Exercises

1. Use a direct proof to show that the sum of two odd integers is even.
2. Use a direct proof to show that the sum of two even integers is even.
3. Show that the square of an even number is an even number using a direct proof.
4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.
5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind of proof did you use?
6. Use a direct proof to show that the product of two odd numbers is odd.
7. Use a direct proof to show that every odd integer is the difference of two squares.
8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.
9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.
10. Use a direct proof to show that the product of two rational numbers is rational.
11. Prove or disprove that the product of two irrational numbers is irrational.
12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
13. Prove that if x is irrational, then $1/x$ is irrational.
14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.
15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.
16. ☞ Prove that if m and n are integers and mn is even, then m is even or n is even.
17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
    a) a proof by contraposition.
    b) a proof by contradiction.
18. Prove that if n is an integer and $3n + 2$ is even, then n is even using
    a) a proof by contraposition.
    b) a proof by contradiction.
19. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?
20. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?
21. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?
22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.
23. Show that at least ten of any 64 days chosen must fall on the same day of the week.
24. Show that at least three of any 25 days chosen must fall in the same month of the year.
25. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]
26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.
27. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.
28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.
29. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.
30. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.
31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.
32. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, (*iii*) $3x - 1$ is rational.
33. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.
34. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (*2*) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.
35. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (*1*) $\sqrt{x + 3} = 3 - x$ is given; (2) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.
36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.
37. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \rightarrow p_4$, $p_3 \rightarrow p_1$, $p_4 \rightarrow p_2$, $p_2 \rightarrow p_5$, and $p_5 \rightarrow p_3$ are true.

38. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

39. Prove that at least one of the real numbers $a_1, a_2, \dots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

40. Use Exercise 39 to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

41. Prove that if n is an integer, these four statements are equivalent: (*i*) n is even, (*ii*) $n + 1$ is odd, (*iii*) $3n + 1$ is odd, (*iv*) $3n$ is even.

42. Prove that these four statements about the integer n are equivalent: (*i*) n^2 is odd, (*ii*) $1 - n$ is even, (*iii*) n^3 is odd, (*iv*) $n^2 + 1$ is even.

1.8 Proof Methods and Strategy

Introduction

Assessment

In Section 1.7 we introduced many methods of proof and illustrated how each method can be used. In this section we continue this effort. We will introduce several other commonly used proof methods, including the method of proving a theorem by considering different cases separately. We will also discuss proofs where we prove the existence of objects with desired properties.

In Section 1.7 we briefly discussed the strategy behind constructing proofs. This strategy includes selecting a proof method and then successfully constructing an argument step by step, based on this method. In this section, after we have developed a versatile arsenal of proof methods, we will study some aspects of the art and science of proofs. We will provide advice on how to find a proof of a theorem. We will describe some tricks of the trade, including how proofs can be found by working backward and by adapting existing proofs.

When mathematicians work, they formulate conjectures and attempt to prove or disprove them. We will briefly describe this process here by proving results about tiling checkerboards with dominoes and other types of pieces. Looking at tilings of this kind, we will be able to quickly formulate conjectures and prove theorems without first developing a theory.

We will conclude the section by discussing the role of open questions. In particular, we will discuss some interesting problems either that have been solved after remaining open for hundreds of years or that still remain open.

Exhaustive Proof and Proof by Cases

Sometimes we cannot prove a theorem using a single argument that holds for all possible cases. We now introduce a method that can be used to prove a theorem, by considering different cases separately. This method is based on a rule of inference that we will now introduce. To prove a conditional statement of the form

$$(p_1 \vee p_2 \vee \cdots \vee p_n) \to q$$

the tautology

$$[(p_1 \vee p_2 \vee \cdots \vee p_n) \to q] \leftrightarrow [(p_1 \to q) \wedge (p_2 \to q) \wedge \cdots \wedge (p_n \to q)]$$

can be used as a rule of inference. This shows that the original conditional statement with a hypothesis made up of a disjunction of the propositions $p_1, p_2, \dots, p_n$ can be proved by proving each of the n conditional statements $p_i \to q$, $i = 1, 2, \dots, n$, individually. Such an argument is called a **proof by cases**. Sometimes to prove that a conditional statement $p \to q$ is true, it is convenient to use a disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ instead of p as the hypothesis of the conditional statement, where p and $p_1 \vee p_2 \vee \cdots \vee p_n$ are equivalent.

Exercises

1. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.

2. Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

3. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. [*Hint:* Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.]

4. Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.

5. Prove using the notion of without loss of generality that $\min(x, y) = (x + y - |x - y|)/2$ and $\max(x, y) = (x + y + |x - y|)/2$ whenever x and y are real numbers.

6. Prove using the notion of without loss of generality that $5x + 5y$ is an odd integer when x and y are integers of opposite parity.

7. Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).

8. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?

9. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?

10. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?

11. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.

12. Show that the product of two of the numbers $65^{1000} - 8^{2001} + 3^{177}$, $79^{1212} - 9^{2399} + 2^{2001}$, and $24^{4493} - 5^{8192} + 7^{1777}$ is nonnegative. Is your proof constructive or nonconstructive? [*Hint:* Do not try to evaluate these numbers!]

13. Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

14. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

15. Show that each of these statements can be used to express the fact that there is a unique element x such that $P(x)$ is true. [Note that we can also write this statement as $\exists! x P(x)$.]

a) $\exists x \forall y (P(y) \leftrightarrow x = y)$

b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$

c) $\exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y))$

16. Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

17. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a - c| = |b - c|$.

18. Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is less than $1/2$.

19. Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

20. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n + \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

21. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n - \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

22. Use forward reasoning to show that if x is a nonzero real number, then $x^2 + 1/x^2 \geq 2$. [*Hint:* Start with the inequality $(x - 1/x)^2 \geq 0$ which holds for all nonzero real numbers x.]

23. The **harmonic mean** of two real numbers x and y equals $2xy/(x + y)$. By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

24. The **quadratic mean** of two real numbers x and y equals $\sqrt{(x^2 + y^2)/2}$. By computing the arithmetic and quadratic means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

***25.** Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

***26.** Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [*Hint:* Work backward, assuming that you did end up with nine zeros.]

27. Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.

28. Formulate a conjecture about the final two decimal digits of the square of an integer. Prove your conjecture using a proof by cases.

29. Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

30. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

31. Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.

32. Prove that there are infinitely many solutions in positive integers x, y, and z to the equation $x^2 + y^2 = z^2$. [*Hint:* Let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$, where m and n are integers.]

33. Adapt the proof in Example 4 in Section 1.7 to prove that if $n = abc$, where a, b, and c are positive integers, then $a \leq \sqrt[3]{n}$, $b \leq \sqrt[3]{n}$, or $c \leq \sqrt[3]{n}$.

34. Prove that $\sqrt[3]{2}$ is irrational.

35. Prove that between every two rational numbers there is an irrational number.

36. Prove that between every rational number and every irrational number there is an irrational number.

***37.** Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

a) Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).

b) Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

38. Prove or disprove that if you have an 8-gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, then you can measure 4 gallons by successively pouring some of or all of the water in a jug into another jug.

39. Verify the $3x + 1$ conjecture for these integers.

a) 6 **b)** 7 **c)** 17 **d)** 21

40. Verify the $3x + 1$ conjecture for these integers.

a) 16 **b)** 11 **c)** 35 **d)** 113

41. Prove or disprove that you can use dominoes to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite).

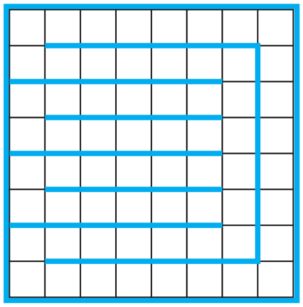
42. Prove or disprove that you can use dominoes to tile a standard checkerboard with all four corners removed.

43. Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares.

44. Prove or disprove that you can use dominoes to tile a 5×5 checkerboard with three corners removed.

45. Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist. [*Hint:* First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers squares 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

***46.** Prove that when a white square and a black square are removed from an 8×8 checkerboard (colored as in the text) you can tile the remaining squares of the checkerboard using dominoes. [*Hint:* Show that when one black and one white square are removed, each part of the partition of the remaining cells formed by inserting the barriers shown in the figure can be covered by dominoes.]



47. Show that by removing two white squares and two black squares from an 8×8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

***48.** Find all squares, if they exist, on an 8×8 checkerboard such that the board obtained by removing one of these square can be tiled using straight triominoes. [*Hint:* First use arguments based on coloring and rotations to eliminate as many squares as possible from consideration.]

***49.** **a)** Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.

b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard using these tetrominoes.

***50.** Prove or disprove that you can tile a 10×10 checkerboard using straight tetrominoes.

Key Terms and Results

TERMS

proposition: a statement that is true or false

propositional variable: a variable that represents a proposition

truth value: true or false

¬ *p* (negation of *p*): the proposition with truth value opposite to the truth value of p

logical operators: operators used to combine propositions

compound proposition: a proposition constructed by combining propositions using logical operators

truth table: a table displaying all possible truth values of propositions

$p \vee q$ (disjunction of *p* and *q*): the proposition "p or q," which is true if and only if at least one of p and q is true

$p \wedge q$ (conjunction of p and q): the proposition "p and q," which is true if and only if both p and q are true

$p \oplus q$ (exclusive or of p and q): the proposition "p *XOR* q," which is true when exactly one of p and q is true

$p \rightarrow q$ (p implies q): the proposition "if p, then q," which is false if and only if p is true and q is false

converse of $p \rightarrow q$: the conditional statement $q \rightarrow p$

contrapositive of $p \rightarrow q$: the conditional statement $\neg q \rightarrow \neg p$

inverse of $p \rightarrow q$: the conditional statement $\neg p \rightarrow \neg q$

$p \leftrightarrow q$ (biconditional): the proposition "p if and only if q," which is true if and only if p and q have the same truth value

bit: either a 0 or a 1

Boolean variable: a variable that has a value of 0 or 1

bit operation: an operation on a bit or bits

bit string: a list of bits

bitwise operations: operations on bit strings that operate on each bit in one string and the corresponding bit in the other string

logic gate: a logic element that performs a logical operation on one or more bits to produce an output bit

logic circuit: a switching circuit made up of logic gates that produces one or more output bits

tautology: a compound proposition that is always true

contradiction: a compound proposition that is always false

contingency: a compound proposition that is sometimes true and sometimes false

consistent compound propositions: compound propositions for which there is an assignment of truth values to the variables that makes all these propositions true

satisfiable compound proposition: a compound proposition for which there is an assignment of truth values to its variables that makes it true

logically equivalent compound propositions: compound propositions that always have the same truth values

predicate: part of a sentence that attributes a property to the subject

propositional function: a statement containing one or more variables that becomes a proposition when each of its variables is assigned a value or is bound by a quantifier

domain (or universe) of discourse: the values a variable in a propositional function may take

$\exists x P(x)$ (existential quantification of $P(x)$): the proposition that is true if and only if there exists an x in the domain such that $P(x)$ is true

$\forall x P(x)$ (universal quantification of $P(x)$): the proposition that is true if and only if $P(x)$ is true for every x in the domain

logically equivalent expressions: expressions that have the same truth value no matter which propositional functions and domains are used

free variable: a variable not bound in a propositional function

bound variable: a variable that is quantified

scope of a quantifier: portion of a statement where the quantifier binds its variable

argument: a sequence of statements

argument form: a sequence of compound propositions involving propositional variables

premise: a statement, in an argument, or argument form, other than the final one

conclusion: the final statement in an argument or argument form

valid argument form: a sequence of compound propositions involving propositional variables where the truth of all the premises implies the truth of the conclusion

valid argument: an argument with a valid argument form

rule of inference: a valid argument form that can be used in the demonstration that arguments are valid

fallacy: an invalid argument form often used incorrectly as a rule of inference (or sometimes, more generally, an incorrect argument)

circular reasoning or begging the question: reasoning where one or more steps are based on the truth of the statement being proved

theorem: a mathematical assertion that can be shown to be true

conjecture: a mathematical assertion proposed to be true, but that has not been proved

proof: a demonstration that a theorem is true

axiom: a statement that is assumed to be true and that can be used as a basis for proving theorems

lemma: a theorem used to prove other theorems

corollary: a proposition that can be proved as a consequence of a theorem that has just been proved

vacuous proof: a proof that $p \rightarrow q$ is true based on the fact that p is false

trivial proof: a proof that $p \rightarrow q$ is true based on the fact that q is true

direct proof: a proof that $p \rightarrow q$ is true that proceeds by showing that q must be true when p is true

proof by contraposition: a proof that $p \rightarrow q$ is true that proceeds by showing that p must be false when q is false

proof by contradiction: a proof that p is true based on the truth of the conditional statement $\neg p \rightarrow q$, where q is a contradiction

exhaustive proof: a proof that establishes a result by checking a list of all possible cases

proof by cases: a proof broken into separate cases, where these cases cover all possibilities

without loss of generality: an assumption in a proof that makes it possible to prove a theorem by reducing the number of cases to consider in the proof

counterexample: an element x such that $P(x)$ is false

constructive existence proof: a proof that an element with a specified property exists that explicitly finds such an element

nonconstructive existence proof: a proof that an element with a specified property exists that does not explicitly find such an element

rational number: a number that can be expressed as the ratio of two integers p and q such that $q \neq 0$

uniqueness proof: a proof that there is exactly one element satisfying a specified property

EXAMPLE 12 Determine whether 2 and 3 are primitive roots modulo 11.

Solution: When we compute the powers of 2 in $\mathbf{Z}_{11}$, we obtain $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 5$, $2^5 = 10$, $2^6 = 9$, $2^7 = 7$, $2^8 = 3$, $2^9 = 6$, $2^{10} = 1$. Because every element of $\mathbf{Z}_{11}$ is a power of 2, 2 is a primitive root of 11.

