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Information Theoretical Aspects of Complex Systems

Lecture 2.09

EEU45C09 / EEP55C09
Self Organising Technological Networks

CONNECT
Networks of the Future

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Problem 1

(a) Calculate the state probability distribution of the Markov process in Fig. Q1.

(b) Calculate the Shannon entropy of the Markov process in Fig. Q1.

- Note: Assume that when there is a choice for the transitions from a state, all such transitions are equiprobable.

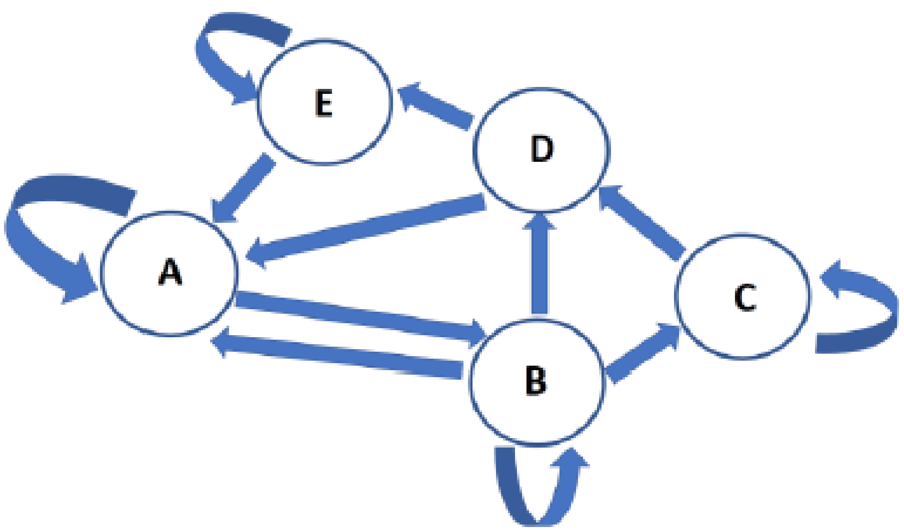


Fig. Q1

(a)

$$P_{AA} = P_{AB} = P_{CC} = P_{CD} = P_{DA} = P_{DE} = P_{EA} = P_{EE} = 0.5$$
$$P_{BA} = P_{BB} = P_{BC} = P_{BD} = 0.25$$
$$P_{AC} = P_{AD} = P_{AE} = P_{BE} = P_{CA} = P_{CB} = P_{CE} = P_{DB} = P_{DC} = P_{DD} = 0$$

$$P_A = P_{AA}P_A + P_{BA}P_B + P_{DA}P_D + P_{EA}P_E$$
$$P_B = P_{AB}P_A + P_{BB}P_B$$
$$P_C = P_{BC}P_B + P_{CC}P_C$$
$$P_D = P_{BD}P_B + P_{CD}P_C$$
$$P_E = 1 - P_A - P_B - P_C - P_D$$

$$P_A = 0.5P_A + 0.25P_B + 0.5P_D + 0.5P_E$$
$$P_B = 0.5P_A + 0.25P_B$$
$$P_C = 0.25P_B + 0.5P_C$$
$$P_D = 0.25P_B + 0.5P_C$$
$$P_E = 1 - P_A - P_B - P_C - P_D$$

$P_A = 3/8$
$P_B = 1/4$
$P_C = 1/8$
$P_D = 1/8$
$P_E = 1/8$

(b)

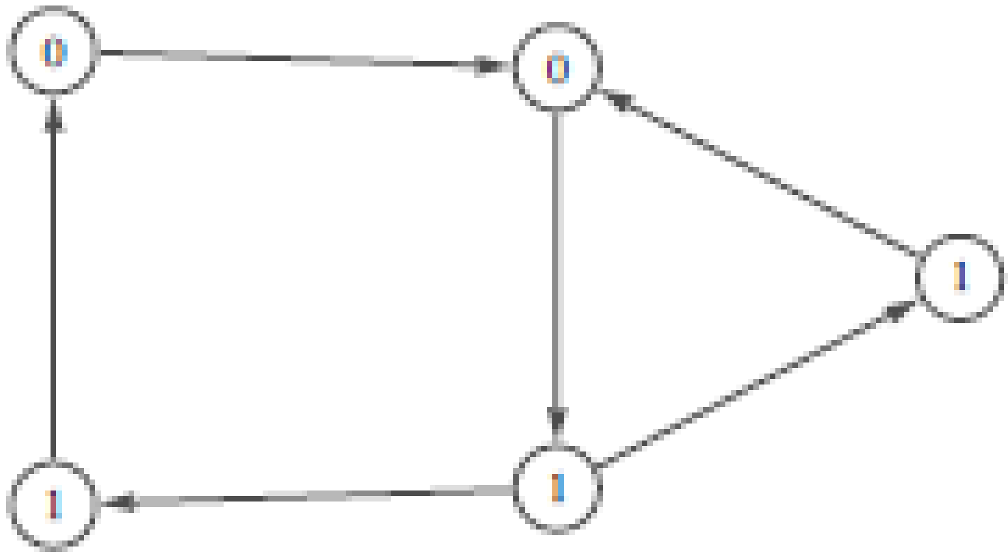
$$s = P_A I_A + P_B I_B + P_C I_C + P_D I_D + P_E I_E =$$

