1. (Definition) Initial functions: zero function (constant), successor function, projection functions.

2. (Definition) primitive recursion.

3. (Definition) A primitive function is either an initial function or obtained by applying composition or primitive recursion to primitive functions.

4. (Theorem) Every primitive function is provably recursive, i.e. there is a $\Sigma_1$-formula $\varphi(v^n, u)$ such that $\Phi_{\mathrm{PA}} \vdash \varphi(\underline{a}^n, \underline{f(a^n)})$ and $\Phi_{\mathrm{PA}} \vdash \forall v^n \exists^{=1} u \varphi(v^n, u)$.

5. (Theorem) $\Sigma_1$-completeness: For any $\Sigma_1$-formula $\varphi$, if $\mathfrak{N} \models \varphi$ then $\Phi_{\mathrm{PA}} \vdash \varphi$.

6. (Theorem) Formalized $\Sigma_1$-completeness: For any $\Sigma_1$-formula $\varphi$, $\Phi_{\mathrm{PA}} \vdash (\varphi \to \mathrm{der}(\underline{n^\varphi}))$.

   Proof. Use induction to show

   $$\Phi_{\mathrm{PA}} \vdash \forall v(\varphi \to \mathrm{der}(\underline{n^{subst(n^\varphi, f(v))}})),$$

   where *subst* is a function that takes codes of $\varphi$ and of the term represented by $v$ and returns the code of formula that is $\varphi$ with its $v$ substituted by the term represented by $v$; $f$ takes a variable-free term and returns its code.

7. (L1) If $\Phi \vdash \varphi$ then $\Phi \vdash \mathrm{der}(\underline{n^\varphi})$.

8. (L2) $\Phi \vdash (\mathrm{der}(\underline{n^\varphi}) \wedge \mathrm{der}(\underline{n^{(\varphi \to \psi)}}) \to \mathrm{der}(\underline{n^\psi}))$.

9. (L3) $\Phi \vdash (\mathrm{der}(\underline{n^\varphi}) \to \mathrm{der}(\underline{n^{\mathrm{der}(\underline{n^\varphi})}}))$.

10. $\Phi_{\mathrm{PA}}$ satisfies (L1): Suppose $\Phi_{\mathrm{PA}} \vdash \varphi$. Then there is a derivation for this. Encode this derivation with a number $p$. Then $\mathfrak{N} \models \mathrm{prvs}(\underline{p}, \underline{n^\varphi})$. By $\Sigma_1$-completeness, it follows that $\Phi_{\mathrm{PA}} \vdash \mathrm{prvs}(\underline{p}, \underline{n^\varphi})$.

    Alternatively, one could obtain this fact by this line of reasoning: $\Phi_{\mathrm{PA}} \vdash \varphi \Rightarrow \mathfrak{N} \models \mathrm{der}(\underline{n^\varphi}) \Rightarrow \Phi_{\mathrm{PA}} \vdash \mathrm{der}(\underline{n^\varphi})$ (by $\Sigma_1$-completeness) $\Rightarrow \Phi_{\mathrm{PA}} \vdash \mathrm{der}(\underline{n^{\mathrm{der}(\underline{n^\varphi})}})$ (by L3) $\Rightarrow \mathfrak{N} \models \mathrm{der}(\underline{n^{\mathrm{der}(\underline{n^\varphi})}}) \Rightarrow \Phi_{\mathrm{PA}} \vdash \mathrm{der}(\underline{n^\varphi})$.

11. $\Phi_{\mathrm{PA}}$ satisfies (L3) since it is a special case of Formalized $\Sigma_1$-completeness.