Basic Structures: Sets, Functions, Sequences, and Summation

Chapter 2

Edited by: Dr. Meshal Alfarhood

Functions

Section 2.3

Section Summary

Definition of a Function

- Domain, Codomain
- Image, Preimage

Types of Functions

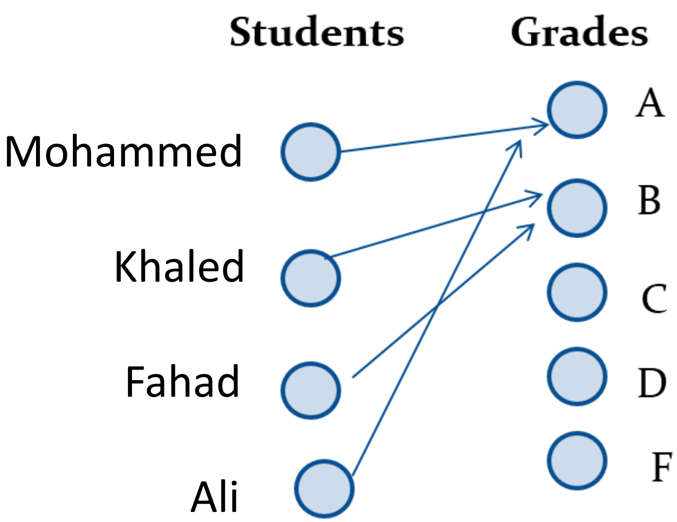
- One-to-one
- Onto
- One-to-one correspondence

Inverse Function

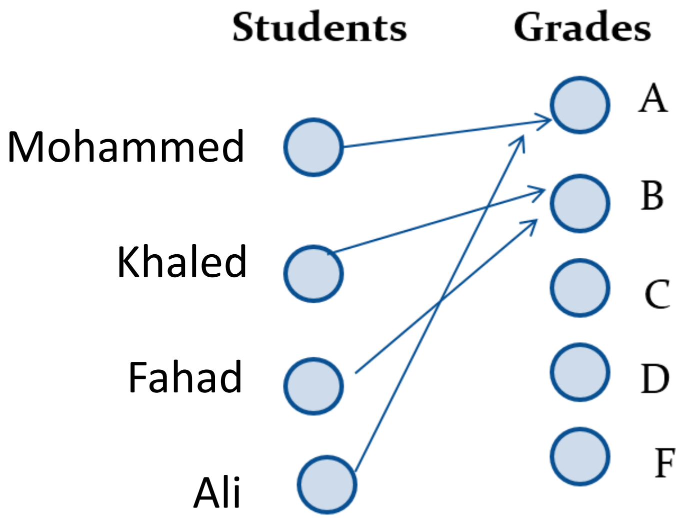
Function Composition

Functions[1]

Definition: Let A and B be nonempty sets. A ***function*** f from A to B, denoted $f: A \to B$ is an assignment of exactly one element of B to each element of A.



Functions $_2$

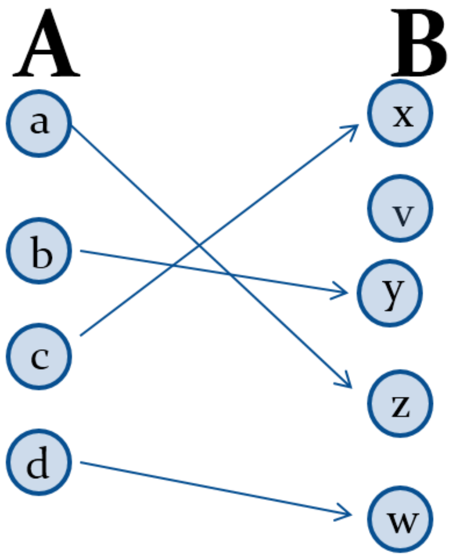


Given a function $f: A \rightarrow B$:

- $\boldsymbol{A}$ is called the ***domain*** of f.
- $\boldsymbol{B}$ is called the ***codomain*** of f.
- If $f(a) = b$,
  - b is called the ***image*** of a under f.
  - a is called the ***preimage*** of b.
- The ***range*** of f is the set of all images of points in **A** under f.
  - We denote it by $f(\boldsymbol{A})$.
  - Range is always a subset of the codomain.

