CSC 311 – Winter 2022-2023
Analysis and Design of Algorithms
8. Graph Algorithms

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Outline

- Representing graphs
- Graph searching
- Breadth-First Search
- Depth-First Search

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Graphs

- A graph $G = (V, E)$
  - V = set of vertices
  - E = set of edges = subset of $V \times V$
  - Thus $|E| = O(|V|^2)$

Graph Variations

- Variations:
  - A *connected graph* has a path from every vertex to every other
  - In an *undirected graph*
    - edge (u,v) = edge (v,u)
    - No self-loops
  - In a *directed* graph:
    - edge (u,v) goes from vertex u to vertex v, notated $u \rightarrow v$

Graph Variations

- More variations:
  - A *weighted graph* associates weights with either the edges or the vertices
    - e.g., a road map: edges might be weighted w/ distance
  - A *multigraph* allows multiple edges between the same vertices
    - e.g., the call graph in a program (a function can get called from multiple points in another function)

Graphs

- We will typically express running times in terms of $|E|$ and $|V|$ (often dropping the |'s)
  - If $|E| \approx |V|^2$ the graph is *dense*
  - If $|E| \approx |V|$ the graph is *sparse*
- If you know you are dealing with dense or sparse graphs, different data structures may make sense

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Representing Graphs

- Assume $V = \{1, 2, \ldots, n\}$
- An *adjacency matrix* represents the graph as a n x n matrix A:
  - $A[i, j]$ = 1 if edge $(i, j) \in E$ (or weight of edge)
    = 0 if edge $(i, j) \notin E$

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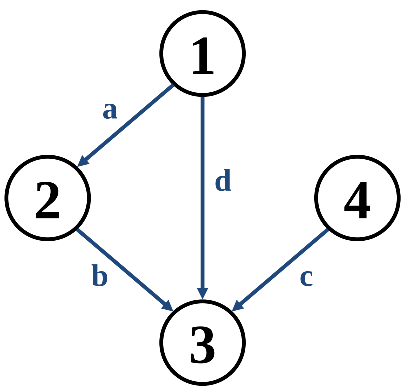
Graphs: Adjacency Matrix

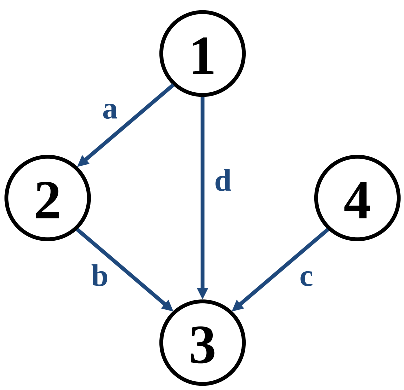
- Example:



A	1	2	3	4
1				
2				
3			??	
4				

Graphs: Adjacency Matrix

- Example:



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

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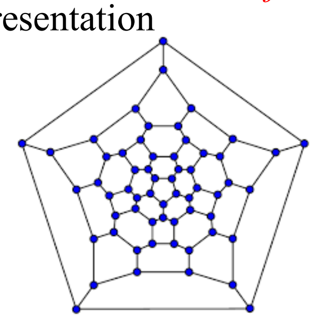
Graphs: Adjacency Matrix

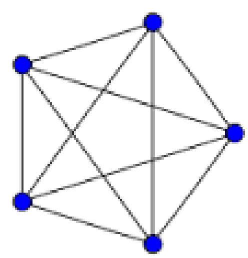
- *How much storage does the adjacency matrix require?*
- A: $O(V^2)$
- *What is the minimum amount of storage needed by an adjacency matrix representation of an undirected graph with 4 vertices?*
- A: 6 bits
  - Undirected graph $\rightarrow$ matrix is symmetric
  - No self-loops $\rightarrow$ don't need diagonal

Graphs: Adjacency Matrix

- The adjacency matrix is a dense representation
  - Usually too much storage for large graphs
  - But can be very efficient for small graphs
- Most large interesting graphs are sparse
  - e.g., planar graphs, in which no edges cross, have $|E| = O(|V|)$ by Euler's formula
  - For this reason the *adjacency list* is often a more appropriate representation



Planar graph



Non planar graph

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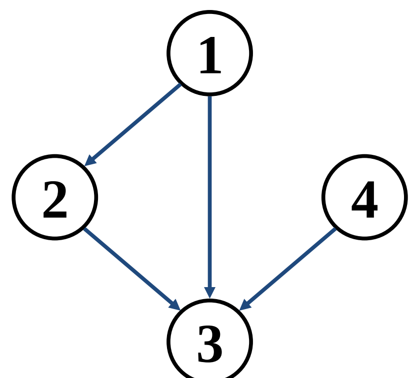
Graphs: Adjacency List

- Adjacency list: for each vertex $v \in V$, store a list of vertices adjacent to v
- Example:
  - Adj[1] = {2,3}
  - Adj[2] = {3}
  - Adj[3] = {}
  - Adj[4] = {3}
- Variation: can also keep a list of edges coming *into* vertex



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Graphs: Adjacency List

- How much storage is required?
  - The *degree* of a vertex v = # incident edges
    - Directed graphs have in-degree, out-degree
  - For directed graphs, # of items in adjacency lists is
    Σ out-degree(v) = $|E|$
    takes $\Theta(V + E)$ storage
  - For undirected graphs, # items in adjacency lists is
    Σ degree(v) = 2 $|E|$ (*handshaking lemma*)
    also $\Theta(V + E)$ storage
  - (In a party of people some of whom shake hands, an even number of people must have shaken an odd number of other people's hands).
- So: Adjacency lists take $O(V+E)$ storage

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Graph Searching

- Given: a graph $G = (V, E)$, directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree

Breadth-First Search

- "Explore" a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find ("discover") its children, then their children, etc.

Breadth-First Search

- Will associate vertex colors to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Grey vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - Black vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of grey vertices

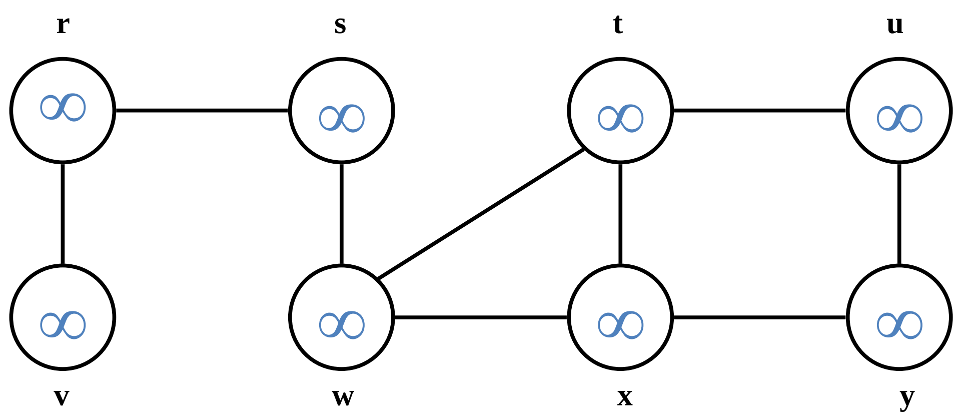
Breadth-First Search

```
BFS(G, s) {
   initialize vertices;
   Q = {s};                    // Q is a FIFO queue; initialize to s
   while (Q not empty) {
      u = Dequeue(Q);
      for each v ∈ u->adj {
         if (v->color == WHITE){
            v->color = GREY;
            v->d = u->d + 1;
            v->p = u;
            Enqueue(Q, v);}
      }
      u->color = BLACK;
   }
}
```

