Question 1 :

1. In the Deutsch algorithm, when we consider U_f as a single-qubit operator $\hat{U}_{f(x)}$, $(|0\rangle - |1\rangle)/\sqrt{2}$ is an eigenstate of $\hat{U}_{f(x)}$, whose associated eigenvalue gives us the answer to the Deutsch problem. Suppose we were not able to prepare this eigenstate directly. Show that if we instead input $|0\rangle$ to the target qubit, and otherwise run the same algorithm, we get an algorithm that gives the correct answer with probability ¾.

Answer : Divided into two cases

① when $f(x)$ is constant :

Start with $|0\rangle$ then it's superposition state :

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Apply U_f , this situation will not change state

$$U_f\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = e^{i\theta}\frac{|0\rangle+|1\rangle}{\sqrt{2}}$$

Apply the Hadamard gate H

$$H\left(e^{i\theta}\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = |0\rangle$$

So the probability of measuring $|0\rangle$ is 1 .

② when $f(x)$ is Balanced .

same as ①. after Apply H gate

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

Apply U_f , this time the oracle will flip the state

$$U_f\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

$$H\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = |1\rangle$$

So it's like correct half of the time

Conclution :

$$P = (1 + 1/2)/2 = 3/4$$

Question 2:

pure state : $|\psi_i\rangle$

density operator ρ, each state $|\psi_i\rangle$ occurs with a probability p_i : $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$

for p_i should satisfied :

$$\sum_i p_i = 1$$

Due to the formula that we used to calculate diagonal elements:

$$Tr(X) = \sum_j \langle j|X|j\rangle$$

So change ρ into formula:

$$Tr(\rho) = Tr(\sum_i p_i |\psi_i\rangle\langle\psi_i|)$$

$$\Rightarrow Tr(\rho) = \sum_i p_i Tr(|\psi_i\rangle\langle\psi_i|)$$

$$\Rightarrow Tr(\rho) = \sum_i p_i \langle\psi_i|\psi_i\rangle$$

$$= \sum_i p_i = 1$$

Question 3:

To prove if a set of operators P_i satisfy the conditions of being self-adjoint ($P_i = P_i^\dagger$) and idempotent ($P_i^2 = P_i$), then $P_i P_j = 0$ ($i \neq j$)

Answer: Given $P_i = P_i^\dagger \Rightarrow P_i$ is Hermitian

Also because Idempotent property so $P_i^2 = P_i$

$(P_i P_j)^\dagger = P_j^\dagger P_i^\dagger$

Since $P_i = P_i^\dagger$ && $P_j = P_j^\dagger$

So $(P_i P_j)^\dagger = P_j P_i$

Given : $P_i P_i = P_i$ && $P_j P_j = P_j$

$P_i P_j P_i = P_i^2 P_j = P_i P_j$

$(P_i P_j)^\dagger = (P_i P_j P_i)^\dagger = P_i^\dagger P_j^\dagger P_i^\dagger = P_i P_j P_i$

$P_j P_i = P_i P_j P_i$

Multiply $P_j \Rightarrow P_j P_i P_j = P_i P_j P_i P_j = P_i P_j$ $\because i \neq j$ $\therefore P_j P_i = 0$

$0 = P_i P_j$

Question 4:

$$QFT|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i x k / N} |k\rangle$$

$\because$ 3-qubit system $\therefore N = 2^3 = 8$

$$\therefore \quad QFT|x\rangle = \frac{1}{\sqrt{8}} \sum_{k=0}^{7} e^{2\pi i x k/8} |k\rangle$$

Second time QFT:

$$QFT(QFT|x\rangle) = \frac{1}{8} \sum_{k=0}^{7} e^{2\pi i x k/8} \sum_{j=0}^{7} e^{2\pi i k j/8} |j\rangle$$

So two times apply QFT

equal to a bit-reversal operation. means actually back to state $|x\rangle$ but with a global phase factor.

To summarize: two consecutive apply of QFT will bring back to original state $|x_1, x_2, x_3\rangle$, but in a global phase.