The Foundations: Logic and Proofs

Chapter 1

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Propositional Equivalences

Section 1.3

Section Summary

Tautology, Contradiction, and Contingency.

Logical Equivalence

- Important Logical Equivalences
- Showing Logical Equivalence

Satisfiability

Tautology, Contradiction, and Contingency

A ***tautology*** is a proposition which is always **true**.

- Example: $p \lor \neg p$

A ***contradiction*** is a proposition which is always **false**.

- Example: $p \land \neg p$

A ***contingency*** is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \lor \neg p$	$p \land \neg p$
T	F	T	F
F	T	T	F

Logical Equivalence

- Two propositions *p* and *q* are **logically equivalent** if and only if the values of the columns in their truth table agree.
- We write this as $p \equiv q$
- **Example**: show that $\neg p \vee q$ is equivalent to $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	**T**	**T**
T	F	F	**F**	**F**
F	T	T	**T**	**T**
F	F	T	**T**	**T**

$$\boxed{p \rightarrow q \equiv \neg p \vee q}$$

- Two propositions p_1 and p_2 are **logically equivalent** if $p_1 \leftrightarrow p_2$ is a tautology.

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Example: Show that De Morgan's Second Law holds.

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	**F**	**F**
T	F	F	T	T	**F**	**F**
F	T	T	F	T	**F**	**F**
F	F	T	T	F	**T**	**T**

Important Logical Equivalences[1]

Identity Laws: $\boxed{p \wedge T \equiv p, \qquad p \vee F \equiv p}$

Domination Laws: $\boxed{p \vee T \equiv T, \qquad p \wedge F \equiv F}$

Idempotent laws: $\boxed{p \vee p \equiv p, \qquad p \wedge p \equiv p}$

Double Negation Law: $\boxed{\neg(\neg p) \equiv p}$

Negation Laws: $\boxed{p \vee \neg p \equiv T, \qquad p \wedge \neg p \equiv F}$

Important Logical Equivalences 2

Commutative Laws: $\boxed{p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p}$

Associative Laws:

$$\boxed{\begin{aligned}(p \wedge q) \wedge r &\equiv p \wedge (q \wedge r)\\ (p \vee q) \vee r &\equiv p \vee (q \vee r)\end{aligned}}$$

Distributive Laws:

$$\boxed{\begin{aligned}(p \vee (q \wedge r)) &\equiv (p \vee q) \wedge (p \vee r)\\ (p \wedge (q \vee r)) &\equiv (p \wedge q) \vee (p \wedge r)\end{aligned}}$$

Absorption Laws: $\boxed{p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p}$

Biconditional:

$$\boxed{\begin{aligned}p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p)\\ p \leftrightarrow q &\equiv \neg p \leftrightarrow \neg q\end{aligned}}$$

Showing Logical Equivalences

- We can show that two expressions are logically equivalent ($A \equiv B$):
  1. By using truth tables.
  2. By developing a series of logically equivalent statements.
     - To prove that $A \equiv B$ we produce a series of equivalences beginning with A and ending with B.

$$\begin{gathered} A \equiv A_1 \\ \vdots \\ A_n \equiv B \end{gathered}$$

Equivalence Proofs [1]

Example 1: Show that $\neg(p \rightarrow q) \equiv p \wedge \neg q$

Solution:

$$\begin{aligned} \neg(p \rightarrow q) &\equiv \neg(\neg p \vee q) \\ &\equiv p \wedge \neg q \end{aligned}$$

Equivalence Proofs 2

Example 2: Show that $\neg\left(p \vee \left(\neg p \wedge q\right)\right) \equiv \neg p \wedge \neg q$

Solution:

$$\equiv \neg p \wedge \left[\neg\left(\neg p\right) \vee \neg q\right]$$

$$\equiv \neg p \wedge \left(p \vee \neg q\right)$$

$$\equiv \left(\neg p \wedge p\right) \vee \left(\neg p \wedge \neg q\right)$$

$$\equiv F \vee \left(\neg p \wedge \neg q\right)$$

$$\equiv \left(\neg p \wedge \neg q\right)$$

Equivalence Proofs 3

Example 3: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution:

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) \\
&\equiv T \vee T \\
&\equiv T
\end{aligned}
$$

Propositional Satisfiability

- A compound proposition is ***satisfiable*** if there is an assignment of truth values to its variables that make it true.
- When no such assignments exist, the compound proposition is ***unsatisfiable***.

Propositional Satisfiability

Example: Determine the satisfiability of the following compound propositions:

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

Solution: Satisfiable. Assign **T** to *p, q,* and *r*.

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Satisfiable. Assign **T** to *p* and ***F*** to *q*.

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

Solution: Unsatisfiable. Check each possible assignment of truth values to the propositional variables and none will make the proposition true.