Counting

Chapter 6

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Permutations and Combinations

Section 6.3

Section Summary

Permutations (without Repetition)

Combinations (without Repetition)

Permutations

Definition: A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects.

- An ordered arrangement of r elements of a set is called an *r-permutation*.
- $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$
- P(n,r) = nPr

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

Example: How many ways can we select three students from a group of five students to stand in line for a picture?

- #ways = $5 \times 4 \times 3 = \mathbf{60}$.
- Using the formula: $\mathrm{P}(5{,}3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1} = 5 \times 4 \times 3 = \mathbf{60}$.

Counting Permutations

Examples:

1. How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?
   - $P(100,3) = 100 \cdot 99 \cdot 98 =$ **970,200**
2. Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?
   - The first city is chosen. Thus we have $P(7,7) = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$ **5040** paths.
3. How many permutations of the letters **ABCDEFGH** contain the string **ABC** ?
   - We solve this problem by counting the permutations of six objects, ABC, D, E, F, G, and H. Thus, we have $P(6,6) = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 =$ **720**

Combinations

Definition: An ***r-combination*** of elements of a set is an unordered selection of *r* elements from the set.

C(n,r) = nCr

$$\boxed{C(n,r)=\frac{n!}{(n-r)!r!}.}$$

$C(n, r)$ is also denoted by $\binom{n}{r}$

Example: Let *S* be the set {*a*, *b*, *c*, *d*}.

- The 3-combination of S is all 3-element subsets of S. It is the same as ***C*(4,3) = 4** → **{*a, b, c*}, {*a, b, d*}, {*a, c, d*}, {*b, c, d*}**.
- The 2-combination of S is all 2-element subsets of S. It is the same as ***C*(4,2) = 6** → **{*a, b*}, {*a, c*}, {*a, d*}, {*b, c*}, {*b, d*}, {*c, d*}**.

Counting Combinations

Examples:

1. How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

$$\boxed{C(10,5) = \frac{10!}{5!5!} = 252.}$$

2. How many different committees of three students can be formed from a group of four students?

$$\boxed{C(4,3) = \frac{4!}{1!\,3!} = 4.}$$

3. Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

$$\boxed{C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.}$$

Exercise

- **Find the value of the following:**
  1. $P(n, 0) = 1$
  2. $P(n, 1) = n$
  3. $P(n, n) = n!$
  4. $C(n, 0) = 1$
  5. $C(n, 1) = n$
  6. $C(n, n) = 1$

- **True or False:**
  1. $P(n, r) = C(n,r) \cdot r!$ True
  2. $C(n, r) = C(n, n-r)$ True
  3. $P(n, n) = 2 \cdot P(n, n-2)$ True

More Examples

Examples:

1. How many bit strings of length 10 that has exactly 3 zeros?

$$C(10,3) = \binom{10}{3}$$

2. How many bit strings of length 10 that has at least 3 zeros?

$$\binom{10}{3} + \binom{10}{4} + \cdots + \binom{10}{10} = \sum_{k=3}^{10} \binom{10}{k}$$

3. Suppose we have 4 books in Arabic, 3 books in Math, and 3 books in CS. How many ways to arrange them?

$$10!$$

4. Suppose we have 4 books in Arabic, 3 books in Math, and 3 books in CS. How many ways to arrange if we want each subject books together?

$$3!\,(4! \cdot 3! \cdot 3!)$$

Generalized Permutations and Combinations

Section 6.5

Section Summary

Permutations (with Repetition)

Combinations (with Repetition)

Permutations with Indistinguishable Objects

Permutations with Repetition

Definition: The number of r-permutations of a set of n objects <u>with repetition allowed</u> is n^r.

Examples:

1. How many strings of length five can be formed from the uppercase letters of the English alphabet?
   - **Solution**: The number of such strings is 26^5.
2. From the set of first 10 natural numbers (0-9), you are asked to make a four-digit number (PIN). How many different permutations are possible?
   - **Solution**: The total number of such PINs is 10^4.

Combinations with Repetition

Definition: The number of r-combinations from a set with n elements when repetition is allowed is:

$$\boxed{C(n+r-1,r)=C(n+r-1,\ n-1).}$$

Examples:

1. How many ways are there to select five bills from a box containing at least five of each of the following: \$1, \$2, \$5, \$10, \$20, \$50, \$100?
   - **Solution**: [n=7, r=5] $\boxed{C(11,5)=\frac{11!}{5!6!}=462}$
2. Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?
   - **Solution**: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. [n=4, r=6]

$$\boxed{C(9,6)=C(9,3)=\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}=84}$$

Combinations with Repetition 2

Examples:

1. How many solutions does the equation have, where x_1 , x_2 and x_3 are nonnegative integers? $x_1 + x_2 + x_3 = 11$

   - **Solution:** Each solution corresponds to a way to select 11 items from a set with three elements; x_1 items of type one, x_2 items of type two, and x_3 items of type three. [n=3, r=11]

$$\boxed{C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13\cdot 2}{1\cdot 2} = 78}$$

2. How many solutions for the same equation in question 1, where $x_1 \geq 1$, $x_2 \geq 2$ and $x_3 \geq 3$?

   - **Solution:** 1+2+3=6 → 11-6 =5. Therefore, each solution corresponds to a way to select 5 items from a set with three elements. [n=3, r=5]

$$\boxed{C(3+5-1,5) = C(7,5) = C(7,2) = \frac{7\cdot 6}{1\cdot 2} = 21}$$

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 Combinations and Permutations With and Without Repetition.

Type	***Repetition Allowed?***	***Formula***
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!(n-1)!}$

Permutations with Indistinguishable Objects

Theorem: **The number of different permutations of *n* objects**, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Example:

1. How many different strings can be made by reordering the letters of the word ***SUCCESS***.
   - n = 7 (#letters)
   - # strings = $\frac{\mathbf{7!}}{\mathbf{3!\,2!}} = 420$
2. Same as (1), but must starts with letter "**S**".
   - n = 6 (#letters)
   - # strings = $\frac{\mathbf{6!}}{\mathbf{2!\,2!}} = 180$