Basic Structures: Sets, Functions, Sequences, and Summation

Chapter 2

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Sequences and Summations

Section 2.4

Section Summary

Sequences

- Arithmetic Sequence
- Geometric Sequence
- Recurrence Relations
  - Example: Fibonacci Sequence

Summations

Sequences

Definition: A ***sequence*** is a function ***f*: *A*→*B*,** where ***A*** is a subset of $\mathbb{Z}$**.**

- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{\mathbf{1}}{\mathbf{5}} \ldots\ldots$ $\boxed{a_n = \frac{1}{n}}$
- 1, 3, 9, 27, 81, 243 ……. $\boxed{a_n = 3^n}$

$\boldsymbol{a_n}$ is called <u>*term* of the sequence</u>.

$\boldsymbol{n}$ is the index.

$\boldsymbol{\{a_n\}}$ is the entire sequence.

Arithmetic Sequence

Definition: An arithmetic sequence has a <u>constant difference</u> between each term.

$$a, a+d, a+2d, \ldots, \boldsymbol{a+nd}$$

Examples: Consider the following sequences. What is the term of the sequence (a_n)?

1. $\{1,3,5,7,9,\ldots\}$
   - $a=1, d=2$ $\quad \boxed{a_n = 1 + 2n}$
2. $\{7,4,1,-2,-5,\ldots\}$
   - $a=7, d=-3$ $\quad \boxed{a_n = 7 - 3n}$

Geometric Sequence

Definition: A geometric sequence has a constant ratio between each term.

$$a, ar, ar^2, \ldots, ar^n$$

Examples: What is the term of the sequence (a_n)?

1. $\{1,-1,1,-1,1,\ldots\}$
   - $a=1, r=-1$ $\boxed{a_n = 1(-1)^n}$
2. $\{2,10,50,250,1250, \ldots\}$
   - $a=2, r=5$ $\boxed{a_n = 2(5)^n}$
3. $\{6,2,\frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$
   - $a=6, r=\frac{1}{3}$ $\boxed{a_n = 6(1/3)^n}$

Exercise

Exercise 1: Let $\{a_n\} = \{7,13,19,25,\ldots\}$ for $n \geq 0$, Find a_{100}?

Solution: It's arithmetic sequence, a=7 and d=6

$$\mathbf{a_n = a_0 + d(n)}$$

when the index starts with 0

$$\mathbf{= 7 + 6n}$$

So, $a_{100} = 7 + 6(100) = 607$

Exercise 2: Let $\{a_n\} = \{7,13,19,25,\ldots\}$ for $n \geq 1$, Find a_{100}?

Solution: It's arithmetic sequence, a=7 and d=6

$$\mathbf{a_n = a_1 + d(n-1)}$$

when the index starts with 1

$$\mathbf{= 7 + 6(n-1)}$$

So, $a_{100} = 7 + 6(100-1) = 601$

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

Example: *Fibonacci sequence.* Let $f_n = f_{n-1} + f_{n-2}$ and $f_0 = 0$, $f_1 = 1$. How the sequence looks like?

Solution:

{0,1,1,2,3,5,8,13,21,…}

Guessing Sequences

Given a few elements of a sequence, try to identify the sequence or the formula .

Some questions to ask?

- Are there repeated terms of the same value?
- Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
- Can you obtain a term by combining the previous terms in some way?
- Do the terms match those of a well known sequence?

Exercise

Example: Find the term of the sequence (a_n) for:

1. $\{a_n\}=\{1, {}^1/_2, {}^1/_4, {}^1/_8, {}^1/_{16}\}$.
   - This is a geometric sequence with $a = 1$ and $r = ½$. Thus, $\boldsymbol{a_n = 1/2^n}$
2. $\{a_n\}=\{1,7,25,79,241,727,2185,6559,19681,59047\}$.
   - **Solution**: Note the ratio of each term to the previous ≈ 3. So now compare with the sequence of 3^n . We notice that the n^{th} term is 2 less than the corresponding power of 3. Thus, $\boldsymbol{a_n = 3^n - 2}$.

n	1	2	3	4	5
3^n	3	9	27	81	243
a_n	1	7	25	79	241

Some Useful Sequences

TABLE: Some Useful Sequences.	
n^{th} Term	***First 10 Terms***
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,…
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,…
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,…
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89,…
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,…
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,…
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,…

Summations[1]

$$\boxed{\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ \ldots + a_n}$$

The variable i is called the *index of summation*. It runs through all the **integers** starting with its *lower limit* (1) and ending with its *upper limit* (n).