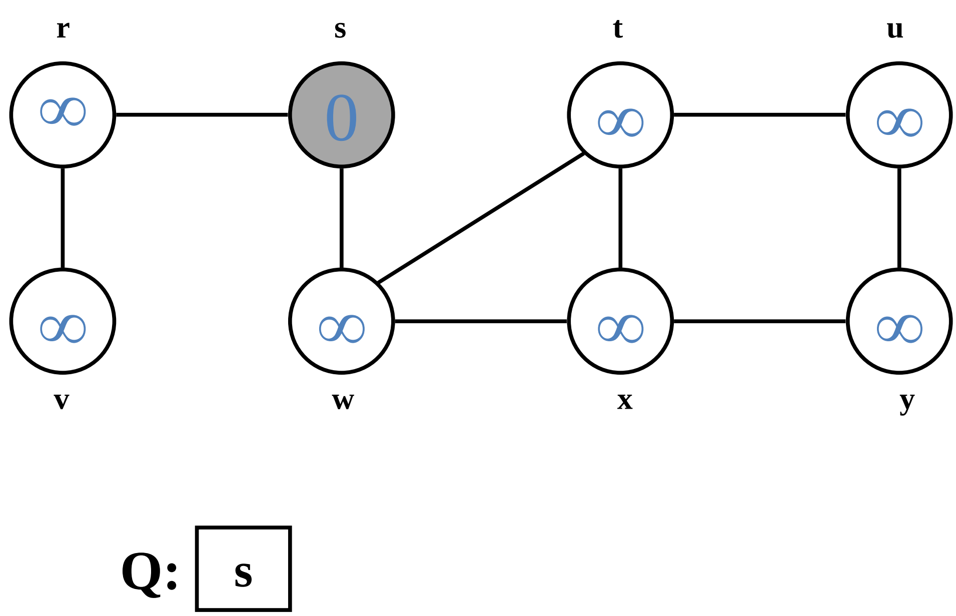
What does `v->d` represent?

What does `v->p` represent?

Breadth-First Search: Example



Breadth-First Search: Example



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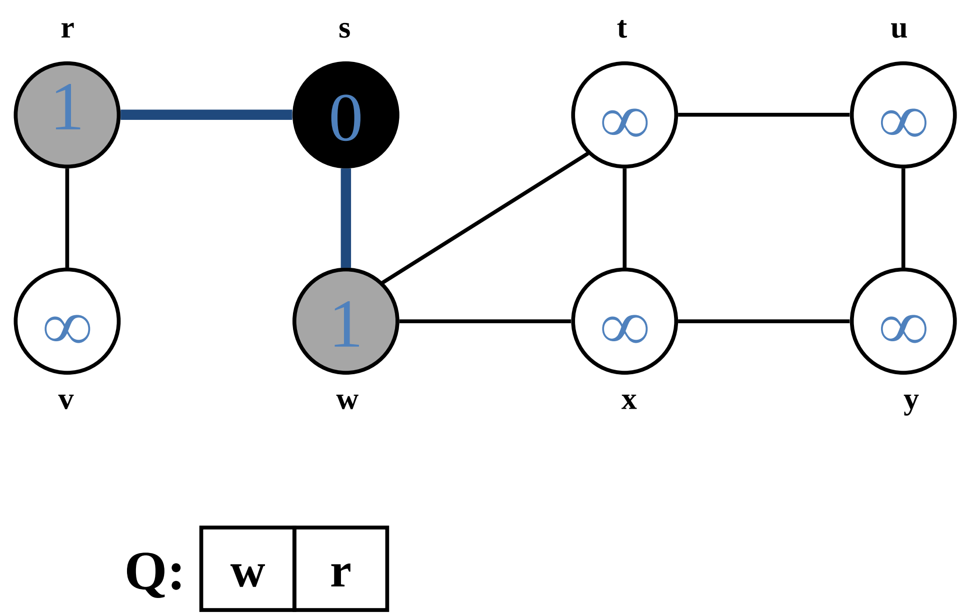
Breadth-First Search: Example



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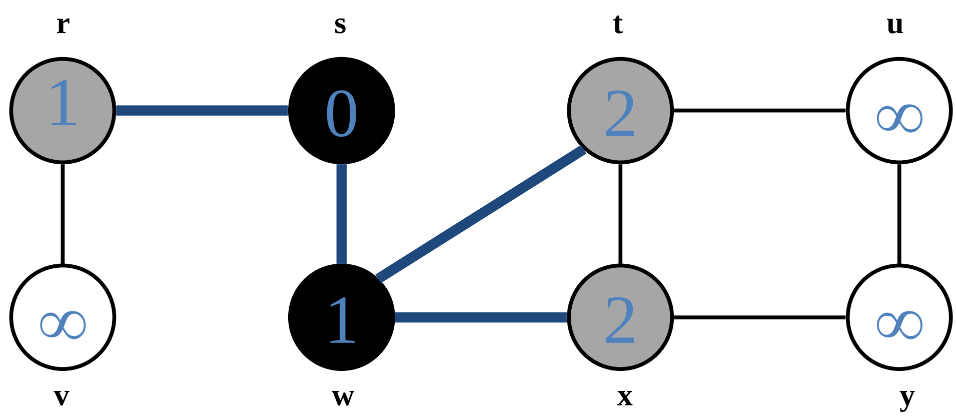
Breadth-First Search: Example



Q:	r	t	x

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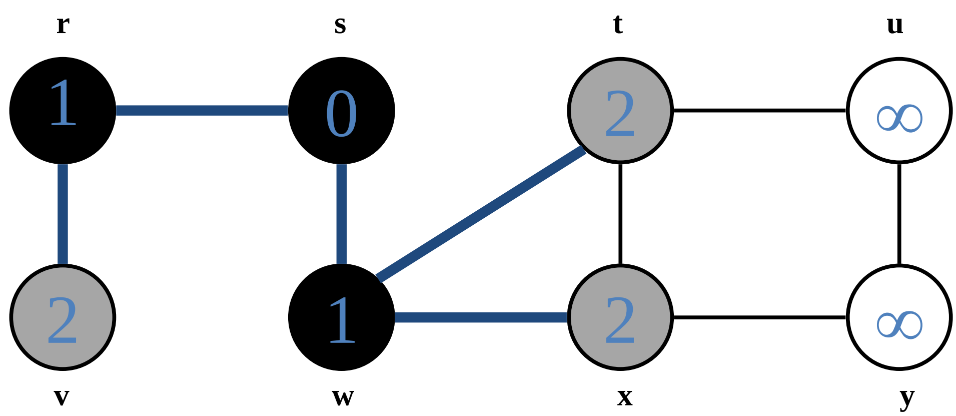
Breadth-First Search: Example



