$$\ln(y) = \beta_0 + \beta_1 \cdot time$$

Problem Set 4

if reg e $Y = \beta_0 \cdot e^{\beta_1 \cdot time}$, get β_0, β_1.

Due Wednesday February 5, 4pm

$\Downarrow$

$\ln y = \underbrace{\ln(\beta_0)}_{\alpha} + \beta_1 \cdot time$,

Data Exercises

(1) Extract data on the aggregate level of U.S. quarterly imports, seasonally adjusted, in real 2017 chained dollars, currently available for 1947Q1 through 2024Q4.[1] The FRED label is IMPGSC1. Create a time index.

(a) For the period 1947 – 2005, plot the level of imports, and its natural log, against time.

(b) By inspection, determine if the imports series is better represented using a linear or exponential trend.

(c) Estimate an exponential trend model using the estimation period 1947 – 2005. Report results. $y_t = T_t + \varepsilon_t$, $\quad T_t = e^{\beta_0 + \beta_1 \cdot t} \Leftrightarrow \ln(T_t) = \beta_0 + \beta_1 t$

(d) Generate point and 90% interval forecasts for the log-level of imports for 2006Q1-2024Q4. Plot the data (log-levels) up to 2005Q4 and your forecast for 2006Q1-2024Q4. Comment. (residual) $\hat{e}_t = y_t - \hat{T}_t$

(e) Plot your forecasts against the actual data (log-levels) for 2006Q1-2024Q4. How did the forecast perform?

(f) Generate point and 90% interval forecasts for the level of imports for 2006 – 2024Q4. Plot the data (levels) up to 2005Q4 and your forecast for 2006Q1-2024Q4. Comment.

(g) Plot your forecasts against the actual data for 2006Q1-2024Q4. How did the forecast perform?

(h) Now re-estimate using the full sample 1947 – 2024Q4. Report results. Generate point and 90% interval forecasts for the level of imports for the next 16 quarters (4 years).

(i) Do the forecasts appear reliable or unreliable?

[1] In matlab you can do this via `fred(...)` and `fetch(.,.)` commands.

Theoretical Questions

(2) *Although this problem can be solved on a computer, try to solve it with a pen and paper (and maybe a calculator) to get a better feel for what is behind trend estimation.* Your friend has a knitting business and she want to estimate a linear trend model without an intercept for her annual profit. She has five years of data:

Time	Profit
2020	300
2021	350
2022	630
2023	780
2024	1020

(a) Estimate a model

$$y_t = \beta \cdot t + \varepsilon_t,\ t = 1, \ldots, 5$$

by minimizing the sum of squared residuals (i.e., by OLS) based on observed data.

(b) Make a 5-year-ahead point forecast of your friend's profit.

(3) In the trend model

$$y_t = T_t + \varepsilon_t, \quad T_t = \beta_0 + \beta_1 Time_t$$

suppose $\beta_1 > 0$.

$y_t = \beta_0 + \beta_1 \text{ Time } t + \varepsilon_t.$

(a) Does this mean that the series y_t is expected to grow or decline in subsequent periods?

When $\beta_1 > 0$, y_t is expected to grow, in subsequent periods. Because $E[y_{T+h} | y_1 \cdots y_T] = \beta_0 + \beta_1 \cdot \text{Time}_{t+h}$, Expected y_t increase as t increase,

(b) Does this mean that the series y_t will grow with certainty in every period?

No, Since ε_t is i.i.d, for example, it's possible that the random ε_{t+1} randonly take a negative value with large absolute value and ε_t randomly take a positive value at t, That may make $y_{t+1} < y_t$, meaning that y_t will not grow. In essense, since ε_t is i.i.d. random variable, it's possible that. $y_t = T_t + \varepsilon_t > y_{t+h} = T_{t+h} + \varepsilon_{t+h}$ even though $T_{t+h} > T_t$. Thus, y_t will not grow with certainty in every point.

(2) *Although this problem can be solved on a computer, try to solve it with a pen and paper (and maybe a calculator) to get a better feel for what is behind trend estimation.* Your friend has a knitting business and she want to estimate a linear trend model without an intercept for her annual profit. She has five years of data:

Time	Profit
2020	300
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$y_t = \beta_1 \cdot t_1 + \varepsilon_t$

(a) Estimate a model

Scaler

$$y_t = \beta \cdot t + \varepsilon_t,\ t = 1, \ldots, 5$$

by minimizing the sum of squared residuals (i.e., by OLS) based on observed data.

(b) Make a 5-year-ahead point forecast of your friend's profit.

(a) Set time =2020 as 1, the base year, then all following years are 2, 3, 4, 5,

For the model $y_t = \beta \cdot t + \varepsilon_t$, we use b to forecast it. Then, we have $y_t = b \cdot t + e_t$,

To min SSR, we want $\min SSR(b) = \min_b \sum_{t=1}^{5} e_t^2 = \sum_{i=1}^{5} (y_t - b \cdot t)^2 =$

$\min_b \ (300 - b)^2 + (350 - 2b)^2 + (630 - 3b)^2 + (780 - 4b)^2 + (1020 - 5b)^2$

F.o.c.: $\frac{\partial SSR(b)}{\partial b} = -2(300 - b) - 4(350 - 2b) - 6(630 - 3b) - 8(780 - 4b) - 10(1020 - 5b) = 0$

$\Rightarrow \quad -600 + 2b - 1400 + 8b - 3780 + 18b - 6240 + 32b - 10200 + 50b = 0$

$110b - 2000 - 10020 - 10200 = 0$

$110b = 22220 \qquad$ So $b = 202$,

Check S.o.c. $\frac{\partial^2 SSR(b)}{\partial b^2} = 110 > 0$, which means the optimal point is a minimizer,

So the optimal b is $\hat{\beta}$ and $\hat{\beta} = 202$

(b) To forecast y_{5+h} we have. $E[y_{5+h} | y_1, y_2, y_3, y_4, y_5] = E[\beta \cdot (5+h) + \varepsilon_{5+h} | y_1 \cdots y_5]$

$= \beta \cdot (5+h) + E[\varepsilon_{5+h} | y_1 \cdots y_5] = \beta \cdot (5+h)$. Since we don't know β, we use $\hat{\beta}$ to estimate

The fitted value, is $\hat{y}_{T+h|h} = \hat{\beta} \cdot (5+h) = 202(5+h)$

$\hat{Y}_{T+1|T} = \hat{\beta} \cdot (T+1) = 202 \cdot 6 = 1212$ (year 6)

$\hat{Y}_{T+2|T} = \hat{\beta} \cdot (T+2) = 202 \cdot 7 = 1414$ (year 7)

$\hat{Y}_{T+3|T} = \hat{\beta} \cdot (T+3) = 202 \cdot 8 = 1616$ (year 8)

$\hat{Y}_{T+4|T} = \hat{\beta} \cdot (T+4) = 202 \cdot 9 = 1818$ (year 9)

$\hat{Y}_{T+5|T} = \hat{\beta} \cdot (T+5) = 202 \cdot 10 = 2020$ (year 10)

(3) In the trend model

$$y_t = T_t + \varepsilon_t, \quad T_t = \beta_0 + \beta_1 Time_t$$

suppose $\beta_1 > 0$.

(a) Does this mean that the series y_t is expected to grow or decline in subsequent periods?

(b) Does this mean that the series y_t will grow with certainty in every period?