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- *Divide and Conquer*: Break a problem into sub-problems, solve each independently, and combine the solution.
- *Dynamic Programming*: Solve a problem by breaking it into *overlapping* subproblems and solving them. Invented by Richard Bellman in the 1950s.
- The word *Programming* here does not mean what you think.

- Dynamic Programming is applicable to problems that have:
  - Optimal substructure.
  - Overlapping subproblems.

Overlapping subproblems

- Consider algorithm to recursively calculate the n-th fibonacci number:

Input : Integer $n \geq 0$
Output: The n-th fibonacci number.

```
if n ≤ 1 then
    return 1
return fibonacciBad(n − 1) + fibonacciBad(n − 2)
```

Algorithm 1: *fibonacciBad*(n): calculates the n-th fibonacci number.

- How many calls result from *fibonacciBad*(5)?

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- How many are unique?

- How many calls result from *fibonacciBad*(5)?
- How many are unique?
- How can we avoid repeatedly solving *overlapping subproblems*?.

Input : Integer $n \geq 0$
Output: The n-th fibonacci number.

$M \leftarrow$ new array of size $n + 1$
for $i \leftarrow 0$ to n **do**
    $M[i] \leftarrow 0$
return $\mathit{fibRec}(n, M)$

Algorithm 2: $\mathit{fib}(n)$: Top down with *Memoization*

Input : Integer $n \geq 0$ and array $M[0 \ldots n]$
Output: The n-th fibonacci number.

if $M[n] = 0$ **then**
  if $n \leq 1$ **then**
    $M[n] = 1$
  else
    $M[n] \leftarrow \mathit{fibRec}(n-1) + \mathit{fibRec}(n-2)$
return $M[n]$

Algorithm 3: $\mathit{fibRec}(n)$: Top down with *Memoization*

Input : Integer $n \geq 0$
Output: The n-th fibonacci number.

```
M ← new array of size n + 1
M[0] ← M[1] ← 1
if n ≤ 1 then
    return M[n]
for i ← 2 to n do
    M[i] ← M[i − 1] + M[i − 2]
return M[n]
```

Algorithm 4: *fib*(n): Bottom-up.

Input : Integer $n \geq 0$
Output: The n-th fibonacci number.

$M \leftarrow$ new array of size $n + 1$
$M[0] \leftarrow M[1] \leftarrow 1$
if $n \leq 1$ **then**
    return $M[n]$
for $i \leftarrow 2$ to n **do**
    $M[i] \leftarrow M[i-1] + M[i-2]$
return $M[n]$

Algorithm 5: *fib*(n): Bottom-up.

- Time complexity?

The rod cutting problem

- Given a rod of length n, and a table showing the prices p_i for rods of sizes $1 \leq 1 \leq n$.
- What is the maximum revenue, r_n, from cutting up a rod of length n and selling the pieces.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

- Strategies?

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- Highest price?

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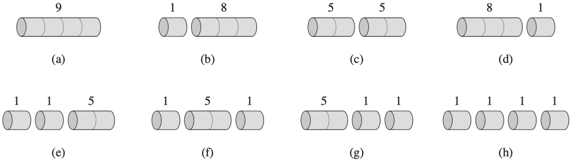
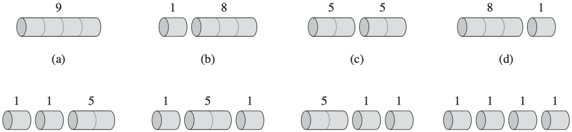
- Brute force? How many ways are there to cut a rod of size 4?



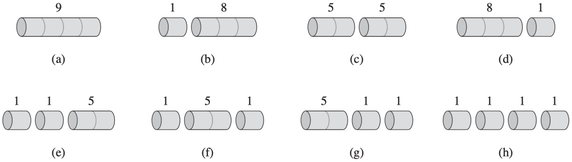
- $4 = 4$



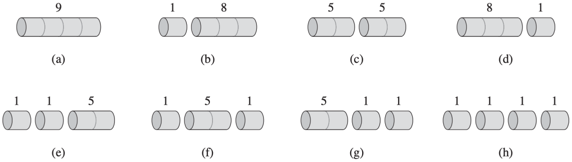
(a) (b) (c) (d)

(e) (f) (g) (h)

