①

Big-O

① **Def** $g(n)$ is Big O for $f(n)$ **iff** $\exists$ two constants c, n_0 such that

$$f(n) \le c\, g(n)$$

$$\forall \quad n \geq n_0$$

c, n_0 positive

Examples

①

$f(n) = n^2 + 3$

$g(n) = n^3$

$f(n) \le g(n)$

$n^2+3 \le$ 1, 2, 3, 4 n^3

$c=1, n_0=2$

$c=2, n_0=1$

$c=3, n_0=2$

$c=4, n_0=1$

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2

$f(n) = n^2 + 3$

$g(n) = n^2$

is $f(n) \simeq O(g(n))$

$n^2 + 3 \leq 4n^2$

$c = 4, \quad n_0 = 1$

$1n^2 + 3 \leq 2n^2$

$c = 2, \; n_0 = 2$

$n^2 + 3 \leq 3n^2$ $\qquad c = 3; \; n_0 = 2$

كيف نوجد ال c اذا كانت الدالة تربيعية الدرجة

To find c for: $an^2 + bn + k$

$c = a + 1$

or $c = a + 2$

$c = a + 3$

$c = a + b + r$

2

2

$f(n) = n^2 + 3$

$g(n) = n^2$

is $f(n) \simeq O(g(n))$

$n^2 + 3 \leq 4n^2$

$c = 4, \quad n_0 = 1$

$1n^2 + 3 \leq 2n^2$

$c = 2, \quad n_0 = 2$

$n^2 + 3 \leq 3n^2$ $\qquad c = 3, \quad n_0 = 2$

كيف نوجد ال c اذا كانت الدالة مربعة الدرجة

To find c for: $an^2 + bn + K$

$c = a + 1$

or $c = a + 2$

$c = a + 3$

$c = a + b + r$

3

$f(n) = 3n^2 + 5n$

$\boxed{g(n) = n^3}$

$3n^2 + 5n \leq 1 (n^3)$

$c = 1, \quad n_0 = 5$

$3n^2 + 5n$ is $O(n^3)$

④

Compare n^3 _ 2^n

n	n^3	2^n
1	1	2
2	8	4
3	27	8
⋮		
		1024

4

Compare n^3 _ 2^n

n	n^3	2^n
1	1	2
2	8	4
3	27	8
⋮		
no ← 10	1000	1024
11	12 —	2048
12	17 ...	4096

5

Ex:

Show that: $f(x) = x^2 + 2x + 1$

is $O(x^2)$

≥ 1

$$\boxed{x^2 + 2x + 1} \leq 1x^2 + 2x^2 + 1x^2$$

$$\leq 4x^2$$

$c = 4$, $k_0 = 1$

التوضيح

$$\begin{cases} x^2 \leq x^2 \\ 2x \leq 2x^2 \\ 1 \leq x^2 \end{cases}$$

$$\therefore\ x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2$$

$$\leq 4x^2$$

$O(n^n)$

$O(n!)$

$\vdots$

$O(3^n)$

$O(2^n)$

$O(n^4)$

$O(n^3)$

Little o

↓

upper bound

$O(n^2)$	$\Omega(n^2)$	$\Theta(n^2)$

tight bound

$T(n) = 3n^2 + 5n + 2$

$\Omega(n)$

$\Omega(\sqrt{n})$

$\Omega(\log n)$

$\Omega(\log\log n)$

$\Omega(1)$

Little ω

lower bound

Big O

$3n^2 + 5n + 2 \leq \cancel{10n^2}$

↓

$\leq 3n^2 + 5n^2 + 2n^2 \leq 10n^2$

$c = 10, \; n_0 = 1$

Big Ω

~~f~~ $f(n)$ is $\Omega(g(n))$ iff

$\exists$ positive constants c, n_0

s.t $\quad f(n) \geq c \cdot g(n) \quad \forall\, n \geq n_0$

Ex: $f(n) = 6n + 11$ find Ω

الحل

$$6n + 11 \geq 5n$$

$$\therefore \Omega(n), \quad c = 5$$

$$6n - 5n \geq -11$$

$$n \geq -11$$

$$\boxed{\therefore n_0 = 1}$$

n_0 ليها

يوجد قانون لاختيار n_0

$$n_0 = \frac{\sum_{i=0}^{k-1} |a_i|}{a_k - c} = \frac{11}{6-5} = 11$$

Big-O

$f(n)$ is $O(\underline{g(n)})$ iff $\exists$ positive c, n_0, $n_0 \in Z^+$

s.t $\boxed{f(n) \leq c\, g(n)}$ $\forall\, n \geq n_0$

E(x) : $f(n) = (4n^3 - 2n^2 + 1)(7n^4 + 5n^3 - 1)$

find Big O, c, n_0

$4n^3 - 2n^3 + 1$	$7n^4 + 5n^3 - 1$
$\leq 4n^3 + 2n^3 + 1n^3$	$\leq 7n^4 + 5n^4 + 1n^4$
$\leq 7n^3$	$\leq 13n^4$
$n_0 = 1$	$n_0 = 1$

$\therefore f(n) \leq 7n^3 * 13n^4$

$\leq 91n^7$

$\therefore O(n^7)$, $n_0 = 1$, $c = 91$

Ex: find Big Θ with proof

$f(n) = 5n + 3$

① Big O

$5n + 3 \le 5n + 3n \le 8n$

$\therefore \boxed{O(n)}$, $c = 8$, $n_0 = 1$

② Ω

$5n + 3 \ge 4n$, $c = 4$, $n_0 = 1$

$\boxed{\Omega(n)}$

$n_0 = \frac{3}{5-4} = 3$

③ $f(n)$ is $\Theta(n)$

$c_1 = 4$, $c_2 = 8$, $n_0 = 1$

$4n \le 5n + 3 \le 8n$

, $n_0 = 1$

~~(47)~~ (53)

