Question 1. Show that the sequence a_n is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if:

a)$a_n = -n + 2$
b)$a_n = 5(-1)^n - n + 2$

Solution. a)replace n in $a_n = -n + 2$ by n-1 $a_{n-1} = -(n-1) + 2 = -n + 1 + 2 = -n + 3$
replace n in $a_n = -n + 2$ by n-2 $a_{n-2} = -(n-2) + 2 = -n + 2 + 2 = -n + 4$

starting from the expression $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ proof this term equal to a_n
$a_n = a_{n-1} + 2a_{n-2} + 2n - 9$
$= (-n + 3) + 2(-n + 4) + 2n - 9$
$= -n + 2$
$= a_n$

b)replace n in a_n by n-1 $a_{n-1} = 5(-1)^{n-1} + 2 = 5(-1)^{n-1} + 3$
replace by n-2 $5(-1)^{n-2} - (n-2) + 2 = 5(-1)^{n-2} - n + 4$
starting from the expression $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ proof this term equal to a_n
$a_n = a_{n-1} + 2a_{n-2} + 2n - 9$
$= 5(-1)^{n-1} + 3 + 2[5(-1)^{n-2} - n + 4] + 2n - 9$
$= 5(-1)^{n-1} + 3 + 10(-1)^{n-2} - 2n + 8 + 2n - 9$
$= 5(-1)^{n-1} + 10(-1)^{n-2} - n + 2$
$= 5(-1)(-1)^{n-2} + 10(-1)^{n-2} - n + 2$
factor out $(-1)^{n-2}$
$= (-1)^{n-2}(5(-1) + 10) - n + 2$
$= (-1)^{n-2}(-5 + 10) - n + 2$
$= 5(-1)^{n-2} - n + 2$
mult. by 1:
$= 5(1)(-1)^{n-2} - n + 2$
since $(-1)^2 = 1$:
$= 5(-1)^2(-1)^{n-2} - n + 2$
$= 5(-1)^n - n + 2$
$= a_n$

Question 2. Find the solution to each of these recurrent relations and initial conditions. Use iterative aproach.

a)$a_n = 3a_{n-1}$, $a_0 = 2$
b)$a_n = a_{n-1} + 2$, $a_0 = 3$

*Solution.*1.$a_n = 2.3^n$
2.$a_n = 2n + 3$ ∎

Question 3. Suppose that the number of bacteria in a colony triples every hour.

a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

Solution.

a) $a_n = 3a_{n-1}$
b)

$$\begin{aligned} a_0 &= 100 \\ a_1 &= 3 \cdot 100 \\ a_2 &= 3 \cdot 3 \cdot 100 = 3^2 \cdot 100 \\ &\vdots \\ a_n &= 3^n \cdot 100 \end{aligned}$$

So, $a_{10} = 3^{10} \cdot 100 = 5,904,900$

Question 4. Let $\{a_n\}$ be an arithmetic sequence:

a) What's the value of a_{30} if the value of the initial term is -23 and the common difference is 7?
b) Express the sequence as a recurrence relation.
c) Express a summation for this sequence, and find a closed form formula for it.

Solution.

a) Since it is an arithmetic sequence, we know $a_n = a + nd$, so $a_{30} = -23 + 30(7) = 187$
b) $a_n = a_{n-1} + 7$
c)

$$\Sigma_{i=0}^{n}(a + id) = \Sigma_{i=0}^{n}a + \Sigma_{i=0}^{n}id = (n+1)a + \frac{n(n+1)}{2}d$$

Question 5. Show that if $a|b$ and $b|a$,where a and b are integers, then $a = b$ or $a = -b$.

Solution.$a|b$, b=ak
$b|a$, a=bc
(ab)=(ab) ck
ck=1
so either c=1, k=1 or c=-1 k=-1

Another solution
$a|b$, b=ak, a=b/k
$b|a$, a=bc
bc=b/k
c=1/k
ck=1

Question 6. Show that if a, b, and c are integers, where $a \neq 0$ and $c \neq 0$,such that ac|bc ,then a|b.

Since $a|c$, there exists an integer f such that:

$$c = af$$

Since $b|d$, there exists an integer g such that:

$$d = bg$$

Multiply these two equations:

$$cd = (af)(bg) = afbg = abfg = (ab)(fg)$$

Since f and g are integers, their product fg is an also integer.

By the definition of **divides**, we have then shown that ab divides cd.

$$ab|cd$$

Question 7. Show that if $n|m$, where n and m are integers greater than 1, and if $a \equiv b$ *(mod m)*, where a and b are integers, then $a \equiv b$ *(mod n)*.

Let $n|m$ that means $m = sn$ for some integer s
Since $a \equiv b(mod\ m)$ then $a = b + km$ for some integer k
a = b + ksn
$a = b + qn$ where $q = ks$
$a \equiv b(mod\ n)$

Question 8. Find counterexamples to each of these statements about congruences.

a)If $ac \equiv bc\ (mod\ m)$,where a,b,c,and m are integers with $m \geq 2$, then $a \equiv b\ (mod\ m)$.
b)If $a \equiv b\ (mod\ m)$ and $c \equiv d\ (mod\ m)$, where a, b, c, d, and m are integers with c and d positive and $m \geq 2$,then $ac \equiv bd\ (mod\ m)$.

Solution. a)Let us choose $m = 3$, $a = 1$, and $c = 3$
We then obtain $ac \equiv bc\ (mod\ m)$, because $ac = 3$, and $bc = 6$ are both multiples of 3.
However, $a \equiv b\ (mod\ m)$, is not true, because 1 mod 3 is different from 2 mod 3.