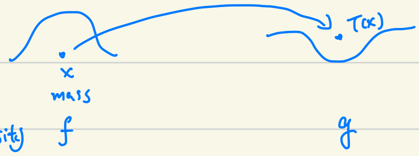
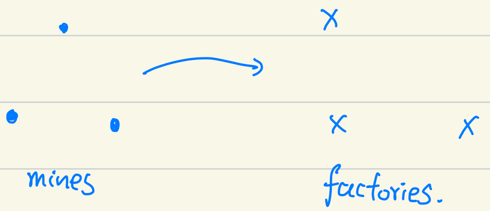
Intro to Optimal Transport.

- Monge pbm :



minimize $\int |x - T(x)| f(x) dx$

- e.g. mines to factories.



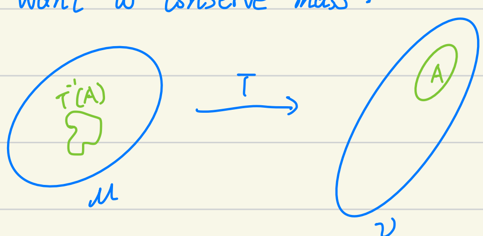
- Can consider other costs $c(x,y)$ minimise $\int c(x, T(x)) f(x) dx$
- we want to work with measures. • we have a source measure call μ • target measure ν

$\mu(E)$ tells us how much mass we have in the set E.

- we need mass balance : $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$. , i.e. $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n) = 1$ for probability mass.

we seek a transpostation map $T(x)$. • Source is supported on set X ; target is supported on set Y.

$T: X \to Y$. , we want to conserve mass :

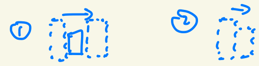


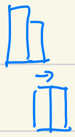
Require. $\mu(T^{-1}(A)) = \nu(A) \quad \forall A \subset Y$

$\mu(T^{-1}(A))$ is called the push-forward of μ through T, denote $T_{\#}\mu$

- mass conservation : $T_{\#}\mu = \nu$.
- The Monge formulation of O.T. is $\min \left\{ \int_{\mathbb{R}^n} c(x, T(x)) \, d\mu(x) \;\middle|\; T_{\#}\mu = \nu \right\}$.

- Is there a minimizer? - Can we compute the solution ?
- Is it unique?
- Stability? well-postness.
- Is it feasible.

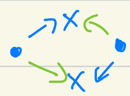


- Book moving Problem  $C(x,y) = |x-y|$. 2 mass-preserving plans.

Both have cost of 2. non-uniqueness.

e.g. $c(x,y) = \frac{1}{2}|x-y|^2$. unique solution : ②.

Cost of ① : $\frac{1}{2} \cdot 2^2 = 2$. Cost of ② : $2 \cdot (\frac{1}{2} \cdot 1^2) = 1$.

e.g.  non-uniqueness.

$x \leftarrow$ mass $= 1/2$

e.g. 

$x \leftarrow$ mass $= 1/2$.

mass $=1$.
Source.

no feasible mapping need allow mass to split.

- we generalize via. Kantorovich formulation. We seek a transport plan that allows mass to split. Again we have a source measure μ on set X and a target measure ν support on set Y. We want to know: how much mass gets moved from x to y. We store this in a measure $\pi := X \times Y$.

mine at $x=0$ with 1 unit of resource, factories $y = 0, 1$ with $1/3$, $2/3$ unit respectively. $\pi(0,0) = \frac{1}{3}$, $\pi(0,1) = 2/3$. $\pi(0, \mathbb{R}) = 1$. In general: If $A \subset X$, $B \subset Y$ $\pi(A,B)$ tells us how much mass is transported from A to B.

all mass from point x.

We need to conserve mass, choose some $x \in X$ Consider $\pi(x, Y) = \mu(x)$. (total mass coming from X)
More generally: if $A \subset X$, $\pi(A, Y) = \mu(A)$. We say that μ is the marginal of π on X. Also, we need ν to be the marginal of π on Y. If $B \subset Y$ then $\pi(X, B) = \nu(B)$.