Put ✓ or ✗

$3n^2+5$ is $\Theta(n^2)$

✓ $3n^2+5$ is not $O(n)$

∴ $3n^2+5$ is not $\Theta(n)$

Prove $3n^2+5$ is not $O(n)$

~~(43)~~ (53)

put ✓ or ✗

$3n^2+5$ is $\Theta(n^{2})$

$3n^2+5$ is not $O(n)$

$3n^2+5$ is not $\Theta(n)$

$\therefore$ $3n^2+5$ is not $\Omega(n)$

---

prove $3n^2+5$ $\in \Omega(n)$

~~(47)~~ (53)

Ex: Find Big O for $f(n)$

$f(n) = 6n+11$, and c, n_0

Find many solutions for c, n_0

حل

Ex: Find Big O for $f(n)$

$f(n) = 3n^2 + 8n + 1$, c, n_0

حل

$3n^2+8n+1 \le c\,n^2$:

$3n^2+8n+1 \le$

$\ge 2n^2$

Take $c = 4$

$3n^2 + 8n + 1 \le 4n^2$

$n^2 - 8n - 1 \ge 0$

$an^2 + bn + c = 0$

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$\cancel{1747}$ $(54$

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$n = \frac{8 + \sqrt{64 + 4}}{2}$$

2	68
2	34
17	17
	1

$$n = \frac{8 + 2\sqrt{17}}{2}$$

$$n_0 = 4 + \sqrt{17}$$

$f(n) = 6n^5 - 3n + 10n^2 - 1 \quad O(n^5)$

$6 \quad -3 \quad 10 \quad -1 \quad 12$

$c = |6| + |-3| + |10| + |-1|$

$= 6 + 3 + 10 + 1 = 20$

$$\boxed{n_0 = 1}$$

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find

Ex: $\sum_{i=1}^{n} \log(i)$

الحل

$$\sum_{i=1}^{n} \log(i) = \underbrace{\log(1) + \log(2) + \log(3) + \dots + \log(n)}_{n \text{ عددهم}}$$

$$\leq \log(n) + \log(n) + \dots + \log(n)$$

$$\leq n \log(n)$$

$\therefore \ O(n \log n)$

$c = 1 \ , \ n_0 = 1$

$\log 1 = 0$
$\log 2 = 1$
$\log 4 = 2$
$\log 8 = 3$

n	$n+10$	$\leq$	$2n$
1	11		2
2	12		4
3	13		6
4	14		8
5	15		10
6	16		12
8	18		16
9	19		18
10	20		20

$c=2$

$n_0=10$

Ex $\log(3n^4-5n^2+1)$ find Big O

Sol

$$\log(3n^4-5n^2+1) \le \log(3n^4+5n^4+1n^4)$$
$$\le \log(9 \cdot n^4)$$
$$\le \log 9 + \log n^4$$
$$\le \log 9 + 4\underline{\log n}$$
$$\le \log n + 4\log n$$
$$\log(3n^4-5n^2+1) \le 5\log n$$

$n_0 = 9$, $c = 5$

$\log 9 \le \log n$

$n_0 = 9$

Ex $\log(3n^4 - 5n^2 + 1)$ find Big O

Sol

$$\log(3n^4 - 5n^2 + 1) \le \log(3n^4 + 5n^4 + 1n^4)$$
$$\le \log(9 \cdot n^4)$$
$$\le \log 9 + \log n^4$$
$$\le \log 9 + 4 \underline{\log n}$$
$$\le \log n + 4 \log n$$
$$\log(3n^4 - 5n^2 + 1) \le 5 \log n$$

$n_0 = 9$, $c = 5$

$\log 9 \le \log n$

$n_0 = 9$

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$\sqrt{5n^3 - 2n^2 \cancel{+} 1} \leq \sqrt{5n^3 + 2n^3 + n^3}$

$\leq \sqrt{8n^3}$

$\sqrt{5n^3 - 2n^2 - 1} \leq \sqrt{8} * \sqrt{n^3}$

$\leq \sqrt{8} * n^{\frac{2}{3}}$

$O(n^{\frac{2}{3}})$, $c = \sqrt{8}$,

$n_0 = 1$

$\sqrt{n^2} = n$

$\sqrt{n^2 + n}$

$1n^7 - \underline{\underline{5}}n^6 - \underline{\underline{10}}n^3 + \underline{n}^2 - \underline{1}$

find Ω

Sol

$\underline{n^7} - 5n^6 - 10n^3 + n^2 - 1$

$\leq \frac{1}{2}n^7$

$c = \frac{1}{2}$

$$n_0 = \frac{5+10+1+1}{a_k - c}$$

$$= \frac{17}{1-\frac{1}{2}} = \frac{17}{\frac{1}{2}}$$

$$\boxed{n_0 = 34}$$

$\therefore \Omega(n^7),\ c = \frac{1}{2},\ \underline{\underline{n_0 = 34}}$

n	2^n	n^3
1	2	1
2	4	8
3	8	27
4	16	64
⋮		
10	1024	1000
11	2048	1313 تقريبا
12	4096	1713

$\therefore\ n^3 < 2^n \quad \forall n \geq \boxed{10}$

$\therefore \boxed{n_0 = 10}$

$\therefore\ n^3$ is $O(2^n)$

with $c=1$, $n_0=10$

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6

theorems

$f_1(n)$ is $O(g_1(n))$

$f_2(n)$ is $O(g_2(n))$

$f_1(n) + f_2(n)$ is $O(\max(g_1(n), g_2(n)))$

Ex: $f_1(n) = O(2^n)$

$f_2(n) = O(n^3)$

$\therefore (f_1 + f_2)(n)$ is $O(2^n)$

7

~~Proble~~: if $f_1(n)$ is $O(g_1(n))$
$f_2(n)$ is $O(g_2(n))$

Then

$(f_1+f_2)(n)$ is $O(\max(g_1(n), g_2(n)))$

g_1

proof: $\because f_1(n)$ is $O(g_1(n))$

$\therefore \exists\, c_1, n_0,\ f_1(n) \le c_1 g_1(n)$

$\because f_2(n)$ is $O(g_2(n))$.

