[INFO-F409] Learning Dynamics
First assignment

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1 The Hawk-Dove game

Conventions and notations

First of all, the Hawk-Dove game is modeled by the matrix presented in the Table 1.

	Hawk	Dove
Hawk	$\frac{V-D}{2}$, $\frac{V-D}{2}$	V , 0
Dove	0 , V	$\frac{V}{2}-T$, $\frac{V}{2}-T$

Table 1 – Payoff matrix of the Hawk-Dove game

The different actions of a player $i \in \{1, 2\}$ are denoted by the set $\mathcal{A} = \{H, D\}$ where H is the hawk action and D the dove action. Moreover, to denote the different actions payoff, we need an utility function of those actions. This utility function is then written :

$$u_i(a_i, a_{-i})$$

such that a_i is an action of player i and a_{-i} is the action of the other player where player i choses a_i. For instance, $u_1(Hawk, Dove) = V$ and $u_2(Hawk, Dove) = 0$).

In the case of mixed strategies, the notion of **expected value** of a payoff function is used and is written in the general case :

$$U_i(p_1, ..., p_k) = p_k u(a_k)$$

where i is a player, p_k is the probability that the other player chose the action a_k. In the case of the Hawk-Dove game, the expected value formula would be written such that $k = 2$ because it exists only 2 actions, i.e. :

$$U_i(p_1, p_2) = p_1 u(a_1) + p_2 u(a_2)$$

Furthermore, to find Nash equilibria, best responses have to be found. Those one will be highlighted in red for the line player considered as player one and green for the column player considered as player two.

1.1 Question 1 - Nash equilibrium

Statement

Find all the (mixed strategy) Nash equilibria of this game. How do the results change when the order of the parameters V, D and T is changed (V>D, D>T, etc.)?

Sign of V, D and T

Before starting the analysis, we should remind that V and D are representing respectively the fitness value of winning resources in fight and costs of injury thereby it would not make sense having negative values. V and D are then always positive values. Same for T which is the cost of wasting time. The time wasted is obviously always a positive value. To summarize,

$$V, D, T \geq 0$$

1.1.1 First case : $V > D$

In the first case, we consider that $V > D$. Therefore, we know that the value of $u_i(Hawk, Hawk) = \frac{V-D}{2}$ will be strictly positive. In this case, the choice of both players are quiet easy. As illustrated in Table 2, the Nash equilibrium is $(Hawk, Hawk) \in \mathcal{A}$. Indeed, both players are chosing $Hawk$ because if one of them is switching to $Dove$ then $u_i(Hawk, Hawk)$ which was a positive value becomes $u_1(Dove, Hawk)$ for player 1 or $u_2(Hawk, Dove)$ for player 2 which equals zero. Thus, they obviously prefer to pick $(Hawk, Hawk)$.

$\boxed{\text{Nash equilibrium when } (V > D) = \{(Hawk, Hawk)\}}$

	Hawk	Dove
Hawk	$\frac{V-D}{2} > 0$ $\frac{V-D}{2} > 0$	0 V
Dove	V 0	$\frac{V}{2} - T$ $\frac{V}{2} - T$

Table 2 – Nash equilibrium/Best responses in case 1

1.1.2 Second case : $V = D$

The second case considers the same value for V and D which means that $\frac{V-D}{2} = 0$. The Table 3 shows the Nash equilibria for that case. We can see that there doesn't exist only one Nash equilibrium, but three. $(Hawk, Hawk)$ stays a Nash equilibrium in this case, but $(Dove, Hawk)$ and $(Hawk, Dove)$ are now also Nash equilibria. In other words, if player 2 is chosing $Hawk$, the best response to it is whether $(Hawk, Hawk)$ or $(Dove, Hawk)$ because $u_1(Hawk, Hawk) = u_1(Dove, Hawk) = 0$. On the other hand, if player 2 is chosing $Dove$, the best response is only $(Hawk, Dove)$ since $u_1(Hawk, Dove) > u_1(Dove, Dove) \equiv V > \frac{V}{2} - T$. By symmetry, it is also valid for the opposite case where player 1's choice is known, so that the best responses are (symmetrically) equivalent which means $(Hawk, Hawk)$,$(Hawk, Dove)$ and $(Dove, Hawk)$ are the best responses for player 2.

