dynamical system ($f : X \mapsto X$) that are sometimes taken as defining features:

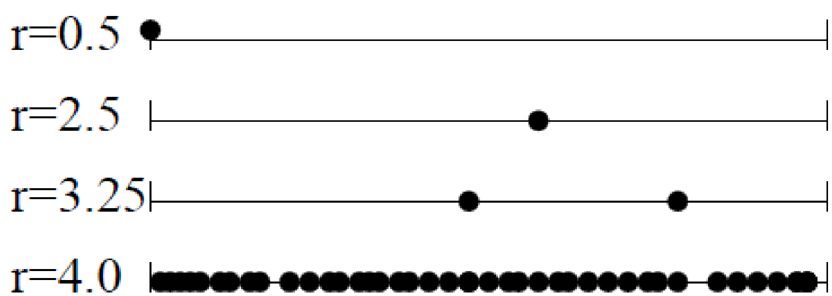
1. **Dense periodic points:** The periodic points of f are dense in X.
2. **Topological transitivity:** For all open sets $U, V \in X$, there exists an $x \in U$ such that, for some $n < \infty$, $f_n(x) \in V$. I.e., in any set there exists a point that will get arbitrarily close to any other set of points.

Bifurcation Diagram

We have seen several possible long-term behaviors for the logistic equation:

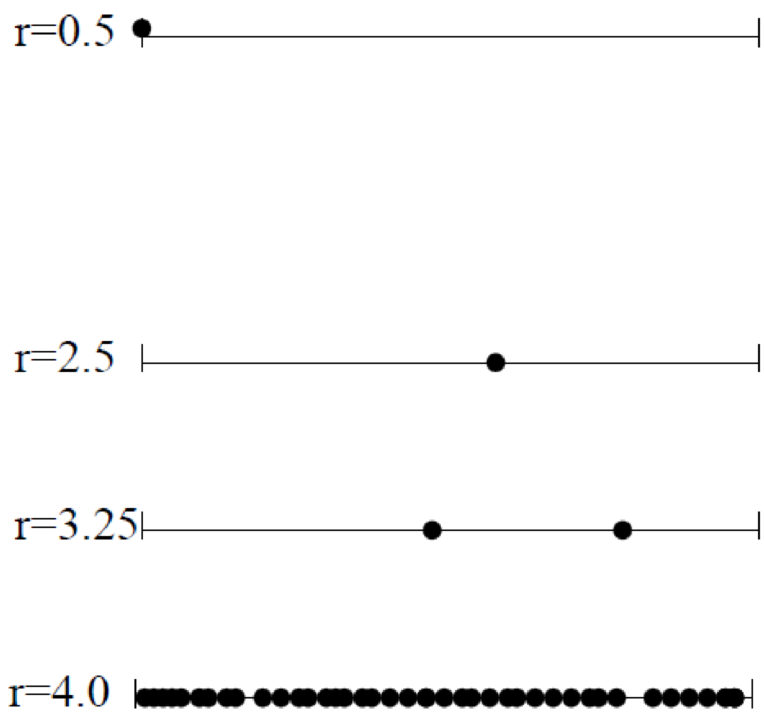
1. $r = 0.5$: attracting fixed point at 0.
2. $r = 2.5$: attracting fixed point at 0.6.
3. $r = 3.25$: attracting cycle of period 2.
4. $r = 4.0$: chaos.

Graphically, we can illustrate this as follows:



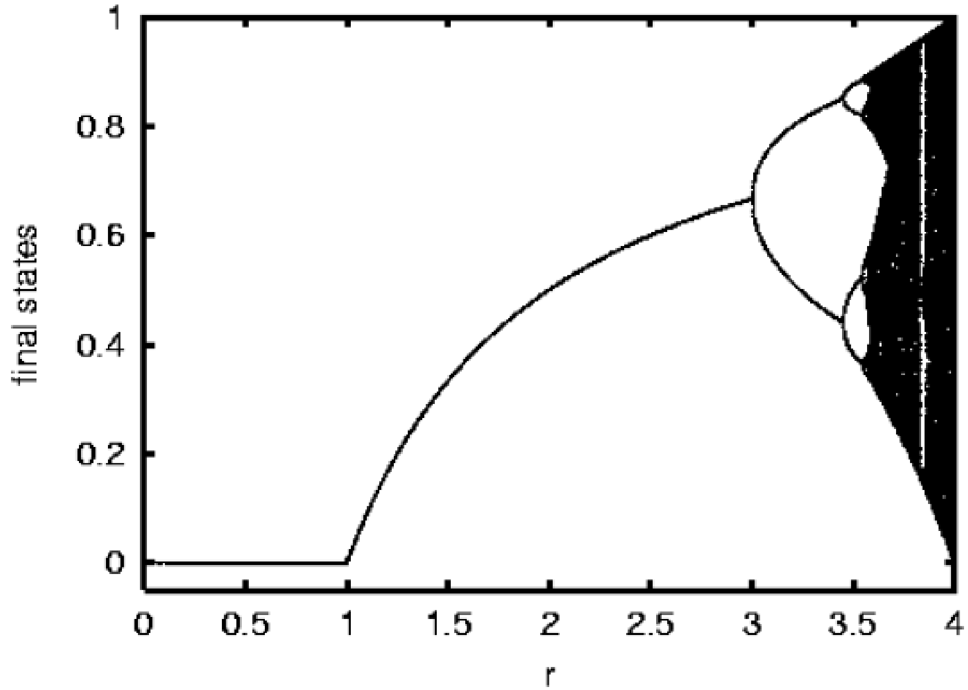
- I.e., for each r, iterate and plot the final x values as dots on the number line.
- What else can the logistic equation do??

