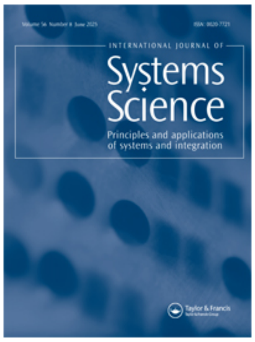

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ABSTRACT
The problem of eigenstructure assignment in linear systems via state-derivative feedback is considered. A new derivative feedback design framework named complementary system framework is proposed. By using this framework, notions of complementary controllability and complementary controllability indices of linear systems are introduced. A necessary and sufficient condition for the solvability of the eigenstructure assignment problem in complementary S-controllable (C-controllable) linear systems is then given by inequalities which involve the complementary controllability indices of a linear system, and a list of the degrees of invariant polynomials and two lists of non-negative integers to represent the finite non-zero, zero and infinite eigenvalue structure of the closed-loop system. Based on a simple parametric solution to a group of recursive equations, a complete parametric approach for solving the eigenstructure assignment problem is proposed. General parametric expressions for the closed-loop eigenvectors and the feedback gain matrix are established in terms of a group of parameter vectors. The approach generalises and improves the previous results in this area. Besides, the combined problem of simultaneously assigning dynamical order and finite eigenstructure in linear systems by state-derivative feedback is also considered. Application of the proposed approach to control of a three degrees of freedom mass-spring-dashpot system is discussed.

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1. Introduction

Eigenstructure assignment in linear systems is a very important problem in linear system theory. The eigenstructure assignment problem is to seek a feedback control law such that the closed-loop system has given eigenvalues and eigenvectors. Assigning eigenvalues can alter the stability and the speed of the response of the system, while assigning eigenvectors can alter the transient response and sensitivities of the eigenvalues of the system. Consequently, as one of the important problems in linear system theory, the problem of eigenstructure assignment has been intensively and widely studied by many researchers during the past several decades for both normal and descriptor linear systems, see for example, Fahmy and O'Reilly (1982), Kwon and Youn (1987), G. R. Duan (1993, 1999), Zhang (2008, 2019), Kaczorek (2005, 2023) and the references therein.

It is well known from classical control theory that derivative feedback can be extremely useful, and even in some cases essential to achieve desired performance (Lewis & Syrmos, 1991). The use of proportional and derivative (PD) feedback controllers has had a long history in industrial practice where derivative controls are employed to provide anticipatory action for overshoot reduction in the responses (Kuo, 1980). In recent years, pure derivative feedback has received significant attention (Araújo, 2019; Arthur & Yoon, 2020; Beteto et al., 2022, 2023; Gonzalez et al., 2019; Leandro & Pereira, 2023; Rossi et al., 2018; Sever & Yazici, 2021; Thabet, 2019; Thabet et al., 2019, 2024, 2021; Tseng et al., 2018; Vogás et al., 2024; Yazici & Sever, 2017, 2018a, 2018b; Zaghdoud et al., 2018; Zaheer et al., 2021). The motivation for derivative feedback comes from several practical applications, for instance, controlled vibration suppression of mechanical systems. The main sensors used in these problems are accelerometers and from the accelerometers signals it is possible to reconstruct the velocities by integration with reasonable accuracy. Although it is also possible to reconstruct the displacements by integration

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from the accelerometers signals, due to involving two integrations it may increase the cost and complexity of the implementation as well as the errors associated to bias in the acceleration measurements. Therefore, the available signals for feedback are accelerations and velocities only, that is, the state-derivative signals (defining the velocities and displacements as the state variables). In such situations, the state or PD feedback controller is no longer available, and derivative feedback can be utilised as an alternative for state or PD feedback control. The derivative feedback control has been shown to be a feasible method and applied in the design of many practical control systems, such as aircraft roll maneuver control (Kwak & Yedavalli, 2001; Yedavalli et al., 1998), control of active suspension systems (Assunção et al., 2007; Beteto et al., 2022; da Silva et al., 2013; Reithmeier & Leitmann, 2003; Sever & Yazici, 2021; Vogás et al., 2024; Yazici & Sever, 2017, 2018b), vibration control of landing gear components (Kwak et al., 2002b), vehicle stability enhancement (Fallah et al., 2013), control of cantilevered beams (Araújo, 2019; Kwak et al., 2002a), control of vibration absorber systems (Abdelaziz, 2007, 2009, 2011; Abdelaziz & Valášek, 2004, 2005), control of bridge cable vibration (Y. F. Duan et al., 2005), steel jacket platforms (Yazici & Sever, 2018a), control of cart-pendulum system (Zaheer et al., 2021), control of magnetic levitation system (Arthur & Yoon, 2020; Zaheer et al., 2021) and disturbance rejection of torsion bar (Gonzalez et al., 2019).

In the literature, derivative feedback has been utilised to address various control problems, such as linear quadratic regulator (Y. F. Duan et al., 2005; Kwak et al., 2002a, 2002b; Yazici & Sever, 2018a), pole assignment (Abdelaziz & Valášek, 2004, 2005; Mirassadi & Tehrani, 2017; Zaghdoud et al., 2018), robust pole assignment (Abdelaziz, 2007, 2009; Araújo, 2019; Faria et al., 2009), eigenstructure assignment (Abdelaziz, 2011), tracking control (Tseng, 2010; Tseng & Wang, 2012; Tseng et al., 2018), robust predictive control (Rossi et al., 2018), variable structure control (Tseng & Wang, 2010), sliding-mode control (Thabet et al., 2021; Tseng & Wang, 2013), feedback stabilisation (Arthur & Yoon, 2020; Assunção et al., 2007; Leandro & Pereira, 2023; Thabet, 2019; Thabet et al., 2019; Vogás et al., 2024; Zaghdoud et al., 2018; Zaheer et al., 2021), H_2/H_∞ gain scheduling control (Beteto et al., 2022, 2023), L_2 gain control (Yazici & Sever, 2017, 2018b), disturbance rejection control (Gonzalez et al., 2019), adaptive control (Basturk & Krstic, 2014; Thabet et al., 2024), etc.

This paper deals with the problem of eigenstructure assignment in the following linear system

$$E\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

via the state-derivative feedback control

$$u(t) = -K\dot{x}(t) \tag{2}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^r$ are, respectively, the state vector and the input vector; $E, A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times r}$ are known matrices with $\operatorname{rank} E = n_E \leq n$, $\operatorname{rank} A = n_A \leq n$, and $\operatorname{rank} B = r$; $K \in \mathbb{R}^{r\times n}$ is the state-derivative feedback gain matrix. The system (1) is called a normal system if the derivative matrix E is non-singular. Specially, when the matrix E is an identity matrix, the system is called a standard state space system. The system (1) is called a descriptor system if the derivative matrix E is singular.

Firstly, to deal with the eigenstructure assignment problem, we need to establish a new framework for derivative feedback. There are already some good design frameworks in the literature, and the reciprocal state space (RSS) framework is the most popular one, which was informally introduced by Tseng (1997). The RSS framework hinges on mathematically switching the state with the derivative of the state and then the relevant theories of state feedback control can be applied. Many derivative feedback control problems have been successfully solved in the RSS framework, see for example, Kwak et al. (2000), Y. F. Duan et al. (2005), Tseng (2010), Tseng and Wang (2010, 2012, 2013), Tseng et al. (2018), Thabet (2019), Thabet et al. (2019, 2024, 2021), and Sever and Yazici (2021). However, the main drawback of the RSS framework is that it can be applied only for systems that have nonsingular state matrices. Furthermore, when the state matrix is ill-conditioned, large computational errors will be produced. To overcome this problem, analogous to switching idea of the RSS framework, we propose a new derivative feedback design framework named complementary system (CS) framework, which hinges on mathematically switching the original system with its complementary system.

Secondly, we solve the existence part of the eigenstructure assignment problem. The problem is to find a condition that identifies whether a state-derivative feedback (2) exists such that the closed-loop system $(E + KB)\dot{x}(t) = Ax(t)$ has a desired eigenstructure. For this, we need to study the limits of state-derivative feedback (2) in altering the dynamics of system (1). For linear systems using state feedback, this problem has been thoroughly studied in literature. The pioneering work on this aspect is due to Rosenbrock (1970), who obtained the limits of state feedback in altering the eigenstructure of standard state space systems in terms of inequalities which involve the controllability indices of the systems and the degrees of the desired invariant polynomials. This result is referred to as the fundamental theorem of state feedback, and it was extended by Kučera and Zagalak (1988), Özcaldiran (1990), and Zagalak and Kučera (1991, 1992, 1995) to descriptor systems. One extreme was treated in Kučera and Zagalak (1988) where the desired eigenstructure is specified solely by a list of invariant polynomials, while the other extreme was treated in Zagalak and Kučera (1991) where the desired eigenstructure is specified solely by a list of positive integers. The problem of simultaneously assigning the finite and infinite eigenstructure was solved in Özcaldiran (1990) and Zagalak and Kučera (1992, 1995) where the desired eigenstructure is given by a list of invariant polynomials and a list of positive integers. For linear systems using state-derivative feedback, this problem has also been studied by a number of researchers. Abdelaziz and Valášek (2004) restricted their attention to single-input normal linear systems, while Abdelaziz and Valášek (2005), Mirassadi and Tehrani (2017) and Araújo (2019) focussed on multi-input normal and descriptor linear systems. However, they address just the aspect of positioning but say nothing about the achievable structure of repeated eigenvalues. Abdelaziz (2009) attempted to established a similar result to the fundamental theorem of state feedback for state-derivative feedback in multi-input standard state space systems. Several necessary conditions which ensure solvability via state-derivative were given. He proved that, when assigning zero eigenvalues, the limits of state-derivative feedback in altering the eigenstructure of a standard state space system are given in terms of inequalities which involve the controllability indices of the system and the degrees of the desired invariant polynomials with the restriction that one closed-loop eigenvalue should be zero and the other $(n-1)$ closed-loop eigenvalues are arbitrary finite values. Unfortunately, this result may not necessarily hold true. In fact, the system with a singular state matrix A has $(n - \text{rank}A)$ uncontrollable zero eigenvalues by derivative feedback, since a derivative feedback controller has no influence on the rank of A (the matrix A is the same in both open-loop and closed-loop systems). Therefore, the zero eigenvalue must be assigned with multiplicity at least $(n - \text{rank}A)$. This indicates that neither the controllability indices nor the invariant polynomials can identify the $(n - \text{rank}A)$ uncontrollable zero eigenvalues of linear systems that have singular state matrices A by derivative feedback. Besides, the infinite eigenstructure assignment was not investigated in Abdelaziz and Valášek (2004, 2005), Abdelaziz (2009), Mirassadi and Tehrani (2017) and Araújo (2019).

Thus, the solvability of the state-derivative feedback eigenstructure assignment problem has only been partially solved and has to be studied. By using the CS framework, state-derivative feedback design of system (1) can be converted into state feedback design of its complementary system. With the help of the complementary system, the notions of complementary controllability (including complementary S-controllability and complementary C-controllability) and controllability indices of the system (1) are introduced. Then the limits of state-derivative feedback in altering the dynamics of linear systems are studied using the CS framework. A necessary and sufficient condition is given for a list of invariant polynomials and two lists of non-negative integers to represent the finite non-zero (when assigning finite non-zero eigenvalues), zero (when assigning zero eigenvalues), and infinite (when assigning infinite eigenvalues) eigenvalue structure of the closed-loop system. The condition consists of inequalities which involve the complementary controllability indices of the linear system, a list of the degrees of the invariant polynomials and two lists of non-negative integers. This result is named the fundamental theorem of state-derivative feedback.

Thirdly, we study the solution of the eigenstructure assignment problem. With the development and application of derivative feedback control theory, the problem of eigenstructure assignment in both normal and descriptor linear systems via state-derivative feedback has been a subject of intensive research (Abdelaziz, 2007, 2009, 2011; Abdelaziz

& Valášek, 2004, 2005; Araújo, 2019; Lewis & Syrmos, 1991; Mirassadi & Tehrani, 2017). The pioneering work on this aspect is due to Lewis and Syrmos (1991), who established a geometric theory for derivative feedback and studied the problem for descriptor linear systems with an emphasis on eigenvalue assignment. A procedure for solving the eigenvalue assignment problem using state-derivative feedback in normal linear systems was obtained by Abdelaziz and Valášek (2004), in the single input case, which was then generalised to the multi-input case by the same authors Abdelaziz and Valášek (2005). In addition, algorithms for the eigenvalue assignment problem in normal single input linear systems and the robust eigenvalue assignment problem in normal multi-input linear systems using state-derivative feedback were proposed by Abdelaziz (2007, 2009). Abdelaziz (2011) considered the eigenstructure assignment problem via state-derivative feedback in normal linear systems. A parametric approach for solving the problem was presented. Mirassadi and Tehrani (2017) and Araújo (2019) studied the partial eigenvalue assignment problem in descriptor and normal linear systems via state-derivative feedback. However, these existing solutions to the problem of eigenstructure assignment in linear systems by state-derivative feedback are subject to the following limitations:

(a) The approach for eigenstructure assignment in Lewis and Syrmos (1991) cannot assign zero eigenvalues of closed-loop systems (if the state matrices A are singular), since the eigenvalues of closed-loop systems are given by the reciprocals of the eigenvalues of a single matrix F, in which there is no zero.
(b) The approaches for eigenstructure assignment in Abdelaziz and Valášek (2004, 2005), Abdelaziz (2007, 2009, 2011), Mirassadi and Tehrani (2017) and Araújo (2019) apply only for systems that have nonsingular derivative matrices E and/or nonsingular state matrices A.
(c) The development of the techniques for systems with singular state matrices A in Abdelaziz (2009, 2011) depends crucially on the condition that the multiplicity of the zero eigenvalues of A is equal to one. Besides, the approaches for eigenstructure assignment in Abdelaziz (2007, 2009, 2011) impose the condition that the number of the zero eigenvalues of the closed-loop systems is equal to one. But these requirements are too restrictive in many situations.
(d) The development of the techniques in Abdelaziz and Valášek (2004, 2005), Abdelaziz (2007, 2009, 2011), Mirassadi and Tehrani (2017) and Araújo (2019) depends crucially on invertibility of closed-loop derivative matrices $(I + BK)$ or $(E + BK)$. As a result, the approaches for eigenstructure assignment proposed by Abdelaziz and Valášek (2004, 2005), Abdelaziz (2007, 2009, 2011), Mirassadi and Tehrani (2017) and Araújo (2019) can assign only finite eigenvalues of closed-loop systems, but cannot assign infinite eigenvalues of closed-loop systems.
(e) When the state matrices A are singular, the approaches for eigenstructure assignment in Abdelaziz (2009, 2011) impose the condition that $K = \tilde{F}\tilde{U}^{\mathrm{T}}, \tilde{F} = \begin{bmatrix} 0_{m\times 1} & F \end{bmatrix}, F \in \mathbb{R}^{m\times(n-1)}$ (lost some degrees of freedom in K). Thus, these approaches do not reveal all the degrees of freedom existing in the eigenstructure assignment problem.

Thus new techniques for solving this eigenstrucure assignment problem should be developed. Based on a simple complete explicit parametric solution to a group of recursive equations, presented by Zhang (2011), a complete parametric approach for eigenstrucure assignment in complementary S-controllable (complementary C-controllable) linear systems via state-derivative feedback is proposed. General parametric expressions for the closed-loop eigenvectors and the feedback gain matrix are established in terms of certain parameter vectors. It is shown that, if A is non-singular, then the approach assigns arbitrarily n non-zero closed-loop eigenvalues which may include infinite eigenvalues, and that, if A is singular, then the approach assigns arbitrarily rank$A$ closed-loop eigenvalues which may include zero and infinite eigenvalues, while the $(n - \mathrm{rank}A)$ un-controllable zero eigenvalues remain unchanged. In addition, the approach guarantees the closed-loop regularity. Our approach generalises and improves the existing results. It has the following advantages over the existing ones:

(1) It removes the restriction required by Abdelaziz and Valášek (2004, 2005), Abdelaziz (2007, 2009, 2011), Mirassadi and Tehrani (2017) and Araújo

(2019) that the derivative matrices E and/or the state matrices A are nonsingular.

