PERFORMANCE ANALYSIS

CS212: Data Structure

Outline

- Performance analysis
  - By experiment
  - By analysis
- Growth rate of a function
- Big O notation

Introduction

- We saw two implementations of the ADT List, a linked implementation and an array implementation.
  - The question that we want to answer now: which implementation gives a better performance?
- In general, when having different algorithms that solve the same problem, how to compare their performances? Which one is **better**?

- we need to define what we mean by **best**.
  - time complexity—the time it takes to execute
  - space complexity—the memory it needs to execute.

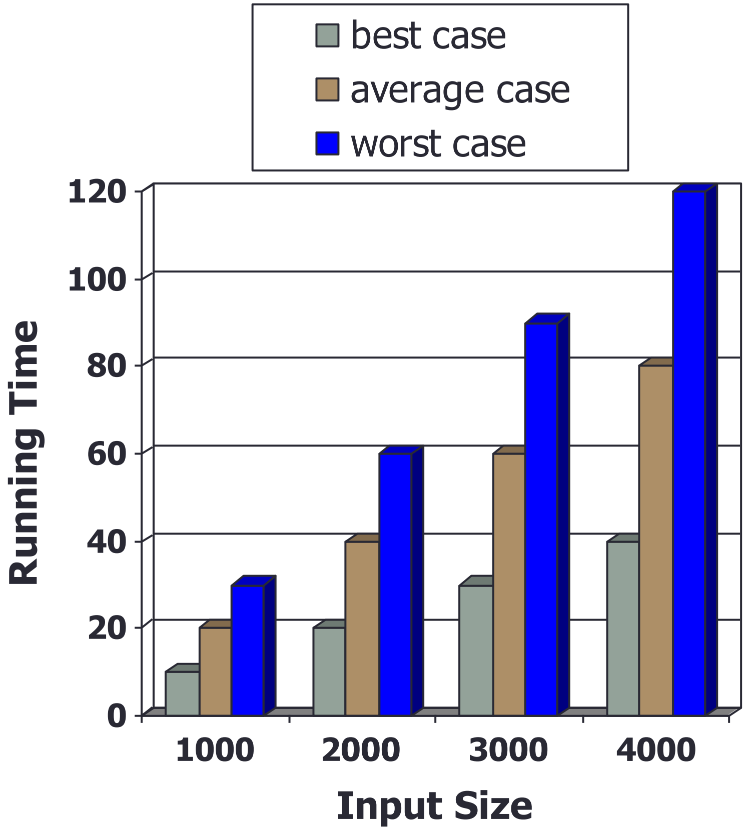
Introduction

Performance analysis :

- The process of measuring the complexity of algorithms
  - We will concentrate on the time complexity of algorithms,
- There are two ways to compare algorithms:
  - **Experimental analysis** : compare the running time for different input sizes.
  - **Theoretical analysis** : analyze the algorithms independently of the implementation (hardware/software).

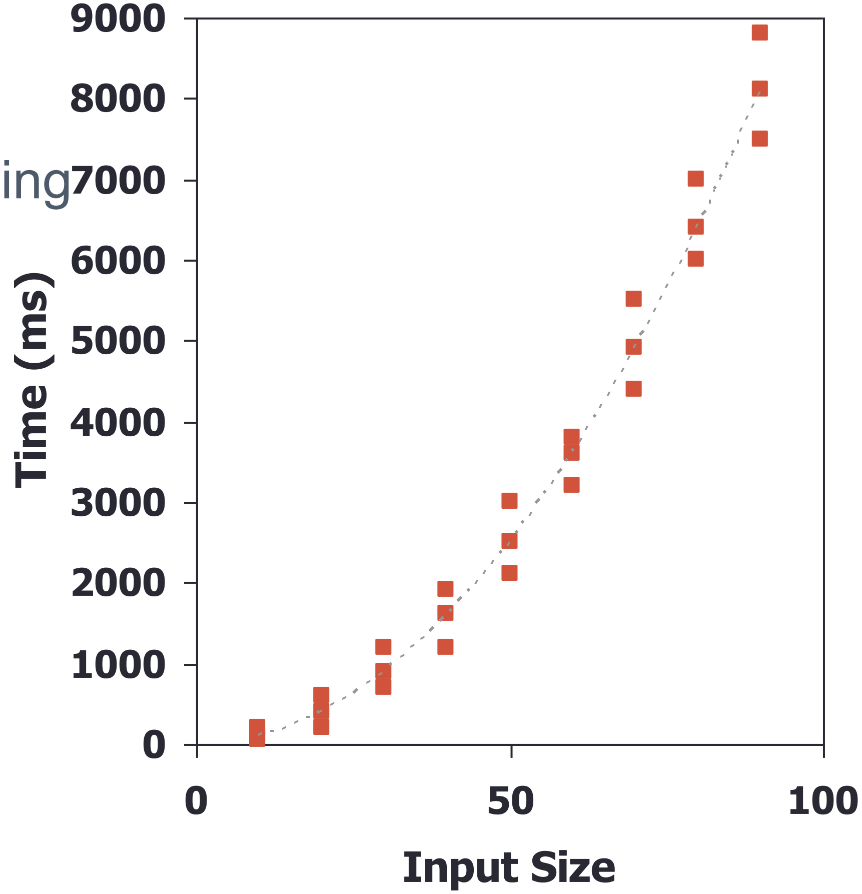
Experimental analysis

- Most algorithms transform input objects into output objects.
- The running time of an algorithm typically grows with the input size.
- Average case time is often difficult to determine.
- We focus on the worst case running time.
  - Easier to analyze
  - Crucial to applications such as games, finance and robotics



Experimental analysis

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition.
- Use a method like System.currentTimeMillis() to get an accurate measure of the actual running time.
- Plot the results



Experimental analysis

Limitations:

- It is necessary to implement the algorithm, which may be difficult.
- Results may not be indicative of the running time on other inputs not included in the experiment.
- In order to compare two algorithms, the same hardware and software environments must be used (depends on them).

