HW 2

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1 Part A:

1. If both a and b are concave, and from the Problem description, we know both sequence of a and sequences of b are all non-decreasing sequence. Therefore,$z_i = a_i + b_{k-i}$, and $z_{i+1} = a_{i+1} + b_{k-(i+1)}$, $z_{i-1} = a_{i-1} + b_{k-(i-1)}$. In order to prove z_i is a concave function as a function of i. From the definition of concave, if $z_i - z_{i-1} \geq z_{i+1} - z_i$, then z_i is concave.
   Therefore, $z_i - z_{i-1} = (a_i + b_{k-i}) - (a_{i-1} + b_{k-i+1}) = (a_i - a_{i-1}) + (b_{k-i} - b_{k-i+1})$ and $z_{i+1} - z_i = (a_{i+1} + b_{k-i-1}) - (a_i + b_{k-i}) = (a_{i+1} - a_i) + (b_{k-i-1} - b_{k-i})$. Because sequence a is a non-decreasing concave sequence, it follows the definition that states: $a_i - a_{i-1} \geq a_{i+i} - a_i$. Assume $k - i = x$, and because sequence b is also a non-decreasing concave sequence, from the description: $b_x - b_{x-1} \geq b_{x+1} - b_x$. Add the b_{x-1} on both side and subtract b_{x+1} on both side would result in $b_x - b_{x+1} \geq b_{x-1} - b_x$.
   $\because a_i - a_{i-1} \geq a_{i+i} - a_i$
   $\because b_x - b_{x+1} \geq b_{x-1} - b_x \Rightarrow b_{k-i} - b_{k-i+1} \geq b_{k-i-1} - b_{k-i}$
   $\therefore$ add the previous two equation $(a_i - a_{i-1}) + (b_{k-i} - b_{k-i+1} \geq (a_{i+1} - a_i) + (b_{k-i-1} - b_{k-i})$
   $\therefore (a_i + b_{k-i}) - (a_{i-1} + b_{k-i+1}) \geq (a_{i+1} + b_{k-i-1}) - (a_i + b_{k-i})$
   $\therefore z_i - z_{i-1} \geq z_{i+1} - z_i$
   Therefore by definition, the sequence $z_i = a_i + b_{k-i}$ is also concave function as i.

2. In order to find c_k, by the definition of c_k, we need to find the largest value of $a_i + b_{k-i}$ which as 1 indicates, is the largest value in the sequence z. Assuming both sequence a and sequence b is concave, then as proved in Part A: 1, sequence z is also concave.
   First, we need to find the middle element of sequence z at index $[\frac{k}{2}]$, name it z_a. Then compare z_a with two element next to itself. Name the element at index $[\frac{k}{2} - 1]$, z_l and the element at index $[\frac{n}{2} + 1]$,z_r. If z_a is the largest of the three or z_a is equal to either z_i or z_r, return z_a to be the largest element in the sequence. If z_a is not the largest of the three, but z_l is the largest. The maximum number of the whole sequence must be located on the left half. Otherwise z_r is the largest of the three, the maximum number would locate on the right half of the sequence.
   Therefore, we recurse on the left half or the right half of the sequence. Since there are two calls to the subproblems(right part of the sequence and left part of the sequence), we have two $\mathrm{T}(\frac{k}{2})$ but we are executing on side on each recursive call. We also have to check if z_l is the largest element or equal to z_l or z_r, that takes constant O(1) time. So, the runtime is T(n) = $\mathrm{T}(\frac{k}{2})$ + O(1) = $\mathrm{O}(\log_2 k)$.

   Proof:
   Base Case: When k = 2, There are only 2 c_k, z_a is at index 1 and there is no element on the left, which means there is no z_l, we just need to compare it with the z_r and return the larger one to be c_k.
   Other cases: Since z is a concave sequence, the slope of z is non-increasing over time. So when we choose the middle point of the whole sequence z — z_a. If one of z_l and z_r or both are equal to z_a, then the slope at z_a is equal to 0, and as we already know the slope is non-increasing, there cannot be any value larger than z_a. Therefore, z_a is the largest element in the sequence and we return z_a. If none of z_l and z_r is equal to z_a. There is two conditions:

   - If both z_l and z_r are smaller than z_a, since the slope is non-increasing, $z - a$ is the point for the slope to turn from positive to negative. Otherwise, z_a and z_r cannot be both smaller than z_a.

Because z_i is a concave function, slope is decreasing on the left side of z_a but are all positive, every point before z_l cannot be larger than z_l. Slope on the right side of z_a is decreasing and negative, therefore every point beyond z_r cannot be larger than z_r. Therefore, we return z_a as the largest value of the whole sequence.

- If one of z_l and z_r is smaller and one is larger, we just take the part where has the larger part. Since the concave function suggests that the slope before is non-increasing and the slope after is non-increasing. Either way is going to yield a result that elements on one side is larger than all the element on the other side. So we choose the side with the larger value and call recursive function on that side.

2 Part B:

4. As we know i(k) is the largest index that maximizes the sum $a_i + b_{k-i}$, let the index be indicated by i_k. We now know $a_{i_k} + b_{k-i_k}$ is the biggest for every i.

