Graph Algorithms

Graphs

- A graph G = (V, E)
  - V = set of vertices
  - E = set of edges = subset of $V \times V$
  - Thus $|E| = O(|V|^2)$

Graph Variations

- Variations:
  - A *connected graph* has a path from every vertex to every other
  - In an *undirected graph:*
    - Edge (u,v) = edge (v,u)
  - In a *directed* graph:
    - Edge (u,v) goes from vertex u to vertex v, notated u→v

Graph Variations

- More variations:
  - A *weighted graph* associates weights with either the edges or the vertices
    - E.g., a road map: edges might be weighted w/ distance

Graphs

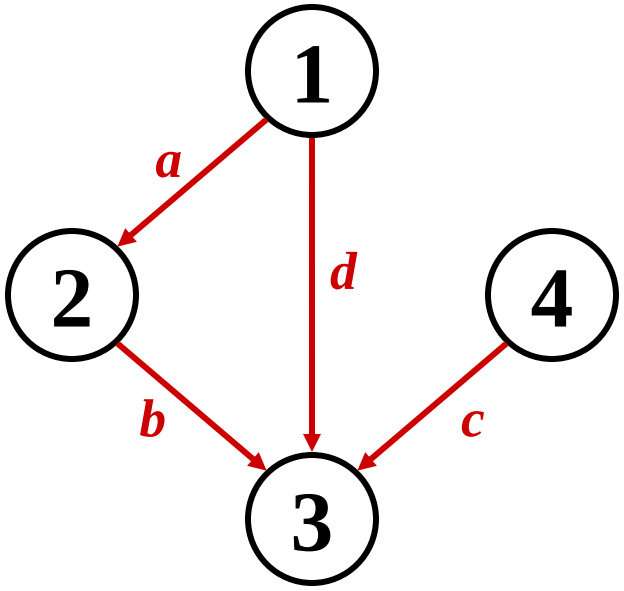
- We will typically express running times in terms of |E| and |V|
  - If $|E| \approx |V|^2$ the graph is *dense*
  - If $|E| \approx |V|$ the graph is *sparse*
- If you know you are dealing with dense or sparse graphs, different data structures may make sense

Representing Graphs

- Assume V = {1, 2, …, n}
- An *adjacency matrix* represents the graph as a n x n matrix A:
  - A[i, j] = 1 if edge (i, j) $\in$ E (or weight of edge)
    = 0 if edge (i, j) $\notin$ E

Graphs: Adjacency Matrix

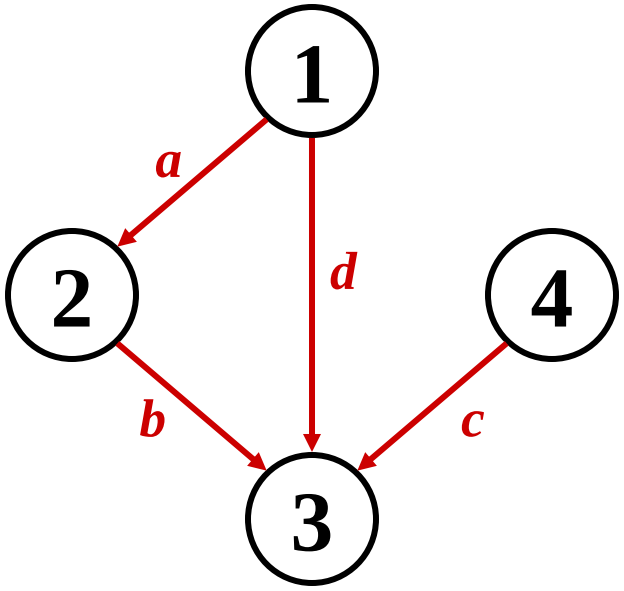
- Example:



A	1	2	3	4
1				
2				
3			??	
4				

Graphs: Adjacency Matrix

- Example:



A	1	2	3	4
1	0	1	1	0
2	0	0	1	0
3	0	0	0	0
4	0	0	1	0

Graphs: Adjacency Matrix

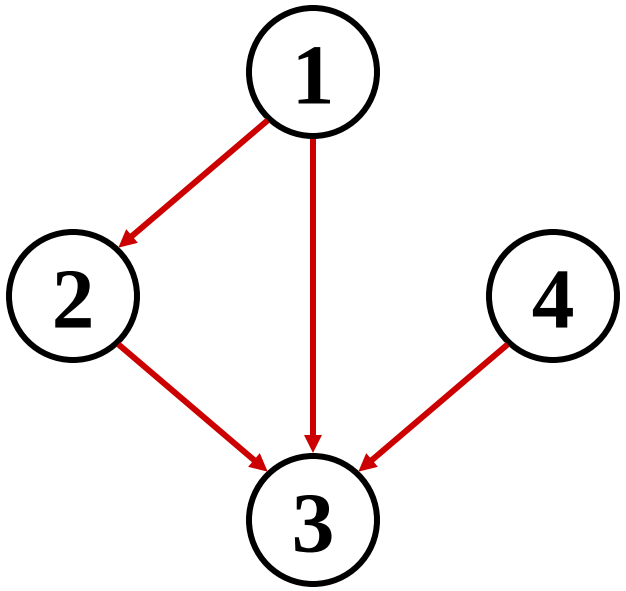
- *How much storage does the adjacency matrix require?*
- A: $O(V^2)$
- *What is the minimum amount of storage needed by an adjacency matrix representation of an undirected graph with 4 vertices?*
- A: 6 bits
  - Undirected graph $\rightarrow$ matrix is symmetric
  - No self-loops $\rightarrow$ don't need diagonal

Graphs: Adjacency Matrix

- The adjacency matrix is a dense representation
  - Usually too much storage for large graphs
  - But can be very efficient for small graphs
- Most large interesting graphs are sparse
  - For this reason the *adjacency list* is often a more appropriate respresentation

Graphs: Adjacency List

- Adjacency list: for each vertex $v \in V$, store a list of vertices adjacent to v
- Example:
  - Adj[1] = {2,3}
  - Adj[2] = {3}
  - Adj[3] = {}
  - Adj[4] = {3}
- Variation: can also keep a list of edges coming *into* vertex



Graphs: Adjacency List

- How much storage is required?
  - The *degree* of a vertex v = # incident edges
    - Directed graphs have in-degree, out-degree
  - For directed graphs, # of items in adjacency lists is
    $$\Sigma \text{ out-degree}(v) = |E|$$
    takes $\Theta(V + E)$ storage
  - For undirected graphs, # items in adj lists is
    $$\Sigma \text{ degree}(v) = 2\,|E|$$
    also $\Theta(V + E)$ storage
- So: Adjacency lists take O(V+E) storage

Graph Searching

- Given: a graph G = (V, E), directed or undirected
- Goal: methodically explore every vertex and every edge
- Ultimately: build a tree on the graph
  - Pick a vertex as the root
  - Choose certain edges to produce a tree
  - Note: might also build a *forest* if graph is not connected

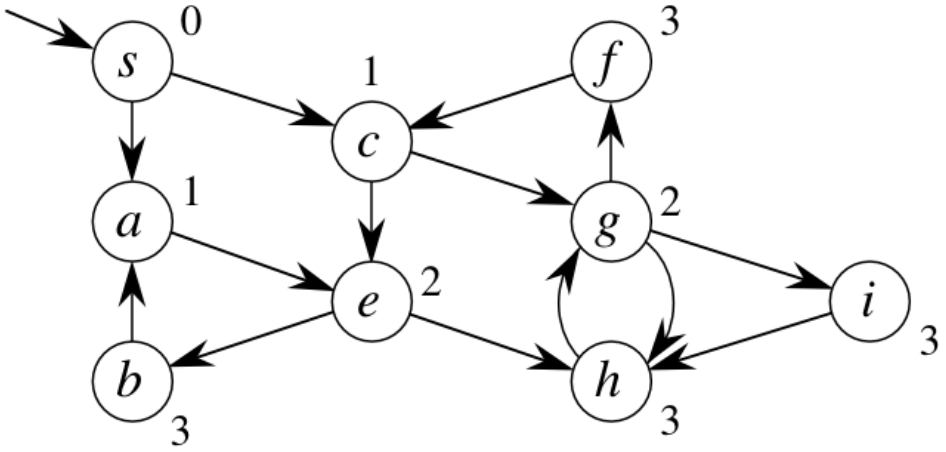
Traverse of Graph

- Given graph $G = (V, E)$
- Given a source node $s \in V$
  - Visit all nodes in V reachable from s

- Need an efficient strategy to achieve that
  - BFS: Breadth-first search

Intuition

- Starting from source node *s*,
  - Spread a wavefront to visit other nodes
- Example
  - Imagine given a tree
  - What if a graph?



Intuition cont.

- $\delta(u, v)$:
  - Distance (smallest # edges) from node u to v in G
- Goal:
  - Start from source s, first visit those nodes of distance 1 from s, then 2, and so on.
  - *More formally: BFS*
    - *Input: Graph $G=(V, E)$ and source node $s \in V$*
    - *Output: $\delta(s, u)$ for every $u \in V$*
      - *$\delta(s,u) = \infty$ if u is unreachable from s*
  - *Assume adjacency list representation for G*

Breadth-First Search

- "Explore" a graph, turning it into a tree
  - One vertex at a time
  - Expand frontier of explored vertices across the *breadth* of the frontier
- Builds a tree over the graph
  - Pick a *source vertex* to be the root
  - Find ("discover") its children, then their children, etc.

Breadth-First Search

- Will associate vertex “colors” to guide the algorithm
  - White vertices have not been discovered
    - All vertices start out white
  - Grey vertices are discovered but not fully explored
    - They may be adjacent to white vertices
  - Black vertices are discovered and fully explored
    - They are adjacent only to black and gray vertices
- Explore vertices by scanning adjacency list of grey vertices