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation.
- Characterizes running time as a function of the input size, n.
- Takes into account all possible inputs.
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment.

Theoretical Analysis

Pseudocode:

- High-level description of an algorithm.
- More structured than English prose.
- Less detailed than a program.
- Preferred notation for describing algorithms.
- Hides program design issues.

Example: find max element of an array

Algorithm ***arrayMax***(***A***, ***n***)
  Input array ***A*** of ***n*** integers
  Output maximum element of ***A***

  currentMax ← ***A***[0]
  for ***i*** ← 1 **to** ***n*** − 1 **do**
    if ***A***[*i*] > ***currentMax*** **then**
      currentMax ← ***A***[*i*]
  return ***currentMax***

Theoretical Analysis

Pseudocode Details:

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces

- Method declaration

  Algorithm ***method*** (***arg*** [, ***arg***…])

  Input …

  Output …

- Method call

  var.method (***arg*** [, ***arg***…])

- Return value

  return *expression*

- Expressions

  $\leftarrow$ Assignment (like $=$ in Java)

  $=$ Equality testing (like $==$ in Java)

  n^2 Superscripts and other mathematical formatting allowed

Theoretical Analysis

Counting Primitive Operations

- Primitive operation corresponds to a low-level instruction with a constant execution time.
  - Examples: Evaluating an expression, Assigning a value to a variable, Indexing into an array, Calling a method, Returning from a method
- Instead of determine the specific execution time of each primitive operation, simply ***count*** how many primitive operation are executed.
- This operation count will correlate to an actual running time in a specific computer.
- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size.

Theoretical Analysis

Counting Primitive Operations

	Statements	Step/ Exec utio	Freq .	Total
1	Algorithm Sum1(a[],n)			
2	{			
3	S = 0.0;			
4	for i=1 to n do			
5	s = s+a[i];			
6	return s;			
7	}			

Theoretical Analysis

Counting Primitive Operations

	Statements	S/E	Freq.	Total
1	Algorithm Sum1(a[],n)	0	-	0
2	{	0	-	0
3	S = 0.0;	1	1	1
4	$\boxed{\text{for i=1 to n do}}^{n+1}$	1	n+1	n+1
5	n s = s+a[i];	1	n	n
6	return s;	1	1	1
7	}	0	-	0

$\boxed{2n+3}$

Theoretical Analysis

Counting Primitive Operations

	Statements	S/E	Freq.	Total
1	Algorithm Sum2(a[],n,m)			
2	{			
3	for i=1 to n do;			
4	for j=1 to m do			
5	s = s+a[i][j];			
6	return s;			
7	}			

Theoretical Analysis

Counting Primitive Operations

	Statements	S/E	Freq.	Total
1	Algorithm Sum2(a[],n,m)	0	-	0
2	{	0	-	0
3	for i=1 to n do (n+1)	1	n+1	n+1
4	for j=1 to m do (m+1)	1	n(m+1)	n(m+1)
5	s = s+a[i][j];	1	nm	nm
6	return s;	1	1	1
7	}	0	-	0
				2nm+2n+2

Estimating Running Time

- Algorithm ***Sum1*** executes $2n + 3$ primitive operations in the worst case.
- Define:
  - a = Time taken by the fastest primitive operation
  - b = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of ***Sum1.*** Then

$$a\,(2n + 3) \le T(n) \le b(2n + 3)$$

- Hence, the running time $T(n)$ is bounded by two linear functions.

Growth Rate of Running Time

- Changing the hardware/ software environment
  - Affects ***T***(***n***) by a constant factor, but
  - Does not alter the growth rate of ***T***(***n***)

- The linear growth rate of the running time ***T***(***n***) is an intrinsic property of algorithm ***Sum1.***

Why Growth Rate Matters

if runtime is...	time for n + 1	time for 2 n	time for 4 n
$c \log n$	$c \log (n + 1)$	$c (\log n + 1)$	$c(\log n + 2)$
$c\, n$	$c\, (n + 1)$	$2c\, n$	$4c\, n$
$c\, n \log n$	$\sim c\, n \log n + c\, n$	$2c\, n \log n + 2cn$	$4c\, n \log n + 4cn$
$c\, n^2$	$\sim c\, n^2 + 2c\, n$	**$4c\, n^2$**	$16c\, n^2$
$c\, n^3$	$\sim c\, n^3 + 3c\, n^2$	$8c\, n^3$	$64c\, n^3$
$c\, 2^n$	$c\, 2^{n+1}$	$c\, 2^{2n}$	$c\, 2^{4n}$

