Stokes' Theorem

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1 Stokes' Theorem.

For surface S in R^3 space is an oriented surface in x,y,z coordinate with boundary ∂S . Let R be abounded, open region in s, t, plane with smooth boundary ∂R . Suppose that F is a continuously differentiable vector field. $\vec{r}$ is a smooth parametrization that maps R to S, and $\partial R to \partial S$. Then

$$\iint_S curl\ \vec{F}\ \cdot\ d\vec{S} = \iint_S \left(\nabla \times \vec{F}\right)\ \cdot\ \vec{n}\ dS = \int_{\partial S} \vec{F}\ \cdot\ \vec{T}\ ds = \int_{\partial S} \vec{F}\ \cdot\ d\vec{r}$$

2 Proof.

Let

$$\vec{r}\,(s,t) = \begin{bmatrix} x\,(s,t) \\ y\,(s,t) \\ z\,(s,t) \end{bmatrix}\ : R \to S.$$

Then.

$$d\vec{r}\mid_{\partial R} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{bmatrix} \begin{bmatrix} ds \\ dt \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s}ds + \frac{\partial x}{\partial t}dt \\ \frac{\partial y}{\partial s}ds + \frac{\partial y}{\partial t}dt \\ \frac{\partial z}{\partial s}ds + \frac{\partial z}{\partial t}dt \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{bmatrix} ds + \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix} dt\ = \frac{\partial \vec{r}}{\partial s}ds + \frac{\partial \vec{r}}{\partial t}dt$$

Hence.

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\partial S} \left(\vec{F} \cdot \frac{\partial \vec{r}}{\partial s}ds + \vec{F} \cdot \frac{\partial \vec{r}}{\partial s}\right)$$

We define a 2-dimensional vector field $G = (G1, G2)$ on the s,t, plane by

$$G_1 = \vec{F}\ \cdot\ \frac{\partial \vec{r}}{\partial s}\ and\ G_2 = \vec{F}\ \cdot\ \frac{\partial \vec{r}}{\partial t}$$

Therefore, we put G into the original line integral

$$\int_{\partial S} \vec{F} \ \cdot \ d\vec{r} = \int_{\partial R} (G_1 ds + G_2 dt) = \int_R \ \left(\tfrac{\partial G_2}{\partial s} - \tfrac{\partial G_1}{\partial t}\right) dsdt \ ,$$

$$\int_{\partial S} \vec{F} \ \cdot \ d\vec{r} = \int_{\partial R} (G_1 ds + G_2 dt)$$

On the other hand.

$$\iint_S curl \ \vec{F} \ \cdot \ d\vec{S} \ = \ \iint_R curl \ \vec{F} \mid_{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} dsdt$$

We expand it, get

$$curl \ \vec{F} \mid_{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \ = \begin{vmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{vmatrix}$$

$$= \frac{\partial \vec{F}}{\partial s} \cdot \frac{\partial \vec{r}}{\partial t} - \frac{\partial \vec{F}}{\partial t} \cdot \frac{\partial \vec{r}}{\partial s} \ = \frac{\partial G_2}{\partial s} - \frac{\partial G_1}{\partial t}$$

Hence.

$$\int_R curl \ F \mid_{\vec{r}} \cdot \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \ dsdt \ = \ \int_R \ \left(\tfrac{\partial G_2}{\partial s} - \tfrac{\partial G_1}{\partial t}\right) dsdt$$

For the line integral part,we have

$$\int_{\partial S} \vec{F} \ \cdot \ d\vec{r} = \int_{\partial R} \ (G_1 ds + G_2 dt)$$

Use Green's Theorem, we know

$$\int_{\partial R} \ (G_1 ds + G_2 dt) \ = \int_R \ \left(\tfrac{\partial G_2}{\partial s} - \tfrac{\partial G_1}{\partial t}\right) dsdt \ ,$$

So, we can conclude that

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_R \ \left(\tfrac{\partial G_2}{\partial s} - \tfrac{\partial G_1}{\partial t}\right) dsdt = \int_R curl \ F \mid_{\vec{r}} \cdot \tfrac{\partial \vec{r}}{\partial s} \times \tfrac{\partial \vec{r}}{\partial t} \ dsdt = \iint_S curl \ F \ \cdot \ d\vec{S}$$

Thus, we finished our proof of Stokes' Theorem.

References

[1] Fichtengoltz, G. M. (1959). Course on the Differential and Integral.