

## Counting

Chapter 6

Edited by: Dr. Meshal Alfarhood

# Binomial Coefficients and Identities

Section 6.4

## **Section Summary**

Pascal's Identity and Triangle

The Binomial Theorem

Identities Involving Binomial Coefficients

## Pascal's Identity

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \qquad 1$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \qquad 1 \qquad 1$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \text{By Pascal's identity:} \qquad 1 \qquad 2 \qquad 1$$

$$\begin{pmatrix} 3 \\ 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \qquad 1 \qquad 3 \qquad 3 \qquad 1$$

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \qquad \qquad \qquad 1 \qquad 4 \qquad 6 \qquad 4 \qquad 1$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \end{pmatrix} \qquad \qquad \qquad 1 \qquad 5 \qquad 10 \quad 10 \quad 5 \qquad 1$$

$$\begin{pmatrix} 6 \\ 0 \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 6 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \begin{pmatrix} 6 \\ 6 \end{pmatrix} \qquad \qquad 1 \qquad 6 \qquad 15 \qquad 20 \quad 15 \qquad 6 \qquad 1$$

$$\begin{pmatrix} 7 \\ 0 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \end{pmatrix} \begin{pmatrix} 7 \\ 2 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \end{pmatrix} \begin{pmatrix} 7 \\ 4 \end{pmatrix} \begin{pmatrix} 7 \\ 5 \end{pmatrix} \begin{pmatrix} 7 \\ 6 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix} \qquad \qquad 1 \qquad 7 \qquad 21 \quad 35 \quad 35 \quad 21 \qquad 7 \qquad 1$$

$$\begin{pmatrix} 8 \\ 0 \end{pmatrix} \begin{pmatrix} 8 \\ 1 \end{pmatrix} \begin{pmatrix} 8 \\ 2 \end{pmatrix} \begin{pmatrix} 8 \\ 3 \end{pmatrix} \begin{pmatrix} 8 \\ 4 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} \begin{pmatrix} 8 \\ 8 \end{pmatrix} \qquad 1 \qquad 8 \qquad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Other observations from Pascal's triangle:

$$C(n,r) = C(n,n-r)$$

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

#### The Binomial Theorem

1. 
$$(x + y)^2 = x^2 + 2xy + y^2$$

2. 
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

3. 
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

4. 
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

**Binomial Theorem**: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j$$

## Using the Binomial Theorem

#### **Examples:**

- 1. What is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(x + y)^{25}$ ?
  - **Solution:** the coefficient of  $x^{12}y^{13}$  in the expansion is obtained when j = 13.

$$\binom{25}{13} = \frac{25!}{13! \, 12!} = 5,200,300.$$

- 2. What is the coefficient of  $x^{12}y^{13}$  in the expansion of  $(2x 3y)^{25}$ ?
  - **Solution**: We view the expression as  $(2x + (-3y))^{25}$ . By the binomial theorem:

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} {25 \choose j} 2x^{25-j} (-3y)^j.$$

Consequently,

$$\binom{25}{13} 2^{12} (-3)^{13} = -\frac{25!}{13! \ 12!} 2^{12} 3^{13}.$$

## A Useful Identity

**Theorem 1**: With  $n \ge 0$ ,  $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ .

<u>Proof</u> (using binomial theorem): With x = 1 and y = 1, from the binomial theorem we see that:

$$2^{n} = (1+1)^{n} = \sum_{k=0}^{n} {n \choose k} 1^{k} 1^{(n-k)} = \sum_{k=0}^{n} {n \choose k}.$$

**Theorem 2:** With  $n \ge 0$ ,  $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ 

<u>Proof</u> (using binomial theorem): With x = -1 and y = 1, from the binomial theorem we see that:

$$0 = 0^n = ((-1) + 1)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k} (-1)^k.$$