Bifurcation Diagram



- Do this for more and more r values and "glue" the lines together.
- Turn sideways and ...

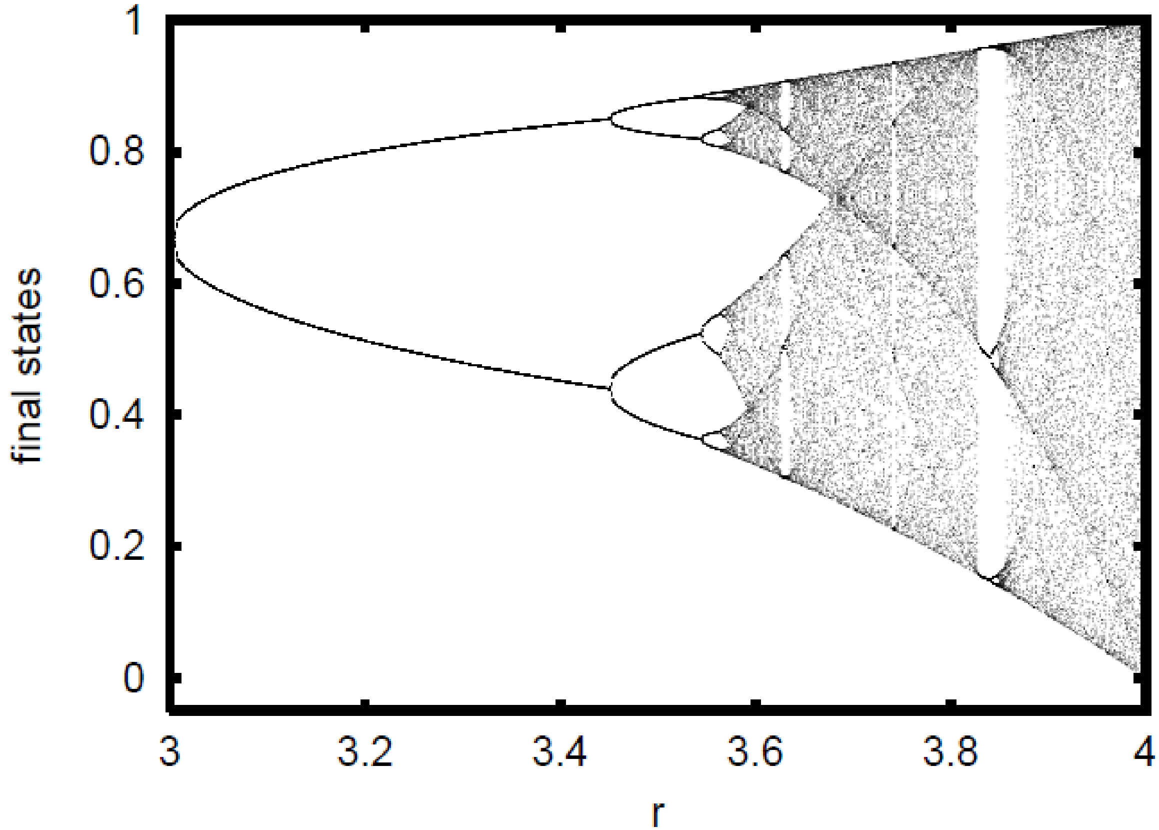
Bifurcation Diagram



- The bifurcation diagram shows all the the possible long-term behaviors for the logistic map.
- $0 < r < 1$, the orbits are attracted to zero.
- $1 < r < 3$, the orbits are attracted to a non-zero fixed point.
- $3 < r < 3.45$, orbits are attracted to a cycle of period 2.
- Chaotic regions appear as dark vertical lines.

Bifurcation Diagram

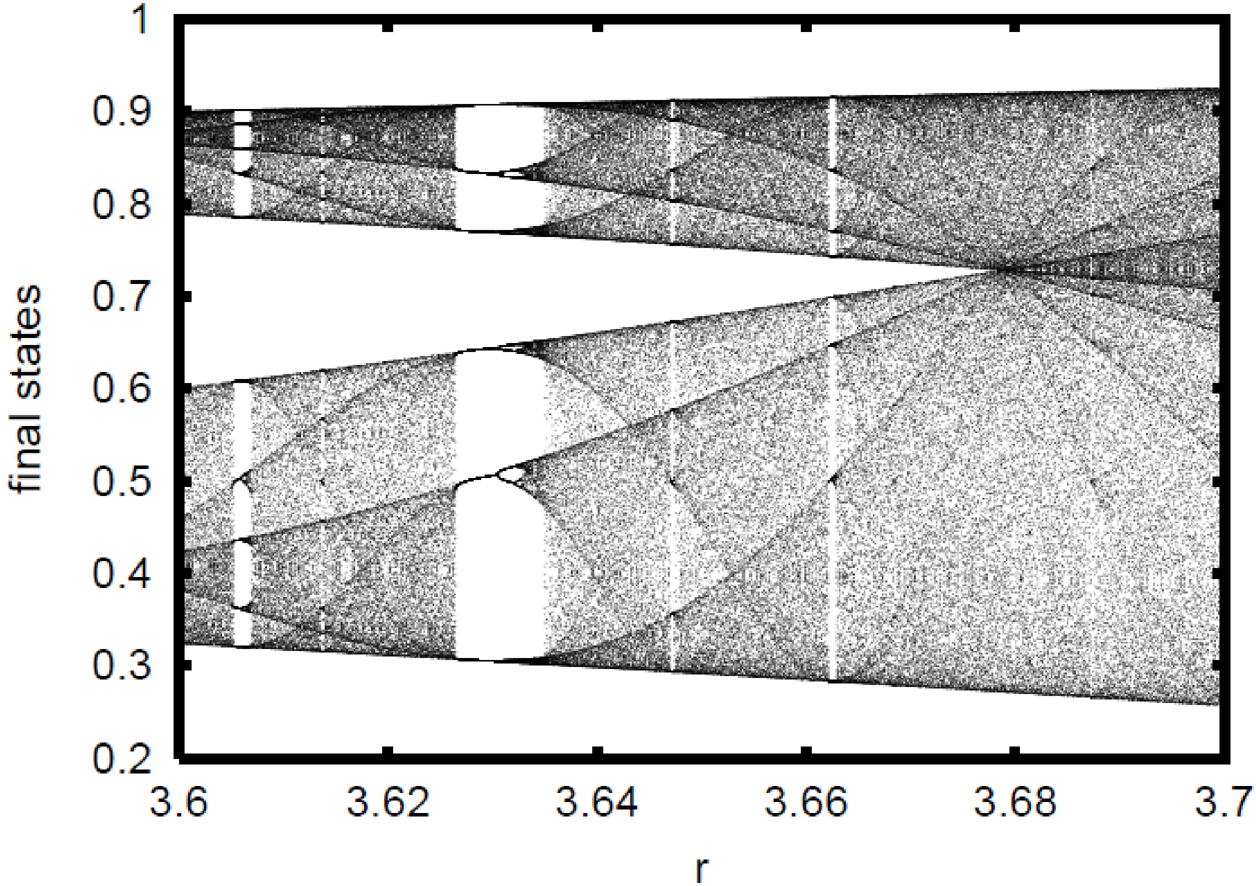
Let's zoom in on a region of the bifurcation diagram:



- The sudden qualitative changes are known as **bifurcations**.
- There are **period-doubling bifurcations** at $r \approx 3.45$, $r \approx 3.544$, etc.
- Note the window of period 3 near $r = 3.83$.

Bifurcation Diagram

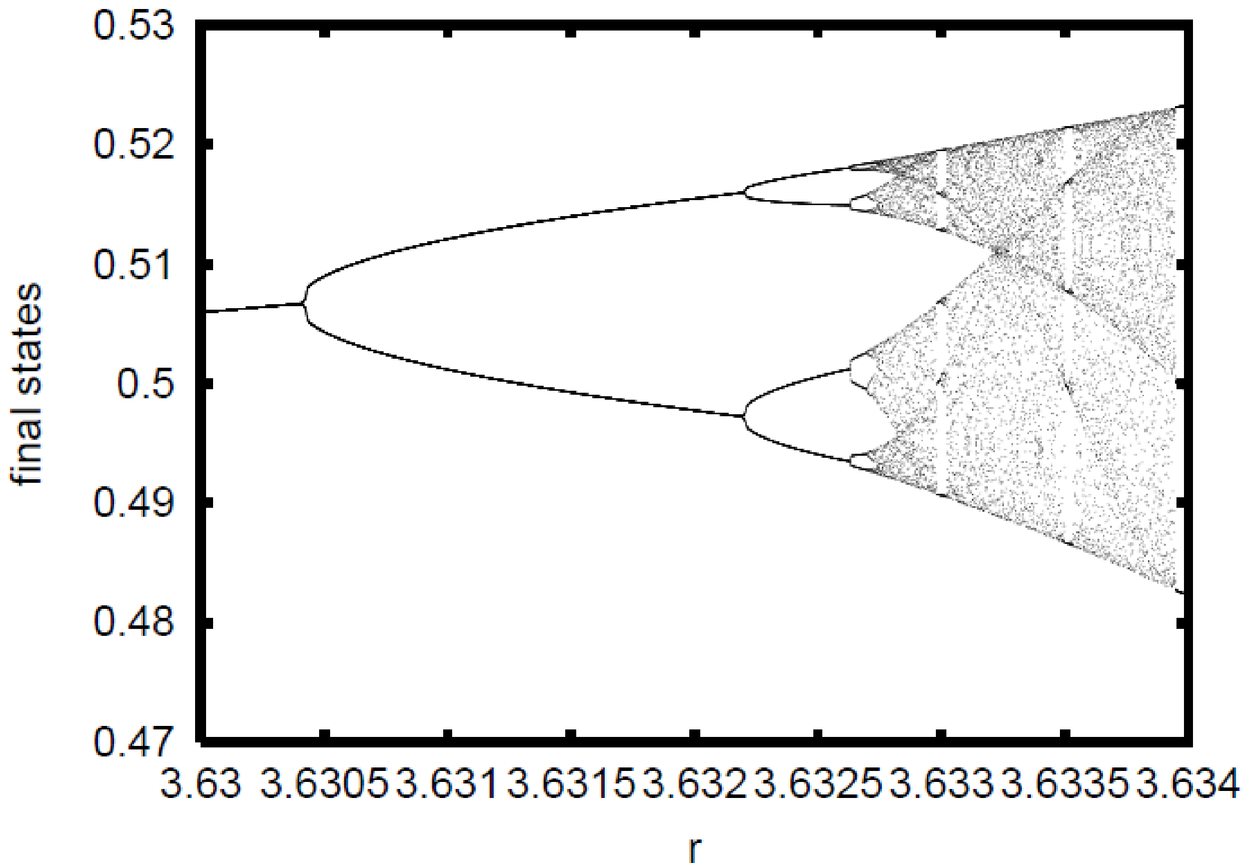
Let's zoom in again:



- Note the sudden changes from chaotic to periodic behavior.

