dynamic programming

Matrix - Multiplication

CSC	311

0508793242 [illegible] / P

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Matrix-chain multiplication - DP

slides math 69 → 70, 73,

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \end{bmatrix}$$

$3 \times 2 \qquad 2 \times 4$

$$C = A \cdot B = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{bmatrix}$$

3×4

$$c_{23} = a_{21} \cdot b_{13} + a_{22} \cdot b_{23}$$

A · B	عدد مرات الضرب
3×2 2×4	$3 \cdot 2 \cdot 4 = 24$

Example

Consider A B C

10×100 $\quad 100 \times 5$ $\quad 5 \times 50$

Find least number of multiplication cost of multiplying $A \cdot B \cdot C$

Sol

① $(A \cdot B) \cdot C$

$D = A \cdot B$ 10×5	# of cost $= 10 \cdot 100 \cdot 5$ $= 5000$
$D \times C$ $10 \times 5 \quad 5 \times 50$	# of cost $= 10 \cdot 5 \cdot 50$ $= 2500$

Total cost $= 5000 + 2500 = 7500$

② $A \cdot (B \cdot C)$

Multi'	cost
$B \times C$ $100 \times 5 \quad 5 \times 50$	$100 \cdot 5 \cdot 50 = 25000$
$A \cdot (B \cdot C)$ (10×100) (100×50) ~~100×5~~	$(10)(100)(50) = 50000$

Total cost $= 25000 + 50000 = 75000$

Denote $\langle A_1, A_2, \ldots, A_n \rangle$ by $\langle P_0, P_1, P_2, \ldots, P_n \rangle$

حيث :

$A_1 \quad A_2 \quad A_3 \quad \ldots, \quad A_n$

$P_0 \times P_1 \quad P_1 \times P_2 \quad P_2 \times P_3 \quad \ldots \quad P_{n-1} \times P_n$

أو $(P_0, P_1) \quad (P_1, P_2) \quad (P_2, P_3) \quad \ldots \quad (P_{n-1}, P_n)$

افترض انا نريد حساب . . . $A_1 \cdot A_2 \cdot A_3 \cdot A_4$

<u>نوجد عندنا 5 احتمالات</u>

① $(A_1 \cdot (A_2 \cdot (A_3 \cdot A_4)))$

② $(A_1 \cdot ((A_2 \cdot A_3) \cdot A_4))$

③ $((A_1 \cdot A_2) \cdot (A_3 \cdot A_4))$

④ $(((A_1 \cdot A_2) \cdot A_3) \cdot A_4)$

⑤ $((A_1 \cdot (A_2 \cdot A_3)) \cdot A_4)$

م/عاطف
0508793242

Matrix multiplication

A_1 $\quad$ A_2 $\quad$ A_3

10×20 $\quad$ 20×5 $\quad$ 5×15

$P_0 \times P_1$ $\quad$ P_1 $\quad$ P_2

$(A_1 \cdot A_2) \cdot A_3$

$C \cdot A_3$

10×5 $\quad$ 5×15

اختيار المصفوفة:

$C = A_1 \cdot A_2$

or

$A_1 \cdot (A_2 \cdot A_3)$

$A_1 \cdot D$

10×20 $\quad$ 20×15

العملية	عدد عمليات الضرب	الناتج
$C = A_1 \cdot A_2$	$10 \cdot 20 \cdot 5$	1000
$C \cdot A_3$	$10 \cdot 5 \cdot 15$	750
Total		1750

العملية	عدد عمليات الضرب	الناتج
$D = A_2 \cdot A_3$	$20 \cdot 5 \cdot 15$	150
$A_1 \cdot D$	$10 \cdot 20 \cdot 15$	300
Total		4500

يوجد عندنا استنتاجين

① اقل عدد من عمليات الضرب

= 1750

minimum cost

② الحل الأمثل الذي يعطي هذه عمليات الضرب السابقة

1750

هو

$(A_1 \cdot A_2)\ A_3$

optimal solution

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المشكلة المراد حلها هي وضع الأقواس بحيث نحصل على أقل عمليات الضرب