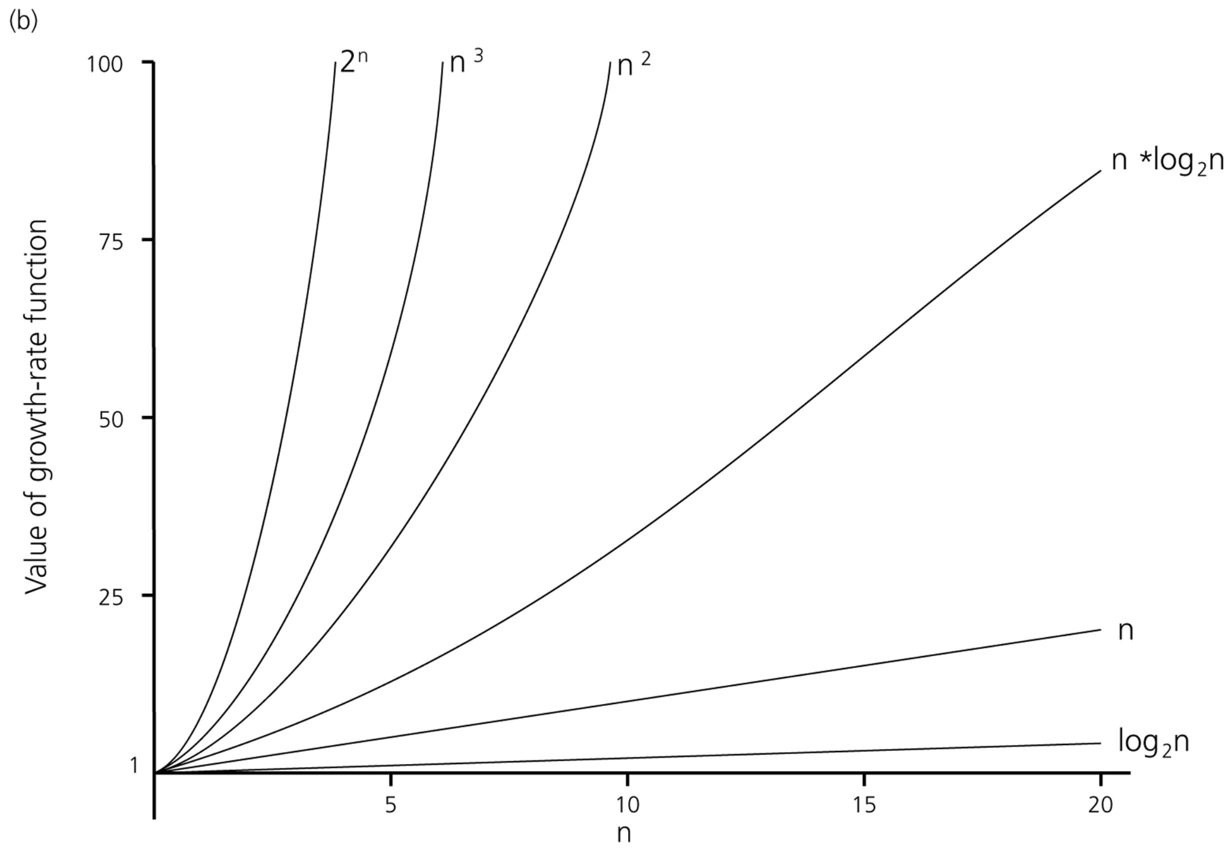
runtime quadruples when problem size doubles

Comparing Growth Rate in Tabular form

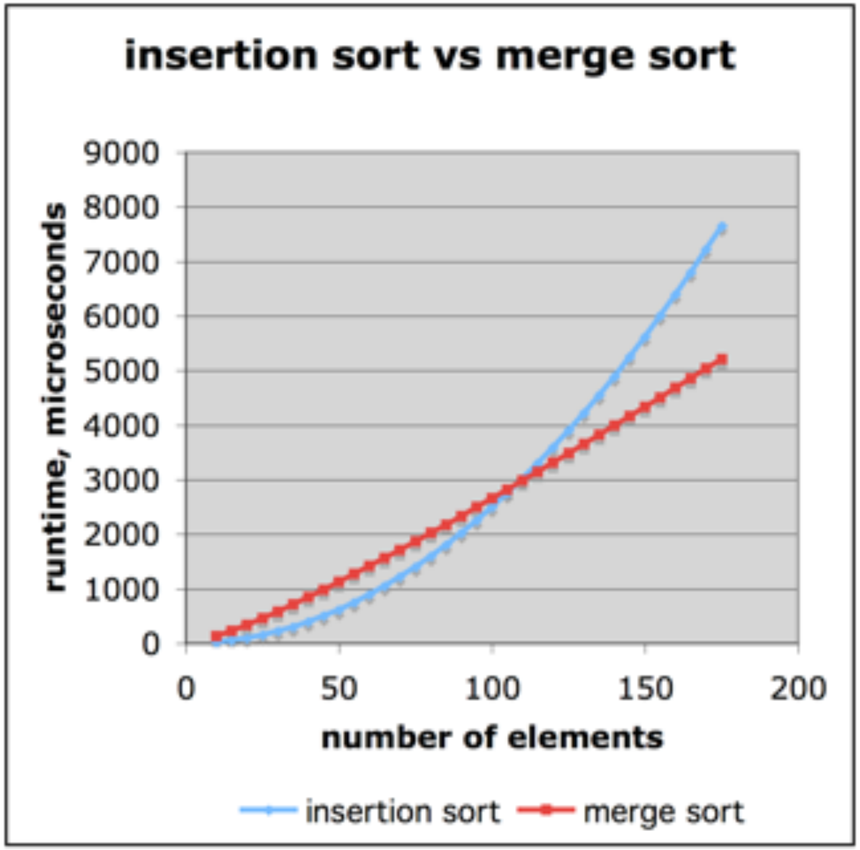
(a)