Bifurcation Diagram

Let's zoom in once more:



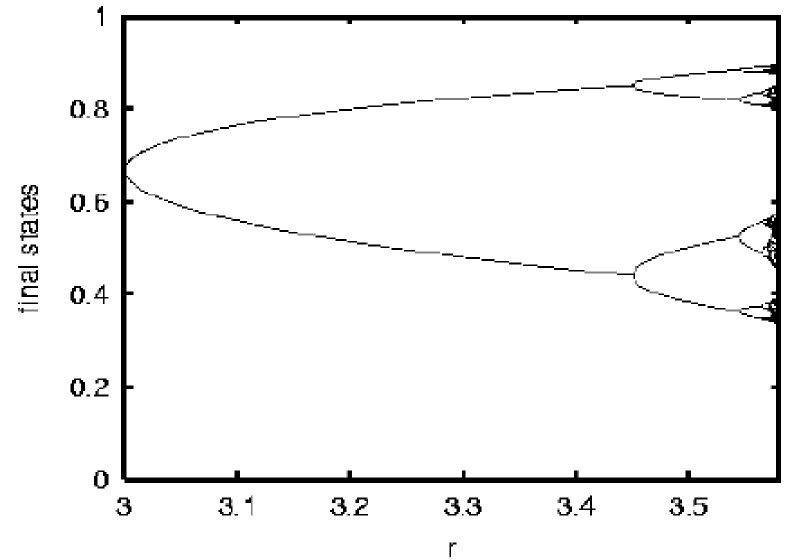
- Note the small scales on the vertical axis, and the tiny scale on the y axis.
- Note the self-similar structure. As we zoom in we keep seeing pitchforks.

Bifurcation Diagram - Summary

- As we vary r, the logistic equation shuffles suddenly between chaotic and periodic behaviors, but the bifurcation diagram reveals that these transitions appear in a structured, or regular, way.
- This is an example of a sort of “order within chaos.”
- Bifurcations—a sudden, qualitative change in behavior as a parameter is continuously varied—is a generic feature of non-linear systems.
- In the next few slides we’ll examine one of the regularities in the bifurcation diagram: The **period-doubling route to chaos**.

Period-Doubling Route to Chaos

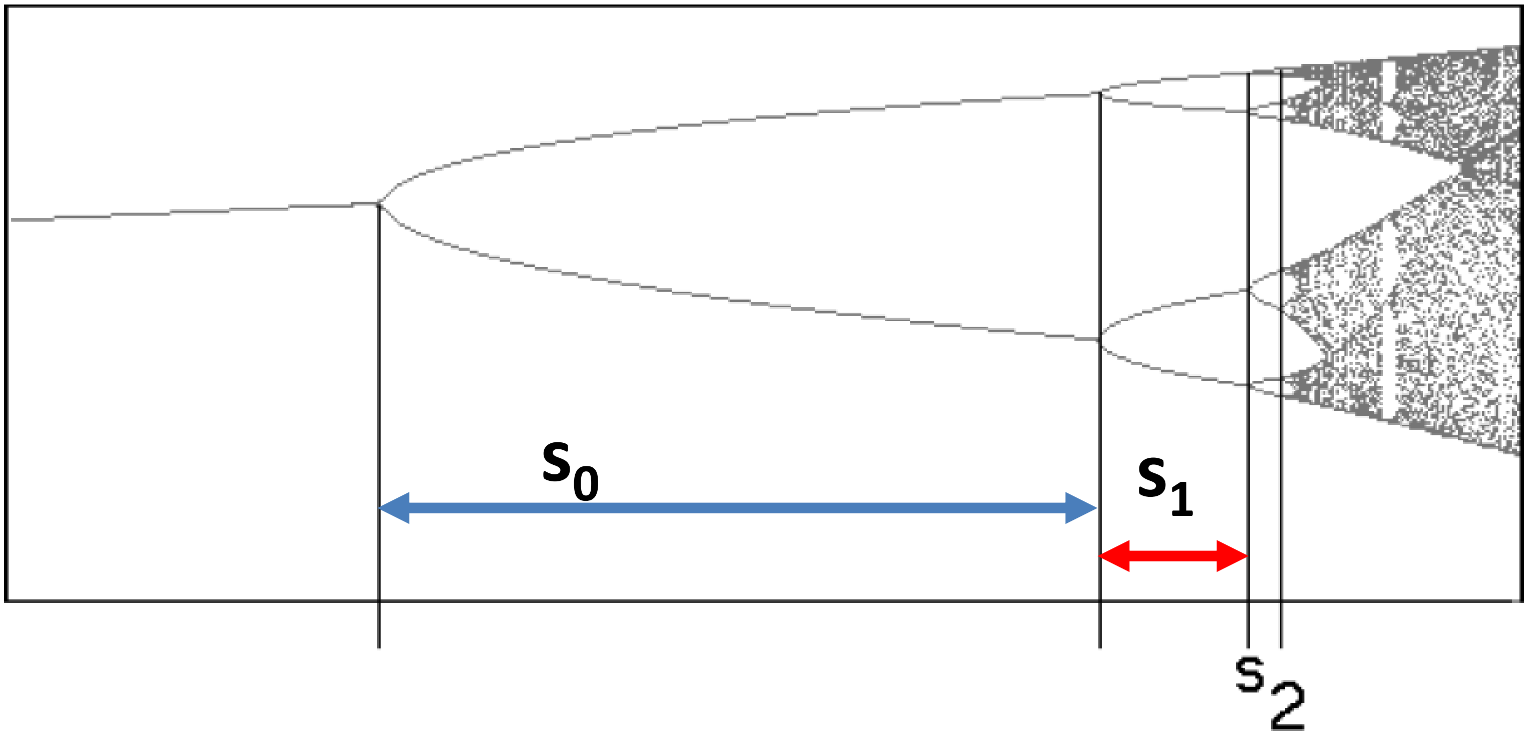
- As r is increased from 3, a sequence of period doubling bifurcations occur.