Q: | t | x | v |

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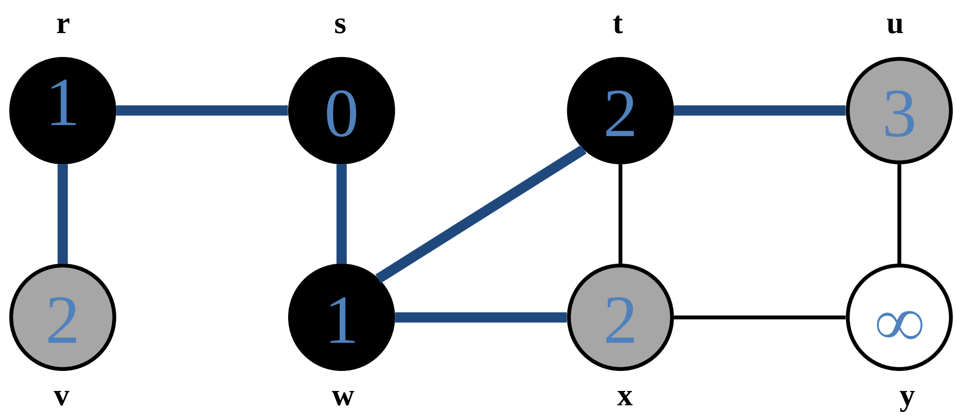
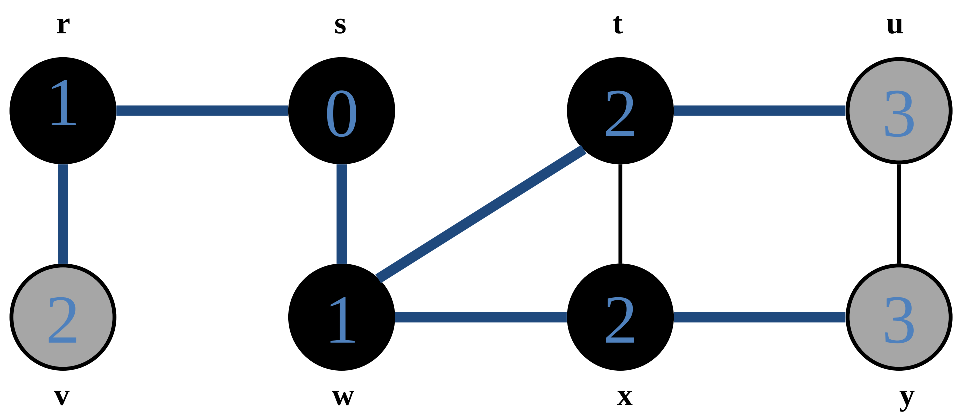
Breadth-First Search: Example



Q:

x	v	u

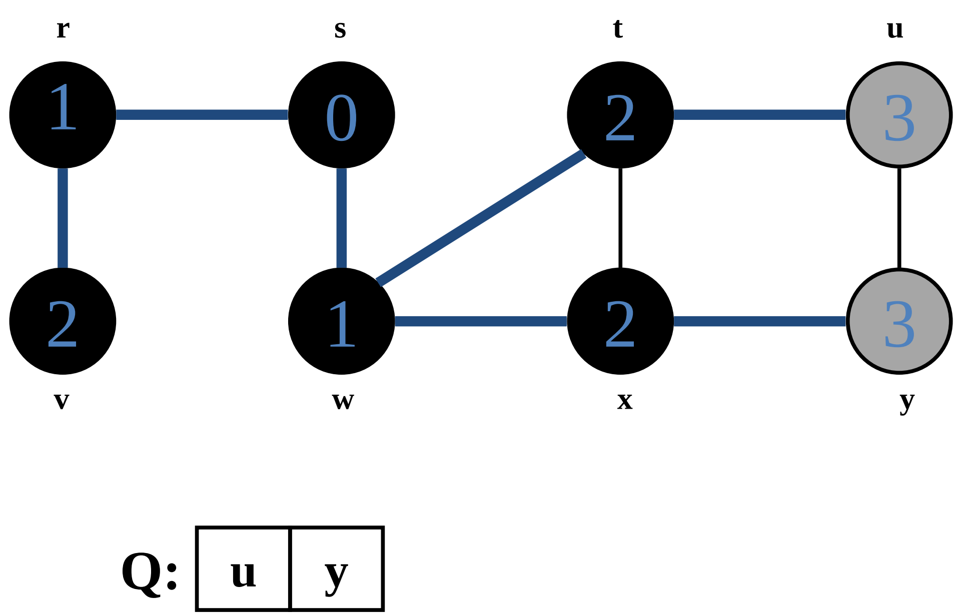
Breadth-First Search: Example



Q:

v	**u**	**y**

Breadth-First Search: Example



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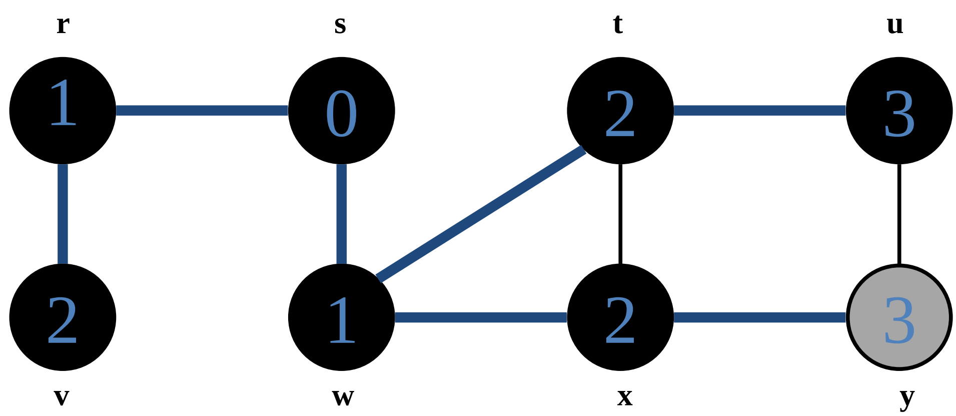
Breadth-First Search: Example



Q: y

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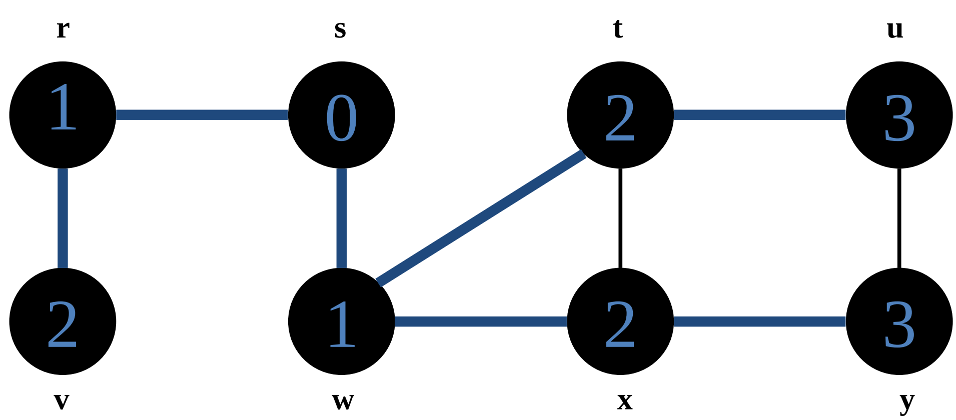
Breadth-First Search: Example



Q: Ø

BFS: The Code Again

```
BFS(G, s) {
   initialize vertices;
   Q = {s};
   while (Q not empty) {
      u = Dequeue(Q);
      for each v ∈ u->adj {
         if (v->color == WHITE)
            v->color = GREY;
            v->d = u->d + 1;
            v->p = u;
            Enqueue(Q, v);
      }
      u->color = BLACK;
   }
}
```

Touch every vertex: $O(V)$

u = every vertex, but only once

v = every vertex that appears in some other vertex's adjacency list

What will be the running time?