$\therefore \exists\, c_2, n_0',\ f_2(n) \le c_2 g_2(n)$

$\therefore f_1(n) + f_2(n) \le c_1 g_1(n) + c_2 g_2(n)$

$\therefore f_1(n) + f_2(n) \le c_1\, g(n) + c_2\, g(n)$

$\le (c_1 + c_2)\, g(n)$

$f_1(n) + f_2(n) \le c\ \ g(n)$

~~47~~ 57 8

Theory

if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$

then:

① $f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$

② $f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$ $n_0 = \max(n_{10}, n_{20})$

proof

$f_1(n) = O(g_1(n)) \Leftrightarrow f_1(n) \le c_1 g_1(n) \quad \forall n \ge n_{01}$

$f_2(n) = O(g_2(n)) \Leftrightarrow f_2(n) \le c_2 g_2(n) \quad \forall n \ge n_{02}$

① $f_1(n) + f_2(n) \le c_1 g_1(n) + c_2 g_2(n)$

$\forall n \ge \max(n_{01}, n_{02})$

let $g(n) = \max(g_1(n), g_2(n))$

$f_1(n) + f_2(n) \le c_1 g(n) + c_2 g(n)$

$f_1(n) + f_2(n) \le \underbrace{(c_1 + c_2)}_{\boxed{c}} g(n)$

② $f_1(n) \cdot f_2(n) \le c_1 g_1(n) \cdot c_2 g_2(n)$

$\le c_1 \cdot c_2 \, g_1(n) \cdot g_2(n)$

$f_1(n) \cdot f_2(n) = O(g_1(n) \cdot g_2(n))$

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Example Find Big-O , c and n_0

$$F(n) = (4n^3 - 2n^2 + 1)(7n^4 + 5n^3 - 1)$$

sol :

$$F(n) = \underbrace{(4n^3 - 2n^2 + 1)}_{O(n^3)} \underbrace{(7n^4 + 5n^3 - 1)}_{O(n^4)}$$

$O(n^3)$	$O(n^4)$
$c = 4+2+1 = 7$	$c = 7+5+1 = 13$
$n_0 = 1$	$n_0 = 1$

$$O(n^3 \cdot n^4) = O(n^7)$$

$$c = c_1 \cdot c_2 = 7 \cdot 13 = 91$$

$$n_0 = 1$$

ويمكن الحل
بفك الأقواس
تحصل ع أكبر حد
وتكمل عادي

Example

$$f(n) = \underbrace{(4n^3 + 2n^2 + 1) + (7n^4 + 5n^3 - 1)}_{O(n^4)}$$

$\cancel{c = 9+2+7 = 18}$ $\quad c = 18$

$n_0 = 1$

EX:- $f_1 + f_2$

$(3n^2-7n+1) + (5n^3+10n^2-1)$

$O(\max(n^2, n^3)) = O(n^3)$

$C = 11 + 16 = 27$

$n_0 = 1$

Ex: the th. is not work on:.

Find Big O

$\log(3n^4 - 5n^2 + 1)$

$3n^4 - 5n^2 + 1 \le 9n^4$

$O(n^4)$, $c = 9$, $n_0 = 1$

$\log(3n^4 - 5n^2 + 1) \le \log(9n^4)$

$\le \log 9 + \log n^4$

$\le \log 9 + 4 \log n$

$\le 5 \log n$

$O(\log n)$, $c = 5$.

n_0 لايجاز

$\log 9 + 4 \log n \le 5 \log n$

$\log 9 \le 5 \log n - 4 \log n$

$\log 9 \le \log n \Rightarrow n \ge 9$

$\therefore n_0 = 9$

Ex: Find Big O ~~57~~ (61)

$$\sqrt{5n^3 - 2n^2 - 1}$$

Sol

$$5n^3 - 2n^2 - 1 \le 8n^3$$

$$\Theta(n^3), \quad c = 8, \quad n_0 = 1$$

$$\therefore \sqrt{5n^3 - 2n^2 - 1} \le \sqrt{8n^3} = \sqrt{8}\, n^{\frac{3}{2}} = \sqrt{8}\, n^{1.5}$$

$$\therefore O(n^{\frac{3}{2}}), \quad c = \sqrt{8}, \quad n_0 = 1$$

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62 ~~52~~ ~~48~~

Ex:

$$\sqrt{5n^3-2n^2-1} \leq \sqrt{8n^3} = \sqrt{8}\cdot n^{3/2}$$

$O(n^3)$	$O(n^{3/2})$
$c=8$	$c=\sqrt{8}$
$n_0=1$	$n_0=1$

Ω

Def:

$f(n) = \Omega(g(n))$ iff there $\exists$ positive constant c and n_0 $\ni$

$$f(n) \geq c\,g(n) \quad \forall n \geq n_0$$

Ex:

$f(n) = 6n+11$

- $\Omega(n)$
  - $c=6$, $n_0=1$ — $6n+11 \geq 6n$
  - $c=5$, $n_0=1$ — $6n+11 \geq 5n$
  - $c=1$, $n_0=1$
- $\Omega(1)$ — correct but not acceptable
  - $c=10$, $n_0=1$
  - $c=100$, $n_0=15$

~~52~~ (63)

Find Ω for $6n+11$

sol

طل $\quad 6n+11 \geq 6n$

$\Omega(n) \quad c = 6, \quad n_0 = 1$

طلاتم $\quad 6n+11 \geq 5n$

$\Omega(n), \quad c = 5, \quad n_0 = 1.$

طلاتم $\quad 6n+11 \geq n$

$c = 1, \quad n_0 = 1$

طلاتم $\quad 6n+11 \geq 100$

$\Omega(1) \quad c = 100, \quad n_0 = 15$

صحيح لكنه غير مقبول

~~49~~ ~~53~~ 64

Ex: $f(n) = 6n - 11$

$\Omega(n)$ —— $c \neq 6$

1. $c = 5$, $n_0 = 11$
2. $c = 4$, $n_0 = 6$
3. $c = 3$, $n_0 = 4$
4. $c = 1$, $n_0 = 2$

$$\boxed{\begin{aligned} 6n - 11 &\geq 5n \\ 6n - 5n &\geq 11 \\ n &\geq 11 \end{aligned}}$$

Th.

if $f(n)$ is polynomial of degree k and $a_k > 0$ then $f(n) = \Omega(n^k)$

th: ~~f(n)~~ is a polynomial of degree k

then $g(n) = O(n^k)$

let $g(n) = a_0 + a_1 n^1 + a_2 n^2 + a_3 n^3 + \cdots a_{k-1} n^{k-1} + a_k n^k$

$$= n^k \left(\frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \frac{a_2}{n^{k-2}} + \cdots + \frac{a_{k-1}}{n} + a_k \right)$$

$$\le n^k \left(|a_0| + |a_1| + \cdots + |a_k| \right)$$

$$\le c \cdot n^k$$

$c = |a_0| + |a_1| + \cdots |a_k|$

$\therefore g(n)$ is $O(n^k)$

$c = |a_0| + |a_1| + \cdots + |a_k|$

, $n_0 = 1$

$$\frac{a_2 n^2}{n^k}$$

$$\frac{a_2}{n^{k-2}}$$

$$\frac{5}{2} < 5$$

$$\frac{5}{2} < 5$$

Th: $f(n)$ is a polynomial of degree k

then $g(n) = O(n^k)$

let $g(n) = a_0 + a_1 n^1 + a_2 n^2 + a_3 n^3 + \cdots a_{k-1} n^{k-1} + a_k n^k$

$$= n^k \left(\frac{a_0}{n^k} + \frac{a_1}{n^{k-1}} + \frac{a_2}{n^{k-2}} + \cdots + \frac{a_{k-1}}{n} + a_k \right)$$

$$\frac{a_2 n^2}{n^k}$$

$$\frac{a_2}{n^{k-2}}$$

$$\leq n^k \left(|a_0| + |a_1| + \cdots + |a_k| \right)$$

$$\leq c \cdot n^k$$

$c = |a_0| + |a_1| + \cdots |a_k|$

$\therefore g(n)$ is $O(n^k)$

$c = |a_0| + |a_1| + \cdots + |a_k|$

, $n_0 = 1$

$$\frac{5}{2} < 5$$

$$\frac{5}{2} < 5$$

~~46~~ 56

if $n \geq 1$

$$\leq \underbrace{\left[|a_0| + |a_1| + |a_2| \dots + |a_k| \right]}_{\text{constant } c} \cdot n^k$$

Ex:

$$f(n) = 6n^5 - 10n^4 + 7n^3 - 1 = O(n^5)$$

$c = 6 + 10 + 7 + 1$

$= 24$

$n_0 = 1$

Ex:

$$f(n) = 3n^{10} - 7n^8 + n^3 - 10n^2 + 1 = O(n^{10})$$

$c = 3 + 7 + 1 + 10 + 1$

$= 22$

$n_0 = 1$

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$\boxed{}$ (65) ~~53~~

Ω

Th:

$f(n)$ is a polynomial of degree k and a

then $f(n) = \Omega(n^k)$

$$f(n) = 6n^5 + 3n^4 + 10n + 1 = \Omega(n^5)$$

$c = 6, \; n_0 = 1$

Q: $f(n) = 6n^5 - 3n^4 - 10n^2 - 1 = \Omega$

Hent:

$$f(n) = \sum_{i=0}^{k} a_i n^i$$

$a_k > 0$

$a_i \in R$

→ ① $\Omega(n^k)$

→ ② c pick any $0 < c < a_k$

→ ③ $n_0 = \dfrac{\sum_{i=0}^{k-1} |a_i|}{a_k - c}$ ← مجموع كل المعاملات عدا معامل اكبر اس مع تجاهل الاشارات

answer$_1$: $c = 5, \quad n_0 = \dfrac{14}{6-5} = 14$

$$f(n) = \log(\boxed{n^3 - 10n^2 - 1})$$

$\Omega(n^3), \; c = \frac{1}{2}, \; n_0 = \dfrac{11}{\frac{1}{2}} =$

$n^3 - 10n^2 - 1 \geq \frac{1}{2} n^3$

$$f(n) = \log(n^3 - 10n^2 - 1)^2 \geq \log(\tfrac{1}{2}n^3) =$$

$\log\frac{1}{2} + \log n^3 = \log\frac{1}{2} + 3\log n \geq 2\log n$

$\log\frac{1}{2} = -1$

$= \Omega(\log n), \; c = 2, \; n_0 = 2$

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67 ~~5~~ ~~15~~

$$\cancel{x = 5 + \begin{cases} \left(\frac{n+1}{2}\right)^2 & n \text{ odd} \\ \left(\frac{n}{2}\right)^2 & n \text{ even} \end{cases}}$$