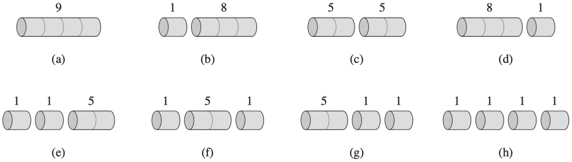
- $4 = 4$
- $4 = 1 + 3$



- $4 = 4$
- $4 = 1 + 3$
- $4 = 2 + 2$



- $4 = 4$
- $4 = 1 + 3$
- $4 = 2 + 2$
- $4 = 3 + 1$



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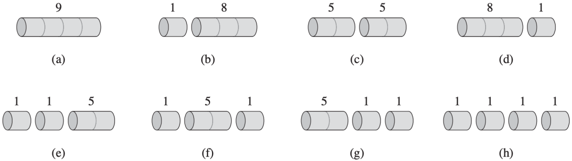

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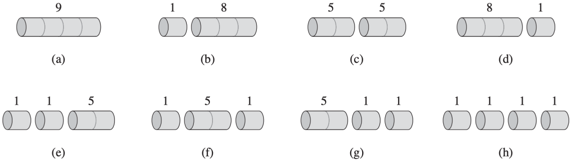

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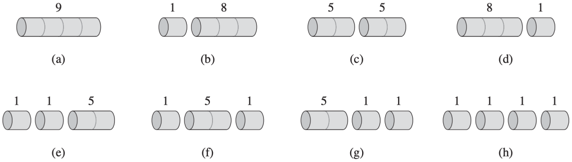

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  - If we cut at 2 meters: we split the rod into 2 meters and 3 meters.
    - Maximum revenue possible $= r_2 + r_3$.

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  - If we cut at 3 meters: we split the rod into 3 meters and 2 meters.

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  - If we cut at 3 meters: we split the rod into 3 meters and 2 meters.
    - Maximum possible revenue $= r_3 + r_2$

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- $r_5 = max(p_5, r_1 + r_4, r_2 + r_3, r_3 + r_2, r_4 + r_1)$

- $r_n = max(p_n, r_{n-1} + r_1, r_{n-2} + r_2, \ldots, r_1 + r_{n-1})$.

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- $r_n = max_{1 \leq i \leq n}(p_i + r_{n-i})$
- r_n spawns n subproblems.

- Recursive algorithm to find r_n:

Input : Size of the rod, n, and p, an array of prices
Output: The maximum revenue r_n from breaking and selling pieces

if $n = 0$ **then**
  return 0
$q \leftarrow -\infty$
for $i \leftarrow 1$ to n **do**
  $q \leftarrow max(q, p[i] + CUTROD(p, n - i))$
return q

Algorithm 6: $CUTROD(p, n)$: calculates the maximum revenue. Top-down, recursive.

- Running time:

$$T(n) = \begin{cases} 1 + \sum_{j=0}^{n-1} T(j) & n > 0 \\ 1 & n = 0 \end{cases}$$

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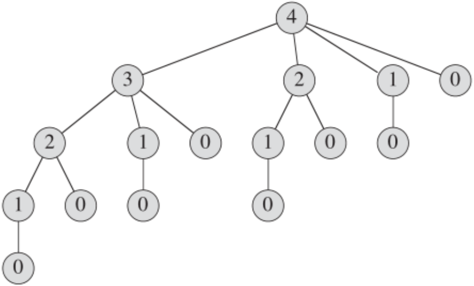
- Is this good?
- It is actually equal to 2^n.
- Why is it so bad?


4
3
2
1
0
2
1
0
1
0
0
1
0
0
0
0

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- Can either do it top-down or bottom-up.
  - Bottom-up: calculate r_1, then r_2, then $r_3, \ldots$.
  - Top-down: Very similar to *CUTROD*, but with *memoization*.

Memoization

Input : Size of the rod, n, and p, an array of prices
Output: The maximum revenue r_n from breaking and selling pieces

```
r[0 ... n] a new array
for i ← 0 to n do
    r[i] ← −∞
return MemoizedCutRodAux(p, n, r)
```

Algorithm 7: *MemoizedCutRod*(p, n)

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Input : Size of the rod n, p an array of prices, and r an array of revenues

Output: The maximum revenue r_n from breaking and selling pieces

if $r[n] \geq 0$ **then**
    return $r[n]$
if $n = 0$ **then**
    $q \leftarrow 0$
else
    $q \leftarrow -\infty$
    for $i \leftarrow 1$ to n **do**
        $q \leftarrow max(q, p[i] + MemoizedCutRodAux(p, n - i, r))$
$r[n] \leftarrow q$
return q

Algorithm 8: $MemoizedCutRodAux(p, n, r)$: Memoized. Recursive.