When we compute the powers of 3 modulo 11, we obtain $3^1 = 3$, $3^2 = 9$, $3^3 = 5$, $3^4 = 4$, $3^5 = 1$. We note that this pattern repeats when we compute higher powers of 3. Because not all elements of $\mathbf{Z}_{11}$ are powers of 3, we conclude that 3 is not a primitive root of 11. ◀

An important fact in number theory is that there is a primitive root modulo p for every prime p. We refer the reader to [Ro10] for a proof of this fact. Suppose that p is prime and r is a primitive root modulo p. If a is an integer between 1 and $p - 1$, that is, an element of $\mathbf{Z}_p$, we know that there is an unique exponent e such that $r^e = a$ in $\mathbf{Z}_p$, that is, $r^e \textbf{ mod } p = a$.

DEFINITION 4 Suppose that p is a prime, r is a primitive root modulo p, and a is an integer between 1 and $p - 1$ inclusive. If $r^e \textbf{ mod } p = a$ and $0 \le e \le p - 1$, we say that e is the *discrete logarithm* of a modulo p to the base r and we write $\log_r a = e$ (where the prime p is understood).

EXAMPLE 13 Find the discrete logarithms of 3 and 5 modulo 11 to the base 2.

Solution: When we computed the powers of 2 modulo 11 in Example 12, we found that $2^8 = 3$ and $2^4 = 5$ in $\mathbf{Z}_{11}$. Hence, the discrete logarithms of 3 and 5 modulo 11 to the base 2 are 8 and 4, respectively. (These are the powers of 2 that equal 3 and 5, respectively, in $\mathbf{Z}_{11}$.) We write $\log_2 3 = 8$ and $\log_2 5 = 4$ (where the modulus 11 is understood and not explicitly noted in the notation). ◀

The discrete logarithm problem is hard!

The **discrete logarithm problem** takes as input a prime p, a primitive root r modulo p, and a positive integer $a \in \mathbf{Z}_p$; its output is the discrete logarithm of a modulo p to the base r. Although this problem might seem not to be that difficult, it turns out that no polynomial time algorithm is known for solving it. The difficulty of this problem plays an important role in cryptography, as we will see in Section 4.6

Exercises

1. Show that 15 is an inverse of 7 modulo 26.

2. Show that 937 is an inverse of 13 modulo 2436.

3. By inspection (as discussed prior to Example 1), find an inverse of 4 modulo 9.

4. By inspection (as discussed prior to Example 1), find an inverse of 2 modulo 17.

5. Find an inverse of a modulo m for each of these pairs of relatively prime integers using the method followed in Example 2.

a) $a = 4, m = 9$
b) $a = 19, m = 141$
c) $a = 55, m = 89$
d) $a = 89, m = 232$

6. Find an inverse of a modulo m for each of these pairs of relatively prime integers using the method followed in Example 2.

a) $a = 2, m = 17$
b) $a = 34, m = 89$
c) $a = 144, m = 233$
d) $a = 200, m = 1001$

***7.** Show that if a and m are relatively prime positive integers, then the inverse of a modulo m is unique modulo m. [*Hint:* Assume that there are two solutions b and c of the congruence $ax \equiv 1 \pmod{m}$. Use Theorem 7 of Section 4.3 to show that $b \equiv c \pmod{m}$.]

8. Show that an inverse of a modulo m, where a is an integer and $m > 2$ is a positive integer, does not exist if $\gcd(a, m) > 1$.

9. Solve the congruence $4x \equiv 5 \pmod 9$ using the inverse of 4 modulo 9 found in part (a) of Exercise 5.

10. Solve the congruence $2x \equiv 7 \pmod{17}$ using the inverse of 2 modulo 7 found in part (a) of Exercise 6.

11. Solve each of these congruences using the modular inverses found in parts (b), (c), and (d) of Exercise 5.

a) $19x \equiv 4 \pmod{141}$
b) $55x \equiv 34 \pmod{89}$
c) $89x \equiv 2 \pmod{232}$

12. Solve each of these congruences using the modular inverses found in parts (b), (c), and (d) of Exercise 6.

a) $34x \equiv 77 \pmod{89}$

b) $144x \equiv 4 \pmod{233}$

c) $200x \equiv 13 \pmod{1001}$

13. Find the solutions of the congruence $15x^2 + 19x \equiv 5 \pmod{11}$. [*Hint:* Show the congruence is equivalent to the congruence $15x^2 + 19x + 6 \equiv 0 \pmod{11}$. Factor the left-hand side of the congruence; show that a solution of the quadratic congruence is a solution of one of the two different linear congruences.]

14. Find the solutions of the congruence $12x^2 + 25x \equiv 10 \pmod{11}$. [*Hint:* Show the congruence is equivalence to the congruence $12x^2 + 25x + 12 \equiv 0 \pmod{11}$. Factor the left-hand side of the congruence; show that a solution of the quadratic congruence is a solution of one of two different linear congruences.]

***15.** Show that if m is an integer greater than 1 and $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m/\gcd(c, m)}$.

16. a) Show that the positive integers less than 11, except 1 and 10, can be split into pairs of integers such that each pair consists of integers that are inverses of each other modulo 11.

b) Use part (a) to show that $10! \equiv -1 \pmod{11}$.

17. Show that if p is prime, the only solutions of $x^2 \equiv 1 \pmod{p}$ are integers x such that $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

***18. a)** Generalize the result in part (a) of Exercise 16; that is, show that if p is a prime, the positive integers less than p, except 1 and $p - 1$, can be split into $(p - 3)/2$ pairs of integers such that each pair consists of integers that are inverses of each other. [*Hint:* Use the result of Exercise 17.]

b) From part (a) conclude that $(p - 1)! \equiv -1 \pmod{p}$ whenever p is prime. This result is known as **Wilson's theorem**.

c) What can we conclude if n is a positive integer such that $(n - 1)! \not\equiv -1 \pmod{n}$?

***19.** This exercise outlines a proof of Fermat's little theorem.

a) Suppose that a is not divisible by the prime p. Show that no two of the integers $1 \cdot a, 2 \cdot a, \ldots, (p - 1)a$ are congruent modulo p.

b) Conclude from part (a) that the product of $1, 2, \ldots, p - 1$ is congruent modulo p to the product of $a, 2a, \ldots, (p - 1)a$. Use this to show that

$$(p - 1)! \equiv a^{p-1}(p - 1)! \pmod{p}.$$

c) Use Theorem 7 of Section 4.3 to show from part (b) that $a^{p-1} \equiv 1 \pmod{p}$ if $p \not| a$. [*Hint:* Use Lemma 3 of Section 4.3 to show that p does not divide $(p - 1)!$ and then use Theorem 7 of Section 4.3. Alternatively, use Wilson's theorem from Exercise 18(b).]

d) Use part (c) to show that $a^p \equiv a \pmod{p}$ for all integers a.

20. Use the construction in the proof of the Chinese remainder theorem to find all solutions to the system of congruences $x \equiv 2 \pmod{3}$, $x \equiv 1 \pmod{4}$, and $x \equiv 3 \pmod{5}$.

21. Use the construction in the proof of the Chinese remainder theorem to find all solutions to the system of congruences $x \equiv 1 \pmod{2}$, $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$, and $x \equiv 4 \pmod{11}$.

22. Solve the system of congruence $x \equiv 3 \pmod{6}$ and $x \equiv 4 \pmod{7}$ using the method of back substitution.

23. Solve the system of congruences in Exercise 20 using the method of back substitution.

24. Solve the system of congruences in Exercise 21 using the method of back substitution.

25. Write out in pseudocode an algorithm for solving a simultaneous system of linear congruences based on the construction in the proof of the Chinese remainder theorem.

***26.** Find all solutions, if any, to the system of congruences $x \equiv 5 \pmod{6}$, $x \equiv 3 \pmod{10}$, and $x \equiv 8 \pmod{15}$.

***27.** Find all solutions, if any, to the system of congruences $x \equiv 7 \pmod{9}$, $x \equiv 4 \pmod{12}$, and $x \equiv 16 \pmod{21}$.

28. Use the Chinese remainder theorem to show that an integer a, with $0 \le a < m = m_1 m_2 \cdots m_n$, where the positive integers $m_1, m_2, \ldots, m_n$ are pairwise relatively prime, can be represented uniquely by the n-tuple $(a \textbf{ mod } m_1, a \textbf{ mod } m_2, \ldots, a \textbf{ mod } m_n)$.

***29.** Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime integers greater than or equal to 2. Show that if $a \equiv b \pmod{m_i}$ for $i = 1, 2, \ldots, n$, then $a \equiv b \pmod{m}$, where $m = m_1 m_2 \cdots m_n$. (This result will be used in Exercise 30 to prove the Chinese remainder theorem. Consequently, do not use the Chinese remainder theorem to prove it.)

***30.** Complete the proof of the Chinese remainder theorem by showing that the simultaneous solution of a system of linear congruences modulo pairwise relatively prime moduli is unique modulo the product of these moduli. [*Hint:* Assume that x and y are two simultaneous solutions. Show that $m_i \mid x - y$ for all i. Using Exercise 29, conclude that $m = m_1 m_2 \cdots m_n \mid x - y$.]

31. Which integers leave a remainder of 1 when divided by 2 and also leave a remainder of 1 when divided by 3?

32. Which integers are divisible by 5 but leave a remainder of 1 when divided by 3?

33. Use Fermat's little theorem to find 7^{121} **mod** 13.

34. Use Fermat's little theorem to find 23^{1002} **mod** 41.

35. Use Fermat's little theorem to show that if p is prime and $p \not| a$, then a^{p-2} is an inverse of a modulo p.

36. Use Exercise 35 to find an inverse of 5 modulo 41.

37. a) Show that $2^{340} \equiv 1 \pmod{11}$ by Fermat's little theorem and noting that $2^{340} = (2^{10})^{34}$.

b) Show that $2^{340} \equiv 1 \pmod{31}$ using the fact that $2^{340} = (2^5)^{68} = 32^{68}$.

c) Conclude from parts (a) and (b) that $2^{340} \equiv 1 \pmod{341}$.

38. a) Use Fermat's little theorem to compute 3^{302} **mod** 5, 3^{302} **mod** 7, and 3^{302} **mod** 11.

b) Use your results from part (a) and the Chinese remainder theorem to find 3^{302} **mod** 385. (Note that $385 = 5 \cdot 7 \cdot 11$.)

39. a) Use Fermat's little theorem to compute 5^{2003} **mod** 7, 5^{2003} **mod** 11, and 5^{2003} **mod** 13.

b) Use your results from part (a) and the Chinese remainder theorem to find 5^{2003} **mod** 1001. (Note that $1001 = 7 \cdot 11 \cdot 13$.)

40. Show with the help of Fermat's little theorem that if n is a positive integer, then 42 divides $n^7 - n$.

41. Show that if p is an odd prime, then every divisor of the Mersenne number $2^p - 1$ is of the form $2kp + 1$, where k is a nonnegative integer. [*Hint:* Use Fermat's little theorem and Exercise 37 of Section 4.3.]

42. Use Exercise 41 to determine whether $M_{13} = 2^{13} - 1 = 8191$ and $M_{23} = 2^{23} - 1 = 8{,}388{,}607$ are prime.

43. Use Exercise 41 to determine whether $M_{11} = 2^{11} - 1 = 2047$ and $M_{17} = 2^{17} - 1 = 131{,}071$ are prime.

Let n be a positive integer and let $n - 1 = 2^s t$, where s is a nonnegative integer and t is an odd positive integer. We say that n passes **Miller's test for the base *b*** if either $b^t \equiv 1$ (mod n) or $b^{2^j t} \equiv -1 \pmod{n}$ for some j with $0 \le j \le s - 1$. It can be shown (see [Ro10]) that a composite integer n passes Miller's test for fewer than $n/4$ bases b with $1 < b < n$. A composite positive integer n that passes Miller's test to the base b is called a **strong pseudoprime to the base *b***.

***44.** Show that if n is prime and b is a positive integer with $n \nmid b$, then n passes Miller's test to the base b.

45. Show that 2047 is a strong pseudoprime to the base 2 by showing that it passes Miller's test to the base 2, but is composite.

46. Show that 1729 is a Carmichael number.

47. Show that 2821 is a Carmichael number.

***48.** Show that if $n = p_1 p_2 \cdots p_k$, where $p_1, p_2, \ldots, p_k$ are distinct primes that satisfy $p_j - 1 \mid n - 1$ for $j = 1, 2, \ldots, k$, then n is a Carmichael number.

49. a) Use Exercise 48 to show that every integer of the form $(6m + 1)(12m + 1)(18m + 1)$, where m is a positive integer and $6m + 1$, $12m + 1$, and $18m + 1$ are all primes, is a Carmichael number.

b) Use part (a) to show that 172,947,529 is a Carmichael number.

50. Find the nonnegative integer a less than 28 represented by each of these pairs, where each pair represents (a **mod** 4, a **mod** 7).

a) $(0, 0)$ **b)** $(1, 0)$ **c)** $(1, 1)$
d) $(2, 1)$ **e)** $(2, 2)$ **f)** $(0, 3)$
g) $(2, 0)$ **h)** $(3, 5)$ **i)** $(3, 6)$

51. Express each nonnegative integer a less than 15 as a pair (a **mod** 3, a **mod** 5).

52. Explain how to use the pairs found in Exercise 51 to add 4 and 7.

53. Solve the system of congruences that arises in Example 8.

54. Show that 2 is a primitive root of 19.

55. Find the discrete logarithms of 5 and 6 to the base 2 modulo 19.

56. Let p be an odd prime and r a primitive root of p. Show that if a and b are positive integers in $\mathbf{Z}_p$, then $\log_r(ab) \equiv \log_r a + \log_r b \pmod{p - 1}$.

