CS 577: Introduction to Algorithms — Randomness Review Solutions

Solutions

1. Let T_i be the event that the first tails is flipped on the i^{th} flip. P(T_i) = $(\frac{1}{2})^{i-1} \times \frac{1}{2}$ where $(\frac{1}{2})^{i-1}$ is the probability the previous $i-1$ flips are heads and $\frac{1}{2}$ is the probability i^{th} is tails. Using the fact that $(1-r)^{-n} = (1 + \binom{n}{1}r + \binom{n+1}{2}r^2 + \binom{n+2}{3}r^3 + ...)$.

   $\mathbb{E}[T]$ = $\sum_{i=1}^{\inf} P(H_i) \times i$ = $\frac{1}{2}(1 + 2\frac{1}{2} + 3(\frac{1}{2})^2 + 4(\frac{1}{2})^3 + ...)$ = $\frac{1}{2}(1 - \frac{1}{2})^{-2}$ = 2.

   This is rather complicated so another way to calculate $\mathbb{E}[T]$ easier would be to use conditioning:

   $\mathbb{E}[T]$ = $\mathbb{E}[\mathbb{E}[T|I_1]]$ = $E[T|I_1 = 1] \times P(I_1 = 1) + \mathbb{E}[T|I_1 = 0] \times P(I_1 = 0)$

   $\mathbb{E}[T|I_1 = 1] = 1$, since the first flip is tails, $\mathbb{E}[T|I_1 = 0] = 1 + \mathbb{E}[T]$ since flips are independent and the first flip is heads. $P(I_1 = 0) = P(I_1 = 1) = \frac{1}{2}$ since it is a fair coin.

   So, $\mathbb{E}[T] = 1 \times \frac{1}{2} + (1 + \mathbb{E}[T]) \times \frac{1}{2} \implies \mathbb{E}[T] = 2$ A similar arguement can be made for heads.

   (a) $\mathbb{E}[T|I_1 = 0] = 1 + E[T] = 1 + 2 = 3$ since the first flip is heads and the rest of the flips are independent. Same argument for $\mathbb{E}[H|I_1 = 1]$.

   (b) Using the idea of conditioning, $\mathbb{E}[X] = P(I_1 = 0) \times \mathbb{E}[X|I_1 = 0] + P(I_1 = 1) \times \mathbb{E}[X|I_1 = 1]$.
   $\mathbb{E}[X|I_1 = 0] = 1 + E[T]$ since the first flip is heads all that is needed is a tails, so the expected number of flips for a head tail sequence is $1 + \mathbb{E}[T]$. $\mathbb{E}[X|I_1 = 1] = 1 + \mathbb{E}[X]$ since the first flip is tails we still need a heads for the head tail sequence which is just like adding one flip to the expected value to get a head tail sequence. So, $\mathbb{E}[X] = \frac{1}{2} \times (1 + \mathbb{E}[T]) + \frac{1}{2} \times (1 + \mathbb{E}[X]) \implies \mathbb{E}[X] = 4$.

   (c) The integers $\{1, ..., n\}$ can be represented using $logn$ bits. So an algorithm using $O(logn)$ flips to generate a random number:

   RNG(n)

   Generate a vector of bits using a coin to randomly determine the values of the bits
   RETURN the corresponding integer from the vector of bits

   (d) A permutation is just an ordering of the set $\{1, ..., n\}$. So an algorithm using in expectation $O(nlogn)$:

   Permute(Set S)

   if (S has one element) **then** return S
   end if
   For Each element in S generate a random bit
   if bit = 0 **then** put the element in S_1
   else, put the element in S_2
   end if
   RETURN $\{Permute(S_1), Permute(S_2)\}$

2. Let B_i be the probability that the algorithm is called i times (including the initial call). So P(B_i) = $(\frac{1}{2})^{i-1} \times \frac{1}{2}$ where $(\frac{1}{2})^{i-1}$ is the probability the previous $i-1$ values are 1 and $\frac{1}{2}$ is the probability i^{th} value is 0.