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Input : Size of the rod, n, and p, an array of prices
Output: The maximum revenue r_n from breaking and selling pieces

```
let r[0 ... n] be a new array
r[0] ← 0
for j ← 1 to n  do
    q ← −∞
    for i = 1 to j do
        q ← max(q, p[i] + r[j − i])
    r[j] = q
return  r[n]
```

Algorithm 9: *BottomUpCutRod*(p, n)

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- Let's see how to do it for *BottomUpCutRod*:

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

Input : Size of the rod, n, and p, an array of prices
Output: The maximum revenue r_n from breaking and selling pieces

let $r[0 \dots n]$ and $s[0 \dots n]$ be new arrays
$r[0] \leftarrow 0$
for $j \leftarrow 1$ to n **do**
    $q \leftarrow -\infty$
    for $i = 1$ to j **do**
        if $q < p[i] + r[j-i]$ **then**
            $q \leftarrow p[i] + r[j-i]$
            $s[j] \leftarrow i$
    $r[j] = q$
return r and s

Algorithm 10: *ExtendedBottomUpCutRod*(p, n)

i	0	1	2	3	4	5	6	7	8	9	10
$r[i]$	0	1	5	8	10	13	17	18	22	25	30
$s[i]$	0	1	2	3	2	2	6	1	2	3	10

Input : Size of the rod, n, and p, an array of prices
Output: How to cut up the rod to achieve the maximum revenue.

$(r, s) \leftarrow \mathit{ExtendedBottomUpCutRod}(p, n)$
while $n > 0$ **do**
    print $s[n]$
    $n \leftarrow n - s[n]$

Algorithm 11: $\mathit{PrintCutRodSolution}(p, n)$: prints the way to optimally cut the rod

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- Reminder: Two matrics A and B can be multiplied only if A has dimensions $n \times m$ and B has dimensions $m \times k$.

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$$A_1 A_2 A_3 \cdots A_{n-1} A_n$$

- Where A_1 has dimension $p_0 \times p_1$, A_2 has dimension $p_1 \times p_2$ and so on.

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- In what order do you perform these multiplications?
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- With dimensions:
  - $A : 10 \times 100$
  - $B : 100 \times 5$
  - $C : 5 \times 50$

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  - $A(BC)$:
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- How many ways are there to multiply (or parenthesize) this matrix product?
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- Brute force?

- How many ways are there to multiply n matrices?

$$P(n) = \begin{cases} 1 & n = 1 \\ \sum_{k=1}^{n-1} P(k)P(n-k) & n > 1 \end{cases}$$

which is $\Omega(2^n)$, as you saw in HW 3.

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- So brute force is not feasible...

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- Contradiction.
- This *Optimal substructure* helps us construct optimal solutions to the problem by using optimal solutions to smaller subproblems.

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$$m[i,j] = \begin{cases} 0 & i = j \\ max_{i \le k < j}(m[i,k] + m[k+1,j] + p_{i-1}p_k p_j) & i \ne j \end{cases}$$

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- We define $s[i,j] =$ the value of k for A_{ij}
- We can now write a recursive algorithm to find $m[1,n]$ and $s[1,n]$.

Input : List of matrices $A_i \cdots A_j$, and dimensions array $p[i-1 \ldots j]$. Matrix A_r is of dimensions $p[r-1] \times p[r]$

Output: Minimum cost of multiplying the chain of matrices

1: **if** $j \leq i$ **then**
2:     **return** 0
3: $q \leftarrow \infty$
4: **for** $k \leftarrow i$ to $j-1$ **do**
5:     $q \leftarrow max(q, parenthesize1(A, i, k, p) + parenthesize1(A, k+1, j, p) + p[i-1] * p[k] * p[j]$
6: **return** q

Algorithm 12: parenthesize1(A,i,j,p)

- How bad is this?

- $$T(n) = \begin{cases} \Theta(1) & n = 1 \\ \sum_{k=1}^{n} T(i) + T(n-i) + \Theta(1) & n > 1 \end{cases}$$

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- Many subproblems are solved over and over!.

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- So we can greatly improve $\Omega(2^n)$ if we don't *repeat* solving the same subproblems.
- Overlapping subproblems!