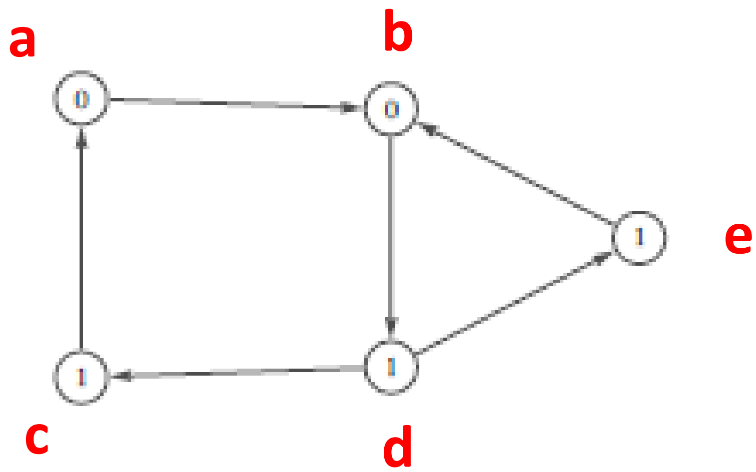
$$= (3/8) \cdot 1\ bit + \left(\frac{1}{4}\right) \cdot 2\ bits + \left(\frac{1}{8}\right) \cdot 1\ bit + \left(\frac{1}{8}\right) \cdot 1\ bit + \left(\frac{1}{8}\right) \cdot 1\ bit = 1.25\ bits$$

Problem 2

Given the below hidden Markov model, when two arcs leave a node, it is assumed that they have the same probability.

Calculate the correlation complexity η .






```
p(a) = p(c)
p(b) = p(a)+p(e)
p(c) = p(d) * 0.5   (since P_dc = P_de = 0.5)
p(d) = p(b)
p(e) = 1-p(a)-p(b)-p(c)-p(d)
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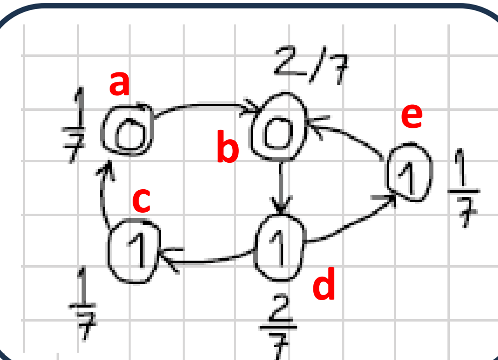
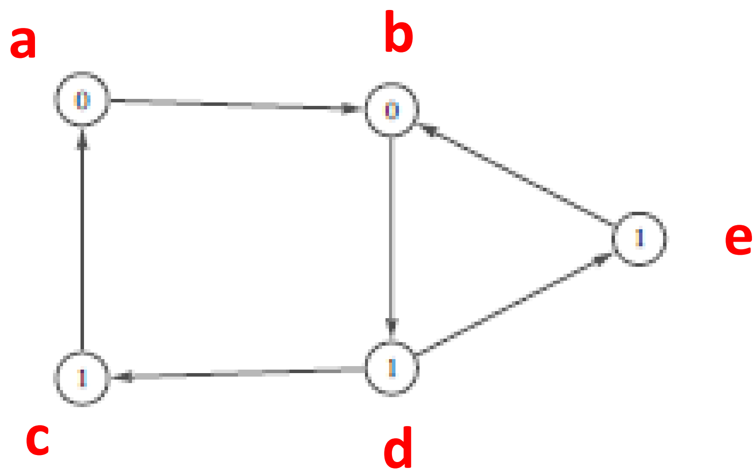


$\Rightarrow P_1 \quad p(0) = \frac{3}{7}, \; p(1) = \frac{4}{7}$

$P_2 \quad p(00) = \frac{1}{7}, \; p(01) = \frac{2}{7}, \; p(10) = \frac{2}{7}, \; p(11) = \frac{2}{7}$

$$P_n = \{p_n(x_1, \dots, x_n)\}_{x_1, \dots, x_n \in \Lambda^n} \quad (n = 1, 2, \dots)$$



Only uncertainty is in node d:

- going to the left we have: 100
- going to the right we have: 101

==> m=3

Correlation length tells how long we have to observe before resolving the uncertainty on which state path the system followed.

This suggests we can calculate $\eta = \sum_{m=1}^{\infty}(m-1)k_m$. Therefore,

$$\eta = \sum_{m=2}^{\infty}(m-1)k_m = k_2 + 2k_3$$

S_1 = p(0) $\log_2$ [1/p(0)] + p(1) $\log_2$ [1/p(1)] = 0.9852

$$k_1 = K[P_1^{(0)}; P_1] = \sum_{x_1} p(x_1) \log \frac{p(x_1)}{1/\nu} = \log \nu - S_1$$

$k_1 = \log 2 - S_1$ = 1 - S_1 = 0.0148

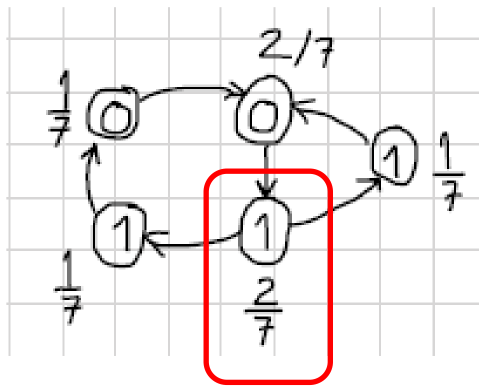
$$S_n = S[P_n] = \sum_{\sigma_n} p(\sigma_n) \log \frac{1}{p(\sigma_n)}$$

S_2 = p(00) $\log_2$ [1/p(00)] + p(01) $\log_2$ [1/p(01)] + ... = 1.9502

$$k_n = -S_n + 2S_{n-1} - S_{n-2}$$

we define $S_0 = 0$

$$k_2 = -S_2 + 2S_1 = 0.0202$$



s = 1 bit x 2/7 = 2/7

$$k_{\text{corr}} = \sum_{m=1}^{\infty} k_m$$

$$S_{\max} = \log \nu = (\log \nu - \Delta S_{\infty}) + \Delta S_{\infty} = k_{\text{corr}} + s$$

$$\log 2 = s + k_1 + k_2 + k_3$$

$k_3 = \log_2(2) - s - k_1 - k_2 = 0.6793$

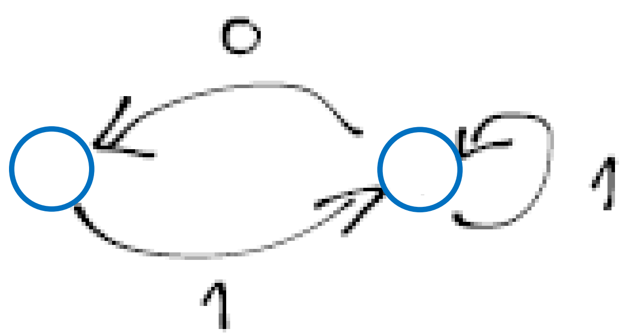
$$\eta = \sum_{m=2}^{\infty} (m-1) k_m = k_2 + 2k_3$$

$= 1.3788$

Problem 3

Consider the process defined by the finite automaton below. When two arcs leave a node, it is assumed they have the same probability.

Calculate the correlation complexity η .



Since it is a Markov process, then m = 2

$$\eta = \sum_{m=1}^{\infty} (m-1) k_m \qquad k_{corr} = \sum_{m=1}^{\infty} k_m \qquad S_{max} = k_{corr} + s$$

$$\eta = k_2 = \log 2 - s - k_1$$