One-to-one

Definition: A function f is called ***one-to-one*** if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f.

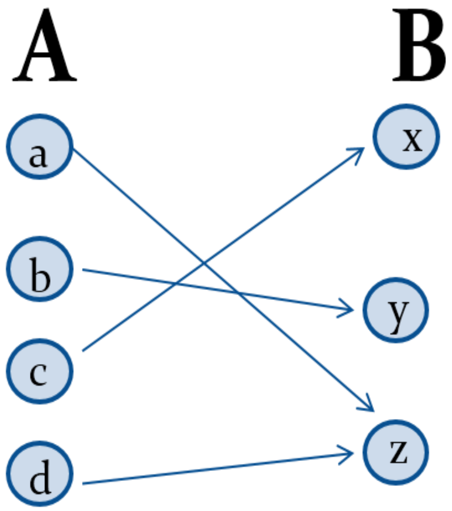


|A|≤|B|

Each b ∈ B receives <u>at most</u> 1 arrow.

Onto

Definition: A function *f* is called ***onto*** if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

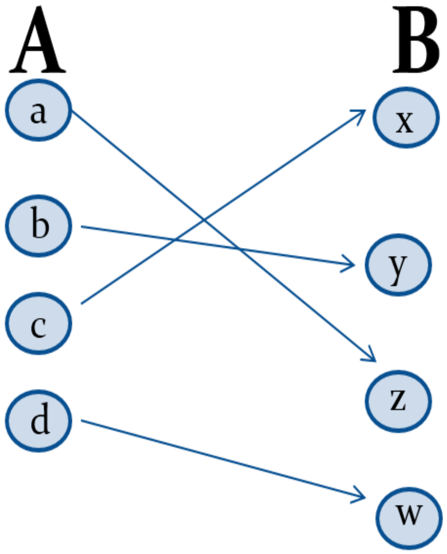


|A|≥|B|

Each b ∈ B receives at least 1 arrow.

One-to-one correspondence

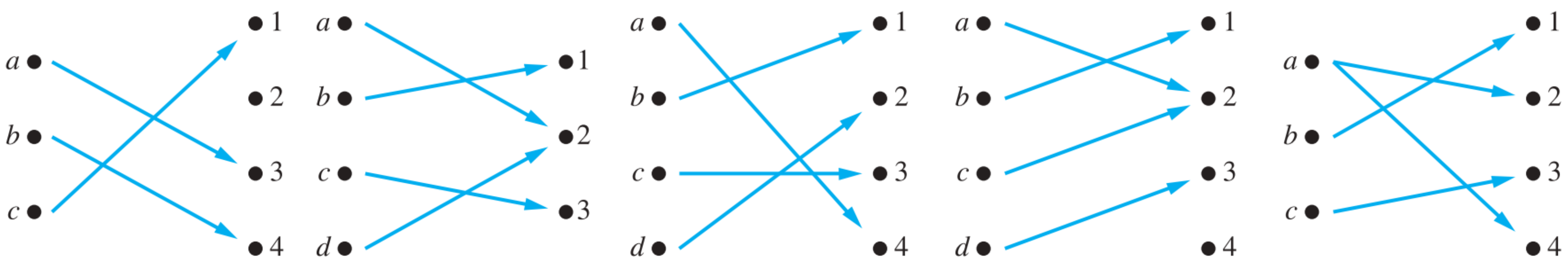
Definition: A function f is called a ***one-to-one correspondence*** if it is both one-to-one and onto.



$|A| = |B|$

Each b ∈ B receives <u>exactly</u> 1 arrow.

