Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Information Theoretical Aspects of Complex Systems

Lecture 2.08

EEU45C09 / EEP55C09
Self Organising Technological Networks

CONNECT
Networks of the Future

Nicola Marchetti
nicola.marchetti@tcd.ie

Cellular automata

- Cellular automata (CA) are simple models of *dynamical systems that are discrete in space and time*

- The number of states per lattice site, or cell, is finite and usually small

- A CA’s time development is governed by a local updating rule that is applied in parallel over the whole lattice

- John von Neumann introduced cellular automata in the 1950’s, and he wanted to use these models in his study of self-reproduction and noise-sensitivity of computation

- One purpose was to demonstrate the existence of objects capable of complex behaviour combined with the capability of self-reproduction

Cellular automata

- This work led to the design of a CA rule on a 2D lattice with 29 states per cell

- The designed object capable of making a copy of itself in this space, also had the capability to simulate any computational process, usually termed a computationally universal system

- In this way *complex behaviour* of the object was said to be guaranteed

- This work was completed and published by Arthur Burks after von Neumann’s death (von Neumann & Burks, 1966)

- There are several possibilities to construct CA rules, even with only 2 states per cell, that demonstrate various examples of complex behaviour

Computation: Information is

- input
- stored
- transferred
- combined (or "processed")
- output

Universal Computation (= *Programmable* Computers):

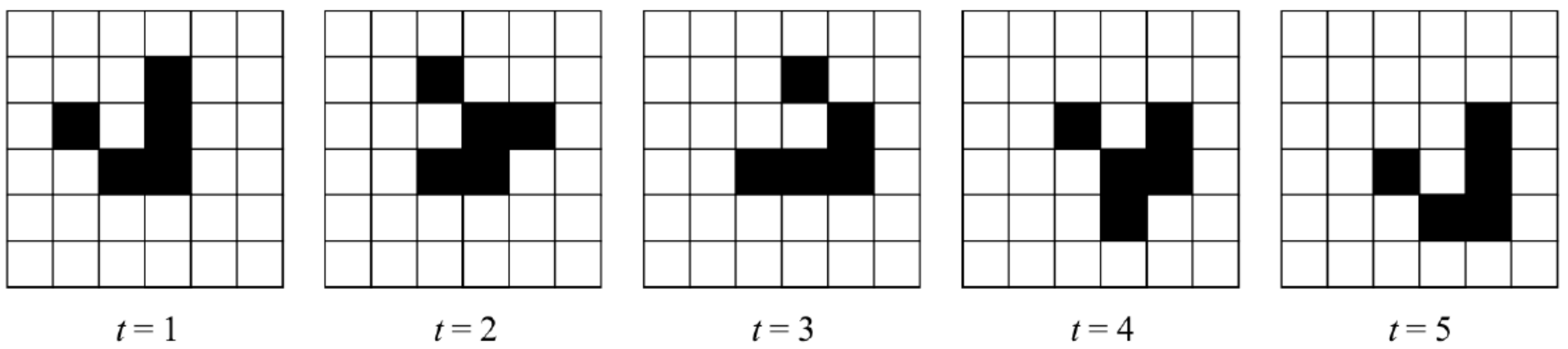
Only a small set of logical operations is needed to support universal computation!

Game of Life

- One well-known is the “Game of Life” (GOL) that was introduced by John H. Conway in 1970

- This rule has been studied extensively, mainly because of its capability of producing complex behaviour like propagating spatio-temporal structures from random initial states, exemplified by the simple “glider” in Fig. 1

Fig. 1



GOL exemplified by 5-time steps of the “glider”, a propagating object with an internal cycle of 5-time steps

Game of Life

- GOL rule is based on a local configuration involving the cell and its 8 neighbours

- Cells can be “alive” or “dead”, represented by black and white in Fig. 1

- A “dead” cell becomes “alive” in the next time step if exactly 3 neighbours are “alive”, while a cell that is “alive” remains so only if 2 or 3 neighbours are “alive”

- One can show that GOL also has the capability of universal computation

- This can be done, for example, by constructing structures in the lattice that serves as wires that allow for propagating signals that may interact through structures serving as logical gates