Ex in Big-O ::

find Big-O of

$3n^8 - 7n^3 + 1$	$+ \log n^3$
$O(n^8)$	$O(\log n)$
$C_1 = 11$	$C_2 = 3$
$n'_0 = 1$	$n''_0 = 1$

$$O(n^8)$$

$$C = 11 + 3 = 14$$

$$n_0 = 1$$

constant $< \log < n <$ ~~logn~~ $n \log n < n^2 < 2^n < 3^n < \dots < n! < n^n$

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~~54~~ 68

<u>Given List of Functions</u>

3^n, $\log(n!)$, $\sqrt{n}$, $\log n^3$, $\sqrt{n}^{\,n}$, $n^{\log n}$, n

order on growth ascending:

$\log n^3 = 3\log n$

$n! = 1\cdot 2\cdot 3\cdots\cdots n \le n\cdot n\cdots\cdots n = n^n$

$n! \le n^n$

$\log(n!) < \log n^n$

$\log(n!) < n\log n$

$\boxed{\sqrt{n}^{\,n},\ n^{\log n}}$??

$\sqrt{n}^{\,n} = n^{\frac{1}{2}(n)} = n^{\frac{n}{2}} > n^{\log n} \Rightarrow \boxed{\sqrt{n}^{\,n} > n^{\log n}}$ True

$\boxed{3^n,\ n^{\log n}}$??

$3^n \overset{?}{>} n^{\log n}$ Take log

$\sqrt{n}\cdot\sqrt{n}$

$n\log 3 \overset{?}{>} \log n\cdot\log n$ put $n=32$

$32\log 3 > 25$ True

$\therefore \boxed{3^n > n^{\log n}}$ True

~~55~~ 69

$3^n \overset{??}{>} \sqrt{n}^{\,n}$

$3^n \overset{?}{>} n^{\frac{1}{2}n}$ Take log

$n \log 3 \overset{?}{>} \frac{1}{2} n \log n$

$n \log 3 \overset{?}{>} n\left(\frac{1}{2}\log\right)$ false

$\boxed{\therefore\ 3^n < \sqrt{n}^{\,n}} \longrightarrow$ True

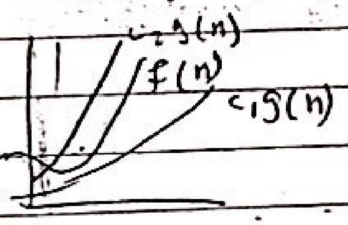
∴ ordering in ascending :-

$\log n^3 < \sqrt{n} < n < \log(n!) < n^{\log n} < 3^n < \sqrt{n}^{\,n}$

Def Θ:

$f(n) = \Theta(g(n))$ iff there $\exists$ Positive constants $c_1, c_2, n_0 \ni c_1 g(n) \le f(n) \le c_2 g(n)$ $\forall\, n \ge n_0$

Th:
$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$
then $f(n) = \Theta(g(n))$ and vice versa.



Th: $f(n) = a_k n^k + a_{k-1} n^{k-1} + \dots a_1 n^1 + a_0$
$f(n)$ Polynomial of degree k and $a_k > 0$ then $f(n) = \Theta(n^k)$

Ex: $f(n) = 1n^7 - 5n^6 - 10n^3 + n^2 - 1 = \Theta(n^7)$

$\Omega(n^7)$: $c_1 = \frac{1}{2}$ $\quad n_0 = \frac{17}{\frac{1}{2}} = 34$

$O(n^7)$; $c_2 = 18$ $\quad n_0 = 1$

التوضيح خلف الورقة

72 ~~60~~

prove $f(n) = 5n+3$ is $\Theta(n)$

Sol

1) Big O

$$5n+3 \le 6n$$

$$c_2 = 6, \quad n_0 = 3 \Rightarrow O(n)$$

2) $\Omega(n)$

$$5n+3 \ge 4n$$

$$c_1 = 4, \quad n_0 = 1 \quad \boxed{\Omega(n)}$$

$$4n \le 5n+3 \le 6n$$

$$c_1 = 4, \quad c_2 = 6, \quad n_0 = 3$$

$$\therefore \boxed{5n+3 \text{ is } \Theta(n)}$$

73

Given $f(n), g(n)$ find how are thay rela

$$f(n) = ?\ g(n)$$

$O \quad \Theta \quad \Omega$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 \Rightarrow f(n) = O(g(n)) \\ \text{Pos. constant} \Rightarrow f(n) = \Theta(g(n)) \\ \infty \Rightarrow f(n) = \Omega(g(n)) \end{cases}$$