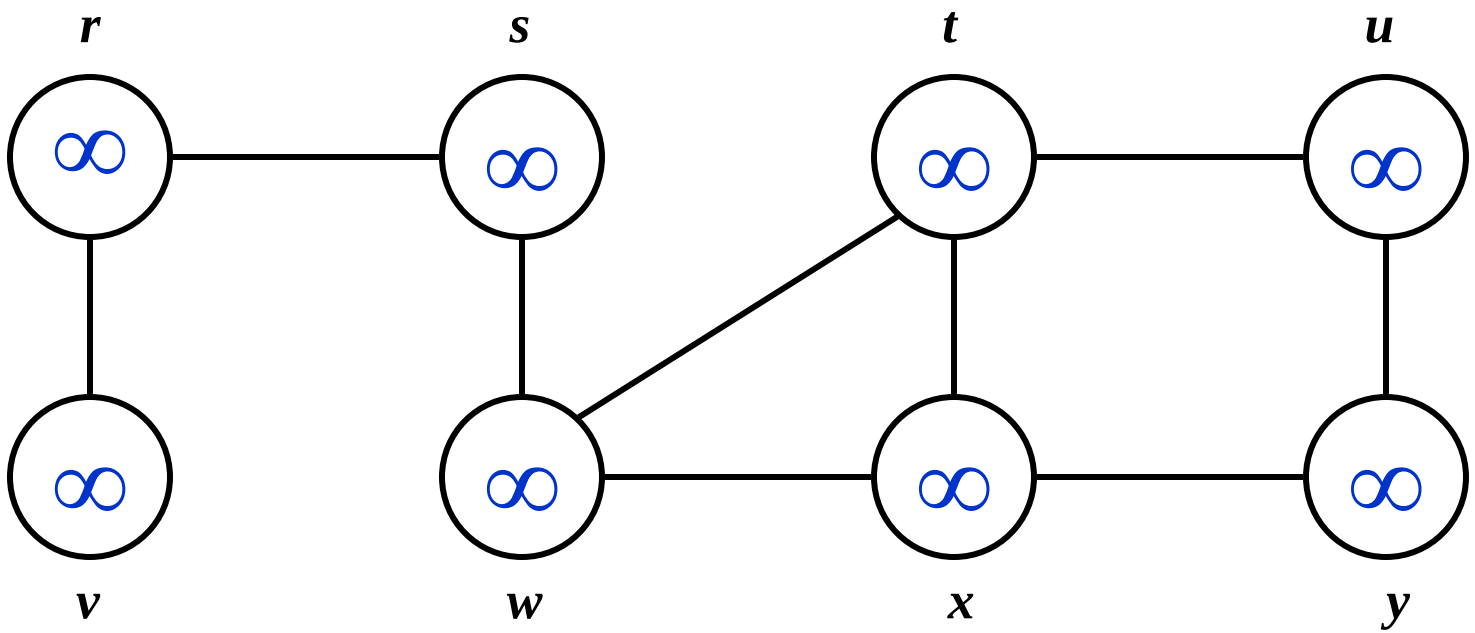
Breadth-First Search

```
BFS(G, s) {
    initialize vertices;
    Q = {s};          // Q is a queue ; initialize to s
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

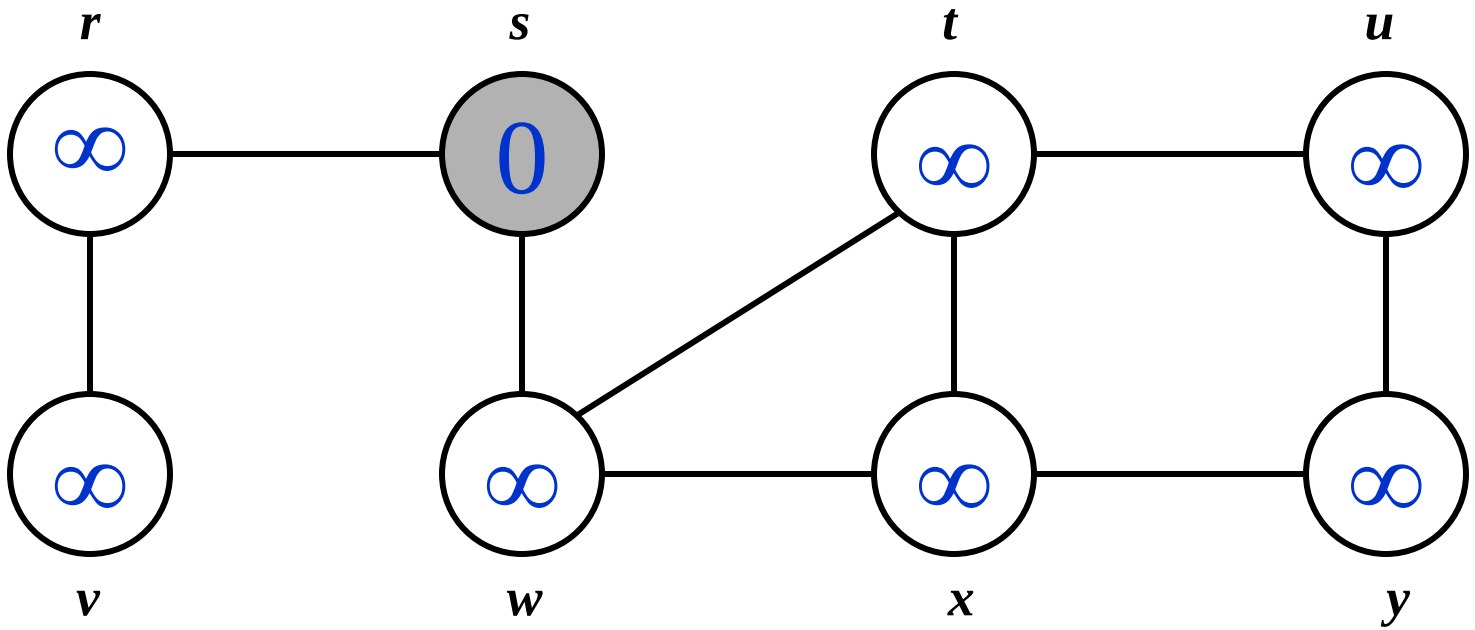
What does `v->d` represent?

What does `v->p` represent?

Breadth-First Search: Example

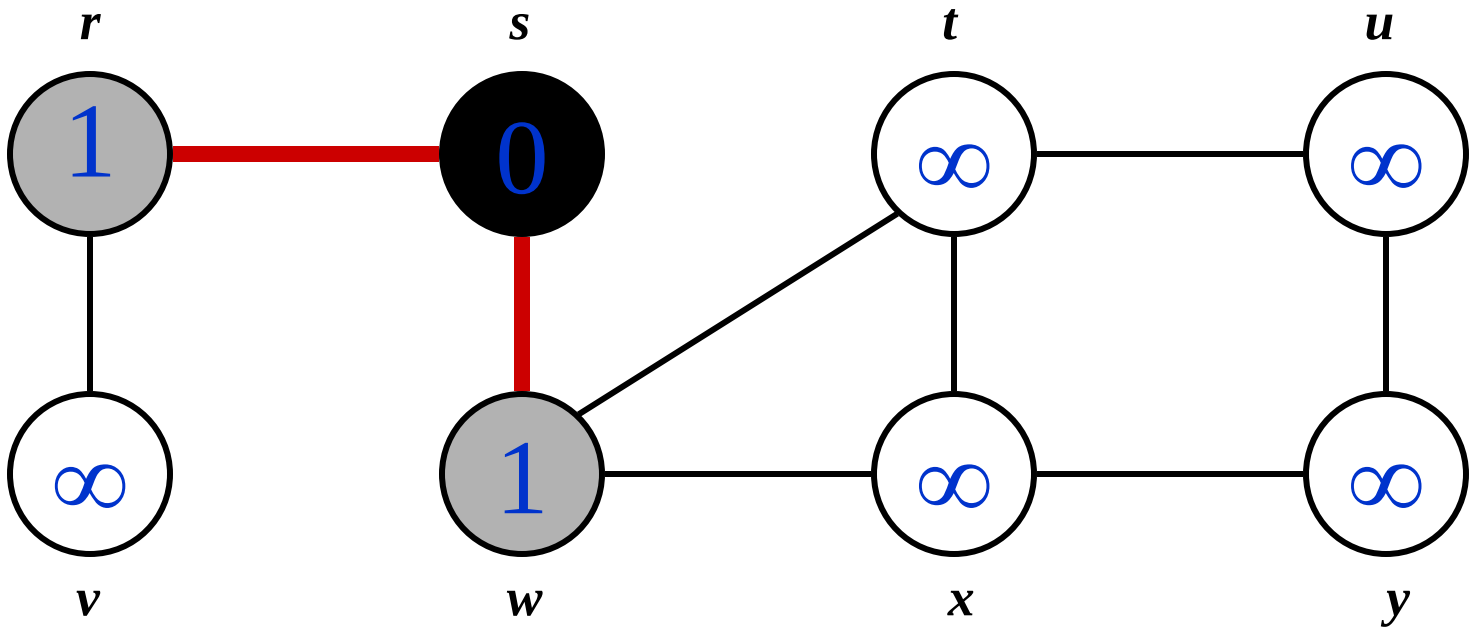


Breadth-First Search: Example



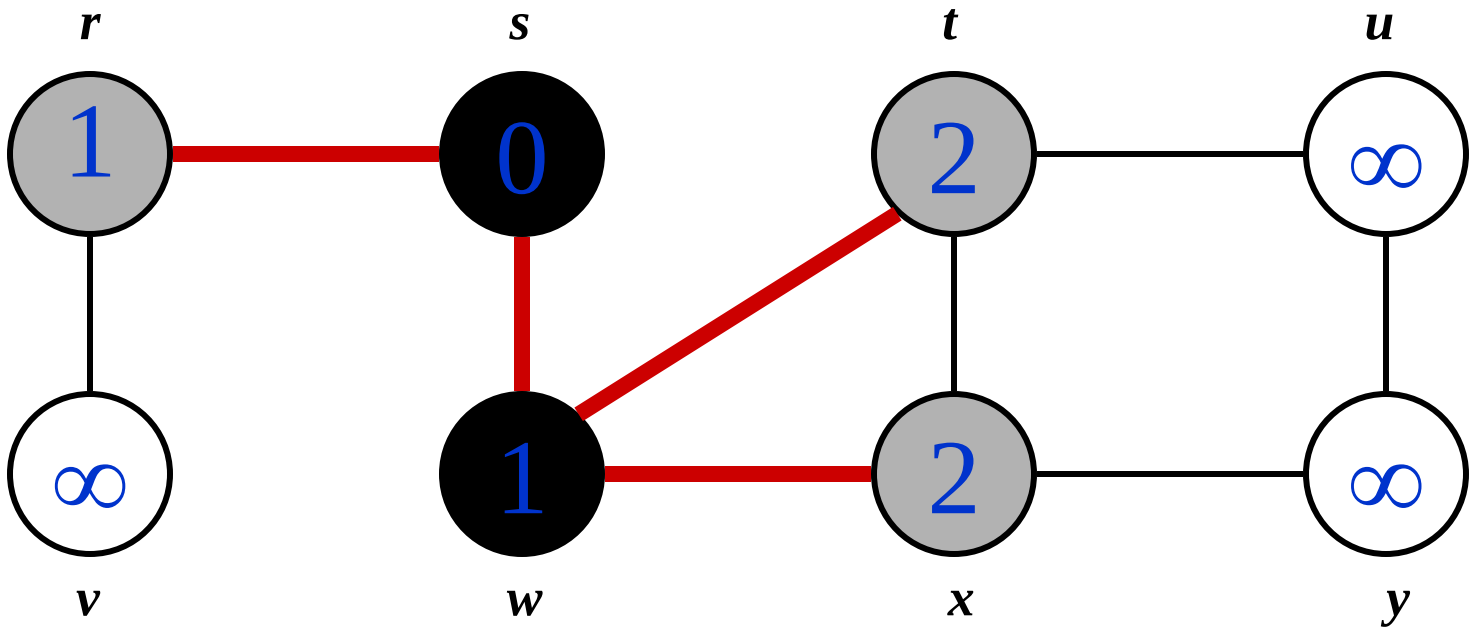
Q: | *s* |

Breadth-First Search: Example



Q: | *w* | *r* |

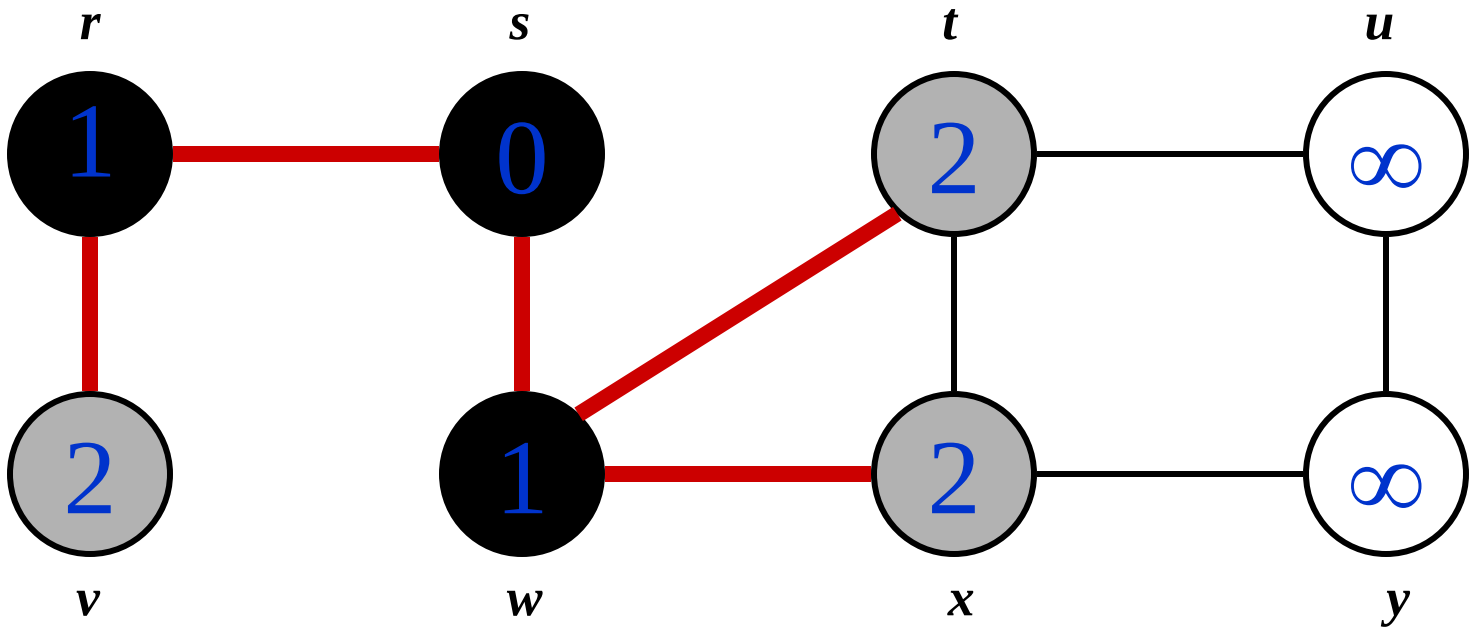
Breadth-First Search: Example



Q:

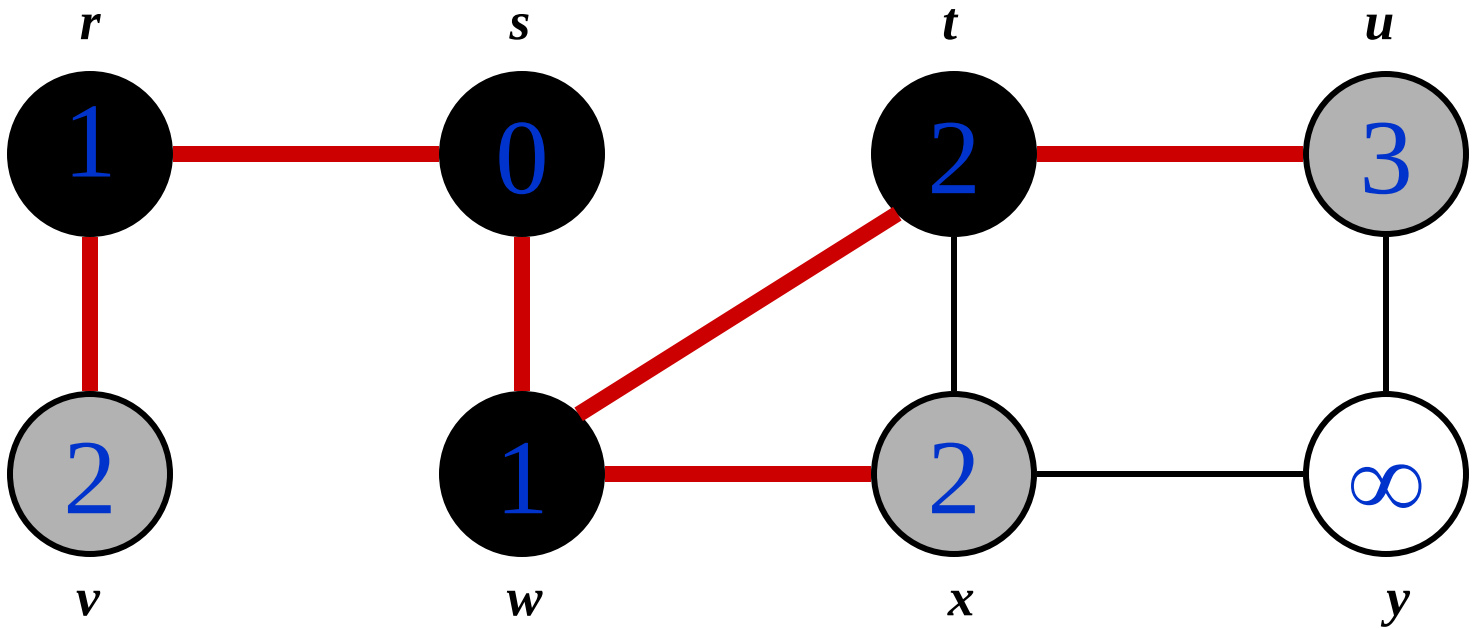
r	t	x

Breadth-First Search: Example



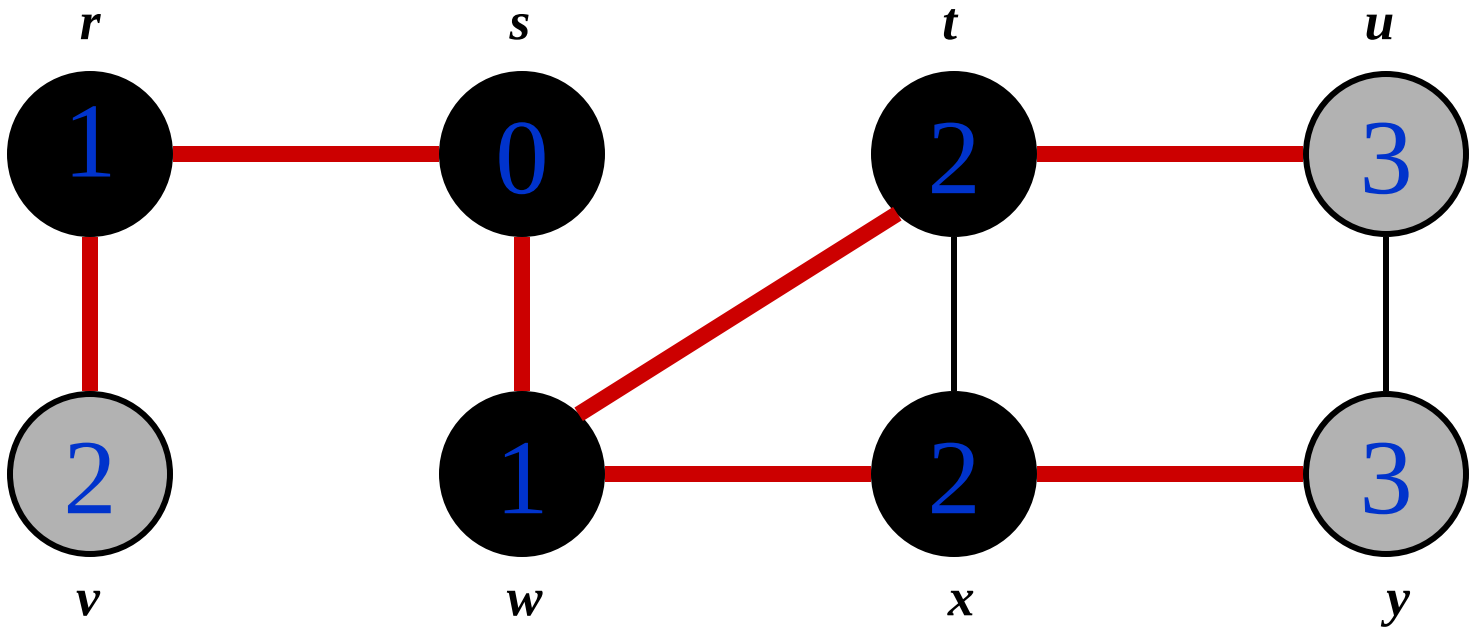
Q: | *t* | *x* | *v* |

Breadth-First Search: Example



Q: | *x* | *v* | *u* |

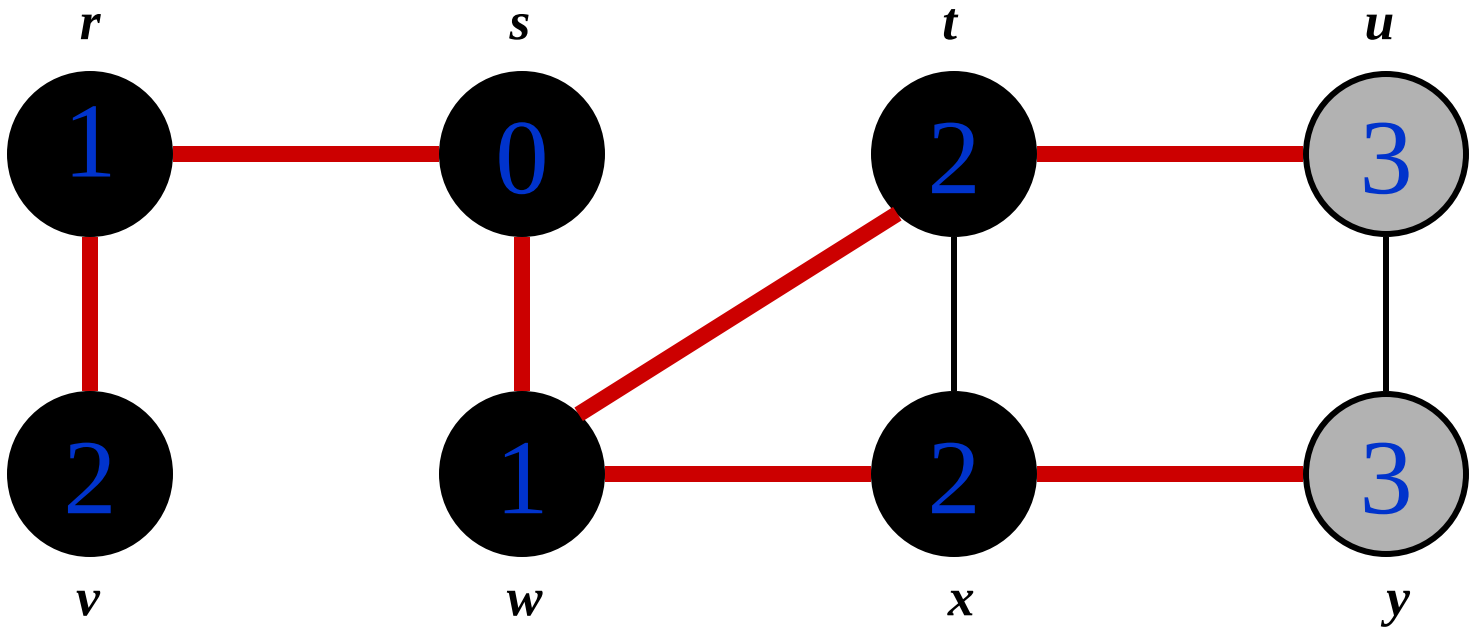
Breadth-First Search: Example



Q:

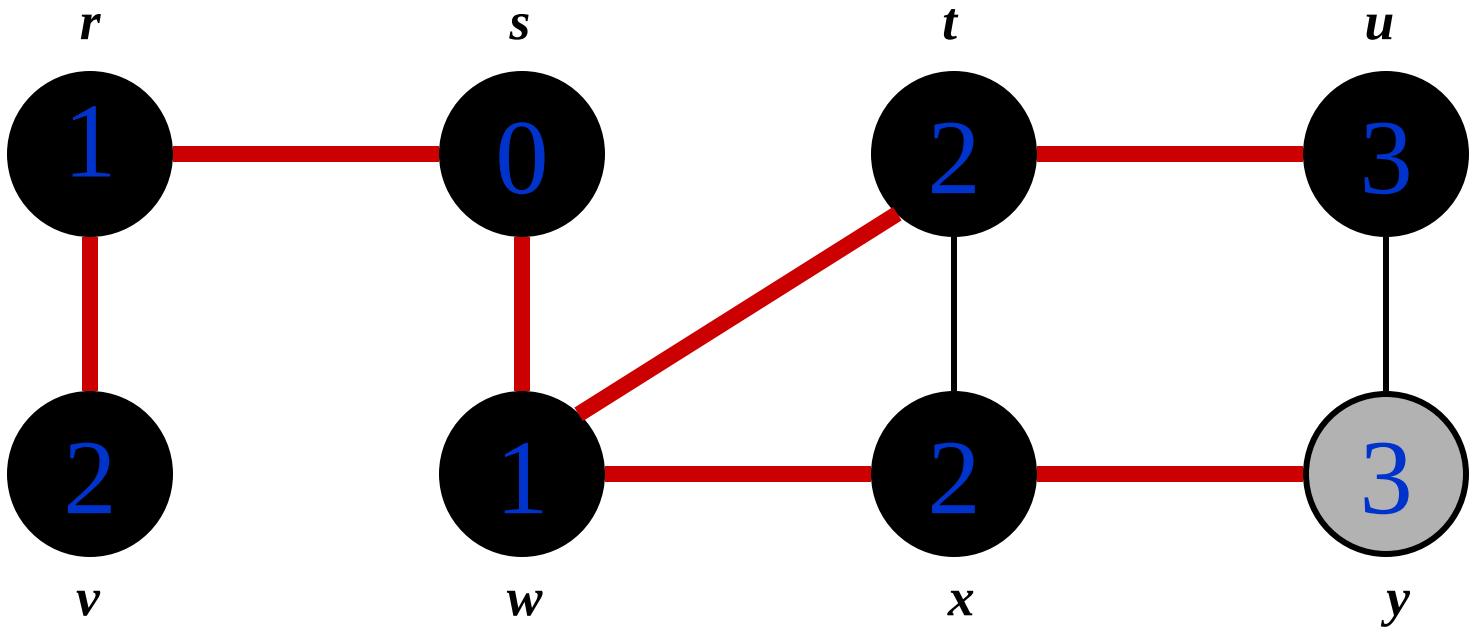
v	u	y

Breadth-First Search: Example



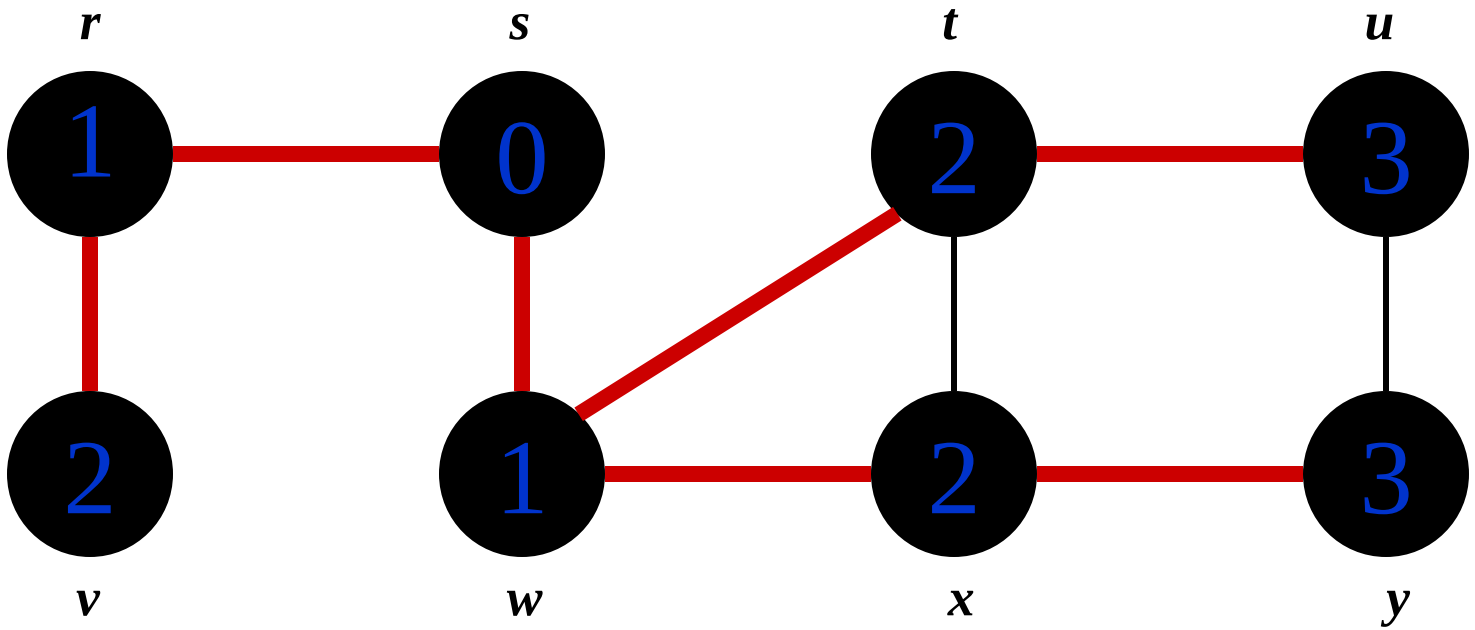
Q: | u | y |

Breadth-First Search: Example



Q: | *y* |

Breadth-First Search: Example



$Q: \quad \emptyset$

BFS: The Code Again