(2) It removes the restriction required by Abdelaziz and Valášek (2004, 2005), Abdelaziz (2007, 2009, 2011), Mirassadi and Tehrani (2017) and Araújo (2019) that the closed-loop derivative matrices $(I + BK)$ or $(E + BK)$ are non-singular. Unlike the approaches for eigenstructure assignment in Abdelaziz and Valášek (2004, 2005), Abdelaziz (2007, 2009, 2011), Mirassadi and Tehrani (2017) and Araújo (2019), it can not only assign finite closed-loop eigenvalues, but also infinite closed-loop eigenvalues.

(3) It removes the restriction required by Abdelaziz (2009, 2011) that the multiplicity of the zero eigenvalues of the state matrix A (if A is singular) is equal to one. It also relaxes the restriction required by Abdelaziz (2007, 2009, 2011) that the number of the zero eigenvalues of the closed-loop system (if A is singular) is equal to one. It will be seen that, to ensure the existence of solutions to the eigenstructure assignment problem, the number of the zero eigenvalues of the closed-loop systems should be at least $(n - \mathrm{rank}A)$.

(4) It removes the restriction on the gain matrix K that $K = \tilde{F}\tilde{U}^{\mathrm{T}}, \tilde{F} = \begin{bmatrix} 0_{m\times 1} & F \end{bmatrix}, F \in \mathbb{R}^{m\times(n-1)}$ (when assigning one zero eigenvalue), which was required by Abdelaziz (2009, 2011). Consequently, unlike the approaches of Abdelaziz (2009, 2011), our approach provides all the degrees of freedom existing in the problem.

Finally, the combined problem of simultaneously assigning dynamical order and finite eigenstructure in linear systems by state-derivative is considered. It is well known that derivative feedback can be used to change the dynamical order of a system. Increasing the dynamical order enables the designer to implement specific otherwise unrealiseable control strategies, on the other hand, dynamical order reduction simplifies the design procedures for a large scale system (Fahmy & Tantawy, 1990). The problem of dynamical order assignment has been studied by several researchers (G. R. Duan & Zhang, 2002; Fahmy & Tantawy, 1990; Owens & Askarpour, 2001). In these results, the dynamical order assignment problem is converted to the eigenstructure assignment problem of the matrix $(E + BK)$. By assigning a certain number of zero eigenvalues to $(E + BK)$ such that it possesses the specified rank, the solution to the dynamical order assignment problem can be obtained. However, the approaches in Fahmy and Tantawy (1990), Owens and Askarpour (2001), and G. R. Duan and Zhang (2002) can-not be applied to the combined problem, since these techniques only design derivative matrix $(E + BK)$. To the best of our knowledge, there is no technique in the literature that can simultaneously assign arbitrary dynamical order and arbitrary finite eigenstructure only using pure state-derivative feedback. Because of the importance of dynamical order assignment and eigenstructure assignment, the problem of eigenstructure assignment with arbitrary (allowable) dynamical order in linear systems by state-derivative feedback has to be studied.

It is shown that the combined problem can be attributed to a special case of the aforementioned eigenstructure assignment problem. Therefore, the approach for eigenstructure assignment above can be applied.

The main contributions of this study are summarised as follows:

(a) The paper introduces a new derivative feedback design framework named complementary system (CS) framework. Unlike the reciprocal state space (RSS) framework, the CS framework does not involve the inverse operation of state matrix A, thus effectively avoiding the numerical instability problem that may arise from the inverse operation of state matrix A. Because it does not require state matrices to be non-singular, the CS framework can be used for the analysis and design of derivative feedback related problems in systems with singular state matrices.

(b) The paper introduces notions of complementary controllability (including complementary S- controllability and complementary C-controllability) and complementary controllability indices for linear systems, and then establishes the fundamental theorem for state-derivative feedback which provides a necessary and sufficient condition for the solvability of state-derivative feedback eigenstructure assignment problem. Unlike the results in Abdelaziz (2009, 2011) where only several necessary conditions for solvability of the problem are provided and strict conditions are imposed on the open- and closed-loop zero eigenvalues, our result fully characterise the solvability of the state-derivative feedback eigenstructure assignment problem.

(c) The paper proposes a new approach for eigenstructure assignment in linear systems via state-derivative feedback. General complete parametric expressions for the closed-loop eigenvectors and the feedback gain matrix are presented in a group of parameter vectors which represent the degrees of the design freedom. The approach assigns arbitrarily all closed-loop eigenvalues except for uncontrollable zero eigenvalues, and guarantees the closed-loop regularity. It generalises and improves the existing results.

(d) The paper considers the combined problem of assigning both arbitrary (allowable) dynamical order and arbitrary (assignable) finite eigenstructure in linear systems by state-derivative feedback. It is shown that the combined problem can be attributed to a special case of the eigenstructure assignment problem. The approach for solving this problem is then established based on the proposed eigenstructure assignment approach. The approach provides maximum design flexibility including full-order eigenstructure assignment, reduced-order eigenstructure assignment and equal-order eigenstructure assignment. This is the first work on this issue.

To demonstrate the effect of the proposed approach, state-derivative feedback control of a three degrees of freedom mass-spring-dashpot system is considered. General complete parametric expression for state-derivative gain matrix K is given which makes the closed-loop system has the desired eigenstructure. Using the parametric expression, the spectral norm of the gain matrix K is minimised by making full use of the freedom provided by eigenstructure assignment.

The paper is divided into six sections. In the next section, the problem of eigenstructure assignment in linear systems via state-derivative feedback, to be solved in this paper, is formulated. The combined problem of assigning dynamical order and assigning finite eigenstructure in linear systems by state-derivative is considered. In Section 3, a new derivative feedback design framework named complementary system framework is proposed. The notions of complementary controllability (including complementary S-controllable and complementary C-controllable) and complementary controllability indices of linear systems are introduced. The fundamental theorem of state-derivative feedback for the proposed eigenstructure assignment problem is established. In Section 4, the proposed eigenstructure assignment problem is treated. General parametric expressions for the closed-loop eigenvectors and the feedback gain matrix are established. In Section 5, an illustrative example is examined. Conclusions are made in Section 6.

2. Problem formulation

If the state-derivative feedback control law (2) is applied to the system (1), then a closed-loop system is obtained as

$$E_c\dot{x}(t) = Ax(t), \quad E_c = E + BK \tag{3}$$

Let $\Lambda = \{\lambda_i, i = 0, 1, 2, \ldots, \nu, \infty, 1 \le \nu \le n_A\}$ be the set of eigenvalues of the matrix pair (E_c, A), where $\lambda_0 = 0, \lambda_\infty = \infty$ and $\lambda_i, i = 1, 2, \ldots, \nu$, are a group of distinct self-conjugate non-zero complex numbers. Denote the algebraic and geometric multiplicities of λ_i by m_i and q_i respectively. Then there are q_i chains of generalised eigenvalues of (E_c, A) associated with λ_i. If A is singular, then $q_0 = n - \operatorname{rank}A$, since the nullity of A remains unchanged for derivative feedback. If A is non-singular, then we make convention that $m_0 = q_0 = 0$, since (E_c, A) has no zero eigenvalue $\lambda_0 = 0$, and if E_c is non-singular, then we make convention that $m_\infty = q_\infty = 0$, since (E_c, A) has no infinite eigenvalue $\lambda_\infty = \infty$. Denote the lengths of these q_i chains by $p_{ij}, j = 1, 2, \ldots, q_i$. Then the following relation holds:

$$p_{i1} + p_{i2} + \cdots + p_{iq_i} = m_i \tag{4}$$

$$m_0 + m_1 + m_2 + \cdots + m_\nu + m_\infty = n \tag{5}$$

Let the eigenvector chains of the matrix pair (E_c, A) associated with finite eigenvalue λ_i be denoted by $v_{ij}^k \in \mathbb{C}^n$, $k = 1, 2, \ldots, p_{ij}$, $j = 1, 2, \ldots, q_i$. Then we have the following equations

$$\begin{gathered}[A - \lambda_i(E + BK)]\, v_{ij}^k = (E + BK)v_{ij}^{k-1}, \quad v_{ij}^0 = 0 \\ k = 1, 2, \ldots, p_{ij}, \quad j = 1, 2, \ldots, q_i, \quad i = 0, 1, 2, \ldots, \nu \end{gathered} \tag{6}$$

Further, let the eigenvector chains of the matrix pair (E_c, A) associated with infinite eigenvalue λ_∞ be denoted by $v_{\infty j}^k \in \mathbb{R}^n, k = 1, 2, \ldots, p_{\infty j}, j = 1, 2, \ldots,$

q_∞. Then we have the following equations

$$(E+BK)v_{\infty j}^{k} = Av_{\infty j}^{k-1}, \quad v_{\infty j}^{0} = 0$$
$$k = 1,2,\ldots,p_{\infty j}, \quad j = 1,2,\ldots,q_\infty \tag{7}$$

The closed-loop system (3) is called regular if the characteristic polynomial $\det(sE_c - A)$ is not identically zero. Closed-loop regularity is an essential requirement in the eigenstructure assignment problem for linear systems since it ensures the existence and uniqueness of solutions of the systems.

We now formulate the problem of eigenstructure assignment (ESA) via state-derivative feedback controller (2) for the linear system (1) as follows:

Problem ESA. Given the linear descriptor system (1), the closed-loop eigenvalue set Λ as described previously, and integers m_i, q_i, p_{ij}, $j = 1,2,\ldots,q_i$, $i = 0,1,2,\ldots,\nu,\infty$ satisfying (4) and (5), find a matrix $K \in \mathbb{R}^{r\times n}$ and a group of vectors $v_{ij}^{k} \in \mathbb{C}^{n}$, $k = 1,2,\ldots,p_{ij}$, $j = 1,2,\ldots,q_i$, $i = 0,1,2,\ldots,\nu,\infty$ such that the following three requirements are simultaneously satisfied:

(a) all the equations in (6) and (7) hold;
(b) vectors $v_{ij}^{k}, k = 1,2,\ldots,p_{ij}, j = 1,2,\ldots,q_i$, $i = 0,1,2,\ldots,\nu,\infty$ are linearly independent;
(c) the characteristic polynomial $\det[s(E+BK) - A]$ is not identically zero.

As a consequence of the above requirements (a)–(c), the resulting closed-loop system is regular; it inherently possesses $\text{rank}E_c = n - q_\infty$ dynamical modes (finite eigenvalues and/or dynamical infinite eigenvalues) and q_∞ non-dynamical modes (non-dynamical infinite eigenvalues); it inherently possesses $\deg[\det(sE_c - A)] = n - m_\infty$ finite eigenvalues and m_∞ infinite eigenvalues.

Let the dynamical order of the closed-loop system (3) be $\rho = \text{rank}E_c$. When the system (1) is reachable at infinity (Lewis, 1986), i.e.

$$\text{rank}\begin{bmatrix} E & B \end{bmatrix} = n \tag{8}$$

then the dynamical order ρ of the closed-loop system (3) can be arbitrarily assigned in the range

$$n - \text{rank}B \le \rho \le n \tag{9}$$

Now we consider the combined problem of simultaneously assigning dynamical order and finite eigenstructure in the linear system (1) by state-derivative feedback controller (2). We aim to assign ρ finite closed-loop eigenvalues. Then we have

$$\rho = \text{rank}E_c = n - q_\infty = \deg\left[\det(sE_c - A)\right]$$
$$= n - m_\infty$$

Therefore, we have $m_\infty = q_\infty = n - \rho$, $p_{\infty j} = 1$, $j = 1,2,\ldots,q_\infty$. This shows that the combined problem is actually a special case of Problem ESA. If $\rho = n$, then this special eigenstructure assignment is said to be full order, and if $\rho < n$, then this special eigenstructure assignment is said to be reduced order, and if $\rho = \text{rank}E$, then this special eigenstructure assignment is said to be equal order.

3. Complementary system framework and fundamental theorem for state-derivative feedback

The problem of eigenvalue assignment by state feedback in descriptor linear systems is closely related to the notions of controllability. The descriptor linear system (1) is said to be strongly controllable (S-controllable) if the matrix pencil $[E - A \quad B]$ has no finite or infinite zeros (Verghese et al., 1981). The descriptor linear system (1) is said to be completely controllable (C-controllable) if one can reach any state from any initial state (Yip & Sincovec, 1981).

Lemma 3.1 (G. R. Duan, 2010): *Given the descriptor linear system* (1), *then*

(1) *the system* (1) *is S-controllable if and only if the following conditions hold:*

$$\begin{aligned} &\text{rank}\begin{bmatrix} sE - A & B \end{bmatrix} \\ &\quad = n, \quad \forall\, s \in \mathbb{C} \qquad (\text{R} - \text{controllable}) \\ &\text{rank}\begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} \\ &\quad = n + \text{rank}E \quad (\text{I} - \text{controllable}) \end{aligned} \tag{10}$$

(2) *the system* (1) *is C-controllable if and only if the conditions* (8) *and* (10) *hold.*

The authors recognise the limitations of applying these controllability concepts to derivative feedback, since a derivative feedback controller has no influence on the rank of A.

The reciprocal state space (RSS) framework is a very popular derivative feedback design framework. However, this type of techniques requires the state matrices of linear systems to be non-singular. To overcome this limitation, we propose a new framework. To develop our theory, we introduce the following system

$$A\dot{z}(t) = Ez(t) + Bv(t) \tag{11}$$

where $z(t) \in \mathbb{R}^n, v(t) \in \mathbb{R}^r$ are, respectively, the state vector and the input vector. We say that system (1) and system (11) are complementary in the sense of interchanging the roles of E and A. When a state feedback control law

$$v(t) = Kz(t) \tag{12}$$

is applied to the system (11), a closed-loop system is obtained as

$$A\dot{z}(t) = E_c z(t), \quad E_c = E + BK \tag{13}$$

Then we have the following observations:

(1) if λ is a non-zero finite eigenvalue of the closed-loop system (3), then $1/\lambda$ is a non-zero finite eigenvalue of the closed-loop system (13), and the algebraic and geometric multiplicities of λ and $1/\lambda$ are the same, vice versa;
(2) if the closed-loop system (3) has a zero eigenvalue $\lambda_0 = 0$, then the closed-loop system (13) has an infinite eigenvalue $1/\lambda_0 = \infty$, and the algebraic and geometric multiplicities of $\lambda_0 = 0$ and $1/\lambda_0 = \infty$ are the same, vice versa;
(3) if the closed-loop system (3) has an infinite eigenvalue $\lambda_\infty = \infty$, then the closed-loop system (13) has a zero eigenvalue $1/\lambda_\infty = 0$, and the algebraic and geometric multiplicities of $\lambda_\infty = \infty$ and $1/\lambda_\infty = 0$ are the same, vice versa.

Thus, finding state-derivative feedback gain matrix K for the system (1) such that the closed-loop system (3) has the eigenvalue set $\Lambda = \{\lambda_i, i = 0, 1, 2, \ldots, \nu, \infty, 1 \le \nu \le n_A\}$ is equivalent to finding state feedback gain matrix K for the complementary system (11) such that the closed-loop system (13) has the eigenvalue set $\Lambda^{-1} := \{1/\lambda_i, i = 0, 1, 2, \ldots, \nu, \infty, 1 \le \nu \le n_A\}$.

We call the above new framework the complementary system (CS) framework. The CS framework is useful because it allows control techniques originally developed for state feedback control can be directly employed to design the state-derivative feedback law.

By using CS framework, we can introduce several new controllability notions for the system (1), which are suitable for use in derivative feedback. Due to complementarity of the system (1) and the system (11), we shall use the following controllability terminology.

Definition 3.1: The linear system (1) is called complementary R-controllable (complementary I-controllable, complementary S-controllable, complementary C-controllable) if and only if its complementary system (11) is R-controllable (I-controllable, S-controllable, C-controllable).

Using Definition 3.1 and Lemma 3.1, we can obtain the following lemma.

