The Foundations: Logic and Proofs

Chapter 1

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Predicates and Quantifiers

Section 1.4

Section Summary

Predicates

Quantifiers

- Universal Quantifier
- Existential Quantifier

Negating Quantifiers

- De Morgan’s Laws for Quantifiers

Propositional Logic Not Enough

- We cannot express the following using propositions:
  - “<u>X</u> is greater than -1”.

- *Propositional functions* are a generalization of propositions.
  - We define propositional function as P(x)=“x is greater than -1”

- Propositional functions become propositions by:

1. assigning values.
2. using quantifiers.

Propositional Functions

- Example 1: Let $P(x)$="$x < 5$". Find the truth value of:
  - $P(x)$ has no truth value (not proposition)
  - $P(1)$ true
  - $P(10)$ false
- Example 2: Let $P(x,y,z)$="$x + y = z$". Find the truth value of:
  - $P(2,-1,5)$ false
  - $P(3,4,7)$ true
  - $P(x, 3, z)$ has no truth value (not proposition)

Quantifiers

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "**All** computers in CS department is protected by intrusion detection system."
- "**There exists** at least one student who has a brown hair."

The two most important quantifiers are:

- ***Universal Quantifier***, **"**For all," symbol: $\forall$
  - $\forall x\, P(x)$ means "for all values of x, $P(x)$ is true in a particular domain (U)"
- ***Existential Quantifier***, "There exists," symbol: $\exists$
  - $\exists x\, P(x)$ means "for at least one value of x, $P(x)$ is true in a particular domain (U)"
  - $\exists x\, P(x)$ means "There exists an x such that $P(x)$ is true in a particular domain (U)"

U → domain (universe of discourse)

Universal Quantifier

Assume $U=\{x_1,x_2,..,x_n\}$ .. The Universal quantifier, $\forall x\, P(x)$,implies:

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

Examples:

P(x)	**Domain**	**Truth value of $\forall x\, P(x)$**
$\boldsymbol{x > 0}$	$\mathbb{Z}$ (all integers)	False Counterexample: x=0
$\boldsymbol{x > 0}$	$\mathbb{Z}^+$ (positive integers)	True
***x* is even**	$\mathbb{Z}$ (all integers)	False Counterexample: x=1
$\boldsymbol{x^2 > 0}$	$\mathbb{Z}$ (all integers)	False Counterexample: x=0
$\boldsymbol{3x \le 4x}$	$\mathbb{Z}$ (all integers)	False Counterexample: x=-1

Existential Quantifier

Assume $U=\{x_1,x_2,..,x_n\}$ .. The Existential quantifier, $\exists x\, P(x)$,implies:

$$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Examples:

P(x)	Domain	Truth value of $\exists x\, P(x)$
$x > 0$	$\mathbb{Z}$ (all integers)	True Example: x=1
$x < 0$	$\mathbb{Z}^+$ (positive integers)	False
x is even	$\mathbb{Z}$ (all integers)	True Example: x=2
$x^4 < x^2$	$\mathbb{Z}$ (all integers)	False
$x^4 < x^2$	$\mathbb{R}$ (Real numbers)	True Example: x=0.5

Negating Quantifiers

Let's define *J(x)* as "x has taken a course in Java"

1. "Every student in your class has taken a course in Java." $\forall x\, J(x)$

- Negation: "It is not the case that every student in your class has taken a course in Java."
  - This implies that "There is a student in your class who has not taken Java."
- $\neg\forall x\, J(x) \equiv \exists x\, \neg J(x)$

2. "There is a student in this class who has taken a course in Java." $\exists x\, J(x)$

- Negation: "It is not the case that there is a student in this class who has taken Java."
  - This implies that "Every student in this class has not taken Java."
- $\neg\exists x\, J(x) \equiv \forall x\, \neg J(x)$

$$\boxed{\begin{aligned} \neg\forall x P(x) &\equiv \exists x \neg P(x) \\ \neg\exists x P(x) &\equiv \forall x \neg P(x) \end{aligned}}$$

Exercise

Let's define: $T(x,y)$="x has taken subject y" and $S(x)$="person x is in this class". Assume the domain all students. Express the following statements using predicates and quantifiers:

1. Fahad has taken CSC281.
   - **$T(\text{Fahad}, \text{CSC281})$**
2. Every student has taken CSC111.
   - **$\forall x\ T(x, \text{CSC111})$**
3. Every student in this class has taken CSC111.
   - **$\forall x\ (\ S(x) \rightarrow T(x, \text{CSC111})\ )$**
4. There is a student in this class who has taken CSC111.
   - **$\exists x\ (\ S(x) \land T(x, \text{CSC111})\ )$**
5. There is a student in this class who has not taken CSC111.
   - (Negating #3)
   - **$\exists x\ (\ S(x) \land \neg T(x, \text{CSC111})\ )$**

Nested Quantifiers

Section 1.5

Section Summary

Nested Quantifiers

Order of Quantifiers

Translating English into Nested Quantifiers

Nested Quantifiers

Example 1: "Every real number has an additive inverse."

$$\forall x \, \exists y \, (x + y = 0)$$

Example 2: "Every real number except zero has a multiplicative inverse."

$$\forall x \, \exists y \, ((x \neq 0) \rightarrow (xy = 1))$$

Example 3: "The product of a positive real number and a negative real number is always a negative real number."

$$\forall x \, \forall y \, (\, (x>0) \wedge (y<0) \rightarrow (xy<0) \,)$$

Order of Quantifiers$_1$

Statement	***When True?***
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.

When quantifiers are of <u>different types</u>, their <u>order matters</u>.

Order of Quantifiers$_2$

Examples:

1. Let $P(x,y)$=“**x+y = y+x**”. Assume the domain is $\mathbb{R}$ (real numbers).
   - $\forall x \, \forall y \, P(x,y)$ true
   - $\forall y \, \forall x \, P(x,y)$ true

2. Let $Q(x,y)$=“**x+y = 0**”. Assume the domain is $\mathbb{R}$ (real numbers).
   - $\forall x \, \exists y \, Q(x,y)$ true
   - $\exists y \, \forall x \, Q(x,y)$ false

Order of Quantifiers$_3$

Example:

Let P(x,y)="**x · y = 0**". Assume the domain is $\mathbb{R}$ (real numbers). Find the truth value of:

1. $\forall x \forall y P(x, y)$ $\boxed{\textbf{Answer}:\text{False}}$

2. $\forall x \exists y P(x, y)$ $\boxed{\textbf{Answer}:\text{True}}$

3. $\exists x \forall y P(x, y)$ $\boxed{\textbf{Answer}:\text{True}}$

4. $\exists x \exists y P(x, y)$ $\boxed{\textbf{Answer}:\text{True}}$

Translating English Sentences into Nested Quantifiers

Example: Let $F(x,y)$="y is the father of x"

$M(x,y)$="y is the mother of x"

Express the following statements:

1. "Ali is the father of Bilal".
   - F(Bilal, Ali)
2. "Everyone has a father".
   - $\forall x \, \exists y \, F(x,y)$
3. "Everyone has a father and a mother".
   - $\forall x \, \exists y \, \exists z \, F(x,y) \land M(x,z)$
4. "Everyone has a <u>single</u> father".
   - $\forall x \, \exists y \, \forall z \, F(x,y) \land ((y \mathrel{!=} z) \rightarrow \neg F(x,z))$