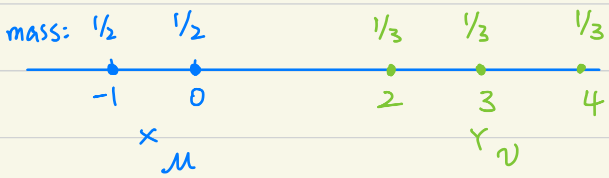
Proof of Thm.

• Def: [Transport plan]: allow mass to split. suppose we have a source measure μ on set X. and a target measure ν supported on set Y. Information that how much mass get moved from x to y, store it in a measure $\pi := X \times Y$.

• e.g.



$$\pi(-1,2) = 1/3, \quad \pi(-1,3) = 1/6, \quad \pi(0,3) = 1/6, \quad \pi(0,4) = 1/3.$$

and $\pi(-1,Y) = \mu(-1) = \pi(-1,2) + \pi(-1,3) + \pi(-1,4) = \frac{1}{3} + \frac{1}{6} + 0 \quad = \frac{1}{3}$

$$\pi(X,2) = \nu(2) = \pi(-1,2) + \pi(0,2) = \frac{1}{3} + 0 = \frac{1}{3}.$$

• for $x \in X$, $\pi(x,Y) = \mu(x)$, if $A \subset X$, $\pi(A,Y) = \mu(A)$. $\Rightarrow$ μ is marginal of π on X.

• Def: [Legendre-Fenchel Transform]:

Denote the L-F transform of f is f^* by

$$f^*(y) = \sup_{x \in \mathrm{dom}(f)} \{xy - f(x)\}.$$

• property 1: for all $x \in Dom(f)$ and $y \in Dom(f^*)$ $\Rightarrow$ $f(x) + f^*(y) \geq x \cdot y$. "=" iff $y \in \partial f(x)$.

proof: $f^*(y) \geq xy - f(x) \Rightarrow f(x) + f^*(y) \geq xy$. $\cdots\cdots$ ①

let $y \in \partial f(x) \Leftrightarrow f(z) \geq f(x) + y(z-x)$ for all $z \in Dom(f)$.

$\Leftrightarrow \quad f(z) \geq f(x) + yz - yx.$ for all z

$\Leftrightarrow \quad xy - f(x) \geq yz - f(z)$ for all z.

$\Leftrightarrow \quad xy - f(x) \geq \sup_z \{z \cdot y - f(z)\} = f^*(y).$

$\Leftrightarrow \quad xy - f(x) \geq f^*(y) \Rightarrow f(x) + f^*(y) \geq xy \cdots\cdots$ ②.

By ① and ② ■

• Thm: Suppose that μ satifies $\int |x|^2 d\mu(x) < \infty$ and u is convex and differentiable μ a.e.

$\mu \in \mathcal{P}(\mathbb{R}^d)$, $u: \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$. let $T = \nabla u$ and suppose $\int |T(x)|^2 d\mu(x) < \infty$.

Then T is optimal for the transport cost $c(x,y) = \frac{1}{2}|x-y|^2$ between μ and $\nu := T\#\mu$.

proof: for convex function u, we have properity I that

$$u(x) + u^*(y) \geq xy \quad \text{for all } x, y \in \mathbb{R}^d, \quad u(x) + u^*(y) = xy \text{ iff } y = \nabla u(x).$$

for any transport plan $\gamma \in \Pi(\mu, \nu)$, $\quad y = T(x)$

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (x \cdot y) \, d\gamma(x,y) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} u(x) + u^*(y) \, d\gamma(x,y)$$

$$= \int_{\mathbb{R}^d} u(x) d\mu(x) + \int_{\mathbb{R}^d} u^*(T(x)) d\mu(x)$$

since $T(x) = \nabla u(x)$.

$$= \int_{\mathbb{R}^d} u(x) + u^*(T(x)) \, d\mu(x) \qquad u(x) + u^*(T(x)) = x \cdot T(x)$$

$$= \int_{\mathbb{R}^d} x \cdot T(x) \, d\mu(x).$$

denote as $\quad = \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot y \, d\gamma_T(x,y)$, where $y = T(x)$.

$$\Leftrightarrow \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot y \, d\gamma(x,y) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} x \cdot y \, d\gamma_T(x,y).$$

for $\gamma \in \Pi$, $\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}(|x|^2 + |y|^2)\, d\gamma(x,y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}(|x|^2 + |y|^2)\, d\gamma_T(x,y)$.

$$\Rightarrow \int \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2\, d\gamma - \int xy\, d\gamma \geq \int \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2\, d\gamma_T - \int xy\, d\gamma_T$$

$$\Rightarrow \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|x-y|^2\, d\gamma \geq \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|x-y|^2\, d\gamma_T .$$

Thus T is optimal $\blacksquare$