Lemma 3.2: *Given the linear system* (1)*, then*

(1) *the system* (1) *is complementary S-controllable if and only if the following conditions hold:*

$$\operatorname{rank}\begin{bmatrix} sA - E & B \end{bmatrix} = n, \quad \forall\, s \in \mathbb{C}$$
$$(\textit{complementary R-controllable}) \tag{14}$$

$$\operatorname{rank}\begin{bmatrix} A & 0 & 0 \\ E & A & B \end{bmatrix} = n + \operatorname{rank} A$$
$$(\textit{complementary I-controllable}) \tag{15}$$

(2) *the system* (1) *is complementary C-controllable if and only if the system* (1) *is complementary S-controllable, that is, the condition* (14) *holds, and the following condition holds:*

$$\operatorname{rank}\begin{bmatrix} A & B \end{bmatrix} = n \tag{16}$$

Obviously, a complementary C-controllable linear system is also complementary S-controllable. If A is non-singular, then the complementary system (11) is a normal linear system, and the conditions (15) and (16) disappear.

Suppose that the system (11) is S-controllable (i.e. controllable in Zagalak and Kučera (1995) and Özcaldiran (1990), which is the same concept as S-controllable). Here we introduce the controllability indexes of the system (11) described in Zagalak and Kučera (1995). Let $H(s), L(s)$ be, respectively, $n \times r, r \times r$ matrices over $\mathbb{R}[s]$, the ring of polynomials in

s with coefficients in $\mathbb{R}$, such that

$$\begin{bmatrix} sA-E & -B \end{bmatrix}\begin{bmatrix} H(s) \\ L(s) \end{bmatrix} = 0 \tag{17}$$

Then the matrices $H(s), L(s)$ are said to form a (right) normal external description of (11) if $\left[\begin{smallmatrix} H(s) \\ L(s) \end{smallmatrix}\right]$ is a decreasingly column-degree, minimal polynomial basis of $\mathrm{Ker}[sA-E \quad -B]$; and $H(s)$ is a minimal polynomial basis of $P_B(sA-E)$, where P_B is a maximal anihilator of B. The controllability indices of system (11) are defined to be the column degrees of any normal external description of (11). See Malabre et al. (1990) for details.

The following result solves the existence part of the eigenstructure assignment problem of the system (11), which was proved by Özcaldiran (1990).

Suppose the system (11) is C-controllable (i.e.reachable in Özcaldiran (1990), which is equivalent to C-controllable) with controllability indices $c_1 \geq c_2 \geq \cdots \geq c_r$. Let $d_1(s), d_2(s), \ldots, d_r(s)$ be monic polynomials such that $d_{i+1}(s)$ divides $d_i(s), i = 1, 2, \ldots, r-1$. Let $h_1 \geq h_2 \geq \cdots \geq h_{q_0}$ be non-negative integers satisfying

$$\sum_{i=1}^{r}(\mathrm{deg}d_i(s) + h_i) = \mathrm{rank}A \tag{18}$$

where, by convention, $h_i = 0$ for $i = q_0+1, q_0+2, \ldots, r$. Then there exists a real state feedback matrix K such that $d_1(s), d_2(s), \ldots, d_r(s)$ are the invariant polynomials and $h_1, h_2, \ldots, h_{q_0}$ are the infinite eigenvalue orders (or, equivalently, $h_1+1, h_2+1, \ldots, h_{q_0}+1$ are the infinite elementary divisor orders) of the closed-loop system (13) if and only if

$$\sum_{i=1}^{j}(\mathrm{deg}d_i(s) + h_i) \geq \sum_{i=1}^{j} c_i, \quad j = 1, 2, \ldots, r \tag{19}$$

and equality holds when $j = r$.

Also from Özcaldiran (1990), if the system (11) is S-controllable but not C-controllable, then the above result still holds true, but the values of the last μ of the h_i's should be fixed at zero, where $\mu = q_0 - \dim(\mathbf{R}^* \bigcap \mathrm{Ker}A)$ ($\mathbf{R}^*$ denotes the reachable subspace of the system (11)).

We call the controllability indices $c_1, c_2, \ldots, c_r$ of the system (11) the complementary controllability indices of the system (1).

Theorem 3.1: *Suppose that the linear system (1) is complementary C-controllable with complementary controllability indices $c_1 \geq c_2 \geq \cdots \geq c_r$. Let $\phi_1(s), \phi_2(s), \ldots, \phi_r(s)$ be a list of monic polynomials such that $\phi_{i+1}(s)$ divides $\phi_i(s), i = 1, 2, \ldots, r-1$ and $\phi_i(0) \neq 0, i = 1, 2, \ldots, r$. Let $h_1 \geq h_2 \geq \cdots \geq h_{q_0}$ and $l_1 \geq l_2 \geq \cdots \geq l_{q_\infty}$ be two lists of non-negative integers satisfying*

$$\sum_{i=1}^{r}\mathrm{deg}\phi_i(s) + \sum_{i=1}^{q_0} h_i + \sum_{i=1}^{q_\infty}(l_i+1) = \mathrm{rank}A \tag{20}$$

Let $\mu_i = h_i + 1,\ i = 1, 2, \ldots, q_0$, and, by convention, $\mu_i = 0$ for $i = q_0+1, q_0+2, \ldots, r$, and let $\rho_i = l_i + 1,\ i = 1, 2, \ldots, q_\infty$, and, by convention, $\rho_i = 0$ for $i = q_\infty+1, q_\infty+2, \ldots, r$. Then there exists a real derivative feedback gain matrix K such that the closed-loop system (3) has the finite non-zero eigenvalue structure given by $\phi_1(s), \phi_2(s), \ldots, \phi_r(s)$, the zero eigenvalue structure given by $s^{\mu_1}, s^{\mu_2}, \ldots, s^{\mu_{q_0}}$ and the infinite eigenvalue structure given by the infinite eigenvalue orders $l_1, l_2, \ldots, l_{q_\infty}$ (or, equivalently, the infinite elementary divisor orders $\rho_1, \rho_2, \ldots, \rho_{q_\infty}$) if and only if

$$\sum_{i=1}^{j}(\mathrm{deg}\phi_i(s) + h_i + \rho_i) \geq \sum_{i=1}^{j} c_i, \quad j = 1, 2, \ldots, r \tag{21}$$

and equality holds when $j = r$.

Proof: We prove this theorem by using CS framework. Note that the eigenvalues of the closed-loop system (3) must be the reciprocals of the eigenvalues of the closed-loop system (13). For the specified finite non-zero eigenvalue structure given by $\phi_1(s), \phi_2(s), \ldots, \phi_r(s)$ and infinite eigenvalue structure given given by $\rho_1, \rho_2, \ldots, \rho_{q_\infty}$ of the system (3), we construct the finite eigenvalue structure given by $d_i(s) = s^{\mathrm{deg}\phi_i(s)+\rho_i}\phi_i(1/s)/\phi_i(0)$ of the system (13). Then $d_1(s), d_2(s), \ldots, d_r(s)$ be monic polynomials such that $d_{i+1}(s)$ divides $d_i(s), i = 1, 2, \ldots, r-1$. For the zero eigenvalue structure given by $s^{\mu_1}, s^{\mu_2}, \ldots, s^{\mu_{q_0}}$ of the system (3), we obtain the infinite eigenvalue structure given by $h_1, h_2, \ldots, h_{q_0}$ of the system (13). Obviously, $\mathrm{deg}d_i(s) = \mathrm{deg}\phi_i(s) + \rho_i, i = 1, 2, \ldots, r$, so (20) becomes

$$\sum_{i=1}^{r}\mathrm{deg}d_i(s) + \sum_{i=1}^{q_0} h_i = \mathrm{rank}A$$

It follows from the Özcaldiran's (1990) result that there exists a real state feedback gain matrix K such that the closed-loop system (13) has the finite eigenvalue structure given by $d_1(s), d_2(s), \ldots, d_r(s)$, and the infinite eigenvalue structure given by the infinite eigenvalue orders $h_1, h_2, \ldots, h_{q_0}$ (or, equivalently, the infinite elementary divisor orders $\mu_1, \mu_2, \ldots, \mu_{q_0}$) if and only if

$$\sum_{i=1}^{j}(\text{deg}d_i(s) + h_i) \geq \sum_{i=1}^{j} c_i, \quad j = 1, 2, \ldots, r$$

Then the theorem follows from the fact that the eigenvalues of the closed-loop system (3) are reciprocals of those of the closed-loop system (13). ■

Theorem 3.1 solves the existence part of the state-derivative feedback eigenstructure assignment problem. This result can be viewed as a generalisation of the fundamental theorems of state feedback (Özcaldiran, 1990; Rosenbrock, 1970; Zagalak & Kučera, 1992, 1995), and can be referred to as the fundamental theorem of state-derivative feedback.

Remark 3.1: If system (1) is complementary S-controllable but not complementary C-controllable, then the result in Theorem 3.1 still holds true, but the values of the last μ of the h_i's should be fixed at zero, where $\mu = q_0 - \dim(\mathbf{R}^* \bigcap \text{Ker}A)$. This result can be obtained directly from the Özcaldiran's (1990) result above and the complementarity of system (1) and system (11).

Remark 3.2: It follows from Theorem 3.1 that, if A is non-singular, then $q_0 = 0$ and therefore all closed-loop eigenvalues should be non-zero; if A is singular, then q_0 is fixed and $q_0 = n - \text{rank}A$ and therefore at least $(n - \text{rank}A)$ closed-loop eigenvalues should be zero.

Remark 3.3: It is easy to verify that, in the special case of $m_i = q_i = p_{i1} = 1, i = 1, 2, \ldots, \text{rank}A$ (or $m_i = q_i = p_{i1} = 1, i = 1, 2, \ldots, \text{rank}A - 1, \infty$) and $m_0 = q_0 = n - \text{rank}A, p_{0j} = 1, j = 1, 2, \ldots, n - \text{rank}A$ (if A is non-singular, then this part does not appear), the inequalities in (21) hold automatically. Therefore, the Problem ESA for this case is always solvable.

4. The general parametric solution to problem ESA

We relate the problem to the following recursive equations

$$Lz_k = Mz_{k-1}, z_0 = 0, \quad k = 1, 2, \ldots, l \tag{22}$$

where $L, M \in \mathbb{C}^{s\times t}$ $(s < t)$ are known matrices and z_k $(k = 1, 2, \ldots, l)$ are to be determined.

Lemma 4.1 (Zhang, 2011): *Suppose that L is of full row-rank. Then all solutions of (22) are given by*

$$z_k = H_1 f_k + H_2 f_{k-1} + \cdots + H_k f_1, \quad k = 1, 2, \ldots, l \tag{23}$$

where $f_k \in \mathbb{C}^{t-s}$, $k = 1, 2, \ldots, l$, are a group of arbitrarily chosen free parameter vectors; H_k, $k = 1, 2, \ldots, l$, are determined by

$$PLQ = \begin{bmatrix} I & 0 \end{bmatrix}, \quad P \in \mathbb{C}^{s\times s}, Q \in \mathbb{C}^{t\times t}$$
$$Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}, \quad Q_1 \in \mathbb{C}^{t\times s}$$
$$H_k = (Q_1 PM)^{k-1} Q_2$$

4.1. Solution of closed-loop eigenvectors associated with finite non-zero eigenvalues

The equations in (6) associated with finite non-zero eigenvalues $\lambda_i, i = 1, 2, \ldots, \nu$, are

$$[A - \lambda_i(E + BK)]\, v_{ij}^k = (E + BK)v_{ij}^{k-1}, \quad v_{ij}^0 = 0$$
$$k = 1, 2, \ldots, p_{ij}, \quad j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, \nu \tag{24}$$

Let

$$w_{ij}^k = Kv_{ij}^k, \quad k = 1, 2, \ldots, p_{ij}, \quad j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, \nu \tag{25}$$

and denote

$$L_i = \begin{bmatrix} A - \lambda_i E & -\lambda_i B \end{bmatrix}, \quad M_i = \begin{bmatrix} E & B \end{bmatrix}, \quad i = 1, 2, \ldots, \nu$$
$$z_{ij}^k = \begin{bmatrix} v_{ij}^k \\ w_{ij}^k \end{bmatrix}, \quad k = 1, 2, \ldots, p_{ij}, \quad j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, \nu \tag{26}$$

Then the equations in (24) can be equivalently written as

$$L_i z_{ij}^k = M_i z_{ij}^{k-1}, \quad z_{ij}^0 = 0, \quad k = 1, 2, \ldots, p_{ij},$$

$$j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, \nu \quad (27)$$

For each i and j, equations in (27) fall into the type of recursive equations in (22).

Suppose that the system (1) is complementary S-controllable. Then, for any $s \neq 0$, we have

$$\begin{bmatrix} -sE & -sB \end{bmatrix} = \begin{bmatrix} \frac{1}{s}A - E & B \end{bmatrix} \begin{bmatrix} sI_n & 0 \\ 0 & -sI_r \end{bmatrix}$$

Under the assumption of controllability, we have

$$\operatorname{rank} \begin{bmatrix} \frac{1}{s}A - E & B \end{bmatrix} = n$$

In the case of $s \neq 0$, $\operatorname{diag}(sI_n, -sI_r)$ is an invertible matrix. Therefore, the following holds

$$\operatorname{rank} \begin{bmatrix} A - sE & -sB \end{bmatrix} = n, \quad \forall\, s \in \mathbb{C}, s \neq 0$$

Subject to this condition, there exist invertible matrices $P(s) \in \mathbb{C}^{n\times n}$ and $Q(s) \in \mathbb{C}^{(n+r)\times(n+r)}$ satisfying

$$P(s) \begin{bmatrix} A - sE & -sB \end{bmatrix} Q(s) = \begin{bmatrix} I_n & 0_{n\times r} \end{bmatrix} \quad (28)$$

For finite non-zero eigenvalues $\lambda_i, i = 1, 2, \ldots, \nu$, let

$$P_i = P(\lambda_i), \quad Q_i = Q(\lambda_i) \quad (29)$$

From (26), (28) and (29), we have

$$P_i L_i Q_i = \begin{bmatrix} I_n & 0_{n\times r} \end{bmatrix} \quad (30)$$

Partition $Q_i,\ i = 1, 2, \ldots, \nu$, into the following form:

$$Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \end{bmatrix}, \quad Q_{i1} \in \mathbb{C}^{(n+r)\times n}, \quad i = 1, 2, \ldots, \nu \quad (31)$$

Let

$$H_{ik} = (Q_{i1} P_i M_i)^{k-1} Q_{i2}, \quad k = 1, 2, \ldots, d_i, \quad i = 1, 2, \ldots, \nu \quad (32)$$

where $d_i = \max_{1\leq j\leq q_i}\{p_{ij}\}$. Partition $H_{ik},\ k = 1, 2, \ldots, d_i,\ i = 1, 2, \ldots, \nu$, as follows:

$$H_{ik} = \begin{bmatrix} N_{ik} \\ D_{ik} \end{bmatrix}, \quad N_{ik} \in \mathbb{C}^{n\times r} k = 1, 2, \ldots, d_i, \quad i = 1, 2, \ldots, \nu \quad (33)$$

By applying Lemma 4.1 to (27), we can obtain the following theorem.

Theorem 4.1: *Suppose that the linear system (1) is complementary S-controllable. Then the general solution for the closed-loop eigenvectors $v_{ij}^k, k = 1, 2, \ldots, p_{ij}$, $j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, \nu$, associated with the finite non-zero closed-loop eigenvalues, together with the corresponding vectors $w_{ij}^k, k = 1, 2, \ldots, p_{ij}, j = 1, 2, \ldots,$ $q_i, i = 1, 2, \ldots, \nu$, is given by*

$$\begin{bmatrix} v_{ij}^k \\ w_{ij}^k \end{bmatrix} = \begin{bmatrix} N_{i1} \\ D_{i1} \end{bmatrix} f_{ij}^k + \begin{bmatrix} N_{i2} \\ D_{i2} \end{bmatrix} f_{ij}^{k-1} + \cdots + \begin{bmatrix} N_{ik} \\ D_{ik} \end{bmatrix} f_{ij}^1$$
$$k = 1, 2, \ldots, p_{ij}, \quad j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, \nu \quad (34)$$

where $f_{ij}^k \in \mathbb{C}^r, k = 1, 2, \ldots, p_{ij}, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, \nu$, are a group of arbitrarily chosen free parameter vectors; $N_{ik}, D_{ik}, k = 1, 2, \ldots, p_{ij}, j = 1, 2, \ldots, q_i$, $i = 1, 2, \ldots, \nu$, are determined by (26), (28)–(33).