$\boxed{\text{Nash equilibria when } (V = D) = \{(Hawk, Hawk), (Hawk, Dove), (Dove, Hawk)\}}$

	Hawk	Dove
Hawk	$\frac{V-D}{2} = 0$ $\frac{V-D}{2} = 0$	0 V
Dove	V 0	$\frac{V}{2} - T$ $\frac{V}{2} - T$

Table 3 – Nash equilibria/Best responses in case 2

1.1.3 Third case : $V < D$

Finally, we are now considering that $V < D$. Therefore, $u_i(Hawk, Hawk) = \frac{V-D}{2}$ is now strictly a negative value. The Table 4 is showing the Nash equilibria for this case. Compared to the previous case, $(Hawk, Hawk)$ is not a Nash equilibrium anymore but $(Hawk, Dove)$ and $(Dove, Hawk)$ keep there. Indeed, when player 2 is playing $Hawk$, player 1 would play $Dove$ since $u_1(Hawk, Hawk) < u_1(Dove, Hawk) \equiv \frac{V-D}{2} < 0$. In the case where player 2 is playing $Dove$, it doesn't change, it means that $u_1(Hawk, Dove) = V$ is still greater than $u_1(Dove, Dove) = \frac{V}{2} - T$. As reasoned in the previous case, this is also valid when player 2 is depending of player 1's choice thanks to the matrix's symmetry.

$$\boxed{\text{Nash equilibria when } (V < D) = \{(Hawk, Dove), (Dove, Hawk)\}}$$

	Hawk	Dove
Hawk	$\frac{V-D}{2} < 0$ $\frac{V-D}{2} < 0$	0 V
Dove	V 0	$\frac{V}{2} - T$ $\frac{V}{2} - T$

Table 4 – Nash equilibria/Best responses in case 3

1.1.4 What about the value of T ?

As you can see, the different cases are not depending of the value of T. In each case, $(Dove, Dove)$ will never be a Nash equilibrium since that $\frac{V}{2} - T$ will always be smaller than V (as reminder, V and $T > 0$), then $(Dove, Dove)$ will never be the best response.

1.1.5 Mixed strategy

To find the mixed strategy Nash equilibrium, we have to take into account the probabilities p and q which are respectively the probabilities of playing a certain action for player 1 and player 2. The general probabilities of a 2x2 game's matrix is shown in the Figure 1.

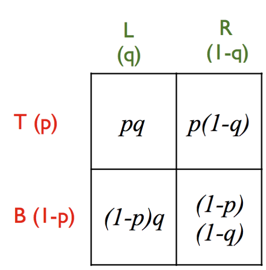

Figure 1 – Probabilities of a 2x2 game's matrix

Thus, we have to compute the expected value of each case which means the player 1's expected payoff for the pure strategy $Hawk(p)$ and $Dove(1-p)$, same for player 2 i.e $Hawk(q)$ and $Dove(1-q)$.

Therefore, for player 1 we obtain those equations [1] :

$$U_1^H = q \cdot (\frac{V-D}{2}) + (1-q) \cdot V \tag{1}$$

$$U_1^D = (1-q) \cdot (\frac{V}{2} - T) \tag{2}$$

To find the value of probability q, we equalize the two equations to isolate q :

$$\begin{aligned} q \cdot (\frac{V-D}{2}) + (1-q) \cdot V &= (1-q) \cdot (\frac{V}{2} - T) \\ \frac{qV}{2} - \frac{qD}{2} + V - pV &= \frac{V}{2} - T - \frac{qV}{2} + qT \\ -\frac{qD}{2} &= -\frac{V}{2} - T + pT \\ \frac{qD}{2} + pT &= \frac{V}{2} + T \\ q \cdot (\frac{D}{2} + T) &= \frac{V}{2} + T \\ q &= \frac{\frac{V}{2} + T}{\frac{D}{2} + T} \\ \Aboxed{q &= \frac{V+2T}{D+2T}} \end{aligned} \tag{3, 4}$$