1. سوف نستخدم الرمز $A_{i..j}$

$\boxed{A_{i..j} = A_i \cdot A_{i+1} \cdots A_j}$

مثال

$\boxed{A_{1..4} = A_1 \cdot A_2 \cdot A_3 \cdot A_4}$

لحساب الضرب $A_1 \cdot A_2 \cdots A_n$

نقسمه لجزئيه

$A_{1..n} = \boxed{A_1 \cdot A_2 \cdots A_k} \cdot \boxed{A_{k+1} \cdots A_n}$

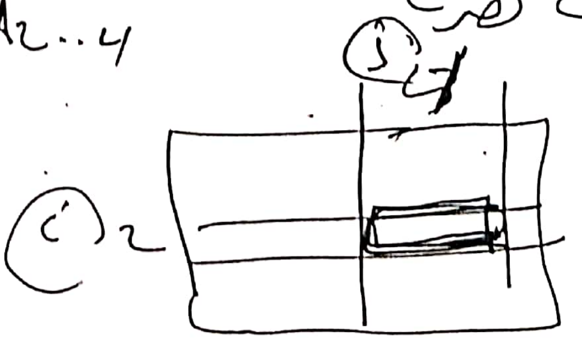
$A_{1..n} = A_{1..k} \cdot A_{k+1..n}$

---

2. نستخدم الرمز $\boxed{m[i,j]}$ وهو اقل عدد من عمليات الضرب لحساب $A_{i..j}$

3. اقل عدد من العمليات لحساب $A_{1..n}$ هو $m[1,n]$

$m[2,4] \rightarrow$ اقل عدد عمليات لضرب $A_{2..4}$



~~22~~ (7)

(4)

cost of computing $A_{i..j}$ = cost of computing $A_{i..k}$ + cost of computing $A_{k+1..j}$ +

cost of multiplying $A_{i..k}$ and $A_{k+1..j}$

$$\boxed{m[i,j] = \underset{i \le k < j}{\min} \left\{ m[i,k] + m[k+1,j] + p_{i-1} \cdot p_k \cdot p_j \right.}$$

$m[i,i] = 0 \qquad$ for $i = 1, 2, \dots, n$

$$\therefore m[i,j] = \begin{cases} 0 & \text{if } i = j \\ \underset{i \le k < j}{\text{Min}} \left\{ m[i,k] + m[k+1,j] + p_{i-1} \cdot p_k \cdot p_j \right. & \text{if } i < j \end{cases}$$

تجاهل slide 43 → 44

الكود

$m[5,5] = A_{5..5} = A_5$

Example

matrix	A_1	A_2	A_3	A_4	A_5	A_6
Dimension	10×20	20×5	5×15	15×50	50×10	10×15

P_0 P_1 P_1 P_2 P_2 P_3 لذا P_3 P_4 P_4 P_5 P_5 P_6

سوف نستخدم جدولين

- الجدول الأول به عدد عمليات الضرب الناتجة من ضرب المصفوفات

- الجدول الثاني تخزين قيمة ال K التي تعطي الحل الأمثل لخانة محددة

نرسم الجدولين في الصفحة التالية

لكن الحسابات سوف تكون بعد رسم الجدول

~~4~~ 9

$$m[i,j] = \begin{cases} 0 & \text{if } (i = j) \\ \min\limits_{i \le k < j} \left\{ m[i,k] + m[k+1,j] + P_{i-1} P_k P_j \right\} & \text{if } i < j \end{cases}$$

optimal → minimum

$m[1,4]$ = cost of multiplying $A_{1..4}$

$A_1 \cdot A_2 \cdot A_3 \cdot A_4$

$s[i,j]$ = k

$i \le k < j$

الناتج من ~~[illegible]~~ المتتابعة

والتي تحقق أقل cost

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~~5~~ 10

P_0	P_1	P_2	P_3	P_4	P_5	P_6
10	20	5	15	50	10	15

m[i,j]

i \ j	1	2	3	4	5	6
1	0	1000	1750	7250	7750	8750
2		0	1500	8750	7250	8500
3			0	3750	6250	7000
4				0	7500	9750
5					0	7500
6						0

min $A_{1\cdots 6}$

s[i,j]

i \ j	1	2	3	4	5	6
1	1	1	2	2	2	2 → K
2		2	2	2	2	2
3			3	3	4	5
4				4	4	5
5					5	5
6						6