Function	10	100	1,000	10,000	100,000	1,000,000
1	1	1	1	1	1	1
$\log_2 n$	3	6	9	13	16	19
n	10	10^2	10^3	10^4	10^5	10^6
$n * \log_2 n$	30	664	9,965	10^5	10^6	10^7
n^2	10^2	10^4	10^6	10^8	10^{10}	10^{12}
n^3	10^3	10^6	10^9	10^{12}	10^{15}	10^{18}
2^n	10^3	10^{30}	10^{301}	$10^{3,010}$	$10^{30,103}$	$10^{301,030}$

Comparing Growth Rate in Tabular form



Comparison of Two Algorithms (an example)



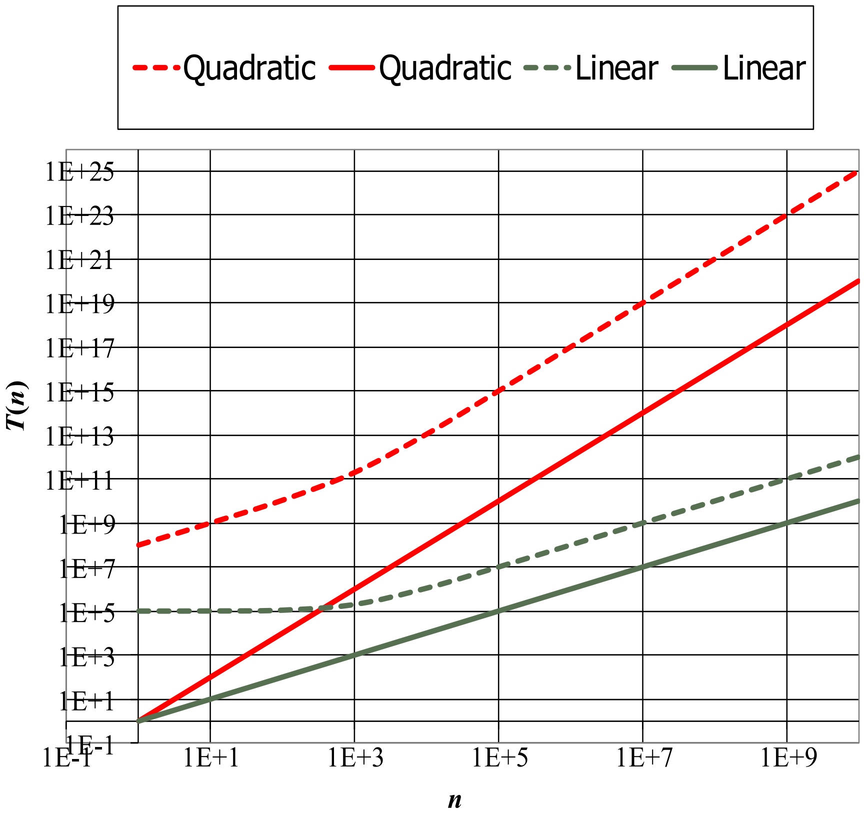
insertion sort is
$n^2 / 4$

merge sort is
$2 n \log n$

sort a million items using a basic PC?
insertion sort takes roughly 70 hours
while
merge sort takes roughly 40 seconds

Constant Factors

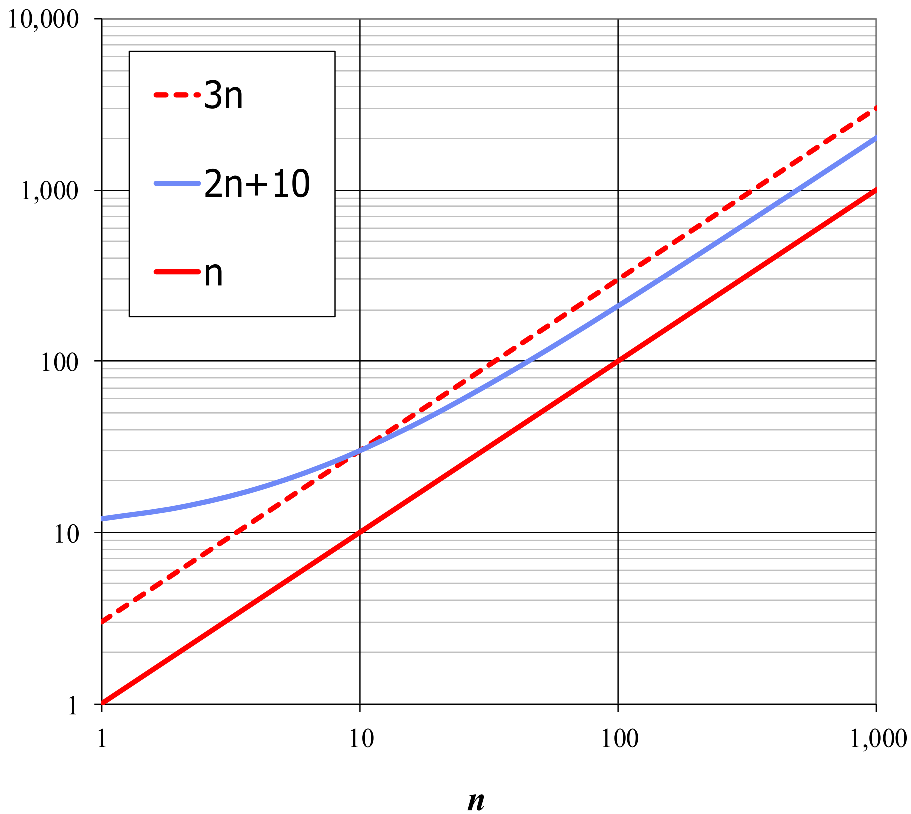
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Big-Oh Notation

- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \le cg(n)$ for $n \ge n_0$
- Example: $2n + 10$ is $O(n)$
  - $2n + 10 \le cn$
  - $10 \le cn - 2n$
  - $10 \le (c - 2)\, n$
  - $10/(c - 2) \le n$
  - Pick $c = 3$ and $n_0 = 10$



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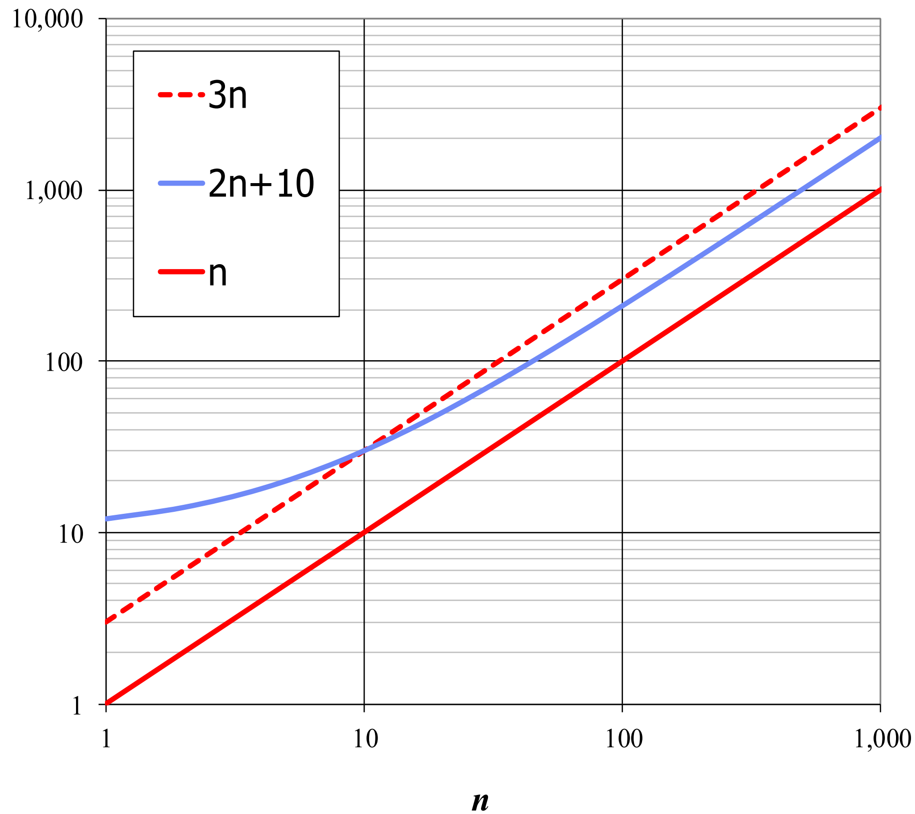