Example of different types



(a) One-to-one, not onto

(b) Onto, not one-to-one

(c) One-to-one and onto

(d) Neither one-to-one nor onto

(e) Not a function

Exercise

Determine if the following functions are **one-to-one** or **onto**, where $f: \mathbb{Z} \to \mathbb{Z}$:

1. $f(x) = x+1$
   - It's one-to-one.
   - It's onto.
2. $f(x) = x^2$
   - It's NOT one-to-one; because 1 and -1 give the same result.
   - It's NOT onto; because no x such that $x^2 = -1$.
3. $f(x) = x^3$
   - It's one-to-one.
   - It's NOT onto; because no x such that $x^3 = 2$.

Exercise

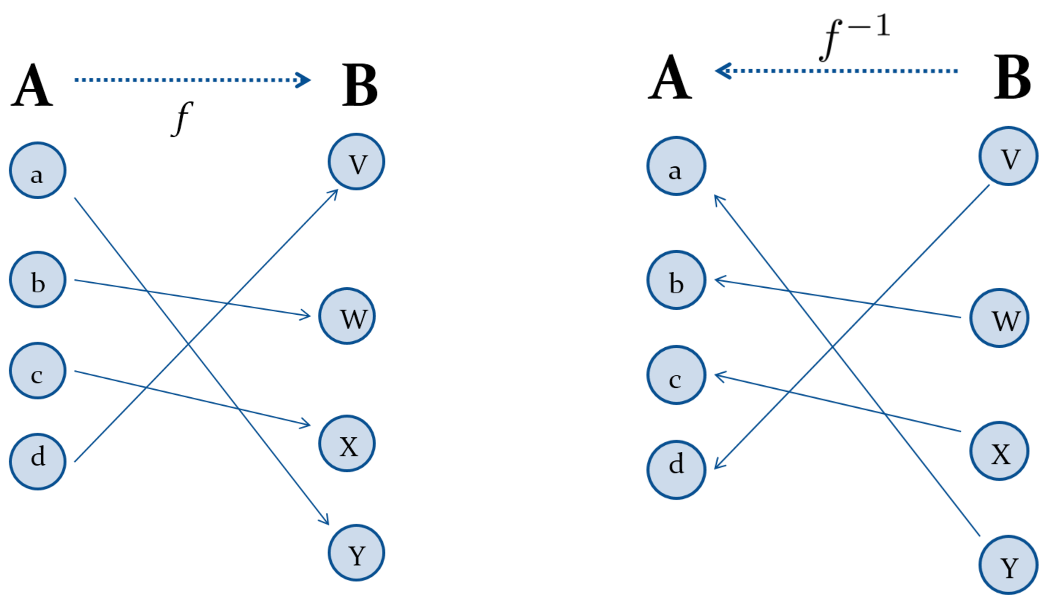
Determine if the following functions are **one-to-one** or **onto**, where *f*: $\mathbb{Z} \to \mathbb{Z}$:

Ceiling Function:	$\lceil 1.2 \rceil = 2$
Floor Function:	$\lfloor 1.2 \rfloor = 1$

4. $f(x) = \left\lceil \frac{x}{2} \right\rceil$
   - It’s NOT one-to-one; because 1 and 2 give the same result.
   - It’s onto.
5. $f(x) = 2\left\lceil \frac{x}{2} \right\rceil$
   - It’s NOT one-to-one; because 1 and 2 give the same result.
   - It’s NOT onto; because we can’t reach odd numbers.

Inverse Functions

Definition: Let ***f*** be a **1-1 correspondence** from *A* to *B*. Then the ***inverse*** of *f*, $\boxed{f^{-1},}$ is the function from *B* to *A* defined as $f^{-1}(y) = x \text{ iff } f(x) = y$



No inverse exists unless *f* is a 1-1 correspondence

Exercise

Is the following functions invertible?