   (a) Whenever i is odd the value returned from *BiasedBit* is 0, and when i is even the value returned is 1.

   P($B = 0$) = $(B_1 + B_3 + B_5 + ...)$ = $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + ...$ = $\frac{\frac{1}{2} - 0}{1 - \frac{1}{4}}$ = $\frac{2}{3}$.

A similar argument can be made for P($B = 1$) which = $\dfrac{1}{3}$

(b) Let k be a 2-bit number, we can generate the bits from *BiasedBit*. P(k = 0) =$\dfrac{2(2)}{3(3)}$ = $\dfrac{4}{9}$, P(k = 1) = P(k = 2) = $\dfrac{2}{9}$, P(k = 3) = $\dfrac{1}{9}$. An algorithm that mimics *FairCoin* that only uses *BiasedBit* for it's randomness:

FairBit

generate k a 2-bit number
if (k = 0) **then** return 0
end if
if (k = 1 OR k = 2) **then** return 1
end if
RETURN *FairBit*

3. (a) Let A be the event that some users pick duplicated number. Let A_{ij} be the event that user i and user j pick the same number. Then we can upper bound $P(A)$ by:

$$\begin{aligned} P(A) &= P(\cup_{i\neq j} A_{ij}) \\ &\leq \sum_{i \neq j} P(A_{ij}) \\ &= \binom{n}{2} \frac{1}{2^b} \\ &= \frac{n(n-1)}{2^b \cdot 2} \end{aligned}$$

where the second line is the union bound, the third line holds since for all $i \neq j$, $P(A_{ij})$ should be the same: user i can pick anything, but j has to choose whatever i has chosen, which gives $\frac{1}{2^b}$. The problem boils down to counting how many A_{ij} exist: exactly $\binom{n}{2}$.

Since $b \geq 6 + 2\log n$, we have:

$$P(A) \leq \frac{n(n-1)}{2^b \cdot 2} \leq \frac{n(n-1)}{2^6 \cdot 2^{2\log n} \cdot 2} \leq \frac{n^2}{2^6 \cdot n^2 \cdot 2} \leq 0.008$$

Finally this gives:

$$P(\text{No collision}) = 1 - P(A) \geq 1 - 0.008 \geq 0.992$$

(b) Let A be the event in which a unique leader is elected. Let A_i denote the event that the unique leader picks the number i, where $i \in \{1, 2, \ldots, 2^b - 1\}$. Then we get the following equations, these are explained below:

$$\begin{aligned} P(A) &= P(\cup_{i=1}^{2^b-1} A_i) \\ &= \sum_{i=1}^{2^b-1} P(A_i) \\ &= \sum_{i=1}^{2^b-1} n \cdot \frac{1}{2^b} \cdot \left(1 - \frac{i}{2^b}\right)^{n-1} \tag{1} \\ &= \frac{n}{2^b} \sum_{i=1}^{2^b-1} \left(\frac{i}{2^b}\right)^{n-1} \\ &= \frac{n}{2^{bn}} \sum_{i=1}^{2^b-1} i^{n-1} \end{aligned}$$

The first equality is just by the definition of A and the A_i's. The second equality follows by noting that the A_i's are disjoint events. Let us look at the third equality: since the leader uniquely picks number i, he or she only has one choice out of the 2^b candidates, that is how we get the term $\frac{1}{2^b}$. The rest of the users have to pick a number bigger than the one that the leader selected. Since the leader selected number i, the others have pick one of $(2^b - i)$ numbers. So the probability that the rest of the users (there are $n-1$ of them) do not pick the same number as the leader does is $\left(1 - \frac{i}{2^b}\right)^{n-1}$. Since each of the n users could be the leader, that is why we multiply n in the beginning.
Recall that $\sum_{x=m}^{n} p(x) \geq \int_{m-1}^{n} p(x)dx$. In particular, let $m = 1$, $n = 2^b - 1$, and $p(x) = x^{n-1}$, we have:

$$\sum_{i=1}^{2^b-1} i^{n-1} \geq \int_{0}^{2^b-1} i^{n-1} di = \left.\frac{i^n}{n}\right|_{i=0}^{2^b-1} = \frac{(2^b-1)^n}{n}$$

substitute this result back to Eq. 1, we get:

$$P(A) \geq \frac{n}{2^{bn}} \frac{(2^b-1)^n}{n} = \left(1 - \frac{1}{2^b}\right)^n$$

