Counting

Chapter 6

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The Basics of Counting

Section 6.1

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Section Summary

The Product Rule

The Sum Rule

The Subtraction Rule

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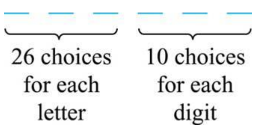
The Product Rule

The Product Rule: If there are n_1 ways to do something and there are n_2 ways to do another thing, then there are $n_1 \cdot n_2$ ways to do both actions.

- Do task1 AND task2

Examples:

1. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?



   - **Solution**: There are 26 x 26 x 26 x 10 x 10 x 10 = **17,576,000** different possible license plates.

2. How many bit strings of length seven are there?
   - **Solution**: Since each of the seven bits is either a 0 or a 1, the answer is $2^7 = $ **128**.

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The Sum Rule

The Sum Rule: If there are n_1 ways to do something and there are n_2 ways to do another thing, and both actions cannot be done at the same time, then there are $n_1 + n_2$ ways to choose one of the actions.

- Do task1 OR task2 (not both)

Examples:

1. The mathematics department must choose either a student or a faculty member as a representative for a university committee. How many choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student.
   - **Solution**: There are 37 + 83 = **120** possible ways to pick a representative.
2. A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?
   - **Solution**: There are 23 + 15 + 19 = **57** ways to choose a project.

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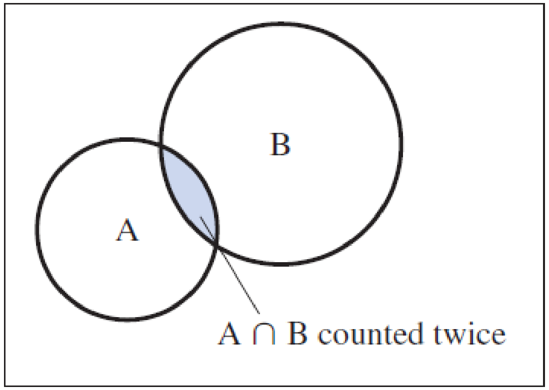
The principle of inclusion-exclusion

- Generalization of the sum rule.

The Subtraction Rule: If a task can be done either in one of n_1 ways or in one of n_2 ways, then the total number of ways to do the task is $n_1 + n_2$ – (the number of ways to do the task that are common to the two different ways).

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$



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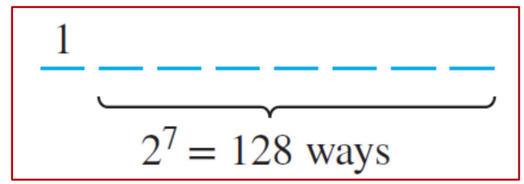
The principle of inclusion-exclusion$_2$

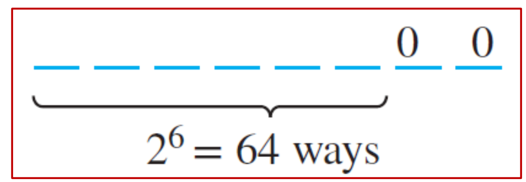
Example: How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

Solution: Using the subtraction rule:

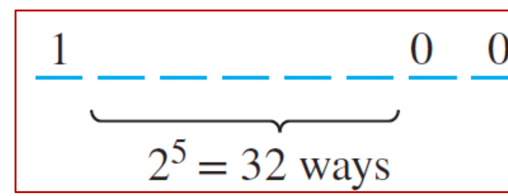
- Number of bit strings of length eight that start with a 1 bit: $2^7 = 128$



- Number of bit strings of length eight that end with bits 00: $2^6 = 64$



- Number of bit strings of length eight that start with a 1 bit and end with bits 00 : $2^5 = 32$



Hence, the number is $128 + 64 - 32 =$ **160**.

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Combining the Sum and Product Rule

Examples:

1. Suppose labeling chairs can be either a single letter or a letter followed by a digit. Find the number of possible labels?
   - 26 + (26 x 10) = **286.**
2. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
   - let P_6, P_7, and P_8 be the passwords of length 6, 7, and 8
   - $P_6 = 36^6 - 26^6 = 1{,}867{,}866{,}560.$
   - $P_7 = 36^7 - 26^7 = 70{,}332{,}353{,}920.$
   - $P_8 = 36^8 - 26^8 = 2{,}612{,}282{,}842{,}880.$
   - Consequently, Possible passwords = $P_6 + P_7 + P_8$ = **2,684,483,063,360**.

$$\boxed{\sum_{i=6}^{8}(36^i - 26^i)}$$

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Example: Counting Functions

Examples:

1. How many **functions** are there from a set with *m* elements to a set with *n* elements?
   - **Solution**: There are $n \cdot n \cdot .. \cdot n$ = $\mathbf{n^m}$ such functions.
2. How many **one-to-one** functions are there from a set with *m* elements to one with *n* elements?
   - **Solution**: Let m ≤ n, then there are $\boldsymbol{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-m+1)}$ such functions.
   - If m > n, then the number of one-to-one functions becomes **0**.
3. How many **one-to-one correspondence** functions are there from a set with *m* elements to one with *n* elements?
   - **Solution**: Let m=n, then there are **m!** such functions.