Bottom up solution

Input : List of matrices $A_1 \cdots A_n$, and dimensions array $p[0 \ldots n]$. Matrix A_r is of dimensions $p[r-1] \times p[r]$

Output: Minimum cost of multiplying the chain of matrices

let $m[1 \cdots n, 1 \cdots n]$ and $s[1 \cdots n, 1 \cdots n]$ be new arrays
for $i \leftarrow 1$ to n **do**
    $m[i,i] \leftarrow 0$
for $l \leftarrow 2$ to n **do**
    for $i \leftarrow 1$ to $n - l + 1$ **do**
        $j \leftarrow i + l - 1$
        $m[i,j] \leftarrow \infty$
        for $k \leftarrow i$ to $j - 1$ **do**
            $q \leftarrow m[i,k] + m[k+1,j] + p[i-1] \times p[k] \times p[j]$
            if $q < m[i,j]$ **then**
                $m[i,j] \leftarrow q$
                $s[i,j] \leftarrow k$
return m,s

Algorithm 13: parenthesizeBottomUp(A,n,p)

- Time complexity?

- Time complexity?
- $\Theta(n^3)$

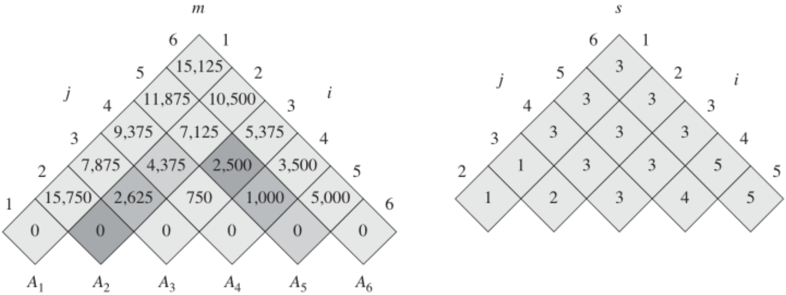

Figure 15.5 The m and s tables computed by MATRIX-CHAIN-ORDER for $n = 6$ and the following matrix dimensions:

matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

The tables are rotated so that the main diagonal runs horizontally. The m table uses only the main diagonal and upper triangle, and the s table uses only the upper triangle. The minimum number of scalar multiplications to multiply the 6 matrices is $m[1, 6] = 15{,}125$. Of the darker entries, the pairs that have the same shading are taken together in line 10 when computing

$$
\begin{aligned}
m[2,5] &= \min \begin{cases} m[2,2] + m[3,5] + p_1 p_2 p_5 &= 0 + 2500 + 35 \cdot 15 \cdot 20 &= 13{,}000 , \\ m[2,3] + m[4,5] + p_1 p_3 p_5 &= 2625 + 1000 + 35 \cdot 5 \cdot 20 &= 7125 , \\ m[2,4] + m[5,5] + p_1 p_4 p_5 &= 4375 + 0 + 35 \cdot 10 \cdot 20 &= 11{,}375 \end{cases} \\
&= 7125 .
\end{aligned}
$$

```
MEMOIZED-MATRIX-CHAIN(p)
n = p.length - 1
let m[1..n, 1..n] be a new table
for i = 1 to n
    for j = i to n
        m[i, j] = ∞
return LOOKUP-CHAIN(m, p, 1, n)
```

```
LOOKUP-CHAIN(m, p, i, j)
if m[i, j] < ∞
    return m[i, j]
if i == j
    m[i, j] = 0
else for k = i to j - 1
        q = LOOKUP-CHAIN(m, p, i, k)
            + LOOKUP-CHAIN(m, p, k + 1, j) + p_{i-1} p_k p_j
        if q < m[i, j]
            m[i, j] = q
return m[i, j]
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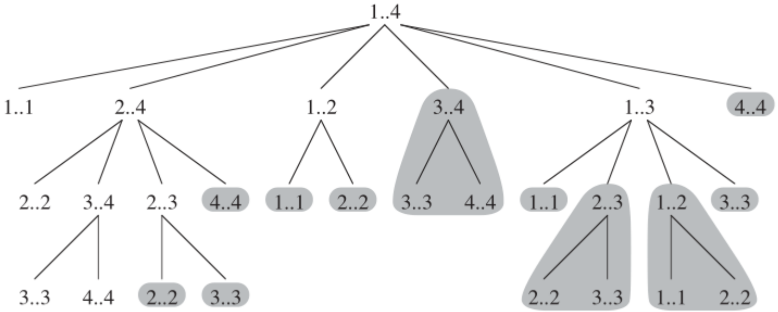
- We can conclude the algorithm is $O(n^3)$.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- The multiplications previous to that? at $s[1, s[1, n]]$ and $s[s[1, n] + 1, n]$.
- Recursively:

Input : Table s and indices i and j.
Output: Prints parenthesization.

if $i = j$ **then**
    print A_i
else
    print (
    printParens(s, i, s[i,j])
    printParens(s, s[i,j]+1, j)
    print)

Algorithm 14: printParens(s, i, j)

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  - The cost of solving one subproblem.

Building a DP algorithm

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4. Identify the dimensions and values for a table with one entry for each subproblem.
5. Give either a bottom up or top-down memoization algorithm.

Building a DP algorithm

1. Describe the problem's solution in a general way that can also describe the solutions to subproblems.
2. Find the optimal substructure and define it concretely.
3. Use the optimal substructure to recursively define the optimal solution to the problem in terms of subproblems.
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8. Using the second table, construct an optimal solution.

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  - The total stolen value $\sum_{i \in S} v_i$ is maximized.
  - The total weight of stolen goods $sum_{i \in S} w_i$ does not exceed the knapsack's capacity, W.

- Consider the example:

	Item 1	Item 2	Item 3
v_i	4	4	6
w_i	1	2	4

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- In other words, $m(S, W) = m(S \setminus \{v_k\}, W - w_k) + v_k$

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- We need to take a step back and think of a better way to represent the subproblems and their solution.

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- $m(n, W)$: *value* of the optimal solution for items $\{1, 2, \ldots, n\}$ and knapsack capacity W.

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- We can use this to define $m(i, j)$ recursively:

$$m(i,j) = \begin{cases} 0 & i = 0 \\ m(i-1,j) & w_i > j \\ max\{v_i + m(i-1, j-w_i), m(i-1,j)\} & w_i \leq j \end{cases}$$

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- Bottom up algorithm to find $m(n, W)$?
- How many rows and columns in m?

Input : List of values $\{v_1, v_2, \ldots, v_n\}$,integer weights $\{w_1, w_2, \ldots, w_n\}$, and integer capacity W

Output: Value of optimal solution to the 0,1-knapsack problem.