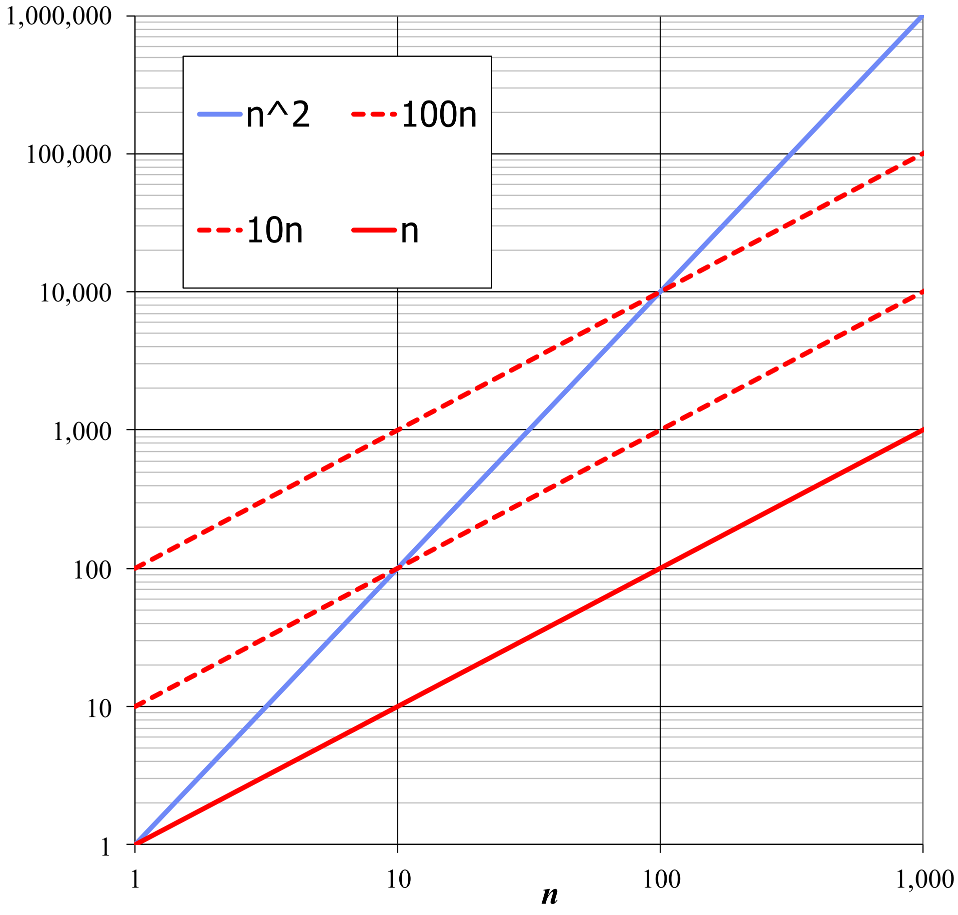
$2n + 10 \le 2n + 10n = 12n$

$n_0 = 1$

$c = 12$

Big-Oh Example

- Example: the function n^2 is not $O(n)$
  - $n^2 \leq cn$
  - $n \leq c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- **7n-2**

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq cn$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- **$3n^3 + 20n^2 + 5$**

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq cn^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- **3 log n + 5**

  3 log n + 5 is O(log n)

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

More Big-Oh Examples

- **7n-2**

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq cn$ for $n \geq n_0$

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  this is true for $c = 8$ and $n_0 = 2$

$7n - 2 \leq 7n$

$n_0 = 1$

$c = 7$

More Big-Oh Examples

- **7n-2**

  $7n-2$ is $O(n)$

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq cn$ for $n \geq n_0$

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  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq cn^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

$3n^3 + 20n^2 + 5 \leq 3n^3 + 20n^3 + 5n^3 = 28n^3$

$n_0 = 1$

$c = 28$

- **3 log n + 5**

  $3 \log n + 5$ is $O(\log n)$

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

More Big-Oh Examples

- **7n-2**

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq cn$ for $n \geq n_0$

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- **$3n^3 + 20n^2 + 5$**

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  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c\log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

$3 \log n + 5 \leq 3 \log n + 5 \log n = 8 \log n$

$n_0 = 2$

$c = 8$

Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function.

- The statement "$\boldsymbol{f}(\boldsymbol{n})$ is $\boldsymbol{O}(\boldsymbol{g}(\boldsymbol{n}))$" means that the growth rate of $\boldsymbol{f}(\boldsymbol{n})$ is no more than the growth rate of $\boldsymbol{g}(\boldsymbol{n})$.

- We can use the big-Oh notation to rank functions according to their growth rate.

Big-Oh Rules

- If $f(n)$ is a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors

- Use the smallest possible class of functions

  Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$".

- Use the simplest expression of the class

  Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

- The asymptotic analysis of an algorithm determines the running time in big-Oh notation.

- To perform the asymptotic analysis
    - We find the worst-case number of primitive operations executed as a function of the input size.
    - We express this function with big-Oh notation.

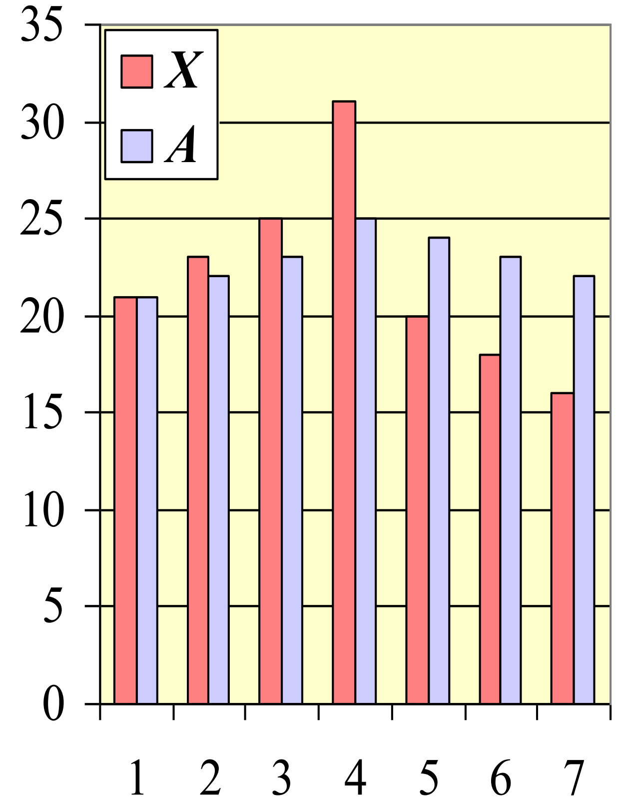
- Example:
    - We determine that algorithm ***Sum1*** executes at most $2n + 3$ primitive operations
    - We say that algorithm ***Sum1*** "runs in $O(n)$ time"

- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Computing Prefix Averages (an example)

- We further illustrate asymptotic analysis with two algorithms for prefix averages.
- The ***i***-th prefix average of an array ***X*** is average of the first (***i*** + 1) elements of ***X***:

  $A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$
- Computing the array ***A*** of prefix averages of another array ***X*** has applications to financial analysis



Computing Prefix Averages (an example)

- The following algorithm computes prefix averages in quadratic time.

