EEU44C04 / CS4031 / CS7NS3 / EEP55C27
Next Generation Networks

M/M/1 queue

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Deriving queuing theory results

- Poisson assumptions for the arrival of customers to a queue make the analysis tractable
  - Markov chains

- Useful references:
  - T. G. Robertazzi, *Computer Networks and Systems*, 3rd edition, Springer, 2000
  - L. Kleinrock, *Computer Applications, Volume 2, Queuing Systems*, Wiley, 1976

Little's law

- The expected number in the system (queue) is the product of the arrival rate and the average time spent in the system (queue)

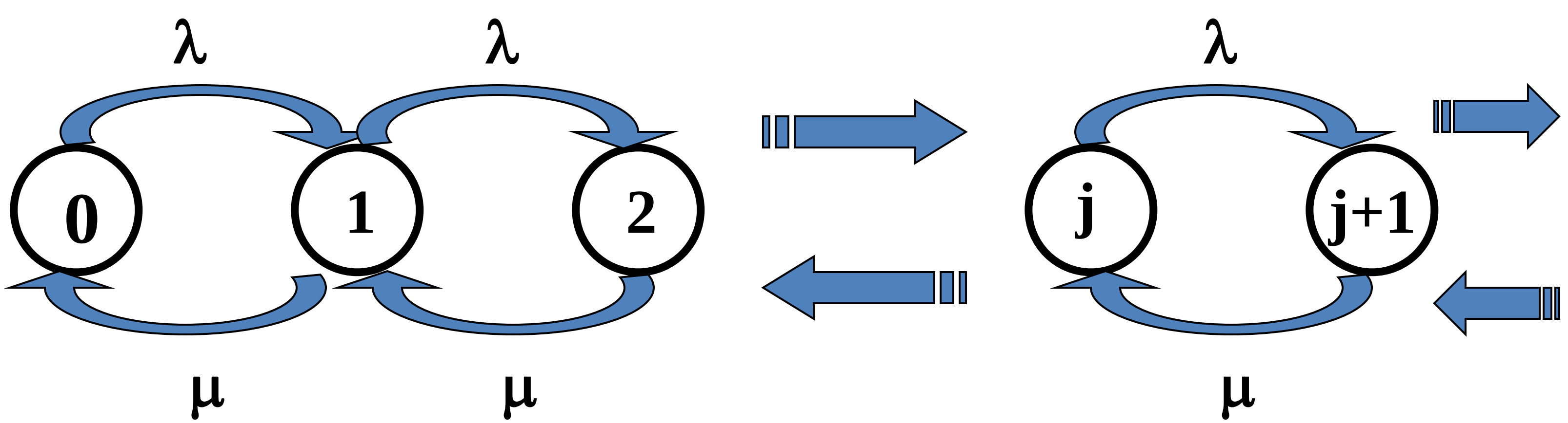
$$E[n] = \lambda E[\tau]$$

The M/M/1 model

- Single-server system
- Arrivals according to Poisson process of rate λ
- Inter-arrival times exponential r.v.'s
- Service times exponential r.v.'s with mean $1/\mu$
- Infinite buffers

Markov chain

- Due to the **memoryless property** of exponential r.v.'s, the number of customers in the system can be expressed as a continuous-time Markov chain
  - States represent the number of customers (packets, users) in the queuing system



Probability mass function

- Let $\rho = \lambda / \mu < 1$ (called **utilization**)
- We can then determine the probability mass function for the number of customers currently in the system
  - Let p_n denote the probability that n customers are currently in the system

- Results:

$$p_0 = (1 - \rho)$$

$$p_n = \rho^n p_0 = \rho^n (1 - \rho)$$

Average number in the system

- We can calculate the **average number** of customers (packets, calls,…) in the system as

$$E[n]=\sum_{j=0}^{\infty} j p_j = \frac{\rho}{1-\rho}$$

Average delay

- We can use Little's law to get the **average delay** through the system:

$$E[\tau] = \frac{1}{\lambda}\frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda}$$

- Also, the average waiting time is:

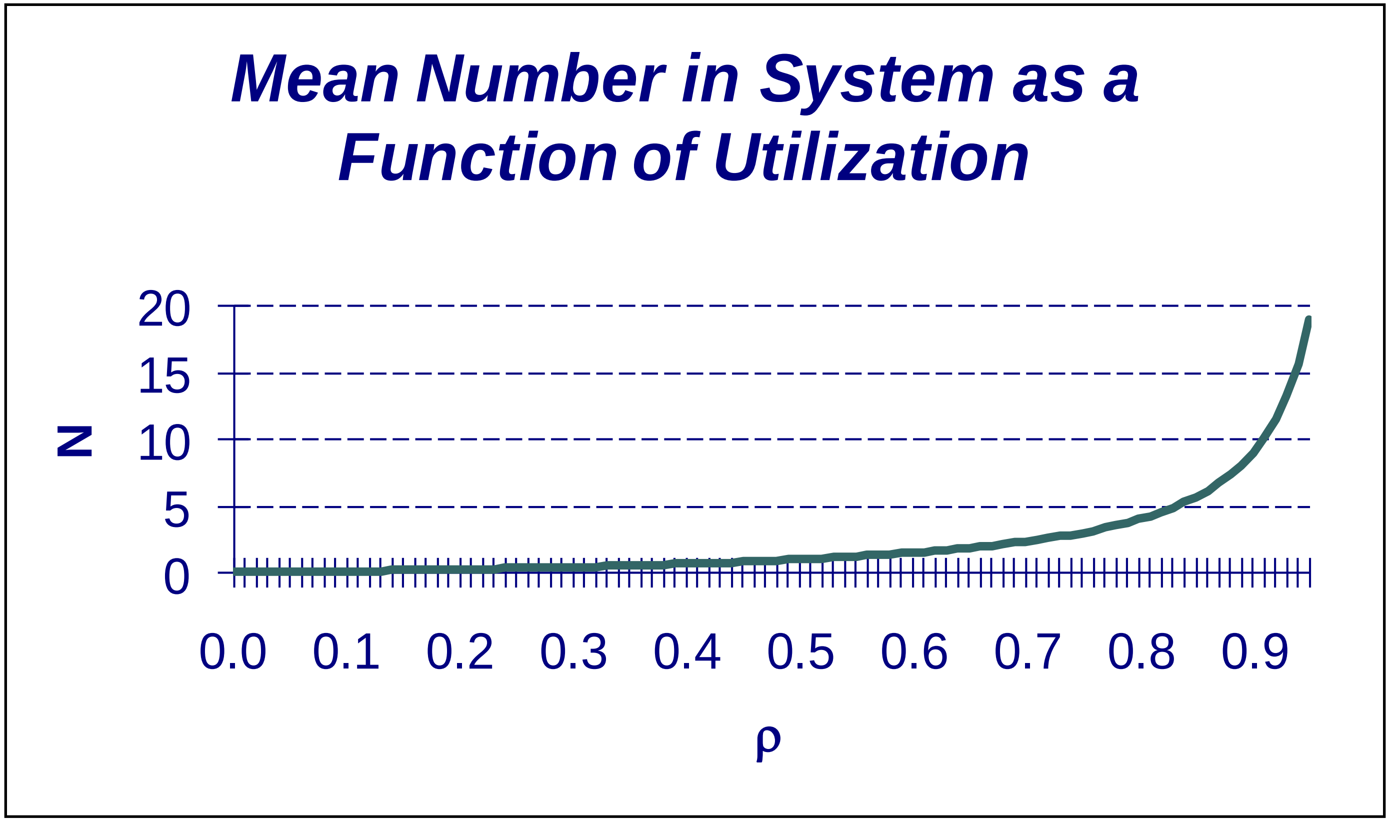
$$E[\tau_Q] = E[\tau] - E[s] = E[\tau] - \frac{1}{\mu} = \frac{1}{\mu}\frac{\rho}{1-\rho}$$

Average number in the queue

- We can use Little's law again to get the average number of customers in the queue:

$$E[n_q] = \lambda E[\tau_Q] = \frac{\rho^2}{1-\rho}$$

M/M/1 queue graphical results



Problem

A concentrator receives messages from a group of terminals and transmits them over a single transmission line. Suppose that messages arrive according to a Poisson process at a rate of one message every 4 ms, and that message transmission times are exponentially distributed with mean 3 ms.

What is the utilization of the system ?

What is the average delay through the system, expressed in ms?

- **M/M/1**
- **arrival rate** $\lambda = 1/4$ msg/msec
- **service rate** $\mu = 1/3$ msg/msec

What is the utilization of the system ?

$$\rho = \frac{\lambda}{\mu} = 3/4$$

What is the average delay through the system, expressed in ms?

$$E[\tau] = \frac{1}{\mu - \lambda} = \frac{1}{1/3 - 1/4} = 12$$

Problem

You have designed a first-come first-served data processing system. You based the design on an M/M/1 queuing model. The log of the system shows that there are on average 19 jobs in the system and that the average processing time (service time) for one job is 2 seconds.

Estimate the system utilization, the job arrival intensity and the average waiting time.

$$E[n] = \frac{\rho}{1-\rho}$$

$$19 = \frac{\rho}{1-\rho} \Rightarrow 19(1-\rho) = \rho \Rightarrow \rho = \frac{19}{20} = 0.95$$

$$\lambda = \rho\mu = 0.95 \times \frac{1}{2} = 0.475\, jobs/\sec$$

$$E[\tau_Q] = \frac{1}{\mu}\frac{\rho}{1-\rho} = \frac{1}{1/2}\frac{0.95}{1-0.95} = 38\sec$$

Problem

In a single server system, arrivals can be modeled as a Poisson process with rate of 3 min^{-1} and the service times are exponentially distributed with mean 10 s.

Give the Kendall's notation (motivating your choice) and calculate:

a) the average number of customers in the system;
b) the average delay and the average waiting time;
c) the average number of customers in the queue.

Kendall's notation: M / M / 1

Poisson arrivals

Exponential service times

One server

$$\rho = \frac{\lambda}{\mu} = \frac{3/60}{1/10} = 0.5$$

a) $$E[n] = \frac{\rho}{1-\rho} = 1$$

b) $$\begin{cases} E[\tau] = \dfrac{1}{\lambda} E[n] = 20\,\mathrm{s} \\ E[\tau_q] = E[\tau] - \dfrac{1}{\mu} = 10\,\mathrm{s} \end{cases}$$

c) $$E[n_q] = \lambda E[\tau_q] = 0.5$$