- Given Data $x = \{x_1, \dots, x_n\}$, parameters $\{z_1, \dots, z_m\}$. a family of Distribution Q.

want to find $q^*(z) = \underset{q(z) \in Q}{\arg\min}\ KL(q(z) \,\|\, p(z|x))$.

and $KL(q(z) \,\|\, p(z|x)) = -\int_z q(z) \log\left[\frac{p(z|x)}{q(z)}\right] dz$

$$= \int_z q(z) \log q(z)\, dz - \int_z \log p(z|x)\ q(z)\, dz$$

$$= \mathbb{E}_q[\log q(z)] - \mathbb{E}_q\left[\log\left(\frac{p(x|z)\cdot p(z)}{p(x)}\right)\right]$$

$$= \mathbb{E}_q[\log q(z)] - \mathbb{E}_q[\log(p(x|z)\cdot p(z))] + \mathbb{E}_q[\log p(x)]$$

$$= \underbrace{\mathbb{E}_q[\log q(z)] - \mathbb{E}_q[\log(p(x|z)\cdot p(z))]}_{= -ELBO(q(z)).} + \log p(x).$$

$\Rightarrow KL(q(z) \,\|\, p(z|x)) = -ELBO(q(z)) + \log p(x)$.

Suppose Q is a family that $q^* \in Q$ and q^* maximize $ELBO(q^*(z))$

for all $q \in Q$, $KL(q(z) \,\|\, p(z|x)) \geq 0$

$$\Rightarrow \quad \log p(x) \geq ELBO(q(z)).$$

and q^* satisfies $\log p(x) = ELBO(q^*(z))$.

$$\log p(x) = \text{ELBO}(q^*(z))$$

$$= \mathbb{E}_{q^*}[\log(p(x|z) \cdot p(z))] - \mathbb{E}_{q^*}[\log q^*(z)]$$

$$\mathbb{E}_{q^*}[\log q^*(z)] = \mathbb{E}_{q^*}\left[\log\left(\frac{p(x|z) \cdot p(z)}{p(x)}\right)\right]$$

$\Leftrightarrow \quad q^*(z) = p(z|x) =$ posterior Distribution. $\blacksquare$