CA and complex behaviour

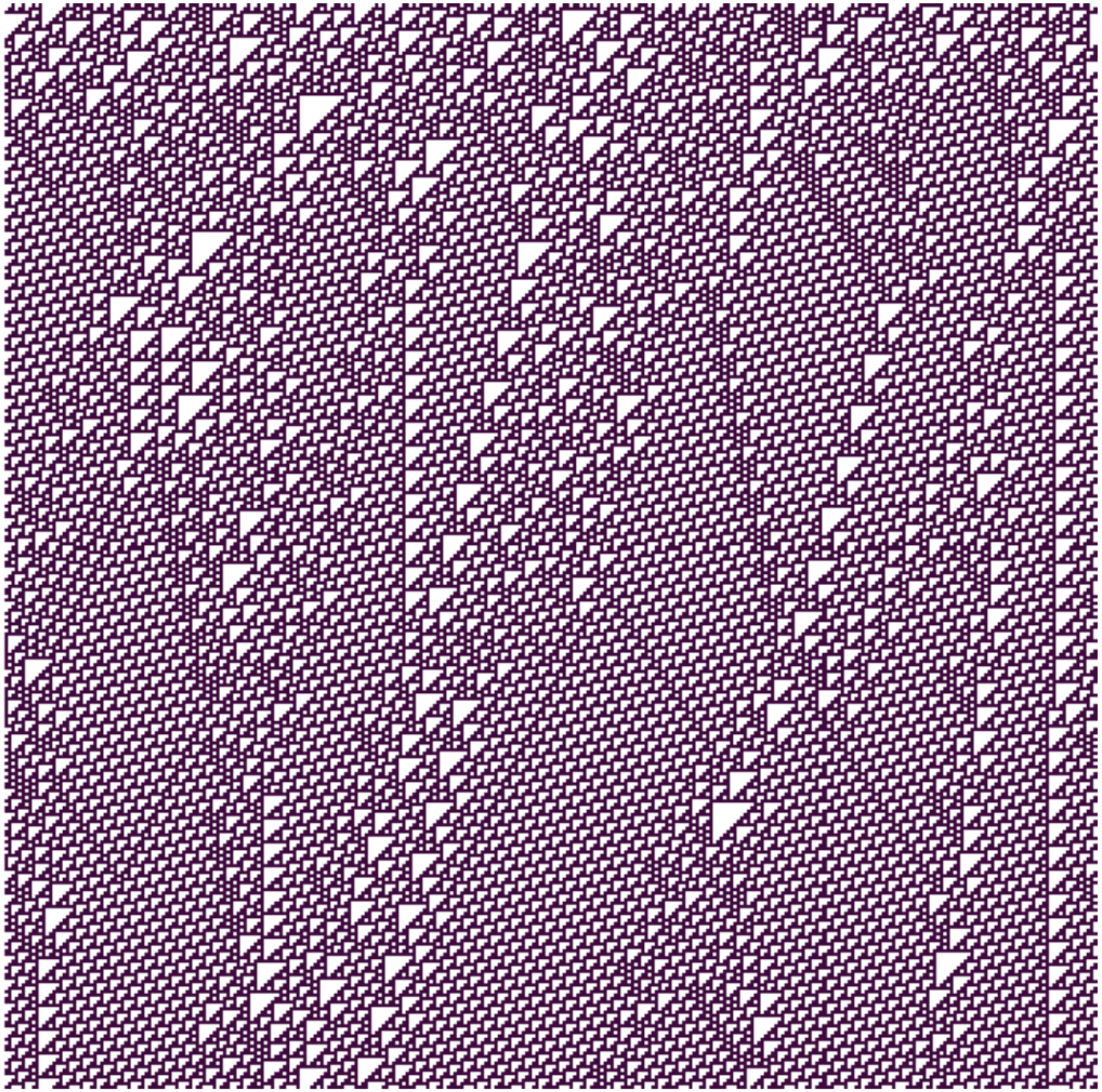
- Already in the simplest class of CA, where CA rules in 1D have local interaction depending on nearest neighbours only, there are examples of various types of complex behaviour

- For example, it was proved that the simple CA rule R110, depending on only two states per cell, is computationally universal (Cook, 2004)

- The space-time pattern of R110 is exemplified in Fig. 2

CA and complex behaviour

Fig. 2



The time evolution of CA rule R110 starting with an "random" initial state (in the top row) shows how a periodic background pattern is built up at the same time as complex structures propagate and interact

This is a good example of the complexity that simple CA rules may exhibit

Elementary cellular automata

- The simplest class of CA is based on a 1D lattice with 2 states per cell and a nearest-neighbour rule for the dynamics

- We call this class of CA, Elementary Cellular Automata (ECA)

- Such a rule is fully determined by specifying the next state of a cell for each of the 8 possible local states that describes the present state of the local neighbourhood

- An example of such a specification is shown in the following Table

t	111	110	101	100	011	010	001	000
$t+1$	0	1	1	0	1	1	1	0

Elementary cellular automata

- The time evolution of this rule, starting from a random sequence of 0’s and 1’s, is exemplified in Fig. 2