57. Write out a table of discrete logarithms modulo 17 with respect to the primitive root 3.

If m is a positive integer, the integer a is a **quadratic residue** of m if $\gcd(a, m) = 1$ and the congruence $x^2 \equiv a \pmod{m}$ has a solution. In other words, a quadratic residue of m is an integer relatively prime to m that is a perfect square modulo m. If a is not a quadratic residue of m and $\gcd(a, m) = 1$, we say that it is a **quadratic nonresidue** of m. For example, 2 is a quadratic residue of 7 because $\gcd(2, 7) = 1$ and $3^2 \equiv 2 \pmod{7}$ and 3 is a quadratic nonresidue of 7 because $\gcd(3, 7) = 1$ and $x^2 \equiv 3 \pmod{7}$ has no solution.

58. Which integers are quadratic residues of 11?

59. Show that if p is an odd prime and a is an integer not divisible by p, then the congruence $x^2 \equiv a \pmod{p}$ has either no solutions or exactly two incongruent solutions modulo p.

60. Show that if p is an odd prime, then there are exactly $(p - 1)/2$ quadratic residues of p among the integers $1, 2, \ldots, p - 1$.

If p is an odd prime and a is an integer not divisible by p, the **Legendre symbol** $\left(\dfrac{a}{p}\right)$ is defined to be 1 if a is a quadratic residue of p and -1 otherwise.

61. Show that if p is an odd prime and a and b are integers with $a \equiv b \pmod{p}$, then

$$\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right).$$

62. Prove **Euler's criterion**, which states that if p is an odd prime and a is a positive integer not divisible by p, then

$$\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} \pmod{p}.$$

[*Hint:* If a is a quadratic residue modulo p, apply Fermat's little theorem; otherwise, apply Wilson's theorem, given in Exercise 18(b).]

63. Use Exercise 62 to show that if p is an odd prime and a and b are integers not divisible by p, then

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).$$

64. Show that if p is an odd prime, then -1 is a quadratic residue of p if $p \equiv 1 \pmod{4}$, and -1 is not a quadratic residue of p if $p \equiv 3 \pmod{4}$. [*Hint:* Use Exercise 62.]

65. Find all solutions of the congruence $x^2 \equiv 29 \pmod{35}$. [*Hint:* Find the solutions of this congruence modulo 5 and modulo 7, and then use the Chinese remainder theorem.]

Exercises

1. Encrypt the message DO NOT PASS GO by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.

a) $f(p) = (p + 3)$ **mod** 26 (the Caesar cipher)
b) $f(p) = (p + 13)$ **mod** 26
c) $f(p) = (3p + 7)$ **mod** 26

2. Encrypt the message STOP POLLUTION by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.

a) $f(p) = (p + 4)$ **mod** 26
b) $f(p) = (p + 21)$ **mod** 26
c) $f(p) = (17p + 22)$ **mod** 26

3. Encrypt the message WATCH YOUR STEP by translating the letters into numbers, applying the given encryption function, and then translating the numbers back into letters.

a) $f(p) = (p + 14)$ **mod** 26
b) $f(p) = (14p + 21)$ **mod** 26
c) $f(p) = (-7p + 1)$ **mod** 26

4. Decrypt these messages that were encrypted using the Caesar cipher.

a) EOXH MHDQV
b) WHVW WRGDB
c) HDW GLP VXP

5. Decrypt these messages encrypted using the shift cipher $f(p) = (p + 10)$ **mod** 26.

a) CEBBOXNOB XYG
b) LO WI PBSOXN
c) DSWO PYB PEX

6. Suppose that when a long string of text is encrypted using a shift cipher $f(p) = (p + k)$ **mod** 26, the most common letter in the ciphertext is X. What is the most likely value for k assuming that the distribution of letters in the text is typical of English text?

7. Suppose that when a string of English text is encrypted using a shift cipher $f(p) = (p + k)$ **mod** 26, the resulting ciphertext is DY CVOOZ ZOBMRKXMO DY NBOKW. What was the original plaintext string?

8. Suppose that the ciphertext DVE CFMV KF NFEUVI, REU KYRK ZJ KYV JVVU FW JTZVETV was produced by encrypting a plaintext message using a shift cipher. What is the original plaintext?

9. Suppose that the ciphertext ERC WYJJMGMIRXPC EHZERGIH XIGLRSPSKC MW MRHMWXM-RKYMWLEFPI JVSQ QEKMG was produced by encrypting a plaintext message using a shift cipher. What is the original plaintext?

10. Determine whether there is a key for which the enciphering function for the shift cipher is the same as the deciphering function.

11. What is the decryption function for an affine cipher if the encryption function is $c = (15p + 13)$ **mod** 26?

***12.** Find all pairs of integers keys (a, b) for affine ciphers for which the encryption function $c = (ap + b)$ **mod** 26 is the same as the corresponding decryption function.

13. Suppose that the most common letter and the second most common letter in a long ciphertext produced by encrypting a plaintext using an affine cipher $f(p) = (ap + b)$ **mod** 26 are Z and J, respectively. What are the most likely values of a and b?

14. Encrypt the message GRIZZLY BEARS using blocks of five letters and the transposition cipher based on the permutation of $\{1, 2, 3, 4, 5\}$ with $\sigma(1) = 3$, $\sigma(2) = 5$, $\sigma(3) = 1$, $\sigma(4) = 2$, and $\sigma(5) = 4$. For this exercise, use the letter X as many times as necessary to fill out the final block of fewer then five letters.

15. Decrypt the message EABW EFRO ATMR ASIN which is the ciphertext produced by encrypting a plaintext message using the transposition cipher with blocks of four letters and the permutation σ of $\{1, 2, 3, 4\}$ defined by $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 4$, and $\sigma(4) = 2$.

***16.** Suppose that you know that a ciphertext was produced by encrypting a plaintext message with a transposition cipher. How might you go about breaking it?

17. Suppose you have intercepted a ciphertext message and when you determine the frequencies of letters in this message, you find the frequencies are similar to the frequency of letters in English text. Which type of cipher do you suspect was used?

The **Vigenère cipher** is a block cipher, with a key that is a string of letters with numerical equivalents $k_1k_2 \ldots k_m$, where $k_i \in \mathbf{Z}_{26}$ for $i = 1, 2, \ldots, m$. Suppose that the numerical equivalents of the letters of a plaintext block are $p_1p_2 \ldots p_m$. The corresponding numerical ciphertext block is $(p_1 + k_1)$ **mod** 26 $(p_2 + k_2)$ **mod** 26 $\ldots$ $(p_m + k_m)$ **mod** 26. Finally, we translate back to letters. For example, suppose that the key string is RED, with numerical equivalents 17 4 3. Then, the plaintext ORANGE, with numerical equivalents 14 17 00 13 06 04, is encrypted by first splitting it into two blocks 14 17 00 and 13 06 04. Then, in each block we shift the first letter by 17, the second by 4, and the third by 3. We obtain 5 21 03 and 04 10 07. The cipherext is FVDEKH.

18. Use the Vigenère cipher with key BLUE to encrypt the message SNOWFALL.

19. The ciphertext OIKYWVHBX was produced by encrypting a plaintext message using the Vigenère cipher with key HOT. What is the plaintext message?

20. Express the Vigenère cipher as a cryptosystem.

To break a Vigenère cipher by recovering a plaintext message from the ciphertext message without having the key, the first step is to figure out the length of the key string. The second step is to figure out each character of the key string by determining the corresponding shift. Exercises 21 and 22 deal with these two aspects.

21. Suppose that when a long string of text is encrypted using a Vigenère cipher, the same string is found in the ciphertext starting at several different positions. Explain how this information can be used to help determine the length of the key.

22. Once the length of the key string of a Vigènere cipher is known, explain how to determine each of its characters. Assume that the plaintext is long enough so that the frequency of its letters is reasonably close to the frequency of letters in typical English text.

***23.** Show that we can easily factor n when we know that n is the product of two primes, p and q, and we know the value of $(p-1)(q-1)$.

In Exercises 24–27 first express your answers without computing modular exponentiations. Then use a computational aid to complete these computations.

24. Encrypt the message ATTACK using the RSA system with $n = 43 \cdot 59$ and $e = 13$, translating each letter into integers and grouping together pairs of integers, as done in Example 8.

25. Encrypt the message UPLOAD using the RSA system with $n = 53 \cdot 61$ and $e = 17$, translating each letter into integers and grouping together pairs of integers, as done in Example 8.

26. What is the original message encrypted using the RSA system with $n = 53 \cdot 61$ and $e = 17$ if the encrypted message is 3185 2038 2460 2550? (To decrypt, first find the decryption exponent d, which is the inverse of $e = 17$ modulo $52 \cdot 60$.)

27. What is the original message encrypted using the RSA system with $n = 43 \cdot 59$ and $e = 13$ if the encrypted message is 0667 1947 0671? (To decrypt, first find the decryption exponent d which is the inverse of $e = 13$ modulo $42 \cdot 58$.)

***28.** Suppose that (n, e) is an RSA encryption key, with $n = pq$ where p and q are large primes and $\gcd(e, (p-1)(q-1)) = 1$. Furthermore, suppose that d is an inverse of e modulo $(p-1)(q-1)$. Suppose that $C \equiv M^e \pmod{pq}$. In the text we showed that RSA decryption, that is, the congruence $C^d \equiv M \pmod{pq}$ holds when $\gcd(M, pq) = 1$. Show that this decryption congruence also holds when $\gcd(M, pq) > 1$. [*Hint:* Use congruences modulo p and modulo q and apply the Chinese remainder theorem.]

29. Describe the steps that Alice and Bob follow when they use the Diffie-Hellman key exchange protocol to generate a shared key. Assume that they use the prime $p = 23$ and take $a = 5$, which is a primitive root of 23, and that Alice selects $k_1 = 8$ and Bob selects $k_2 = 5$. (You may want to use some computational aid.)

30. Describe the steps that Alice and Bob follow when they use the Diffie-Hellman key exchange protocol to generate a shared key. Assume that they use the prime $p = 101$ and take $a = 2$, which is a primitive root of 101, and that Alice selects $k_1 = 7$ and Bob selects $k_2 = 9$. (You may want to use some computational aid.)

In Exercises 31–32 suppose that Alice and Bob have these public keys and corresponding private keys: $(n_{\text{Alice}}, e_{\text{Alice}}) = (2867, 7) = (61 \cdot 47, 7)$, $d_{\text{Alice}} = 1183$ and $(n_{\text{Bob}}, e_{\text{Bob}}) = (3127, 21) = (59 \cdot 53, 21)$, $d_{\text{Bob}} = 1149$. First express your answers without carrying out the calculations. Then, using a computational aid, if available, perform the calculation to get the numerical answers.

31. Alice wants to send to all her friends, including Bob, the message "SELL EVERYTHING" so that he knows that she sent it. What should she send to her friends, assuming she signs the message using the RSA cryptosystem.

32. Alice wants to send to Bob the message "BUY NOW" so that he knows that she sent it and so that only Bob can read it. What should she send to Bob, assuming she signs the message and then encrypts it using Bob's public key?

33. We describe a basic key exchange protocol using private key cryptography upon which more sophisticated protocols for key exchange are based. Encryption within the protocol is done using a private key cryptosystem (such as AES) that is considered secure. The protocol involves three parties, Alice and Bob, who wish to exchange a key, and a trusted third party Cathy. Assume that Alice has a secret key k_{Alice} that only she and Cathy know, and Bob has a secret key k_{Bob} which only he and Cathy know. The protocol has three steps:

(i) Alice sends the trusted third party Cathy the message "request a shared key with Bob" encrypted using Alice's key k_{Alice}.

(ii) Cathy sends back to Alice a key $k_{\text{Alice,Bob}}$, which she generates, encrypted using the key k_{Alice}, followed by this same key $k_{\text{Alice,Bob}}$, encrypted using Bob's key, k_{Bob}.

(iii) Alice sends to Bob the key $k_{\text{Alice,Bob}}$ encrypted using k_{Bob}, known only to Bob and to Cathy.

Explain why this protocol allows Alice and Bob to share the secret key $k_{\text{Alice,Bob}}$, known only to them and to Cathy.

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

It is worthwhile to revisit each of the mathematical induction proofs in Examples 1–14 to see how these steps are completed. It will be helpful to follow these guidelines in the solutions of the exercises that ask for proofs by mathematical induction. The guidelines that we presented can be adapted for each of the variants of mathematical induction that we introduce in the exercises and later in this chapter.

Exercises

1. There are infinitely many stations on a train route. Suppose that the train stops at the first station and suppose that if the train stops at a station, then it stops at the next station. Show that the train stops at all stations.

2. Suppose that you know that a golfer plays the first hole of a golf course with an infinite number of holes and that if this golfer plays one hole, then the golfer goes on to play the next hole. Prove that this golfer plays every hole on the course.

Use mathematical induction in Exercises 3–17 to prove summation formulae. Be sure to identify where you use the inductive hypothesis.

3. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

4. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.

a) What is the statement $P(1)$?

b) Show that $P(1)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step, identifying where you use the inductive hypothesis.

f) Explain why these steps show that this formula is true whenever n is a positive integer.

5. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ whenever n is a nonnegative integer.

6. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.

7. Prove that $3 + 3 \cdot 5 + 3 \cdot 5^2 + \cdots + 3 \cdot 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

8. Prove that $2 - 2 \cdot 7 + 2 \cdot 7^2 - \cdots + 2(-7)^n = (1 - (-7)^{n+1})/4$ whenever n is a nonnegative integer.

9. **a)** Find a formula for the sum of the first n even positive integers.

b) Prove the formula that you conjectured in part (a).

10. **a)** Find a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

11. **a)** Find a formula for

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$$

by examining the values of this expression for small values of n.

b) Prove the formula you conjectured in part (a).

12. Prove that

$$\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j=\frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$$

whenever n is a nonnegative integer.