By symmetry, we can deduct the equations for player 2 :

$$U_2^H = p \cdot (\frac{V-D}{2}) + (1-p) \cdot V \tag{5}$$

$$U_2^D = (1-p) \cdot (\frac{V}{2} - T) \tag{6}$$

therefore, the value of p will be the same as q too :

$$p = \frac{V+2T}{D+2T} \tag{7}$$

which means that :

$$p = q = \frac{V+2T}{D+2T} \in [0,1] \tag{8}$$

Therefore, for player one (two), when :

- $q(p) > \frac{V+2T}{D+2T}$, the best response set is $Hawk$ or $p(q) = 1$
- $q(p) = \frac{V+2T}{D+2T}$, the best response set is the set of all $p(q)$ values in [0,1]
- $q(p) < \frac{V+2T}{D+2T}$, the best response set is $Dove$ or $p(q) = 0$

[1] U_i^j means the player i's expected payoff for the pure strategy j

1.2 Question 2 - Mixed strategy drawing

Statement

Under which conditions does displaying become more beneficial than escalating? Draw the set of all mixed strategies.

Graph of all mixed strategies

Based on the calculations of probabilities p and q in the subsubsection 1.1.5, the graph of all mixed strategies for the Hawk-Dove game is presented in the Figure 2. The green lines are the set of player 1's best mixed strategies while the red ones are the set of player 2's best mixed strategies. The blue circles are the mixed strategies Nash equilibria of the game. Thus, we have three possible Nash equilibria that are defined by the sets $\{(1,0);(1,0)\}$, $\{(0,1),(0,1)\}$ and $\{(\frac{V+2T}{D+2T}, \frac{V+2T}{D+2T}), (\frac{V+2T}{D+2T}, \frac{V+2T}{D+2T})\}$.

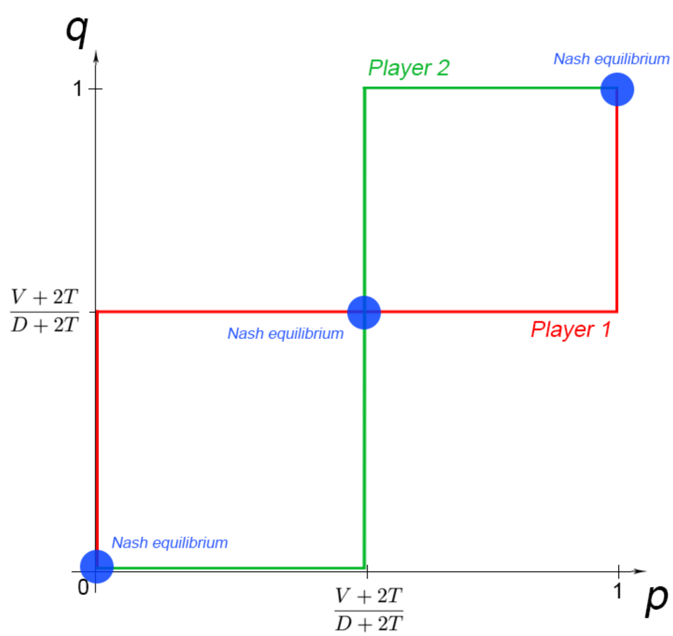

Figure 2 – Mixed strategies graph of the Hawk-Dove game

Remark Note that $\frac{V+2T}{D+2T}$ has to be a value between 0 and 1 otherwise it wouldn't be a valid probability. Therefore, to respect this condition, necessarily $V \leq D$.

To conclude, as calculated in the subsubsection 1.1.5 and represented in the Figure 2, if a player i plays $Hawk$ with a probability $> \frac{V+2T}{D+2T}$ then the opponent player would play $Hawk$ because the expected value for playing $Hawk$ is higher than the expected value of playing $Dove$. Similarly, playing $Dove$ is then an optimal choice when the player i chooses $Hawk$ with a probability $< \frac{V+2T}{D+2T}$.