```
p(R) = p(L) + 0.5 * p(R)
p(L) = 1 - p(R)

==> p(L) = 1/3 , p(R) = 2/3
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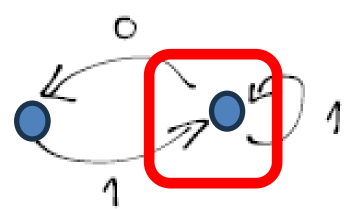

s = 1 bit * p(R) = 2/3

S_1 = p(L) $\log_2$ [1/p(L)] + p(R) $\log_2$ [1/p(R)] = 0.9183

$$k_1 = \log 2 - S_1 = 0.0817$$

$$\eta = k_2 = \log 2 - s - k_1 = 0.2516$$

Problem 4

Which one among the following systems cannot be considered as complex?

(a) A system designed according to cellular automata principles
(b) A system designed according to top-down optimisation
(c) A system achieving self-synchronisation
(d) A system with nonlinear interaction among its components

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Problem 5

Which one among the following message sources has the highest Shannon information content?

(a) A fair die
(b) A biased die
(c) A fair coin
(d) A biased coin

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Problem 6

Which one among the following statements about cellular automata is incorrect?

(a) For reversible cellular automata, entropy remains constant in time
(b) For irreversible cellular automata, entropy remains constant in time
(c) For irreversible cellular automata, entropy decreases in time
(d) Computation rules are forces, and forces do work

Which one among the following statements about cellular automata is incorrect?

- **(a)** For reversible cellular automata, entropy remains constant in time
- **(b)** For irreversible cellular automata, entropy remains constant in time
- **(c)** For irreversible cellular automata, entropy decreases in time
- **(d)** Computation rules are forces, and forces do work

Problem 7

For a cellular automaton using an alphabet of size 5 to represent its cell states, and a neighbourhood of size 7 (including the cell under consideration), how many rules are possible?

(a) 7^{7^5}
(b) $5^{5 \cdot 7}$
(c) 5^{5^7}
(d) None of the above is the correct answer

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(b) $5^{5 \cdot 7}$
(c) 5^{5^7}
(d) None of the above is the correct answer

Explanation:

Given that we have

- ***5 possible state values***
- ***Neighbourhood size = 7***

Then the number of possible rules is 5^{5^7}.

Ackowledgement

- Kristian Lindgren