Problem Set 3

Due Wednesday January 29, 4pm

Data Exercises

(1) The Census Bureau (www.census.gov) released data on Housing Starts (look for the report on "New Residential Construction" under Topics/Business and Economy/Economic Indicators) for December 2024 on January 17, 2025.

  (a) What was the number of seasonally adjusted units of "New Privately-Owned Housing Units Started " in December 2024 for the U.S.?

  (b) What was the estimated percentage change for December 2024 from December 2023 for the U.S.?

  (c) What was the estimated percentage change for the South region?

  (d) Notice that the Census Bureau gives confidence intervals for these percentages. Why do you think they do this? What was the confidence interval for the South region? How does this change your interpretation of the estimate?

(2) The Census Bureau released data on Home Ownership Rates on October 29. (You can find the release using the same steps as above.)

  (a) What was the rental vacancy rate in the third quarter of 2024?

  (b) Using Table 2 from this release, what is the "margin of error" for this estimate? What is the meaning of the "margin of error"?

  (c) Calculate a 90% confidence interval for the vacancy rate.

(3) The Excel file "realgdpgrowth.xlsx" is posted on Canvas. It contains quarterly series of U.S. national accounts data, from 1947q2 through 2024q3. All are real percent changes from the previous period, seasonally adjusted. The time index is "date".

  (a) The series "pce_nondurables" is personal consumption expenditures on non-durable goods. Estimate the mean of the series, and plot the series along with the fitted mean.

  (b) Using the constant mean model, generate point and interval forecasts for non-durables for the next 4 quarters. Plot your forecasts. Discuss.

Theoretical Questions

(4) Let y_t be a time series with a constant mean $\mathbb{E}(y_t) = \mu$. Show that the sample mean $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$ is an unbiased estimator for μ. Does the proof change in the time-series case compared to the i.i.d. data setting?

(5) (a) Let y_t be a time series with a constant mean $\mathbb{E}(y_t) = \mu$, a constant variance $\mathbb{V}(y_t) = \sigma^2$, and $Cov(y_t, y_s) = 0$ for $t \neq s$. Show that the variance of the sample mean $\bar{y}_T$ is σ^2/T. Does the proof change in the time-series case compared to the i.i.d. data setting?

(b) Let z_t be a time series with a constant mean $\mathbb{E}(z_t) = \mu$ and a constant variance $\mathbb{V}(z_t) = \sigma^2$. Does $\mathbb{V}\bar{z}_T = \sigma^2/T$? Prove or provide a counterexample.

(6) Suppose that (X, Y) are jointly normal with $\mathbb{E}X = \mathbb{E}Y = 0$, $\mathbb{E}X^2 = 2$, $\mathbb{E}Y^2 = 1$, $\mathbb{E}XY = 1$. That is the joint density of (X, Y) is

$$f_{X,Y}(x, y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

and the density of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} y^2 \right).$$

Find the conditional density of X given $Y = y$, $f_{X|Y}(x|y)$. Interpret.

(4) Let y_t be a time series with a constant mean $\mathbb{E}(y_t) = \mu$. Show that the sample mean $\bar{y}_T = \frac{1}{T}\sum_{t=1}^{T} y_t$ is an unbiased estimator for μ. Does the proof change in the time-series case compared to the i.i.d. data setting?

(a) To show $\bar{Y}_T$ is an unbiased estimator for μ, we WTS $E[\bar{Y}_T] = \mu$.

$E[\bar{Y}_T] = E[\frac{1}{T}\sum_{t=1}^{T} Y_t] = \frac{1}{T}\cdot\sum_{t=1}^{T} E[Y_t]$, since we know $E(Y_t) = \mu$

$\frac{1}{T}\cdot\sum_{t=1}^{T} E[Y_t] = \frac{1}{T}\cdot\sum_{t=1}^{T}\cdot\mu = \frac{1}{T}\cdot T\cdot\mu = \mu$, Thus, $E[\bar{Y}_T] = \mu$ and therefore $\bar{Y}_T$ is unbiased estimator

(b). The proof doesn't change in time-series because above proof only utilize the linearity of expectation and the fact that $E(Y_t) = \mu$ as we stated. It's not about the property from some data type (i.i.d or time series).

