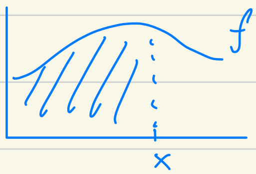
Cyclical Monotonicity & Kantorovich Pbm.

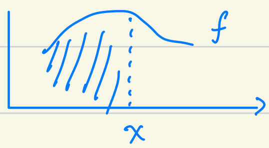
- 1-Dim. : Solution is monotone Can we construct it?

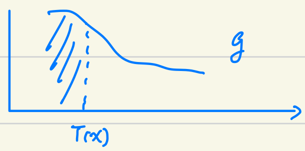
  use cumulative Dist fun.



$F(x) = \int_{-\infty}^{x} f(t)\,dt.$ $F(x)$ is monotone.

$G(y) = \int_{-\infty}^{y} g(t)\,dt.$





Expect $F(x) = G(T(x))$

Get an exact solution : $T(x) = G^{-1}(F(x))$

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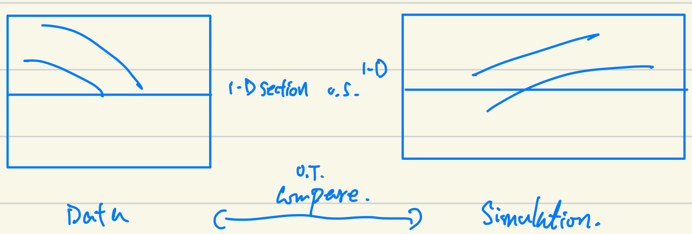


- Let's think about cts pbm in $\mathbb{R}^n$. Goal: To find a map $T(x)$ to push forward density.

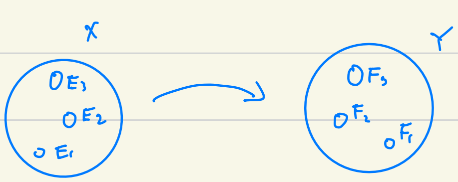
minimize $\frac{1}{2}\int_X |x - T(x)|^2 f(x)\,dx$ s.t. $\int_{T^{-1}(A)} f(x)\,dx = \int_A g(y)\,dy.$

Do a change of variables. $y = T(x) \Rightarrow \int_{T^{-1}(A)} f(x)\,dx = \int_{T^{-1}(A)} g(T(x))\det(\nabla T(x))\,dx.$ $\forall A$.

Expect : $g(T(x))\det(\nabla T(x)) = f(x).$

Choose some $x_1, x_2, \dots, x_N \in X$. Suppose T is optimal. let $y_i = T(x_i)$.

let E_i be a ball center at x_i. s.t. $\int_{E_i} f(x)\,dx = \varepsilon$ and let $F_i = T(E_i)$.



let's create a new map $\hat{T}$ measure preserving.

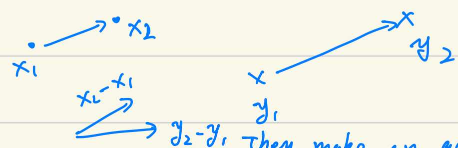
and we want $\hat{T}(x_i) = y_{i+1}$. $\hat{T}(E_i) = F_{i+1}$, $\hat{T}(x) = T(x)$ if $x \notin \bigcup_{i=1}^{N} E_i$.

T is optimal so $\frac{1}{2}\int_X |x - T(x)|^2 f(x)\,dx \le \frac{1}{2}\int_X |x - \check{T}(x)|^2 f(x)\,dx$.

As in 1-D: $\frac{1}{\varepsilon}\sum_{i=1}^{N}\int_{E_i} X \cdot (\check{T}(x) - T(x)) \cdot f(x)\,dx \le 0$.

Take $\varepsilon \downarrow 0 \Rightarrow \sum_{i=1}^{N} X_i \cdot (y_{i+1} - y_i) \le 0$. This condition is <u>cyclical monotonicity</u>.

If $N=2$: $X_1 \cdot (y_2 - y_1) + x_2 \cdot (y_1 - y_2) < 0. \Rightarrow (x_2 - x_1)(y_2 - y_1) \ge 0$



They make an accute angle.
This has hold for all N.
Restrict our ability to "twist" mass.

