CSC281: Discrete Math for Computer Science

Computer Science Department — Second Semester 1441/1442

King Saud University — Tutorial 11: The Basics of Counting+ The Pigeonhole Principle

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Question 1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

Question 2. How many strings of five ASCII characters contain the character @ (at sign) at least once? [Note: There are 128 different ASCII characters.

1

(a) We need to use the **product rule**, because the first event is picking a mathematics major and the second event is picking a computer science major

$$18 \cdot 325 = 5850$$

(b) We need to use the **sum rule**, because the event is picking a mathematics major OR picking a computer science major (while nobody falls in both categories, thus non-overlapping)

$$18 + 325 = 343$$

Question 3. How many positive integers less than 1000*

a) are divisible by 7?

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11?

d) are divisible by either 7 or 11?

e) are divisible by exactly one of 7 and 11?

f) are divisible by neither 7 nor 11?

g) have distinct digits?

h) have distinct digits and are even?

2

There are 128 different ASCII characters

Strings of length 5 We need to use the **product rule**, because the first event is picking the first character, the second event is picking the second character, ..., the 5th event is picking the 5th character.

$$128 \cdot 128 \cdot 128 \cdot 128 \cdot 128 = 128^5$$

Strings of length 5 without an @ We need to use the **product rule**, because the first event is picking the first bit, the second event is picking the second bit, ..., the 4th event is picking the 4th bit. Note: When the string cannot contain an @, then there are 127 possible ASCII characters

$$127 \cdot 127 \cdot 127 \cdot 127 \cdot 127 = 127^5$$

Strings of length 5 with at least one @ Strings of length 5 with at least one @ are strings of length 5 that are not strings of length 5 without an @

$$128^5 - 127^5 = 34,359,738,368 - 33,038,369,407 = 1,321,368,961$$

Question 4. How many license plates can be made using either three digits followed by three uppercase English letters or three Arabic letters followed by three digits?*

Question 5. How many strings of eight English letters are there

a) that contain no vowels, if letters can be repeated?

b) that contain no vowels, if letters cannot be repeated?

c) that start with a vowel, if letters can be repeated?

d) that start with a vowel, if letters cannot be repeated?

e) that contain at least one vowel, if letters can be repeated?

f) that contain exactly one vowel, if letters can be repeated?

g) that start with X and contain at least one vowel, if letters can be repeated?

h) that start and end with X and contain at least one vowel, if letters can be repeated?

4

There are 26 possible letters and 10 possible digits.

First letter= 26 ways
Second letter= 26 ways
Third letter= 26 ways
First digit= 10 ways
Second digit= 10 ways
Third digit= 10 ways
Order = 2 ways (because letters are before the digits or the digits are before the letters)

Using the product rule:

$$26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 2 = 26^3 \cdot 10^3 \cdot 2 = 35,152,000$$

Question 6. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.*

Question 7. A room contains 10 men and 10 women. A manger selects members of team at random without looking at them.

a) How many member must the manger select to be sure of having at least three members of the same gender?

b) How many member must the manger select to be sure of having at least three women in the team?*

Question 8. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

3

(a) Let A be the positive integers less than 1000.

We then note that A contains 999 integers, while we are interested in integers divisible by 7.

$$|A| = 999$$
$$d = 7$$

Using the quotient rule (round down!):

$$n_7 = \frac{|A|}{d} = \frac{999}{7} \approx 142.7143 \approx 142$$

Thus 142 integers are divisible by 7.

3 (b) **Number of integers divisible by 7 and by 11**

Let A be the positive integers less than 1000.

We then note that A contains 999 integers, while we are interested in integers divisible by 7 and divisible by 11, thus divisible by $7 \cdot 11 = 77$.

$$|A| = 999$$
$$d = 77$$

Using the quotient rule (round down!):

$$n_{77} = \frac{|A|}{d} = \frac{999}{77} \approx 12.9740 \approx 12$$

Thus 12 integers are divisible by 77.

Number of integers divisible by 7 but not by 11

Integers divisible by 7 but not by 11 are integers divisible by 7 that are not integers that are divisible by 7 and 11.

$$n_{7,\text{ not }11} = n_7 - n_{11} = 142 - 12 = 130$$

4 (c) Let A be the positive integers less than 1000.

We then note that A contains 999 integers, while we are interested in integers divisible by 7 and divisible by 11, thus divisible by $7 \cdot 11 = 77$.

$$|A| = 999$$
$$d = 77$$

Using the quotient rule (round down!):

$$n_{77} = \frac{|A|}{d} = \frac{999}{77} \approx 12.9740 \approx 12$$

Thus 12 integers are divisible by 77.

5 (d) **Number of integers divisible by 11**

Let A be the positive integers less than 1000.

We then note that A contains 999 integers, while we are interested in integers divisible by 11.

$$|A| = 999$$
$$d = 11$$

Using the quotient rule (round down!):

$$n_{11} = \frac{|A|}{d} = \frac{999}{11} \approx 90.8182 \approx 90$$

Thus 90 integers are divisible by 11.

