CSC281: Discrete Math for Computer Science

Computer Science Department | Structural Induction | First Semester 1442

King Saud University | Tutorial 10: ~~Strong~~ Induction + Recursive Definitions

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Question 1. Find the recursive definition for each of the following sequences $a_n,\ n = 1, 2...$ if:

a) $a_n = 8n$
b) $a_n = 3n + 2$
c) $a_n = 7^n$
d) $a_n = 30$

Question 2. Fine the recursive definition for each of the following

a) the set of all positive integers power of 5.
b) the set of positive integers congruent to 2 modulo 3.

Question 3. Let S be the subset of the set of ordered pairs of integers defined recursively by:
Basis step: $(0, 0) \in S$.
Recursive step: If $(a, b) \in S$, then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

a) List the elements of S produced by the first five applications of the recursive definition.
b) Use structural induction to show that $5|a + b$ when $(a, b) \in S$.

Q3

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a) List the elements of S produced by the first five applications of the recursive definition.
b) Use structural induction to show that $5|a + b$ when $(a, b) \in S$.

Solution. a) Given:

$$\begin{aligned}(0,0) &\in S\\ (a+2,b+3) \text{ whenever } (a,b) &\in S\\ (a+3,b+2) \text{ whenever } (a,b) &\in S\end{aligned}$$

First application: Apply the recursive step on (0,0):

$$\begin{aligned}(0+2,0+3) = (2,3) \in S\\ (0+3,0+2) = (3,2) \in S\end{aligned}$$

Second application: Apply the recursive step on (2,3) and (3,2)

$$\begin{aligned}(2+2,3+3) = (4,6) \in S\\ (2+3,3+2) = (5,5) \in S\\ (3+2,2+3) = (5,5) \in S\\ (3+3,2+2) = (6,4) \in S\end{aligned}$$

Third application: Apply the recursive step on (4,6),(5,5) and (6,4)

$$\begin{aligned}(4+2,6+3) = (6,9) \in S\\ (4+3,6+2) = (7,8) \in S\\ (5+2,5+3) = (7,8) \in S\\ (5+3,5+2) = (8,7) \in S\\ (6+2,4+3) = (8,7) \in S\\ (6+3,4+2) = (9,6) \in S\end{aligned}$$

Fourth application: Apply the recursive step on (6,9), (7,8), (8,7) and (9,6)

$$\begin{aligned}(6+2,9+3) = (8,12) \in S\\ (6+3,9+2) = (9,11) \in S\\ (7+2,8+3) = (9,11) \in S\\ (7+3,8+2) = (10,10) \in S\\ (8+2,7+3) = (10,10) \in S\\ (8+3,7+2) = (11,9) \in S\\ (9+2,6+3) = (11,9) \in S\\ (9+3,6+2) = (12,8) \in S\end{aligned}$$

Fifth application: Apply the recursive step on (8,12), (9,11), (10,10), (11,9), and (12,8)

$$\begin{aligned}(8+2,12+3) = (10,15) \in S\\ (8+3,12+2) = (11,14) \in S\\ (9+2,11+3) = (11,14) \in S\\ (9+3,11+2) = (12,13) \in S\\ (10+2,10+3) = (12,13) \in S\\ (10+3,10+2) = (13,12) \in S\\ (11+2,9+3) = (13,12) \in S\\ (11+3,9+2) = (14,11) \in S\\ (12+2,8+3) = (14,11) \in S\\ (12+3,8+2) = (15,10) \in S\end{aligned}$$

b) To proof: $5|(a+b)$ whenever $(a,b) \in S$
Proof by Structural Induction
Basis step $n = 0$
$(0,0) \in S$

$$5|0 + 0 \text{since } 0 \text{ is divisible by any integer}$$

Thus the property is true for the basis step
Recursive step Assume that $(a,b) \in S$ with $5|a+b$. The new elements formed from (a,b) in the recursive step are:

$$\begin{aligned}(a+2,b+3)\\ (a+3,b+2)\end{aligned}$$

Since $(a+2)+(b+3) = a+b+5$ and since $a+b$ and 5 are both divisible by5:

$$5|(a+2)+(b+3)$$

Since $(a+3)+(b+2) = a+b+5$ and since $a+b$ and 5 are both divisible by5:

$$5|(a+3)+(b+2)$$

Thus the property is true for the recursive step.
Conclusion By the principle of structural induction, $5|(a,b)$ whenever $(a,b) \in S$