• irrotational: This is a stronger condition than being irrotational.

• Thm (Rockefeller): A cyclically monotone map can be expressed as the gradient of a convex function.

• Convex: A function f is convex if $\forall x, y \in dom(f)$, and $\lambda \in [0,1]$. $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$.

• We can write $T(x) = \nabla u(x)$ where $u(x)$ is convex. From mass-preservation: $\det(\nabla T(x)) = f(x)/g(T(x))$.

$\Rightarrow$ $\boxed{\det(\underbrace{D^2 u(x)}_{\text{Hesrian}}) = f(x)/g(\nabla u(x))}$ $\leftarrow$ Monge-Ampere equation. PDE.

• [Kantorovich Pbm]: $\inf\left\{ \int_{X\times Y} \overset{\text{cost fun.}}{c(x,y)}\, d\gamma(x,y). \mid \gamma \in \overset{\text{marginal}}{\Pi(\mu,\nu)} \right\}$. $\Pi(\mu,\nu)$ is the set of measures whose marginals on X and Y are μ, ν. IS This Pbm feasible? It is feasible if we have mass balance. can split masses, as long as $c(\cdot,\cdot)$ bdd from below and X, Y are bdd. then we have an inf that is finite.

- Can we find an actual minimizer?

we need a compactness arguement.

• Thm [Weierstruss] If $f: U \to \mathbb{R}$. is continuous. and if U is compact then f attains a minimum on U.

Thm: Suppose $X, Y \subseteq \mathbb{R}^n$ are compact. and that $c(x,y)$ is continuous. Then the Kantorovich pbm has minimum.

proof: Before we talk about compactness, we need a notion of convergence. we identify $\mathcal{U} = \Pi(\mu,\nu)$

W.L.O.G there are prob meas. we say that a sequence of measures $\gamma_n \to \gamma$ if

$\int_{X \times Y} g(x,y)\, d\gamma_n(x,y) \longrightarrow \int_{X \times Y} g(x,y)\, d\gamma \quad \forall g \in C^0(X \times Y)$ (g is cont).

Q1: Is $\Pi(\mu,\nu)$ compact? Choose any sequence $\pi_n \in \Pi(\mu,\nu)$ we need to extract a convergence subseq.

Since the π_n is a prob meas, we can extract a sub-seq $\pi_{n_k} \to \pi$, which is also a prob meas.

Need to check: Does $\pi \in \Pi(\mu,\nu)$? check marginals. choose any $g \in C(X)$.

$$\int_{X \times Y} g(x)\, d\pi(x,y) = \lim_{k \to \infty} \int_{X \times Y} g(x)\, d\pi_{n_k}(x,y) \quad \text{since } \pi_{n_k} \to \pi.$$

$$= \int_X g(x)\, d\mu(x) \quad \text{since } \pi_{n_k} \in \Pi(\mu,\nu) \text{ so the marginal over } X \text{ is } \mu.$$

Same argument $\Rightarrow$ the marginal of π over y is ν. $\Rightarrow \pi \in \Pi(\mu,\nu)$. $\Rightarrow \Pi(\mu,\nu)$ is compact.

Q2: Is my value of f cont? where $f(\pi) = \int_{X \times Y} c(x,y)\, d\pi(x,y)$ for $\pi \in \mathcal{U}$.

choose any $\pi_n \in \Pi(\mu,\nu)$ s.t. $\pi_n \to \pi \in \Pi(\mu,\nu)$ need to show that $f(\pi_n) \to f(\pi)$.

$$f(\pi_n) = \int_{X \times Y} \overset{\text{cont}}{c(x,y)}\, d\overset{\pi_n \to \pi}{\pi_n(x,y)} \to \int_{X \times Y} c(x,y)\, d\pi(x,y) \quad \text{since } \pi_n \to \pi \text{ and } c \text{ is cont.}$$

$$= f(\pi) \quad \text{So } f \text{ is continuous.}$$

So, the Kantorovich pbm seeks to minimise a real-valued, continuous function over a compact set. This is possible by Weierstrass! So, the Kantorovich pbm admits a minimizer.