Total running time: $O(V+E)$

BFS: The Code Again

```
BFS(G, s) {
   initialize vertices;
   Q = {s};
   while (Q not empty) {
      u = RemoveTop(Q);
      for each v ∈ u->adj {
         if (v->color == WHITE)
            v->color = GREY;
            v->d = u->d + 1;
            v->p = u;
            Enqueue(Q, v);
      }
      u->color = BLACK;
   }
}
```

What will be the storage cost in addition to storing the tree?

Total space used:
$O(\max(\text{degree}(v))) = O(E)$

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source node
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - We can use BFS to calculate shortest path from one vertex to another in $O(V+E)$ time

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore deeper in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex v that still has unexplored edges
  - When all of v's edges have been explored, backtrack to the vertex from which v was discovered

Depth-First Search

- Vertices initially colored white
- Then colored gray when discovered
- Then black when finished

Depth-First Search: The Code

```
DFS(G)
{
  for each vertex u ∈ G->V
  {
    u->color = WHITE;
  }
  time = 0;
  for each vertex u ∈ G->V
  {
    if (u->color == WHITE)
      DFS_Visit(u);
  }
}
```

```
DFS_Visit(u)
{
  u->color = GREY;
  time = time+1;
  u->d = time;
  for each v ∈ u->Adj[]
  {
    if (v->color == WHITE)
      DFS_Visit(v);
  }
  u->color = BLACK;
  time = time+1;
  u->f = time;
}
```

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Depth-First Search: The Code

```
DFS(G)
{
  for each vertex u ∈ G->V
  {
    u->color = WHITE;
  }
  time = 0;
  for each vertex u ∈ G->V
  {
    if (u->color == WHITE)
      DFS_Visit(u);
  }
}
```

```
DFS_Visit(u)
{
  u->color = GREY;
  time = time+1;
  u->d = time;
  for each v ∈ u->Adj[]
  {
    if (v->color == WHITE)
      DFS_Visit(v);
  }
  u->color = BLACK;
  time = time+1;
  u->f = time;
}
```

What does u->d represent?

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Depth-First Search: The Code

```
DFS(G)
{
  for each vertex u ∈ G->V
  {
    u->color = WHITE;
  }
  time = 0;
  for each vertex u ∈ G->V
  {
    if (u->color == WHITE)
      DFS_Visit(u);
  }
}
```

```
DFS_Visit(u)
{
  u->color = GREY;
  time = time+1;
  u->d = time;
  for each v ∈ u->Adj[]
  {
    if (v->color == WHITE)
      DFS_Visit(v);
  }
  u->color = BLACK;
  time = time+1;
  u->f = time;
}
```

What does `u->f` represent?

Depth-First Search: The Code

- u->d: timestamp that records when u is first discovered (and then grayed).
- u->f: timestamp that records when the search finishes examining adjacency list of u (and blackens)

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Depth-First Search: The Code

```
DFS(G)
{
  for each vertex u ∈ G->V
  {
    u->color = WHITE;
  }
  time = 0;
  for each vertex u ∈ G->V
  {
    if (u->color == WHITE)
      DFS_Visit(u);
  }
}
```

```
DFS_Visit(u)
{
  u->color = GREY;
  time = time+1;
  u->d = time;
  for each v ∈ u->Adj[]
  {
    if (v->color == WHITE)
      DFS_Visit(v);
  }
  u->color = BLACK;
  time = time+1;
  u->f = time;
}
```

What will be the running time?

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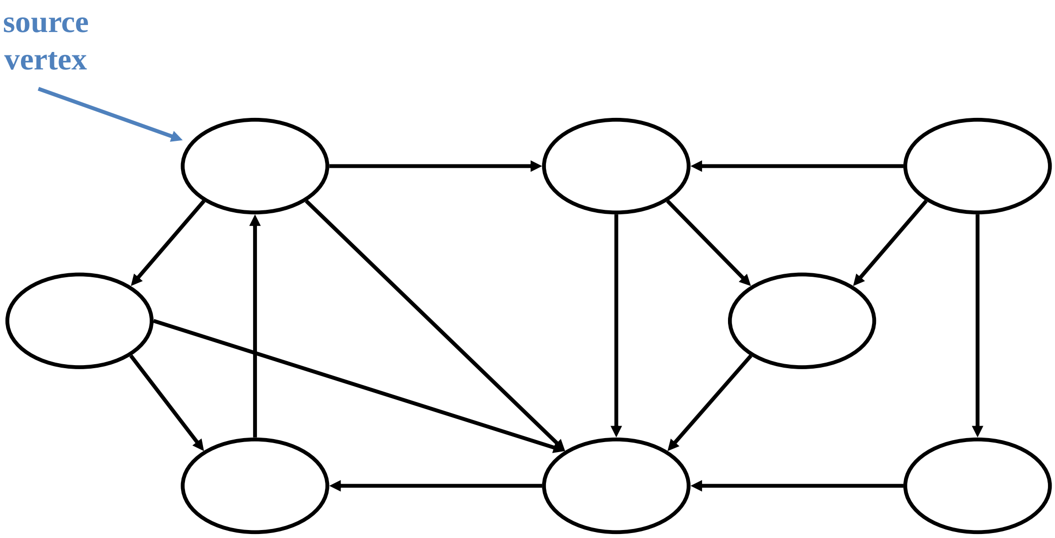
Depth-First Search Analysis

- Running time:
  - The exploration of edge to edge:
    - Each loop in DFS_Visit can be attributed to an edge in the graph
    - Runs once/edge if directed graph, twice if undirected
    - Thus loop will run in $O(E)$ time, algorithm $O(V+E)$

