1)

Solution. a) $P(1) = 1^3 = (\frac{1(1+1)}{2})^2$

b) Basis step:
for $n = 1$
$1^3 = (\frac{2}{2})^2$
$1 = 1^2 = 1$ QED ($P(1)$ is true)

c) Inductive step:
Assume $P(k)\ for\ k \geq 1 : 1^3 + 2^3 + ... + k^3 = (\frac{k(k+1)}{2})^2$

d) Show $P(k+1)$:

e)

$$(\frac{(k+1)((k+1)+1)}{2})^2 = 1^3 + 2^3 + ... + k^3 + (k+1)^3 \quad (1)$$
$$(\frac{(k+1)((k+1)+1)}{2})^2 = (\frac{k(k+1)}{2})^2 + (k+1)^3 \quad (2)$$
$$(\frac{(k+1)((k+2)}{2})^2 = (\frac{k^2+k}{2})^2 + \frac{4(k+1)^3}{4} \quad (3)$$
$$\frac{(k+1)((k+2)}{2} . \frac{(k+1)((k+2)}{2} = \frac{k^2+k}{2} . \frac{k^2+k}{2} + \frac{4(k+1)^3}{4} \quad (4)$$
$$\frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \frac{k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 4}{4} \quad (5)$$
$$\frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \quad (6)$$
$$QED \quad (7)$$

f) - $P(b)$ for constant b
$\forall k \geq b(P(k) \rightarrow P(k+1)$
Therefor, $\forall n \geq b\ P(n)$ by induction

2)

Question 2. Prove that $3 + 3 \times 5 + 3 \times 5^2 + ... + 3 \times 5^n = 3(5^{n+1} - 1)/4$ whenever n is a nonnegative integer.

*Solution.*let $P(n)$ be $3 + 3 \times 5 + 3 \times 5^2 + ... + 3 \times 5^n = \frac{3(5^{n+1}-1)}{4}$
Basis Step: $n = 0$

$$3 \times 5^0 = 3 \times 1 = 3 \quad (8)$$
$$\frac{3(5^{0+1} - 1)}{4} = \frac{3(5^1 - 1)}{4} = \frac{3(5-1)}{4} = \frac{3(4)}{4} = 3 \quad (9)$$
$$(10)$$

we note $P(0)$ is true, as both sides of the equations is equal 3.
Inductive Step: Let $P(k)$ be true.
$3 + 3 \times 5 + 3 \times 5^2 + ... + 3 \times 5^k = \frac{3(5^{k+1}-1)}{4}$
we need to prove that $P(k+1) is true$.

$$3 + 3 \times 5 + 3 \times 5^2 + ... + 3 \times 5^k + 3 \times 5^{k+1} \quad (11)$$
$$= \frac{3(5^{k+1} - 1)}{4} + 3 \times 5^{k+1} \quad (12)$$
$$= 3(\frac{(5^{k+1} - 1)}{4} + 5^{k+1}) \quad (13)$$
$$= \frac{3}{4}((5^{k+1} - 1) + 4 \times 5^{k+1}) \quad (14)$$
$$= \frac{3}{4}((1+4)5^{k+1} - 1) \quad (15)$$
$$= \frac{3}{4}(5 \times 5^{k+1} - 1) \quad (16)$$
$$= \frac{3}{4}(5^{k+2} - 1) \quad (17)$$
$$= \frac{3(5^{k+2} - 1)}{4} \quad (18)$$
$$= \frac{3(5^{(k+1)+1} - 1)}{4} \quad (19)$$

we note that $P(k+1)$ is also true.
conclusion by the principle of mathematical induction, $P(n)$ is true for all non negative integers n.

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3)

Question 3. Prove that $2^n > n^2$ if n is an integer greater than 4.

*Solution.*Let $P(n)$ be $2^n > n^2$
Basis Step: $n = 5$
$2^5 = 32 > 25 = 5^2$
we then note $P(5)$ is true.
Inductive Step: Let $P(k)$ be true. (k is greater than 4).
$2^k > k^2$

we need to prove that $P(k+1)$ is also true.

$$2^{k+1} = 2 \times 2^k \quad (20)$$
$$= 2^k + 2^k \quad (21)$$
$$> k^2 + k^2 \quad (22)$$
$$= k^2 + k \times k \quad (23)$$
$$> k^2 + 4k \qquad \text{since } k > 4 \quad (24)$$
$$> k^2 + 2k + 1 \qquad \text{since} 4k > 2k + 1 \text{when } k > 4 \quad (25)$$
$$= (k+1)^2 \quad (26)$$

we note that $P(k+1)$ is also true.
conclusion by the principle of mathematical induction, $P(n)$ is true for all positive integers n greater than4.

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4)

Solution. a) The postages that can be formed using 3-cent and 10-cent stamps are all possible linear combinations $3x + 10y$ that can be formed (with x and y nonnegative integers).

$$3, 6, 9, 10, 12, 13, 15, 16, 18, 19, 20, 21, 22, ...$$

Note: All stamps of more than or equal to 18 cents can be formed using 3-cent and 10-cent stamps.

b) To proof: A postage of n cents can be formed using just 3-cent and 10-cent stamps with $n \geq 18$.
Proof by Induction
Let $P(n)$ be " A postage of n cents can be formed using just 3 cent and 10 cent stamps ".
Basis Step $n = 18$
$P(18)$ is true, because 18 cents can be formed using six 3-cent stamps.

$$6 \times 3 = 18$$

Inductive Step We assume that $P(k)$ is true, thus a postage of k cents can be formed using just 3-cent and 10-cent stamps.
We then need to prove that $P(k+1)$ is true. If k cents could be formed using three or more 3-cent stamps, then we replace three 3cent stamps with one 10-cent stamps to obtain $k+1$ cents.

$$3 \times 3 + 1 = 9 + 1 = 10$$

If k cents could be formed using less than three 3-cent stamp, then the k cents was formed with at least two 10-cent stamps (since $k \geq 18, k = 18$ and $k = 19$ both use at least three 3cent stamps). If we then replace two 10-cent stamps with seven 3-cent stamps, then we obtain $k+1$ cents.

$$2 \times 10 + 1 = 20 + 1 = 21 = 3 \times 7$$

Thus we then note that $P(k+1)$ is true (in all cases).
Conclusion By the principle of mathematical induction, $P(n)$ is true for all positive integers n.

c) To proof: A postage of n cents can be formed using just 3-cent and 10-cent stamps with $n \geq 18$.
Proof by Strong Induction
Let $P(n)$ be " A postage of n cents can be formed using just 3 cent and 10 cent stamps ".
Basis Step $n = 18$ and $n = 19$ and $n = 20$
$P(18)$ is true, because 18 cents can be formed using six 3-cent stamps.

$$6 \times 3 = 18$$

$P(19)$ is true, because 19 cents can be formed using one 10-cent stamps and three 3-cent stamps.

$$1 \times 10 + 3 \times 3 = 10 + 9 = 19$$

$P(20)$ is true, because 20 cents can be formed using two 10-cent stamps.

$$2 \times 10 = 20$$

Inductive step We assume that $P(18), P(19), ..., P(k)$ are all true, thus all postage from 18 to k cents can be formed using 10-cent and 3-cent stamps.
We then need to prove that $P(k+1)$ is true. Since $P(k-2)$ is true, $P(k+1)$ is also true (because we add one 3-cent stamp to the formation for $k-2$).
Conclusion By the principle of strong induction, $P(n)$ is true for all positive integers n.

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