Remark 4.1: In the special case of $m_i = q_i, p_{ij} = 1, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, \nu$, (34) has the following simple form:

$$\begin{bmatrix} v_{ij}^1 \\ w_{ij}^1 \end{bmatrix} = \begin{bmatrix} N_{i1} \\ D_{i1} \end{bmatrix} f_{ij}^1, \quad j = 1, 2, \ldots, q_i, \quad i = 1, 2, \ldots, \nu$$

Remark 4.2: At the first step toward the solution to the closed-loop eigenvectors with finite non-zero eigenvalues, we need to obtain the polynomial matrices $P(s)$ and $Q(s)$ which are determined by (28). This polynomial form solution is theoretically convenient to use, however, it is not desirable to use in high dimension cases since polynomial matrix decomposition (28) is not economic and not in general numerically reliable. Noting that the solution (34) is constructed from $P(s_i)$ and $Q(s_i), i = 1, 2, \ldots, \nu$, thus instead of solving $P(s)$ and $Q(s)$, we may directly solve $P(s_i)$ and $Q(s_i), i = 1, 2, \ldots, \nu$ by transforming the matrices $[A - s_iE \quad -s_iB] (s_i \neq 0), i = 1, 2, \ldots, \nu$ into the form of $[I \quad 0]$ using some numerically stable algorithms. Applying a series of singular value decompositions for the matrices $[A - s_iE \quad -s_iB], i = 1, 2, \ldots, \nu$, we obtain

$$\hat{P}_i \begin{bmatrix} A - s_iE & -s_iB \end{bmatrix} \hat{Q}_i = \begin{bmatrix} \Sigma_i & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, \nu$$

where $\hat{P}_i \in \mathbb{C}^{n\times n}, \hat{Q}_i \in \mathbb{C}^{(n+r)\times(n+r)}, i = 1, 2, \ldots, \nu$, are orthogonal matrices, and $\Sigma_i, i = 1, 2, \ldots, \nu$, are non-singular diagonal matrices formed by the singular values of $[A - s_iE \quad -s_iB], i = 1, 2, \ldots, \nu$. Rewrite the above equation as

$$\Sigma_i^{-1}\hat{P}_i\begin{bmatrix}A-s_iE & -s_iB\end{bmatrix}\hat{Q}_i = \begin{bmatrix}I & 0\end{bmatrix}, \quad i=1,2,\ldots,\nu$$

Then we obtain $P(s_i)=\Sigma_i^{-1}\hat{P}_i$, $Q(s_i)=\hat{Q}_i$, $i=1,2,\ldots,\nu$. It is not difficult to see that the solution for the closed-loop eigenvectors with finite nonzero eigenvalues constructed from the matrices $P(s_i)$, $Q(s_i)$, $i=1,2,\ldots,\nu$ is numerically reliable because it is mainly based on singular value decompositions.

4.2. Solution of closed-loop eigenvectors associated with zero eigenvalue

In this subsection, we assume that the state matrix A is singular, then the matrix pair (E_c, A) must have zero eigenvalues. In this case, the equations in (6) associated with zero eigenvalue $\lambda_0=0$ are

$$\begin{gathered}[A-\lambda_0(E+BK)]\,v_{0j}^k=(E+BK)v_{0j}^{k-1}, \quad v_{0j}^0=0\\ k=1,2,\ldots,p_{0j},\ j=1,2,\ldots,q_0\end{gathered} \tag{35}$$

Let

$$w_{0j}^k=Kv_{0j}^k, \quad k=1,2,\ldots,p_{0j},\ j=1,2,\ldots,q_0 \tag{36}$$

Then the equations in (35) can be equivalently written as

$$\begin{gathered}\begin{bmatrix}A-\lambda_0E & -\lambda_0B\end{bmatrix}\begin{bmatrix}v_{0j}^k\\ w_{0j}^k\end{bmatrix} = \begin{bmatrix}E & B\end{bmatrix}\begin{bmatrix}v_{0j}^{k-1}\\ w_{0j}^{k-1}\end{bmatrix}, \quad \begin{bmatrix}v_{0j}^0\\ w_{0j}^0\end{bmatrix}=0\\ k=1,2,\ldots,p_{0j},\ j=1,2,\ldots,q_0\end{gathered} \tag{37}$$

Since $\operatorname{rank}[A-\lambda_0E \quad -\lambda_0B]<n$ and $1/\lambda_0=\infty$, vectors v_{0j}^k, w_{0j}^k, $k=1,2,\ldots,p_{0j}$, $j=1,2,\ldots,q_0$, can be regarded as the infinite eigenvectors and the corresponding vectors of the closed-loop system (13). Then the equations in (37) can be regarded as infinite eigenstructure assignment equations for the system (11), which fall into the type of recursive equations studied by Zhang (2010).

Suppose that the system (1) is complementary S-controllable, that is, the system (11) is S-controllable. Since $\operatorname{rank}A=n_A<n$, we can always find two invertible matrices $R\in\mathbb{R}^{n\times n}$ and $F\in\mathbb{R}^{n\times n}$ by applying matrix elementary transformation to the matrix A, such that the following relation holds:

$$RAF=\begin{bmatrix}I_{n_A} & 0\\ 0 & 0\end{bmatrix} \tag{38}$$

Partition the matrices R and F as follows:

$$\begin{gathered}R=\begin{bmatrix}R_1\\ R_2\end{bmatrix}, \quad R_1\in\mathbb{R}^{n_A\times n},\\ F=\begin{bmatrix}F_1 & F_2\end{bmatrix}, \quad F_1\in\mathbb{R}^{n\times n_A}\end{gathered} \tag{39}$$

Further, partition the matrices REF_2 and RB as follows:

$$\begin{gathered}REF_2=\begin{bmatrix}\Pi_1\\ \Pi_2\end{bmatrix}, \quad \Pi_2\in\mathbb{R}^{(n-n_A)\times(n-n_A)},\\ RB=\begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix}, \quad \Theta_2\in\mathbb{R}^{(n-n_A)\times r}\end{gathered} \tag{40}$$

Under the controllability assumption, it is shown in Zhang (2010) that $\operatorname{rank}[\Pi_2 \quad \Theta_2]=n-n_A$. Then, applying matrix elementary transformation to the matrix $[\Pi_2 \quad \Theta_2]$, we can obtain two invertible matrices $\Phi\in\mathbb{R}^{(n-n_A)\times(n-n_A)}$ and $\Psi\in\mathbb{R}^{(n-n_A+r)\times(n-n_A+r)}$ such that the following relation holds:

$$\Phi\begin{bmatrix}\Pi_2 & \Theta_2\end{bmatrix}\Psi=\begin{bmatrix}I_{n-n_A} & 0\end{bmatrix} \tag{41}$$

Partition the matrix Ψ as follows:

$$\Psi=\begin{bmatrix}\Psi_1 & \Psi_2\end{bmatrix}, \quad \Psi_2\in\mathbb{R}^{(n-n_A+r)\times r} \tag{42}$$

Denote

$$S=\begin{bmatrix}F_1\\ 0\end{bmatrix}R_1\begin{bmatrix}E & B\end{bmatrix}, \quad T=\begin{bmatrix}F_2 & 0\\ 0 & I_r\end{bmatrix} \tag{43}$$

and

$$U=S-T\Psi_1\Phi R_2\begin{bmatrix}E & B\end{bmatrix}S, \quad G=T\Psi_2 \tag{44}$$

Let

$$H_{0j}^{kl}=\begin{cases}U^{l-1}G, \quad l=1,2,\ldots,k,\\ \qquad k=1,2,\ldots,p_{0j}-1,\ j=1,2,\ldots,q_0\\ SU^{l-2}G, \quad l=2,3,\ldots,k,\ k=p_{0j},\\ \qquad j=1,2,\ldots,q_0\\ T, \quad l=1,\ k=p_{0j},\ j=1,2,\ldots,q_0\end{cases} \tag{45}$$

Partition the matrices $H_{0j}^{kl}, l=1,2,\ldots,k, k=1,2,\ldots,p_{0j}, j=1,2,\ldots,q_0$, as follows:

$$H_{0j}^{kl}=\begin{bmatrix}N_{0j}^{kl}\\ D_{0j}^{kl}\end{bmatrix}, \quad l=1,2,\ldots,k,\ k=1,2,\ldots,p_{0j},$$

$$j = 1, 2, \ldots, q_0 \tag{46}$$

where $N_{0j}^{kl} \in \mathbb{R}^{n\times(n-n_A+r)}$ for $l = 1, k = p_{0j}$, $j = 1, 2, \ldots, q_0$, and $N_{0j}^{kl} \in \mathbb{R}^{n\times r}$ otherwise.

By applying the general solution of a type of recursive equations given by Zhang (2010) to (37), we can prove the following theorem.

Theorem 4.2: *Suppose that the linear system (1) is complementary S-controllable. Then the general solution for the closed-loop eigenvectors v_{0j}^k, $k = 1, 2, \ldots, p_{0j}$, $j = 1, 2, \ldots, q_0$, associated with the zero closed-loop eigenvalue, together with the corresponding vectors w_{0j}^k, $k = 1, 2, \ldots, p_{0j}$, $j = 1, 2, \ldots, q_0$, is given by*

$$\begin{bmatrix} v_{0j}^k \\ w_{0j}^k \end{bmatrix} = \begin{bmatrix} N_{0j}^{k1} \\ D_{0j}^{k1} \end{bmatrix} f_{0j}^k + \begin{bmatrix} N_{0j}^{k2} \\ D_{0j}^{k2} \end{bmatrix} f_{0j}^{k-1} + \cdots + \begin{bmatrix} N_{0j}^{kk} \\ D_{0j}^{kk} \end{bmatrix} f_{0j}^1 \tag{47}$$

$$k = 1, 2, \ldots, p_{0j}, \quad j = 1, 2, \ldots, q_0$$

where $f_{0j}^k \in \mathbb{R}^r$, $k = 1, 2, \ldots, p_{0j} - 1$, $j = 1, 2, \ldots, q_0$, $f_{0j}^k \in \mathbb{R}^{n-n_A+r}$, $k = p_{0j}$, $j = 1, 2, \ldots, q_0$, are a group of arbitrarily chosen free parameter vectors; N_{0j}^{kl}, D_{0j}^{kl}, $l = 1, 2, \ldots, k$, $k = 1, 2, \ldots, p_{0j}$, $j = 1, 2, \ldots, q_0$, are determined by (38)–(46).

Remark 4.3: In the special case of $m_0 = q_0, p_{0j} = 1, j = 1, 2, \ldots, q_0$, (47) has the following simple form:

$$\begin{bmatrix} v_{0j}^1 \\ w_{0j}^1 \end{bmatrix} = \begin{bmatrix} N_{0j}^{11} \\ D_{0j}^{11} \end{bmatrix} f_{0j}^1, \quad j = 1, 2, \ldots, q_0$$

Remark 4.4: Notice that the main work for the solution to the closed-loop eigenvectors with zero eigenvalues is to solve Equation (38). Instead of applying matrix elementary transformation, we may apply singular value decomposition for the matrix A and obtain

$$\hat{R}A\hat{F} = \begin{bmatrix} \Sigma_{n_A} & 0 \\ 0 & 0 \end{bmatrix}$$

where $\hat{R} \in \mathbb{R}^{n\times n}$ and $\hat{F} \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and Σ_{n_A} is a non-singular diagonal matrix formed by the non-zero singular values of A. Rewrite the above equation as

$$\begin{bmatrix} \Sigma_{n_A}^{-1} & 0 \\ 0 & I_{n-n_A} \end{bmatrix} \hat{R}A\hat{F} = \begin{bmatrix} I_{n_A} & 0 \\ 0 & 0 \end{bmatrix}$$

Then we obtain $R = \mathrm{diag}(\Sigma_{n_A}^{-1}, I_{n-n_A})\hat{R}, F = \hat{F}$. It is easy to see that the solution for the closed-loop eigenvectors with zero eigenvalues constructed from the matrices R, F is numerically reliable because it is mainly based on a singular value decomposition.

4.3. Solution of closed-loop eigenvectors associated with infinite eigenvalue

Let

$$w_{\infty j}^k = K v_{\infty j}^k, \quad k = 1, 2, \ldots, p_{\infty j}, \quad j = 1, 2, \ldots, q_\infty \tag{48}$$

Then the equations in (7) can be equivalently written as

$$E v_{\infty j}^k + B w_{\infty j}^k = A v_{\infty j}^{k-1}, \quad v_{\infty j}^0 = 0,\ w_{\infty j}^0 = 0$$
$$k = 1, 2, \ldots, p_{\infty j}, \quad j = 1, 2, \ldots, q_\infty \tag{49}$$

Denote

$$L_\infty = \begin{bmatrix} E & B \end{bmatrix}, \quad M_\infty = \begin{bmatrix} A & 0 \end{bmatrix} \tag{50}$$

and

$$z_{\infty j}^k = \begin{bmatrix} v_{\infty j}^k \\ w_{\infty j}^k \end{bmatrix}, \quad k = 1, 2, \ldots, p_{\infty j}, \quad j = 1, 2, \ldots, q_\infty$$

Then the equations in (49) can be equivalently written as

$$L_\infty z_{\infty j}^k = M_\infty z_{\infty j}^{k-1}, \quad z_{\infty j}^0 = 0, \quad k = 1, 2, \ldots, p_{\infty j},$$
$$j = 1, 2, \ldots, q_\infty \tag{51}$$

which, for each j, fall into the type of recursive equations in (22).

Suppose that the system (1) is complementary S-controllable. Then, from (50), we can obtain $\mathrm{rank} L_\infty = \mathrm{rank}\begin{bmatrix} E & B \end{bmatrix} = n$ by setting $s = 0$ in (14). Therefore, there exist invertible matrices $P_\infty \in \mathbb{R}^{n\times n}$ and $Q_\infty \in$

$\mathbb{R}^{(n+r)\times(n+r)}$ satisfying

$$P_\infty L_\infty Q_\infty = \begin{bmatrix} I_n & 0_{n\times r} \end{bmatrix} \tag{52}$$

Partition Q_∞ into the following form:

$$Q_\infty = \begin{bmatrix} Q_{\infty 1} & Q_{\infty 2} \end{bmatrix}, \quad Q_{\infty 1} \in \mathbb{R}^{(n+r)\times n} \tag{53}$$

Let

$$H_{\infty k} = (Q_{\infty 1} P_\infty M_\infty)^{k-1} Q_{\infty 2}, \quad k = 1, 2, \ldots, d_\infty \tag{54}$$

where $d_\infty = \max_{1\le j\le q_\infty}\{p_{\infty j}\}$. Partition $H_{\infty k}, k = 1, 2, \ldots, d_\infty$, as follows:

$$H_{\infty k} = \begin{bmatrix} N_{\infty k} \\ D_{\infty k} \end{bmatrix}, \quad N_{\infty k} \in \mathbb{C}^{n\times r}, \; k = 1, 2, \ldots, d_\infty \tag{55}$$

By applying Lemma 4.1 to (51), we can prove the following theorem.

Theorem 4.3: *Suppose that the linear system* (1) *is complementary S-controllable. Then the general solution for the closed-loop eigenvectors* $v_{\infty j}^k, k = 1, 2, \ldots, p_{\infty j}, j = 1, 2, \ldots, q_\infty$, *associated with the infinite closed-loop eigenvalue, together with the corresponding vectors* $w_{\infty j}^k, k = 1, 2, \ldots, p_{\infty j}, j = 1, 2, \ldots, q_\infty$, *is given by*

$$\begin{bmatrix} v_{\infty j}^k \\ w_{\infty j}^k \end{bmatrix} = \begin{bmatrix} N_{\infty 1} \\ D_{\infty 1} \end{bmatrix} f_{\infty j}^k + \begin{bmatrix} N_{\infty 2} \\ D_{\infty 2} \end{bmatrix} f_{\infty j}^{k-1} + \cdots + \begin{bmatrix} N_{\infty k} \\ D_{\infty k} \end{bmatrix} f_{\infty j}^1 \tag{56}$$

$$k = 1, 2, \ldots, p_{\infty j}, \quad j = 1, 2, \ldots, q_\infty$$

where $f_{\infty j}^k \in \mathbb{R}^r, k = 1, 2, \ldots, p_{\infty j}, j = 1, 2, \ldots, q_\infty$, *are a group of arbitrarily chosen free parameter vectors;* $N_{\infty k}, D_{\infty k}, k = 1, 2, \ldots, p_{\infty j}, j = 1, 2, \ldots, q_\infty$, *are determined by* (50), (52)–(55).

