Basic Structures: Sets, Functions, Sequences, and Summation

Chapter 2

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Sets

Section 2.1

Section Summary

Definition of sets

Describing Sets

- Roster Method
- Set-Builder Notation

Some Important Sets in Mathematics

Empty Set and Universal Set

Subsets and Set Equality

Cardinality of Sets

Cartesian Product

Sets

- A ***set*** is an unordered collection of objects.
  - Example: A={1,3,5,7,9}, B={a, 2, Fahad, Riyadh}
    - $1 \in A$ means that 1 is an element of the set *A*.
    - $1 \notin B$ means that 1 is **not** a member of *B*.
  - About sets:
    - Order not important.
      - $S = \{a,b,c,d\} = \{b,c,a,d\}$
    - Listing more than once does not change the set.
      - $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$
    - Sets can be elements of sets.
      - $\{\{1,2,3\},a,\{b,c\}\}$

Describing a Set

- **There are two ways to describe sets:**
  1. Roster Method
  2. Set-Builder Notation
- **Examples:**
  - Set of all odd positive integers less than 10
    1. A = {1,3,5,7,9}
    2. A = {x | x is an odd positive integer less than 10}
  - Set of all positive integers less than 100
    1. B = {1,2,3,...,99}
    2. B = {x | x is a positive integer less than 100}

Some Important Sets

N = set of *natural numbers* = {0,1,2,3….}

Z = set of *integers* = {…,-3,-2,-1,0,1,2,3,…}

$\mathbf{Z}^{+}$ = set of *positive integers* = {1,2,3,…..}

R = set of *real numbers*

$\mathbf{R}^{+}$ = set of *positive real numbers*

Q = set of rational numbers

Universal Set and Empty Set

- The ***universal set*** U is the set containing everything currently under consideration.
- The ***empty set*** is the set with no elements. $\emptyset$, or $\{\}$
- The empty set is different from a set containing the empty set. $\emptyset \neq \{\emptyset\}$

Set Equality

Definition: Two sets are ***equal*** if and only if they have the same elements.

- A and B are equal iff $\boxed{\forall x\left(x \in A \leftrightarrow x \in B\right)}$
- We write $A = B$
- **Examples:**
  - {1,3,5} = {3,5,1}
  - {1,5,5,5,3,3,1} = {1,3,5}

Subsets

Definition: The set A is a ***subset*** of B, if and only if every element of A is also an element of B.

- A is a subset of the set B iff $\boxed{\forall x\left(x \in A \rightarrow x \in B\right)}$
- We write $A \subseteq B$
- Any set is considered to be a subset of itself.
- **Examples:**
  1. $\emptyset \subseteq S$, for every set S.
  2. $\{1,3,5,7,9\} \subseteq \{1,2,3,4,5,6,7,8,9\}$
  3. $\{1,3,5,7,9\} \subseteq \{1,3,5,7,9\}$

Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a ***proper subset*** of B.

- A is a proper subset of the set B iff

$$\boxed{\forall x(x \in A \rightarrow x \in B) \land \exists x(x \in B \land x \notin A)}$$

- We write $A \subset B$
- No set is a proper subset of itself.
- **Examples:**
  1. $\emptyset \subset S$, for every set S, except the empty set.
  2. $\{1,3,5,7,9\} \subset \{1,2,3,4,5,6,7,8,9\}$

Set Cardinality

Definition: The ***cardinality*** of a finite set A is the number of (distinct) elements of A.

- We write $|A|$

Examples:

1. $|\emptyset| = 0$
2. $|\{\emptyset\}| = 1$
3. Let S be the letters of the English alphabet. Then $|S| = 26$
4. $|\{1,2,3\}| = 3$
5. The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set A is called the ***power set*** of A.

- We write P(A)

Examples:

1. P({a,b}) = {Ø, {a},{b},{a,b}}
2. P(Ø) = {Ø}
3. P({Ø}) = {Ø, {Ø}}

The cardinality of the power set is 2^n, where n *is* #elements.