Q2

Question 2. Fine the recursive definition for each of the following

a) the set of all positive integers power of 5.
b) the set of positive integers congruent to 2 modulo 3.

Solution. a) S is the set of positive integer powers of 5.
The first positive integer power of 5 is $5^1 = 5$.

$$5 \in S$$

Every positive integer powers of 5 is the previous integer power of 5 multiplied by 5

$$5s \in S \text{ whenever } s \in S$$

it is like:
if $s \in S \rightarrow 5s \in S$

b) S is the set of positive integer congruent to 2 modulo 3 (which are numbers with reminder 2 when divided by 3).
BS: The first positive integer congruent to 2 modulo 3 is 2.

$$2 \in S$$

RS: Every positive integer congruent to 2 modulo 3 is the previous positive integer congruent to 2 modulo 3 increased by 3.

$$s + 3 \in S \text{ whenever } s \in S$$

$2+3 = 5 \equiv 2 (\mathrm{mod}\ 3)$
$5+3 = 8 \equiv 2 (\mathrm{mo}\ 3)$
and so on...

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Q1

Question 1. Find the recursive definition for each of the following sequences $a_n,\ n = 1, 2...$ if:

a) $a_n = 8n$
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c) $a_n = 7^n$
d) $a_n = 30$

Solution. a) Given

$$a_n = 8n$$

a_{n-1}
$\cancel{8}n - \cancel{1} \to 0$
$= 8(n)$

Let us first determine the first value at $n = 1$:

$$a_1 = 8(1) = 8$$

Next we determine the recursive definition by writing a_n in terms of a_{n-1}.

$$a_n = 8n = 8n - 8 + 8 = 8(n-1) + 8 = a_{n-1} + 8$$

Thus the recursive definition is then:

$$\begin{aligned}a_1 &= 8\\ a_n &= a_{n-1} + 8 \quad \text{when } n \geq 2\end{aligned}$$

b) Given

$$a_n = 3n + 2$$

Let us first determine the first value at $n = 1$:

$$a_1 = 3(1) + 2 = 3 + 1 = 5$$

Next we determine the recursive definition by writing a_n in terms of a_{n-1}.

$$a_n = 3n + 2 = 3n - 3 + 3 + 2 = 3(n-1) + 3 + 2 = \underbrace{[3(n-1) + 2]}_{3(n)+2} + 3 = a_{n-1} + 3$$

a_n

Thus the recursive definition is then:

$$\begin{aligned}a_1 &= 5\\ a_n &= a_{n-1} + 3 \quad \text{when } n \geq 2\end{aligned}$$

c) Given

$$a_n = 7^n$$

Let us first determine the first value at $n = 1$:

$$a_1 = 7^1 = 7$$

Next we determine the recursive definition by writing a_n in terms of a_{n-1}.

$$a_n = 7^n = 7 \times 7^{n-1} = 7 \times a_{n-1}$$

Thus the recursive definition is then:

$$\begin{aligned}a_1 &= 7\\ a_n &= 7 \times a_{n-1} \quad \text{when } n \geq 2\end{aligned}$$

d) Given

$$a_n = 30$$

Let us first determine the first value at $n = 1$:

$$a_1 = 30$$

Next we determine the recursive definition by writing a_n in terms of a_{n-1}.

$$a_n = 30 = a_{n-1}$$

Thus the recursive definition is then:

$$\begin{aligned}a_1 &= 30\\ a_n &= a_{n-1} \quad \text{when } n \geq 2\end{aligned}$$