Remark 4.5: In the special case of $m_\infty = q_\infty, p_{\infty j} = 1, j = 1, 2, \ldots, q_\infty$, (56) has the following simple form:

$$\begin{bmatrix} v_{\infty j}^1 \\ w_{\infty j}^1 \end{bmatrix} = \begin{bmatrix} N_{\infty 1} \\ D_{\infty 1} \end{bmatrix} f_{\infty j}^1, \quad j = 1, 2, \ldots, q_\infty$$

Remark 4.6: Notice that the main work for the solution to the closed-loop eigenvectors with infinite eigenvalues is to solve Equation (52). Instead of applying matrix elementary transformation, we may apply singular value decompositions for the matrix $L_\infty = [E \quad B]$ and obtain

$$\hat{P}_\infty \begin{bmatrix} E & B \end{bmatrix} \hat{Q}_\infty = \begin{bmatrix} \Sigma & 0_{n\times r} \end{bmatrix}$$

where $\hat{P}_\infty \in \mathbb{R}^{n\times n}$ and $\hat{Q}_\infty \in \mathbb{R}^{(n+r)\times(n+r)}$ are orthogonal matrices, and Σ is a non-singular diagonal matrix formed by the singular values of $[E \quad B]$. Rewrite the above equation as

$$\Sigma^{-1}\hat{P}_\infty \begin{bmatrix} E & B \end{bmatrix} \hat{Q}_\infty = \begin{bmatrix} I_n & 0_{n\times r} \end{bmatrix}$$

Then we obtain $P_\infty = \Sigma^{-1}\hat{P}_\infty, Q_\infty = \hat{Q}_\infty$. Thus the solution for the closed-loop eigenvectors with infinite eigenvalues constructed from the matrices P_∞, Q_∞ is numerically reliable because it is mainly based on a singular value decomposition.

4.4. Solution of problem ESA

With the set $\Lambda = \{\lambda_i, i = 0, 1, 2, \ldots, \nu, \infty, 1 \le \nu \le n_A\}$ described previously, the Jordan form J_f, determined by the finite non-zero closed-loop eigenvalues $\lambda_i, i = 1, 2, \ldots, \nu$, is in the following form

$$J_f = \operatorname{diag}(J_1, J_2, \ldots, J_\nu)$$
$$J_i = \operatorname{diag}\left(J_{i1}, J_{i2}, \ldots, J_{iq_i}\right)$$
$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & \lambda_i \end{bmatrix}_{p_{ij}\times p_{ij}}$$

and the Jordan form J_0, determined by the zero closed-loop eigenvalues $\lambda_0 = 0$, is in the following form

$$J_0 = \operatorname{diag}\left(J_{01}, J_{02}, \ldots, J_{0q_0}\right)$$
$$J_{0j} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{p_{0j}\times p_{0j}}$$

and the Jordan form N, determined by the infinite closed-loop eigenvalues $\lambda_\infty = \infty$, is in the following form

$$N = \operatorname{diag}\left(N_1, N_2, \ldots, N_{q_\infty}\right)$$

$$N_j = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}_{p_{\infty j} \times p_{\infty j}}$$

For vectors $v_{ij}^k, w_{ij}^k, k = 1, 2, \ldots, p_{ij}, j = 1, 2, \ldots, q_i, i = 1, 2, \ldots, \nu$, let the matrices V_f and W_f be defined by

$$\begin{aligned} V_f &= \begin{bmatrix} V_1 & V_2 & \cdots & V_\nu \end{bmatrix}, \\ W_f &= \begin{bmatrix} W_1 & W_2 & \cdots & W_\nu \end{bmatrix} \\ V_i &= \begin{bmatrix} V_{i1} & V_{i2} & \cdots & V_{iq_i} \end{bmatrix}, \\ W_i &= \begin{bmatrix} W_{i1} & W_{i2} & \cdots & W_{iq_i} \end{bmatrix} \\ V_{ij} &= \begin{bmatrix} v_{ij}^1 & v_{ij}^2 & \cdots & v_{ij}^{p_{ij}} \end{bmatrix}, \\ W_{ij} &= \begin{bmatrix} w_{ij}^1 & w_{ij}^2 & \cdots & w_{ij}^{p_{ij}} \end{bmatrix} \end{aligned}$$

Then the equations in (24) and (25) can be equivalently written in the unified matrix forms

$$AV_f = (E + BK)V_f J_f \tag{57}$$

$$W_f = KV_f \tag{58}$$

For vectors $v_{0j}^k, w_{0j}^k, k = 1, 2, \ldots, p_{0j}, j = 1, 2, \ldots, q_0$, let the matrices V_0 and W_0 be defined by

$$\begin{aligned} V_0 &= \begin{bmatrix} V_{01} & V_{02} & \cdots & V_{0q_0} \end{bmatrix}, \\ W_0 &= \begin{bmatrix} W_{01} & W_{02} & \cdots & W_{0q_0} \end{bmatrix} \\ V_{0j} &= \begin{bmatrix} v_{0j}^1 & v_{0j}^2 & \cdots & v_{0j}^{p_{0j}} \end{bmatrix}, \\ W_{0j} &= \begin{bmatrix} w_{0j}^1 & w_{0j}^2 & \cdots & w_{0j}^{p_{0j}} \end{bmatrix} \end{aligned}$$

Then the equations in (35) and (36) can be equivalently written in the unified matrix forms

$$AV_0 = (E + BK)V_0 J_0 \tag{59}$$

$$W_0 = KV_0 \tag{60}$$

For vectors $v_{\infty j}^k, w_{\infty j}^k, k = 1, 2, \ldots, p_{\infty j}, j = 1, 2, \ldots, q_0$, let the matrices V_∞ and W_∞ be defined by

$$\begin{aligned} V_\infty &= \begin{bmatrix} V_{\infty 1} & V_{\infty 2} & \cdots & V_{\infty q_\infty} \end{bmatrix}, \\ W_\infty &= \begin{bmatrix} W_{\infty 1} & W_{\infty 2} & \cdots & W_{\infty q_\infty} \end{bmatrix} \\ V_{\infty j} &= \begin{bmatrix} v_{\infty j}^1 & v_{\infty j}^2 & \cdots & v_{\infty j}^{p_{\infty j}} \end{bmatrix}, \\ W_{\infty j} &= \begin{bmatrix} w_{\infty j}^1 & w_{\infty j}^2 & \cdots & w_{\infty j}^{p_{\infty j}} \end{bmatrix} \end{aligned}$$

Then the equations in (7) and (48) can be equivalently written in the unified matrix forms

$$(E + BK)V_\infty = AV_\infty N \tag{61}$$

$$W_\infty = KV_\infty \tag{62}$$

By combining (58), (60) and (62), we have

$$\begin{bmatrix} W_f & W_0 & W_\infty \end{bmatrix} = K \begin{bmatrix} V_f & V_0 & V_\infty \end{bmatrix} \tag{63}$$

In order that real matrix K to be solved from (63), we choose $v_{ij}^k \in \mathbb{R}^n, w_{ij}^k \in \mathbb{R}^r$ for a real eigenvalue λ_i and $v_{\infty j}^k \in \mathbb{R}^n, w_{\infty j}^k \in \mathbb{R}^r$ for the infinite eigenvalue λ_∞, whereas $v_{lj}^k = \bar{v}_{ij}^k \in \mathbb{C}^n, w_{lj}^k = \bar{w}_{ij}^k \in \mathbb{C}^r$ for a complex conjugate pair of eigenvalues $\lambda_i, \lambda_l = \bar{\lambda}_i$. From (34), (47) and (56), it is easy to see that this condition can be equivalently converted into the following constraint on the group of parameter vectors $f_{ij}^k, k = 1, 2, \ldots, p_{ij}, j = 1, 2, \ldots, q_i, i = 0, 1, \ldots, \nu, \infty$.

Constraint 4.1: $f_{ij}^k \in \mathbb{R}^r$ for a non-zero real eigenvalue λ_i, $f_{0j}^k \in \mathbb{R}^r (k \neq p_{0j})$ and $f_{0j}^{p_{0j}} \in \mathbb{R}^{n - n_A + r}$ for the zero eigenvalue $\lambda_0 = 0$, and $f_{\infty j}^k \in \mathbb{R}^r$ for the infinite eigenvalue $\lambda_\infty = \infty$, and $f_{lj}^k = \bar{f}_{ij}^k \in \mathbb{C}^r$ for a complex conjugate pair of eigenvalues λ_i, $\lambda_l = \bar{\lambda}_i$.

To ensure the requirement (b) in Problem ESA, we need to supply the following constraint on parameter vectors $f_{ij}^k, k = 1, 2, \ldots, p_{ij}, j = 1, 2, \ldots, q_i, i = 0, 1, \ldots, \nu, \infty$.

Constraint 4.2: $\det[V_f \quad V_0 \quad V_\infty] \neq 0$.

Because of Constraint 4.2, we can obtain the general form of the state-derivative feedback gain matrix K from (63) as follows:

$$K = \begin{bmatrix} W_f & W_0 & W_\infty \end{bmatrix} \begin{bmatrix} V_f & V_0 & V_\infty \end{bmatrix}^{-1} \tag{64}$$

As for the closed-loop regularity, we have the following lemma.

Lemma 4.2: *Let V_f, V_0, V_∞ and K be matrices satisfying Equations* (57), (59) *and* (61) *and Constraints* 4.1 *and* 4.2. *Then the real gain matrix K makes the closed-loop system* (3) *regular if and only if*

$$\det \begin{bmatrix} (E + BK)V_f & (E + BK)V_0 & AV_\infty \end{bmatrix} \neq 0 \tag{65}$$

Proof: From (57), (59) and (61), we obtain

$$\begin{aligned}&[s(E+BK)-A]\begin{bmatrix}V_f & V_0 & V_\infty\end{bmatrix}\\&=\begin{bmatrix}(E+BK)V_f & (E+BK)V_0 & AV_\infty\end{bmatrix}\\&\times\begin{bmatrix}sI-J_f & 0 & 0\\ 0 & sI-J_0 & 0\\ 0 & 0 & sN-I\end{bmatrix}\end{aligned}$$

Thus, we have

$$\begin{aligned}&\det\,[s(E+BK)-A]\det\begin{bmatrix}V_f & V_0 & V_\infty\end{bmatrix}\\&=(-1)^{m_\infty}s^{m_0}\det\left(sI-J_f\right)\det\\&\times\begin{bmatrix}(E+BK)V_f & (E+BK)V_0 & AV_\infty\end{bmatrix}\end{aligned}\tag{66}$$

Since $\det[V_f\,V_0\,V_\infty]\neq 0$ and $(-1)^{m_\infty}s^{m_0}\det(sI-J_f)$ is not identically zero, it follows from (66) that $\det[s(E+BK)-A]$ is not identically zero if and only if the condition (65) holds. The proof is done. ■

Using (34), (47), (56), (58), (60) and (62), condition (65), which ensures the closed-loop regularity, can be turned into the following constraint on the group of parameter vectors $f_{ij}^k, k=1,2,\ldots,p_{ij}, j=1,2,\ldots,q_i, i=0,1,\ldots,\nu,\infty$.

Constraint 4.3: $\det\begin{bmatrix}EV_f+BW_f & EV_0+BW_0 & AV_\infty\end{bmatrix}\neq 0$.

Summarizing the above results, we have the following theorem for solution to Problem ESA.

Theorem 4.4: *Suppose that the linear system* (1) *is complementary S-controllable. Problem ESA has solutions if and only if there exists a group of parameter vectors* $f_{ij}^k, k=1,2,\ldots,p_{ij}, j=1,2,\ldots,q_i, i=0,1,\ldots,\nu,\infty$, *satisfying Constraints* 4.1–4.3. *When this condition is met, the group of closed-loop eigenvectors* $v_{ij}^k, k=1,2,\ldots,p_{ij}, j=1,2,\ldots,q_i, i=0,1,\ldots,\nu,\infty$, *is given by* (34), (47) *and* (56), *and the feedback gain matrix* K *is given by* (64) *with Constraints* 4.1–4.3 *satisfied.*

It follows from Theorem 3.1 that the geometric multiplicity q_∞ of the infinite eigenvalues in Theorem 4.4 can be arbitrarily assigned in the range

$$0\leq q_\infty\leq \mathrm{rank}B$$

or, equivalently, the dynamical order $\mathrm{rank}E_c$ of the closed-loop system (3) can be arbitrarily assigned in the range

$$n-\mathrm{rank}B\leq\rho=\mathrm{rank}E_c=n-q_\infty\leq n$$

This is consistent with the dynamical order assignment condition (9).

Recall the combined problem mentioned before, that is, the problem of finite eigenstructure assignment with an arbitrary specified dynamical order ρ via state-derivative controller (2) in the linear system (1). Then the solution to the combined problem is given by the corollary of Theorem 4.4 as follows.

Corollary 4.1: *Suppose that the linear system* (1) *is complementary S-controllable. Given positive integer* ρ *satisfying* $n-\mathrm{rank}B\leq\rho\leq n$ *and let* $m_\infty=q_\infty=n-\rho$, $p_{i\infty}=1$, $j=1,2,\ldots,q_\infty$ *in Problem ESA. Then the combined problem has solutions if and only if there exists a group of parameter vectors* f_{ij}^k, $k=1,2,\ldots,p_{ij}$, $j=1,2,\ldots,q_i$, $i=0,1,\ldots,\nu$, $f_{\infty j}^1$, $j=1,2,\ldots,n-\rho$, *satisfying Constraints* 4.1–4.3. *When this condition is met, the group of closed-loop eigenvectors* $v_{ij}^k, k=1,2,\ldots,p_{ij}, j=1,2,\ldots,q_i, i=0,1,\ldots,\nu$; $v_{\infty j}^1$, $j=1,2,\ldots,n-\rho$, *is given by* (34), (47) *and* (56), *and the feedback gain matrix* K *is given by* (64) *with Constraints* 4.1–4.3 *satisfied. In this case, the closed-loop system* (3) *possesses the dynamical order* ρ *and finite eigenstructure given by* $\hat{\Lambda}=\{\lambda_i, i=0,1,2,\ldots,\nu,1\leq\nu\leq n_A\}$, m_i, q_i, p_{ij}, $j=1,2,\ldots,q_i$, $i=0,1,\ldots,\nu$, *and* $[V_f\,V_0]$.

Remark 4.7: The assignment of closed-loop eigenvectors with finite non-zero eigenvalues is mainly based on a series of matrix decompositions for the matrices $[A-s_iE \quad -s_iB]$, $i=1,2,\ldots,\nu$, while the assignment of closed-loop eigenvectors with infinite eigenvalues is mainly based on a matrix decomposition for the matrix $[E \quad B]$. Obvioluy, the latter is simpler than the former. Thus, compared to traditional full-order design, the reduced-order design of the state-derivative feedback eigenstructure assignment problem is simpler and needs less computational work, especially for high dimensional systems. It is well known that the closed-loop dominant pole determines the dominant component of the system response. Therefore, the dynamical order is preferably reduced in an attempt to faciliate the design algorithms. In system design, we may focus on assigning

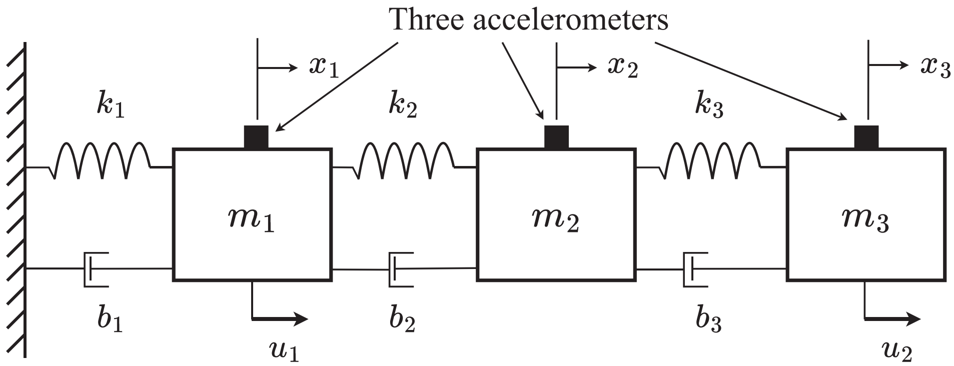

Figure 1. Three-mass-spring-dashpot system.

important eigenvalues, and eliminating some unimportant eigenvalues by reducing the dynamical order of the system.

