CSC281: Discrete Math for Computer Science

Computer Science Department
King Saud University

First Semester 1443

Tutorial 12: Counting (6.3, 6.4, 6.5, and 8.5)

Question 1. How many bit strings of length 12 contain

a) exactly three 1s?

b) at most three 1s?

c) at least three 1s?

d) an equal number of 0s and 1s?

Solution.

a) The order is not important thus we use combination.
$n = 12$ and $r = 3$

$$C(12,3) = \frac{12!}{3!(12-3)!} = \frac{12!}{3!9!} = 220$$

b) (combination) $n = 12$ and $r \leq 3$

$$\begin{aligned} C(12,3) &= \frac{12!}{3!(12-3)!} = \frac{12!}{3!9!} = 220 \\ C(12,2) &= \frac{12!}{2!(12-2)!} = \frac{12!}{2!10!} = 66 \\ C(12,1) &= \frac{12!}{1!(12-1)!} = \frac{12!}{1!11!} = 12 \\ C(12,0) &= \frac{12!}{0!(12-0)!} = \frac{12!}{0!12!} = 1 \end{aligned}$$

Add the numbers of bit strings for each value of r:

$$220 + 66 + 12 + 1 = 299$$

c) $n = 12$ and $r \geq 3$

$$C(12,3) = \frac{12!}{3!(12-3)!} = \frac{12!}{3!9!} = 220$$
$$C(12,4) = \frac{12!}{4!(12-4)!} = \frac{12!}{4!8!} = 495$$
$$C(12,5) = \frac{12!}{5!(12-5)!} = \frac{12!}{5!7!} = 792$$
$$C(12,6) = \frac{12!}{6!(12-6)!} = \frac{12!}{6!6!} = 924$$
$$C(12,7) = \frac{12!}{7!(12-7)!} = \frac{12!}{7!5!} = 792$$
$$C(12,8) = \frac{12!}{8!(12-8)!} = \frac{12!}{8!8!} = 495$$
$$C(12,9) = \frac{12!}{9!(12-9)!} = \frac{12!}{9!3!} = 220$$
$$C(12,10) = \frac{12!}{10!(12-10)!} = \frac{12!}{10!2!} = 66$$
$$C(12,11) = \frac{12!}{11!(12-11)!} = \frac{12!}{11!1!} = 12$$
$$C(12,12) = \frac{12!}{12!(12-12)!} = \frac{12!}{12!0!} = 1$$

Add the number of bit strings for each value of r:
220+495+792+924+792+495+220+66+12+1=4017

d) $n = 12$ and $r = 6$

$$C(12,6) = \frac{12!}{6!(12-6)!} = \frac{12!}{6!6!} = 924$$

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Question 2. Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?

b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?

c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

Solution. a) We have the total number of people =13, and we want to find the total ways of choosing 10 players out of 13:

$$
\begin{aligned}
&=C(13,10) .. \text{ (order is not important)}\\
&=C(13,3) .. \ (C(n,r)=C(n,n-r))\\
&=\frac{13\times 12\times 11}{3\times 2\times 1}\\
&=13\times 11\times 2\\
&=286
\end{aligned}
$$

b) (Order matters) Total number of assigning 10 positions out of 13 people:

$$
\begin{aligned}
&=P(13,10)\\
&=\frac{13!}{(13-10)!}\\
&=\frac{13!}{3!}\\
&=13\times 12\times 11\times ...\times 4\\
&=1037836800
\end{aligned}
$$

c) Total ways of choosing 10 players out of 13 players where there must be at least one woman player:

$$
\begin{aligned}
&=C(3,1)\times C(10,9)+C(3,2)\times C(10,8)+C(3,3)\times C(10,7) ... \ \text{ (here we do not care about the order)}\\
&=3\times C(10,1)+3\times C(10,2)+1\times C(10,3) .. \ \text{ (we know} C(n,r)=C(n,n-r))\\
&=3\times 10+3\times \frac{10\times 9}{2\times 1}+\frac{10\times 9\times 8}{3\times 2\times 1}\\
&=30+135+10\times 3\times 4\\
&=30+135+120\\
&=285
\end{aligned}
$$

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Question 3. What is the coefficient of x^7 in $(1+x)^{11}$?