قيم K

$A_{1..1}$

القطر فوق الرئيسي

$m[i, i+1]$

$m[1,2] = m[1,1] + m[2,2] + P_0 \cdot P_1 \cdot P_2$ $k=1$

$\boxed{k=1}$ $= 1000$

$\boxed{i=1, j=2}$

$m[2,3] = P_1 \cdot P_2 \cdot P_3$ $k=2$

$\boxed{i=2, j=3 \quad k=2}$ $= 20 \cdot 5 \cdot 15 = 1500$

$m[3,4] = P_2 \cdot P_3 \cdot P_4$ $k=3$

$= 5 \cdot 15 \cdot 50 = 3750$

$m[4,5] = P_3 \cdot P_4 \cdot P_5 = 7500$ $k=4$

$m[5,6] = P_4 \cdot P_5 \cdot P_6 = 7500$ $k=5$

~~7~~ 12

$m(i, j]$

$m[i, i+2] \rightarrow$ الصفر فوق القطر الـ بعد

$i\ j$

$m[1, 3] = ??$ $\quad k = 1, 2$ $\quad i = 1, j = 3$

$= \min$

$k = 1 \rightarrow$ $m[1, 1] + m[2, 3] + P_0 \cdot P_1 \cdot P_3$

$= 0 + 1500 + 10 \cdot 20 \cdot 15$

$= 1500 + 3000$

$= \boxed{4500}$

$k = 2 \rightarrow$

$m[1, 2] + m[3, 3] + P_0 \cdot P_2 \cdot P_3$

$= 1000 + 0 + 10 \cdot 5 \cdot 15$

$= 1000 + 750$

$= \boxed{1750} \leftarrow \min$

$\boxed{\text{best } k = 2}$

$\cancel{8}$ (13)

$m[2,4] = ??$ (i over 2, j over 4)

$k = 2, 3$

$\boxed{i=2}$, $\boxed{j=4}$

$$= \min \begin{cases} k=2 \rightarrow & m[2,2] + m[3,4] + P_1 \cdot P_2 \cdot P_4 \\ & = 0 + 3750 + 20 \cdot 5 \cdot 50 \\ & = 3750 + 5000 \\ & = 8750 \rightarrow \min \\ & \boxed{\text{best } k = 2} \\ k=3 & m[2,3] + m[4,4] + P_1 \cdot P_3 \cdot P_4 \\ & = 1500 + 0 + 20 \cdot 15 \cdot 50 \end{cases}$$

$= 1500 + \text{\_\_\_\_}$

$= 16500$

الحل الأمثل هو 8

~~12~~ 18

minimum cost of multiplying $A_{1..6}$

$= 8750$

عملية ضرب

$A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6 = ??$

كيف نضع الأقواس

حد الجزء الأول

د الأقواس

$A_{1..6} \longrightarrow k = 2$

$k+1$

$A_{1..6} = A_{1..2} \cdot A_{3..6}$

$$\boxed{A_{1..6} = (A_1 \cdot A_2) \cdot (A_3 \cdot A_4 \cdot A_5 \cdot A_6)}$$

$\longrightarrow 1$

$A_{3..6} \longrightarrow k = 5$

$A_{3..6} = A_{3..5} \cdot A_{6..6}$

$$\boxed{A_{3..6} = (A_3 \cdot A_4 \cdot A_5) \cdot A_6} \longrightarrow 2$$

$A_{3..5}$

~~14~~ = 19

$A_{3..5} \longrightarrow k = 4$

$$A_{3..5} = A_{3..4} \cdot A_{5..5}$$

$$\boxed{A_{3..5} = (A_3 \cdot A_4) \cdot A_5} \longrightarrow 3$$

~~نرجع بالعكس~~

وصلنا صدر التحليل اللي حطناله

$$A_{1..6} = (A_1 \cdot A_2) \cdot (A_3 \cdot A_4 \cdot A_5 \cdot A_6)$$

$$A_{1..6} = (A_1 \cdot A_2) \cdot (A_{3..6}) \rightarrow \text{ص } 2$$

$$A_{1..6} = (A_1 \cdot A_2) \cdot \left(\boxed{(A_3 \cdot A_4 \cdot A_5)} \cdot A_6\right) \rightarrow \text{ص } 3$$