Definition 4.1 (Potter et al., 1979): A subset of $\mathbb{R}^{p\times q}$ is a Zarisk open set of $\mathbb{R}^{p\times q}$ if it is nonempty and its complement is the set of solutions in $\mathbb{R}^{p\times q}$ to a finite set of polynomial equations.

The first author proved the following lemma.

Lemma 4.3 (Zhang, 2013): *Let $X \in \mathbb{R}^{p\times q}$ and $F(X)$ be a $s \times t$ matrix function whose elements are rational functions of the elements of X. Define*

$$\mathcal{S} = \Big\{X \,|\, X \in \mathbb{R}^{p\times q},\ F(X) \quad \text{is well defined and} \quad \operatorname{rank}F(X) = \min\{s,t\}\Big\}$$

If $\mathcal{S}$ is nonempty, then it is a Zarisk open set of $\mathbb{R}^{p\times q}$.

Denote a pair of complex eigenvalue s_i and s_l of the closed-loop system (3) by $s_i = \bar{s}_j = \sigma_i + \sigma_l \mathrm{i}$, and the corresponding parameter vectors by $f_{ij}^k = \bar{f}_{lj}^k = \xi_{ij}^k + \xi_{lj}^k \mathrm{i}$, where σ_i, σ_l, ξ_{ij}^k and ξ_{lj}^k are real. It is easy to see that the total number of the real parameters in f_{ij}^k, $k = 1, 2, \ldots, p_{ij}$, $j = 1, 2, \ldots, q_i$, $i = 0, 1, \ldots, \nu, \infty$ (If $f_{ij}^k = \bar{f}_{lj}^k = \xi_{ij}^k + \xi_{lj}^k \mathrm{i}$, we replace $f_{ij}^k, \bar{f}_{lj}^k$ by ξ_{ij}^k, ξ_{lj}^k.) are $\left(nr + (n - n_A)^2\right)$. Let $X \in \mathbb{R}^{nr+(n-n_A)^2}$ be a vector composed of these real parameters. Obviously, the elements of the matrices $[V_f \quad V_0 \quad V_\infty]$ and $[EV_f + BW_f \quad EV_0 + BW_0 \quad AV_\infty]$ in Constraints 4.2 and 4.3 are rational functions of the elements of X. Let

$$\mathcal{S}_1 = \Big\{X \,|\, X \in \mathbb{R}^{nr+(n-n_A)^2}, \text{ and } \operatorname{rank}\left[V_f \quad V_0 \quad V_\infty\right] = n\Big\}$$

$$\mathcal{S}_2 = \Big\{X \,|\, X \in \mathbb{R}^{nr+(n-n_A)^2}, \text{ and } \operatorname{rank}\left[EV_f + BW_f \quad EV_0 + BW_0 \quad AV_\infty\right] = n\Big\}$$

It follows from Lemma 4.3 that, if the solution to Problem ESA exists, then $\mathcal{S}_1$ and $\mathcal{S}_2$ are Zarisk open sets of $\mathbb{R}^{nr+(n-n_A)^2}$. Therefore, Constraints 4.2 and 4.3 hold for almost all parameter vectors f_{ij}^k, $k = 1, 2, \ldots, p_{ij}$, $j = 1, 2, \ldots, q_i$, $i = 0, 1, \ldots, \nu, \infty$.

Based on Theorem 4.4, when the complementary S-controllable (complementary C-controllable) linear system (1), set Λ, and integers $m_i, q_i, p_{ij}, j = 1, 2, \ldots, q_i, i = 0, 1, \ldots, \nu, \infty$, satisfying (4) and (5), are all provided, an algorithm for solving Problem ESA can be given as follows.

5. An illustrative example

In this section, we will use a practical example to illustrate the advantages of the proposed approach.

Consider a three degrees of freedom mass-spring-dashpot system shown in Figure 1.

The system dynamic equation can be written as a first order linear system of the form (1) with the matrices E, A and B given by

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_3 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -k_1-k_2 & k_2 & 0 & -b_1-b_2 & b_2 & 0 \\ k_2 & -k_2-k_3 & k_3 & b_2 & -b_2-b_3 & b_3 \\ 0 & k_3 & -k_3 & 0 & b_3 & -b_3 \end{bmatrix}$$

Algorithm 1 Algorithm ESA

1: Solve polynomial matrices $P(s)$ and $Q(s)$ satisfying (28) (when assigning finite non-zero eigenvalues), constant matrices R and F satisfying (38) (when assigning zero eigenvalues), and constant matrices P_∞ and Q_∞ satisfying (52) (when assigning infinite eigenvalues).

2: Compute matrices $N_{ik}, D_{ik}, k = 1, 2, \cdots, p_{ij}, j = 1, 2, \cdots, q_i, i = 1, 2, \cdots, \nu$, satisfying (26), (28)-(33) (when assigning finite non-zero eigenvalues), matrices $N_{0j}^{kl}, D_{0j}^{kl}, l = 1, 2, \cdots, k, k = 1, 2, \cdots, p_{0j}, j = 1, 2, \cdots, q_0$, satisfying (38)-(46) (when assigning zero eigenvalues), and matrices $N_{\infty k}, D_{\infty k}, k = 1, 2, \cdots, p_{\infty j}, j = 1, 2, \cdots, q_\infty$, satisfying (50), (52)-(55) (when assigning infinite eigenvalues).

3: Construct the parametric expressions of matrices $V_f, W_f, V_0, W_0, V_\infty, W_\infty$ as shown above.

4: Find a group of parameter vectors $f_{ij}^k, k = 1, 2, \cdots, p_{ij}, j = 1, 2, \cdots, q_i, i = 0, 1, \cdots, \nu, \infty$, satisfying Constraints 4.1–4.3 (or a set of parameter vectors f_{ij}^k, $k = 1, 2, \cdots, p_{ij}, j = 1, 2, \cdots, q_i, i = 0, 1, \cdots, \nu, \infty$, satisfying Constraints 4.1–4.3).

5: Compute matrices $V_f, W_f, V_0, W_0, V_\infty, W_\infty$ based on the parameter vectors f_{ij}^k, $k = 1, 2, \cdots, p_{ij}$, $j = 1, 2, \cdots, q_i, i = 0, 1, \cdots, \nu, \infty$, obtained in Step 4.

6: Compute the gain matrix K according to (64) based on the matrices $V_f, W_f, V_0, W_0, V_\infty, W_\infty$ obtained in Step 5.

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

where m_1, m_2 and m_3 are the masses, k_1, k_2 and k_3 are the spring constants, b_1, b_2 and b_3 are the damper constants, $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the mass displacements, $u_1(t)$ and $u_2(t)$ are control inputs, the state vector is $x(t) = [x_1(t)\ x_2(t)\ x_3(t)\ \dot{x}_1(t)\ \dot{x}_2(t)\ \dot{x}_3(t)]^{\mathrm{T}}$, and the input vector is $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^{\mathrm{T}}$.

5.1. *Both derivative matrix E and state matrix A are non-singular*

If the system parameters are taken as $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$, $m_3 = 3\,\text{kg}$, $b_1 = b_3 = 2\,\text{Ns/m}$, $b_2 = 0.5\,\text{Ns/m}$, $k_1 = k_2 = 5\,\text{kN/m}$, and $k_3 = 20\,\text{kN/m}$, then

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 5 & 0 & -2.5 & 0.5 & 0 \\ 5 & -25 & 20 & 0.5 & -2.5 & 2 \\ 0 & 20 & -20 & 0 & 2 & -2 \end{bmatrix}$$

It is easy to verify that, both E and A are non-singular, the system is complementary C-controllable, and the open-loop eigenvalues are $\{-0.0588 \pm 0.6764\mathrm{i}, -1.0061 \pm 4.2215\mathrm{i}, -1.1435 \pm 2.8794\mathrm{i}\}$.

In the following, we consider the assignment of two different closed-loop eigenstructures.

Case 1. (full-order design) $\Lambda = \{\lambda_{1,2} = -2 \pm \mathrm{i}, \lambda_3 = -4, \lambda_4 = -5, \lambda_{5,6} = -3 \pm 4\mathrm{i}\}$, $\quad m_i = q_i = p_{i1} = 1,\ i = 1, 2, \ldots, 6$

The full-order eigenstructure assignment is considered in this case. The desired dynamical order of the closed-loop system is equal to the dynamical order of the open-loop system which is $\rho = 6$. By the algorithm in Subsection 4.1, we obtain

$$N_{i1} = \begin{bmatrix} 4\lambda_i^3 + \lambda_i^2 + 10\lambda_i & 4\lambda_i \\ \lambda_i^2 + 10\lambda_i & 0 \\ \lambda_i^2 + 10\lambda_i & -\lambda_i \\ 4\lambda_i^4 + \lambda_i^3 + 10\lambda_i^2 & 4\lambda_i^2 \\ \lambda_i^3 + 10\lambda_i^2 & 0 \\ \lambda_i^3 + 10\lambda_i^2 & -\lambda_i^2 \end{bmatrix},$$

$$D_{i1} = \begin{bmatrix} -4\lambda_i^4 - 11\lambda_i^3 - 52\lambda_i^2 & \\ -25\lambda_i - 50 & -4\lambda_i^2 - 10\lambda_i - 40 \\ -3\lambda_i^3 - 30\lambda_i^2 & 3\lambda_i^2 + 2\lambda_i + 20 \end{bmatrix},$$

$$i = 1, 2, \ldots, 6$$

Then the closed-loop eigenvectors v_{i1}^1, $i = 1, 2, \ldots, 6$, and the corresponding vectors w_{i1}^1, $i = 1, 2, \ldots, 6$, are given by

$$v_{i1}^1 = N_{i1} f_{i1}^1, \quad w_{i1}^1 = D_{i1} f_{i1}^1, \quad i = 1, 2, \ldots, 6$$

From Subsection 4.4, we know that Constraints 4.2 and 4.3 hold for almost all parameter vectors f_{ij}^k, $k = 1, 2, \ldots, p_{ij}$, $j = 1, 2, \ldots, q_i$, $i = 1, 2, \ldots, 6$. We specially choose

$$f_{11}^1 = f_{21}^1 = f_{51}^1 = f_{61}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_{31}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$f_{41}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

then Constraints 4.1–4.3 are satisfied and the matrices V and W are obtained as

$$V = \begin{bmatrix} v_{11}^1 & v_{21}^1 & v_{31}^1 & v_{41}^1 & v_{51}^1 & v_{61}^1 \end{bmatrix}$$
$$= \left[\begin{matrix} -25+50\mathrm{i} & -25-50\mathrm{i} & -16 \\ -17+6\mathrm{i} & -17-6\mathrm{i} & 0 \\ -17+6\mathrm{i} & -17-6\mathrm{i} & 4 \\ -125\mathrm{i} & 125\mathrm{i} & 64 \\ 28-29\mathrm{i} & 28+29\mathrm{i} & 0 \\ 28-29\mathrm{i} & 28+29\mathrm{i} & -16 \end{matrix}\right.$$
$$\left.\begin{matrix} -545 & 431+192\mathrm{i} & 431-192\mathrm{i} \\ -25 & -37+16\mathrm{i} & -37-16\mathrm{i} \\ -20 & -37+16\mathrm{i} & -37-16\mathrm{i} \\ 2725 & -2061+1148\mathrm{i} & -2061-1148\mathrm{i} \\ 125 & 47-196\mathrm{i} & 47+196\mathrm{i} \\ 100 & 47-196\mathrm{i} & 47+196\mathrm{i} \end{matrix}\right]$$
$$W = \begin{bmatrix} w_{11}^1 & w_{21}^1 & w_{31}^1 & w_{41}^1 & w_{51}^1 & w_{61}^1 \end{bmatrix}$$
$$= \left[\begin{matrix} -106+158\mathrm{i} & -106-158\mathrm{i} & -64 \\ -84+87\mathrm{i} & -84-87\mathrm{i} & 60 \end{matrix}\right.$$
$$\left.\begin{matrix} -2440 & 1210-680\mathrm{i} & 1210+680\mathrm{i} \\ -290 & -141+588\mathrm{i} & -141-588\mathrm{i} \end{matrix}\right]$$

According to Theorem 4.4, the feedback gain matrix K is given by

$$K = WV^{-1}$$
$$= \begin{bmatrix} -0.9225 & -48.6875 & 52.31 & -0.9 & -16.81 & 14.4 \\ 0 & -7 & 7 & 0 & -1 & -2 \end{bmatrix}$$

Case 2.(reduced-order design) $\Lambda = \{\lambda_{1,2} = -2 \pm \mathrm{i}, \lambda_3 = -4, \lambda_4 = -5, \lambda_\infty = \infty\}$, $m_i = q_i = p_{i1} = 1$, $i = 1, \ldots, 4$, $m_\infty = q_\infty = 2$, $p_{\infty 1} = p_{\infty 2} = 1$.

The reduced-order eigenstructure assignment is considered in this case. The desired dynamical order of the closed-loop system is equal to the minimum dynamical order which is $\rho = 4$ (According to Corollary 4.1, the permissible range for q_∞ is $0 \le q_\infty \le 2$, so the minimum dynamical order is $\rho = n - q_\infty = 6 - 2 = 4$). There is no available method in the existing literature to deal with this case.

Our approach can be utilised. The closed-loop eigenvectors v_{i1}^1, $i = 1, 2, 3, 4$ and the corresponding vectors w_{i1}^1, $i = 1, 2, 3, 4$ are taken to be the same as those in Case 1. By the algorithm in Subsection 4.3, we obtain

$$N_{\infty 1} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^{\mathrm{T}},$$
$$D_{\infty 1} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Then the closed-loop eigenvectors $v_{\infty 1}^1$, $v_{\infty 2}^1$ and the corresponding vectors $w_{\infty 1}^1$, $w_{\infty 2}^1$ are given by

$$v_{\infty 1}^1 = N_{\infty 1} f_{\infty 1}^1, \quad v_{\infty 2}^1 = N_{\infty 1} f_{\infty 2}^1,$$
$$w_{\infty 1}^1 = D_{\infty 1} f_{\infty 1}^1, \quad w_{\infty 2}^1 = D_{\infty 1} f_{\infty 2}^1$$

Specially choosing

$$f_{\infty 1}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_{\infty 2}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then Constraints 4.1–4.3 are satisfied and the matrices V and W are obtained as

$$V = \begin{bmatrix} v_{11}^1 & v_{21}^1 & v_{31}^1 & v_{41}^1 & v_{\infty 1}^1 & v_{\infty 2}^1 \end{bmatrix}$$
$$= \left[\begin{matrix} -25+50\mathrm{i} & -25-50\mathrm{i} & -16 \\ -17+6\mathrm{i} & -17-6\mathrm{i} & 0 \\ -17+6\mathrm{i} & -17-6\mathrm{i} & 4 \\ -125\mathrm{i} & 125\mathrm{i} & 64 \\ 28-29\mathrm{i} & 28+29\mathrm{i} & 0 \\ 28-29\mathrm{i} & 28+29\mathrm{i} & -16 \end{matrix}\right.$$
$$\left.\begin{matrix} -545 & 0 & 0 \\ -25 & 0 & 0 \\ -20 & 0 & 0 \\ 2725 & -1 & 0 \\ 125 & 0 & 0 \\ 100 & 0 & -1 \end{matrix}\right]$$
$$W = \begin{bmatrix} w_{11}^1 & w_{21}^1 & w_{31}^1 & w_{41}^1 & w_{\infty 1}^1 & w_{\infty 2}^1 \end{bmatrix}$$
$$= \left[\begin{matrix} -106+158\mathrm{i} & -106-158\mathrm{i} & -64 \\ -84+87\mathrm{i} & -84-87\mathrm{i} & 60 \end{matrix}\right.$$
$$\left.\begin{matrix} -2440 & 1 & 0 \\ -290 & 0 & 3 \end{matrix}\right]$$

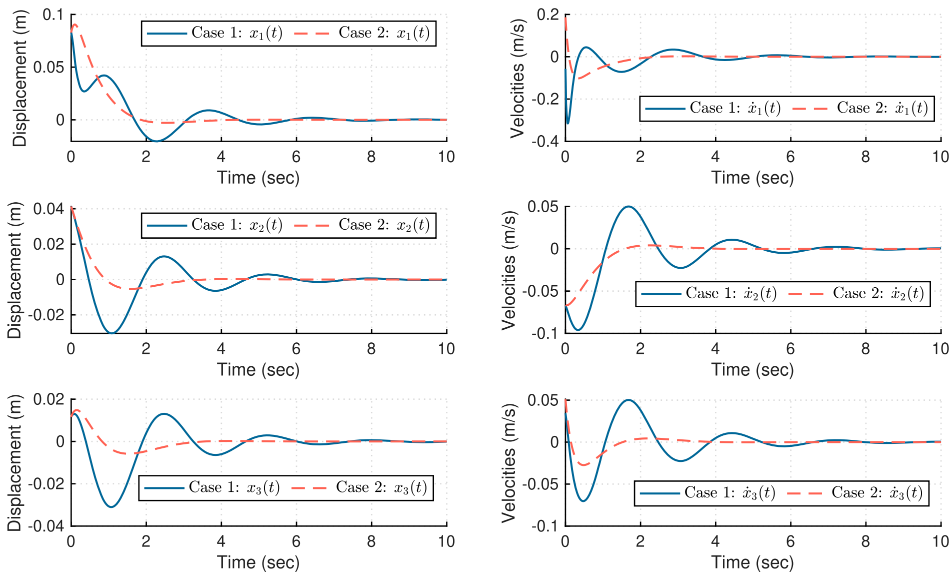

Figure 2. The responses for the closed-loop systems in Cases 1 and 2.