- The top row of black (1) and white (0) dots represent the initial state, and the following rows represent the development in time when the rule in the table above is applied in parallel over the whole row

- In this CA class the rules can thus be described by the binary digits in the second row of the table, which means that eight binary symbols determine the rule

- In the example, we have the rule number $(01101110)_2$ which in decimal form is 110 — the rule R110 mentioned above being computationally universal

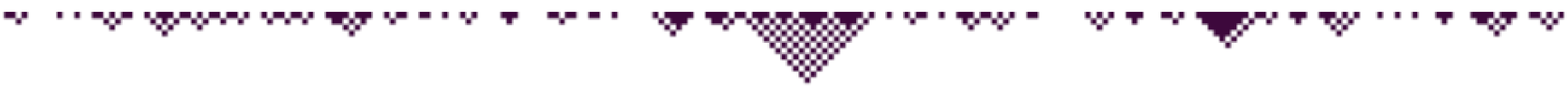
- There are $2^8 = 256$ elementary CA rules, but several of these are equivalent (by symmetries, such as changing symbols and direction), which results in 88 different rules

Dynamic behaviour of CA

- The dynamic behaviour of elementary CA can differ a lot from one rule to another

- Wolfram suggested a classification with four types

- The simplest CAs, *class I*, approach a homogenous fixed point as is exemplified by the rule in Fig. below



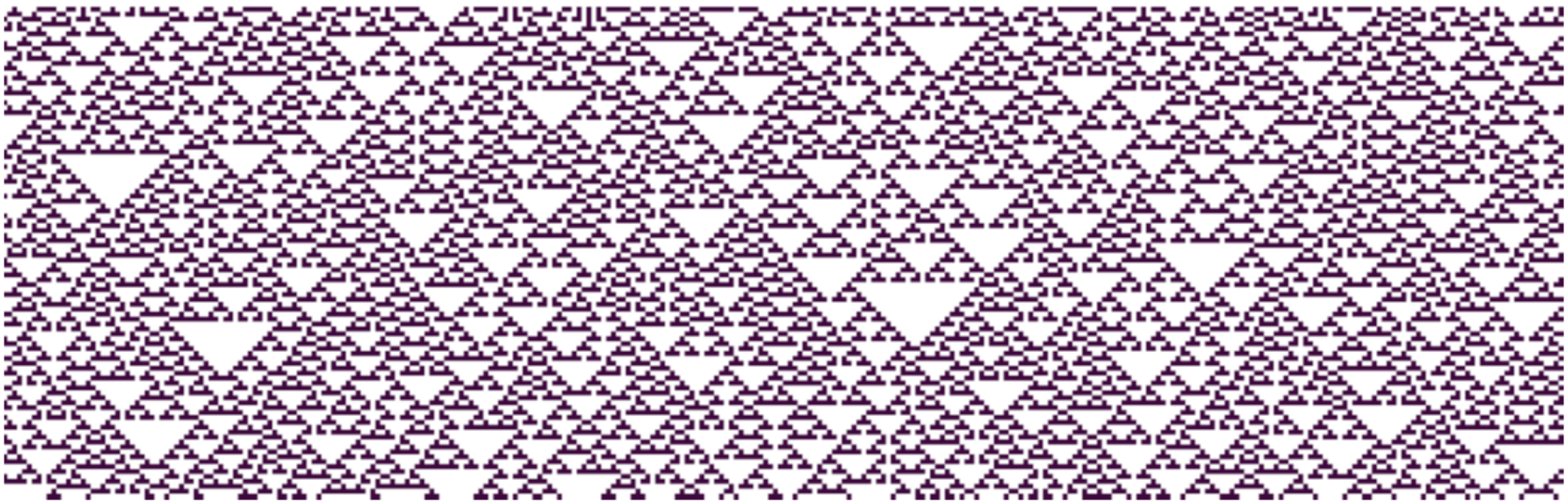
Class I rules approach a fixed point (e.g., R160)

- A *class II* rule develops into an inhomogenous fixed point or to a periodic and/or simple shift of the pattern like in Fig. below

Class II rules develop a periodic pattern in space and/or time (e.g., R213)

Dynamic behaviour of CA

- The *class III* rules never seem to approach an ordered state, but their space-time patterns continue to look disordered, as illustrated in Fig. below



Chaotic or class III rules are characterised by a continuous change of the patterns at the same time as a high disorder is kept (e.g., R22)

