[INFO-F409] Learning Dynamics
First assignment

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1 The Hawk-Dove game

Conventions and notations

First of all, the Hawk-Dove game is modeled by the matrix presented in the Table 1.

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V, 0
Dove	0, V	$\frac{V}{2}-T$, $\frac{V}{2}-T$

Table 1 – Payoff matrix of the Hawk-Dove game

The different actions of a player $i \in \{1,2\}$ are denoted by the set $\mathcal{A} = \{H, D\}$ where H is the hawk action and D the dove action. Moreover, to denote the different actions payoff, we need an utility function of those actions. This utility function is then written :

$$u_i(a_i, a_{-i})$$

such that a_i is an action of player i and a_{-i} is the action of the other player where player i choses a_i. For instance, $u_1(Hawk, Dove) = V$ and $u_2(Hawk, Dove) = 0$).

In the case of mixed strategies, the notion of **expected value** of a payoff function is used and is written in the general case :

$$U_i(p_1, ..., p_k) = p_k u(a_k)$$

where i is a player, p_k is the probability that the other player chose the action a_k. In the case of the Hawk-Dove game, the expected value formula would be written such that $k = 2$ because it exists only 2 actions, i.e. :

$$U_i(p_1, p_2) = p_1 u(a_1) + p_2 u(a_2)$$

Furthermore, to find Nash equilibria, best responses have to be found. Those one will be highlighted in red for the line player considered as player one and green for the column player considered as player two.

1.1 Question 1 - Nash equilibrium

Statement

Find all the (mixed strategy) Nash equilibria of this game. How do the results change when the order of the parameters V, D and T is changed (V>D, D>T, etc.)?

Sign of V, D and T

Before starting the analysis, we should remind that V and D are representing respectively the fitness value of winning resources in fight and costs of injury thereby it would not make sense having negative values. V and D are then always positive values. Same for T which is the cost of wasting time. The time wasted is obviously always a positive value. To summarize,

$$V, D, T \geq 0$$

1.1.1 First case : $V > D$

In the first case, we consider that $V > D$. Therefore, we know that the value of $u_i(Hawk, Hawk) = \frac{V-D}{2}$ will be strictly positive. In this case, the choice of both players are quiet easy. As illustrated in Table 2, the Nash equilibrium is $(Hawk, Hawk) \in \mathcal{A}$. Indeed, both players are chosing $Hawk$ because if one of them is switching to $Dove$ then $u_i(Hawk, Hawk)$ which was a positive value becomes $u_1(Dove, Hawk)$ for player 1 or $u_2(Hawk, Dove)$ for player 2 which equals zero. Thus, they obviously prefer to pick $(Hawk, Hawk)$.

Nash equilibrium when $(V > D) = \{(Hawk, Hawk)\}$

	Hawk	Dove
Hawk	$\frac{V-D}{2} > 0$ $\frac{V-D}{2} > 0$	0 V
Dove	V 0	$\frac{V}{2} - T$ $\frac{V}{2} - T$

Table 2 – Nash equilibrium/Best responses in case 1

1.1.2 Second case : $V = D$

The second case considers the same value for V and D which means that $\frac{V-D}{2} = 0$. The Table 3 shows the Nash equilibria for that case. We can see that there doesn't exist only one Nash equilibrium, but three. $(Hawk, Hawk)$ stays a Nash equilibrium in this case, but $(Dove, Hawk)$ and $(Hawk, Dove)$ are now also Nash equilibria. In other words, if player 2 is chosing $Hawk$, the best response to it is whether $(Hawk, Hawk)$ or $(Dove, Hawk)$ because $u_1(Hawk, Hawk) = u_1(Dove, Hawk) = 0$. On the other hand, if player 2 is chosing $Dove$, the best response is only $(Hawk, Dove)$ since $u_1(Hawk, Dove) > u_1(Dove, Dove) \equiv V > \frac{V}{2} - T$. By symmetry, it is also valid for the opposite case where player 1's choice is known, so that the best responses are (symmetrically) equivalent which means $(Hawk, Hawk)$,$(Hawk, Dove)$ and $(Dove, Hawk)$ are the best responses for player 2.