According to Theorem 4.4, the feedback gain matrix K is given by

$$K = WV^{-1} = \begin{bmatrix} -1.37 & 9.25 & -5.48 & -1 & -2.72 & 0 \\ 0.01625 & -3.03125 & 3.065 & 0 & 0.035 & -3 \end{bmatrix}$$

Dynamical order assignment is one of the issues that we study. Related to our approach are the works of Fahmy and Tantawy (1990), Owens and Askarpour (2001) and G. R. Duan and Zhang (2002). Stability is a prerequisite for ensuring the normal operation of a control system. The approaches in Fahmy and Tantawy (1990), Owens and Askarpour (2001) and G. R. Duan and Zhang (2002) focus only on assigning dynamical order of closed-loop systems, but say nothing about eigenvalues of closed-loop systems. Therefore, the closed-loop systems obtained based on these approaches may be unstable. Different from the approaches in Fahmy and Tantawy (1990), Owens and Askarpour (2001) and G. R. Duan and Zhang (2002), our approach assigns not only the dynamical order, but also finite eigenstructure of the closed-loop system. Thus, our approach can design stable closed-loop systems. This is highly desirable from the perspective of practical application.

The responses of both the closed-loop systems are shown in Figure 2.

In both cases, the initial condition is taken to be $x(0) = \begin{bmatrix} x_1(0) & x_2(0) & x_3(0) & \dot{x}_1(0) & \dot{x}_2(0) & \dot{x}_3(0) \end{bmatrix}^{\mathrm{T}}$, where $x_1(0) = 0.0828\,\mathrm{m}$, $x_2(0) = 0.0413\,\mathrm{m}$, $x_3(0) = 0.0116\,\mathrm{m}$, $\dot{x}_1(0) = -0.0573\,\mathrm{m/s}$, $\dot{x}_2(0) = -0.0668\,\mathrm{m/s}$, $\dot{x}_3(0) = 0.0345\,\mathrm{m/s}$. It can be seen from the figure that all the displacements $x_i, i = 1, 2, 3$, and the velocities $\dot{x}_i, i = 1, 2, 3$, of both the closed-loop systems asymptotically approach zero. This indicates that both full-order design and reduced-order design can make the closed-loop system stable. Compared with full-order design, reduced-order design is simpler and requires less computational work.

5.2. Derivative matrix E is singular and state matrix A is non-singular

If the mass parameters m_3 is taken as $m_3 = 0\,\mathrm{kg}$ and all the other system parameters are not changed (compared to those in Subsection 5.1), then

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 5 & 0 & -2.5 & 0.5 & 0 \\ 5 & -25 & 20 & 0.5 & -2.5 & 2 \\ 0 & 20 & -20 & 0 & 2 & -2 \end{bmatrix}$$

It is easy to verify that, E is singular and A is non-singular, the system is complementary C-controllable, and the open-loop eigenvalues are $\{-1.2169 \pm 3.1023\mathrm{i}, -0.1581 \pm 1.0491\mathrm{i}, -10, \infty\}$.

This system is a descriptor linear system. Related to our approach is the work of Mirassadi and Tehrani (2017). Mirassadi and Tehrani (2017) considered the problem of eigenvalue assignment for descriptor linear systems via state-derivative feedback and provided an algorithm with which only one single solution of the feedback gain matrix can be obtained. Unlike their approach, our approach can give general parameter characterisation of the feedback gain matrix. Besides, their approach requires that both the state matrix A and the closed-loop derivative matrix $(E + BK)$ are non-singular and addresses just the aspect of positioning but say nothing about the achievable structure of repeated eigenvalues.

Let us consider the assignment of the following two different closed-loop eigenstructures.

Case 3. (full-order design) $\{\lambda_{1,2} = -2 \pm \mathrm{i},\ \lambda_{3,4} = -3 \pm 4\mathrm{i},\ \lambda_5 = -4,\ \lambda_6 = -5\}, m_i = q_i = p_{i1} = 1,\ i = 1, 2, \ldots, 6$.

The full-order eigenstructure assignment is considered in this case. The desired dynamical order of the closed-loop system is equal to the maximum dynamical order which is $\rho = 6$ (According to Corollary 4.1, the permissible range for q_∞ is $0 \le q_\infty \le 2$, so the maximum dynamical order is $\rho = n - q_\infty = 6 - 0 = 6$).

By the algorithm in Subsection 4.1, we obtain

$$N_{i1} = \begin{bmatrix} -4\lambda_i^3 - \lambda_i^2 - 10\lambda_i & 4\lambda_i \\ -\lambda_i^2 - 10\lambda_i & 0 \\ -\lambda_i^2 - 10\lambda_i & -\lambda_i \\ -4\lambda_i^4 - \lambda_i^3 - 10\lambda_i^2 & 4\lambda_i^2 \\ -\lambda_i^3 - 10\lambda_i^2 & 0 \\ -\lambda_i^3 - 10\lambda_i^2 & -\lambda_i^2 \end{bmatrix},$$

$$D_{i1} = \begin{bmatrix} 4\lambda_i^4 + 11\lambda_i^3 + 52\lambda_i^2 + 25\lambda_i + 50 & -4\lambda_i^2 - 10\lambda_i - 40 \\ 0 & 2\lambda_i + 20 \end{bmatrix},$$
$$i = 1, 2, \ldots, 6$$

Then the closed-loop eigenvectors $v_{i1}^1,\ i = 1, 2, \ldots, 6$, and the corresponding vectors $w_{i1}^1,\ i = 1, 2, \ldots, 6$, are given by

$$v_{i1}^1 = N_{i1} f_{i1}^1, \quad w_{i1}^1 = D_{i1} f_{i1}^1, \quad i = 1, 2, \ldots, 6$$

Specially choosing

$$f_{11}^1 = f_{21}^1 = f_{51}^1 = f_{61}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad f_{31}^1 = f_{41}^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

then Constraints 4.1–4.3 are satisfied and the matrices V and W are obtained as

$$V = \begin{bmatrix} v_{11}^1 & v_{21}^1 & v_{31}^1 & v_{41}^1 & v_{51}^1 & v_{61}^1 \end{bmatrix}$$
$$= \begin{bmatrix} 25 - 50\mathrm{i} & 25 + 50\mathrm{i} & -12 + 16\mathrm{i} & -12 - 16\mathrm{i} & 280 & 525 \\ 17 - 6\mathrm{i} & 17 + 6\mathrm{i} & 0 & 0 & 24 & 25 \\ 17 - 6\mathrm{i} & 17 + 6\mathrm{i} & 3 - 4\mathrm{i} & 3 + 4\mathrm{i} & 24 & 25 \\ 125\mathrm{i} & -125\mathrm{i} & -28 - 96\mathrm{i} & -28 + 96\mathrm{i} & -1120 & -2625 \\ -28 + 29\mathrm{i} & -28 - 29\mathrm{i} & 0 & 0 & -96 & -125 \\ -28 + 29\mathrm{i} & -28 - 29\mathrm{i} & 7 + 24\mathrm{i} & 7 - 24\mathrm{i} & -96 & -125 \end{bmatrix}$$

$$W = \begin{bmatrix} w_{11}^1 & w_{21}^1 & w_{31}^1 & w_{41}^1 & w_{51}^1 & w_{61}^1 \end{bmatrix}$$
$$= \begin{bmatrix} 106 - 158\mathrm{i} & 106 + 158\mathrm{i} & 18 + 56\mathrm{i} & 18 - 56\mathrm{i} & 1102 & 2350 \\ 0 & 0 & 14 + 8\mathrm{i} & 14 - 8\mathrm{i} & 0 & 0 \end{bmatrix}$$

According to Theorem 4.4, the feedback gain matrix K is given by

$$K = WV^{-1}$$
$$= \begin{bmatrix} -0.49375 & 5.81875 & -1.575 & -0.875 & -0.55 & -1.1 \\ 0 & -2.8 & 2.8 & 0 & -0.8 & 0.8 \end{bmatrix}$$

Case 4. (reduced-order design) $\Lambda = \{\lambda_{1,2} = -2 \pm \mathrm{i},\ \lambda_{3,4} = -3 \pm 4\mathrm{i},\ \lambda_5 = -4,\ \lambda_\infty = \infty\}, \quad m_i = q_i = p_{i1} = 1,\ i = 1, 2, \ldots, 5, \infty$.

The reduced-order eigenstructure assignment is considered in this case The desired dynamical order of the closed-loop system is equal to the dynamical order of the open-loop system which is $\rho = 5$ (According to Corollary 4.1, the assigned dynamical order is $\rho = n - q_\infty = 6 - 1 = 5$), so this eigenstructure assignment is not only reduced-order but also equal-order.

Also, there is no available method in the existing literature to deal with this case.

Our approach can be utilised. The closed-loop eigenvectors $v_{i1}^1,\ i = 1, 2, \ldots, 5$, and the corresponding vectors $w_{i1}^1,\ i = 1, 2, \ldots, 5$, are taken to be the same as those in Case 3. By the algorithm in Subsection 4.3, we obtain

$$N_{\infty 1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{\infty 1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Then the closed-loop eigenvector $v^1_{\infty 1}$ and the corresponding vector $w^1_{\infty 1}$ are given by

$$v^1_{\infty 1} = N_{\infty 1} f^1_{\infty 1}, \quad w^1_{\infty 1} = D_{\infty 1} f^1_{\infty 1}$$

Specially choosing

$$f^1_{\infty 1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

then Constraints 4.1–4.3 are satisfied and the matrices V and W are obtained as

$$V = \begin{bmatrix} v^1_{11} & v^1_{21} & v^1_{31} & v^1_{41} & v^1_{51} & v^1_{61} \end{bmatrix}$$

$$= \begin{bmatrix} -25+50\mathrm{i} & -25-50\mathrm{i} & -16 & -545 & 431+192\mathrm{i} & 431-192\mathrm{i} \\ -17+6\mathrm{i} & -17-6\mathrm{i} & 0 & -25 & -37+16\mathrm{i} & -37-16\mathrm{i} \\ -17+6\mathrm{i} & -17-6\mathrm{i} & 4 & -20 & -37+16\mathrm{i} & -37-16\mathrm{i} \\ -125\mathrm{i} & 125\mathrm{i} & 64 & 2725 & -2061+1148\mathrm{i} & -2061-1148\mathrm{i} \\ 28-29\mathrm{i} & 28+29\mathrm{i} & 0 & 125 & 47-196\mathrm{i} & 47+196\mathrm{i} \\ 28-29\mathrm{i} & 28+29\mathrm{i} & -16 & 100 & 47-196\mathrm{i} & 47+196\mathrm{i} \end{bmatrix}$$

$$W = \begin{bmatrix} w^1_{11} & w^1_{21} & w^1_{31} & w^1_{41} & w^1_{51} & w^1_{61} \end{bmatrix}$$

$$= \begin{bmatrix} -106+158\mathrm{i} & -106-158\mathrm{i} & -64 & -2440 & 1210-680\mathrm{i} & 1210+680\mathrm{i} \\ -84+87\mathrm{i} & -84-87\mathrm{i} & 60 & -290 & -141+588\mathrm{i} & -141-588\mathrm{i} \end{bmatrix}$$

According to Theorem 4.4, the feedback gain matrix K is given by

$$K = WV^{-1}$$

$$= \begin{bmatrix} 0.69575 & 1.43125 & 3.183 & -0.68 & -0.043 & -0.32 \\ -0.976 & 0.8 & -1.104 & -0.16 & -1.216 & 0.16 \end{bmatrix}$$

The responses of both the closed-loop systems are shown in Figure 3. The initial condition in both cases is taken to be the same as in Subsection 5.1. It can be seen from the figure that all the displacements $x_i,\ i = 1, 2, 3$, and the velocities $\dot{x}_i,\ i = 1, 2, 3$, of both the closed-loop systems asymptotically approach zero. The simulation results indicate that both the closed-loop systems are stable. It is worth noting that, like state feedback eigenstructure assignment approaches, our approach can use state-derivative feedback to achieve equal-order eigenstructure assignment of descriptor linear systems.

5.3. Derivative matrix E is non-singular and state matrix A is singular

If the spring parameter k_3 is taken as $k_3 = 0\,\mathrm{kN/m}$, and all the other system parameters are not changed (compared to those in Subsection 5.1), then

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -10 & 5 & 0 & -2.5 & 0.5 & 0 \\ 5 & -5 & 0 & 0.5 & -2.5 & 2 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

It is easy to verify that, E is non-singular and A is singular, the system is complementary C-controllable, and the open-loop eigenvalues are $\{-1.2858 \pm 3.1026\mathrm{i}, -0.2863 \pm 0.7061\mathrm{i}, -1.2725, 0\}$.

Here we consider the assignment of the following closed-loop eigenstructure. Since A only has one zero eigenvalue, both our approach and the approach in Abdelaziz (2011) can be used. By the way, unlike the approach in Abdelaziz (2011), our approach has no requirement for the number of zero eigenvalues of a state matrix.

Case 5. (full-order design) $\{\lambda_{1,2} = -2 \pm \mathrm{i},\ \lambda_{3,4} = -3 \pm 4\mathrm{i},\ \lambda_5 = -5,\ \lambda_0 = 0\}$, and integers are taken as $m_i = q_i = p_{i1} = 1,\ i = 0, 1, \ldots, 5$ (According to Theorem 3.1, q_0 is fixed and $q_0 = n - \mathrm{rank}A = 6 - 5 = 1$).

The full-order eigenstructure assignment is considered in this case. The desired dynamical order of the

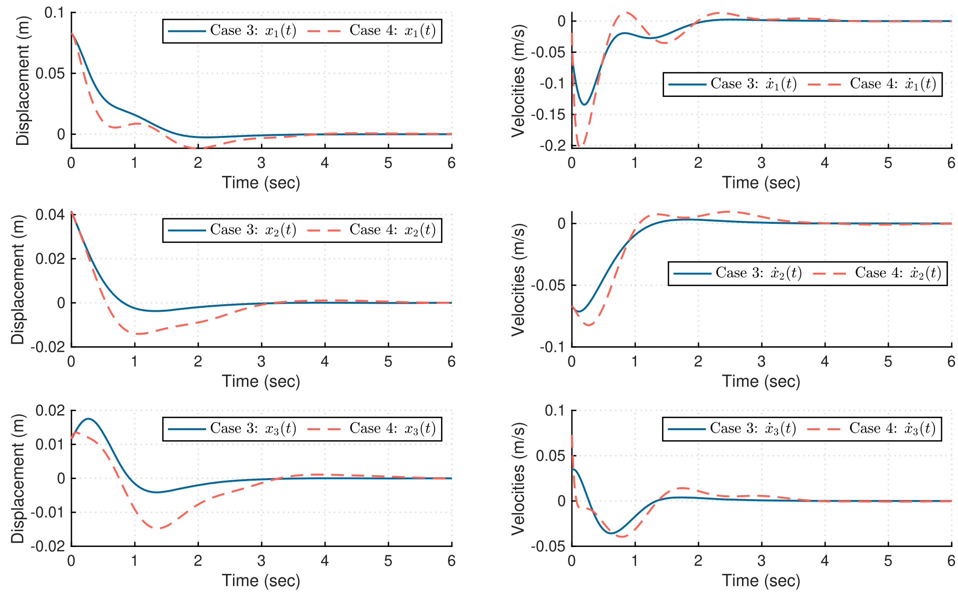

Figure 3. The responses for the closed-loop systems in Cases 3 and 4.

closed-loop system is equal to the dynamical order of the open-loop system which is $\rho = 6$.