Cartesian Product

Definition: The ***Cartesian Product*** of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$$

Examples:

1. A = {a,b}, B = {1,2,3}
   - $A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$
2. A = {0,1}, B = {1,2} and C = {0,1,2}
   - $A \times B \times C$ = {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

Exercise

Suppose A={1,2,{3,4},{5,6,7}}. Determine the truth of the following statements:

1. |A|=4
2. |P(A)|=16
3. $\emptyset \in A$
4. $\emptyset \subseteq A$
5. $\{\emptyset\} \subseteq A$
6. $1 \in A$
7. $1 \subseteq A$
8. $\{1\} \subseteq A$
9. $3 \in A$
10. $\{3,4\} \in A$
11. $\{3,4\} \subseteq A$
12. $\{\{3,4\}\} \subseteq A$

1. True
2. True
3. False
4. True
5. False
6. True
7. False
8. True
9. False
10. True
11. False
12. True

Set Operations

Section 2.2

Section Summary

Set Operations

- Union
- Intersection
- Difference
- Complement

Set Identities

Proving Identities

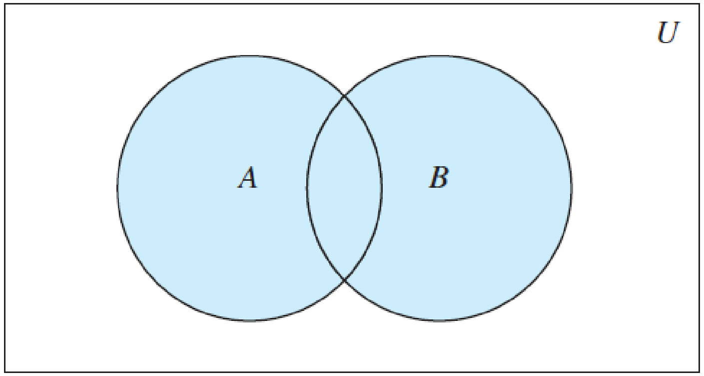
Union

Definition: The ***union*** of sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x \mid x \in A \vee x \in B\}$$

Example: What is $\{1,2,3\} \cup \{3,4,5\}$?

Solution: $\{1,2,3,4,5\}$



$A \cup B$ is shaded.

Intersection

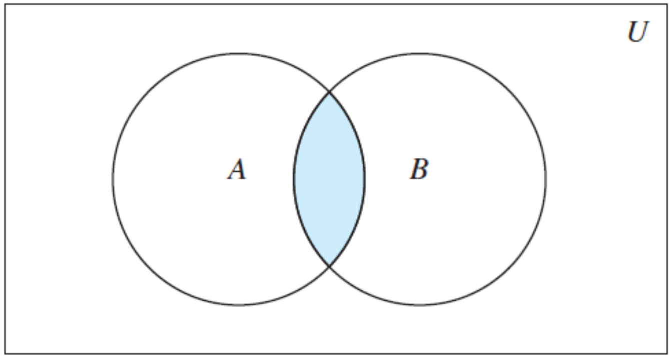
Definition: The ***intersection*** of sets *A* and *B*, denoted by $A \cap B$, is the set: $\{x \mid x \in A \wedge x \in B\}$

Example 1: What is {1,2,3} ∩ {3,4,5} ?

Solution: {3}

Example 2: What is {1,2,3} ∩ {4,5,6} ?

Solution: ∅



$A \cap B$ is shaded.

Note: if the intersection is empty, then *A* and *B* are said to be *disjoint*.

Difference

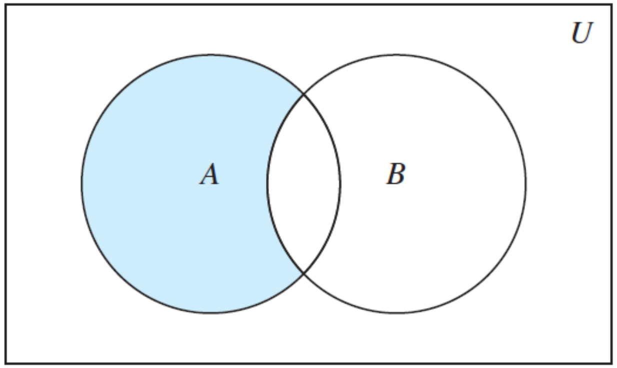
Definition: The ***difference*** of *A* and *B*, denoted by *A* – *B*, is the set containing the elements of *A* that are not in *B*.

$$\boxed{A - B = \{x \mid x \in A \land x \notin B\}}$$

$$\boxed{A - B = A \cap \overline{B}}$$

Example: What is {1,3,5} – {1,2,3}?

Solution: {5}



$A - B$ is shaded.