- At $r = r_\infty \approx 3.569945672$ the periods “accumulate” and the map becomes chaotic.
- For $r > r_\infty$ it has SDIC. For $r < r_\infty$ it does not.
- This is a type of **phase transition**: a sudden qualitative change in a system’s behavior as a parameter is varied continuously.

Period-Doubling Route to Chaos – Geometric Scaling

- Let's examine the ratio of the lengths of the pitchfork tines in the bifurcation diagram.

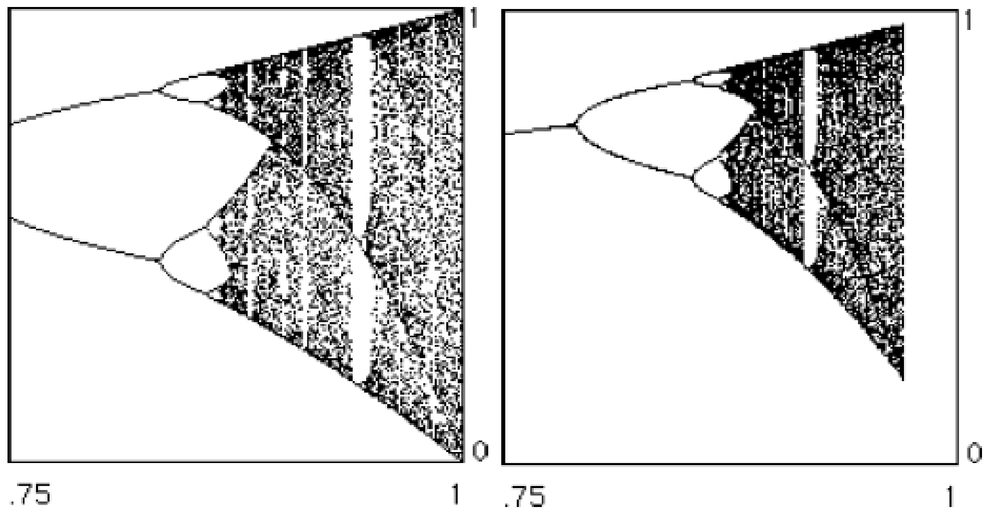


- The first ratio is: $\delta_1 = \frac{s_1 - s_0}{s_2 - s_1}$.
- The n$^{\text{th}}$ ratio is: $\delta_n = \frac{s_n - s_{n-1}}{s_{n+1} - s_n}$.

Feigenbaum's Constant

- This ratio approaches a limit: $\lim_{n\to\infty} \delta_n = 4.669201609\ldots$. This is known as **Feigenbaum's constant** δ.
- This means that the bifurcations occur in a regular way.
- Amazingly, the value of δ is **universal**: it is the same for any period-doubling route to chaos!

Universality



- The figure on the left is the bifurcation diagram for $f(x) = r\sin(\pi x)$.
- The figure on the right is the bifurcation diagram for $f(x) = \frac{27}{4}rx^2(1-x)$.
- The bifurcation diagrams are very similar: **both have** $\delta \approx 4.6692$.
- Mathematically, things are constrained so that there is, in some sense, only one possible way for a system to undergo a period-doubling to chaos.

Experimental Verification of Universality

- Universality isn't just a mathematical curiosity. Physical systems undergo period-doubling order-chaos transitions. Almost miraculously, these systems also appear to have a universal δ.

- Experiments have been done on fluids, circuits, acoustics:
  - Water: $4.3 \pm .8$
  - Mercury: $4.4 \pm .1$
  - Diode: $4.5 \pm .6$
  - Transistor: $4.5 \pm .3$
  - Helium: $4.8 \pm .6$

  Data from Cvitanović, *Universality in Chaos*, World Scientific, 1989.

- A very simple equation, the logistic equation, has produced a quantitative prediction about complicated systems (e.g., fluid turbulence) that has been verified experimentally.

- Nature is somehow constrained.