13. Prove that $1^2-2^2+3^2-\cdots+(-1)^{n-1}n^2=(-1)^{n-1}n(n+1)/2$ whenever n is a positive integer.

14. Prove that for every positive integer n, $\sum_{k=1}^{n}k2^k=(n-1)2^{n+1}+2$.

15. Prove that for every positive integer n,

$$1\cdot 2+2\cdot 3+\cdots+n(n+1)=n(n+1)(n+2)/3.$$

16. Prove that for every positive integer n,

$$1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)$$
$$=n(n+1)(n+2)(n+3)/4.$$

17. Prove that $\sum_{j=1}^{n}j^4=n(n+1)(2n+1)(3n^2+3n-1)/30$ whenever n is a positive integer.

Use mathematical induction to prove the inequalities in Exercises 18–30.

18. Let $P(n)$ be the statement that $n!<n^n$, where n is an integer greater than 1.

a) What is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

19. Let $P(n)$ be the statement that

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2}<2-\frac{1}{n},$$

where n is an integer greater than 1.

a) What is the statement $P(2)$?

b) Show that $P(2)$ is true, completing the basis step of the proof.

c) What is the inductive hypothesis?

d) What do you need to prove in the inductive step?

e) Complete the inductive step.

f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

20. Prove that $3^n<n!$ if n is an integer greater than 6.

21. Prove that $2^n>n^2$ if n is an integer greater than 4.

22. For which nonnegative integers n is $n^2\le n!$? Prove your answer.

23. For which nonnegative integers n is $2n+3\le 2^n$? Prove your answer.

24. Prove that $1/(2n)\le[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)]/(2\cdot 4\cdot\cdots\cdot 2n)$ whenever n is a positive integer.

***25.** Prove that if $h>-1$, then $1+nh\le(1+h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

***26.** Suppose that a and b are real numbers with $0<b<a$. Prove that if n is a positive integer, then $a^n-b^n\le na^{n-1}(a-b)$.

***27.** Prove that for every positive integer n,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}>2(\sqrt{n+1}-1).$$

28. Prove that $n^2-7n+12$ is nonnegative whenever n is an integer with $n\ge 3$.

In Exercises 29 and 30, H_n denotes the nth harmonic number.

***29.** Prove that $H_{2^n}\le 1+n$ whenever n is a nonnegative integer.

***30.** Prove that

$$H_1+H_2+\cdots+H_n=(n+1)H_n-n.$$

Use mathematical induction in Exercises 31–37 to prove divisibility facts.

31. Prove that 2 divides n^2+n whenever n is a positive integer.

32. Prove that 3 divides n^3+2n whenever n is a positive integer.

33. Prove that 5 divides n^5-n whenever n is a nonnegative integer.

34. Prove that 6 divides n^3-n whenever n is a nonnegative integer.

***35.** Prove that n^2-1 is divisible by 8 whenever n is an odd positive integer.

***36.** Prove that 21 divides $4^{n+1}+5^{2n-1}$ whenever n is a positive integer.

***37.** Prove that if n is a positive integer, then 133 divides $11^{n+1}+12^{2n-1}$.

Use mathematical induction in Exercises 38–46 to prove results about sets.

38. Prove that if $A_1,A_2,\ldots,A_n$ and $B_1,B_2,\ldots,B_n$ are sets such that $A_j\subseteq B_j$ for $j=1,2,\ldots,n$, then

$$\bigcup_{j=1}^{n}A_j\subseteq\bigcup_{j=1}^{n}B_j.$$

39. Prove that if $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ are sets such that $A_j \subseteq B_j$ for $j = 1, 2, \ldots, n$, then

$$\bigcap_{j=1}^{n} A_j \subseteq \bigcap_{j=1}^{n} B_j.$$

40. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B \\ = (A_1 \cup B) \cap (A_2 \cup B) \cap \cdots \cap (A_n \cup B).$$

41. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B \\ = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

42. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\ = (A_1 \cap A_2 \cap \cdots \cap A_n) - B.$$

43. Prove that if $A_1, A_2, \ldots, A_n$ are subsets of a universal set U, then

$$\overline{\bigcup_{k=1}^{n} A_k} = \textstyle\bigcap_{k=1}^{n} \overline{A_k}.$$

44. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 - B) \cup (A_2 - B) \cup \cdots \cup (A_n - B) \\ = (A_1 \cup A_2 \cup \cdots \cup A_n) - B.$$

45. Prove that a set with n elements has $n(n-1)/2$ subsets containing exactly two elements whenever n is an integer greater than or equal to 2.

***46.** Prove that a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly three elements whenever n is an integer greater than or equal to 3.

In Exercises 47 and 48 we consider the problem of placing towers along a straight road, so that every building on the road receives cellular service. Assume that a building receives cellular service if it is within one mile of a tower.

47. Devise a greedy algorithm that uses the minimum number of towers possible to provide cell service to d buildings located at positions $x_1, x_2, \ldots, x_d$ from the start of the road. [*Hint:* At each step, go as far as possible along the road before adding a tower so as not to leave any buildings without coverage.]

***48.** Use mathematical induction to prove that the algorithm you devised in Exercise 47 produces an optimal solution, that is, that it uses the fewest towers possible to provide cellular service to all buildings.

Exercises 49–51 present incorrect proofs using mathematical induction. You will need to identify an error in reasoning in each exercise.

49. What is wrong with this "proof" that all horses are the same color?
Let $P(n)$ be the proposition that all the horses in a set of n horses are the same color.

Basis Step: Clearly, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color. Consider any $k+1$ horses; number these as horses $1, 2, 3, \ldots, k, k+1$. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Because the set of the first k horses and the set of the last k horses overlap, all $k+1$ must be the same color. This shows that $P(k+1)$ is true and finishes the proof by induction.

50. What is wrong with this "proof"?
"Theorem" For every positive integer n, $\sum_{i=1}^{n} i = (n + \frac{1}{2})^2/2$.

Basis Step: The formula is true for $n = 1$.

Inductive Step: Suppose that $\sum_{i=1}^{n} i = (n+\frac{1}{2})^2/2$. Then $\sum_{i=1}^{n+1} i = (\sum_{i=1}^{n} i) + (n+1)$. By the inductive hypothesis, $\sum_{i=1}^{n+1} i = (n+\frac{1}{2})^2/2 + n + 1 = (n^2 + n + \frac{1}{4})/2 + n + 1 = (n^2 + 3n + \frac{9}{4})/2 = (n+\frac{3}{2})^2/2 = [(n+1)+\frac{1}{2}]^2/2$, completing the inductive step.

51. What is wrong with this "proof"?
"Theorem" For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.

Basis Step: Suppose that $n = 1$. If $\max(x, y) = 1$ and x and y are positive integers, we have $x = 1$ and $y = 1$.

Inductive Step: Let k be a positive integer. Assume that whenever $\max(x, y) = k$ and x and y are positive integers, then $x = y$. Now let $\max(x, y) = k+1$, where x and y are positive integers. Then $\max(x-1, y-1) = k$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$, completing the inductive step.

52. Suppose that m and n are positive integers with $m > n$ and f is a function from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Use mathematical induction on the variable n to show that f is not one-to-one.

***53.** Use mathematical induction to show that n people can divide a cake (where each person gets one or more separate pieces of the cake) so that the cake is divided fairly, that is, in the sense that each person thinks he or she got at least $(1/n)$th of the cake. [*Hint:* For the inductive step, take a fair division of the cake among the first k people, have each person divide their share into what this person thinks are $k+1$ equal portions, and then have the $(k+1)$st person select a portion from each of the k people. When showing this produces a fair division for $k+1$ people, suppose that person $k+1$ thinks that person i got p_i of the cake where $\sum_{i=1}^{k} p_i = 1$.]

54. Use mathematical induction to show that given a set of $n+1$ positive integers, none exceeding $2n$, there is at least one integer in this set that divides another integer in the set.

***55.** A knight on a chessboard can move one space horizontally (in either direction) and two spaces vertically (in either direction) or two spaces horizontally (in either direction) and one space vertically (in either direction). Suppose that we have an infinite chessboard, made up of all squares (m, n) where m and n are nonnegative integers that denote the row number and the column number of the square, respectively. Use mathematical induction to show that a knight starting at $(0, 0)$ can visit every square using a finite sequence of moves. [*Hint:* Use induction on the variable $s = m + n$.]

56. Suppose that

$$\mathbf{A} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where a and b are real numbers. Show that

$$\mathbf{A}^n = \begin{bmatrix} a^n & 0 \\ 0 & b^n \end{bmatrix}$$

for every positive integer n.

57. (*Requires calculus*) Use mathematical induction to prove that the derivative of $f(x) = x^n$ equals nx^{n-1} whenever n is a positive integer. (For the inductive step, use the product rule for derivatives.)

58. Suppose that **A** and **B** are square matrices with the property $\mathbf{AB} = \mathbf{BA}$. Show that $\mathbf{AB}^n = \mathbf{B}^n\mathbf{A}$ for every positive integer n.

59. Suppose that m is a positive integer. Use mathematical induction to prove that if a and b are integers with $a \equiv b \pmod{m}$, then $a^k \equiv b^k \pmod{m}$ whenever k is a nonnegative integer.

60. Use mathematical induction to show that $\neg(p_1 \vee p_2 \vee \cdots \vee p_n)$ is equivalent to $\neg p_1 \wedge \neg p_2 \wedge \cdots \wedge \neg p_n$ whenever $p_1, p_2, \ldots, p_n$ are propositions.

***61.** Show that

$$[(p_1 \to p_2) \wedge (p_2 \to p_3) \wedge \cdots \wedge (p_{n-1} \to p_n)] \\ \to [(p_1 \wedge p_2 \wedge \cdots \wedge p_{n-1}) \to p_n]$$

is a tautology whenever $p_1, p_2, \ldots, p_n$ are propositions, where $n \geq 2$.

***62.** Show that n lines separate the plane into $(n^2 + n + 2)/2$ regions if no two of these lines are parallel and no three pass through a common point.

****63.** Let $a_1, a_2, \ldots, a_n$ be positive real numbers. The **arithmetic mean** of these numbers is defined by

$$A = (a_1 + a_2 + \cdots + a_n)/n,$$

and the **geometric mean** of these numbers is defined by

$$G = (a_1 a_2 \cdots a_n)^{1/n}.$$

Use mathematical induction to prove that $A \geq G$.

64. Use mathematical induction to prove Lemma 3 of Section 4.3, which states that if p is a prime and $p \mid a_1 a_2 \cdots a_n$, where a_i is an integer for $i = 1, 2, 3, \ldots, n$, then $p \mid a_i$ for some integer i.

65. Show that if n is a positive integer, then

$$\sum_{\{a_1,\ldots,a_k\} \subseteq \{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n.$$

(Here the sum is over all nonempty subsets of the set of the n smallest positive integers.)

***66.** Use the well-ordering property to show that the following form of mathematical induction is a valid method to prove that $P(n)$ is true for all positive integers n.

Basis Step: $P(1)$ and $P(2)$ are true.

Inductive Step: For each positive integer k, if $P(k)$ and $P(k+1)$ are both true, then $P(k+2)$ is true.

67. Show that if $A_1, A_2, \ldots, A_n$ are sets where $n \geq 2$, and for all pairs of integers i and j with $1 \leq i < j \leq n$ either A_i is a subset of A_j or A_j is a subset of A_i, then there is an integer i, $1 \leq i \leq n$ such that A_i is a subset of A_j for all integers j with $1 \leq j \leq n$.

***68.** A guest at a party is a **celebrity** if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party, for if there were two, they would know each other. A particular party may have no celebrity. Your assignment is to find the celebrity, if one exists, at a party, by asking only one type of question—asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with $3(n-1)$ questions. [*Hint:* First ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.]

Suppose there are n people in a group, each aware of a scandal no one else in the group knows about. These people communicate by telephone; when two people in the group talk, they share information about all scandals each knows about. For example, on the first call, two people share information, so by the end of the call, each of these people knows about two scandals. The **gossip problem** asks for $G(n)$, the minimum number of telephone calls that are needed for all n people to learn about all the scandals. Exercises 69–71 deal with the gossip problem.

69. Find $G(1)$, $G(2)$, $G(3)$, and $G(4)$.

70. Use mathematical induction to prove that $G(n) \leq 2n - 4$ for $n \geq 4$. [*Hint:* In the inductive step, have a new person call a particular person at the start and at the end.]

****71.** Prove that $G(n) = 2n - 4$ for $n \geq 4$.

***72.** Show that it is possible to arrange the numbers $1, 2, \ldots, n$ in a row so that the average of any two of these numbers never appears between them. [*Hint:* Show that it suffices to prove this fact when n is a power of 2. Then use mathematical induction to prove the result when n is a power of 2.]

***73.** Show that if $I_1, I_2, \ldots, I_n$ is a collection of open intervals on the real number line, $n \geq 2$, and every pair of these intervals has a nonempty intersection, that is, $I_i \cap I_j \neq \emptyset$ whenever $1 \leq i \leq n$ and $1 \leq j \leq n$, then the intersection of all these sets is nonempty, that is, $I_1 \cap I_2 \cap \cdots \cap I_n \neq \emptyset$. (Recall that an **open interval** is the set of real numbers x with $a < x < b$, where a and b are real numbers with $a < b$.)

Sometimes we cannot use mathematical induction to prove a result we believe to be true, but we can use mathematical induction to prove a stronger result. Because the inductive hypothesis of the stronger result provides more to work with, this process is called **inductive loading**. We use inductive loading in Exercise 74.

74. Suppose that we want to prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$$

for all positive integers n.

a) Show that if we try to prove this inequality using mathematical induction, the basis step works, but the inductive step fails.

b) Show that mathematical induction can be used to prove the stronger inequality

$$\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{3n+1}}$$

for all integers greater than 1, which, together with a verification for the case where $n = 1$, establishes the weaker inequality we originally tried to prove using mathematical induction.

75. Let n be an even positive integer. Show that when n people stand in a yard at mutually distinct distances and each person throws a pie at their nearest neighbor, it is possible that everyone is hit by a pie.

