4.1. C A & B

$\alpha_1 = 0.1722$ $\beta_1 = 0.96983$ 4.148503

$\alpha_2 = 0.3543$ $\beta_2 = 0.93783$ 4.19940

Problem Set 8

$0.3392047 + 0.94057 \times 4.1 = 4.1955$

$\alpha_3 = 0.52216$ $\beta_3 = 0.90835$

Due Wednesday March 19, 4pm

Data Exercises

(1) The FRED label for the US unemployment rate is UNRATE. The newest observation (Feb 2025) is released on March 7. Based on the AR(1) model with an intercept, derive one-, two-, and three-step-ahead point forecasts for the unemployment rate by 3 different methods we considered: plug-in, iterated, direct. Explain your calculations and discuss the results. (For each method you should report forecasts for 3 periods.)

(2) Take the AR(1) model

$$y_t = \alpha + \beta y_{t-1} + \varepsilon_t$$

where the errors ε_t are i.i.d. white noise $\mathcal{N}(0,1)$.

(a) Set $\alpha = 1$ and $\beta = 0.25$.

(i) Calculate the mean $\mu = E(y_t)$. (*For this part you do not need Matlab.*)

(ii) Simulate a series of length $T = 240$. Set the initial value $y_1 = \mu$ to equal the unconditional mean (from part (a)). (This is similar to problem 4 from the last problem set). Create a time-series plot of your series.

(iii) Estimate an AR(1) model. Are your coefficient estimates close to the true values?

(b) Repeat with $\alpha = 10$, $\beta = 0.9$.

(c) Repeat with $\alpha = 0$, $\beta = -0.5$.

Theoretical Questions

(3) Use the iteration rule to forecast the AR(1) process $y_t = \beta y_{t-1} + \varepsilon_t$. Assume that all parameters are known.

(a) Show that the optimal forecasts are

$$\begin{aligned}
y_{T+1|T} &= \beta y_T \\
y_{T+2|T} &= \beta^2 y_T \\
&\cdots \\
y_{T+h|T} &= \beta^h y_T
\end{aligned}$$

(b) Show that the corresponding forecast errors are

$$
\begin{aligned}
u_{T+1|T} &= y_{T+1} - y_{T+1|T} = \varepsilon_{T+1} \\
u_{T+2|T} &= y_{T+2} - y_{T+2|T} = \varepsilon_{T+2} + \beta\varepsilon_{T+1} \\
&\dots \\
u_{T+h|T} &= y_{T+h} - y_{T+h|T} = \varepsilon_{T+h} + \beta\varepsilon_{T+h-1} + \cdots + \beta^{h-1}\varepsilon_{T+1}
\end{aligned}
$$

(c) Show that the forecast error variances are

$$
\begin{aligned}
\sigma_1^2 &= \sigma^2 \\
\sigma_2^2 &= \sigma^2\left(1+\beta^2\right) \\
&\dots \\
\sigma_h^2 &= \sigma^2\left(1+\beta^2+\cdots+\beta^{2h-2}\right) = \sigma^2\sum_{i=0}^{h-1}\beta^{2i}
\end{aligned}
$$

(d) Show that the limiting forecast error variance is

$$\lim_{h\to\infty}\sigma_h^2 = \frac{\sigma^2}{1-\beta^2}$$

the unconditional variance of the AR(1) process.

(4) Take the AR(1) model

$$y_t = \alpha + \beta y_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim i.i.d.(0,\sigma^2).$$

Explain why the variance of the forecast error from a two-step-ahead forecast is larger than the variance of the forecast error from the one-step-ahead forecast.

4. For AR(1), the forecase error for 1-Step and 2-Step, Similar to Q3, we have

$Var(u_{T+1|T}) = Var(y_{T+1} - y_{T+1|T}) = Var(\varepsilon_{T+1}) = \sigma^2$

$Var(u_{T+2|T}) = Var(y_{T+2} - y_{T+2|T}) = \beta^2 \cdot Var(\varepsilon_{T+1}) + Var(\varepsilon_{T+2}) + \underbrace{Cov(\beta\varepsilon_{T+1}, \varepsilon_{T+2})}_{=0 \text{ since } \varepsilon_t \text{ is WN.}}$

$= \beta^2 \cdot \sigma^2 + \sigma^2 = (1+\beta^2)\cdot\sigma^2$

Since $\beta^2 \geq 0$ and $\sigma^2 \geq 0$, $Var(u_{T+2|T}) \geq Var(u_{T+1|T})$.

Thus, the varience of forecast error from 2-Step is larger than 1-Step forecast.

3, AR(1): $y_t = \beta y_{t-1} + \varepsilon_t$.