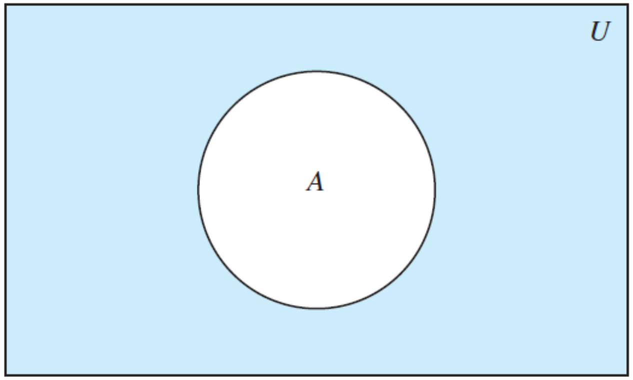
Complement

Definition: If A is a set, then the *complement* of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$.

$$\boxed{\bar{A} = \{x \in U \mid x \notin A\}}$$

Example: If U is the set of all positive integers, and A is the set of positive integers greater than 10. what is the complement of A?

Solution: $\bar{A}=\{1,2,3,4,5,6,7,8,9,10\}$



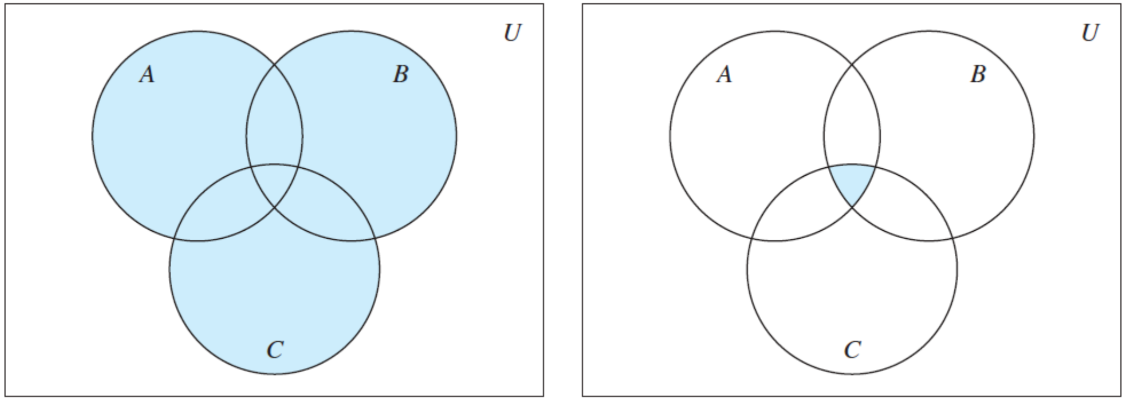
$\bar{A}$ is shaded.

Generalized Unions and Intersections

Let $A_1, A_2, \ldots, A_n$ be an indexed collection of sets.

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

Exercise

- **Exercise 1**: If A−B = {1,5,7,8}, B−A = {2,10}, and A∩B = {3,6,9}. Find sets A and B.
  - **Solution:** A = {1,3,5,6,7,8,9}
    B = {2,3,6,9,10}

- **Exercise 2**: If A = {x,y,z}, B = {1,2}, and C = {x,z}. Find the following sets:
  1. A−C
     = {y}
  2. $|P(A \cup B \cup C)|$
     = 32
  3. $(A \times B) - (B \times C)$
     = {(x,1),(x,2),(y,1),(y,2),(z,1),(z,2)}
  4. $\{(a,b,c) \mid a,b,c \in B \wedge a \neq b \wedge a \neq c\}$
     = {(1,2,2), (2,1,1)}

Set Identities [1]

Identity laws: $A \cup \varnothing = A \qquad A \cap U = A$

Domination laws: $A \cup U = U \qquad A \cap \varnothing = \varnothing$

Idempotent laws: $A \cup A = A \qquad A \cap A = A$

Complementation law: $\left(\overline{\overline{A}}\right) = A$

Set Identities [2]

De Morgan's laws:

$$\boxed{\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}}$$

Absorption laws:

$$\boxed{A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A}$$

Complement laws:

$$\boxed{A \cup \overline{A} = U \qquad A \cap \overline{A} = \varnothing}$$

Set Identities [3]

Commutative laws:

$$\boxed{A \cup B = B \cup A \qquad\qquad A \cap B = B \cap A}$$

Associative laws:

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proving Set Identities

Different ways to prove set identities:

1. Prove by **applying existing identities**.
2. **Membership Tables**: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof Using First Method

Example: Prove that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

Solution:

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A}
\end{aligned}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	B	C	B∩C	A∪ (B∩C)	A∪B	A∪C	(A∪B) ∩ (A∪C)
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0