$$A_{1..6} = (A_1 \cdot A_2) \left(\left((A_3 \cdot A_4) \cdot A_5 \right) \cdot A_6 \right)$$

MCM DP Steps

0:62

7

```
MATRIX-CHAIN-ORDER(p)
n ← length[p] − 1                    → 6
for i ← 1 to n
     do m[i, i] ← 0
for l ← 2 to n                        ▷ l is the chain length.
     do for i ← 1 to n − l + 1        l=2 (1,2,3,4,5)
          do j ← i + l − 1
               m[i, j] ← ∞
               for k ← i to j − 1
                    do q ← m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
                       if q < m[i, j]
                          then m[i, j] ← q
                               s[i, j] ← k
return m and s
```

القطر الرئيسي

الأقطار

$l = 2,3,4,5,6$

طول القطر

n n n K

$O(n^3)$

القطر فوق الرئيسي

$l = 2$ i = 1, 2, 3, 4, 5
j = 2, 3, 4, 5, 6

$l = 3$ i = 1, 2, 3, 4
j = 3, 4, 5, 6

$l = 4$ i = 1, 2, 3
j = 4, 5, 6

$l = 5$ i = 1, 2
j = 5, 6

$l = 6$ i = 1
j = 6

القطر الأطول

MCM DP Example

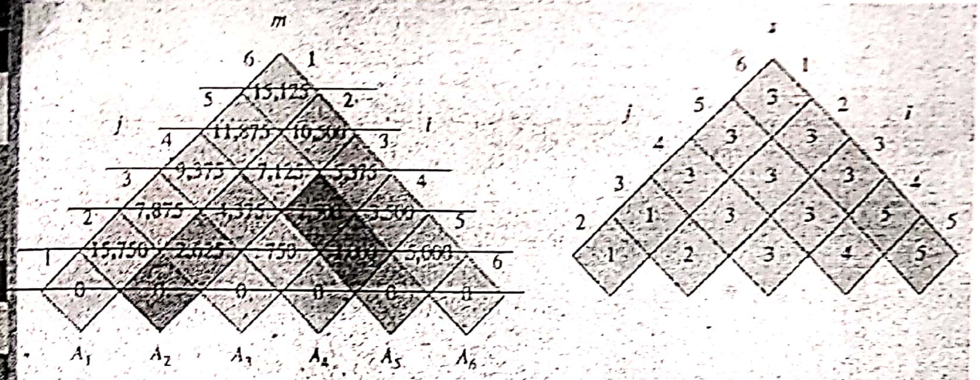


Figure 15.3 The m and s tables computed by MATRIX-CHAIN-ORDER for $n = 6$ and the following matrix dimensions:

matrix	dimension
A_1	30×35
A_2	35×15
A_3	15×5
A_4	5×10
A_5	10×20
A_6	20×25

The tables are rotated so that the main diagonal runs horizontally. Only the main diagonal and upper triangle are used in the m table, and only the upper triangle is used in the s table. The minimum number of scalar multiplications to multiply the 6 matrices is $m[1,6] = 15{,}125$. Of the darker entries, the pairs that have the same shading are taken together in line 9 when computing

$$m[2,5] = \min\begin{cases} m[2,2] + m[3,5] + p_1p_2p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 & = 13000\,, \\ m[2,3] + m[4,5] + p_1p_3p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 & = 7125\,, \\ m[2,4] + m[5,5] + p_1p_4p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 & = 11375 \end{cases}$$

$$= 7125\,.$$

MCM DP Steps

- Step 4, constructing a parenthesization order for the optimal solution.
  - Since $s[1..n,1..n]$ is computed, and $s[i,j]$ is the split position for $A_iA_{i+1}\ldots A_j$, i.e, $A_i\ldots A_{s[i,j]}$ and $A_{s[i,j]+1}\ldots A_j$, thus, the parenthesization order can be obtained from $s[1..n,1..n]$ recursively, beginning from $s[1,n]$.