How do we measure the cost? Now $c(x,y)$ is weighted by the amount of mass moving from x to y.

$\inf \left\{ \int_{X \times Y} c(x,y) \, d\pi(x,y) \;\middle|\; \pi \in \Pi(\mu,\nu) \right\}$. where $\Pi(\mu,\nu)$ consists of measures whose marginals on X and Y are μ and ν respectively.

• Special cases : – Discrete O.T. (Dirac masses $\longrightarrow$ Dirac masses).

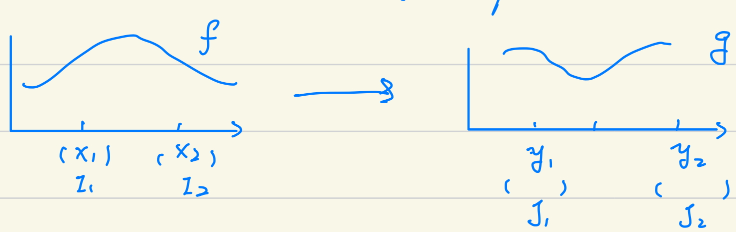
– Continuous O.T. (μ, ν are absolutely continuous with densities f, g).

– Semi-Discrete O.T. (μ is abs cont and ν consist of Diracs).

• Let's start in 1D , Assume things are "nice enough" , Goal : find $T(x)$ to minimize $\frac{1}{2}\int_{\mathbb{R}} (x-T(x))^2 f(x)\,dx$ s.t. $\int_{T^{-1}(A)} f(x)\,dx = \int_A g(y)\,dy \quad \forall A \in \mathbb{R}$.

constraint alternatively, $\int_X h(T(x)) f(x)\,dx = \int_Y h(y) g(y)\,dy \quad \forall h \in \underset{\text{cont fun on } x.}{C^0(x)}$

• let's pick 2 points x_1, x_2 ; $x_1 < x_2$ and $\varepsilon > 0$. and two little open intervals $x_1 \in I_1$, $x_2 \in I_2$ s.t. $\int_{I_1} f(x)\,dx = \varepsilon = \int_{I_2} f(x)\,dx$.



$y_i = T(x_i)$ and $J_i = T(I_i)$.

let's "permute" part of the map and create a new measure-preserving map s.t.

$\tilde{T}(x_1) = y_2$, $\hat{T}(x_2) = y_1$. $\hat{T}(I_1) = J_2$, $\hat{T}(I_2) = J_1$, $\hat{T}(x) = T(x)$ if $x \notin I_1 \cup I_2$.

T was optimal $\Rightarrow \frac{1}{2}\int (x-T(x))^2 f(x)\,dx \le \frac{1}{2}\int_{\mathbb{R}} (x-\hat{T}(x))^2 f(x)\,dx$.

$\Rightarrow -\int_{I_1} x\cdot T(x) f(x)\,dx - \int_{I_2} x\,T(x) f(x)\,dx \le -\int_{I_1} x\cdot \check{T}(x)\cdot f(x)\,dx - \int_{I_2} x\cdot \check{T}(x)\cdot f(x)\,dx$.

$\varepsilon \downarrow 0 \Rightarrow x \to x_1$, $\hat{T}(x) - T(x) \to y_2 - y_1$ and $\int_{I_1} f(x)\,dx = \varepsilon$.

$\Rightarrow \quad \frac{1}{\varepsilon}\int_{I_1} x(\check{T}(x)-T(x)) f(x)\,dx + \frac{1}{\varepsilon}\int_{I_2} x(\check{T}(x)-T(x)) f(x)\,dx \le 0$.

As $\varepsilon \to 0$: $x_1(y_2-y_1) + x_2(y_1-y_2) \le 0$. $\Rightarrow (y_2-y_1)(x_2-x_1) \ge 0$. monotone mapping.

The optimal map is <u>monotone</u>.