Nash equilibria when $(V = D) = \{(Hawk, Hawk), (Hawk, Dove), (Dove, Hawk)\}$

	Hawk	Dove
Hawk	$\frac{V-D}{2} = 0$ $\frac{V-D}{2} = 0$	0 V
Dove	V 0	$\frac{V}{2} - T$ $\frac{V}{2} - T$

Table 3 – Nash equilibria/Best responses in case 2

1.1.3 Third case : $V < D$

Finally, we are now considering that $V < D$. Therefore, $u_i(Hawk, Hawk) = \frac{V-D}{2}$ is now strictly a negative value. The Table 4 is showing the Nash equilibria for this case. Compared to the previous case, $(Hawk, Hawk)$ is not a Nash equilibrium anymore but $(Hawk, Dove)$ and $(Dove, Hawk)$ keep there. Indeed, when player 2 is playing $Hawk$, player 1 would play $Dove$ since $u_1(Hawk, Hawk) < u_1(Dove, Hawk) \equiv \frac{V-D}{2} < 0$. In the case where player 2 is playing $Dove$, it doesn't change, it means that $u_1(Hawk, Dove) = V$ is still greater than $u_1(Dove, Dove) = \frac{V}{2} - T$. As reasoned in the previous case, this is also valid when player 2 is depending of player 1's choice thanks to the matrix's symmetry.

$$\boxed{\text{Nash equilibria when } (V < D) = \{(Hawk, Dove), (Dove, Hawk)\}}$$

	Hawk	Dove
Hawk	$\frac{V-D}{2} < 0$ $\frac{V-D}{2} < 0$	0 V
Dove	V 0	$\frac{V}{2} - T$ $\frac{V}{2} - T$

Table 4 – Nash equilibria/Best responses in case 3

1.1.4 What about the value of T ?

As you can see, the different cases are not depending of the value of T. In each case, $(Dove, Dove)$ will never be a Nash equilibrium since that $\frac{V}{2} - T$ will always be smaller than V (as reminder, V and $T > 0$), then $(Dove, Dove)$ will never be the best response.

1.1.5 Mixed strategy

To find the mixed strategy Nash equilibrium, we have to take into account the probabilities p and q which are respectively the probabilities of playing a certain action for player 1 and player 2. The general probabilities of a 2x2 game's matrix is shown in the Figure 1.

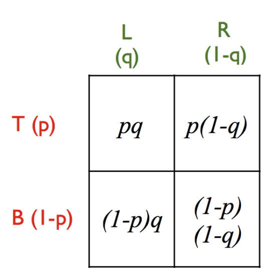

Figure 1 – Probabilities of a 2x2 game's matrix

Thus, we have to compute the expected value of each case which means the player 1's expected payoff for the pure strategy $Hawk(p)$ and $Dove(1-p)$, same for player 2 i.e $Hawk(q)$ and $Dove(1-q)$.
Therefore, for player 1 we obtain those equations [1] :

$$U_1^H = q \cdot (\frac{V-D}{2}) + (1-q) \cdot V \tag{1}$$

$$U_1^D = (1-q) \cdot (\frac{V}{2} - T) \tag{2}$$

To find the value of probability q, we equalize the two equations to isolate q :

$$\begin{aligned} q \cdot (\frac{V-D}{2}) + (1-q) \cdot V &= (1-q) \cdot (\frac{V}{2} - T) \\ \frac{qV}{2} - \frac{qD}{2} + V - pV &= \frac{V}{2} - T - \frac{qV}{2} + qT \\ -\frac{qD}{2} &= -\frac{V}{2} - T + pT \\ \frac{qD}{2} + pT &= \frac{V}{2} + T \\ q \cdot (\frac{D}{2} + T) &= \frac{V}{2} + T \\ q &= \frac{\frac{V}{2} + T}{\frac{D}{2} + T} \end{aligned} \tag{3}$$

$$\boxed{q = \frac{V+2T}{D+2T}} \tag{4}$$

By symmetry, we can deduct the equations for player 2 :

$$U_2^H = p \cdot (\frac{V-D}{2}) + (1-p) \cdot V \tag{5}$$

$$U_2^D = (1-p) \cdot (\frac{V}{2} - T) \tag{6}$$

therefore, the value of p will be the same as q too :

$$p = \frac{V+2T}{D+2T} \tag{7}$$

which means that :

$$p = q = \frac{V+2T}{D+2T} \tag{8}$$

1.2 Question 2 - Mixed strategy drawing

Statement

Under which conditions does displaying become more beneficial than escalating? Draw the set of all mixed strategies.

2 Which social dilemma ?

3 Games in finite population

[1] U_i^j means the player i's expected payoff for the pure strategy j