Number of integers divisible by either 7 or 11

Use the subtraction rule:

$$n_{7\text{ or }11} = n_7 + n_{11} - n_{77} = 142 + 90 - 12 = 220$$

6 (e) **Number of integers divisible by 11, but not by 7**

Integers divisible by 11 but not by 7 are integers divisible by 11 that are not integers that are divisible by 7 and 11.

$$n_{11,\text{ not }7} = n_{11} - n_{77} = 90 - 12 = 78$$

Number of integers divisible by either 7 or 11

Use the subtraction rule:

$$n = n_{7,\text{ not }11} + n_{11,\text{ not }7} = 130 + 78 = 208$$

7 (f) Integers divisible by neither 7 nor 11 is the integers in the set that are not divisible by either 7 or 11

$$n_{\text{not}(7\text{ or }11)} = |A| - n_{7\text{ or }11} = 999 - 220 = 779$$

8 (g) Integers below 1000 have 3 possible digits or less.

1 digit There are 9 possible positive integers less than 1000 with 1 digit: 1, 2, 3, 4, 5, 6, 7, 8, 9

2 digits
First digit: 9 ways (since the first digit cannot be 0)
Second digit: 9 ways (since there are 10 digits, while the digit cannot be the same as the first digit).

Use the product rule: $9 \cdot 9 = 81$ ways

3 digits
First digit: 9 ways (since the first digit cannot be 0)
Second digit: 9 ways (since there are 10 digits, while the digit cannot be the same as the first digit).
Third digit: 8 ways (since there are 10 digits, while the digit cannot be the same as the first nor the second digit).

Use the product rule: $9 \cdot 9 \cdot 8 = 648$ ways

In total Use the sum rule:

$$9 + 81 + 648 = 738$$

9 (h) Integers below 1000 have 3 possible digits or less.

1 digit There are 4 possible even integers: 2, 4, 6, 8

2 digits

First digit: 9 ways (since the first digit cannot be 0) of which 5 are odd and 4 are even

Second digit:
5 ways if the first digit is odd (since the number needs to be even, thus the last digit has to be 0, 2, 4, 6 or 8)
4 ways if the first digit is even (since the number needs to different from the first digit)

Use the product rule and sum rule: $5 \cdot 5 + 4 \cdot 4 = 25 + 16 = 41$ ways

10 **3 digits**
First digit: 9 ways (since the first digit cannot be 0) of which 5 are odd and 4 are even

Second digit: 9 ways (since there are 10 possible digits and the second digit cannot be the same as the first)
If first digit is odd: 4 odd and 5 even
If first digit is even: 5 odd and 4 even

Third digit:

If the first two digits are odd: 5 ways (since the number needs to be even, thus the last digit has to be 0, 2, 4, 6 or 8).
If the first two digits are even: 3 ways
If the first digit is even and the second odd: 4 ways
If the first digit is odd and the second even: 4 ways

Use the product rule: 2, 6, 8, A

First two digits odd: $5 \cdot 4 \cdot 5 = 100$ ways
First two digits even: $4 \cdot 4 \cdot 3 = 48$ ways
First digit is even and the second odd: $4 \cdot 5 \cdot 4 = 80$ ways
First digit is odd and the second even: $5 \cdot 5 \cdot 4 = 100$ ways

Use the sum rule: $100 + 48 + 80 + 100 = 328$ ways

In total Use the sum rule:

$$4 + 41 + 328 = 373$$

5

(a) There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter= 21 ways
Second letter= 21 ways
Third letter= 21 ways
4th letter= 21 ways
5th letter= 21 ways
6th letter= 21 ways
7th letter= 21 ways
8th letter= 21 ways

Using the product rule:

$$21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 = 21^8 = 37,822,859,361$$

(b) There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter= 21 ways
Second letter= 20 ways (since the letters have to be different)
Third letter= 19 ways (since the letters have to be different)
4th letter= 18 ways (since the letters have to be different)
5th letter= 17 ways (since the letters have to be different)
6th letter= 16 ways (since the letters have to be different)
7th letter= 15 ways (since the letters have to be different)
8th letter= 14 ways (since the letters have to be different)

Using the product rule:

$$21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 = 8,204,716,800$$

(c) There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter= 5 ways (since it has to be a vowel)
Second letter= 26 ways
Third letter= 26 ways
4th letter= 26 ways
5th letter= 26 ways
6th letter= 26 ways
7th letter= 26 ways
8th letter= 26 ways

Using the product rule:

$$5 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 5 \cdot 26^7 = 40,159,050,880$$

(d) There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter= 5 ways
Second letter= 25 ways (since the letters have to be different)
Third letter= 24 ways (since the letters have to be different)
4th letter= 23 ways (since the letters have to be different)
5th letter= 22 ways (since the letters have to be different)
6th letter= 21 ways (since the letters have to be different)
7th letter= 20 ways (since the letters have to be different)
8th letter= 19 ways (since the letters have to be different)

Using the product rule:

$$5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 12,113,640,000$$

(e) **Number of possible strings with eight letters**

There are 26 possible letters.