Ex: $\log n = ?(\sqrt{n})$

$$(\log_2 n)' = \frac{1}{n \ln 2}$$

$$\lim_{n\to\infty} \frac{\log_2 n}{\sqrt{n}} = \frac{\infty}{\infty}$$

$$(\sqrt{n})' = \frac{1}{2\sqrt{n}}$$

L'Hospital

$$= \lim_{n\to\infty} \frac{\frac{1}{n\ln 2}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n \ln 2}$$

$$\frac{1}{n\ln 2} \div \frac{1}{2\sqrt{n}}$$

$$n = \sqrt{n}\cdot\sqrt{n}$$

$$= \lim_{n\to\infty} \frac{2}{\ln 2} \cdot \frac{1}{\sqrt{n}} = 0$$

$$\therefore \log n = O(\sqrt{n})$$

$$\log n \leq 1 \sqrt{n}$$

$$c = 1, \quad n_0 = 1$$

Given $f(n), g(n)$ find how are thay rela

$$f(n) = \ ? \ \ g(n)$$

$O \quad \Theta \quad \Omega$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0 \Rightarrow f(n) = O(g(n)) \\ \text{Pos. constant} \Rightarrow f(n) = \Theta(g(n)) \\ \infty \Rightarrow f(n) = \Omega(g(n)) \end{cases}$$

Ex: $\log n = \ ? \ (\sqrt{n})$

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$$(\sqrt{n})' = \frac{1}{2\sqrt{n}}$$

L'Hospital

$$= \lim_{n\to\infty} \frac{\frac{1}{n \ln 2}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n \ln 2}$$

$$\frac{1}{n \ln 2} \cdot \frac{1}{2\sqrt{n}}$$

$$n = \sqrt{n} \cdot \sqrt{n}$$

$$= \lim_{n\to\infty} \frac{2}{\ln 2} \cdot \frac{1}{\sqrt{n}} = 0$$

$$\log n = O(\sqrt{n})$$

$$\log n \leq 1 \sqrt{n}$$

$c = 1, \quad n_0 = 1$

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(1+) ~~64~~ 74

Θ-notation (tight bounds)

$\Theta(g(n)) = \{ f(n)$: there exist positive constants

c_1, c_2, and n_0 such that

$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$

$\forall\, n \ge n_0 \}$

$\boxed{\text{Big-}\Omega}$ $\boxed{\text{Big-}O}$

Example (1):

$3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

rule:

$$\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$$

لو اتنين $g(n)$ ، $f(n)$ بعض (أنا) ...

تكون Big Θ ، Big-O ، Big-Ω

Order of functions

$f(n) = o(g(n)) \approx a \leq b$

$f(n) = \Omega(g(n)) \approx a \geq b$

$f(n) = \Theta(g(n)) \approx a = b$

Th

if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

$f(n) = \Theta(g(n))$

Th

if $f(n)$ is Polynomial of degree k and $a_k > 0$

Then $f(n) = \Theta(n^k)$

ex:

$4n^2 - 10n + 1 = \Theta(n^2)$

show

it is $\Omega(n^2)$

$c_1 = 3$

$n_0 = \frac{10+1}{4-3}$

$= 11$

it $O(n^2)$

$c_2 = 4 + 10 + 1$

$= 15$

$n_0 = 1$

$4n^2 - 10n + 1 = \Theta(n^2)$

$c_1 = 3$ $c_2 = 15$

$n_0 = 11$

Find Big O

$5n^6 - 10n^2 + 1 = \Theta(n^6)$

$5n^6 - 10n^3 + 1 = \Omega(n^6)$

$c_2 = 5 + 10 + 1$

$= 16$

$c_1 = 4$

$4n^3$

$\boxed{n_0 = 1}$

$\boxed{n_0 = \frac{11}{5-4} = 11}$

$\cancel{A}$ $\forall n \geq 1$

$2n^3 = \sqrt{4n^6} \leq \sqrt{5n^6 - 10n^2 + 1} \leq \sqrt{16n^6}$

$4n^3$

هنا (لم كليه متصل) ~~(9)~~ ~~(56)~~ ~~(69)~~ (80)

Ex: show that $\log(n!) = O(n \log n)$

sol

① $\log(n!) = O(n \log n)$?

② $\log(n!) = \Omega(n \log n)$

[illegible]

$\log n! = \log(1 \cdot 2 \cdots n)$

$\log n! = \log(1) + \log(2) + \cdots + \log n$

$\leq \log n + \log n + \cdots + \log n$

$\leq n \log n$

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots n$

$n! \leq n \cdot n \cdots n \cdot n$ $\rightarrow n \geq 1$

$n! \leq n^n$ $\rightarrow n \geq 1$ ((bec $\log 0$ undefined))

$\log n! \leq \log n^n$

$\log n! \leq n \log n$

$\log n!$ is $O(n \log n)$

$n_0 = 1$

$c = 1$

[illegible] ~~57~~ ~~70~~ 81

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots \frac{n}{2} \cdots n$$

$$n! \geq \frac{n}{2} \cdot \left(\frac{n}{2}+1\right) \cdot \left(\frac{n}{2}+2\right) \cdot \left(\frac{n}{2}+3\right) \cdots \left(\frac{n}{2}+\frac{n}{2}\right)$$

$$n! \geq \frac{n}{2} \cdot \frac{n}{2} \cdot \frac{n}{2} \cdots \frac{n}{2}$$

$\frac{n}{2}$ ع [illegible]

$$n! \geq \left(\frac{n}{2}\right)^{\frac{n}{2}}$$

$$n! \geq \left(\frac{n}{2}\right)^{\frac{1}{2}n}$$

$$\log n! \geq \log\left(\frac{n}{2}\right)^{\frac{1}{2}n}$$

$$\log n! \geq \frac{n}{2}\left(\log n - \log 2\right)$$

$$\log n! \geq \frac{n}{2}\left(\log n - 1\right)$$

$\boxed{8 \geq}$ n

$$\log n! \geq \frac{n}{2}\log n - \frac{n}{2}$$

$$\log n! \geq \frac{1}{2}\left(\frac{n}{2}\log n\right) + \boxed{\frac{1}{2}\left(\frac{n}{2}\log n\right) - \frac{n}{2}}$$