   Proof: Assume that i(k) is decreasing as a function of k, i.e. $i(k) > i(k+1)$.
   This statement means that there is a i(k+1) that is smaller than i(k). We are going to name the index i(k+i) as $(i_k - l)$, l is some constant number that is smaller than i_k so that $i_k - l > 0$. Therefore, there exist $a_{i_k-l} + b_{k+1-i_k+l}$ that marks the largest value of sum $a_i + b_{k+1-i}$. Because i(k) is the largest index, i(k) is larger than any other index:

   $$\because a_{i_k} + b_{k-i_k} \geq a_{i_k-l} + b_{k-i_k+l} \tag{1}$$

   When there is k+1 elements, in order for i(k+1) to be smaller than i(k). We are using $>$ sign here because in order to choose the largest index, if $a_{i_k-l} + b_{k+1-i_k+l} = a_{i_k} + b_{k-i_k+l}$, it would choose the $i(k)$ term rather than i(k+1):

   $$a_{i_k-l} + b_{k+1-i_k+l} > a_{i_k} + b_{k-i_k+l} \tag{2}$$

   Because the sequence of b is concave and non decreasing:

   $$(a_{i_k-l} + b_{k+1-i_k+l}) - (a_{i_k-l} + b_{k-i_k+l}) = (b_{k+1-i_k+l} - b_{k-i_k+l}) \tag{3}$$
   $$(a_{i_k} + b_{k-i_k+l}) - (a_{i_k} + b_{k-i_k}) = (b_{k-i_k+l} - b_{k-i_k}) \tag{4}$$

   (4) is greater than (3):

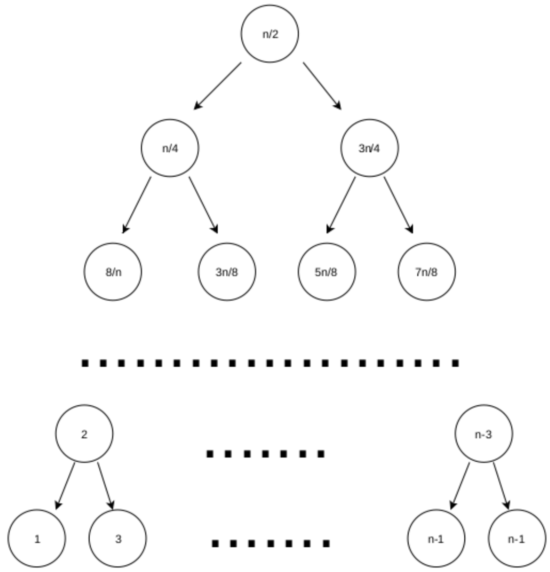
   $$(b_{k-i_k+l} - b_{k-i_k}) \geq (b_{k+1-i_k+l} - b_{k-i_k+l}) \tag{5}$$

   Add (5) to (1):

   $$a_{i_k-l} + b_{k+1-i_k+l} \leq a_{i_k} + b_{k-i_k+l} \tag{6}$$

   As we can see, result (6) is incosistent with assumption at (2). Our premise that $i(k) > i(k+1)$ is false. Therefore by contradiction, $i(k) \leq i(k+1)$.

5. **Algorithm**: First at level 1, we find the c_k at $k = \frac{n}{2}$, name $i(\frac{n}{2})$ as x. Since we have to go through the whole seque to find the c_k at this k, as 3 indicates, this takes a constant n time. Level 2 we find the c_k at the middle of the left side of $\frac{n}{2}$ and at the right side of $\frac{n}{2}$, which are at $\frac{n}{4}$ and $\frac{3n}{4}$. From the proof in number four, we know that $i(\frac{n}{2}) > i(\frac{n}{4})$, and $i(\frac{3n}{4}) \geq i(\frac{n}{2})$. Therefore we only need to compute from $(a_1 + b_{x-1})$ to $(a_{x-1} + b_1)$, a total of$(x-1)$ times to find i($\frac{n}{4}$). Also from $(a_x + b_{n-x})$ to $(a_n + b_x)$, a total of $(n-x)$ times, to find i($\frac{3n}{4}$). So in total, we did $(x-1) + (x-n) = (n-1)$ times, which is O(n) times at this level. For level 3, we would compute $i(\frac{n}{8})$ and $i(\frac{3n}{8})$ on the right and left side of $i(\frac{n}{4})$. Then $0 < i(\frac{n}{8}) \leq i(\frac{n}{4})$, $i(\frac{n}{4}) < i(\frac{3n}{8}) \leq i(\frac{n}{2})$, $i(\frac{n}{2}) < i(\frac{5n}{8}) \leq i(\frac{3n}{4})$, $i(\frac{3n}{4}) < i(\frac{7n}{8}) \leq i(n)$. Which would still result in a O(n) times to compute in this level. We will recurse finding the middle point of right and left part, until we reach the base case 1 and find all the c_k for $k = 0, 1, 2..., n-1$. But since



we cannot devide to n itself. After finish find $i(n)$ with $i(n-1)$, at the last level. Each level has O(n) complexity and there is a total of $log_2 n$ of recursion. When we find all the c_k,algorithm would result in $O(n) * (log_2 n) = O(nlog_2 n)$.

Proof: From the proof in previous question, $i(k) \leq i(k+1)$. Find the middle point of n, and find i($\frac{n}{2}$) at that point. Each recursive would have 2 times the elements as the previous one. However, each one only need to do half the work, since the boundary is halved on each recursion. Each level still does the same work. And we guarantee that every node is found through the recursive call except c_n and c_0. Therefore, determine the base case that find c_{n-1} and find an extra c_n using c_{n-1} and extra c_0 using c_1.