Example: What is the value of $\sum_{i=1}^{5} i^2$?

Solution:

$$\sum_{i=1}^{5} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

Summations [2]

Example: Evaluate the following summations:

$$\boxed{\sum_{i=1}^{n} c = c + c + c + \ldots + c = \boldsymbol{nc}}$$

$$\boxed{\sum_{i=m}^{n} c = \boldsymbol{(n - m + 1)c}}$$

$$\boxed{\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}}}$$

Proof? Next slide

$$\boxed{\sum_{i=m}^{n} i = \sum_{i=1}^{n} i - \sum_{i=1}^{m-1} i = \frac{\boldsymbol{n(n+1)}}{\boldsymbol{2}} - \frac{\boldsymbol{(m-1)m}}{\boldsymbol{2}}}$$

Proof of $\frac{n(n+1)}{2}$

$$\boxed{\sum_{i=1}^{n} i = 1 + 2 + 3 + \ \ldots + n = \frac{n(n+1)}{2}}$$

Proof:

Let $S = 1 + \ \ 2 \ \ + \ \ 3 \ \ + \cdots + n$

add

$$\underline{S = n + (n-1) + (n-2) + \cdots + 1}$$

2S = n (n+1)

S = n (n+1) / 2

Summation of Arithmetic Sequence

$$\boxed{\sum_{i=1}^{n} a + bi = \sum_{i=1}^{n} a + b\sum_{i=1}^{n} i}$$

Example: Evaluate the following summations:

$$\sum_{j=1}^{10} 5\mathrm{j} + 2 = 5\sum_{j=1}^{10} \mathrm{j} + \sum_{j=1}^{10} 2 = 5\frac{10(11)}{2} + 2(10) = 275 + 20 = \mathbf{295}$$

$$\sum_{j=5}^{10} 5\mathrm{j} = 5\sum_{j=1}^{10} \mathrm{j} - 5\sum_{j=1}^{4} \mathrm{j} = 5\frac{10(11)}{2} - 5\frac{4(5)}{2} = 275 - 50 = \mathbf{225}$$

Summation of Geometric Sequence

$$\boxed{\sum_{i=0}^{n} ar^i = a\sum_{i=0}^{n} r^i = a + ar + ar^2 + \ldots + ar^n = a\left[\frac{r^{n+1}-1}{r-1}\right]} \quad r \neq 1$$

Proof:

Let $S = a + ar + ar^2 + \cdots + ar^n$

$r \times S = ar + ar^2 + \ldots + ar^{n+1}$

subtract

$$rS - S = ar^{n+1} - a$$

$$S = \frac{ar^{n+1} - a}{r-1}$$

Example: $\sum_{j=0}^{5} 2(5)^j = 2\left[\frac{5^6-1}{5-1}\right] = 2\left[\frac{15624}{4}\right] = 7812$

Some Useful Summation Formulae

Sum	***Closed From***
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r-1}, \ r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$

Double Summations

Example 1: $\sum_{i=1}^{n}\sum_{j=1}^{m} c = \sum_{i=1}^{n} cm = cmn$

Example 2: $\sum_{i=1}^{n}\sum_{j=1}^{m} i = \sum_{i=1}^{n} mi = m\sum_{i=1}^{n} i = m\frac{n(n+1)}{2}$

Example 3: $\sum_{i=1}^{n}\sum_{j=1}^{m} j = \sum_{i=1}^{n} \frac{m(m+1)}{2} = n\frac{m(m+1)}{2}$

Example 4: $\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4} \frac{3(4)}{2} i = \sum_{i=1}^{4} (6i) = 6\frac{4(5)}{2} = 60$

Product Notation

$$\boxed{\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \cdots \times a_n}$$

Examples:

$$\boxed{\prod_{i=1}^{n} c = c \times c \times c \times \cdots \times c = c^n}$$

$$\boxed{\prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \cdots \times n = n!}$$