```
for i ← 1 to n do
    m[i,0] ← 0
for j ← 1 to W do
    m[0,j] ← 0
for i ← 1 to n do
    for j ← 1 to W do
        if w_j > j then
            m[i,j] ← m[i−1,j]
        else
            m[i,j] ← max{v[i] + m[i−1, j−w[i]], m[i−1,j]}
return m
```

- Consider the exmaple, with capacity $W = 5$:

item	weight	value
1	2	12
2	1	10
3	3	20
4	2	15

		0	1	2	3	4	5
	0						
$w_1 = 2, v_1 = 12$	1						
$w_2 = 1, v_2 = 10$	2						
$w_3 = 3, v_3 = 20$	3						
$w_4 = 2, v_4 = 15$	4						

		0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 12$	1	0					
$w_2 = 1, v_2 = 10$	2	0					
$w_3 = 3, v_3 = 20$	3	0					
$w_4 = 2, v_4 = 15$	4	0					

		0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1, v_2 = 10$	2	0					
$w_3 = 3, v_3 = 20$	3	0					
$w_4 = 2, v_4 = 15$	4	0					

		0	1	2	3	4	5
	0	0	0	0	0	0	0
$w_1 = 2, v_1 = 12$	1	0	0	12	12	12	12
$w_2 = 1, v_2 = 10$	2	0	10	12	22	22	22
$w_3 = 3, v_3 = 20$	3	0					
$w_4 = 2, v_4 = 15$	4	0					

		0	1	2	3	4	5
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$w_4 = 2, v_4 = 15$	4	0					

		0	1	2	3	4	5
	0	0	0	0	0	0	0
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$w_2 = 1, v_2 = 10$	2	0	10	12	22	22	22
$w_3 = 3, v_3 = 20$	3	0	10	12	22	30	32
$w_4 = 2, v_4 = 15$	4	0	10	15	25	30	37

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- This can be worse than exponential!
- Can we improve that? Not by much, but we can.
- Top-down approach with memoization does not solve *all* the subproblems.

if $m[i, j] < 0$ **then**
    if $w[i] > j$ **then**
        $m[i, j] \leftarrow m[i - 1, j]$
    else
        $m[i, j] \leftarrow max\{v[i] + m[i - 1, j - w[i]], m[i - 1, j]\}$
return $m[i, j]$

- How can we construct a solution from m?

$i \leftarrow n$
$j \leftarrow W$
while $i > 0$ **do**
    if $w[i] \leq j$ and $v[i] + m[i-1, j - w[i]] > m[i-1, j]$ **then**
        Print i
        $i \leftarrow i - 1$
        $j \leftarrow j - w[j]$
    else
        $i \leftarrow i - 1$