76. Construct a tiling using right triominoes of the 4×4 checkerboard with the square in the upper left corner removed.

77. Construct a tiling using right triominoes of the 8×8 checkerboard with the square in the upper left corner removed.

78. Prove or disprove that all checkerboards of these shapes can be completely covered using right triominoes whenever n is a positive integer.

a) 3×2^n **b)** 6×2^n
c) $3^n \times 3^n$ **d)** $6^n \times 6^n$

***79.** Show that a three-dimensional $2^n \times 2^n \times 2^n$ checkerboard with one $1 \times 1 \times 1$ cube missing can be completely covered by $2 \times 2 \times 2$ cubes with one $1 \times 1 \times 1$ cube removed.

***80.** Show that an $n \times n$ checkerboard with one square removed can be completely covered using right triominoes if $n > 5$, n is odd, and $3 \nmid n$.

81. Show that a 5×5 checkerboard with a corner square removed can be tiled using right triominoes.

***82.** Find a 5×5 checkerboard with a square removed that cannot be tiled using right triominoes. Prove that such a tiling does not exist for this board.

83. Use the principle of mathematical induction to show that $P(n)$ is true for $n = b, b+1, b+2, \ldots$, where b is an integer, if $P(b)$ is true and the conditional statement $P(k) \rightarrow P(k+1)$ is true for all integers k with $k \geq b$.

5.2 Strong Induction and Well-Ordering

Introduction

In Section 5.1 we introduced mathematical induction and we showed how to use it to prove a variety of theorems. In this section we will introduce another form of mathematical induction, called **strong induction**, which can often be used when we cannot easily prove a result using mathematical induction. The basis step of a proof by strong induction is the same as a proof of the same result using mathematical induction. That is, in a strong induction proof that $P(n)$ is true for all positive integers n, the basis step shows that $P(1)$ is true. However, the inductive steps in these two proof methods are different. In a proof by mathematical induction, the inductive step shows that if the inductive hypothesis $P(k)$ is true, then $P(k+1)$ is also true. In a proof by strong induction, the inductive step shows that if $P(j)$ is true for all positive integers not exceeding k, then $P(k+1)$ is true. That is, for the inductive hypothesis we assume that $P(j)$ is true for $j = 1, 2, \ldots, k$.

The validity of both mathematical induction and strong induction follow from the well-ordering property in Appendix 1. In fact, mathematical induction, strong induction, and well-ordering are all equivalent principles (as shown in Exercises 41, 42, and 43). That is, the validity of each can be proved from either of the other two. This means that a proof using one of these two principles can be rewritten as a proof using either of the other two principles. Just as it is sometimes the case that it is much easier to see how to prove a result using strong induction rather than mathematical induction, it is sometimes easier to use well-ordering than one of the showed that the principle of mathematical induction follows from the well-ordering property. The other parts of this equivalence are left as Exercises 31, 42, and 43.

THE WELL-ORDERING PROPERTY Every nonempty set of nonnegative integers has a least element.

The well-ordering property can often be used directly in proofs.

EXAMPLE 5 Use the well-ordering property to prove the division algorithm. Recall that the division algorithm states that if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \leq r < d$ and $a = dq + r$.

Extra Examples

Solution: Let S be the set of nonnegative integers of the form $a - dq$, where q is an integer. This set is nonempty because $-dq$ can be made as large as desired (taking q to be a negative integer with large absolute value). By the well-ordering property, S has a least element $r = a - dq_0$.

The integer r is nonnegative. It is also the case that $r < d$. If it were not, then there would be a smaller nonnegative element in S, namely, $a - d(q_0 + 1)$. To see this, suppose that $r \geq d$. Because $a = dq_0 + r$, it follows that $a - d(q_0 + 1) = (a - dq_0) - d = r - d \geq 0$. Consequently, there are integers q and r with $0 \leq r < d$. The proof that q and r are unique is left as Exercise 37. ◀

EXAMPLE 6 In a round-robin tournament every player plays every other player exactly once and each match has a winner and a loser. We say that the players $p_1, p_2, \ldots, p_m$ form a *cycle* if p_1 beats p_2, p_2 beats $p_3, \ldots, p_{m-1}$ beats p_m, and p_m beats p_1. Use the well-ordering principle to show that if there is a cycle of length m ($m \geq 3$) among the players in a round-robin tournament, there must be a cycle of three of these players.

Solution: We assume that there is no cycle of three players. Because there is at least one cycle in the round-robin tournament, the set of all positive integers n for which there is a cycle of length n is nonempty. By the well-ordering property, this set of positive integers has a least element k, which by assumption must be greater than three. Consequently, there exists a cycle of players $p_1, p_2, p_3, \ldots, p_k$ and no shorter cycle exists.

Because there is no cycle of three players, we know that $k > 3$. Consider the first three elements of this cycle, p_1, p_2, and p_3. There are two possible outcomes of the match between p_1 and p_3. If p_3 beats p_1, it follows that p_1, p_2, p_3 is a cycle of length three, contradicting our assumption that there is no cycle of three players. Consequently, it must be the case that p_1 beats p_3. This means that we can omit p_2 from the cycle $p_1, p_2, p_3, \ldots, p_k$ to obtain the cycle $p_1, p_3, p_4, \ldots, p_k$ of length $k - 1$, contradicting the assumption that the smallest cycle has length k. We conclude that there must be a cycle of length three. ◀

Exercises

1. Use strong induction to show that if you can run one mile or two miles, and if you can always run two more miles once you have run a specified number of miles, then you can run any number of miles.

2. Use strong induction to show that all dominoes fall in an infinite arrangement of dominoes if you know that the first three dominoes fall, and that when a domino falls, the domino three farther down in the arrangement also falls.

3. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 8$.

a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 10$.

e) Explain why these steps show that this statement is true whenever $n \geq 8$.

4. Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

a) Show statements $P(18)$, $P(19)$, $P(20)$, and $P(21)$ are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

5. **a)** Determine which amounts of postage can be formed using just 4-cent and 11-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

6. **a)** Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps.

b) Prove your answer to (a) using the principle of mathematical induction. Be sure to state explicitly your inductive hypothesis in the inductive step.

c) Prove your answer to (a) using strong induction. How does the inductive hypothesis in this proof differ from that in the inductive hypothesis for a proof using mathematical induction?

7. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.

8. Suppose that a store offers gift certificates in denominations of 25 dollars and 40 dollars. Determine the possible total amounts you can form using these gift certificates. Prove your answer using strong induction.

***9.** Use strong induction to prove that $\sqrt{2}$ is irrational. [*Hint:* Let $P(n)$ be the statement that $\sqrt{2} \neq n/b$ for any positive integer b.]

10. Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

11. Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if $n = 4j$, $4j + 2$, or $4j + 3$ for some nonnegative integer j and the second player wins in the remaining case when $n = 4j + 1$ for some nonnegative integer j.

12. Use strong induction to show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

***13.** A jigsaw puzzle is put together by successively joining pieces that fit together into blocks. A move is made each time a piece is added to a block, or when two blocks are joined. Use strong induction to prove that no matter how the moves are carried out, exactly $n - 1$ moves are required to assemble a puzzle with n pieces.

14. Suppose you begin with a pile of n stones and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have r and s stones in them, respectively, you compute rs. Show that no matter how you split the piles, the sum of the products computed at each step equals $n(n - 1)/2$.

15. Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is square. [*Hint:* Use strong induction to show that this strategy works. For the first move, the first player chomps all cookies except those in the left and top edges. On subsequent moves, after the second player has chomped cookies on either the top or left edge, the first player chomps cookies in the same relative positions in the left or top edge, respectively.]

***16.** Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is two squares wide, that is, a $2 \times n$ board. [*Hint:* Use strong induction. The first move of the first player should be to chomp the cookie in the bottom row at the far right.]

17. Use strong induction to show that if a simple polygon with at least four sides is triangulated, then at least two of the triangles in the triangulation have two sides that border the exterior of the polygon.

***18.** Use strong induction to show that when a simple polygon P with consecutive vertices $v_1, v_2, \ldots, v_n$ is triangulated into $n - 2$ triangles, the $n - 2$ triangles can be numbered $1, 2, \ldots, n - 2$ so that v_i is a vertex of triangle i for $i = 1, 2, \ldots, n - 2$.

***19.** **Pick's theorem** says that the area of a simple polygon P in the plane with vertices that are all lattice points (that is, points with integer coordinates) equals $I(P) + B(P)/2 - 1$, where $I(P)$ and $B(P)$ are the number of lattice points in the interior of P and on the boundary of P, respectively. Use strong induction on the number of vertices of P to prove Pick's theorem. [*Hint:* For the basis step, first prove the theorem for rectangles, then for right triangles, and finally for all triangles by noting that the area of a triangle is the area of a larger rectangle containing it with the areas of at most three triangles subtracted. For the inductive step, take advantage of Lemma 1.]

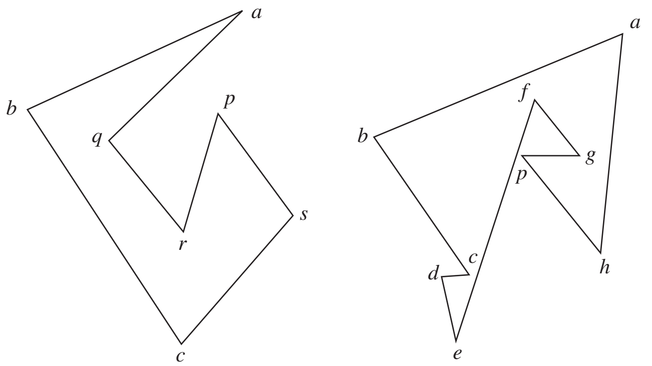
****20.** Suppose that P is a simple polygon with vertices $v_1, v_2, \dots, v_n$ listed so that consecutive vertices are connected by an edge, and v_1 and v_n are connected by an edge. A vertex v_i is called an **ear** if the line segment connecting the two vertices adjacent to v_i is an interior diagonal of the simple polygon. Two ears v_i and v_j are called **nonoverlapping** if the interiors of the triangles with vertices v_i and its two adjacent vertices and v_j and its two adjacent vertices do not intersect. Prove that every simple polygon with at least four vertices has at least two nonoverlapping ears.

21. In the proof of Lemma 1 we mentioned that many incorrect methods for finding a vertex p such that the line segment bp is an interior diagonal of P have been published. This exercise presents some of the incorrect ways p has been chosen in these proofs. Show, by considering one of the polygons drawn here, that for each of these choices of p, the line segment bp is not necessarily an interior diagonal of P.

a) p is the vertex of P such that the angle $\angle abp$ is smallest.

b) p is the vertex of P with the least x-coordinate (other than b).

c) p is the vertex of P that is closest to b.



Exercises 22 and 23 present examples that show inductive loading can be used to prove results in computational geometry.

***22.** Let $P(n)$ be the statement that when nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two vertices of the polygon are not endpoints of any of these diagonals.

a) Show that when we attempt to prove $P(n)$ for all integers n with $n \geq 3$ using strong induction, the inductive step does not go through.

b) Show that we can prove that $P(n)$ is true for all integers n with $n \geq 3$ by proving by strong induction the stronger assertion $Q(n)$, for $n \geq 4$, where $Q(n)$ states that whenever nonintersecting diagonals are drawn inside a convex polygon with n sides, at least two *nonadjacent* vertices are not endpoints of any of these diagonals.

23. Let $E(n)$ be the statement that in a triangulation of a simple polygon with n sides, at least one of the triangles in the triangulation has two sides bordering the exterior of the polygon.

a) Explain where a proof using strong induction that $E(n)$ is true for all integers $n \geq 4$ runs into difficulties.

b) Show that we can prove that $E(n)$ is true for all integers $n \geq 4$ by proving by strong induction the stronger statement $T(n)$ for all integers $n \geq 4$, which states that in every triangulation of a simple polygon, at least two of the triangles in the triangulation have two sides bordering the exterior of the polygon.

***24.** A stable assignment, defined in the preamble to Exercise 60 in Section 3.1, is called **optimal for suitors** if no stable assignment exists in which a suitor is paired with a suitee whom this suitor prefers to the person to whom this suitor is paired in this stable assignment. Use strong induction to show that the deferred acceptance algorithm produces a stable assignment that is optimal for suitors.

25. Suppose that $P(n)$ is a propositional function. Determine for which positive integers n the statement $P(n)$ must be true, and justify your answer, if

a) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+2)$ is true.

b) $P(1)$ and $P(2)$ are true; for all positive integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

c) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(2n)$ is true.

d) $P(1)$ is true; for all positive integers n, if $P(n)$ is true, then $P(n+1)$ is true.

26. Suppose that $P(n)$ is a propositional function. Determine for which nonnegative integers n the statement $P(n)$ must be true if

a) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ is true.

b) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+3)$ is true.

c) $P(0)$ and $P(1)$ are true; for all nonnegative integers n, if $P(n)$ and $P(n+1)$ are true, then $P(n+2)$ is true.

d) $P(0)$ is true; for all nonnegative integers n, if $P(n)$ is true, then $P(n+2)$ and $P(n+3)$ are true.

27. Show that if the statement $P(n)$ is true for infinitely many positive integers n and $P(n+1) \rightarrow P(n)$ is true for all positive integers n, then $P(n)$ is true for all positive integers n.

28. Let b be a fixed integer and j a fixed positive integer. Show that if $P(b), P(b+1), \dots, P(b+j)$ are true and $[P(b) \wedge P(b+1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for every integer $k \geq b+j$, then $P(n)$ is true for all integers n with $n \geq b$.

29. What is wrong with this “proof” by strong induction?

“Theorem” For every nonnegative integer n, $5n = 0$.

Basis Step: $5 \cdot 0 = 0$.

Inductive Step: Suppose that $5j = 0$ for all nonnegative integers j with $0 \leq j \leq k$. Write $k+1 = i+j$, where i and j are natural numbers less than $k+1$. By the inductive hypothesis, $5(k+1) = 5(i+j) = 5i + 5j = 0 + 0 = 0$.

∗30. Find the flaw with the following "proof" that $a^n = 1$ for all nonnegative integers n, whenever a is a nonzero real number.