DFS Example



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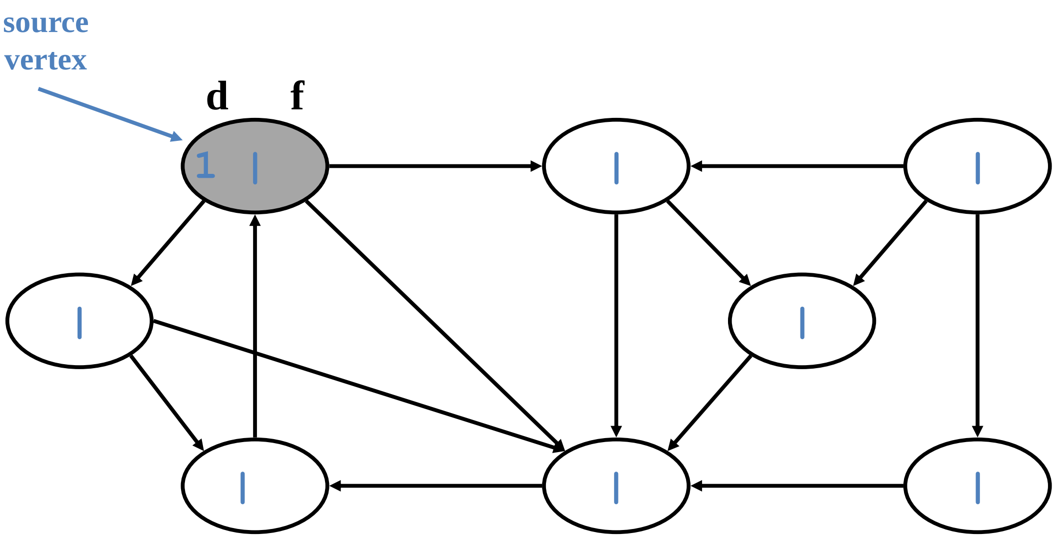
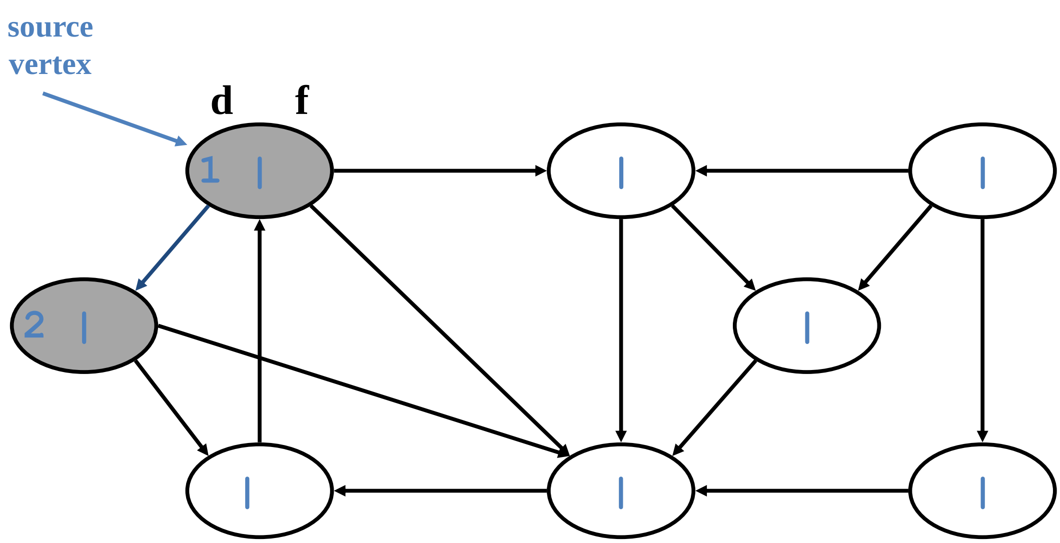
DFS Example



DFS Example



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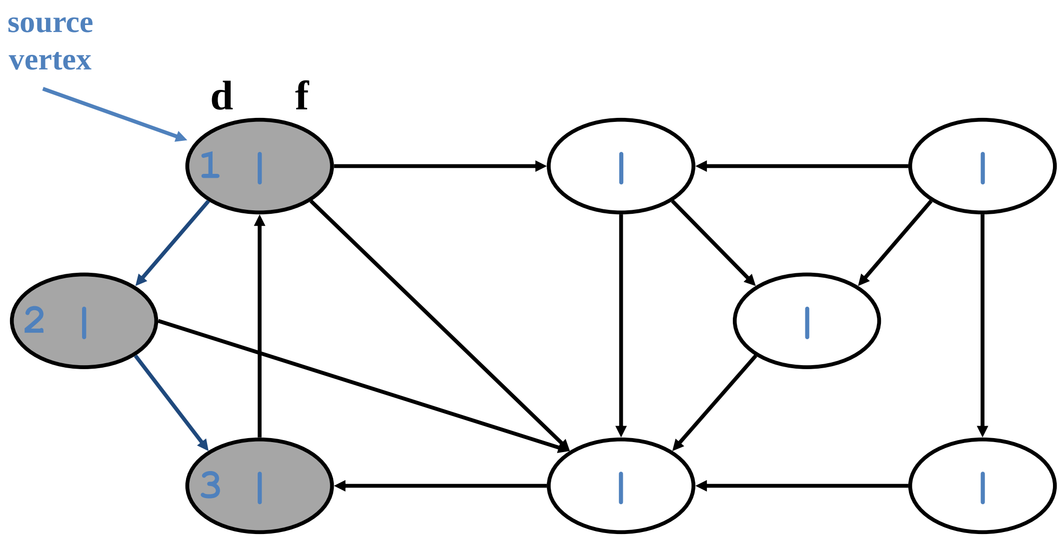
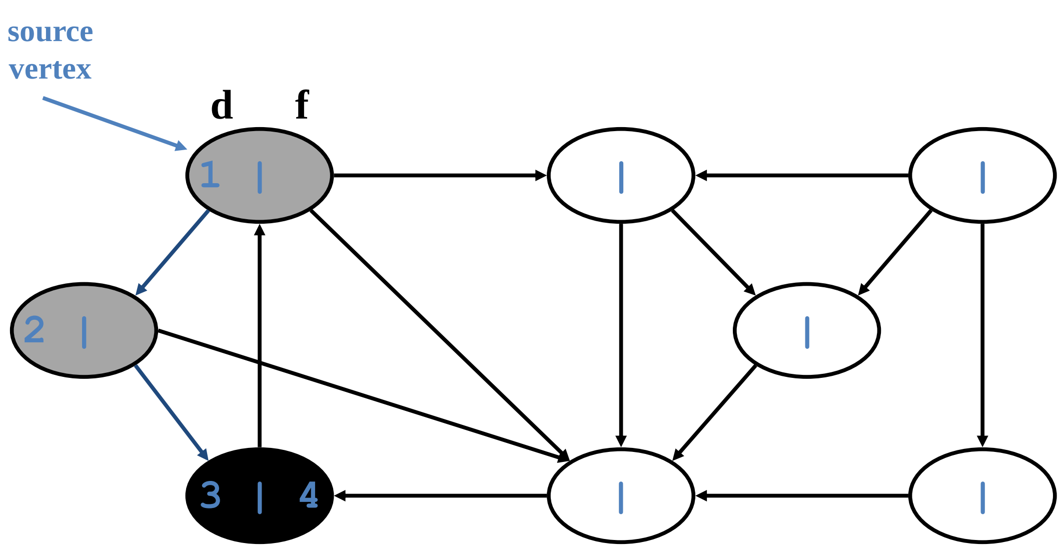
DFS Example



DFS Example



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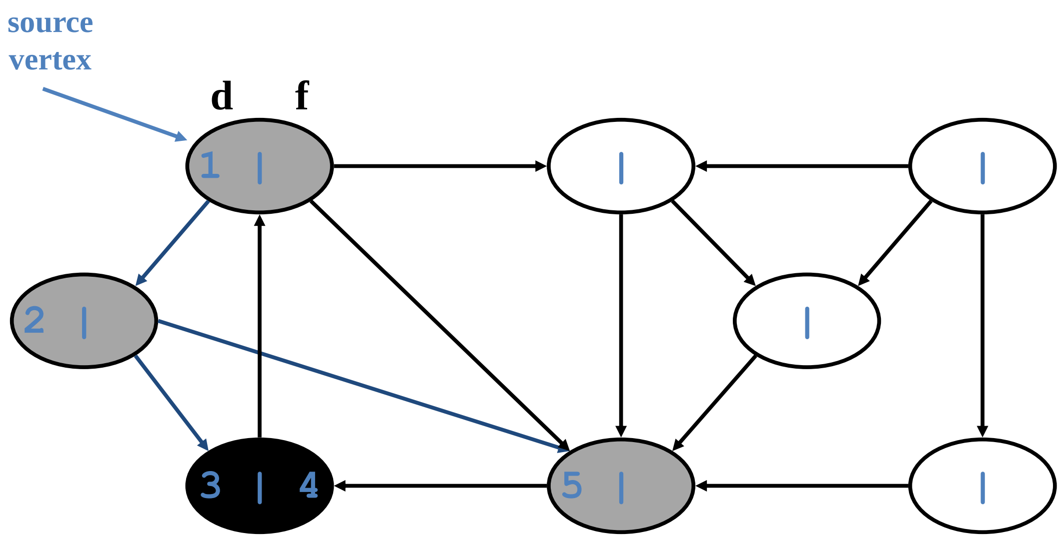
DFS Example