2 Which social dilemma ?

Statement

Player A knows he's confronted with one of three social dilemma's; a prisoner's dilemma, a snowdrift game or stag-hunt game (see above). In each game he needs to decide whether to cooperate (C) or defect (D), yet he is not sure in which he actually is. He's sure that each game is equally likely. The other player, player B, knows in which game he's playing. Determine the pure Nash equilibria using the Bayesian game analysis discussed in the course.

Data (payoff matrix) of each game

The payoff matrix of each game is given in the statement and is taken again in the Figure 3.

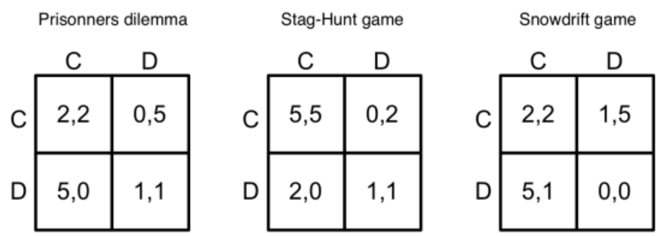

Figure 3 – Payoff matrices of the prisonner's dilemma, stag-hunt and snowdrift game

Resolution

The problem presented here is a *Bayesian* game problem where a player A doesn't know which game will he play while the second player B knows it. For all cases, the player A has the choice between cooperating (C) or defecting (D). To find the Nash equilibria in such a problem, the first thing to do is to enumerate all the possibles combinations of player A's choice in each game. Let's define the set of all player i's action's $\mathcal{A}_i$. In that case, $\mathcal{A}_A$ will then take those values : $\mathcal{A}_A = \{(C,C,C),(C,C,D),(C,D,C),(D,C,C),(C,D,D),(D,C,D),(D,D,C),(D,D,D)\}$. Concerning $\mathcal{A}_B$, player B has only two choices because he knows in which game he's playing, so it's simple as $\mathcal{A}_B = \{C,D\}$.

We know have all the possible actions for both player, the next step is calculating the payoff values for each combinations between an action of player A and player B. To do that, we represent $\mathcal{A}_A$ in the columns of the matrix and $\mathcal{A}_B$ in the lines of the matrix. Moreover, we know that each game is equally likely which means that every game has a probability of $\frac{1}{3}$ to be played by player A. Table 5 is presenting the payoff matrix of player A for the current approached Bayesian problem. Note that the payoff values are multiplied by 3 to avoid fractions to make the matrix more clear.

	CCC	CCD	CDC	DCC	CDD	DCD	DDC	DDD
C	9	8	4	7	3	6	2	1
D	12	7	11	8	6	3	7	2

Table 5 – Payoff matrix of player A for the Bayesian game problem

Let's now evaluate the best responses for player A (thanks to the matrix just shown above) and for player B. To find a Nash equilibrium, the best responses of both players should match. Best responses are illustrated in Figure 4 as well as the matched ones which mean the pure Nash equilibria of the problem.

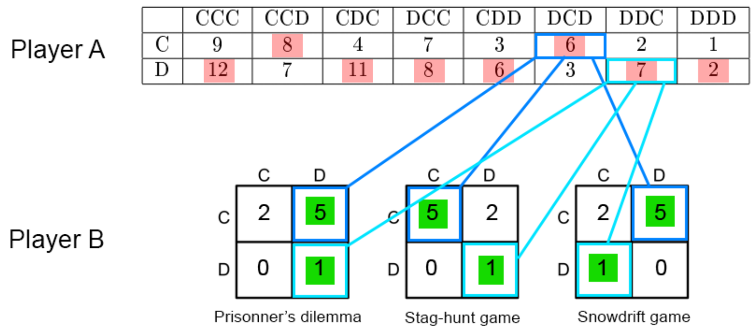

Figure 4 – Nash equilibria and best responses of the problem "Which social dilemma?" – the best responses for player A are highlighted in red, the player B's ones are highlighted in green. The Nash equilibria are brought out in blue and cyan.

To conclude, there exists 2 pure Nash equilibria in this problem. The first one is (C, DCD) (i.e player A cooperating in the second game only while player B is cooperating) and the second one is (D, DDC) (i.e player B cooperating in the last game only while player B is defecting).

3 Games in finite population