Basis Step: $a^0 = 1$ is true by the definition of a^0.

Inductive Step: Assume that $a^j = 1$ for all nonnegative integers j with $j \leq k$. Then note that

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1.$$

∗31. Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

32. Find the flaw with the following "proof" that every postage of three cents or more can be formed using just three-cent and four-cent stamps.

Basis Step: We can form postage of three cents with a single three-cent stamp and we can form postage of four cents using a single four-cent stamp.

Inductive Step: Assume that we can form postage of j cents for all nonnegative integers j with $j \leq k$ using just three-cent and four-cent stamps. We can then form postage of $k + 1$ cents by replacing one three-cent stamp with a four-cent stamp or by replacing two four-cent stamps by three three-cent stamps.

33. Show that we can prove that $P(n, k)$ is true for all pairs of positive integers n and k if we show

a) $P(1, 1)$ is true and $P(n, k) \to [P(n + 1, k) \wedge P(n, k + 1)]$ is true for all positive integers n and k.

b) $P(1, k)$ is true for all positive integers k, and $P(n, k) \to P(n + 1, k)$ is true for all positive integers n and k.

c) $P(n, 1)$ is true for all positive integers n, and $P(n, k) \to P(n, k + 1)$ is true for all positive integers n and k.

34. Prove that $\sum_{j=1}^{n} j(j + 1)(j + 2) \cdots (j + k - 1) = n(n + 1)(n + 2) \cdots (n + k)/(k + 1)$ for all positive integers k and n. [*Hint*: Use a technique from Exercise 33.]

∗35. Show that if $a_1, a_2, \ldots, a_n$ are n distinct real numbers, exactly $n - 1$ multiplications are used to compute the product of these n numbers no matter how parentheses are inserted into their product. [*Hint:* Use strong induction and consider the last multiplication.]

∗36. The well-ordering property can be used to show that there is a unique greatest common divisor of two positive integers. Let a and b be positive integers, and let S be the set of positive integers of the form $as + bt$, where s and t are integers.

a) Show that S is nonempty.

b) Use the well-ordering property to show that S has a smallest element c.

c) Show that if d is a common divisor of a and b, then d is a divisor of c.

d) Show that $c \mid a$ and $c \mid b$. [*Hint:* First, assume that $c \nmid a$. Then $a = qc + r$, where $0 < r < c$. Show that $r \in S$, contradicting the choice of c.]

e) Conclude from (c) and (d) that the greatest common divisor of a and b exists. Finish the proof by showing that this greatest common divisor is unique.

37. Let a be an integer and d be a positive integer. Show that the integers q and r with $a = dq + r$ and $0 \leq r < d$, which were shown to exist in Example 5, are unique.

38. Use mathematical induction to show that a rectangular checkerboard with an even number of cells and two squares missing, one white and one black, can be covered by dominoes.

∗∗39. Can you use the well-ordering property to prove the statement: "Every positive integer can be described using no more than fifteen English words"? Assume the words come from a particular dictionary of English. [*Hint:* Suppose that there are positive integers that cannot be described using no more than fifteen English words. By well ordering, *the smallest positive integer that cannot be described using no more than fifteen English words* would then exist.]

40. Use the well-ordering principle to show that if x and y are real numbers with $x < y$, then there is a rational number r with $x < r < y$. [*Hint:* Use the Archimedean property, given in Appendix 1, to find a positive integer A with $A > 1/(y - x)$. Then show that there is a rational number r with denominator A between x and y by looking at the numbers $\lfloor x \rfloor + j/A$, where j is a positive integer.]

∗41. Show that the well-ordering property can be proved when the principle of mathematical induction is taken as an axiom.

∗42. Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

∗43. Show that we can prove the well-ordering property when we take strong induction as an axiom instead of taking the well-ordering property as an axiom.

5.3 Recursive Definitions and Structural Induction

Introduction

Sometimes it is difficult to define an object explicitly. However, it may be easy to define this object in terms of itself. This process is called **recursion**. For instance, the picture shown in Figure 1 is produced recursively. First, an original picture is given. Then a process of successively superimposing centered smaller pictures on top of the previous pictures is carried out.

Exercises

1. There are 18 mathematics majors and 325 computer science majors at a college.

a) In how many ways can two representatives be picked so that one is a mathematics major and the other is a computer science major?

b) In how many ways can one representative be picked who is either a mathematics major or a computer science major?

2. An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

3. A multiple-choice test contains 10 questions. There are four possible answers for each question.

a) In how many ways can a student answer the questions on the test if the student answers every question?

b) In how many ways can a student answer the questions on the test if the student can leave answers blank?

4. A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

5. Six different airlines fly from New York to Denver and seven fly from Denver to San Francisco. How many different pairs of airlines can you choose on which to book a trip from New York to San Francisco via Denver, when you pick an airline for the flight to Denver and an airline for the continuation flight to San Francisco?

6. There are four major auto routes from Boston to Detroit and six from Detroit to Los Angeles. How many major auto routes are there from Boston to Los Angeles via Detroit?

7. How many different three-letter initials can people have?

8. How many different three-letter initials with none of the letters repeated can people have?

9. How many different three-letter initials are there that begin with an A?

10. How many bit strings are there of length eight?

11. How many bit strings of length ten both begin and end with a 1?

12. How many bit strings are there of length six or less, not counting the empty string?

13. How many bit strings with length not exceeding n, where n is a positive integer, consist entirely of 1s, not counting the empty string?

14. How many bit strings of length n, where n is a positive integer, start and end with 1s?

15. How many strings are there of lowercase letters of length four or less, not counting the empty string?

16. How many strings are there of four lowercase letters that have the letter x in them?

17. How many strings of five ASCII characters contain the character @ ("at" sign) at least once? [*Note:* There are 128 different ASCII characters.

18. How many 5-element DNA sequences

a) end with A?

b) start with T and end with G?

c) contain only A and T?

d) do not contain C?

19. How many 6-element RNA sequences

a) do not contain U?

b) end with GU?

c) start with C?

d) contain only A or U?

20. How many positive integers between 5 and 31

a) are divisible by 3? Which integers are these?

b) are divisible by 4? Which integers are these?

c) are divisible by 3 and by 4? Which integers are these?

21. How many positive integers between 50 and 100

a) are divisible by 7? Which integers are these?

b) are divisible by 11? Which integers are these?

c) are divisible by both 7 and 11? Which integers are these?

22. How many positive integers less than 1000

a) are divisible by 7?

b) are divisible by 7 but not by 11?

c) are divisible by both 7 and 11?

d) are divisible by either 7 or 11?

e) are divisible by exactly one of 7 and 11?

f) are divisible by neither 7 nor 11?

g) have distinct digits?

h) have distinct digits and are even?

23. How many positive integers between 100 and 999 inclusive

a) are divisible by 7?

b) are odd?

c) have the same three decimal digits?

d) are not divisible by 4?

e) are divisible by 3 or 4?

f) are not divisible by either 3 or 4?

g) are divisible by 3 but not by 4?

h) are divisible by 3 and 4?

24. How many positive integers between 1000 and 9999 inclusive

a) are divisible by 9?

b) are even?

c) have distinct digits?

d) are not divisible by 3?

e) are divisible by 5 or 7?

f) are not divisible by either 5 or 7?

g) are divisible by 5 but not by 7?

h) are divisible by 5 and 7?

25. How many strings of three decimal digits

a) do not contain the same digit three times?
b) begin with an odd digit?
c) have exactly two digits that are 4s?

26. How many strings of four decimal digits

a) do not contain the same digit twice?
b) end with an even digit?
c) have exactly three digits that are 9s?

27. A committee is formed consisting of one representative from each of the 50 states in the United States, where the representative from a state is either the governor or one of the two senators from that state. How many ways are there to form this committee?

28. How many license plates can be made using either three digits followed by three uppercase English letters or three uppercase English letters followed by three digits?

29. How many license plates can be made using either two uppercase English letters followed by four digits or two digits followed by four uppercase English letters?

30. How many license plates can be made using either three uppercase English letters followed by three digits or four uppercase English letters followed by two digits?

31. How many license plates can be made using either two or three uppercase English letters followed by either two or three digits?

32. How many strings of eight uppercase English letters are there

a) if letters can be repeated?
b) if no letter can be repeated?
c) that start with X, if letters can be repeated?
d) that start with X, if no letter can be repeated?
e) that start and end with X, if letters can be repeated?
f) that start with the letters BO (in that order), if letters can be repeated?
g) that start and end with the letters BO (in that order), if letters can be repeated?
h) that start or end with the letters BO (in that order), if letters can be repeated?

33. How many strings of eight English letters are there

a) that contain no vowels, if letters can be repeated?
b) that contain no vowels, if letters cannot be repeated?
c) that start with a vowel, if letters can be repeated?
d) that start with a vowel, if letters cannot be repeated?
e) that contain at least one vowel, if letters can be repeated?
f) that contain exactly one vowel, if letters can be repeated?
g) that start with X and contain at least one vowel, if letters can be repeated?
h) that start and end with X and contain at least one vowel, if letters can be repeated?

34. How many different functions are there from a set with 10 elements to sets with the following numbers of elements?

a) 2 **b)** 3 **c)** 4 **d)** 5

35. How many one-to-one functions are there from a set with five elements to sets with the following number of elements?

a) 4 **b)** 5 **c)** 6 **d)** 7

36. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$?

37. How many functions are there from the set $\{1, 2, \ldots, n\}$, where n is a positive integer, to the set $\{0, 1\}$

a) that are one-to-one?
b) that assign 0 to both 1 and n?
c) that assign 1 to exactly one of the positive integers less than n?

38. How many partial functions (see Section 2.3) are there from a set with five elements to sets with each of these number of elements?

a) 1 **b)** 2 **c)** 5 **d)** 9

39. How many partial functions (see Definition 13 of Section 2.3) are there from a set with m elements to a set with n elements, where m and n are positive integers?

40. How many subsets of a set with 100 elements have more than one element?

41. A **palindrome** is a string whose reversal is identical to the string. How many bit strings of length n are palindromes?

42. How many 4-element DNA sequences

a) do not contain the base T?
b) contain the sequence ACG?
c) contain all four bases A, T, C, and G?
d) contain exactly three of the four bases A, T, C, and G?

43. How many 4-element RNA sequences

a) contain the base U?
b) do not contain the sequence CUG?
c) do not contain all four bases A, U, C, and G?
d) contain exactly two of the four bases A, U, C, and G?

44. How many ways are there to seat four of a group of ten people around a circular table where two seatings are considered the same when everyone has the same immediate left and immediate right neighbor?

45. How many ways are there to seat six people around a circular table where two seatings are considered the same when everyone has the same two neighbors without regard to whether they are right or left neighbors?

46. In how many ways can a photographer at a wedding arrange 6 people in a row from a group of 10 people, where the bride and the groom are among these 10 people, if

a) the bride must be in the picture?
b) both the bride and groom must be in the picture?
c) exactly one of the bride and the groom is in the picture?

47. In how many ways can a photographer at a wedding arrange six people in a row, including the bride and groom, if

a) the bride must be next to the groom?
b) the bride is not next to the groom?
c) the bride is positioned somewhere to the left of the groom?

48. How many bit strings of length seven either begin with two 0s or end with three 1s?

49. How many bit strings of length 10 either begin with three 0s or end with two 0s?

***50.** How many bit strings of length 10 contain either five consecutive 0s or five consecutive 1s?

****51.** How many bit strings of length eight contain either three consecutive 0s or four consecutive 1s?

52. Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

53. How many positive integers not exceeding 100 are divisible either by 4 or by 6?

54. How many different initials can someone have if a person has at least two, but no more than five, different initials? Assume that each initial is one of the 26 uppercase letters of the English language.

55. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters $*$, $>$, $<$, !, $+$, and $=$.

a) How many different passwords are available for this computer system?

b) How many of these passwords contain at least one occurrence of at least one of the six special characters?

c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

56. The name of a variable in the C programming language is a string that can contain uppercase letters, lowercase letters, digits, or underscores. Further, the first character in the string must be a letter, either uppercase or lowercase, or an underscore. If the name of a variable is determined by its first eight characters, how many different variables can be named in C? (Note that the name of a variable may contain fewer than eight characters.)

57. The name of a variable in the JAVA programming language is a string of between 1 and 65,535 characters, inclusive, where each character can be an uppercase or a lowercase letter, a dollar sign, an underscore, or a digit, except that the first character must not be a digit. Determine the number of different variable names in JAVA.

58. The International Telecommunications Union (ITU) specifies that a telephone number must consist of a country code with between 1 and 3 digits, except that the code 0 is not available for use as a country code, followed by a number with at most 15 digits. How many available possible telephone numbers are there that satisfy these restrictions?

59. Suppose that at some future time every telephone in the world is assigned a number that contains a country code 1 to 3 digits long, that is, of the form X, XX, or XXX, followed by a 10-digit telephone number of the form NXX-NXX-$XXXX$ (as described in Example 8). How many different telephone numbers would be available worldwide under this numbering plan?

60. A key in the Vigenère cryptosystem is a string of English letters, where the case of the letters does not matter. How many different keys for this cryptosystem are there with three, four, five, or six letters?

61. A wired equivalent privacy (WEP) key for a wireless fidelity (WiFi) network is a string of either 10, 26, or 58 hexadecimal digits. How many different WEP keys are there?

62. Suppose that p and q are prime numbers and that $n = pq$. Use the principle of inclusion–exclusion to find the number of positive integers not exceeding n that are relatively prime to n.

63. Use the principle of inclusion–exclusion to find the number of positive integers less than 1,000,000 that are not divisible by either 4 or by 6.

64. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

65. How many ways are there to arrange the letters a, b, c, and d such that a is not followed immediately by b?

66. Use a tree diagram to find the number of ways that the World Series can occur, where the first team that wins four games out of seven wins the series.

67. Use a tree diagram to determine the number of subsets of $\{3, 7, 9, 11, 24\}$ with the property that the sum of the elements in the subset is less than 28.