1. $\boldsymbol{f(x) = x+1}$, such that $f\colon \mathbb{Z} \rightarrow \mathbb{Z}$.

- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence, so $f^{-1}(y) = y - 1$.

2. $\boldsymbol{f(x) = x^2}$, such that $f\colon \mathbb{Z} \rightarrow \mathbb{Z}$.

- **Solution**: The function f is not invertible because it is not one-to-one correspondence.

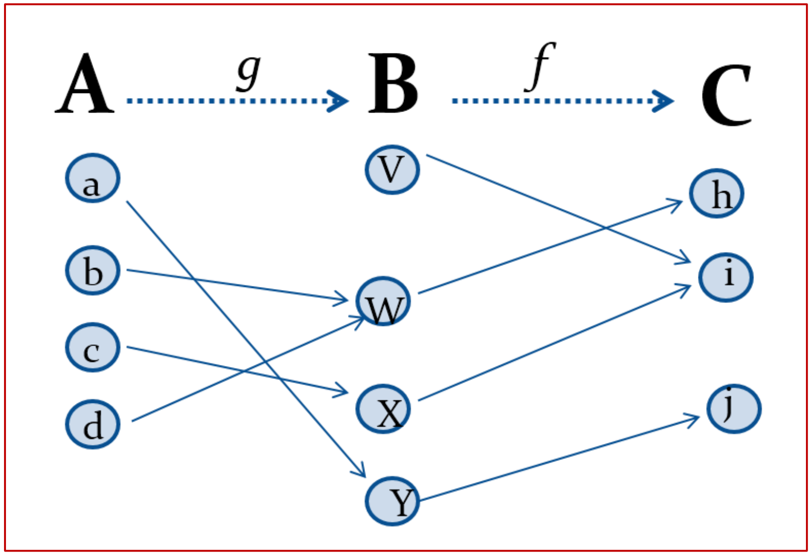
3. $\boldsymbol{f(x) = x^2}$, such that $f\colon \mathbb{R}^+ \rightarrow \mathbb{R}^+$.

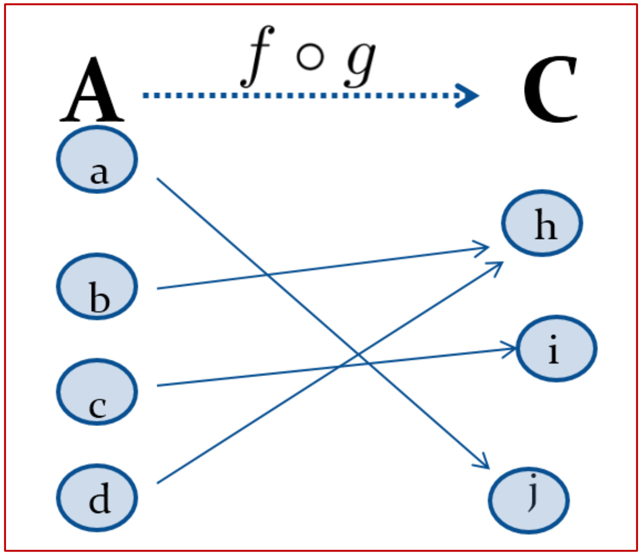
- **Solution**: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence, so $f^{-1}(y) = \sqrt{y}$.

Compositions of Functions

Definition: Let g: $A \to B$, f: $B \to C$. The ***composition*** *of f and g*, denoted $\boxed{f \circ g}$ is the function from A to C defined by

$$f \circ g(x) = f(g(x))$$





The composition $f \circ g$ cannot be defined unless the range of g is a subset of the domain of f.

Exercise

Let f and g be functions from $\mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$f(x) = 2x + 3 \quad \text{and} \quad g(x) = 3x + 2.$$

What is the composition of f and g, and also the composition of g and f ?

Solution:

- $(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)+3 = 6x+7$
- $(g \circ f)(x) = g(f(x)) = g(2x+3) = 3(2x+3)+2 = 6x+11$