Our approach can be used. By the algorithm in Subsection 4.1, we obtain

$$N_{i1} = \begin{bmatrix} 4\lambda_i & 0 \\ 8\lambda_i & 4\lambda_i \\ 8\lambda_i^2 + 9\lambda_i + 10 & 4\lambda_i^2 + 5\lambda_i + 10 \\ 4\lambda_i^2 & 0 \\ 8\lambda_i^2 & 4\lambda_i^2 \\ 8\lambda_i^3 + 9\lambda_i^2 + 10\lambda_i & 4\lambda_i^3 + 5\lambda_i^2 + 10\lambda_i \end{bmatrix}$$

$$D_{i1} = \begin{bmatrix} -4\lambda_i^2 - 6\lambda_i & 2\lambda_i + 20 \\ -24\lambda_i^3 - 43\lambda_i^2 & -12\lambda_i^3 - 23\lambda_i^2 \\ -32\lambda_i - 20 & -32\lambda_i - 20 \end{bmatrix},$$

$$i = 1, 2, \ldots, 5$$

Then the closed-loop eigenvectors $v_{i1}^1,\ i = 1, 2, \ldots, 5$, and the corresponding vectors $w_{i1}^1,\ i = 1, 2, \ldots, 5$, are given by

$$v_{i1}^1 = N_{i1} f_{i1}^1, \quad w_{i1}^1 = D_{i1} f_{i1}^1,\ i = 1, 2, \ldots, 5$$

By the algorithm in Subsection4.2, we obtain

$$N_{01}^{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_{01}^{11} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the closed-loop eigenvector v_{01}^1 and the corresponding vector w_{01}^1 are given by

$$v_{01}^1 = N_{01}^{11} f_{01}^1, \quad w_{01}^1 = D_{01}^{11} f_{01}^1$$

From the above, the general closed-loop eigenvector matrix V and the corresponding vector W are obtained as

$$V = [N_{11} f_{11}^1 \quad N_{21} f_{21}^1 \quad N_{31} f_{31}^1 \quad N_{41} f_{41}^1 \quad N_{51} f_{51}^1 \quad N_{01}^{11} f_{01}^1] \tag{67}$$

$$W = [D_{11} f_{11}^1 \quad D_{21} f_{21}^1 \quad D_{31} f_{31}^1 \quad D_{41} f_{41}^1 \quad D_{51} f_{51}^1 \quad D_{01}^{11} f_{01}^1] \tag{68}$$

According to Theorem 4.4, when the parameter vectors $f_{i1}^1,\ i = 0, 1, 2, 3, 4, 5$ satisfy Constraints 4.1–4.3, the general parametric expression for feedback gain matrix K is given by

$$K = WV^{-1} \tag{69}$$

with V and W given by (67) and (68).

The approach in Abdelaziz (2011) can also be used. Abdelaziz (2011) proved the problem of eigenstructure assignment in a state space system that has one zero eigenvalue via state-derivative can be converted, by the Schur decomposition of the state matrix, into that of eigenstructure assignment via state-derivative feedback in its sub-system, given by the matrix pair $(U^{\mathrm{T}}AU, U^{\mathrm{T}}B)$ (Noting that, here, A and B should be replaced by $E^{-1}A$ and $E^{-1}B$), that has no zero eigenvalue. By the Algorithm 2 in Abdelaziz (2011), the sub-block U of the state orthogonal transformation matrix $\tilde{U}$ is obtained as

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and then we can obtain

$$U^{\mathrm{T}}AU = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 5 & -10 & -2.5 & 0.5 & 0 \\ -2.5 & 2.5 & 0.25 & -1.25 & 1 \\ 0 & 0 & 0 & 2/3 & -2/3 \end{bmatrix},$$

$$U^{\mathrm{T}}B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1/3 \end{bmatrix}$$

The polynomial matrices $\tilde{N}(\lambda)$ and $\tilde{D}(\lambda)$ of the subsystem can be obtained as

$$\tilde{N}(\lambda) = \begin{bmatrix} 4\lambda & -8\lambda \\ 0 & -4\lambda \\ 0 & -4\lambda^2 \\ 4\lambda^2 & -8\lambda^2 \\ 4\lambda^3 + 5\lambda^2 + 10\lambda & -8\lambda^3 - 9\lambda^2 - 10\lambda \end{bmatrix}$$

$$\tilde{D}(\lambda) = \begin{bmatrix} 2\lambda + 20 & 4\lambda^2 + 6\lambda \\ -12\lambda^3 - 23\lambda^2 & 24\lambda^3 + 43\lambda^2 \\ -32\lambda - 20 & +32\lambda + 20 \end{bmatrix}$$

Therefore, the closed-loop eigenvectors $\hat{v}_i$, $i = 1, 2, \ldots, 5$, of the sub-system and the corresponding vectors $\hat{w}_i$, $i = 1, 2, \ldots, 5$, are obtained as

$$\hat{v}_i = \tilde{N}(\lambda_i) f_i, \quad \hat{w}_i = \tilde{D}(\lambda_i) f_i, \quad i = 1, 2, \ldots, 5$$

and then the matrices $\hat{V}$ and $\hat{W}$ are obtained as

$$\hat{V} = [\tilde{N}(\lambda_1) f_1 \quad \tilde{N}(\lambda_2) f_2 \quad \tilde{N}(\lambda_3) f_3 \quad \tilde{N}(\lambda_4) f_4 \quad \tilde{N}(\lambda_5) f_5] \tag{70}$$

$$\hat{W} = [\tilde{D}(\lambda_1) f_1 \quad \tilde{D}(\lambda_2) f_2 \quad \tilde{D}(\lambda_3) f_3 \quad \tilde{D}(\lambda_4) f_4 \quad \tilde{D}(\lambda_5) f_5] \tag{71}$$

Finally, the parametric expression for feedback gain matrix K can be given by

$$K = \hat{W} \hat{V}^{-1} U^{\mathrm{T}} \tag{72}$$

with $\hat{V}$ and $\hat{W}$ given by (70) and (71).

It can be easily seen that the number of free parameters in the solution K given by (69) is $5 \times 2 + 1 \times 3 = 13$, while the number of free parameters in the solution (72) is $5 \times 2 = 10 < 13$. Thus, the solution K given by (72) is not the complete solution of the problem. It is well-known that giving the complete solution to the eigenstructure assignment problem is important because a complete solution provides the most freedom.

The degrees of freedom existing in the solution to the eigenstructure assignment problem can be utilised to achieve some desired system specifications and performances. Here we use them to minimise the spectral norm of feedback gain matrix $\|K\|_2$. The spectral norm of a feedback gain matrix represents the amount of energy required for corresponding control action which we usually want to be as small as possible in order to save energy. In the simulations, the Python function `minimize()` available from the `scipy.optimize` package is utilised.

By minimising $\|K\|_2$ with K given by (69), the optimal parameters are obtained as

$$f_{11}^1 = \bar{f}_{21}^1 = \begin{bmatrix} -0.2878 - 1.6460\mathrm{i} \\ 1.1697 + 2.7945\mathrm{i} \end{bmatrix},$$

$$f_{31}^1 = \bar{f}_{41}^1 = \begin{bmatrix} -1.5480 - 1.2324\mathrm{i} \\ 3.5980 + 1.9452\mathrm{i} \end{bmatrix}$$

$$f_{51}^1 = \begin{bmatrix} -2.0157 \\ 1.2585 \end{bmatrix}, \quad f_{01}^1 = \begin{bmatrix} 3.7133 \\ 7.0392 \\ -1.3672 \end{bmatrix}$$

With these parameters, the feedback gain matrix is computed according to (69) and is given by

$$K_{\text{ours}} = \begin{bmatrix} -0.1949672 & 1.3208829 & 1.8956723 & -0.9766918 & -1.3924052 & -0.0593779 \\ 0.5547405 & 0.1177046 & -0.3681900 & 0.4258856 & -0.6505880 & -2.6847464 \end{bmatrix}$$

which corresponds to $\|K_{\text{ours}}\|_2 = 2.8763$. The errors of the closed-loop system are obtained as $10^{-14} \times \{-0.5773 \pm 0.6217\mathrm{i}, 1.4211 \pm 0.6217\mathrm{i}, -5.9508, 0\}$. This verifies that the eigenvalues of the closed-loop system are correctly assigned.

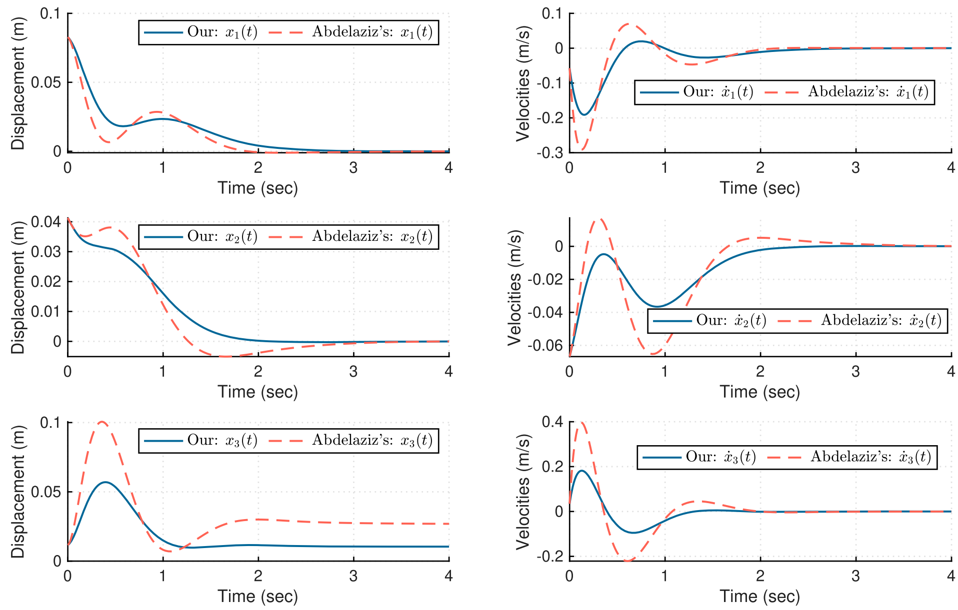

Figure 4. The responses for the closed-loop systems obtained by our approach and Abdelaziz's approach.

By minimising $\|K\|_2$ with K given by (72), the optimal parameters are obtained as

$$f_1 = \bar{f}_2 = -\begin{bmatrix} 7.1467 + 0.8985\mathrm{i} \\ 4.5532 - 2.0624\mathrm{i} \end{bmatrix},$$

$$f_3 = \bar{f}_4 = \begin{bmatrix} 1.9074 - 20.1873\mathrm{i} \\ 3.2646 - 10.0431\mathrm{i} \end{bmatrix}, \quad f_5 = \begin{bmatrix} 82.3049 \\ 53.4502 \end{bmatrix}$$

With these parameters, the feedback gain matrix is computed according to (72) and is given by

$$K_{\text{Abdelaziz}} = \begin{bmatrix} -0.9365290 & 1.5022396 & 0 & -2.2436674 & -0.1588986 & -0.8288533 \\ 1.0747302 & -1.381851 & -2.3158994 & 0.4728714 & 0.4755832 & 0 \end{bmatrix}$$

which corresponds to $\|K_{\text{Abdelaziz}}\|_2 = 2.9800$. The errors of the closed-loop system are obtained as $10^{-12} \times \{-0.0160 \pm 0.1137\mathrm{i}, 0.3411 \pm 0.8118\mathrm{i}, 2.9399, 0\}$. This verifies that the eigenvalues of the closed-loop system are correctly assigned.

By comparison, the norm of the gain matrix K_{ours} is smaller than that of the gain matrix $K_{\text{Abdelaziz}}$. Therefore, our approach possesses better optimality. This is quite reasonable since our approach utilises all the freedom existed in the solution to the eigenstructure assignment problem, but Abdelaziz's approach does not.

The responses of both the closed-loop systems are shown in Figure 4. The initial condition in both cases is taken to be the same as in Subsection 5.1. It can be observed form the figure that, with both the gain matrices, the displacements x_1, x_2 and the velocities $\dot{x}_i, i = 1, 2, 3$, of the closed-loop systems, asymptotically approach zero, while the displacements x_3 of both the closed-loop systems asymptotically approach non-zero steady-state values. This is consistent with the closed-loop systems having zero eigenvalue, which corresponds to the spring parameter $k_3 = 0$.

6. Conclusions

This paper addresses the problem of eigenstructure assignment in linear systems via state-derivative feedback. Firstly, a new derivative feedback design framework named the complementary system (CS) framework is proposed. The CS framework converts the state-derivative feedback related control problem of linear systems into the state feedback related control problem of their complementary systems, and then the relevant theories of state feedback control can be applied. Secondly, using the CS framework, notions of complementary controllability (including complementary S-controllability and complementary C-controllability) and complementary controllability indices of linear systems are introduced, and the limits of state-feedback in altering the dynamics of linear systems are studied. A necessary and sufficient condition for the solvability of the prob-lem of eigenstructure assignment in complementary S-controllable (complementary C-controllable) linear systems, via state-derivative feedback, is given. The condition consists of inequalities which involve the complementary controllability indices of the linear system, a list of the degrees of the invariant polynomials and two lists of non-negative integers, where the list of invariant polynomials and the two lists of non-negative integers represent respectively the finite non-zero (when assigning finite non-zero eigenvalues), zero (when assigning zero eigenvalues), and infinite (when assigning infinite eigenvalues) eigenvalue structure of the closed-loop system. Thirdly, based on a simple complete explicit parametric solution to a group of recursive equations, a complete parametric approach for solving the eigenstructure assignment problem is proposed. General parametric expressions for the closed-loop eigenvectors and the feedback gain matrix are established in terms of certain parameter vectors. The approach assigns arbitrarily rank$A$ closed-loop eigenvalues (If the state matrix A is non-singular, then n closed-loop eigenvalues should be non-zero; if the state matrix A is singular, then $(n - \mathrm{rank}A)$ uncontrollable zero eigenvalues remain unchanged.), and guarantees the closed-loop regularity. The proposed approach generalises and improves the existing results. Finally, the combined problem of simultaneously assigning dynamical order and finite eigenstructure in linear systems by state-derivative feedback is treated. It is shown that the combined problem can be attributed to a special case of the eigenstructure assignment problem. The approach for solving this problem is then established based on the proposed eigenstructure assignment approach. The approach provides maximum design flexibility including full-order eigenstructure assignment, reduced-order eigenstructure assignment and equal-order eigenstructure assignment. The effect of the approach is demonstrated in the design of a three degrees of freedom mass-spring-dashpot system.

The complementary system (CS) framework allows that the designers use the well-known state feedback design methods to directly design state-derivative feedback control systems. It is the authors' belief that the CS framework will be a promising technique for state-derivative feedback related design.

It is shown that the degrees of design freedom existing in the proposed eigenstructure assignment are composed of a group of parameter vectors. By properly restricting these design parameters, closed-loop systems with desired properties can be obtained. Applications of this idea in specific designs of linear control systems, such as robust pole assignment and disturbance attenuation, will be studied in separate papers.

Notice that the expression for the feedback gain matrix K in (64) involves the inverse of the closed-loop eigenvector matrix. When the closed-loop eigenvector matrix is ill-conditioned, using formula (64) to calculate K may result in numerical problems. In the future paper for robust pole assignment, we will improve the condition number of the closed-loop eigenvector matrix by exploring design parameters, and then can improve numerical stability of our approach.

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Disclosure statement

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Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article and its supplementary materials.

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