68. **a)** Suppose that a store sells six varieties of soft drinks: cola, ginger ale, orange, root beer, lemonade, and cream soda. Use a tree diagram to determine the number of different types of bottles the store must stock to have all varieties available in all size bottles if all varieties are available in 12-ounce bottles, all but lemonade are available in 20-ounce bottles, only cola and ginger ale are available in 32-ounce bottles, and all but lemonade and cream soda are available in 64-ounce bottles?

b) Answer the question in part (a) using counting rules.

69. **a)** Suppose that a popular style of running shoe is available for both men and women. The woman's shoe comes in sizes 6, 7, 8, and 9, and the man's shoe comes in sizes 8, 9, 10, 11, and 12. The man's shoe comes in white and black, while the woman's shoe comes in white, red, and black. Use a tree diagram to determine the number of different shoes that a store has to stock to have at least one pair of this type of running shoe for all available sizes and colors for both men and women.

b) Answer the question in part (a) using counting rules.

***70.** Use the product rule to show that there are 2^{2^n} different truth tables for propositions in n variables.

71. Use mathematical induction to prove the sum rule for m tasks from the sum rule for two tasks.

72. Use mathematical induction to prove the product rule for m tasks from the product rule for two tasks.

73. How many diagonals does a convex polygon with n sides have? (Recall that a polygon is convex if every line segment connecting two points in the interior or boundary of the polygon lies entirely within this set and that a diagonal of a polygon is a line segment connecting two vertices that are not adjacent.)

74. Data are transmitted over the Internet in **datagrams**, which are structured blocks of bits. Each datagram contains header information organized into a maximum of 14 different fields (specifying many things, including the source and destination addresses) and a data area that contains the actual data that are transmitted. One of the 14 header fields is the **header length field** (denoted by HLEN), which is specified by the protocol to be 4 bits long and that specifies the header length in terms of 32-bit blocks of bits. For example, if HLEN = 0110, the header is made up of six 32-bit blocks. Another of the 14 header fields is the 16-bit-long **total length field** (denoted by TOTAL LENGTH), which specifies the length in bits of the entire datagram, including both the header fields and the data area. The length of the data area is the total length of the datagram minus the length of the header.

a) The largest possible value of TOTAL LENGTH (which is 16 bits long) determines the maximum total length in octets (blocks of 8 bits) of an Internet datagram. What is this value?

b) The largest possible value of HLEN (which is 4 bits long) determines the maximum total header length in 32-bit blocks. What is this value? What is the maximum total header length in octets?

c) The minimum (and most common) header length is 20 octets. What is the maximum total length in octets of the data area of an Internet datagram?

d) How many different strings of octets in the data area can be transmitted if the header length is 20 octets and the total length is as long as possible?

6.2 The Pigeonhole Principle

Introduction

Links

Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost. Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 19 pigeons, one per hole, could be accommodated. This illustrates a general principle called the **pigeonhole principle**, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it (see Figure 1). Of course, this principle applies to other objects besides pigeons and pigeonholes.

THEOREM 1 **THE PIGEONHOLE PRINCIPLE** If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

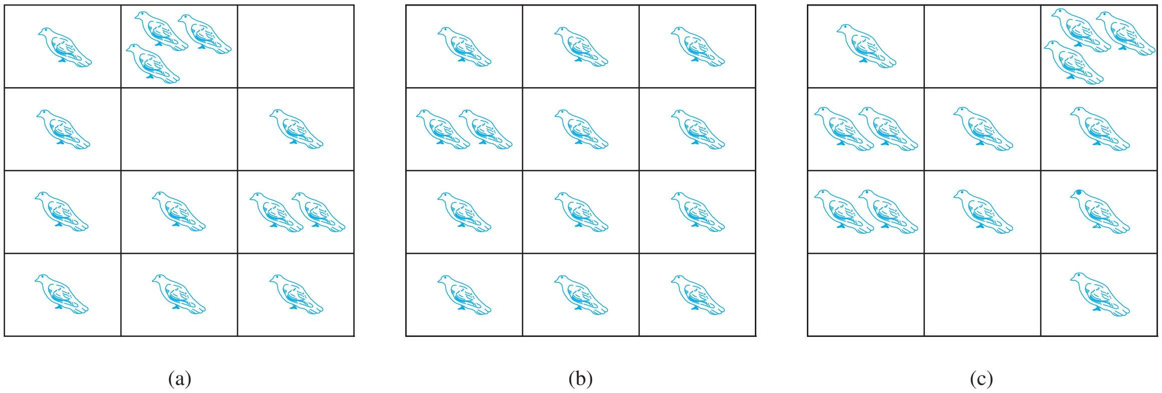


FIGURE 1 **There Are More Pigeons Than Pigeonholes.**

people where every two people are friends or enemies, there may not be three mutual friends or three mutual enemies (see Exercise 26).

It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that $R(m, n) = R(n, m)$ (see Exercise 30). We also have $R(2, n) = n$ for every positive integer $n \geq 2$ (see Exercise 29). The exact values of only nine Ramsey numbers $R(m, n)$ with $3 \leq m \leq n$ are known, including $R(4, 4) = 18$. Only bounds are known for many other Ramsey numbers, including $R(5, 5)$, which is known to satisfy $43 \leq R(5, 5) \leq 49$. The reader interested in learning more about Ramsey numbers should consult [MiRo91] or [GrRoSp90].

Exercises

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.

2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.

3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.

a) How many socks must he take out to be sure that he has at least two socks of the same color?

b) How many socks must he take out to be sure that he has at least two black socks?

4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.

a) How many balls must she select to be sure of having at least three balls of the same color?

b) How many balls must she select to be sure of having at least three blue balls?

5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

6. Let d be a positive integer. Show that among any group of $d + 1$ (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by d.

7. Let n be a positive integer. Show that in any set of n consecutive integers there is exactly one divisible by n.

8. Show that if f is a function from S to T, where S and T are finite sets with $|S| > |T|$, then there are elements s_1 and s_2 in S such that $f(s_1) = f(s_2)$, or in other words, f is not one-to-one.

9. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

***10.** Let $(x_i, y_i), i = 1, 2, 3, 4, 5$, be a set of five distinct points with integer coordinates in the xy plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.

***11.** Let $(x_i, y_i, z_i), i = 1, 2, 3, 4, 5, 6, 7, 8, 9$, be a set of nine distinct points with integer coordinates in xyz space. Show that the midpoint of at least one pair of these points has integer coordinates.

12. How many ordered pairs of integers (a, b) are needed to guarantee that there are two ordered pairs (a_1, b_1) and (a_2, b_2) such that $a_1 \textbf{ mod } 5 = a_2 \textbf{ mod } 5$ and $b_1 \textbf{ mod } 5 = b_2 \textbf{ mod } 5$?

13. a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.

b) Is the conclusion in part (a) true if four integers are selected rather than five?

14. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.

b) Is the conclusion in part (a) true if six integers are selected rather than seven?

15. How many numbers must be selected from the set $\{1, 2, 3, 4, 5, 6\}$ to guarantee that at least one pair of these numbers add up to 7?

16. How many numbers must be selected from the set $\{1, 3, 5, 7, 9, 11, 13, 15\}$ to guarantee that at least one pair of these numbers add up to 16?

17. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse. What is the least number of products the company can have so that at least two products must be stored in the same bin?

18. Suppose that there are nine students in a discrete mathematics class at a small college.

a) Show that the class must have at least five male students or at least five female students.

b) Show that the class must have at least three male students or at least seven female students.

19. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.

a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.

b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.

20. Find an increasing subsequence of maximal length and a decreasing subsequence of maximal length in the sequence 22, 5, 7, 2, 23, 10, 15, 21, 3, 17.

21. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.

22. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.

***23.** Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of whose neighbors are boys.

****24.** Suppose that 21 girls and 21 boys enter a mathematics competition. Furthermore, suppose that each entrant solves at most six questions, and for every boy-girl pair, there is at least one question that they both solved. Show that there is a question that was solved by at least three girls and at least three boys.

***25.** Describe an algorithm in pseudocode for producing the largest increasing or decreasing subsequence of a sequence of distinct integers.

26. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.

27. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.

28. Use Exercise 27 to show that among any group of 20 people (where any two people are either friends or enemies), there are either four mutual friends or four mutual enemies.

29. Show that if n is an integer with $n \geq 2$, then the Ramsey number $R(2, n)$ equals n. (Recall that Ramsey numbers were discussed after Example 13 in Section 6.2.)

30. Show that if m and n are integers with $m \geq 2$ and $n \geq 2$, then the Ramsey numbers $R(m, n)$ and $R(n, m)$ are equal. (Recall that Ramsey numbers were discussed after Example 13 in Section 6.2.)

31. Show that there are at least six people in California (population: 37 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.

32. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars (but at least a penny), then there are two who earned exactly the same amount of money, to the penny, last year.

33. In the 17th century, there were more than 800,000 inhabitants of Paris. At the time, it was believed that no one had more than 200,000 hairs on their head. Assuming these numbers are correct and that everyone has at least one hair on their head (that is, no one is completely bald), use the pigeonhole principle to show, as the French writer Pierre Nicole did, that there had to be two Parisians with the same number of hairs on their heads. Then use the generalized pigeonhole principle to show that there had to be at least five Parisians at that time with the same number of hairs on their heads.

34. Assuming that no one has more than 1,000,000 hairs on the head of any person and that the population of New York City was 8,008,278 in 2010, show there had to be at least nine people in New York City in 2010 with the same number of hairs on their heads.

35. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?

36. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.

37. A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. [*Hint:* It is impossible to have a computer linked to none of the others and a computer linked to all the others.]

38. Find the least number of cables required to connect eight computers to four printers to guarantee that for every choice of four of the eight computers, these four computers can directly access four different printers. Justify your answer.

39. Find the least number of cables required to connect 100 computers to 20 printers to guarantee that 2every subset of 20 computers can directly access 20 different printers. (Here, the assumptions about cables and computers are the same as in Example 9.) Justify your answer.

***40.** Prove that at a party where there are at least two people, there are two people who know the same number of other people there.

41. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 P.M., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.

***42.** Is the statement in Exercise 41 true if 24 is replaced by

a) 2? **b)** 23? **c)** 25? **d)** 30?

43. Show that if f is a function from S to T, where S and T are nonempty finite sets and $m = \lceil |S| / |T| \rceil$, then there are at least m elements of S mapped to the same value of T. That is, show that there are distinct elements $s_1, s_2, \ldots, s_m$ of S such that $f(s_1) = f(s_2) = \cdots = f(s_m)$.

44. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.

***45.** Let x be an irrational number. Show that for some positive integer j not exceeding the positive integer n, the absolute value of the difference between jx and the nearest integer to jx is less than $1/n$.

46. Let $n_1, n_2, \ldots, n_t$ be positive integers. Show that if $n_1 + n_2 + \cdots + n_t - t + 1$ objects are placed into t boxes, then for some i, $i = 1, 2, \ldots, t$, the ith box contains at least n_i objects.

***47.** An alternative proof of Theorem 3 based on the generalized pigeonhole principle is outlined in this exercise. The notation used is the same as that used in the proof in the text.

a) Assume that $i_k \leq n$ for $k = 1, 2, \ldots, n^2 + 1$. Use the generalized pigeonhole principle to show that there are $n + 1$ terms $a_{k_1}, a_{k_2}, \ldots, a_{k_{n+1}}$ with $i_{k_1} = i_{k_2} = \cdots = i_{k_{n+1}}$, where $1 \leq k_1 < k_2 < \cdots < k_{n+1}$.

b) Show that $a_{k_j} > a_{k_{j+1}}$ for $j = 1, 2, \ldots, n$. [*Hint:* Assume that $a_{k_j} < a_{k_{j+1}}$, and show that this implies that $i_{k_j} > i_{k_{j+1}}$, which is a contradiction.]

c) Use parts (a) and (b) to show that if there is no increasing subsequence of length $n + 1$, then there must be a decreasing subsequence of this length.

6.3 Permutations and Combinations

Introduction

Many counting problems can be solved by finding the number of ways to arrange a specified number of distinct elements of a set of a particular size, where the order of these elements matters. Many other counting problems can be solved by finding the number of ways to select a particular number of elements from a set of a particular size, where the order of the elements selected does not matter. For example, in how many ways can we select three students from a group of five students to stand in line for a picture? How many different committees of three students can be formed from a group of four students? In this section we will develop methods to answer questions such as these.

Permutations

We begin by solving the first question posed in the introduction to this section, as well as related questions.

EXAMPLE 1 In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Extra Examples

Solution: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture. ◀

Example 1 illustrates how ordered arrangements of distinct objects can be counted. This leads to some terminology.

A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

Links

EXAMPLE 14 How many bit strings of length n contain exactly r 1s?

Solution: The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \dots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s. ◀

EXAMPLE 15 Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

◀

Exercises

1. List all the permutations of $\{a, b, c\}$.

2. How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?

3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?

4. Let $S = \{1, 2, 3, 4, 5\}$.

a) List all the 3-permutations of S.
b) List all the 3-combinations of S.

5. Find the value of each of these quantities.

a) $P(6, 3)$ **b)** $P(6, 5)$
c) $P(8, 1)$ **d)** $P(8, 5)$
e) $P(8, 8)$ **f)** $P(10, 9)$

6. Find the value of each of these quantities.

a) $C(5, 1)$ **b)** $C(5, 3)$
c) $C(8, 4)$ **d)** $C(8, 8)$
e) $C(8, 0)$ **f)** $C(12, 6)$

7. Find the number of 5-permutations of a set with nine elements.

8. In how many different orders can five runners finish a race if no ties are allowed?

9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?

11. How many bit strings of length 10 contain

a) exactly four 1s?
b) at most four 1s?
c) at least four 1s?
d) an equal number of 0s and 1s?

12. How many bit strings of length 12 contain

a) exactly three 1s?
b) at most three 1s?
c) at least three 1s?
d) an equal number of 0s and 1s?

