The Foundations: Logic and Proofs

Chapter 1

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Proof Methods

Section 1.7 and Section 1.8

Proof Techniques

- Each **theorem** takes the form "*if p, then q*"
- Proof methods:
  - Direct proof $(p \rightarrow q)$
  - Indirect proof $(\neg q \rightarrow \neg p)$
  - Proof by contradiction $(\neg p \rightarrow F)$
  - Proof by cases $(p_1 \rightarrow q) \land (p_2 \rightarrow q) \land \ldots \land (p_n \rightarrow q)$
  - Disproof by counterexample
  - Induction

Some Definitions

Definition: The integer n is **even** if there exists an integer k such that $n=2k$, and n is **odd** if there exists an integer k, such that $n=2k+1$.

Definition: An integer a is a **perfect square** if there is an integer b such that $a = b^2$.

Definition: The real number r is **rational** if there exist integers p and q where $q \neq 0$ such that $r=p/q$ and p and q have no common factors

Direct Proof

***Direct Proof*:** Assume that p is true, and show that q must also be true. $(p \rightarrow q)$

Example 1: Give a direct proof of the theorem "**If *n* is an odd integer, then *n*^{2} is odd**."

Solution: Assume that *n* is odd. Let n=2k+1 for an integer *k*.

Then, $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$=$ odd number

Direct Proof[2]

Example 2: Give a direct proof of the theorem "**if *m* and *n* are both perfect squares, then *nm* is also a perfect square.**"

Solution: Assume that *m and n* are both perfect squares.
Let $m = s^2$ and $n = t^2$.

Then, $mn = s^2 t^2$
$= (ss)(tt)$
$= (st)(st)$
$= (st)^2$
$=$ perfect square

Indirect Proof

***Indirect Proof*: Assume $\neg q$ true and show $\neg p$ is true also. $(\neg q \rightarrow \neg p)$**

Example 1: Prove that if ***3n+2* is an odd integer, then *n* is odd**.

Solution: Assume n is even. So, $n=2k$ for some integer k. $(\neg q)$

Then, $3n+2 = 3(2k) + 2$

$= 6k + 2$

$= 2(3k + 1)$

$=$ even number $(\neg p)$

Indirect Proof[2]

Example 2: Prove that if $\boldsymbol{n^2}$ **is odd, then** $\boldsymbol{n}$ **is odd.**

Solution: Assume n is even. Let $n=2k$ for an integer k. ($\neg q$)

Then, $n^2 = 4k^2$

$= 2\,(2k^2)$

$=$ even number ($\neg p$)

Proof by Contradiction

Proof by Contradiction: Assume $\neg p$ true and derive a contradiction. $(\neg p \rightarrow F)$

Example 1: Use a proof by contradiction to proof that $\sqrt{2}$ is irrational.

Solution: Suppose $\sqrt{2}$ is rational $(\neg \mathrm{p})$. Then there exists integers **a** and **b** with $\sqrt{2}$=a/b, where b≠ 0 and a and b have no common factors.

- Then: $\sqrt{2} = \frac{a}{b} \rightarrow 2 = \frac{a^2}{b^2} \rightarrow a^2 = 2b^2$
- Therefore a^2 must be even. If a^2 is even then a must be even. Since a is even, $a=2c$ for some integer c.
- Then: $2b^2 = 4c^2 \rightarrow b^2 = 2c^2$
- Therefore b^2 is even. Again then b must be even as well.
- But then 2 must divide both a and b. This contradicts our assumption that a and b have <u>no common factors</u>. Therefore, $\sqrt{2}$ is irrational.

Proof by Contradiction$_2$

Another way for *Proof by Contradiction*: Assume $p \wedge \neg q$ true and derive a contradiction.

$$\boldsymbol{p \to q \equiv p \wedge \neg q \to F}$$

Example 2: Prove that if ***3n+2* is an odd integer, then *n* is odd**.

Solution: Assume 3*n*+*2* is odd (p) and n is even ($\neg q$). So, *n* = 2*k* for some integer *k*.

Then, 3*n*+2 = 3(2*k*) + 2

=6*k* +2

= 2(3*k* + 1)

= even number ($\neg\, p$) → <u>contradiction</u>

Proof by Cases

To prove a conditional statement of the form:

$$\left(p_1 \vee p_2 \vee \ldots \vee p_n\right) \rightarrow q$$

Show: $\boxed{\left[\left(p_1 \rightarrow q\right) \wedge \left(p_2 \rightarrow q\right) \wedge \ldots \wedge \left(p_n \rightarrow q\right)\right]}$

Each of the implications $p_i \rightarrow q$ is a *case*.

Proof by Cases[2]

Example: Show that any integer ending with 2 cannot be a perfect square.

Solution: Let n be an integer. We show that n^2 cannot have 2 in units digit.

Let $n = 10a + b$, where $b = 0,1,2,3,4,5,6,7,8,$ or 9

Then, $n^2 = (10a + b)^2$

$= 100a^2 + 20ab + b^2$

$= 10(10a^2 + 2ab) + b^2$

The final decimal digit of n^2 is b^2

Case1: $b=0 \rightarrow b^2 = 0$
Case2: $b=1 \rightarrow b^2 = 1$
Case3: $b=2 \rightarrow b^2 = 4$
Case4: $b=3 \rightarrow b^2 = 9$
Case5: $b=4 \rightarrow b^2 = 16$
Case6: $b=5 \rightarrow b^2 = 25$
Case7: $b=6 \rightarrow b^2 = 36$
Case8: $b=7 \rightarrow b^2 = 49$
Case9: $b=8 \rightarrow b^2 = 64$
Case10: $b=9 \rightarrow b^2 = 81$

Thus, square of an integer ends with 0,1,4,5,6,9

Disproof by Counterexample

Recall: $\boxed{\exists x \neg P(x) \equiv \neg \forall x P(x)}$

Example 1: Prove or disprove that "All prime integers are odd."

Solution: The integer "2" is <u>even</u> and prime. So the claim is false.

Example 2: Prove or disprove that "Every positive integer is the sum of the squares of 2 integers."

Solution: The integer "3" is a counterexample. So the claim is false.