Chaos – Deterministic Source of Randomness

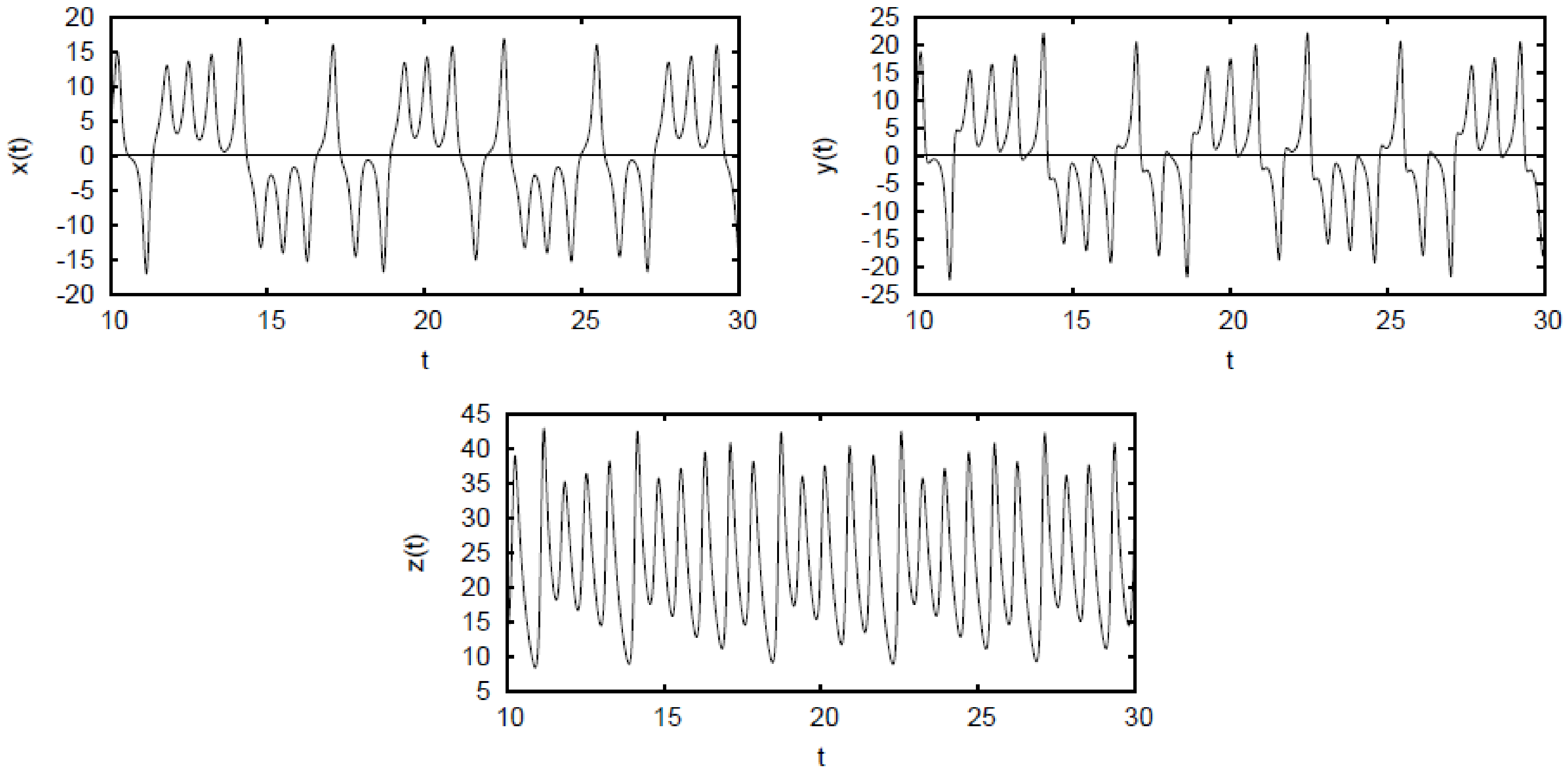
- A chaotic system behaves as if it is random, not governed by a deterministic rule.
- For $r = 4$, the symbolic dynamics of the logistic equation produce a sequence of 0's and 1's that is indistinguishable from a fair coin toss.
- Symbolic dynamics: 0 if $x < \frac{1}{2}$, 1 if $x > \frac{1}{2}$.
- The apparent randomness arises because the system is so deterministic. Determinism gives rise to SDIC.

Strange Attractors

$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= x(\rho - z) - y \\
\frac{dz}{dt} &= xy - \beta z \ . \qquad (1)
\end{aligned}
$$

- The Lorenz Equations: introduced by Edward Lorenz in the early 1960's as a very simple model of a weather system.
- Here there are three variables, x, y, and z that change in time.
- The variables are continuous: defined for every time t, not just discrete times.
- The Lorenz equations are *differential equations*, a type of dynamical system where the rate of change at every instant is specified.
- From this rate of change, one can figure out how the variables themselves change.

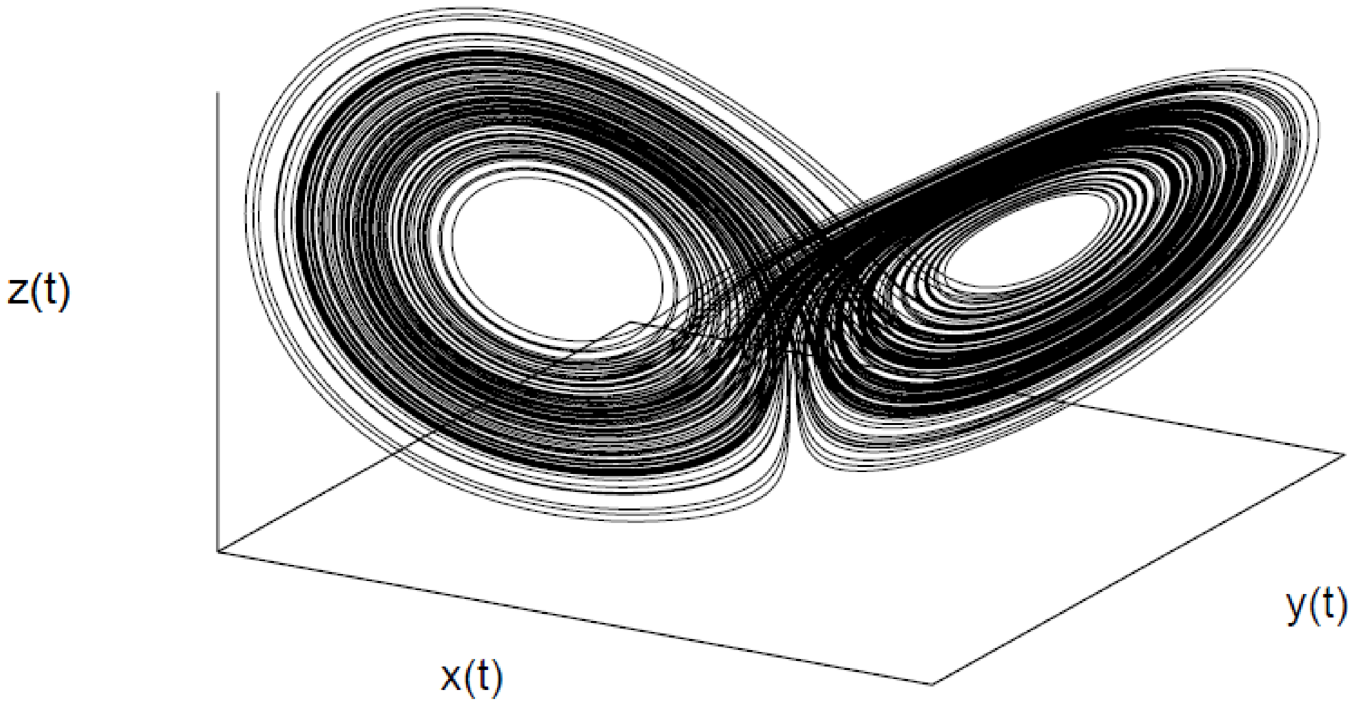
Lorenz Equations – Aperiodic Trajectories