$$\log n! \geq \frac{1}{4} n \log n$$

$n \geq 2$

تاسك عملي 1   11   71   58   81

Prove that $(n+a)^b = \Theta(n^b)$, $b > 0$

① $(n+a)^b \leq c_1 n^b \quad \forall n \geq n_0$

② $(n+a)^b \geq c_2 n^b \quad \forall n \geq n_0$

① $n+a \leq n+|a|$

$\leq n+n \quad \forall n \geq |a|$

$n+a \leq 2n$

$(n+a)^b \leq 2^b n^b$

$c \qquad n_0$

$(n+a)^b$

$O(n^b)$

$n+a \leq 2n$

$2n \geq n+a$

$2n-n \geq a$

$n \geq a$

$n \geq |a|$

② $\boxed{(n+a)^b \geq c_2 n^b}$? $\forall n \geq n_0$

$n+a \geq \frac{1}{2}n+a$

$\frac{1}{2}n+\frac{1}{2}n+a \geq \frac{1}{2}n$

$n+a \geq \frac{1}{2}n$

$(n+a)^b \geq \left(\frac{1}{2}\right)^b n^b$

$c_2 \qquad n_0$

$\therefore n+a \geq \frac{1}{2}n$

$\therefore \frac{1}{2}n+a \geq 0$

$\boxed{n \geq -2a}$

$\frac{1}{2}n \geq -a$

$\boxed{n \geq -2a}$

$\boxed{n_0} = \max(|a|, -2a)$

لم نكملها حضلل (12) ~~(7)~~ (5) (8)

الأزم نعرف الاثباتين تكون صح

$$2^{n+1} = O(2^n)$$

أي

$$2^{n+1} = 2 \times 2^n \leq 3 \cdot 2^n \quad \forall n \geq k$$

$n_0 = 1$ $c = 3$ $n_0 = 1$

c

prove $2^{2n} \neq O(2^n)$?

Assume that $2^{2n} = O(2^n)$ الحل نفرض بالتناقض

$$2^{2n} \leq c\, 2^n$$

$$\cancel{2^n}\, 2^n \leq c\, \cancel{2^n}$$

$$2^n \leq c \rightarrow \log 2^n \leq \log c$$

$$\forall n \leq \log c$$

الأزم ال n تكون أكبر - C وهذا تناقض

المفروض تكون $n \geq n$

prove $6n + 3n \log(n^5) = O(n \log n)$

$$6n + 3n(5 \times \log n) \leq$$

$$6n + 15n \log n \leq 6n\log n + 15n\log n \quad \forall n \geq 1$$

$$\leq 21\, n \log n$$

C n_0

(83)

$2^{n+1} = O(2^n)$

$2^{n+1} = 2^n \cdot 2^1$

$2^{n+1} = 2 \cdot 2^n$

$\leq 3 \cdot 2^n$

$c = \cancel{4}\ 3\ ,\ n_0 = 1$

$2^{2n} \neq O(2^n)$

$2^{2n} \leq c\, 2^n$

$2^n \cdot \cancel{2^n} \leq c \cdot \cancel{2^n}$

$c \geq 2^n$

$2^n \leq c \Rightarrow n \log 2 \leq \log c$

$n \leq \log c$

P. 9, 10

Ex 1 Find ~~Big o~~ Big-o $O(n)$

$300n + 100 \leq 301n$

$300n + 100 \leq 301n$

$100 \leq 301n - 300n$

$100 \leq n$

$n \geq 100$; $\boxed{n_0 = 100}$

$c = 301$

Ex 2 $3n^4 - 2n^3 + 20 \in \Omega(n^4)$

$3n^4 - 2n^3 + 20 \geq 2n^4$

$c = 2$

$3n^4 - 2n^3 + 20 \geq 2n^4$

$n_0 = 1$

$10n^2 + 9 \neq O(n)$

$10n^2 \leq cn - 9$

$10n^2 \leq cn + 9n$

$10n^2 \leq n(c+9) \div n$

$n \leq \frac{c+9}{10}$

we shows there is no such constant c

$10n^2 \leq cn \longrightarrow$ fails if $n > c/10$

EX:

Show

$\log n = O(n^{\varepsilon})$ $\varepsilon > 0$

$\log n < n$

$\log (n^{\varepsilon}) < n^{\varepsilon}$

$\varepsilon \log n < n^{\varepsilon}$

86 ~~73~~ 73 ~~77~~

cont 1

EX : calculate 1) $\sum_{i=1}^{100} O(1) \xrightarrow{c} = O(1)$

2) $\sum_{i=1}^{n} O(1) = O(n)$

1) $\sum_{i=1}^{100} c_i = c_1 + c_2 + c_3 + \dots + c_{100} = c \rightarrow O(1)$

2) $\sum_{i=1}^{n} c_i = c_1 + c_2 + c_3 + \dots + c_n \leq \cancel{max(c_1, c_2, \dots c_n)}$