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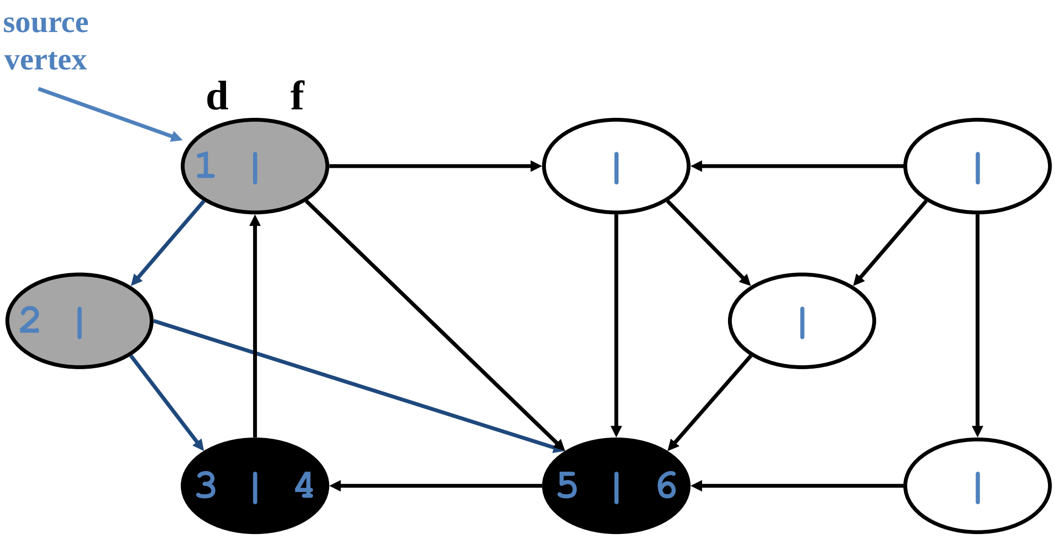
DFS Example



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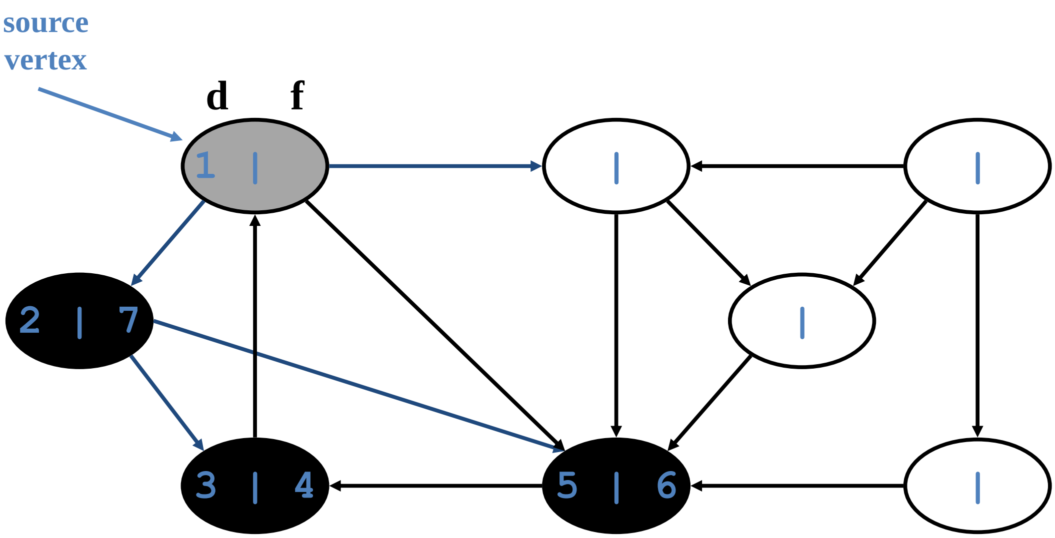
DFS Example



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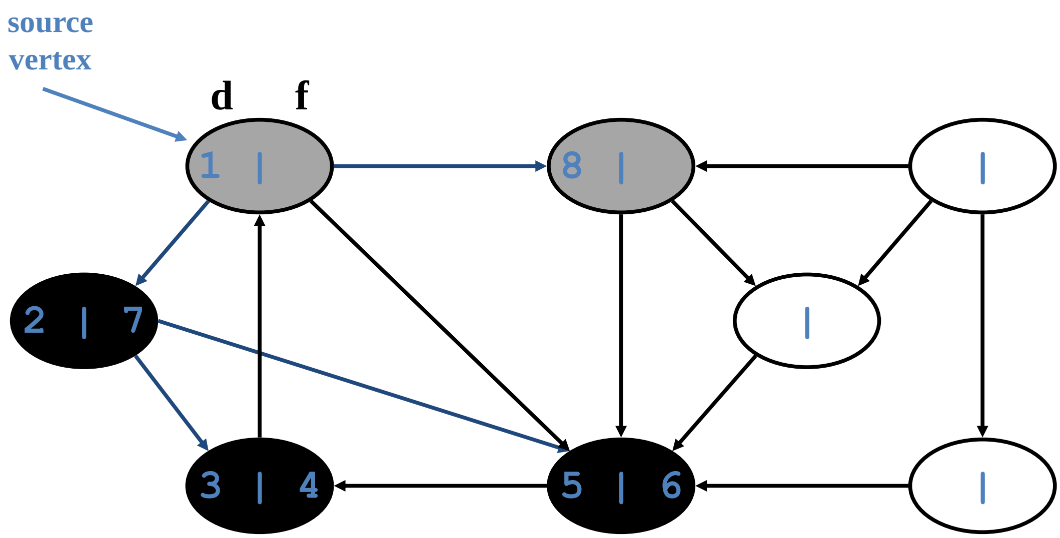
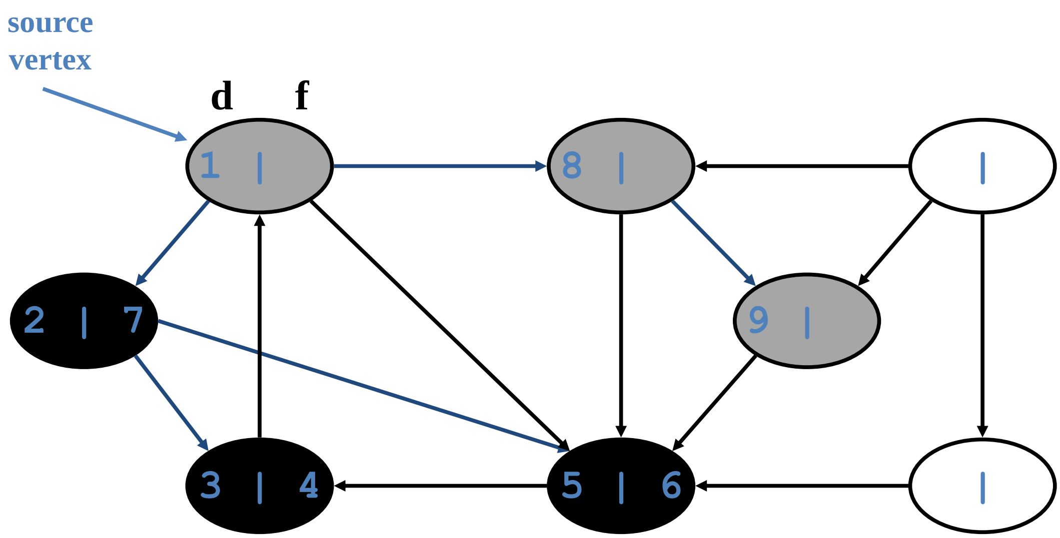
DFS Example