```
BFS(G, s) {
    initialize vertices;
    Q = {s};
    while (Q not empty) {
        u = RemoveTop(Q);
        for each v ∈ u->adj {
            if (v->color == WHITE)
                v->color = GREY;
                v->d = u->d + 1;
                v->p = u;
                Enqueue(Q, v);
        }
        u->color = BLACK;
    }
}
```

Touch every vertex: O(V)

u = every vertex, but only once

So v = every vertex that appears in some other vert's adjacency list

What will be the running time?

Total running time: O(V+E)

Breadth-First Search: Properties

- BFS calculates the *shortest-path distance* to the source node
  - Shortest-path distance $\delta(s,v)$ = minimum number of edges from s to v, or ∞ if v not reachable from s
- BFS builds *breadth-first tree*, in which paths to root represent shortest paths in G
  - Thus can use BFS to calculate shortest path from one vertex to another in O(V+E) time

Depth-First Search

- *Depth-first search* is another strategy for exploring a graph
  - Explore "deeper" in the graph whenever possible
  - Edges are explored out of the most recently discovered vertex *v* that still has unexplored edges
  - When all of *v*'s edges have been explored, backtrack to the vertex from which *v* was discovered

Depth-First Search

- Vertices initially colored white
- Then colored gray when discovered
- Then black when finished

Depth-First Search: The Code

```
DFS(G)
{
    for each vertex u ∈ G->V
    {
        u->color = WHITE;
    }
    time = 0;
    for each vertex u ∈ G->V
    {
        if (u->color == WHITE)
            DFS_Visit(u);
    }
}
```

