CS 577: Introduction to Algorithms — Induction Review

Problems

1. Prove the following formula for the sum of the first n natural numbers:
$$1 + 2 + \ldots + n = \frac{n(n+1)}{2}.$$

2. Prove the following formula for natural numbers:
$$1^2 + 2^2 + \ldots + n^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

3. Show that if set S has n elements, then the number of subsets of S is 2^n.
Note: Both the empty set $\emptyset$ and S itself constitute subsets of S.

4. Show that
$$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n.$$
Hint: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

5. Consider the sequence defined recursively by
$$x_1 = \sqrt{2},$$
$$x_{n+1} = \sqrt{2 + x_n}.$$
Prove that $x_n < 2$ for all n. Then prove that $x_n < x_{n+1}$ for all n.

6. Suppose we have 2^n-by-2^n grid, and an unlimited supply of L-shaped tiles consisting of three unit squares. Show that it is possible to tile the grid such that no tiles overlap and all but one of the four center squares is covered by a tile.
Hint: Show that it is possible to tile the grid such that all but one square is covered, regardless of the location of the hole.

7. Consider the following pseudocode that implements the factorial operation on a nonnegative integer n:
```
function FACTORIAL(n)
    if n == 0 then
        return 1
    else
        return n · Factorial(n − 1)
    end if
end function
```
Prove the correctness of the above program.

8. Recall the binary search algorithm discussed in class: given a sorted list of n numbers and a target value x, the algorithm returns the index of x if it exists, otherwise it returns -1 to indicate nonexistence. Prove the correctness of the algorithm.