KING SAUD UNIVERSITY

Practice problems for final

1. Express the number 442 (base 6) in base 3.

$442 \text{ (base 6)} = 170_{10} = 20022 \text{ (in base 3)}$

2. What is the co-efficient of $x^3y^4z^3$ when expanding $(x+2y+3z)^{10}$.

$$(x+2y+3z)^{10} = \sum_{i_1+i_2+i_3=10} \frac{10!}{i_1! \times i_2! \times i_3!} x^{i_1} \cdot (2y)^{i_2} \cdot (3z)^{i_3}$$

coeff of $x^3y^4z^3 \Rightarrow i_1 = 3, \; i_2 = 4, \; i_3 = 3$

$$= \frac{10!}{3! \cdot 4! \cdot 3!} 2^4 \cdot 3^3$$

3. Calculate the value of $\sum_{k=0}^{n} \prod_{i=0}^{k} 3.$ $= \sum_{k=0}^{n} 3^{k+1}$

$$= 3 \sum_{k=0}^{n} 3^k = 3\left(\frac{3^{n+1}-1}{2}\right)$$

4. Prove using Induction that for all positive integer n, then

$$\sum_{k=0}^{n} k \cdot (k!) = (n+1)! - 1.$$

Base case $(n=0)$

LHS $= 0 \cdot (0!) = 0$

RHS $= (0+1)! - 1 = 1! - 1 = 0$

Inductive case

Assume it is True for some n. For $n+1$ we have

$$\sum_{k=0}^{n+1} k(k!) = \underbrace{\sum_{k=0}^{n} k(k!)}_{= (n+1)! - 1 \text{ by induction hypothesis}} + (n+1)(n+1)!$$

$= (n+1)! \left[1 + (n+1)\right] - 1$

$= (n+2)(n+1)! - 1$

$= (n+2)! - 1$

$=$ RHS

5. Use Induction to show that, $n! > 2^n$ for all $n \geq 4$.

Base case $n=4$ $\quad 4! = 24 > 2^4 = 16$ $\quad$ True

Inductive case

$$(n+1)! = (n+1)\cdot n! > (n+1)\cdot 2^n \quad \text{by induction hypothesis}$$
$$> 2\cdot 2^n$$
$$= 2^{n+1}$$

6. Write generating function in closed form to generate sequence: <5, 3, 1, 1, 1, ...>

$$\langle 1,1,1,1,1,\dots\rangle \longmapsto \frac{1}{1-x}$$

$$\langle 1,1,0,0,0,\dots\rangle \longmapsto \frac{1-x^2}{1-x}$$

$$\langle 2,2,0,0,0,\dots\rangle \longmapsto 2\left(\frac{1-x^2}{1-x}\right)$$

So $\langle 5,3,1,1,1,\dots\rangle \longmapsto 2 + 2\left(\frac{1-x^2}{1-x}\right) + \frac{1}{1-x}$

$$= \frac{2x^2+2x-5}{x-1}$$

7. Solve recurrence relation $a_n = a_{n-1} - 3a_{n-2}$ with initial conditions $a_0 = 1, a_1 = 4$.

$r^2 - r - 3 = 0$ $\quad$ roots $= \frac{1 \pm \sqrt{13}}{2}$

Eq. $a_n = \alpha_1 \left(\frac{1+\sqrt{13}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{13}}{2}\right)^n$

For α_1, α_2 use $\begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{13}}{2} & \frac{1-\sqrt{13}}{2} \end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

8. How many passwords of length 7 can you make using following symbols: a-z, A-Z, @, and 0-9. Each password must have at least one capital letter, and at least one digit.

$$\text{\# passwords} = \sum_{\substack{k_1+k_2+k_3=7 \\ k_1 \geq 1 \\ k_2 \geq 1}} \frac{7!}{k_1!\cdot k_2!\cdot k_3!}\, 26^{k_1}\cdot 10^{k_2}\cdot 63^{k_3}$$

k_1 = # capital letters

k_2 = # digits

k_3 = # any of the symbols a-z, A-Z, @, 0-9

9. Suppose we have three sets: X, Y, and Z of sizes n, m, ℓ respectively. Let set $W = X \times Y$ (cross-product of two sets), and let $E = P(W)$, that is the power set of W. Count the number of functions $f : Z \mapsto E$.

functions $f = |E|^{|Z|} = \left(2^{nm}\right)^{\ell} = 2^{nm\ell}$