```
DFS_Visit(u)
{
    u->color = GREY;
    time = time+1;
    u->d = time;
    for each v ∈ u->Adj[]
    {
        if (v->color == WHITE)
            DFS_Visit(v);
    }
    u->color = BLACK;
    time = time+1;
    u->f = time;
}
```

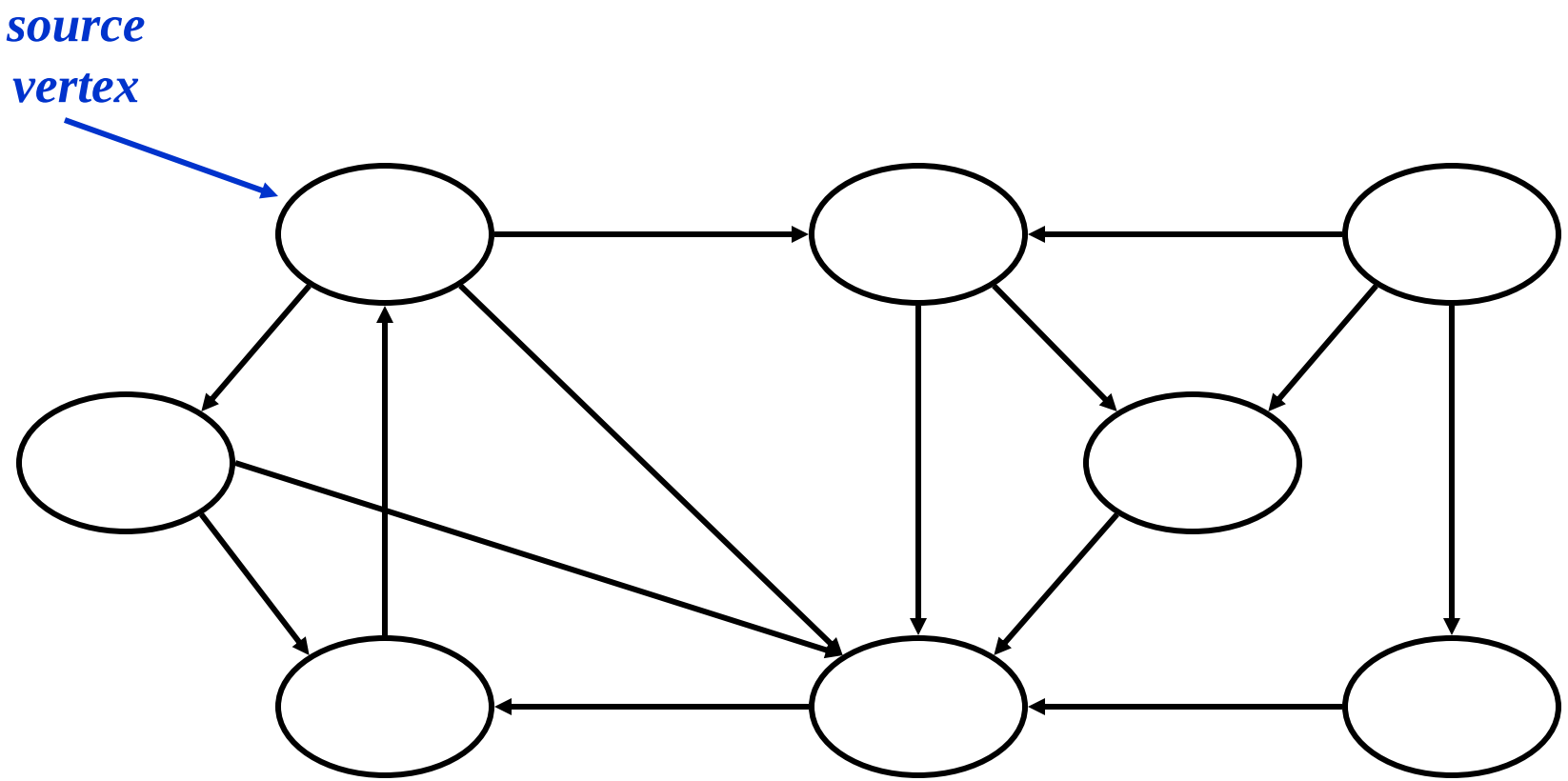
Depth-First Search: Running Time

- *The loops on lines 1-3 and lines 5-7 of DFS take time* ***Θ(V),*** *exclusive of the time to execute the calls to DFS-VISIT.*

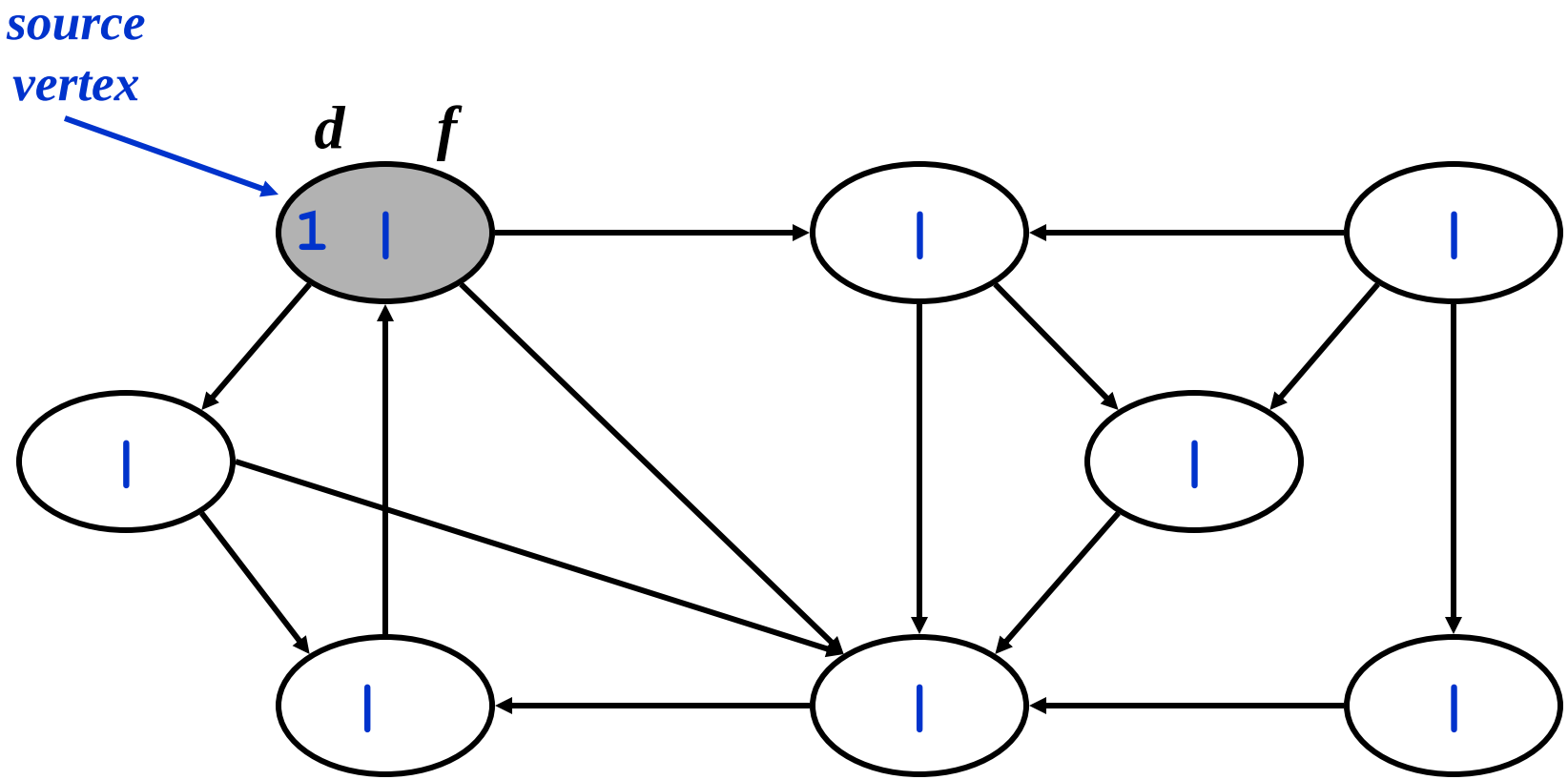
- *The procedure DFS-VISIT is called exactly once for each vertex v* $\in$ *V , since DFS-VISIT is invoked only on white vertices and the first thing it does is paint the vertex gray.*

- *During an execution of DFS-VISIT(v), the loop on lines 4-7 is executed* $|Adj[v]|$ *times. Since the total cost of executing lines 4-7 of DFS-VISIT is* $\Theta(E)$. *The running time of DFS is therefore* $\Theta(V + E)$.

DFS Example



DFS Example



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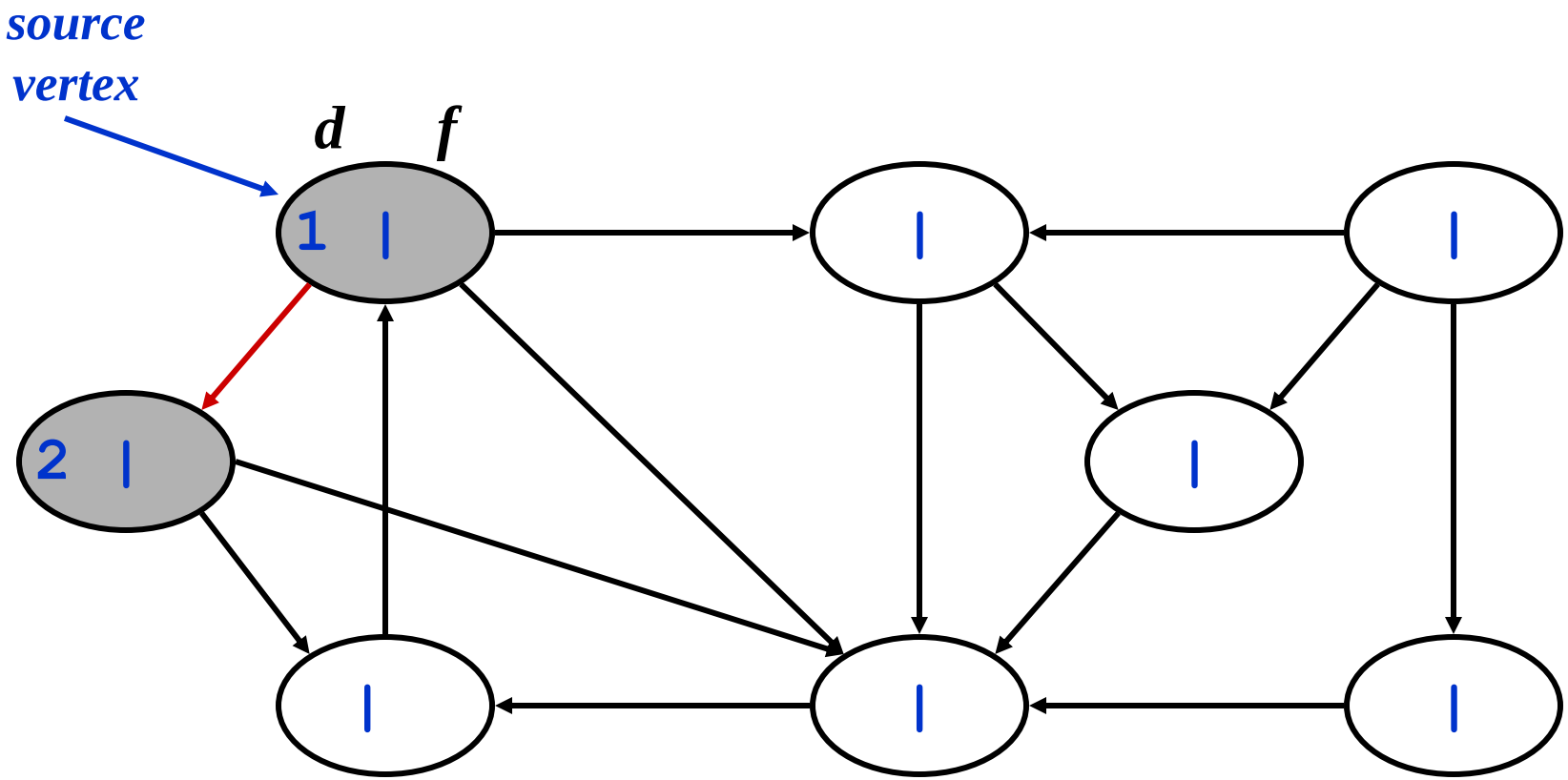

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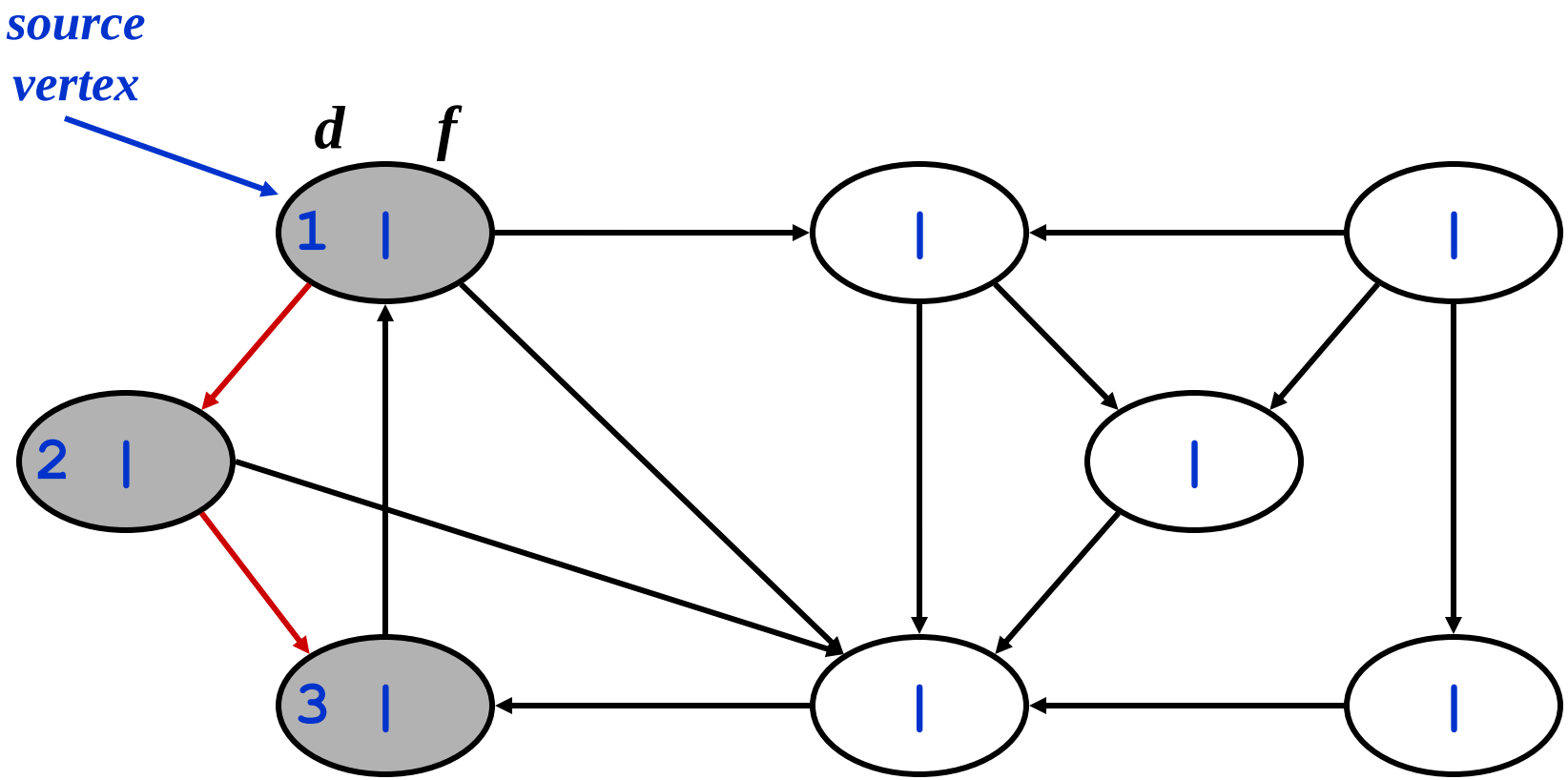

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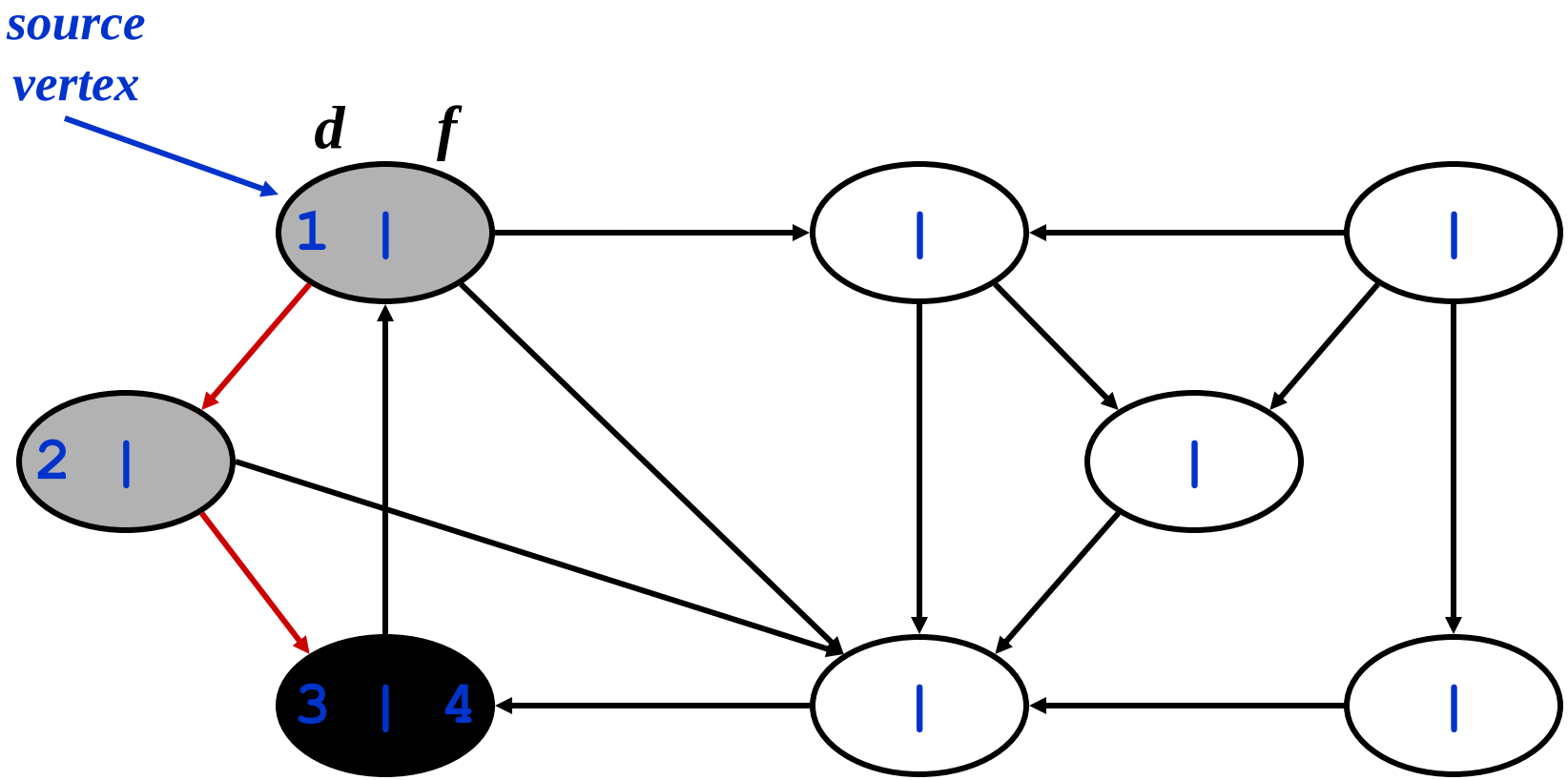