Solution.

$$(x+y)^n=\sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

the term we are looking for is x^7 (1^4x^7) in $(1+x)^{11}$
$n=11$ and $j=7$ is given by $\binom{11}{7}x^{11-7}1^7$ The coefficient of this term is then $\binom{n}{j}=\binom{11}{7}$

$$\binom{n}{j}=\binom{11}{7}=\frac{11!}{7!(11-7)!}=\frac{11!}{7!4!}=330$$

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Question 4. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$

a) using a combinatorial argument.

b) by algebraic manipulation.

Solution.

a) We want to choose a total of 2 objects from two boxes, each containing n . It can be done in $\binom{2n}{2}$ number of ways. Now we can choose both of them from one box one from each box. For the first case there are $\binom{n}{2}$ ways to do it for either of the boxes, whereas in the second case, the product rule implies that the number of ways is $(\binom{n}{1})^2$. Combining both

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

b) $\binom{2n}{2}$= coefficient of x^2 in $(1+x)^{2n}$=coefficient of x^2 in $(1+x)^2(1+x)^2$= 2 coefficient of x^2 in $(1+x)^2$ coefficient of x^0 in $(1+x)^2$ + coefficient of x in $(1+x)^n$ coefficient of x in $(1+x)^2 = 2\binom{n}{2} + (\binom{n}{1})^2$

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Question 5. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 29,$$

where $x_i, i = 1,2,3,4,5,6$, is a nonnegative integer such that

a) $x_i > 1$ for $i = 1,2,3,4,5,6$?

b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$ and $x_6 \geq 6$?

c) $x_1 \leq 5$?

d) $x_1 < 8$ and $x_2 > 8$?

Solution.
The integer solution of the equation $x_1+x_2+x_3+x_4+x_5+x_6 = 29$ can be obtained by selecting r objects from a set with n objects such that there are x_1 chosen from the first type, x_2 are chosen from the second type and so on.Since the order of solutions is not important and the repetition is allowed we use **combination**.

a) $x_i > 1$ for $i = 1,2,3,4,5,6$
since x_i needs to be greater than 1, let us redefine $x'_i = x_i - 2$ with $i = 1,2,3,4,5,6$
$x'_1 + x'_2 + x'_3 + x'_4 + x'_5 + x'_6 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 2 \times 6 = 29 - 12 = 17$
we want to select 17 indistinguishable objects from 6 distinguishable boxes (variables).
$n = 6$ and $r = 17$

$$C(6+17-1, 17) = C(22, 17) = \frac{22!}{17!(22-17)!} = \frac{22!}{17!5!} = 26,334$$

b) $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3, x_4 \geq 4, x_5 > 5$ and $x_6 \geq 6$ Since x_1 needs at least 1, we redefine $x_1' = x_1 - 1$. Since x_2 needs at least 2, we redefine $x_2' = x_2 - 2$. And x_3 to be we redefine $x_3' = x_3 - 3$. And x_4 we redefine $x_4' = x_4 - 4$, And x_5 we redefine $x_5' = x_5 - 6$, Finally we redefine $x_6' = x_6 - 6$
$x_1'+x_2'+x_3'+x_4'+x_5'+x_6' = x_1+x_2+x_3+x_4+x_5+x_6-1-2-3-4-6-6 = 29-22 = 7$
we want to select 7 indistinguishable objects from 6 distinguishable boxes (variables).
$n = 6$ and $r = 7$

$$C(6+7-1,7) = C(12,7) = \frac{12!}{7!(12-17)!} = \frac{12!}{7!5!} = 792$$

c) First we find the number of solutions without restrictions
We want to select 29 indistinguishable objects from 6 distinguishable boxes (variables).
$n = 6$ and $r = 29$

$$C(6+29-1,29) = C(34,29) = \frac{34!}{29!(34-29!)} = \frac{34!}{29!5!} = 278,256$$