	Statements	S/E	Freq.	Total
1	**Algorithm** ***prefixAverages1***(***X, n***)	0	-	0
2	{	0	-	0
3	***A*** ← new array of ***n*** integers	1	n	n
4	**for** ***i*** ← 0 **to** ***n*** – 1 **do**	1	n+1	n+1
5	***s*** ← ***X***[0]	1	n	n
6	**for** ***j*** ← 1 **to** ***i*** **do**	1	1+2+3+… (n)	n(n+1)/2
7	***s*** ← ***s*** + ***X***[***j***]	1	1+2+3+… (n-1)	n(n-1)/2
8	***A***[***i***] ← ***s*** / (***i*** + 1)	1	n	n
9	**return** ***A***	1	1	1
10	}	0	-	0
				n^2+4n+2

Thus, Algorithm ***prefixAverages1*** is $O(n^2)$.

Computing Prefix Averages (an example)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

- The following algorithm computes prefix averages in quadratic time.

	Statements	S/E	Freq.	Total
1	**Algorithm** ***prefixAverages1***(***X, n***)	0	-	0
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3	***A*** ← new array of ***n*** integers	1	n	n
4	**for** ***i*** ← 0 **to** ***n*** – 1 **do**	1	n+1	n+1
5	***s*** ← ***X***[0]	1	n	n
6	**for** ***j*** ← 1 **to** ***i*** **do**	1	1+2+3+… (n)	n(n+1)/2
7	***s*** ← ***s*** + ***X***[***j***]	1	1+2+3+… (n-1)	n(n-1)/2
8	***A***[***i***] ← ***s*** / (***i*** + 1)	1	n	n
9	**return** ***A***	1	1	1
10	}	0	-	0
				n^2+4n+2

n

$\sum_{k=1}^{n} k$

Thus, Algorithm ***prefixAverages1*** is ***O***(***n***^{2}).

Computing Prefix Averages (an example)

- The following algorithm computes prefix averages in linear time by keeping a running sum.

	Statements	S/E	Freq.	Total
1	**Algorithm** ***prefixAverages2***(***X, n***)	0	-	0
2	{	0	-	0
3	***A*** ← new array of ***n*** integers	1	n	n
4	***s ← 0***	1	1	1
5	**for** ***i*** ← 0 **to** ***n*** – 1 **do**	1	n+1	n+1
6	***s ← s + X[i]***	1	n	n
7	***A*[*i*] ← *s* / (*i* + 1)**	1	n	n
8	**return** ***A***	1	1	1
9	}	0	-	0
				4n+3

Thus, Algorithm ***prefixAverages2*** is ***O*(*n*)**.

Computing Prefix Averages (an example)

- The following algorithm computes prefix averages in linear time by keeping a running sum.

	Statements	S/E	Freq.	Total
1	**Algorithm** ***prefixAverages2***(***X, n***)	0	-	0
2	{	0	-	0
3	***A*** ← new array of ***n*** integers	1	n	n
4	***s ← 0***	1	1	1
5	**for** ***i*** ← 0 **to** ***n*** – 1 **do**	1	n+1	n+1
6	**n** ***s ← s + X[i]***	1	n	n
7	***A[i] ← s / (i + 1)***	1	n	n
8	**return** ***A***	1	1	1
9	}	0	-	0
				4n+3

Thus, Algorithm ***prefixAverages2*** is ***O***(***n***).

Big-Oh From Smallest to Largest

$O(1)$		Constant
$O(\log n)$		Logarithmic
$O(n)$		Linear
$O(n \log n)$		n log n
$O(n^c)$	$O(n^2)$, $O(n^3)$, $O(n^{16})$, etc	Polynomial
$O(c^n)$	$O(1.6^n)$, $O(2^n)$, $O(3^n)$, etc	Exponential
$O(n!)$		Factorial
$O(n^n)$		n power n

Big-Oh Examples

O(1)	Push, Pop, Enqueue (if there is a tail reference), Dequeue, Accessing an array element
O(log n) غالبًا تطلع وقت ما نسحب على نص الداتا	Binary search
O(n)	Linear search
O(n log n)	Heap sort, Quick sort (average), Merge sort
$O(n^2)$	Selection sort, Insertion sort, Bubble sort
$O(n^3)$	Matrix multiplication
$O(2^n)$	Towers of Hanoi
O(n!)	All permutation of N elements
$O(n^n)$	….

consider the worst case scinario
to any algorithem we have

Revision

- Each countable step is weighted as 1, anything else is weighted 0 **(S/E)**
- Count the time each step is executed. This can be (1) time, constant time (5, 10, 21,…) or variable time (n, m, n+1, n^2,…) **(Freq)**
- Multiply **(S/E)** by **(Freq)** to get **(Total)**
- Having done all the above for each step, sum **(Total)** for each step together to get the complexity

Revision

- **for/while/do-while** are examples where there are repetition/frequency.
  - **for/while, <, ++ loops**
    - Freq = max – initial
  - **for/while, <=, ++ loops**
    - Freq = max – initial + 1
  - **for/while, >, -- loops**
    - Freq = initial – max
  - **for/while, >=, -- loops**
    - Freq = initial – max + 1

- **IMPORTANT: for/while** checking step/line should add +1 (for last check). Internal loop steps use the above formulas.

Revision

- **do-while, <, ++ loops**
  - Freq = 1+ max – initial
- **do-while, <=, ++ loops**
  - Freq = 1 + max – initial + 1
- **do-while, >, -- loops**
  - Freq = 1+ initial – max
- **do-while, >=, -- loops**
  - Freq = 1 + initial – max + 1

- **IMPORTANT:** both **do-while** checking step/line and internal loop steps use the above formulas.