$\leq max(c_1, c_2, \dots c_n) \sum_{i=1}^{n} 1$

$\leq max(c_1, c_2, \dots c_n) \cdot n$

$O(n)$

~~78~~ ~~79~~ 87

order function according to growth

3^n, $\log(n!)$, $\sqrt{n}^{\,n}$, $n^{\log n}$, n, 100.000

100.000

$n \rightarrow \log(n!) \rightarrow n^{\log n}$

$3^n \rightarrow \sqrt{n}^{\,n}$

$$\log(n!) = \log(1 \times 2 \times 3 \times \dots \times n)$$

$$= \log 1 + \log 2 + \log 3 + \dots + \log n$$

$$\leq n \log n$$

$n^{\frac{n}{2}} \leftarrow \boxed{\sqrt{n}^{\,n} \quad vs \quad n^{\log n}}$

لأن الأس أو النمو أكبر

المقارنة بين الدوال

رتب

n^2 , 3^n , 2^{n^2} , $(2^n)^2$, 10^{100} , n , $\log n$,

$n!$, $(\sqrt{n})^n$, $n^{\sqrt{n}}$

الحل

$$\left.\begin{array}{l} 10^{100} \\ \log n \\ n \\ n^2 \end{array}\right\}$$ قائم وأصغير

$3^n \quad 2^{n^2}$ $\Rightarrow$ $\boxed{3^n < 2^{n^2}}$

$n \log 3 \quad n^2 \log 2$

$$\left.\begin{matrix} 3^n & n^{\sqrt{n}} \\ n\log 3 & \sqrt{n}\log n \end{matrix}\right\} \Rightarrow 3^n < n^{\sqrt{n}}$$

$$3^n < n!$$

$$\left.\begin{matrix} 2^{n^2} & \sqrt{n}^{\,n} \\ 2^{n^2} & n^{\frac{1}{2}n} \\ n^2\log 2 & \frac{1}{2}n\log n \end{matrix}\right\} \sqrt{n}^{\,n} < 2^{n^2}$$

$$\left.\begin{matrix} \sqrt{n}^{\,n} & n^{\sqrt{n}} \\ \therefore n^{\frac{1}{2}n} & n^{\sqrt{n}} \\ \therefore \sqrt{n} < \frac{1}{2}n & \end{matrix}\right\} n^{\sqrt{n}} < \sqrt{n}^{\,n}$$

$$n! = 1.2.3 \ldots \frac{n}{2} \ldots n < n^n$$

$$2^{n^2} = (2.2 \ldots 2)^n = (2^n)^n$$

$$\therefore n < 2^n \Rightarrow \boxed{n! < 2^{n^2}}$$

100

$3^n < n^{\sqrt{n}} < \sqrt{n}^{\,n} < 2^{n^2}$

$n! > \sqrt{n}^{\,n}$

$$\left.\begin{array}{l} n! < n^n \\ n! \quad 2^{n^2} \\ n^n \quad 2^{n^2} \\ n\log n < n^2 \log 2 \end{array}\right\} n! < 2^{n^2}$$

∴ الترتيب النهائي

$100^{100} \le \log n < n < n^2 < 2^n < 3^n < n^{\sqrt{n}} < \sqrt{n}^{\,n} < n! < 2^{n^2}$

Rev:- Ex3:-

* Show that $\log n = O(n^{0.01})$

$\log x < x$

$\log n < n$

$\log n^{0.01} < n^{0.01}$

im Am

$\log n < 100 n^{0.01}$

im fin

$\log n = O(n^{0.01})$

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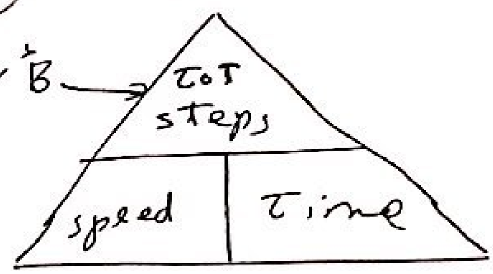
47 ~~34~~ ~~7~~ ~~32~~

22 (B)

قاعدة

B



$$\text{Tot steps} = \text{speed} \times \text{time}$$

$$\text{speed} = \frac{\text{Tot steps}}{\text{time}}$$

$$\text{time} = \frac{\text{Tot steps}}{\text{speed}}$$

Ex:

in last Example Total steps $= 2n+1$

Assume we run code with $n = 1000$, time $= 0.25$

calculate How much time when $n = 2000$, $n = 35000$?

then calculate speed of system

أولاً

$$\text{speed} = \frac{\text{Tot steps}}{\text{time}} = \frac{2(1000)+1}{0.25} \simeq 10.005 \text{ steps/s}$$

$n = 2000 \Rightarrow$ $$\text{time} = \frac{\text{Tot steps}}{\text{speed}} = \frac{2(2000)+1}{10.05} \simeq 0.4 \text{ sec}$$

$n = 35000 \Rightarrow$ $$\text{time} = \frac{\text{Tot steps}}{\text{speed}} = \frac{2(35000)+1}{10.05}$$

$$\simeq 0.7 \text{ sec}$$

47 ~~34~~ ~~7~~ ~~32~~

example :

Let $n = m$

$\# steps = 2n^2 + 2n + 1$

$\boxed{n = 1000 \ldots \quad t = 1.5 \text{ sec}}$ Given

$n = 2000 \quad t = 6 \text{ sec}$

$n = 2500 \quad t = ??$

نحسب سرعتنا بالنسبة n

بجمع عدد الخطوات ..

n	$\# steps = 2n^2 + 2n + 1$	Time
given 1000	$2(1000)^2 + 2(1000) + 1 = 2002001$	1.5
2000	$2(2000)^2 + 2(2000) + 1$	x
2500	$2(2500)^2 + 2(2500) + 1$	y

$$x = \frac{\left(2(2000)^2 + 2(2000) + 1\right) \times 1.5}{2002001} = 6 \text{ sec}$$

[illegible]

$$\text{Speed} = \frac{\# steps}{Time} = \frac{2002001}{1.5} = 1334667 \text{ steps/sec}$$