(a). By default, we have quadratic loss function so the optimal forecast point is conditional expectation,

$$y_{T+1|T} = \underbrace{E[y_{T+1}|\Omega_T]}_{①} = E[\beta y_T + \varepsilon_{T+1}|\Omega_T] = \beta \cdot y_T + 0 = \beta y_T,$$

$$y_{T+2|T} = E[y_{T+2}|\Omega_T] = E[\beta y_{T+1} + \varepsilon_{T+2}|\Omega_T] = \beta \cdot \underbrace{E[y_{T+1}|\Omega_T]}_{①} + 0$$

$$= \beta \cdot \beta y_T = \beta^2 y_T$$

... Inductively, we have

$$y_{T+h|T} = E[y_{T+h}|\Omega_T] = \beta \cdot E[y_{T+h-1}|\Omega_T] + 0 = \beta^2 \cdot E[y_{T+h-2}|\Omega_T] + 0 = \cdots$$

$$= \beta^{h-1} \cdot E[y_{T+h-h+1}|\Omega_T] + 0 = \beta^h \cdot E[y_T|\Omega_T] = \beta^h \cdot y_T. \qquad \square,$$

(b), by (a), we can calculate forecast errors;

$$u_{T+1|T} = y_{T+1} - y_{T+1|T} = (\beta y_T + \varepsilon_{T+1}) - \beta y_T = \varepsilon_{T+1}$$

$$u_{T+2|T} = y_{T+2} - y_{T+2|T} = (\beta^2 y_T + \beta \cdot \varepsilon_{T+1} + \varepsilon_{T+2}) - \beta^2 y_T = \beta \varepsilon_{T+1} + \varepsilon_{T+2}$$

... Inductively, we have

$$u_{T+h|T} = y_{T+h} - y_{T+h|T} = (\beta \cdot y_{T+h-1} + \varepsilon_{T+h}) - \beta^h \cdot y_T = [\beta \cdot (\beta \cdot y_{T+h-2} + \varepsilon_{T+h-1}) + \varepsilon_{T+h}] - \beta^h \cdot y_T$$

$$= [\beta^2 (y_{T+h-3} + \varepsilon_{T+h-2}) + \beta \cdot \varepsilon_{T+h-1} + \varepsilon_{T+h}] - \beta^h \cdot y_T = \cdots$$

$$= [\underline{\beta^h \cdot y_T + \beta^{h-1}} \cdot \varepsilon_{T+1} + \cdots + \beta \varepsilon_{T+h-1} + \varepsilon_{T+h}] - \beta^h \cdot y_T$$

$$= \beta^{h-1} \cdot \varepsilon_{T+1} + \cdots + \beta \varepsilon_{T+h-1} + \varepsilon_{T+h}.$$

(c), WTS Var(u)

$$\sigma_1^2 = Var(u_{T+1|T}) = Var(\varepsilon_{T+1}) = \sigma^2$$

$$\sigma_2^2 = Var(u_{T+2|T}) = Var(\beta \varepsilon_{T+1} + \varepsilon_{T+2}) = \beta^2 \cdot Var(\varepsilon_{T+1}) + Var(\varepsilon_{T+2}) + \underbrace{Cov(\beta \varepsilon_{T+1}, \varepsilon_{T+2})}_{\text{equal 0 when } \varepsilon_t \text{ is WN.}}$$

$$= \beta^2 \cdot \sigma^2 + \sigma^2 = (1+\beta^2) \cdot \sigma^2$$

⋮

$$\sigma_h^2 = Var(u_{T+h|T}) = Var(\beta^{h-1} \cdot \varepsilon_{T+1} + \cdots + \beta \varepsilon_{T+h-1} + \varepsilon_{T+h}) = (\beta^{h-1})^2 \cdot Var(\varepsilon_{T+1}) + \cdots +$$

$$\beta^2 Var(\varepsilon_{T+h-1}) + Var(\varepsilon_{T+h}) = (\beta^{h-1})^2 \cdot \sigma^2 + \cdots + (\beta^1)^2 \cdot \sigma^2 + (\beta^0)^2 \sigma^2 = \sigma^2 \cdot \sum_{i=0}^{h-1} (\beta^i)^2$$

(d). $\sigma_h^2 = \sigma^2 \cdot \sum_{i=0}^{h-1} \beta^{2i}$ by geometric series sum formula, set $r = \beta^2$

$$\sigma_h^2 = \sigma^2 \cdot \sum_{i=0}^{h-1} r^i = \sigma^2 \cdot \frac{1 - r^h}{1 - r} = \sigma^2 \cdot \frac{1 - \beta^{2h}}{1 - \beta^2}$$

As $h \to \infty$ $\sigma_h^2 = \sigma^2 \cdot \frac{1 - \beta^{2h}}{1 - \beta^2} \longrightarrow \sigma^2 \cdot \frac{1}{1 - \beta^2}$ when $|\beta| < 1$. ($\beta^{2h} \to 0$ as $|\beta| < 1$)

Thus. $\lim_{h \to \infty} \sigma_h^2 = \frac{\sigma^2}{1 - \beta^2}$ when $|\beta| < 1$.

Since $Var(u_{T+h|T}) = Var(\beta^{h-1} \cdot \varepsilon_{T+1} + \cdots + \beta \varepsilon_{T+h-1} + \varepsilon_{T+h}) =$

$Var(\beta^h \cdot y_T + \beta^{h-1} \cdot \varepsilon_{T+1} + \cdots + \beta \varepsilon_{T+h-1} + \varepsilon_{T+h}) = Var(y_{T+h|T})$ Since y_T is realization (specific value) $\beta^h \cdot y_T$ is a fixed.

So $\frac{\sigma^2}{1 - \beta^2}$ is the unconditional varience of AR(1) model (y_{T+h} of AR(1) model)