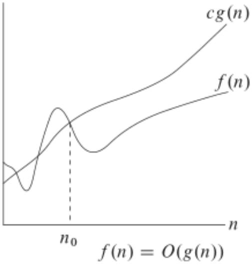
INSERTION-SORT(A)	*cost*	*times*
for $j = 2$ **to** $A.length$	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

$$T(n) = (c_1 + c_2 + c_4 + c_8)n + c_5 \sum_{j=2}^{n} t_j + (c_6 + c_7) \sum_{j=2}^{n} (t_j - 1)$$
$$- (c_2 + c_4 + c_8)$$

$$
\begin{aligned}
T_b(n) &= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8) \\
T_w(n) &= (\frac{c_5}{2} + \frac{c_6}{2} + \frac{c_7}{2})n^2 + (c_1 + c_2 + c_4 + \frac{c_5}{2} - \frac{c_6}{2} - \frac{c_7}{2} + c_8)n \\
&\quad - (c_2 + c_4 + c_5 + c_8) \\
T_{avg}(n) &= (\frac{c_5}{4} + \frac{c_6}{4} + \frac{c_7}{4})n^2 + (c_1 + c_2 + c_4 + \frac{c_5}{4} - \frac{c_6}{4} - \frac{c_7}{4} + c_8)n \\
&\quad - (c_2 + c_4 + \frac{c_5}{2} + c_8)
\end{aligned}
$$

- $T(n)$ grows *slower* than something.
- $T(n)$ grows *faster* than something.
- $T(n)$ grows *similarily* to something.



We represent that $f(n)$ grows *slower (or equal)* to $g(n)$ by the notation $\mathcal{O}(g(n))$.

Definition

We say a function $f(n) = \mathcal{O}(g(n))$ if
$\exists$ real $c > 0$ and integer $n_0 \geq 0$ such that:

$$f(n) \leq cg(n) \qquad\qquad \forall n \geq n_0$$

- $g(n)$ is an *asymptotic* upper bound for $f(n)$

Set definition (CLRS) :

Definition

The notation $O(g(n))$ represents the set of functions:

$$O(g(n)) = \{f(n) : \exists \text{ positive constants } c \text{ and } n_0 \text{ such that } \\ 0 \leq f(n) \leq cg(n)\ \forall n \geq n_0\}$$

- $f(n) = \frac{1}{2}n^2 + 3n$

- $f(n) = 100n + 5$
- Is $f(n) = \mathcal{O}(n)$?
- Is $f(n) = \mathcal{O}(n^2)$?

- $f(n) = 2n\ lg(n+2)^2 + (n+2)^2 lg\frac{n}{2}$
- $O(?)$

- $f(n) = 2n^3 - 8n^2 + n$
- $O(?)$

- $(3n^2 + 6)^5$

- $\sqrt{3n^4 + 2n - 6}$

- $2^{n+3} + 7^{n-2}$

- $f(n) = n! + 5n^3 + 2^n$

- $f(n) = lg(n!)$

Theorem

Suppose $f_1(n) = O(g_1(n))$ *and* $f_2(n) = O(g_2(n))$. *Then,* $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$.

Proof:

- Limit-based definition of O:

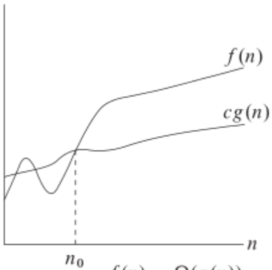
Theorem

$$lim_{n\to\infty}\frac{f(n)}{g(n)} = \begin{cases} 0 & then,\ f(n) = O(g(n)) \\ c & then,\ f(n) = O(g(n))\ \&\ g(n) = O(f(n)) \\ \infty & then,\ g(n) = O(f(n)) \end{cases}$$

- First two cases mean $f(n) = O(g(n))$
- Last two cases mean $g(n) = O(f(n))$

- $lg(n)$ vs $\sqrt{n}$
- Two Proofs.

- $n!$ vs 2^n



$f(n) = \Omega(g(n))$

We use the notation $\Omega(n)$ to say that $f(n)$ grows faster (or equal) to $g(n)$

Definition

We say a function $f(n) = \Omega(g(n))$ if
$\exists$ real $c > 0$ and integer $n_0 \geq 0$ such that:

$$f(n) \geq cg(n) \qquad \forall n \geq n_0$$

- $g(n)$ is an *asymptotic* lower bound for $f(n)$.

- $f(n) = \frac{1}{2}n^2 + 3n$

- $f(n) = 3n\ lgn + 2n$

- $f(n) = 3n \lg n - n$

- $f(n) = 4 \lg n - 8$

- $f(n) = 3n \lg n - 3n$

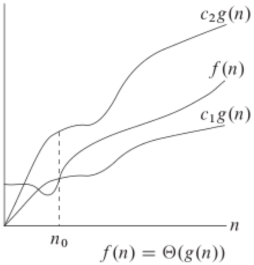
- $f(n) = 6n^4 - 3n^2 + 2n + 7$

- $f(n) = 6n^4 - 4n^2 - 3n + 2$

- Show that $n! = \Omega(2^n)$

- Show that $\lg(n!) = \Omega(n \lg n)$

- $f(n) = 2n^4 - 5n^2 + 15$
- Is $f(n) = \Omega(n^2)$?
- Can we prove that?



If $g(n)$ is *both* an upper and lower asymptotic bound for $f(n)$, we say that $f(n) = \Theta(g(n))$

Definition

We say a function $f(n)$ is in $\Theta(g(n))$ if :

- $\exists$ real constants $c_1, c_2 > 0$
- and integer constant $n_0 \geq 0$.

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \qquad n \geq n_0$$

Theorem

$f(n) = \Theta(g(n))$ *if and only if* $f(n) = \mathcal{O}(g(n))$ *and* $f(n) = \Omega(g(n))$

- Proof should be trivial.

- Show that $f(n) = 3n^2 - n + 4 = \Theta(n^2)$
- Is it also $\Theta(n^3)$?
- Is it also $\Theta(n)$?

- Prove or disprove: $5 \lg(2n^2 - 4n + 2) = \Theta(\lg n)$

- Is $n^3 - 2n^2 + 5n = \Theta(n^4)$
- Prove it.

- Is $n^3 - 2n^2 + 5n = \Theta(n^2 \lg n)$
- Prove it.

- Show that $(n-4)^2 \lg(\lfloor \frac{n}{3} \rfloor) \neq \Theta(n \lg n)$

- Limits, updated:

Theorem

$$lim_{n\to\infty}\frac{f(n)}{g(n)} = \begin{cases} 0 & then,\ f(n) = \mathcal{O}(g(n)) \\ c & then,\ f(n) = \Theta(g(n)) \\ \infty & then,\ f(n) = \Omega(g(n)) \end{cases}$$

- First two cases mean $f(n) = \mathcal{O}(g(n))$
- Last two cases mean $f(n) = \Omega(g(n))$