Substitute $b \geq 8 + \log n$, we have:

$$\left(1 - \frac{1}{2^b}\right)^n \geq \left(1 - \frac{1}{2^8 n}\right)^n$$

From here on we can proceed in two ways:
(i) Use $(1 - \frac{1}{x})^x \geq 0.25$ for $x \geq 2$ (as seen in Section 1), in particular, set $x = 2^8 n = 256n \geq 2$, we get:

$$\left(1 - \frac{1}{2^8 n}\right)^n = \left(\left(1 - \frac{1}{2^8 n}\right)^{256n}\right)^{\frac{1}{256}} \geq 0.25^{1/256} > 0.994$$

(ii) Use $(1 + x) \geq e^{\frac{x}{1+x}}$ (as seen in Section 2), to get:

$$\left(1 - \frac{1}{256n}\right)^n \geq e^{\left(\frac{-\frac{1}{256n}}{1-\frac{1}{256n}}\right)\cdot n} = \left(\frac{1}{e}\right)^{\frac{n}{256n-1}} = \left(\frac{1}{e}\right)^{\frac{1}{256-\frac{1}{n}}} \geq \left(\frac{1}{e}\right)^{\frac{1}{255}} > 0.996$$

4. (a) In the worst case items are visited in decreasing order. $O(n)$

   (b) Looking at the k^{th} iteration we have $\binom{n}{k-1}$ different elements looked at previously, in these cases for the number of cases where the minimum element is in min choose the minimum element to be one of the elements looked at and the other k-1 elements looked at be any other element greater than the minimum or $\binom{n-1}{k-1}$, for the second minimum element there are $\binom{n-2}{k-1}$ cases. Now for any element the probability that line 4 executes is of elements greater than an element/ of remaining elements to look at for element i where $1 \leq i \leq n$ the probability that line 4 executes is $\dfrac{i-1}{n-k+1}$. Using conditional probabilities we have for the k^{th} iteration $\sum_{i=1}^{n} \dfrac{\binom{n-i}{k-1}}{\binom{n}{k-1}} \times \dfrac{i-1}{n-k+1}$ which simplifies to $\dfrac{1}{k}$.

   (c) Expected running time of line 4 is O(logn) since the probability to run line 4 in the k^{th} iteration is $\dfrac{1}{k}$ and we have n iterations the number of executions of line 4 in expectation is $\sum_{k=1}^{n} \dfrac{1}{k}$ which is the harmonic series $O(logn)$.

5. Let X be the random variable representing the number of records in a uniformly random permutation of numbers $\{1, \ldots, n\}$. We need to find $E[X]$. Let X_i be the indicator random variable for the event that i-th element is a

record. Then, $X = \sum_i X_i$. Moreover, for any i, $E[X_i] = Pr[X_i = 1]$ is the probability that the i-th number is greater than all the numbers before it. To find $Pr[X_i = 1]$ we think of first randomly partitioning the n numbers into the "first i" (as a set), the "last $n - i$", and then randomly ordering each of these sets. Whether the largest is placed last doesn't depend on how the numbers are partitioned, nor on the order of the last $n - i$. We can just focus on the random permutation of the first i elements. In a uniformly random permutation of i numbers, the largest number is uniformly distributed; therefore, the probablity it appears last is $\frac{1}{i}$. Thus, $E[X_i] = \frac{1}{i}$, and so $E[X] = \sum_{i=1}^{n} \frac{1}{i}$, the n-th harmonic number.