BTVN T2 ML1
Linh Hoang
ID: 11202127
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Exercise 1
a. Proof that Gaussian distribution is normalized

To prove that the univariate Gaussian distribution is normalized, we will first show that it is normalized for a zero-mean Gaussian and extend that result to show that $N(\mu, \sigma^2)$ is normalized.

The pdf of the zero-mean Gaussian distribution is given by:

$$\varphi(x) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2} x^2\right) \qquad -\infty < x < +\infty$$

To prove that the above expression is normalized, we have to show that:

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2} x^2\right) dx = \sqrt{2\pi\sigma^2}$$

Let:
$$I = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2} x^2\right) dx$$

Squaring the above expression, we have:

$$I = \iint_{-\infty}^{+\infty} \exp\left(-\frac{1}{2\sigma^2} x^2 - \frac{1}{2\sigma^2} y^2\right) dxdy \; (1)$$

We make the transformation from Cartesian coordinates (x, y) to polar coordinates (r, θ), which is defined by:

$$x = rcos(\theta), y = rsin(\theta)$$

And using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we have $x^2 + y^2 = r^2$.

Also the Jacobian of the change of variables is given by:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial(x)}{\partial(r)} & \frac{\partial(x)}{\partial(\theta)} \\ \frac{\partial(y)}{\partial(r)} & \frac{\partial(y)}{\partial(\theta)} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = rcos^2\theta + rsin^2\theta = r$$

Using the same trigonometric identity $\cos^2\theta + \sin^2\theta = 1$. Thus equation (1) can be rewritten as:

$$I^2 = \int_0^{2\pi} \int_0^{+\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r dr d\theta$$

$$= 2\pi \int_0^{+\infty} \exp\left(-\frac{r^2}{2\sigma^2}\right) r dr$$

$$= 2\pi \int_0^{+\infty} 2\pi \int_0^{+\infty} \exp\left(-\frac{u}{2\sigma^2}\right) \frac{1}{2} du$$

$$= 2\pi\sigma^2$$

Where we used the change of variables $r^2 = u$. Thus $I = (2\pi\sigma^2)^{\frac{1}{2}}$.

Finally to prove that $N(\mu, \sigma^2)$ is normalized, we make the transformation $y = x - \mu$ so that:

$$\int_{-\infty}^{+\infty} N(x|\mu, \sigma^2) dx = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy = 1$$

as required.

Therefore, Gaussian distribution is normalized.

b. Expectation of Gaussian distribution is mu (mean)

X has a Gaussian distribution if and only if the probability density function of X is:

$$f_X(x) \quad = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x-\mu^2}{2\sigma^2}\right)$$

From the definition of the expected value of a continuous random variable, we have:

$$E(X) \qquad = \int_{-\infty}^{\infty} x f x(x) dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp(-\frac{x-\mu^2}{2\sigma^2}) dx$$

$$Substituting: t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$
\begin{aligned}
E(X) \quad &= \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma t + \mu)\exp(-t^2)dt \\
&= \frac{1}{\sqrt{\pi}}\left(\sqrt{2}\sigma \int_{-\infty}^{\infty} t\exp(-t^2)dt + \mu \int_{-\infty}^{\infty} \exp(-t^2)dt\right) \\
&= \frac{1}{\sqrt{\pi}}\left(\sqrt{2}\sigma \left[-\frac{1}{2}\exp(-t^2)\right]_{-\infty}^{\infty} + \mu\sqrt{\pi}\right) \\
&= \frac{\mu\sqrt{\pi}}{\sqrt{\pi}} \\
&= \mu
\end{aligned}
$$

c. Variance of Gaussian distribution is sigma^2 (variance)
X has a Gaussian distribution if and only if the probability density function of X is:

$$
f_X(x) \quad = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x-\mu^2}{2\sigma^2}\right)
$$

From Variance as Expectation of Square minus Square of Expectation:

$$
\begin{aligned}
var(X) \quad &= \int_{-\infty}^{\infty} x^2 f_X(x)dx - (E(X))^2 \\
&= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx - \mu^2
\end{aligned}
$$

$$
Substituting: t = \frac{x-\mu}{\sqrt{2}\sigma}
$$

$$
\begin{aligned}
var(X) \quad &= \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\sqrt{2}\sigma t + \mu\right)^2 \exp(-t^2)dt - \mu^2 \\
&= \frac{1}{\sqrt{\pi}}\left(2\sigma^2 \int_{-\infty}^{\infty} t^2\exp(-t^2)dt + 2\sqrt{2}\sigma\mu \int_{-\infty}^{\infty} t\exp(-t^2)dt + \mu^2 \int_{-\infty}^{\infty} \exp(-t^2)dt\right) - \mu^2 \\
&= \frac{1}{\sqrt{\pi}}\left(2\sigma^2 \int_{-\infty}^{\infty} t^2\exp(-t^2)t + 2\sqrt{2}\sigma\mu \left[-\frac{1}{2}\exp(-t^2)\right]_{-\infty}^{\infty} + \mu^2\sqrt{\pi}\right) - \mu^2 \\
&= \frac{1}{\sqrt{\pi}}\left(2\sigma^2 \int_{-\infty}^{\infty} t^2\exp(-t^2)dt + 2\sqrt{2}\sigma\mu \cdot 0\right) + \mu^2 - \mu^2 \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^2\exp(-t^2)dt \\
&= \frac{2\sigma^2}{\sqrt{\pi}}\left(\left[-\frac{t}{2}\exp(-t^2)\right]_{-\infty}^{\infty} + \frac{1}{2}\int_{-\infty}^{\infty} \exp(-t^2)dt\right) \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2}\int_{-\infty}^{\infty} \exp(-t^2)dt \\
&= \frac{2\sigma^2\sqrt{\pi}}{2\sqrt{\pi}} \\
&= \sigma^2
\end{aligned}
$$