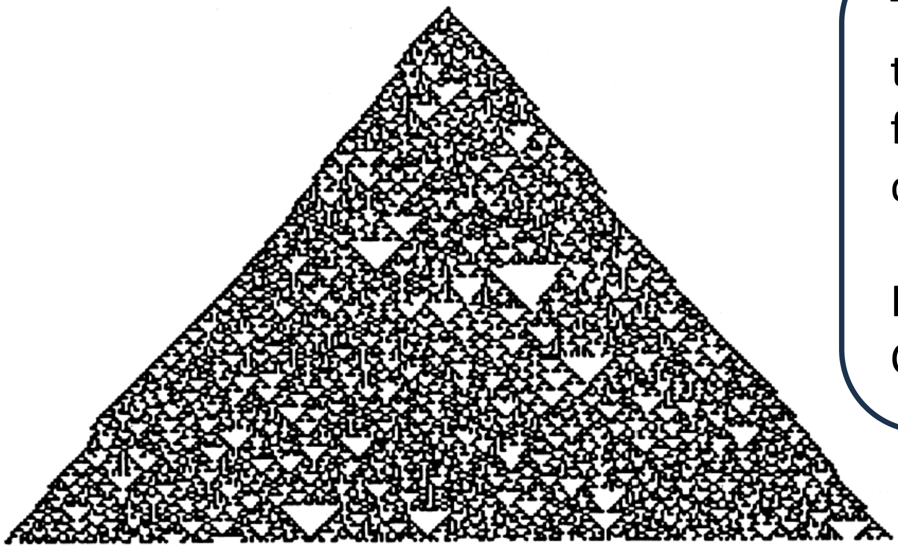
- These rules are often called *chaotic*

Dynamic behaviour of CA

- An important characteristic of the class III (chaotic) rules is that they are sensitive to small perturbations

- If one follows the time evolution starting from two initial states, differing at one position only, the number of differing positions tends to increase linearly in time

- This is illustrated in Fig. below, which shows the differing cells between the space-time patterns of two CAs following rule R22

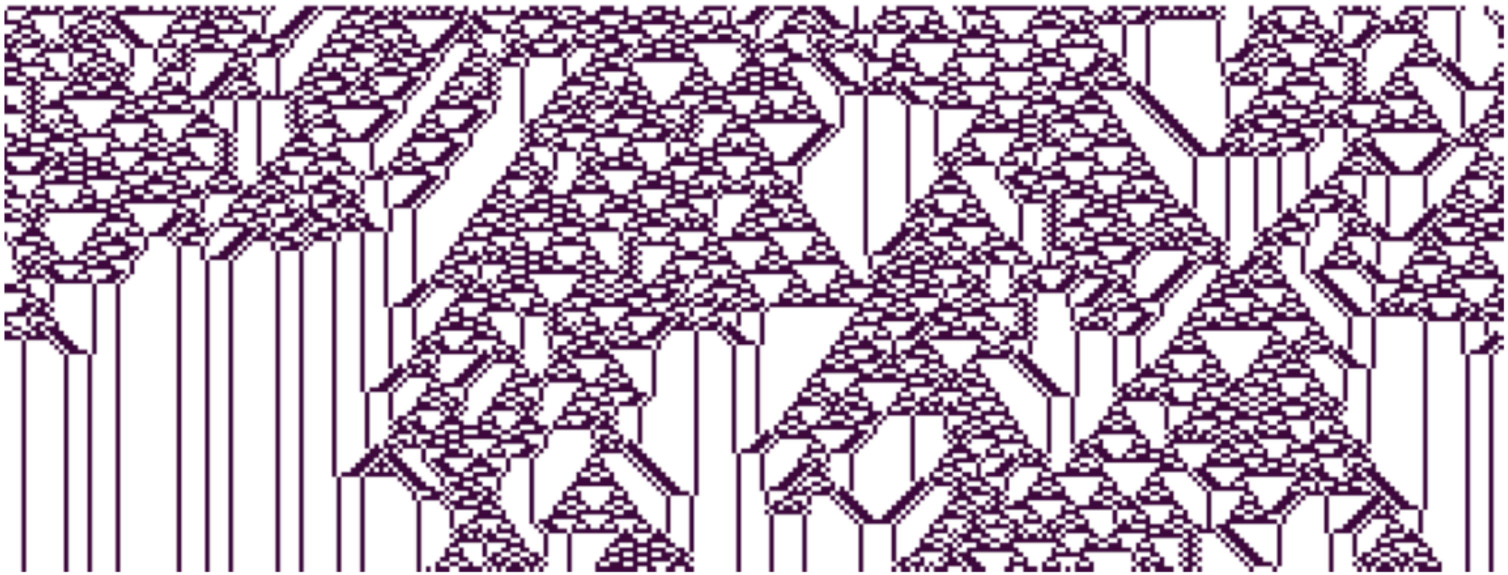


The difference pattern resulting from the time evolution of CA rule R22 starting from two different states, initially differing only by the state in one cell

Here a black dot indicates where the two CA evolutions differ from each other

Dynamic behaviour of CA

- Then there is a *class IV* that is more vaguely defined as a border class between II and III with long complex transients, possibly mixed with a spatio-temporal periodic background pattern, see Fig. below



- Rule R110 in Fig. 2 belongs to this class

Information theory for CA

- What happens with the entropy that describes the internal disorder of the CA state in the time evolution?