(5) (a) Let y_t be a time series with a constant mean $\mathbb{E}(y_t) = \mu$, a constant variance $\mathbb{V}(y_t) = \sigma^2$, and $Cov(y_t, y_s) = 0$ for $t \neq s$. Show that the variance of the sample mean $\bar{y}_T$ is σ^2/T. Does the proof change in the time-series case compared to the i.i.d. data setting?

(b) Let z_t be a time series with a constant mean $\mathbb{E}(z_t) = \mu$ and a constant variance $\mathbb{V}(z_t) = \sigma^2$. Does $\mathbb{V}\bar{z}_T = \sigma^2/T$? Prove or provide a counterexample.

$Y_t = Y_1, Y_2, Y_3, \dots Y_T$

$Var(A, B) = Var(A) + Var(B) + 2\,Cov(A, B)$.

(a) $Var(\bar{Y}_T) = Var(\frac{1}{T}\sum_{i=1}^{T} Y_t) = (\frac{1}{T})^2\cdot Var(\sum_{i=1}^{T} Y_t) = \frac{1}{T^2}\cdot[Var(Y_1) + Var(Y_2) + \cdots + Var(Y_T) + Cov(Y_1, Y_2) + Cov(Y_1, Y_3) + \cdots + Cov(Y_2, Y_1) + Cov(Y_2, Y_3) + \cdots + Cov(Y_T, Y_1) + Cov(Y_T, Y_2) + \cdots + Cov(Y_T, Y_{T-1})]$

Since $Cov(Y_t, Y_s) = 0$ for $t \neq s$, we have $Var(\bar{Y}_T) = \frac{1}{T^2}\cdot\sum_{i=1}^{T} Var(Y_t) = \frac{1}{T^2}\cdot\sum_{t=1}^{T}\sigma^2 = \sigma^2/T$

The proof doesn't change in time series case.

(b) No, prove by counterexample. Based on the result of (a), we have

$Var(\bar{Y}_T) = \frac{1}{T^2}[\sigma^2\cdot T + \sum_{t\neq s} Cov(Z_t, Z_s)]$. Suppose each Z_a are dependent to other Z_b $\forall a \neq b$

Since we have $Var(Z_t) = \sigma^2$, which means for all t, Z_t have the same variance. For example, suppose Z_t, Z_k $\forall t\neq k$ are perfectly correlated, so $Cov(Z_t, Z_s) = 1\cdot\sqrt{Var(Z_t)}\cdot\sqrt{Var(Z_t)} = \sigma^2$

Then, we have $Var(\bar{Y}_T) = \sigma^2/T + \sum_{t\neq s}\sigma^2/T^2 > \sigma^2/T$ since $\sum_{t\neq s}\sigma^2/T^2 > 0$.

Therefore, $Var(\bar{Y}_T) \neq \sigma^2/T$.

(6) Suppose that (X, Y) are jointly normal with $\mathbb{E}X = \mathbb{E}Y = 0$, $\mathbb{E}X^2 = 2$, $\mathbb{E}Y^2 = 1$, $\mathbb{E}XY = 1$. That is the joint density of (X, Y) is

$$f_{X,Y}(x, y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2} \underbrace{\begin{pmatrix} x & y \end{pmatrix}}_{1\times 2} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}_{2\times 2} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{2\times 1} \right)$$

and the density of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}y^2\right).$$

Find the conditional density of X given $Y = y$, $f_{X|Y}(x|y)$. Interpret.

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\cdot(x-y, 2y-x)\cdot\begin{pmatrix} x \\ y \end{pmatrix}\right) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}\cdot(x^2 - xy + 2y^2 - xy)\right) = \frac{1}{2\pi} \exp\left(-\frac{1}{2}x^2 + xy - y^2\right)$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{2\pi}\exp(-\frac{1}{2}x^2 + xy - y^2)}{\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}y^2)} = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{1}{2}x^2 + xy - \frac{1}{2}y^2\right)$$

The pdf of Normal distribution is $\frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-(x-\mu)^2/2\sigma^2\right)$, we write $f_{X|Y}(x|y)$ into

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(x^2 - 2xy + y^2)}{2}\right) = \frac{1}{\sqrt{2\pi}} \cdot \exp\left(-\frac{(x-y)^2}{2}\right)$$

Thus, we can say the conditional density $f_{X|Y}(x|y)$ is the density of a normal random variable, with conditional mean $\mu_{X|Y} = y$ and varience $\sigma^2 = 1$.

When give $Y = y$, we have $X|Y = y \sim N(y, 1)$.