Revision

- **In other word:**
  - ++
    - max – initial
  - --
    - initial – max
  - <=, >=
    - add (+1)
  - If, **do-while**
    - add (+1), for both checking and internal steps
  - Else (**for/while**)
    - add (+1) for checking step only.

- **IMPORTANT:** This only apply for incrementing/decrementing loops with simple checking (<,>,<=,>=). These are generalization (there are **many exceptions**)

Revision

- **One exception example:**

```
for (int i = 0; i < n; i++)
    for(int j = 0; j < i; j++)
        System.out.println(j)
```

  - In this case, the internal loop depends on the external loop (j < i). Therefore, the number of loops is changing each time.
    - The checking line will be executed 1+2+3+…+n
    - The *println* will be executed 0+1+2+…+(n-1)

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2},$$

  - This is an example/approximation for

- Such cases require careful counting

Revision

- **Big-Oh for a function f(n)**
  - **Drop lower-order terms**
  - **Drop constant factors**
  - **Use the smallest possible class**
  - **Use the simplest expression of the class**

- Or, Find highest order term, and drop everything else (including constants)

Revision

- **Proving f(n) is O(g(n))**
  - f(n) is O(g(n)) if there are positive constants c and n_0 such that:

$$f(n) \leq cg(n) \quad \text{for } n \geq n_0$$

- Or:
  - Drop negative terms
  - Upgrade lower-order terms to the same level of the highest-order term level
  - Work it from there to calculate c and n_0

Excercises

a).

```
s = 0;
for (i = 1; i < n-1; i++)
  s = s + 1;
```

b).

```
s = 0;
for (i = n-5; i > 4; i--)
  s = s + i;
```

Excercises

a).

```
s = 0;                          1
for (i = 1; i < n-1; i++)       n-1
  s = s + 1;                    n-2
```

Total: 2n-2 **O(n)**

b).

```
s = 0;
for (i = n-5; i > 4; i--)
  s = s + i;
```

Excercises

a).

s = 0;	**1**	
for (i = 1; i < n-1; i++)	**n-1**	
s = s + 1;	**n-2**	
	Total: 2n-2	**O(n)**

b).

s = 0;	**1**	
for (i = n-5; i > 4; i--)	**n-8**	
s = s + i;	**n-9**	
	Total: 2n-16	**O(n)**

Excercises

c).

```
s = 0;
for (i = 1; i <= n; i++)
  for (j = 1; j <= n; j++)
       s = s + 1;
```

d).

```
i = 0;
while (i <= 10)
  i = i + 1;
```

Excercises

c).

`s = 0;`	**1**	
`for (i = 1; i <= n; i++)`	**n+1**	
`  for (j = 1; j <= n; j++)`	**n(n+1)**	
`      s = s + 1;`	**n^2**	
	Total: $2n^2+2n+2$	**$O(n^2)$**

d).

```
i = 0;
while (i <= 10)
  i = i + 1;
```

Excercises

c).

`s = 0;`		1	
`for (i = 1; i <= n; i++)`	$n+1$		
`  for (j = 1; j <= n; j++)`	$n(n+1)$		
`      s = s + 1;`		n^2	
	Total:	$2n^2+2n+2$	$O(n^2)$

d).

`i = 0;`		1	
`while (i <= 10)`		12	
`  i = i + 1;`	11		
	Total:	24	$O(1)$

Excercises

e).

```
s = 0;
for (i = 1; i <= n; i++)
  for (j = 0; j <= n; j++)
      for (k = 0; k <= n; k++)
            s = s + 1;
```

Excercises

e).

`s = 0;`	**1**
`for (i = 1; i <= n; i++)`	**n+1**
`  for (j = 0; j <= n; j++)`	**n(n+2)**
`      for (k = 0; k <= n; k++)`	**n(n+1)(n+2)**
`            s = s + 1;`	**n(n+1)(n+1)**
	Total: $O(n^3)$

Excercises

f).

```
s = 0;
for (i = 0; i <= n; i++)
  for (j = 0; j < i; j++)
      s = s + 1;
```

g).

```
s = 0;
for (i = 0; i <= n; i++)
  for (j = i+1; j <= n; j++)
      s = s + 1;
```

Excercises

f).

`s = 0;`		**1**	
`for (i = 0; i <= n; i++)`	**n+2**		
`  for (j = 0; j < i; j++)`		**1+2+...+(n+1)**	**~n(n+1)/2**
`      s = s + 1;`		**1+2+...+n**	**=n(n+1)/2**
	Total:	**$O(n^2)$**	

g).

```
s = 0;
for (i = 0; i <= n; i++)
  for (j = i+1; j <= n; j++)
      s = s + 1;
```

Excercises

f).

`s = 0;`		**1**	
`for (i = 0; i <= n; i++)`	**n+2**		
`  for (j = 0; j < i; j++)`		**1+2+...+(n+1)**	**~n(n+1)/2**
`      s = s + 1;`		**1+2+...+n**	**=n(n+1)/2**
	Total:	**$O(n^2)$**	

g).

`s = 0;`		**1**	
`for (i = 0; i <= n; i++)`	**n+2**		
`  for (j = i+1; j <= n; j++)`	**(n+1)+n+...+2+1**		**~n(n+1)/2**
`      s = s + 1;`		**n+(n-1)+...+2+1**	**=n(n+1)/2**
	Total:	**$O(n^2)$**	