Counting

Chapter 6

Edited by: Dr. Meshal Alfarhood

Binomial Coefficients and Identities

Section 6.4

Section Summary

Pascal's Identity and Triangle

The Binomial Theorem

Identities Involving Binomial Coefficients

Pascal's Identity

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$1$$
$$1 \quad 1$$
$$1 \quad 2 \quad 1$$
$$1 \quad 3 \quad 3 \quad 1$$
$$1 \quad 4 \quad 6 \quad 4 \quad 1$$
$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$
$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$
$$1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1$$
$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

Pascal's Identity:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Other observations from Pascal's triangle:

$$C(n,r) = C(n,n-r)$$

$$\sum_{k=0}^{n} C(n,k) = 2^n$$

The Binomial Theorem

1. $(x + y)^2 = x^2 + 2xy + y^2$
2. $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
3. $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
4. $(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Using the Binomial Theorem

Examples:

1. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x + y)^{25}$?

- **Solution:** the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\boxed{\binom{25}{13} = \frac{25!}{13!\,12!} = 5{,}200{,}300.}$$

2. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

- **Solution**: We view the expression as $(2x + (-3y))^{25}$. By the binomial theorem:

$$\left(2x+(-3y)\right)^{25} = \sum_{j=0}^{25}\binom{25}{j}2x^{25-j}(-3y)^{j}.$$

- Consequently,

$$\boxed{\binom{25}{13}2^{12}(-3)^{13} = -\frac{25!}{13!\,12!}2^{12}3^{13}.}$$

A Useful Identity

Theorem 1: With *n* ≥0, $\sum_{k=0}^{n}\binom{n}{k} = 2^n.$

Proof (*using binomial theorem*): With *x* = 1 and *y* = 1, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k 1^{(n-k)} = \sum_{k=0}^{n}\binom{n}{k}.$$

Theorem 2: With *n* ≥0, $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$

Proof (*using binomial theorem*): With *x* = -1 and *y* = 1, from the binomial theorem we see that:

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k 1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}(-1)^k.$$