DFS Example



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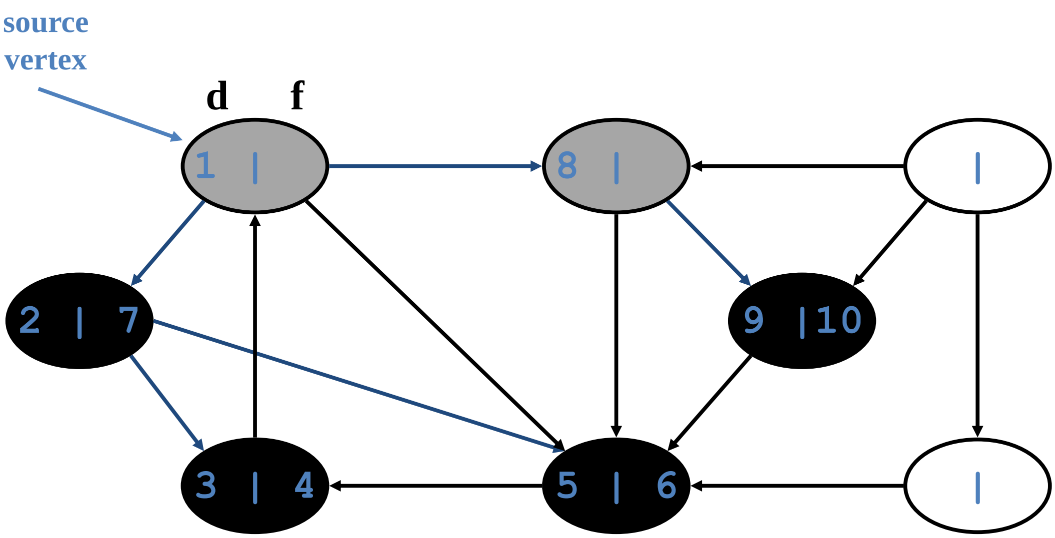
DFS Example



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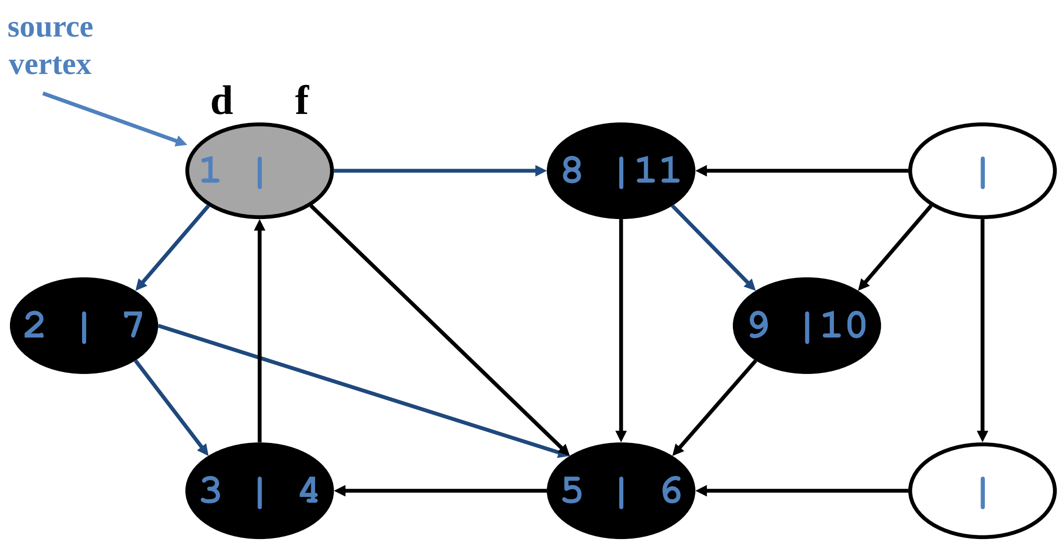
DFS Example



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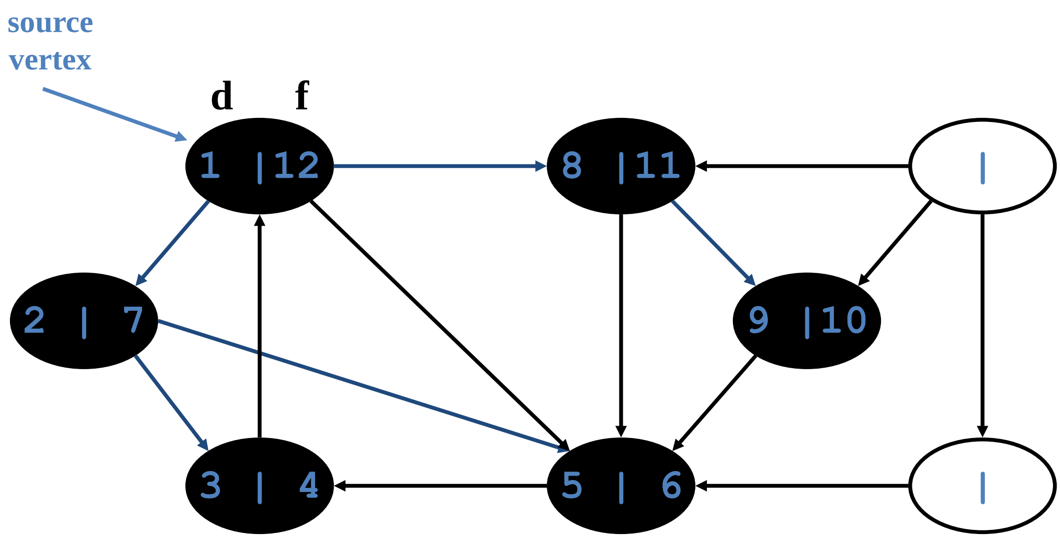
DFS Example