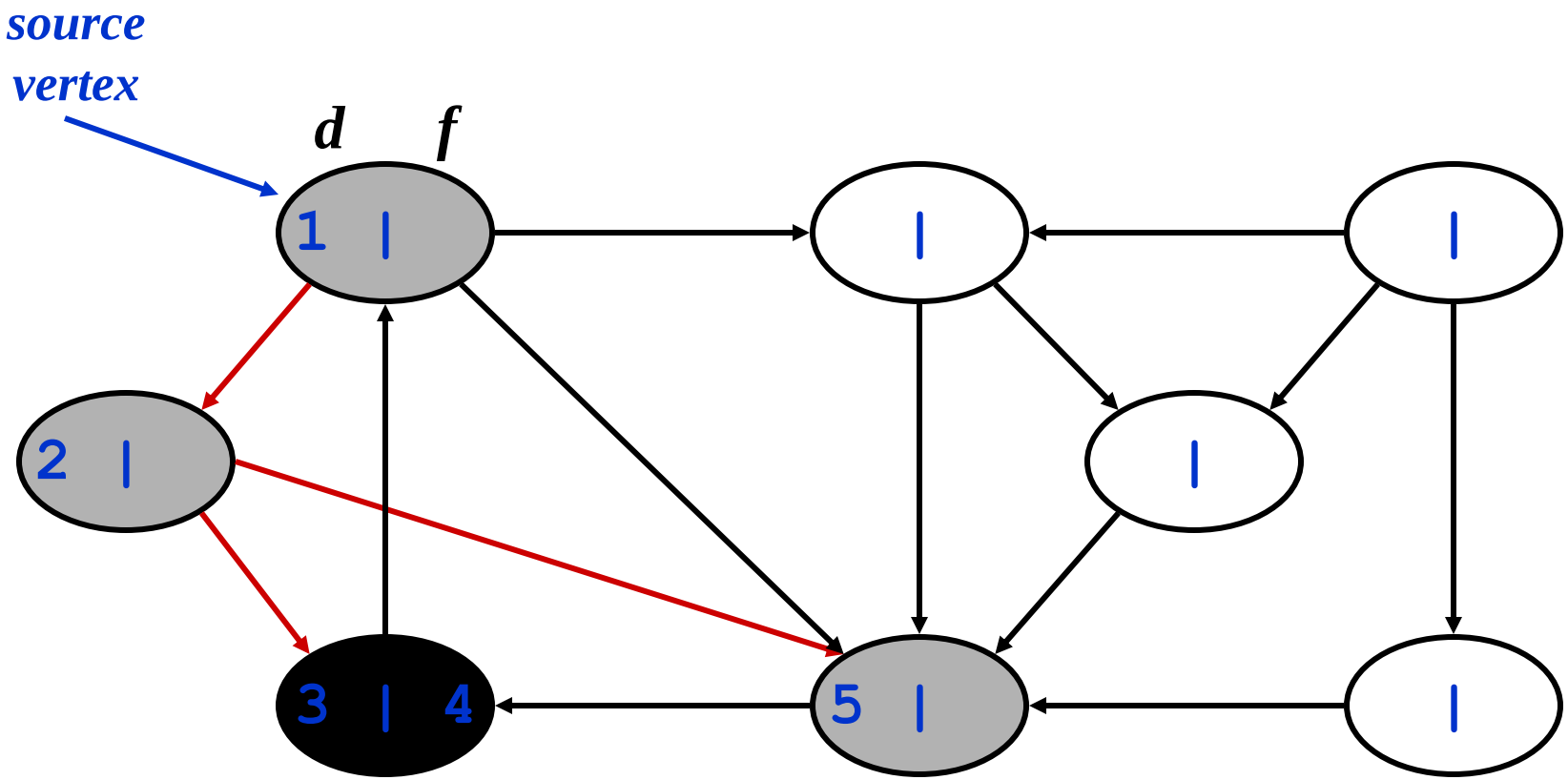


DFS Example



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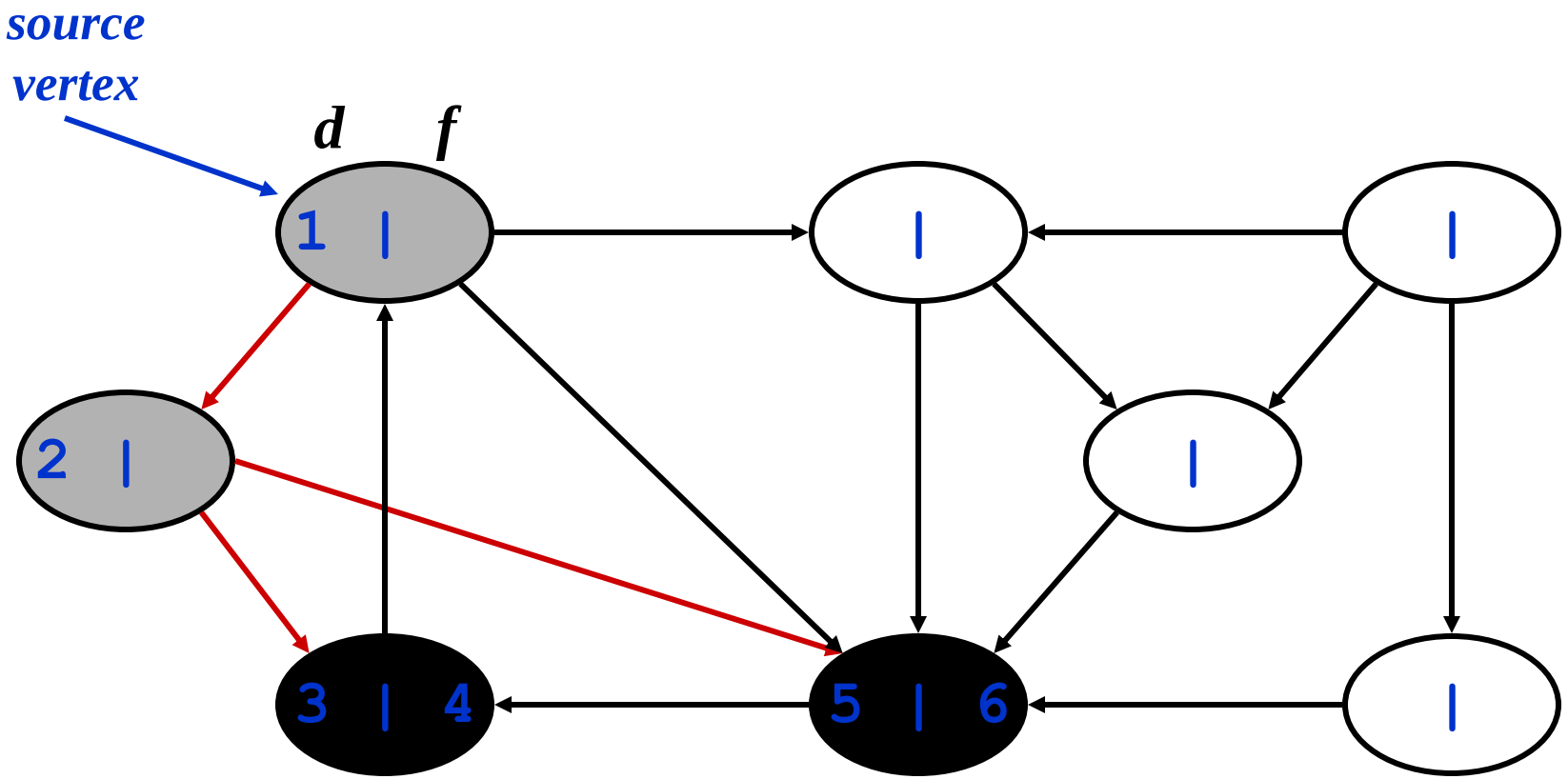

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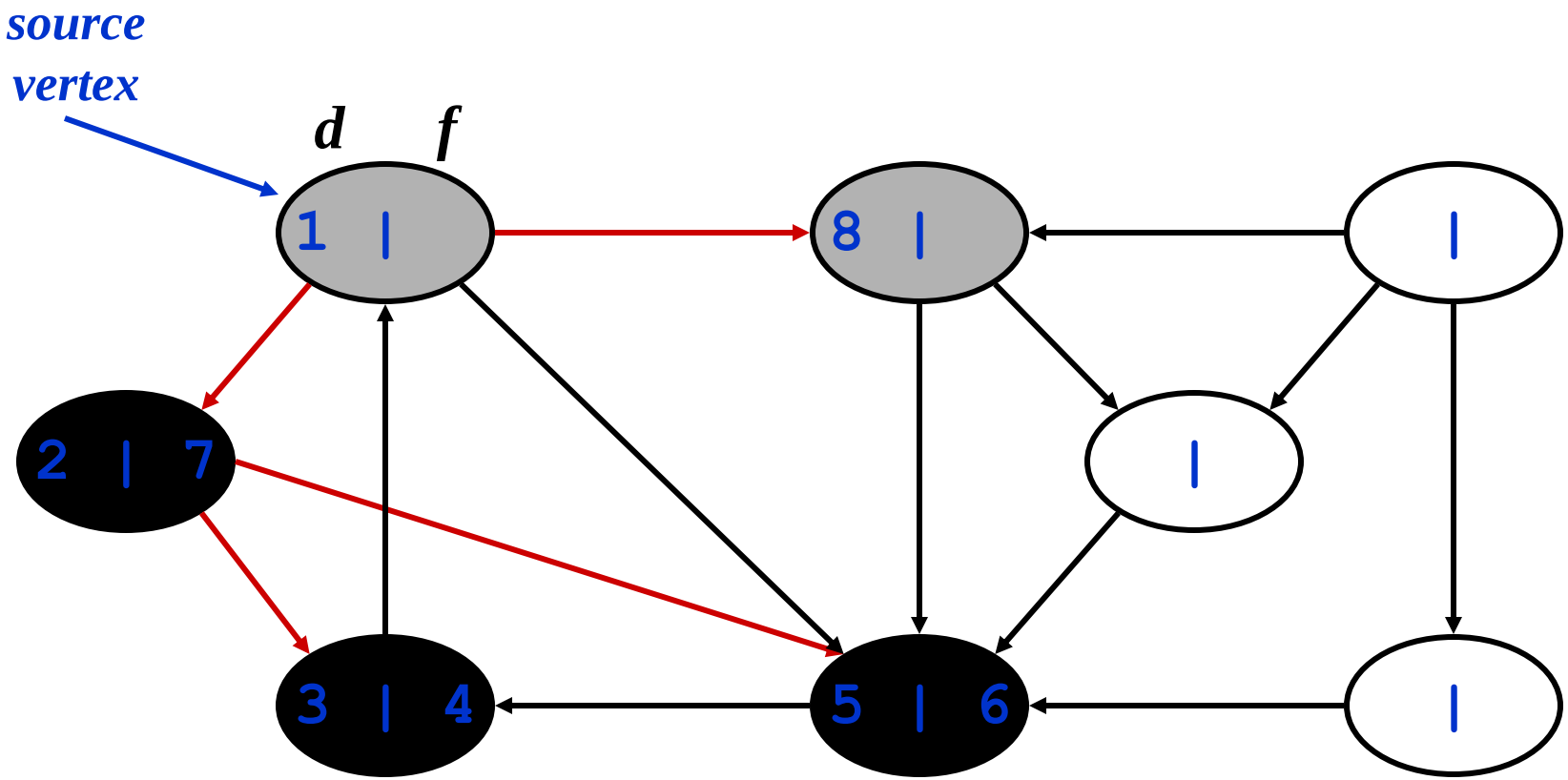

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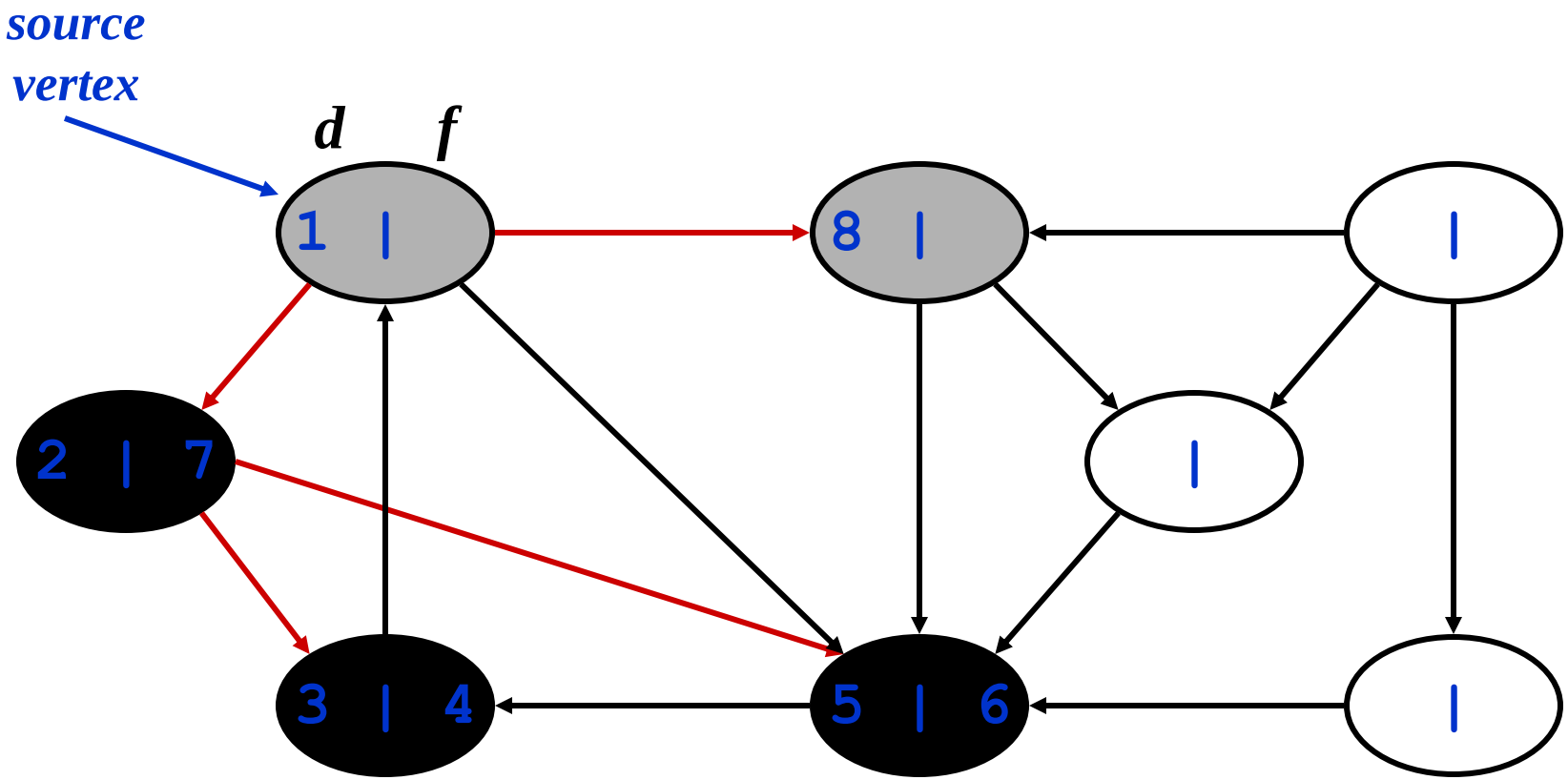

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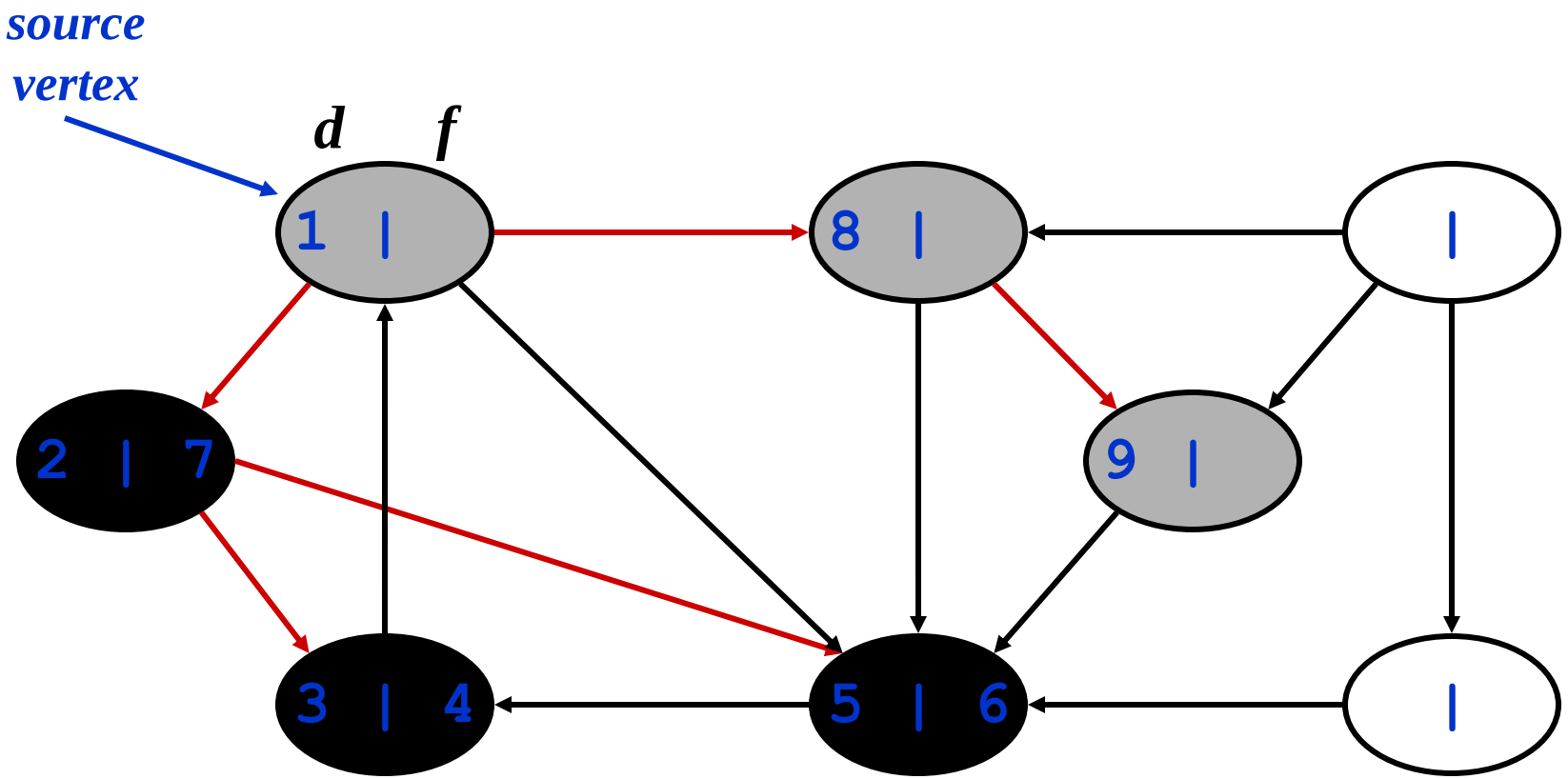

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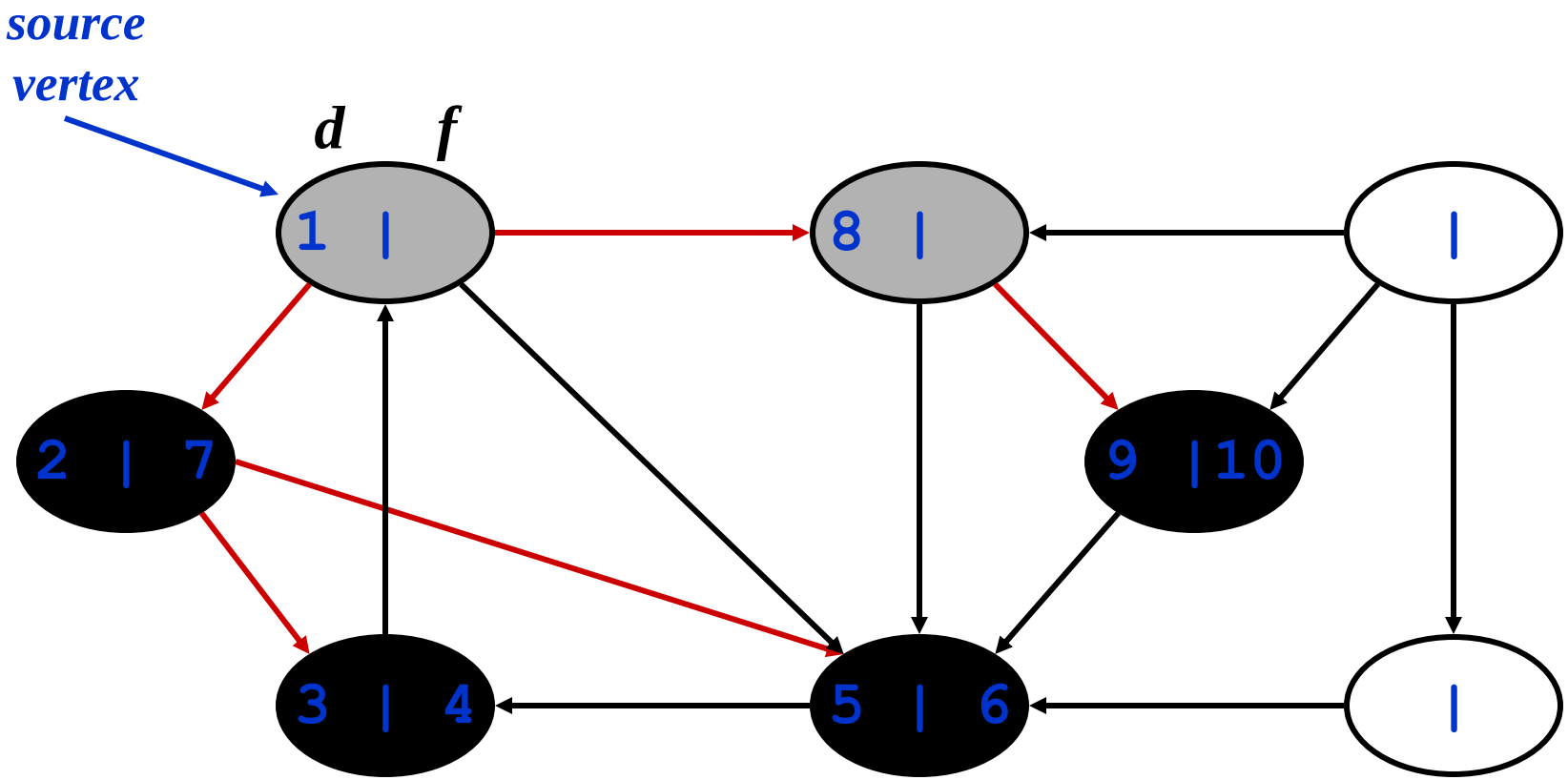

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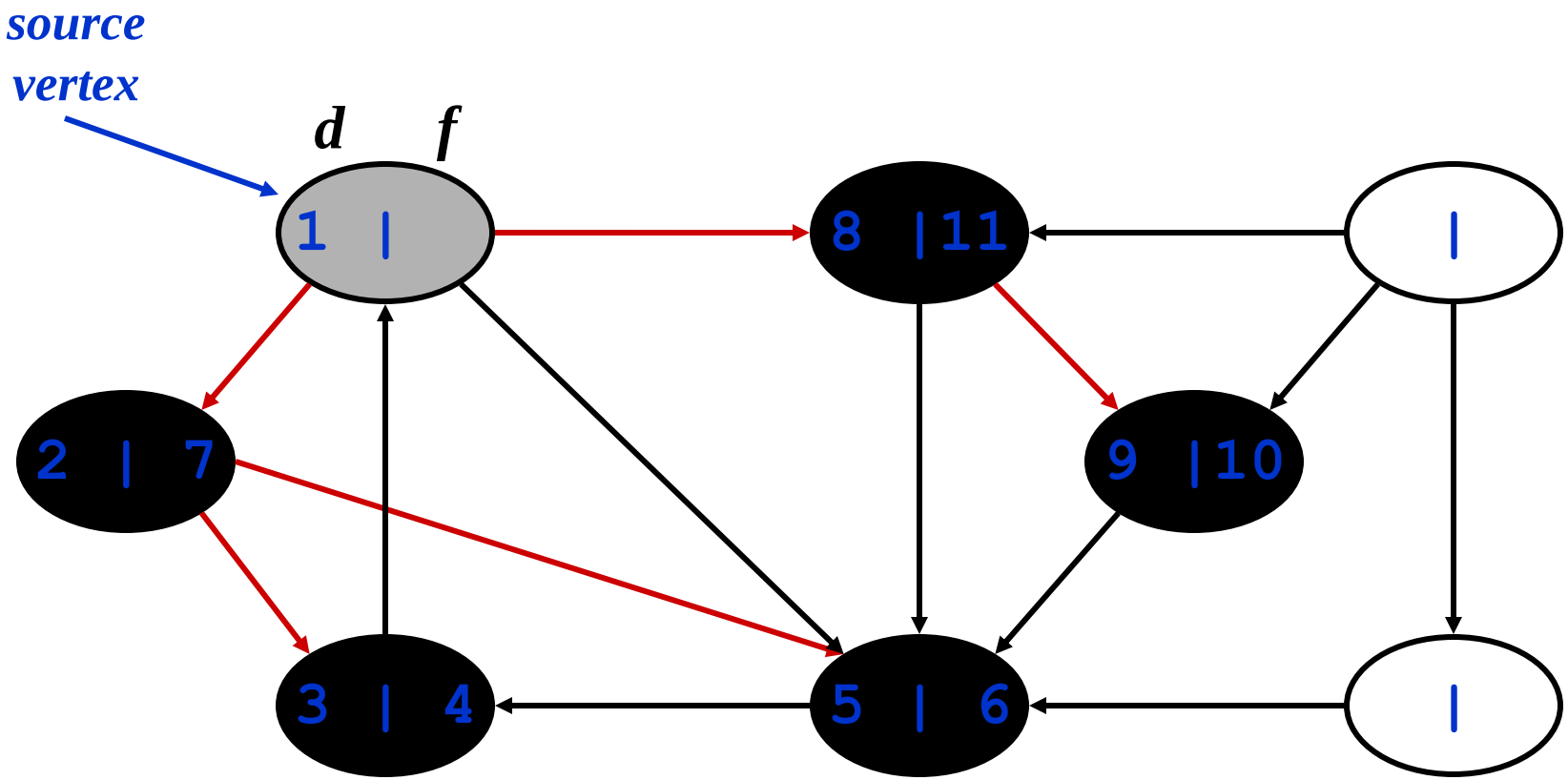

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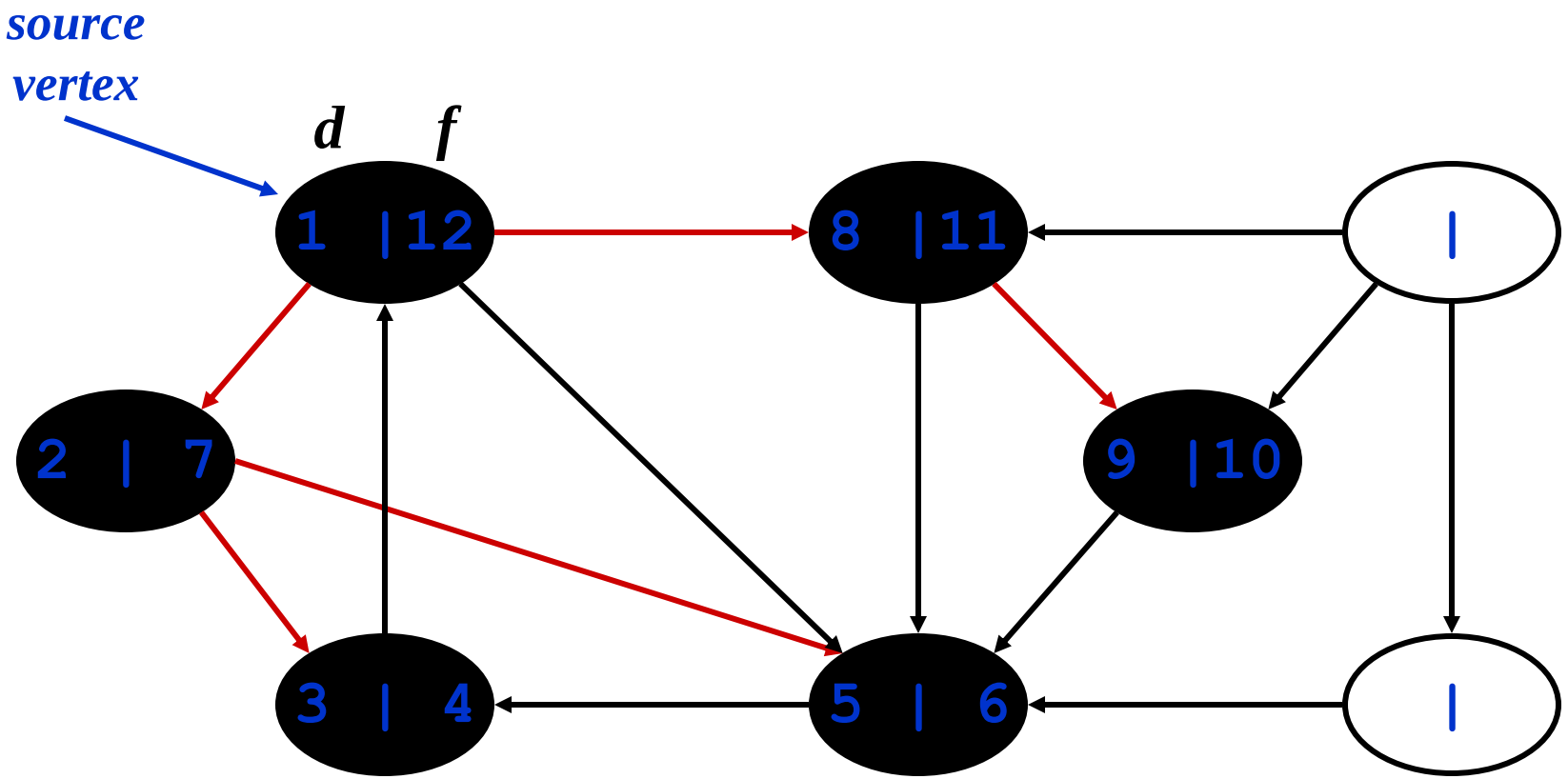

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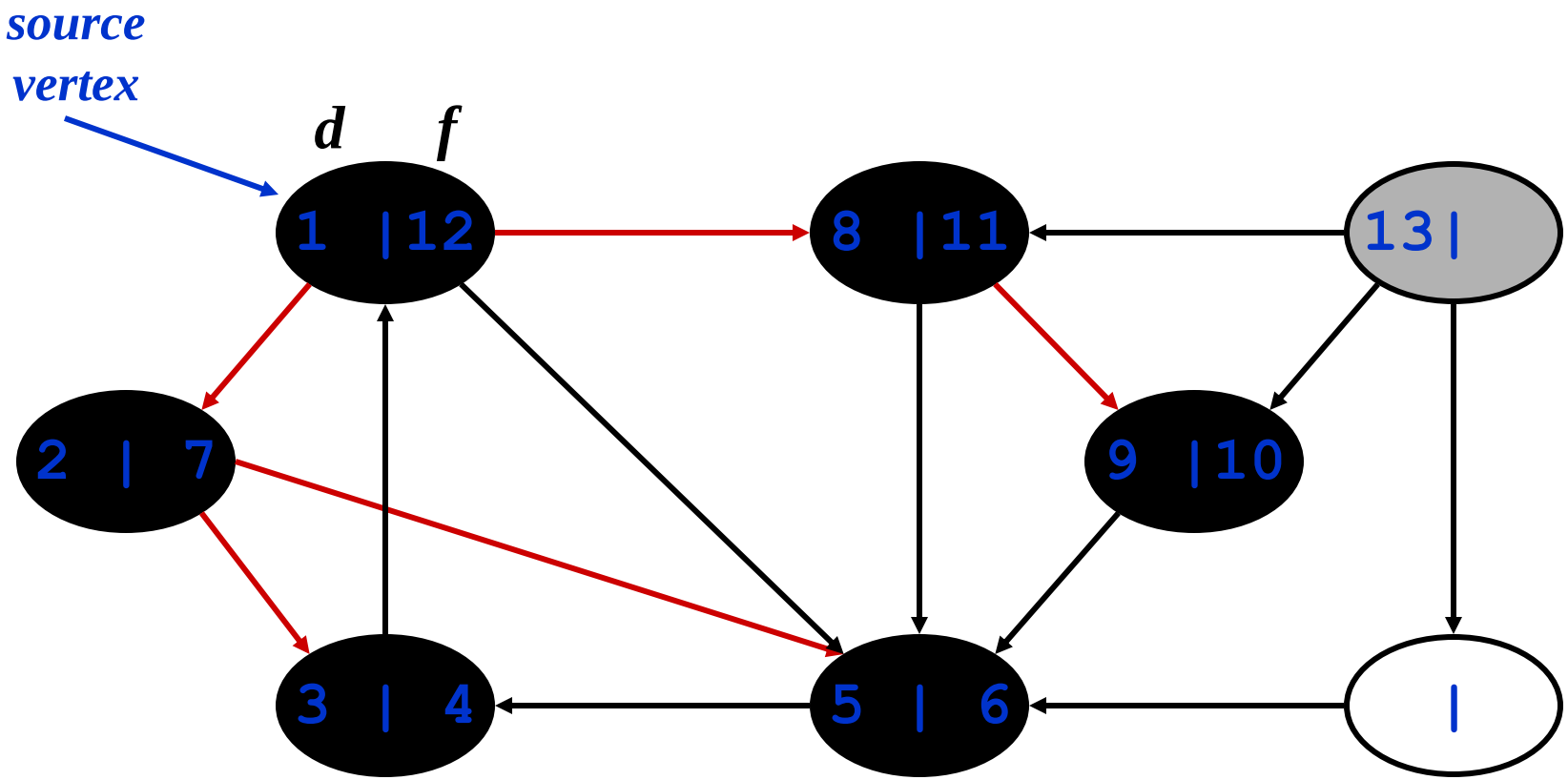

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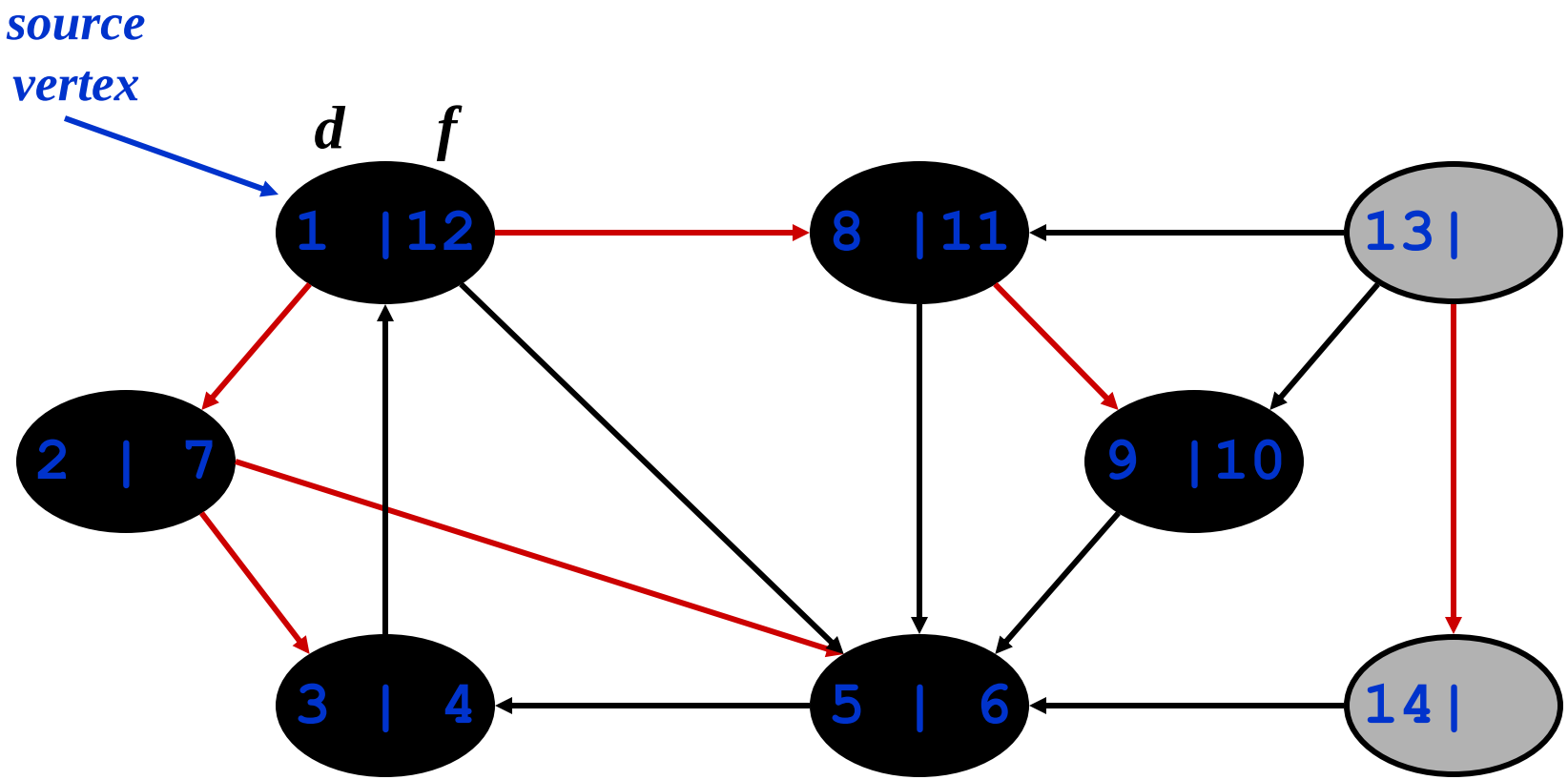
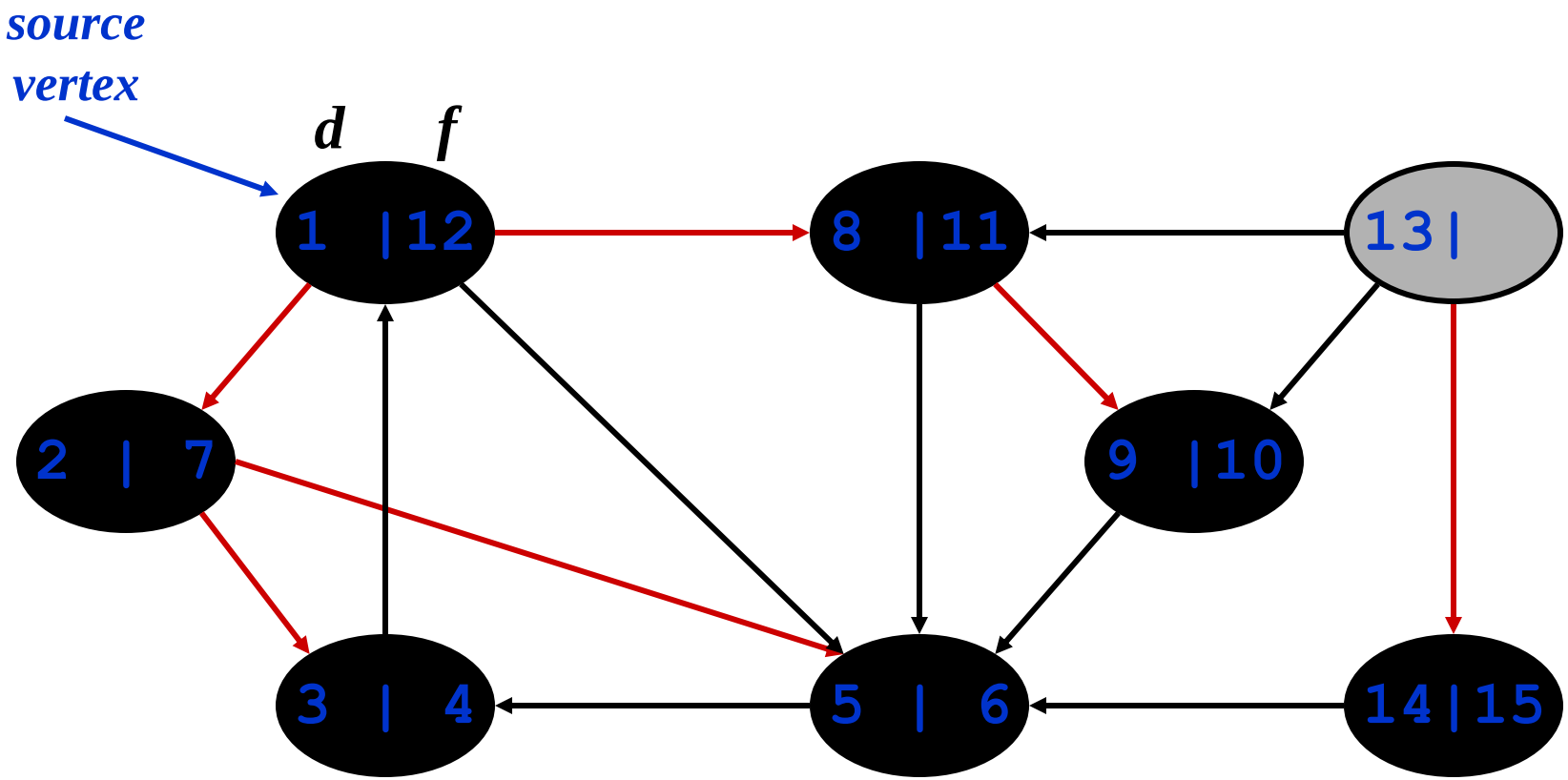
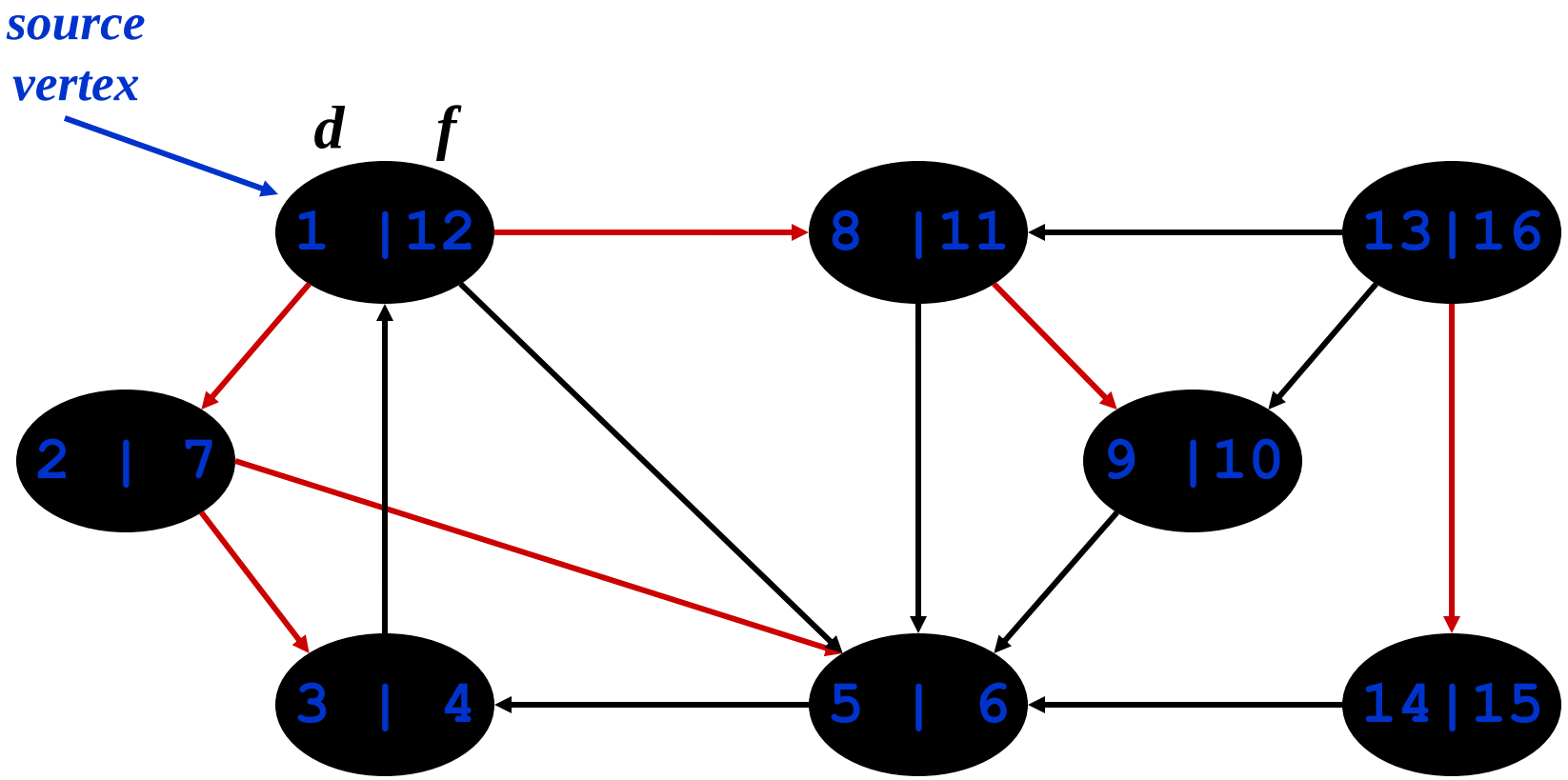


DFS Example



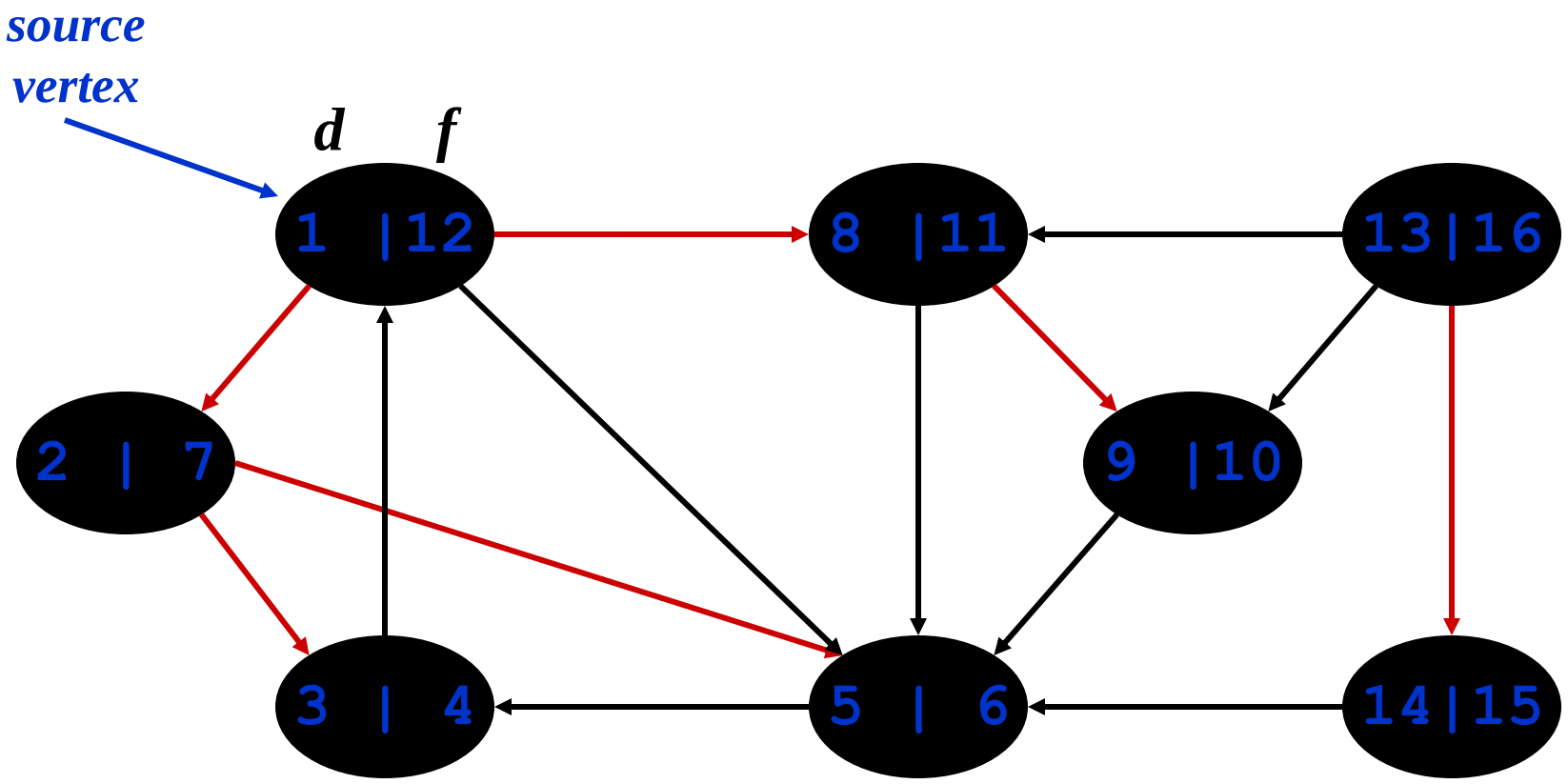
DFS Example



DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex

DFS Example

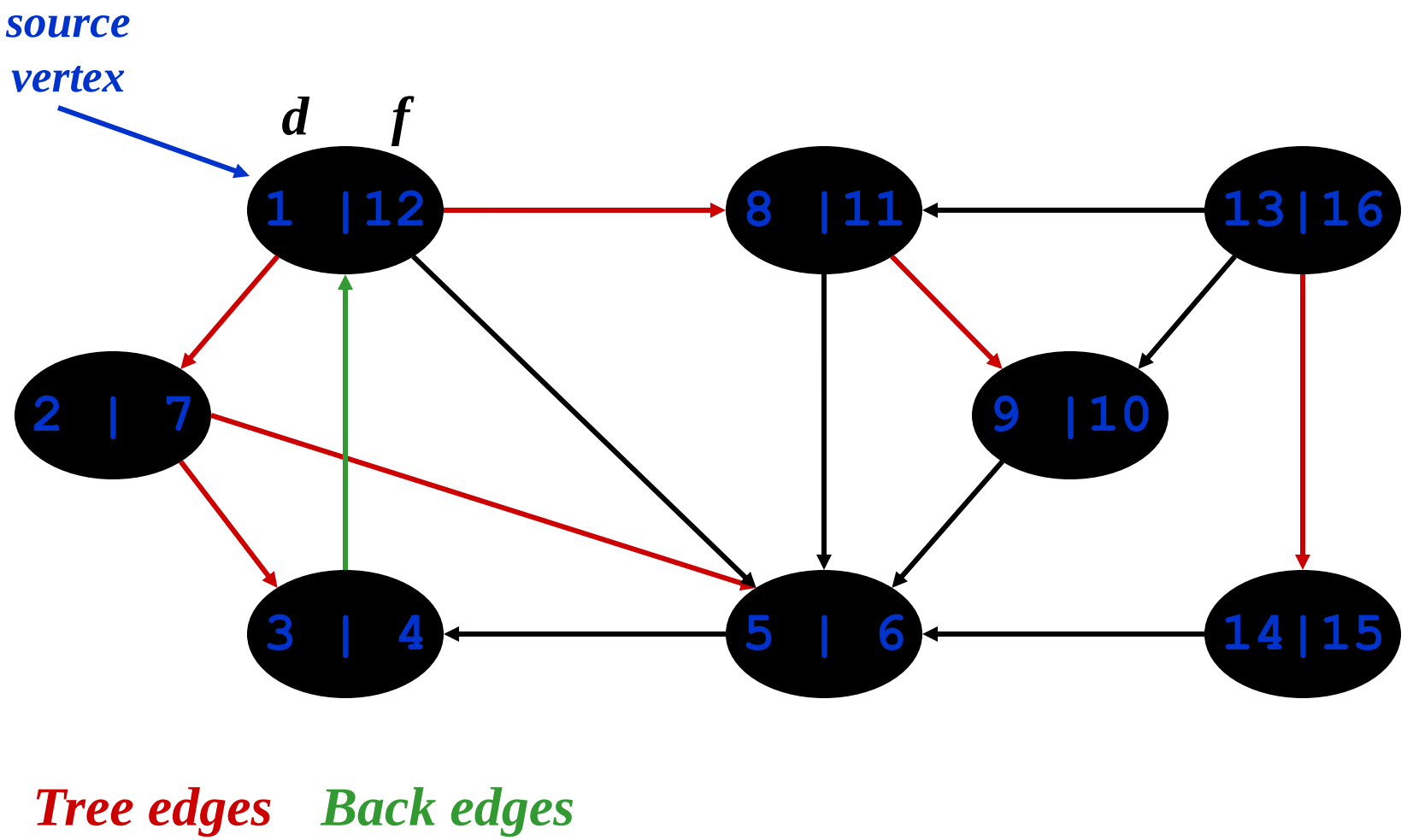