- x, y,and z are all aperiodic. They do not repeat.
- How are x, y, and z related? To see this, let's plot the three variables on the same graph.

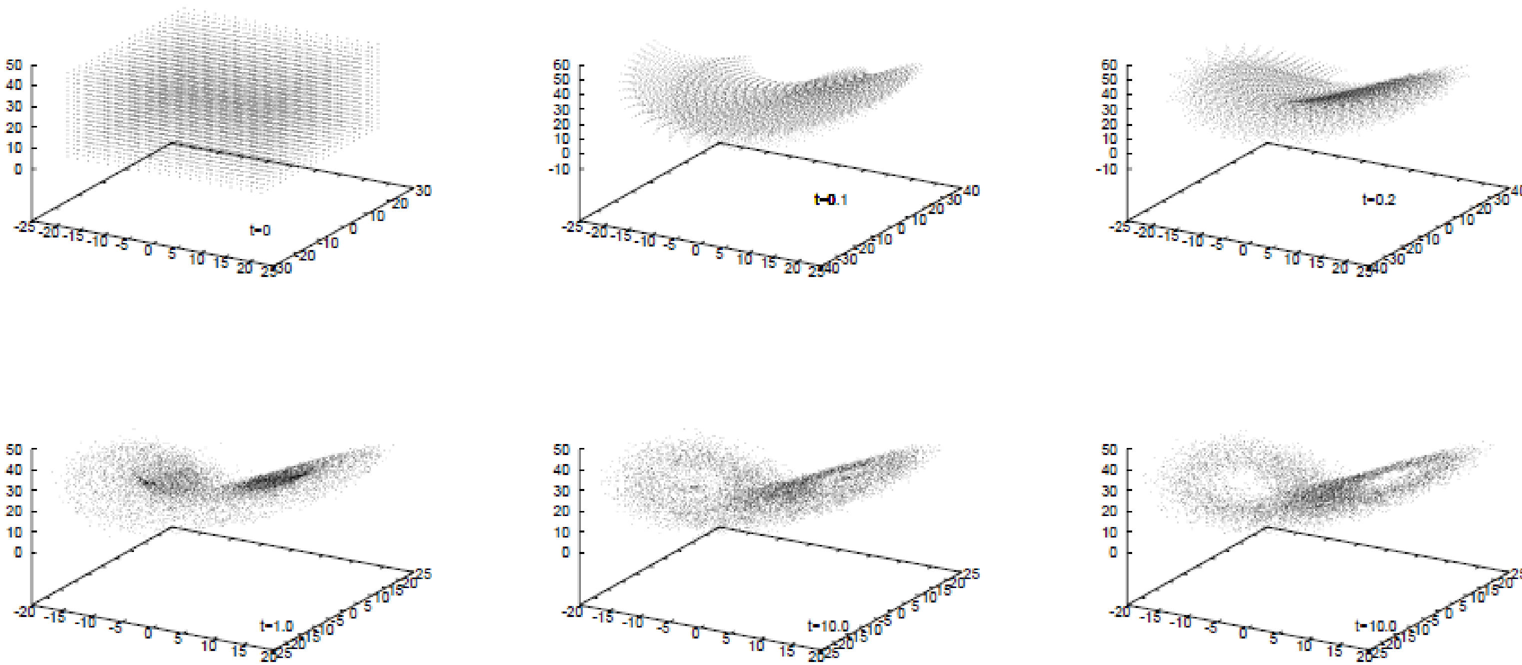
Parameter value: $\sigma = 10$ and $\beta = 2.667$, $\rho = 28$