Number of solutions with $x_1 > 5$
Since x_1 needs at least 6, we redefine $x_1' = x_1 - 6$.
$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 6 = 29 - 6 = 23$
we want to select 23 indistinguishable objects from 6 distinguishable boxes (variables).
$n = 6$ and $r = 23$

$$C(6+23-1,23) = C(28,23) = \frac{28!}{23!(28-23)!} = \frac{28!}{23!5!} = 98,280$$

Number of solutions with $x_1 \leq 5$
Since there are 278,256 solutions without restrictions and 98,280 with $x_1 > 5$, then, there are 278,256-98,280=179,976 solutions with $x_1 \leq 5$

d) Number of solutions with restrictions $x_2 > 8$
redefine $x_2' = x_2 - 9$.
$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 9 = 29 - 9 = 20$
we want to select 20 indistinguishable objects from 6 distinguishable boxes (variables).
$n = 6$ and $r = 20$

$$C(6+20-1,20) = C(25,20) = \frac{25!}{20!(25-20)!} = \frac{25!}{20!5!} = 53,130$$

Number of solutions with restriction $x_1 \geq 8$ and $x_2 > 8$
redefine $x_1' = x_1 - 8$, and redefine $x_2' = x_2 - 9$.
$x_1' + x_2' + x_3' + x_4' + x_5' + x_6' = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 - 8 - 9 = 29 - 17 = 12$
we want to select 12 indistinguishable objects from 6 distinguishable boxes (variables).
$n = 6$ and $r = 12$

$$C(6+12-1,12) = C(17,12) = \frac{17!}{12!(17-12)!} = \frac{17!}{12!5!} = 6188$$

Number of solutions with $x_1 < 8$ and $x_2 > 8$
There are 53,130 solutions with $x_2 > 8$ and 6188 with $x_1 \geq 8$ and $x_2 > 8$, So, there are $53,130 - 6188 = 46,942$ solutions

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Question 6. How many different strings can be made from the letters in MISSISSIPPI, using all the letters?

Solution.
Total number of letters=11
"I" 4 times, "S" 4 times, "P" 2 times, and "M" once.

$$\begin{aligned}\frac{11!}{4! \times 4! \times 2! \times 1!} =& \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4! \times 2! \times 1!}\\ =& 11 \times 10 \times 9 \times 7 \times 5\\ =& 34650\end{aligned}$$

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Question 7. A survey of households in the United States reveals that 96% have at least one television set, 98% have telephone service, and 95% have telephone service and at least one television set. What percentage of households in the United States have neither telephone service nor a television set?

*Solution.*Let V denote the set of households with television sets, and P denote the set of households with telephone service. From the problem statement, we know $|V| = 96,\ |P| = 98$, and $|V \cap P| = 95$.
By inclusion-exclusion,we know that$|V \cup P| = |V| + |P| - |V \cap P| = 96 + 98 - 95 = 99$.
Since were looking for the set of households without telephone service nor television sets, then $100 - |V \cup P| = 100 - 99 = 1$.
Only 1% of households have neither telephone service nor a television set. 1% ■

Question 8. How many students are enrolled in a course either in calculus, discrete mathematics, data structures, or programming languages at a school if there are 507, 292, 312, and 344 students in these courses, respectively; 14 in both calculus and data structures; 213 in both calculus and programming languages; 211 in both discrete mathematics and data structures; 43 in both discrete mathematics and programming languages; and no student may take calculus and discrete mathematics, or data structures and programming languages, concurrently?

*Solution.*Let C denote the set of students that have taken a calculus course , DM denote the set of students that have taken a discrete mathematics course, DS denote the set of students that have taken a data structures , and P denote the set of students that have taken a programming languages course.
$|C| = 507, |DM| = 292, |DS| = 312, |P| = 344, |C \cap DS| = 14, |C \cap P| = 213, |DM \cap DS| = 211, and\ \ |DM \cap P| = 43$
By inclusion-exclusion
$|C \cup DM \cup DS \cup P| = |C| + |DM| + |DS| + |P| - |C \cap DS| - |C \cap P| - |DM \cap DS| - |DM \cap P|$
$|C \cup DM \cup DS \cup P| = 507 + 292 + 312 + 344 - 14 - 213 - 211 - 43$

$|C \cup DM \cup DS \cup P| = 974$ ■