Tree edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
    - Encounter a grey vertex (grey to grey)

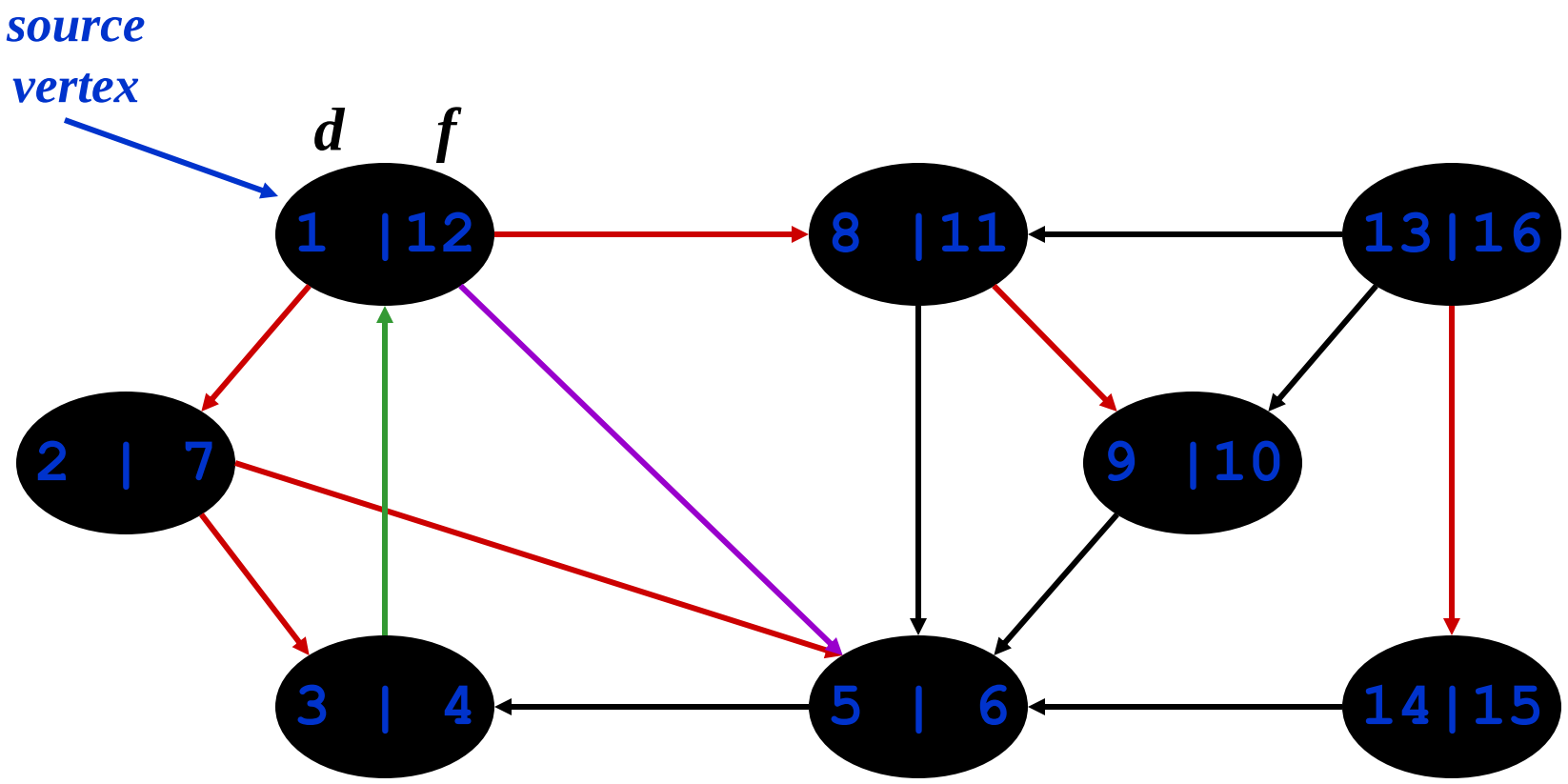
DFS Example


source
vertex
d
f
1 | 12
8 | 11
13 | 16
2 | 7
9 | 10
3 | 4
5 | 6
14 | 15
Tree edges
Back edges

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
    - Not a tree edge, though
    - From grey node to black node

DFS Example

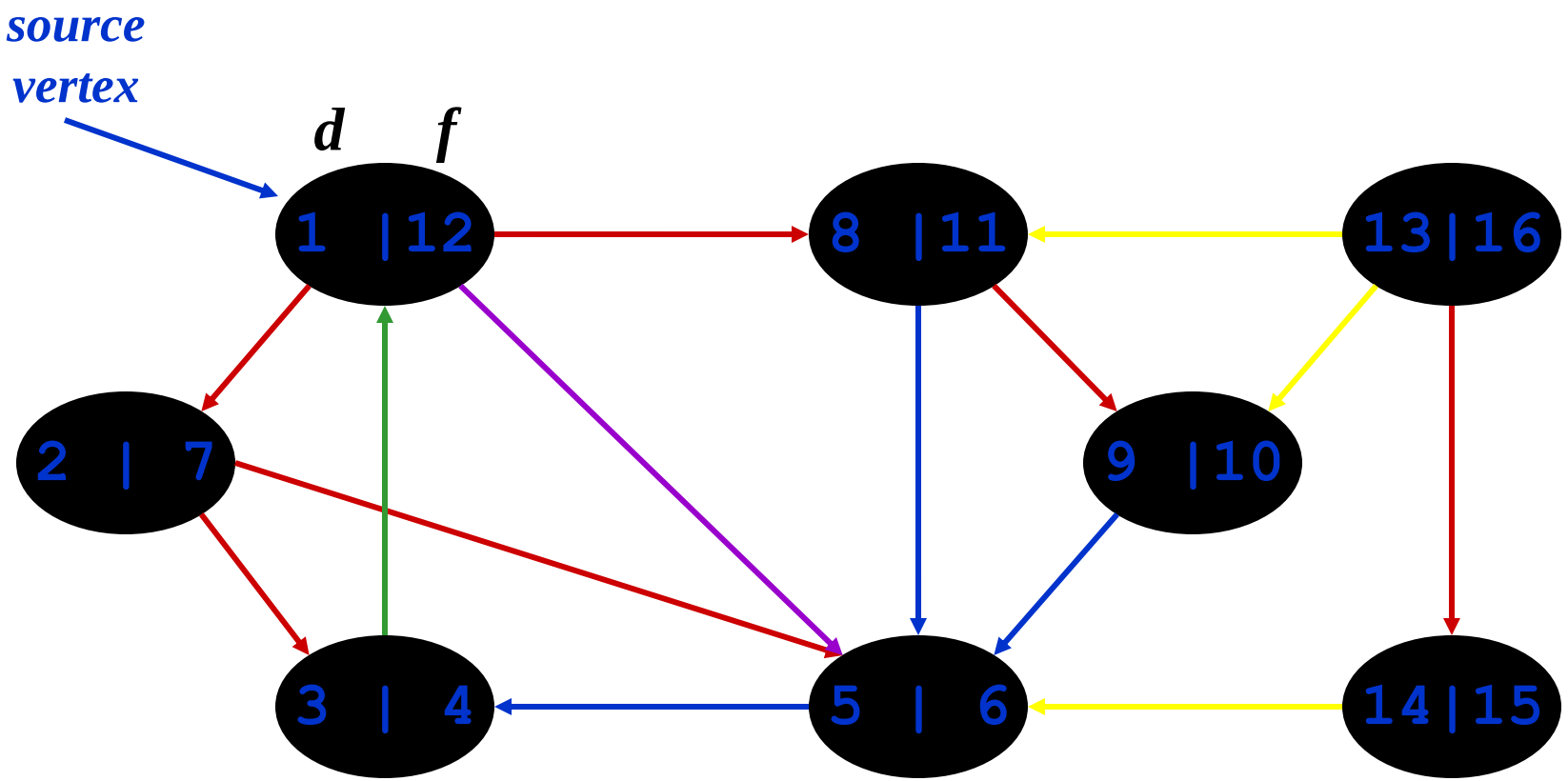


Tree edges *Back edges* *Forward edges*

DFS: Kinds of edges

- DFS introduces an important distinction among edges in the original graph:
  - *Tree edge*: encounter new (white) vertex
  - *Back edge*: from descendent to ancestor
  - *Forward edge*: from ancestor to descendent
  - *Cross edge*: between a tree or subtrees
    - From a grey node to a black node

DFS Example



Tree edges *Back edges* *Forward edges* *Cross edges*

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  - *Tree edge*: encounter new (white) vertex
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  - *Cross edge*: between a tree or subtrees
- Note: tree & back edges are important; most algorithms don’t distinguish forward & cross