13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

14. In how many ways can a set of two positive integers less than 100 be chosen?

15. In how many ways can a set of five letters be selected from the English alphabet?

16. How many subsets with an odd number of elements does a set with 10 elements have?

17. How many subsets with more than two elements does a set with 100 elements have?

18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
b) contain exactly three heads?
c) contain at least three heads?
d) contain the same number of heads and tails?

19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
b) contain exactly two heads?
c) contain at most three tails?
d) contain the same number of heads and tails?

20. How many bit strings of length 10 have

a) exactly three 0s?
b) more 0s than 1s?
c) at least seven 1s?
d) at least three 1s?

21. How many permutations of the letters *ABCDEFG* contain

a) the string *BCD*?
b) the string *CFGA*?
c) the strings *BA* and *GF*?
d) the strings *ABC* and *DE*?
e) the strings *ABC* and *CDE*?
f) the strings *CBA* and *BED*?

22. How many permutations of the letters *ABCDEFGH* contain

a) the string *ED*?
b) the string *CDE*?
c) the strings *BA* and *FGH*?
d) the strings *AB*, *DE*, and *GH*?
e) the strings *CAB* and *BED*?
f) the strings *BCA* and *ABF*?

23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]

25. One hundred tickets, numbered 1, 2, 3, . . . , 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if

a) there are no restrictions?
b) the person holding ticket 47 wins the grand prize?
c) the person holding ticket 47 wins one of the prizes?
d) the person holding ticket 47 does not win a prize?
e) the people holding tickets 19 and 47 both win prizes?
f) the people holding tickets 19, 47, and 73 all win prizes?
g) the people holding tickets 19, 47, 73, and 97 all win prizes?
h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?
i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?
j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

26. Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?
b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

27. A club has 25 members.

a) How many ways are there to choose four members of the club to serve on an executive committee?
b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

28. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?

***29.** How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers k, $k+1$, $k+2$, in the correct order

a) where these consecutive integers can perhaps be separated by other integers in the permutation?

b) where they are in consecutive positions in the permutation?

30. Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

31. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain

a) exactly one vowel?

b) exactly two vowels?

c) at least one vowel?

d) at least two vowels?

32. How many strings of six lowercase letters from the English alphabet contain

a) the letter *a*?

b) the letters *a* and *b*?

c) the letters *a* and *b* in consecutive positions with *a* preceding *b*, with all the letters distinct?

d) the letters *a* and *b*, where *a* is somewhere to the left of *b* in the string, with all the letters distinct?

33. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?

34. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

35. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

36. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

37. How many bit strings of length 10 contain at least three 1s and at least three 0s?

38. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?

39. How many license plates consisting of three letters followed by three digits contain no letter or digit twice?

A **circular *r*-permutation of *n*** people is a seating of r of these n people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table.

40. Find the number of circular 3-permutations of 5 people.

41. Find a formula for the number of circular r-permutations of n people.

42. Find a formula for the number of ways to seat r of n people around a circular table, where seatings are considered the same if every person has the same two neighbors without regard to which side these neighbors are sitting on.

43. How many ways are there for a horse race with three horses to finish if ties are possible? [*Note:* Two or three horses may tie.]

***44.** How many ways are there for a horse race with four horses to finish if ties are possible? [*Note:* Any number of the four horses may tie.)

***45.** There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.)

***46.** This procedure is used to break ties in games in the championship round of the World Cup soccer tournament. Each team selects five players in a prescribed order. Each of these players takes a penalty kick, with a player from the first team followed by a player from the second team and so on, following the order of players specified. If the score is still tied at the end of the 10 penalty kicks, this procedure is repeated. If the score is still tied after 20 penalty kicks, a sudden-death shootout occurs, with the first team scoring an unanswered goal victorious.

a) How many different scoring scenarios are possible if the game is settled in the first round of 10 penalty kicks, where the round ends once it is impossible for a team to equal the number of goals scored by the other team?

b) How many different scoring scenarios for the first and second groups of penalty kicks are possible if the game is settled in the second round of 10 penalty kicks?

c) How many scoring scenarios are possible for the full set of penalty kicks if the game is settled with no more than 10 total additional kicks after the two rounds of five kicks for each team?

6.4 Binomial Coefficients and Identities

As we remarked in Section 6.3, the number of r-combinations from a set with n elements is often denoted by $\binom{n}{r}$. This number is also called a **binomial coefficient** because these numbers occur as coefficients in the expansion of powers of binomial expressions such as $(a + b)^n$. We will discuss the **binomial theorem**, which gives a power of a binomial expression as a sum of terms involving binomial coefficients. We will prove this theorem using a combinatorial proof. We will also show how combinatorial proofs can be used to establish some of the many different identities that express relationships among binomial coefficients.

The Binomial Theorem

Links

The binomial theorem gives the coefficients of the expansion of powers of binomial expressions. A **binomial** expression is simply the sum of two terms, such as $x + y$. (The terms can be products of constants and variables, but that does not concern us here.)

Example 1 illustrates how the coefficients in a typical expansion can be found and prepares us for the statement of the binomial theorem.

EXAMPLE 1 The expansion of $(x + y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out. When $(x + y)^3 = (x + y)(x + y)(x + y)$ is expanded, all products of a term in the first sum, a term in the second sum, and a term in the third sum are added. Terms of the form x^3, x^2y, xy^2, and y^3 arise. To obtain a term of the form x^3, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^3 term in the product has a coefficient of 1. To obtain a term of the form x^2y, an x must be chosen in two of the three sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 2-combinations of three objects, namely, $\binom{3}{2}$. Similarly, the number of terms of the form xy^2 is the number of ways to pick one of the three sums to obtain an x (and consequently take a y

We can prove combinatorial identities by counting bit strings with different properties, as the proof of Theorem 4 will demonstrate.

THEOREM 4 Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Proof: We use a combinatorial proof. By Example 14 in Section 6.3, the left-hand side, $\binom{n+1}{r+1}$, counts the bit strings of length $n+1$ containing $r+1$ ones.

We show that the right-hand side counts the same objects by considering the cases corresponding to the possible locations of the final 1 in a string with $r+1$ ones. This final one must occur at position $r+1, r+2, \ldots,$ or $n+1$. Furthermore, if the last one is the kth bit there must be r ones among the first $k-1$ positions. Consequently, by Example 14 in Section 6.3, there are $\binom{k-1}{r}$ such bit strings. Summing over k with $r+1 \leq k \leq n+1$, we find that there are

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

bit strings of length n containing exactly $r+1$ ones. (Note that the last step follows from the change of variables $j = k-1$.) Because the left-hand side and the right-hand side count the same objects, they are equal. This completes the proof. ◁

Exercises

1. Find the expansion of $(x+y)^4$

a) using combinatorial reasoning, as in Example 1.
b) using the binomial theorem.

2. Find the expansion of $(x+y)^5$

a) using combinatorial reasoning, as in Example 1.
b) using the binomial theorem.

3. Find the expansion of $(x+y)^6$.

4. Find the coefficient of x^5y^8 in $(x+y)^{13}$.

5. How many terms are there in the expansion of $(x+y)^{100}$ after like terms are collected?

6. What is the coefficient of x^7 in $(1+x)^{11}$?

7. What is the coefficient of x^9 in $(2-x)^{19}$?

8. What is the coefficient of x^8y^9 in the expansion of $(3x+2y)^{17}$?

9. What is the coefficient of $x^{101}y^{99}$ in the expansion of $(2x-3y)^{200}$?

***10.** Give a formula for the coefficient of x^k in the expansion of $(x+1/x)^{100}$, where k is an integer.

***11.** Give a formula for the coefficient of x^k in the expansion of $(x^2-1/x)^{100}$, where k is an integer.

12. The row of Pascal's triangle containing the binomial coefficients $\binom{10}{k}$, $0 \leq k \leq 10$, is:

1 10 45 120 210 252 210 120 45 10 1

Use Pascal's identity to produce the row immediately following this row in Pascal's triangle.

13. What is the row of Pascal's triangle containing the binomial coefficients $\binom{9}{k}$, $0 \leq k \leq 9$?

14. Show that if n is a positive integer, then $1 = \binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n} = 1$.

15. Show that $\binom{n}{k} \leq 2^n$ for all positive integers n and all integers k with $0 \leq k \leq n$.

16. a) Use Exercise 14 and Corollary 1 to show that if n is an integer greater than 1, then $\binom{n}{\lfloor n/2 \rfloor} \geq 2^n/n$.
b) Conclude from part (a) that if n is a positive integer, then $\binom{2n}{n} \geq 4^n/2n$.

☞ **17.** Show that if n and k are integers with $1 \leq k \leq n$, then $\binom{n}{k} \leq n^k/2^{k-1}$.

18. Suppose that b is an integer with $b \geq 7$. Use the binomial theorem and the appropriate row of Pascal's triangle to find the base-b expansion of $(11)_b^4$ [that is, the fourth power of the number $(11)_b$ in base-b notation].

19. Prove Pascal's identity, using the formula for $\binom{n}{r}$.

20. Suppose that k and n are integers with $1 \leq k < n$. Prove the **hexagon identity**

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1},$$

which relates terms in Pascal's triangle that form a hexagon.

21. Prove that if n and k are integers with $1 \leq k \leq n$, then $k\binom{n}{k} = n\binom{n-1}{k-1}$,

a) using a combinatorial proof. [*Hint:* Show that the two sides of the identity count the number of ways to select a subset with k elements from a set with n elements and then an element of this subset.]

b) using an algebraic proof based on the formula for $\binom{n}{r}$ given in Theorem 2 in Section 6.3.

22. Prove the identity $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, whenever n, r, and k are nonnegative integers with $r \leq n$ and $k \leq r$,

a) using a combinatorial argument.

b) using an argument based on the formula for the number of r-combinations of a set with n elements.

23. Show that if n and k are positive integers, then

$$\binom{n+1}{k} = (n+1)\binom{n}{k-1} \Big/ k.$$

Use this identity to construct an inductive definition of the binomial coefficients.

24. Show that if p is a prime and k is an integer such that $1 \leq k \leq p-1$, then p divides $\binom{p}{k}$.

25. Let n be a positive integer. Show that

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+2}{n+1}/2.$$

***26.** Let n and k be integers with $1 \leq k \leq n$. Show that

$$\sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1} = \binom{2n+2}{n+1}/2 - \binom{2n}{n}.$$

***27.** Prove the **hockeystick identity**

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

a) using a combinatorial argument.

b) using Pascal's identity.

28. Show that if n is a positive integer, then $\binom{2n}{2} = 2\binom{n}{2} + n^2$

a) using a combinatorial argument.

b) by algebraic manipulation.

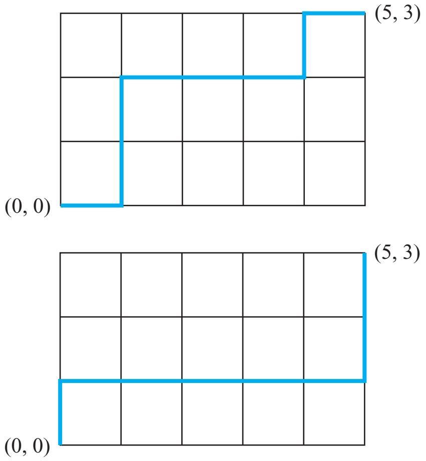
***29.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee and to then select a leader of the committee.]

***30.** Give a combinatorial proof that $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$. [*Hint:* Count in two ways the number of ways to select a committee, with n members from a group of n mathematics professors and n computer science professors, such that the chairperson of the committee is a mathematics professor.]

31. Show that a nonempty set has the same number of subsets with an odd number of elements as it does subsets with an even number of elements.

***32.** Prove the binomial theorem using mathematical induction.

33. In this exercise we will count the number of paths in the *xy* plane between the origin (0, 0) and point (*m*, *n*), where m and n are nonnegative integers, such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.) Two such paths from (0, 0) to (5, 3) are illustrated here.



a) Show that each path of the type described can be represented by a bit string consisting of m 0s and n 1s, where a 0 represents a move one unit to the right and a 1 represents a move one unit upward.

b) Conclude from part (a) that there are $\binom{m+n}{n}$ paths of the desired type.

34. Use Exercise 33 to give an alternative proof of Corollary 2 in Section 6.3, which states that $\binom{n}{k} = \binom{n}{n-k}$ whenever k is an integer with $0 \leq k \leq n$. [*Hint:* Consider the number of paths of the type described in Exercise 33 from $(0, 0)$ to $(n-k, k)$ and from $(0, 0)$ to $(k, n-k)$.]

35. Use Exercise 33 to prove Theorem 4. [*Hint:* Count the number of paths with n steps of the type described in Exercise 33. Every such path must end at one of the points $(n-k, k)$ for $k = 0, 1, 2, \ldots, n$.]

36. Use Exercise 33 to prove Pascal's identity. [*Hint:* Show that a path of the type described in Exercise 33 from $(0, 0)$ to $(n+1-k, k)$ passes through either $(n+1-k, k-1)$ or $(n-k, k)$, but not through both.]

37. Use Exercise 33 to prove the hockeystick identity from Exercise 27. [*Hint:* First, note that the number of paths from $(0, 0)$ to $(n+1, r)$ equals $\binom{n+1+r}{r}$. Second, count the number of paths by summing the number of these paths that start by going k units upward for $k = 0, 1, 2, \ldots, r$.]

38. Give a combinatorial proof that if n is a positive integer then $\sum_{k=0}^{n} k^2\binom{n}{k} = n(n+1)2^{n-2}$. [*Hint:* Show that both sides count the ways to select a subset of a set of n elements together with two not necessarily distinct elements from this subset. Furthermore, express the right-hand side as $n(n-1)2^{n-2} + n2^{n-1}$.]

***39.** Determine a formula involving binomial coefficients for the nth term of a sequence if its initial terms are those listed. [*Hint:* Looking at Pascal's triangle will be helpful.