Lorenz Equations – Strange Attractor



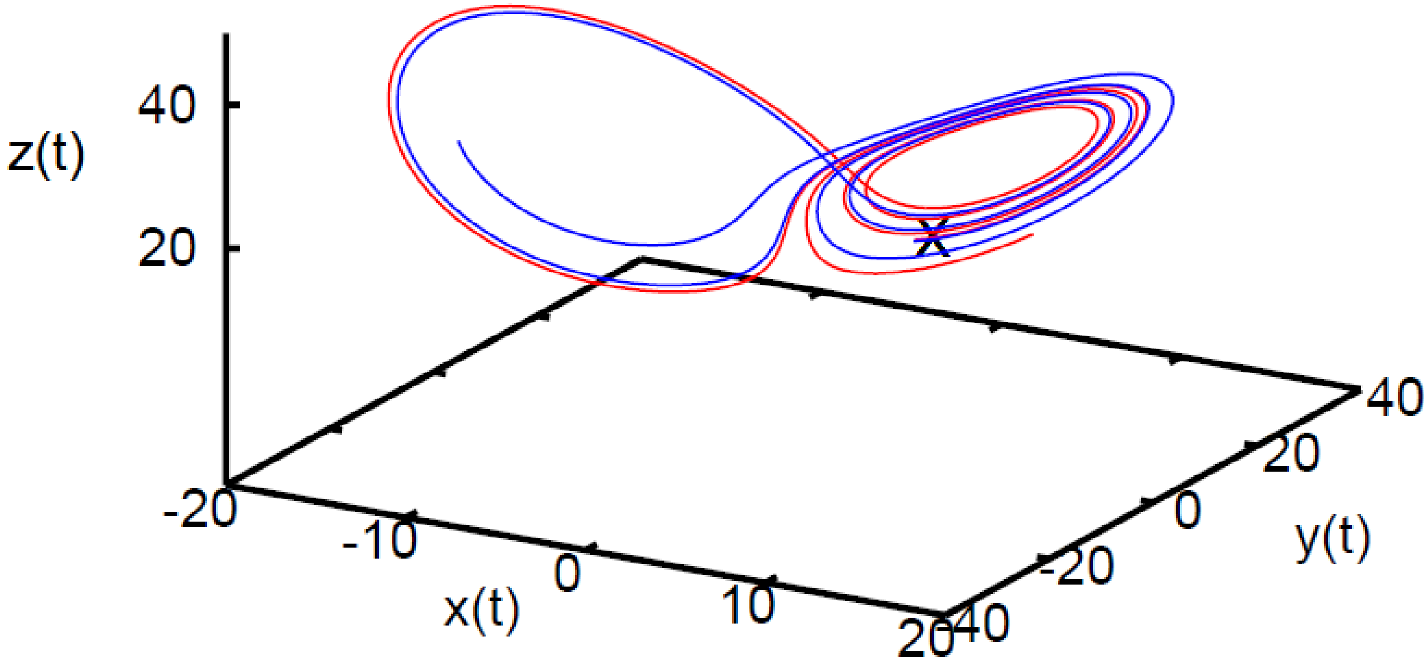
- Although individually x, y, and z move seemingly at random, when plotted together one can see a complicated relationship between them.
- The trajectory weaves through space but never repeats.

Lorenz Equations – Strange Attractor



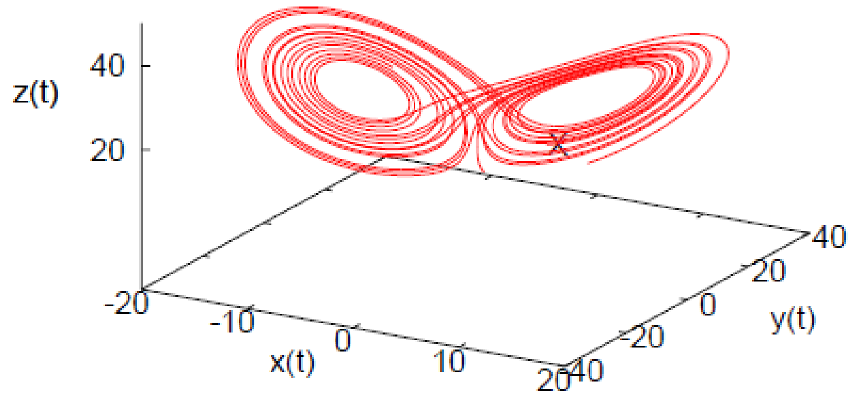
- This shape is an attractor. Orbits get pulled to it.
- Plot of 8000 different initial conditions uniformly distributed in a cube.
- The orbits are pulled to the attractor.

Lorenz Equations – Butterfly Effect



- The Lorenz Equations show the butterfly effect.
- The blue and red orbit start at almost exactly the same point, indicated by X.
- Very quickly the two orbits become quite different.

A Chaotic Attractor



- The attractor is stable; it attracts all orbits.
- But the dynamics on the attractor itself are chaotic.
- The system is a mix of order and unpredictability.
- Roughly speaking, unpredictability $\approx$ weather.
- Global structure, the shape of the attractor $\approx$ climate.
- Strange attractors are a sort of order hidden within chaos.

Chaos - Conclusions

- Deterministic systems can produce random, unpredictable behavior. E.g., logistic equation with $r = 4$.
- Simple systems can produce complicated behavior. E.g., long periodic behavior in logistic equation.
- Some features of dynamical systems are universal—the same for many different systems.
- Chaos and other structures can be stable.
- Aubin and Dahan Dalmedico: [C]haos has definitely blurred a number of old epistemological boundaries and conceptual oppositions hitherto seemingly irreducible such as order/disorder, random/nonrandom, simple/complex, local/global, stable/unstable, ...

Aubin and Dahan Dalmedico, *Historia Mathematica* 29 (2002), 167. doi:10.1006/hmat.2002.2351

Chaos & Complex Systems

- Many researchers who did groundbreaking work in chaos in the 1970s and 1980s are now doing work in complex systems.
- Appreciation that complex behavior can have simple origins.
- Universality gives us some reason to believe that we can understand complicated and complex systems with simple models.
- More generally, order and disorder, simplicity and complexity, are seen to not be opposites or mutually exclusive categories.
- There is a surprising and delightful creativity to simple, iterated systems.
- Chaos and dynamical systems hint at how randomness, complexity, and structure may emerge out of a simple and deterministic(?) world.
- But it is just one thread in the complex systems tapestry.

Ackowledgement

- David Feldman, Santa Fe Institute