Review: Dynamic Programming

- Summary of the basic idea:
  - Optimal substructure: optimal solution to problem consists of optimal solutions to subproblems
  - Overlapping subproblems: few subproblems in total, many recurring instances of each
  - Solve bottom-up, building a table of solved subproblems that are used to solve larger ones

Greedy Algorithms

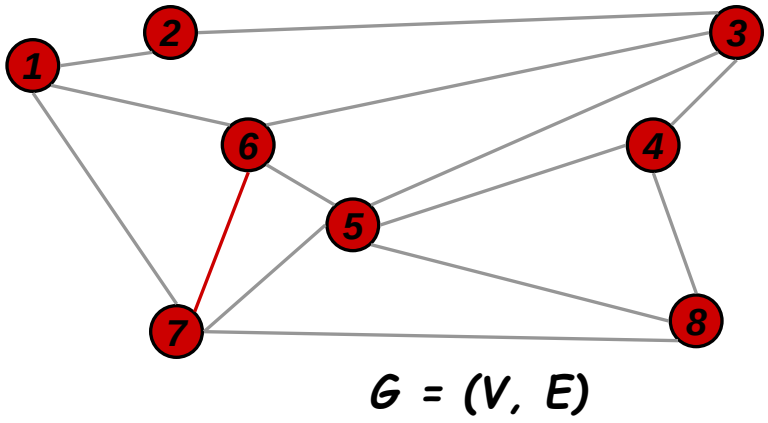
- Like dynamic programming, used to solve optimization problems.
- Problems exhibit optimal substructure (like DP).
- Problems also exhibit the **greedy-choice** property.
  - When we have a choice to make, make the one that looks best *right now*.
  - Make a **locally optimal choice** in hope of getting a **globally optimal solution**.

Greedy Algorithms

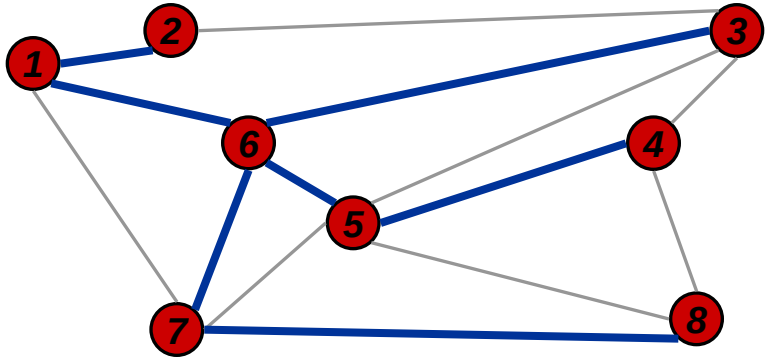
- A *greedy algorithm* always makes the choice that looks best at the moment
  - The hope: a locally optimal choice will lead to a globally optimal solution
  - For some problems, it works
- Dynamic programming can be overkill; greedy algorithms tend to be easier to code

Spanning Tree

- Spanning tree. Let T = (V, F) be a subgraph of G = (V, E).
- T is a spanning tree of G: *T that contains all the vertices* of a graph *G*.