شرح الدكتور

1st EX:

Chain matrix multiplication:

عدد عمليات الضرب

$A \times B \times C \times D$

أحجام المصفوفة:

A	30×1
B	1×40
C	40×10
D	10×25

① $[(A \times B) \times C] \times D$	20700
$(A \times B) \times (C \times D)$	41200
$(A \times (B \times C)) \times D$	8200
$A \times [(B \times C) \times D]$	11,750
$A \times (B \times (C \times D))$	1400

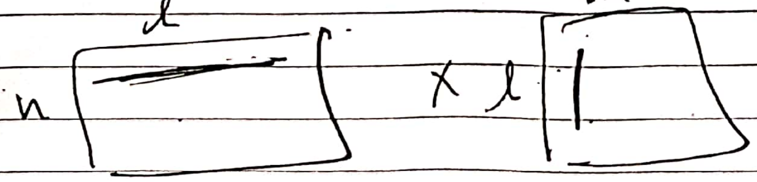
عكس

Prob:
Find cheapest way to multiply $A_1 \times A_2 \times \cdots A_n$
Let C_{ij} = cheapest way to multiply $A_i \times A_{i+1} \cdots A_j$

Solution $= C_{1n}$ $\leftarrow$ $1 \le i \le j \le n$

A_i dimension $= d_{i-1} \times d_i$

$$C_{ij} = \begin{cases} 0 & i = j \\ \min\limits_{i \le k \le j} \{ C_{ik} + C_{k+1\,j} + d_{i-1} d_k d_j \} & o/w \end{cases}$$



$\Rightarrow n \times l \times m$
عملية

3

شرح الدكتور

$\text{cheapest} = C_{ij}$

$$\overline{A_i \times A_{i+1} \times \cdots \times A_j}$$

$(A_i \times \cdots \times A_k) \times (A_{k+1} \times \cdots \times A$

$d_{i-1} \times d_k \quad \ldots \quad d_k \times d_j$

$(A_i) \times (A_{i+1} \times \cdots \times A_j)$

$(A_i \times A_{i+1}) \times (A_{i+2} \cdots A_j)$

$(A_i \times A_{i+1} \times A_{i+2}) \times (A_{i+3} \cdots$

$A_1 \quad d_0 \times d_1$

$A_2 \quad d_1 \times d_2$

$A_3 \quad d_2 \times d_3$

dimension

$A_2 \times A_3 \quad d_1 \times d_3$

cost ~~[illegible]~~ # * $= d_1 \cdot d_2 \cdot d_3$

عدد الضرب

الأحد

(5)

Dynamic Programming

Solves optimization problems

1, 7, 10

15 = 10+1+1+1+1+1

15 = 7+7+1

- buttom-up

- saves intermediate resutte

Fibonacci Sequence:

$$f(n)\begin{cases} 0 & n=0 \\ 1 & n=1 \\ f_{n-1}+f_{n-2} & n>1 \end{cases}$$

Divide and conquer:

tree حل
تريسم

```
T(n) Procedure Fib(n){
        if (n=0 or n=1) return n
        else return Fib(n-1)+Fib(n-2)
}
```

$T(n) = T(n-1) + T(n-2) + c \Rightarrow O(1.58^n)$

$\boxed{\text{Exponention}}$

الثلاثاء (6)

حل الدكتور

Dynamic Pres. Solution :

```
Procedure Fib(n){

    a ← 0
    b ← 1
    for i ← 2 ... n do{

        f ← a+b
        a ← b
        b ← f
    }

    return f
}
```

$\Theta(n)$

	a	b	f
1	0	1	1
2	1	1	2
3	1	2	3
4	2	3	5

2

3