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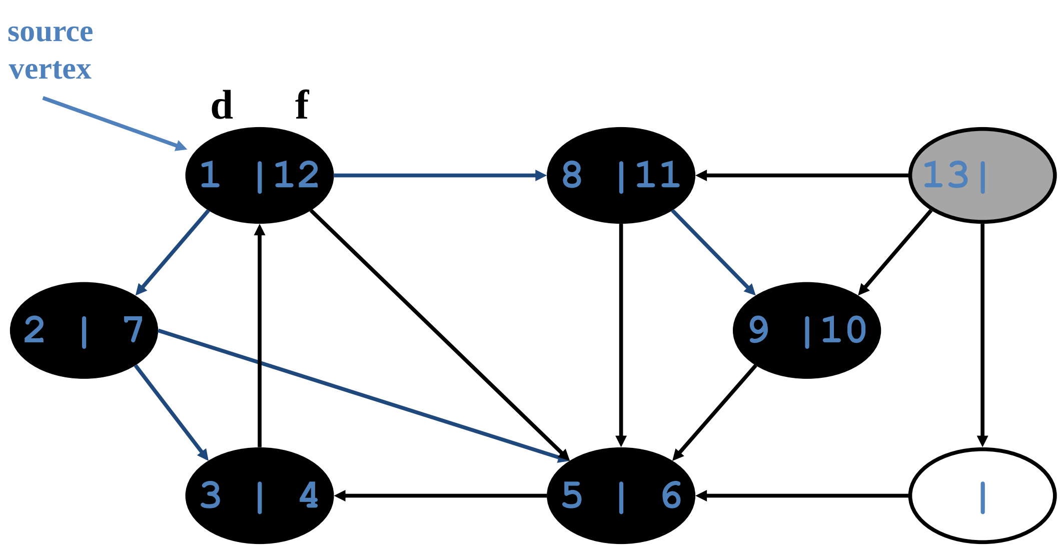
DFS Example



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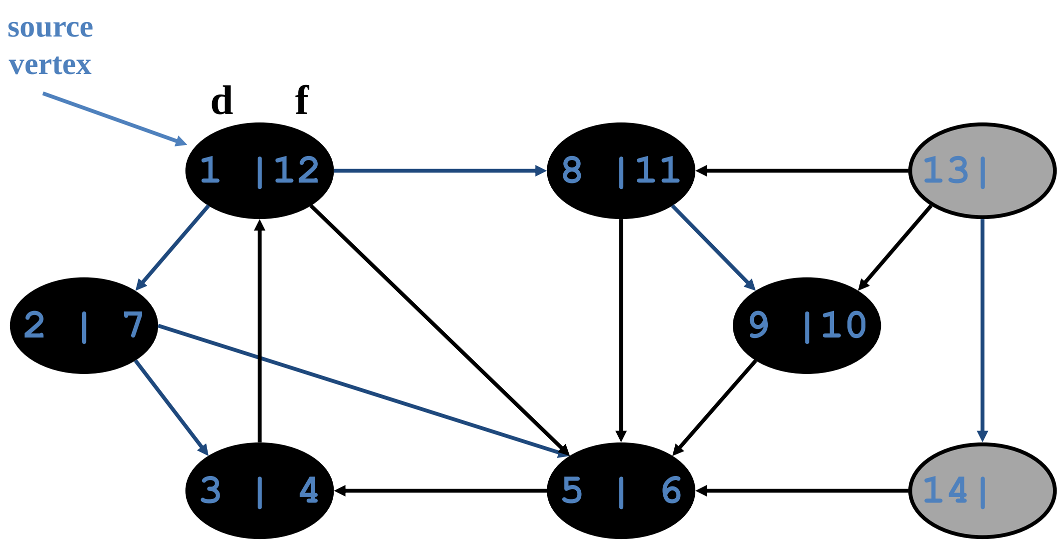
DFS Example



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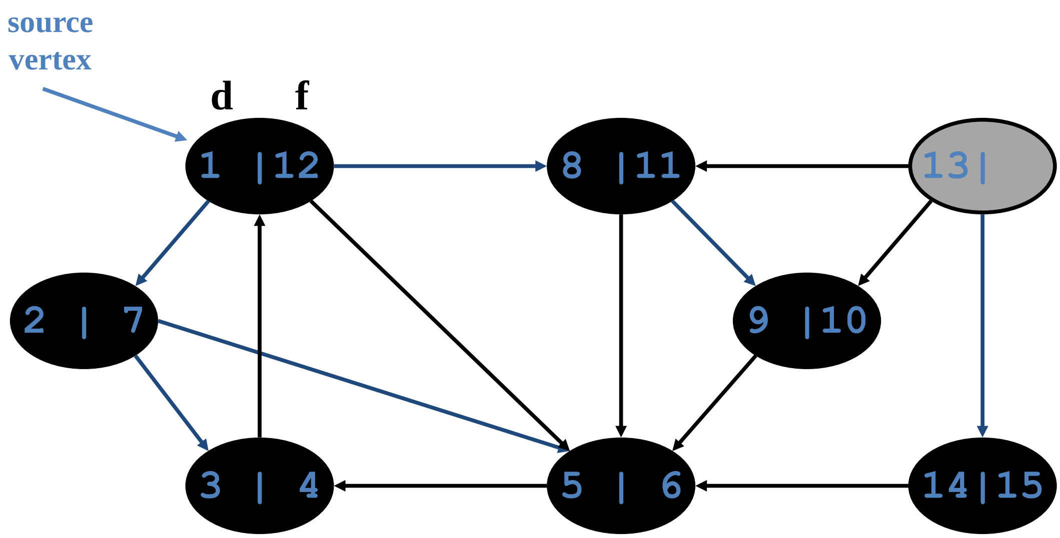
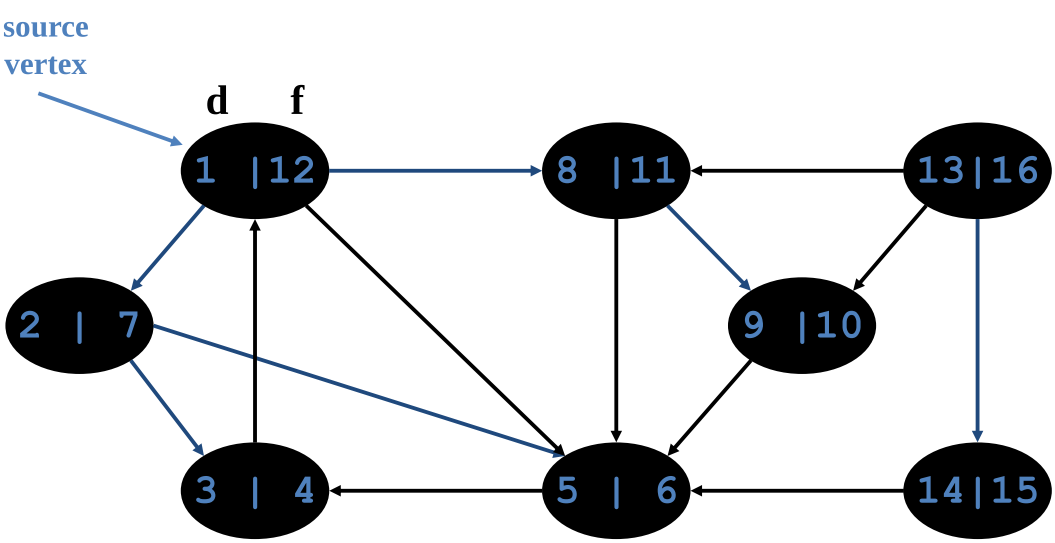
DFS Example



DFS Example



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Reading

Chapter 5 (Sections 5.2, 5.3)
Anany Levitin, Introduction to the design and analysis of algorithms, 3rd Edition, Pearson, 2011.

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