- Initially we may have prepared the system as a maximally disordered sequence of symbols with an entropy of 1 bit per cell

- Under what circumstances will this entropy decrease to create order in the system?

- We shall use the information theory for symbol sequences, to prove relations between the entropy of the CA at time t and the following time step t + 1

Information theory for CA

- Suppose that we have a CA rule with range r, i.e., the local neighbourhood involved in the rule includes 2r + 1 cells (r the left, r to the right, and the cell itself)

- We also assume that there are only two states per cell (0 and 1); the formalism can be extended to more states

- At each time step t, the state of the CA is described as a symbol sequence characterised by its entropy s(t)

- We will also assume that the rules are *deterministic*, which implies that the state at time t fully determines the state at the next time step

Information theory for CA

- Let β_m denote a certain sub-sequence of symbols of length m at time t + 1

- This sequence may have several possible predecessors at time t

- Since the rule has range r, the predecessor sequence has length m + 2r, and we denote such a sequence α_{m+2r}

- The probability distributions that describe the symbol sequences at different times may of course be different, and we use p for probabilities at time t and p′ for probabilities at time t + 1

Information theory for CA

- Then the probability for a sequence β_m at t + 1 can be expressed as the sum of the probabilities for all possible ancestors

$$p'(\beta_m) = \sum_{\alpha_{m+2r} \to \beta_m} p(\alpha_{m+2r})$$

- This relation can be used to establish a connection between block entropies of length m + 2r at time t with block entropies of length m at time t + 1

(***)

(1)

$$S_m(t+1) = \sum_{\beta_m} p'(\beta_m) \log \frac{1}{p'(\beta_m)} = \sum_{\beta_m} \sum_{\alpha_{m+2r} \to \beta_m} p(\alpha_{m+2r}) \log \frac{1}{\sum_{\alpha_{m+2r} \to \beta_m} p(\alpha_{m+2r})} \leq$$

$$\leq \sum_{\beta_m} \sum_{\alpha_{m+2r} \to \beta_m} \left(p(\alpha_{m+2r}) \log \frac{1}{p(\alpha_{m+2r})} \right) = S_{m+2r}(t).$$

Information theory for CA

- Above we have made use of the fact that log(1/x) is a decreasing function of x, when removing the summation in the logarithm, see **(***)**

- Inequality (1) can be used to derive the change in Shannon entropy between two successive time steps in the CA time evolution

$$\Delta s(t+1) = s(t+1) - s(t) = \lim_{m\to\infty}\left(\frac{1}{m}S_m(t+1) - \frac{1}{m+2r}S_{m+2r}(t)\right) =$$

$$= \lim_{m\to\infty}\left(\frac{1}{m}(S_m(t+1) - S_{m+2r}(t)) + \left(\frac{1}{m} - \frac{1}{m+2r}\right)S_{m+2r}(t)\right) \qquad \textbf{(2)}$$

Information theory for CA

- By using Eq. (1) and the fact that the second term in Eq. (2) goes like $2r \cdot s(t)/m$ [**Prove this**], we can conclude that, for deterministic CA rules, the entropy never increases in time

$$\Delta s(t) \leq 0$$

- Wait a minute! *Doesn't this violate the 2nd Law of Thermodynamics ?* [see Lecture 2.01]

> Second law of thermodynamics:
> - *In an isolated system, entropy always increases until it reaches a maximum value*

Information theory for CA

- For reversible CA, $\Delta s(t) = 0$

- For irreversible CA, this can be understood by the fact that deterministic CA rules reduce the number of possible states (symbol sequences) in time

- This decrease in entropy is associated with an increase in total correlation information

- This can be seen in the periodic pattern formed in time by R110 (see Fig. 2), where the initial state is a completely disordered sequence of 0’s and 1’s with an entropy s = 1 bit, but as time goes on then correlation information us built up and entropy decreases

- Also, let us keep in mind that this system is not isolated

Ackowledgement

- Kristian Lindgren, "Information Theory for Complex Systems", pages 39-44