First letter= 26 ways
Second letter= 26 ways
Third letter= 26 ways
4th letter= 26 ways
5th letter= 26 ways
6th letter= 26 ways
7th letter= 26 ways
8th letter= 26 ways

Using the product rule:

$$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^8 = 208,827,064,576$$

Number of possible strings with at least one vowel, if letters can be repeated

By part (a), there are 37,822,859,361 strings with no vowels.

مجموع كل الاسترنقز

اللي اكيد ماعيها

أكيد واحدة منها فيها vowel

$$208,827,064,576 - 37,822,859,361 = 171,004,205,215$$

Thus there are 171,004,205,215 strings with at least one vowel.

(f) There are 26 possible letters, of which 5 are vowels and 21 are consonants.

Position vowel=8 ways (as there are 8 symbols in the string)
Vowel=5 ways
Second letter= 21 ways (not a vowel)
Third letter= 21 ways (not a vowel)
4th letter= 21 ways (not a vowel)
5th letter= 21 ways (not a vowel)
6th letter= 21 ways (not a vowel)
7th letter= 21 ways (not a vowel)
8th letter= 21 ways (not a vowel)

Using the product rule:

$$8 \cdot 5 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 = 8 \cdot 5 \cdot 21^7 = 72,043,541,640$$

(g) **Number of strings starting with X that do not contain a vowel**

There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter=1 way (needs to be X)
Second letter= 21 ways (not a vowel)
Third letter= 21 ways (not a vowel)
4th letter= 21 ways (not a vowel)
5th letter= 21 ways (not a vowel)
6th letter= 21 ways (not a vowel)
7th letter= 21 ways (not a vowel)
8th letter= 21 ways (not a vowel)

Using the product rule:

$$1 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 = 21^7$$

Number of strings starting with X

There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter=1 way (needs to be X)
Second letter= 26 ways
Third letter= 26 ways
4th letter= 26 ways
5th letter= 26 ways
6th letter= 26 ways
7th letter= 26 ways
8th letter= 26 ways

Using the product rule:

$$1 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 26^7$$

Number of strings starting with X with at least one vowel

All strings starting with X that do not contain no vowels will contain at least one vowel. By the subtraction rule:

$$26^7 - 21^7 = 6,230,721,635$$

Thus there are 6,230,721,635 strings starting with an X that contain at least one vowel.

(h) **Number of strings starting and ending with X that do not contain a vowel**

There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter=1 way (needs to be X)
Second letter= 21 ways (not a vowel)
Third letter= 21 ways (not a vowel)
4th letter= 21 ways (not a vowel)
5th letter= 21 ways (not a vowel)
6th letter= 21 ways (not a vowel)
7th letter= 21 ways (not a vowel)
8th letter= 1 way (needs to be X)

Using the product rule:

$$1 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 21 \cdot 1 = 21^6$$

Number of strings starting and ending with X

There are 26 possible letters, of which 5 are vowels and 21 are consonants.

First letter=1 way (needs to be X)
Second letter= 26 ways
Third letter= 26 ways
4th letter= 26 ways
5th letter= 26 ways
6th letter= 26 ways
7th letter= 26 ways
8th letter= 1 way (needs to be X)

Using the product rule:

$$1 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 26 \cdot 1 = 26^6$$

Number of strings starting and ending with X with at least one vowel

All strings starting and ending with X that do not contain no vowels will contain at least one vowel. By the subtraction rule:

$$26^6 - 21^6 = 223,149,655$$

Thus there are 223,149,655 strings starting and ending with an X that contain at least one vowel.

6

There are 30 students in the class. Assume that each student has at least a first name and a last name. Each last name has to begin with a letter of the alphabet.
objects = first letter of each last name = 30 = N
holes = letters of the alphabet = 26 = k

$$\left\lceil \frac{30}{26} \right\rceil = 2$$

So by the Pigeonhole Principle, there are at least two students who have a last name that begin with the same letter.

Question 7. A room contains 10 men and 10 women. A manger selects members of team at random without looking at them.

a) How many member must the manger select to be sure of having at least three members of the same gender?
b) How many member must the manger select to be sure of having at least three women in the team?*

Solution.
There are 20 persons, 10 men and 10 women.

a) The Pigeonhole are the genders. $\lceil \frac{x}{2} \rceil$ must be equal to three, and the least positive integer to satisfy the equation is 5.
b) The first 10 choices may be all men, so the manager needs to choose at least 13 member to be sure that at least 3 of them are women.

Question 8. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Solution.
There are 4 possible values of remainder: 0, 1, 2, 3
And we have 5 integers. Therefore, by pigeonhole principle, at least 2 of them must give the same reminder when divided by 4.