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Exercise

Question: Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters $*$, >, <, !, +, and =.

1. How many different passwords are available for this computer system?

$$\sum_{i=8}^{12} 68^i$$

2. How many of these passwords contain at least one of the six special characters?

$$\sum_{i=8}^{12} (68^i - 62^i)$$

3. Using your answer to part (1), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

$$\sum_{i=8}^{12} 68^i \approx 9.92 \times 10^{21}$$

$$\frac{9.92 \times 10^{21}}{10^9} \approx 9.92 \times 10^{12}\ Seconds \approx 313926 \text{ years!}$$

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The Pigeonhole Principle

Section 6.2

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Section Summary

The Pigeonhole Principle

The Generalized Pigeonhole Principle

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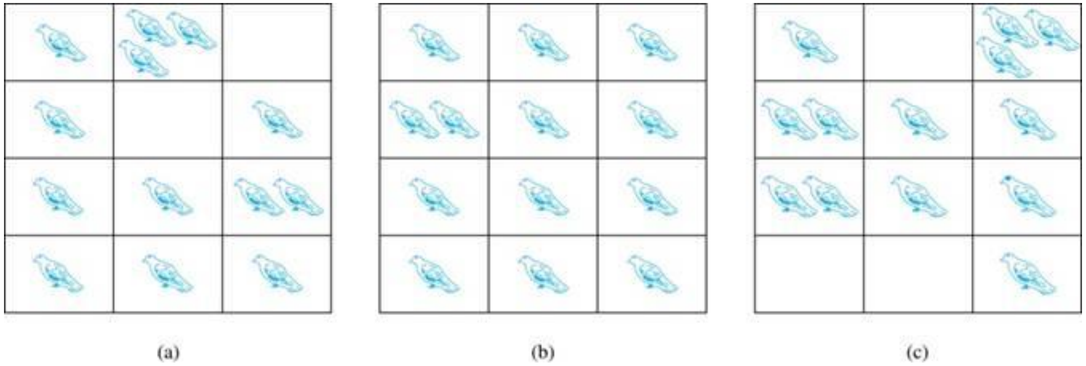
The Pigeonhole Principle

Pigeonhole Principle: If k+1 objects or more are placed into k boxes, then at least one box contains two or more objects.

Proof: We use a proof by contradiction. Suppose each box has one object. Then the total number of objects would be at most k. This contradicts the statement that we have k + 1 objects.

Example: If a flock of 13 pigeons roosts in a set of 12 pigeonholes, one of the pigeonholes must have more than 1 pigeon.



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The Pigeonhole Principle 2

Examples:

1. A function *f* from a set with *k*+1 elements to a set with *k* elements is **not** one-to-one.
2. Among any group of 367 people, there must be at least two with the same birthday.
3. Among a class with 13 students, there must be at least two students born in the same month.
4. In any group of 27 English words, there must be at least two that begin with the same letter.

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The Generalized Pigeonhole Principle

The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

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The Generalized Pigeonhole Principle 2

Examples:

1. This class has 30 students. What is the minimum number of students we guarantee they born in the same month?
   - There are at least $\lceil 30/12 \rceil$ = **3** students born in the same month.
2. Suppose we have 10 black, 10 white, 10 red, and 10 brown socks. All mixed up. How many to pick so we guarantee two socks have the same color?
   - $\lceil N/4 \rceil = 2 \rightarrow N =$ **5**
3. How many socks to pick up so we guarantee we have two reds?
   - Pick **32** socks. → worst-case scenario: first 30 picks are black, white, and brown.
4. How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen?
   - $\lceil N/4 \rceil = 3 \rightarrow N =$ **9**
5. How many must be selected to guarantee that at least three hearts (♥) are selected?
   - Select **42** cards. → worst-case scenario: first 39 picks are clubs (♣), diamonds (♦), and spades (♠).

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Exercise

Question: Given a set of numbers from 1,2,..,25. Show that when pick any 14 numbers in random, there will be at least two numbers sum to 26?

- Worst-case scenario: The first 13 pick are numbers from 1,2,..,13
- The 14th number will be any number between 14,..,25
  - This number will pair with any number from the 13 numbers and sum to 26.

	1	2	3	4	5	6	7	8	9	10	11	12	13
+	25	24	23	22	21	20	19	18	17	16	15	14	
	26	26	26	26	26	26	26	26	26	26	26	26	

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