  - T is acyclic and connected.
  - T is connected and has |V| - 1 arcs.
  - T is acyclic and has |V| - 1 arcs.
  - T is minimally connected: removal of any arc disconnects it.
  - T is maximally acyclic: addition of any arc creates a cycle.
  - T has a unique simple path between every pair of vertices.



G = (V, E)

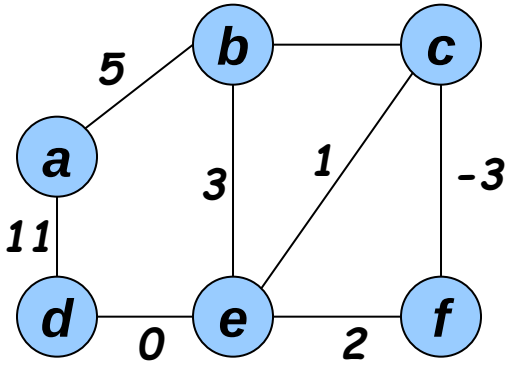


T = (V, F)

Minimum Spanning Trees

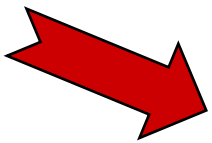
•*A spanning tree for G is a free tree that connects all the vertices in V.*

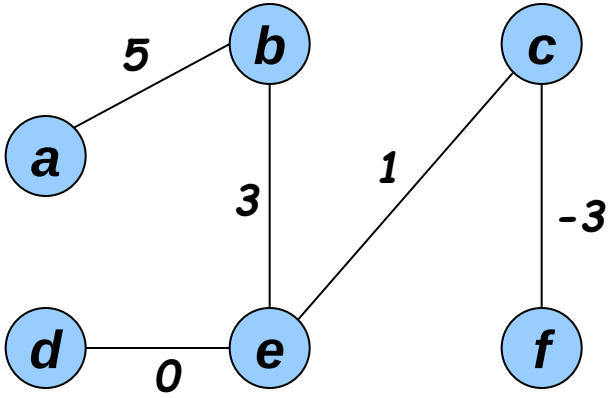
•*The cost of a spanning tree is the sum of the costs of the edges in the tree.*



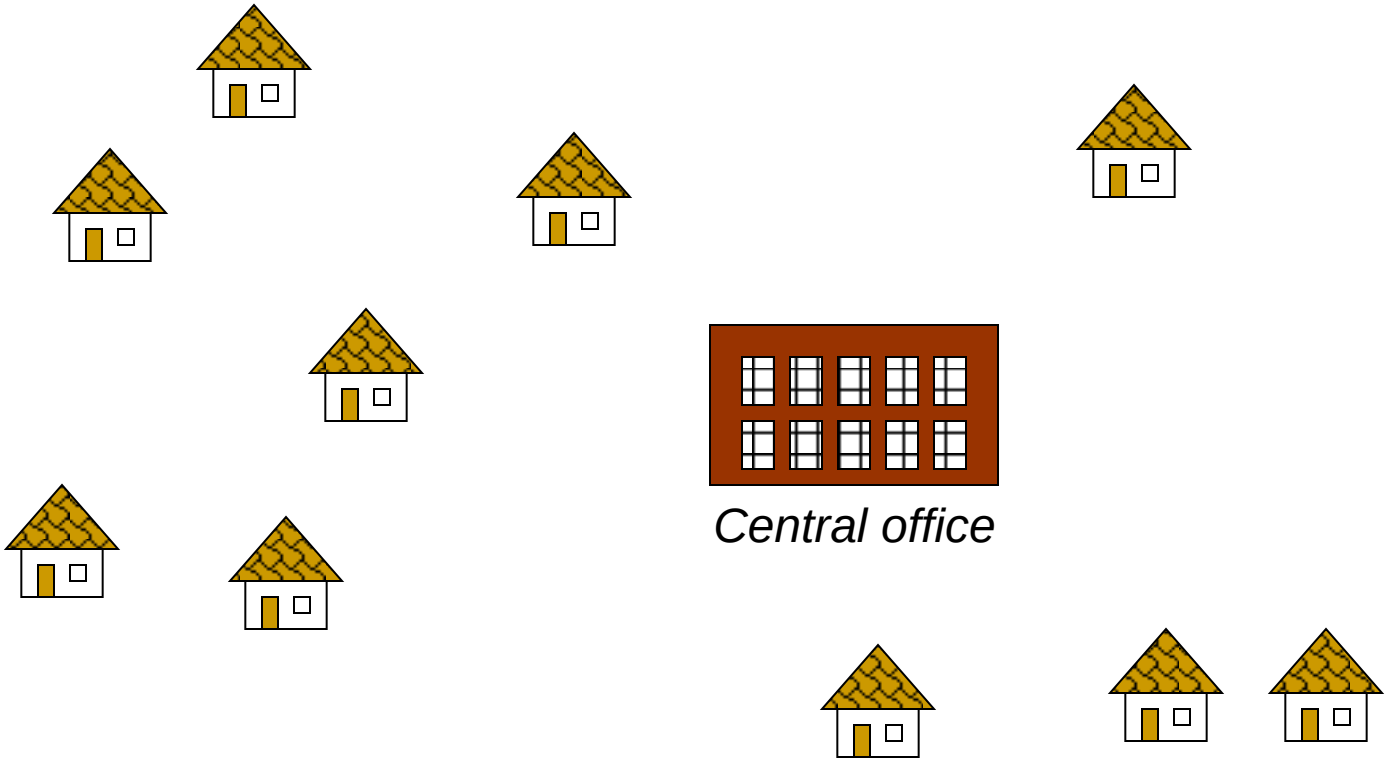
•***Given:*** *a Connected, undirected, weighted graph, G =(V, E) in which each edge (u, v) in E has a cost c (u, v) attached to it.*

•***Find:*** *Minimum - weight spanning tree, T*

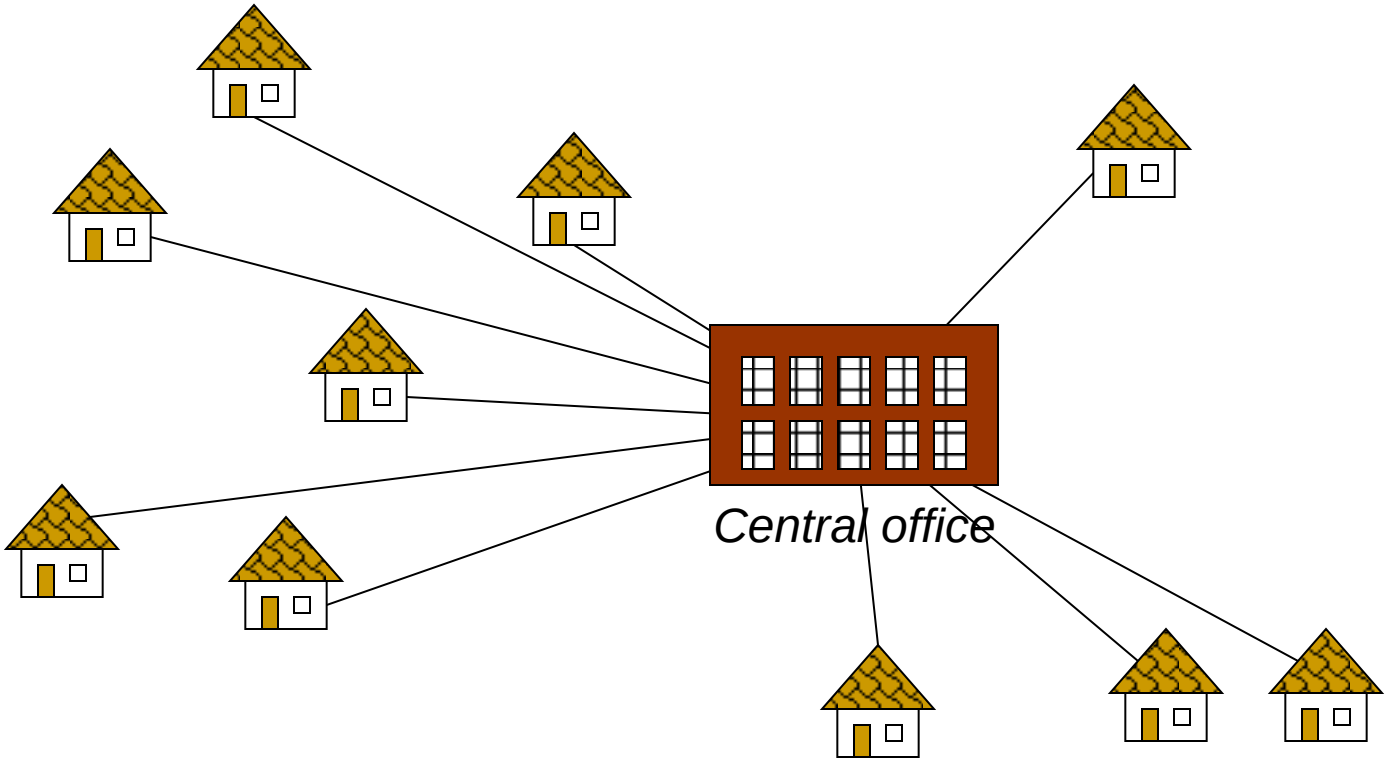




Problem: Laying Telephone Wire

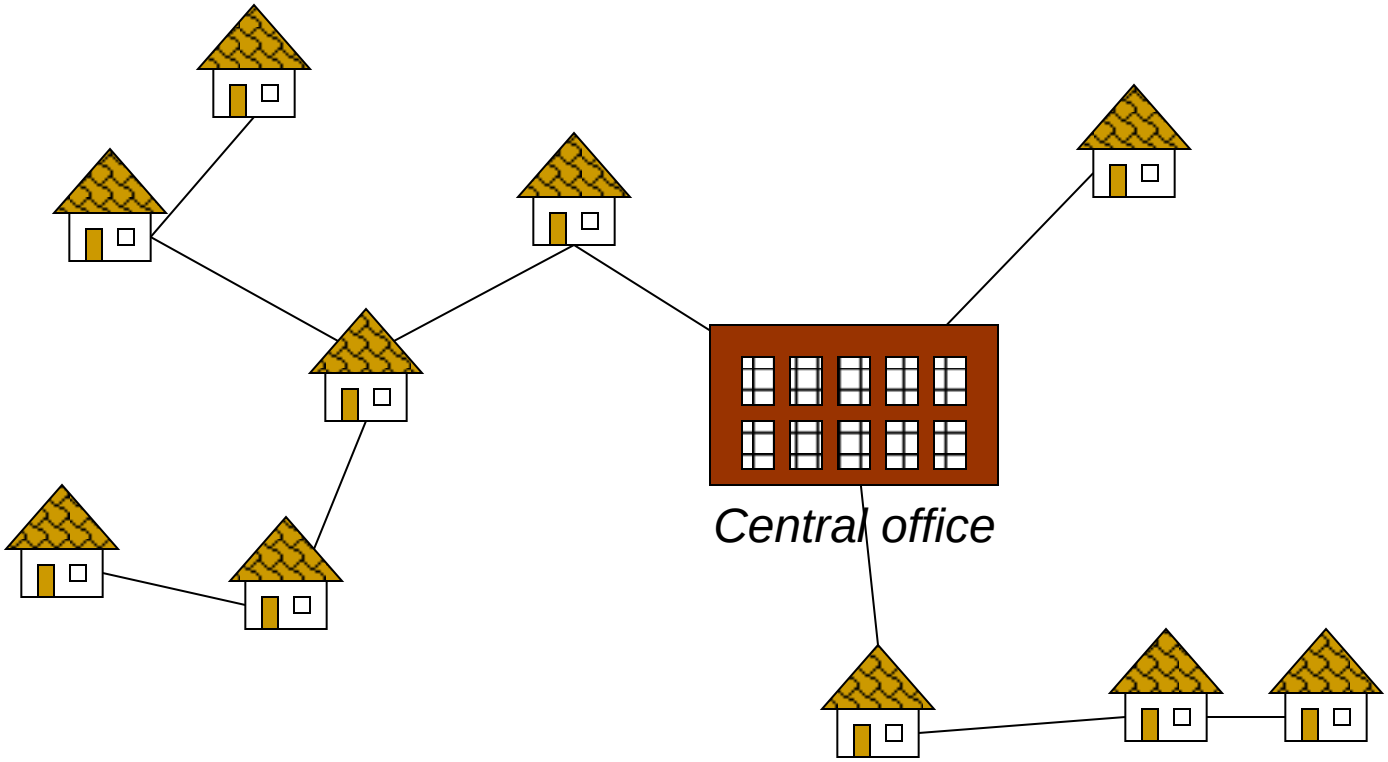


Wiring: Naïve Approach



Expensive!

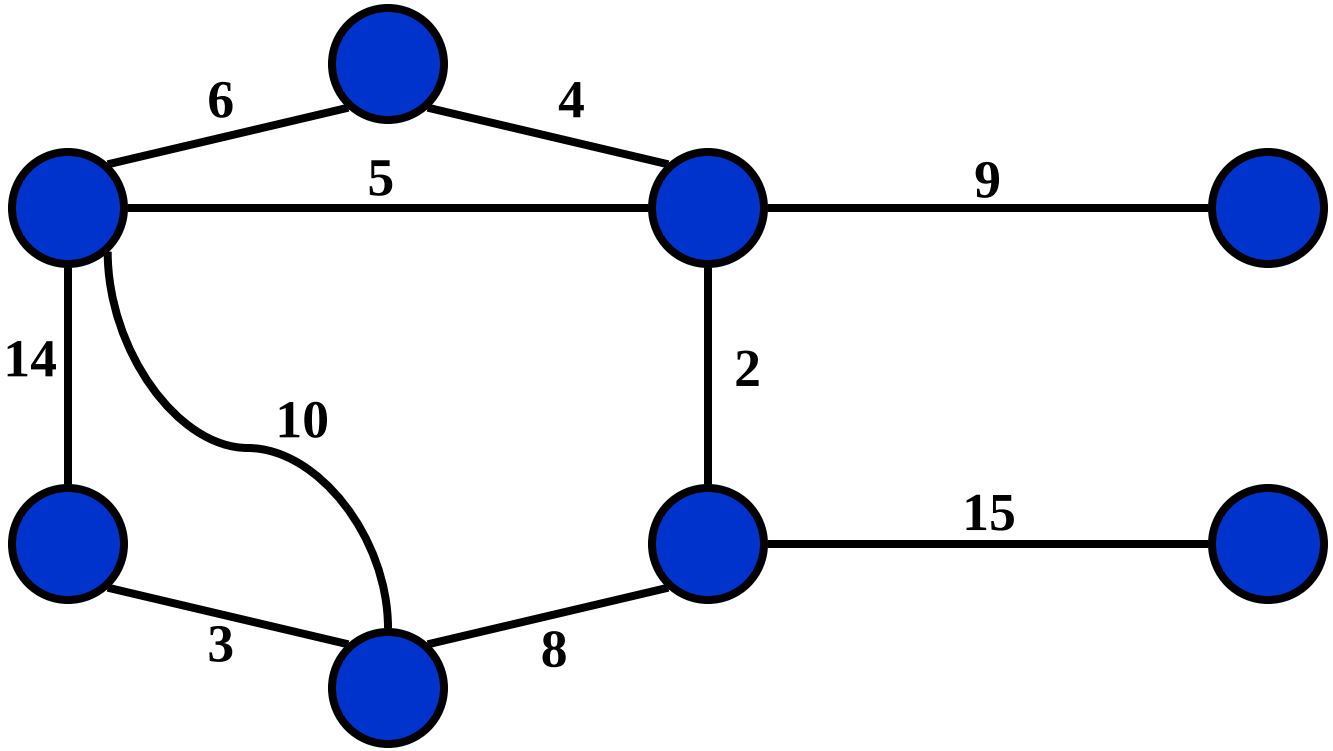
Wiring: Better Approach



Minimize the total length of wire connecting the customers

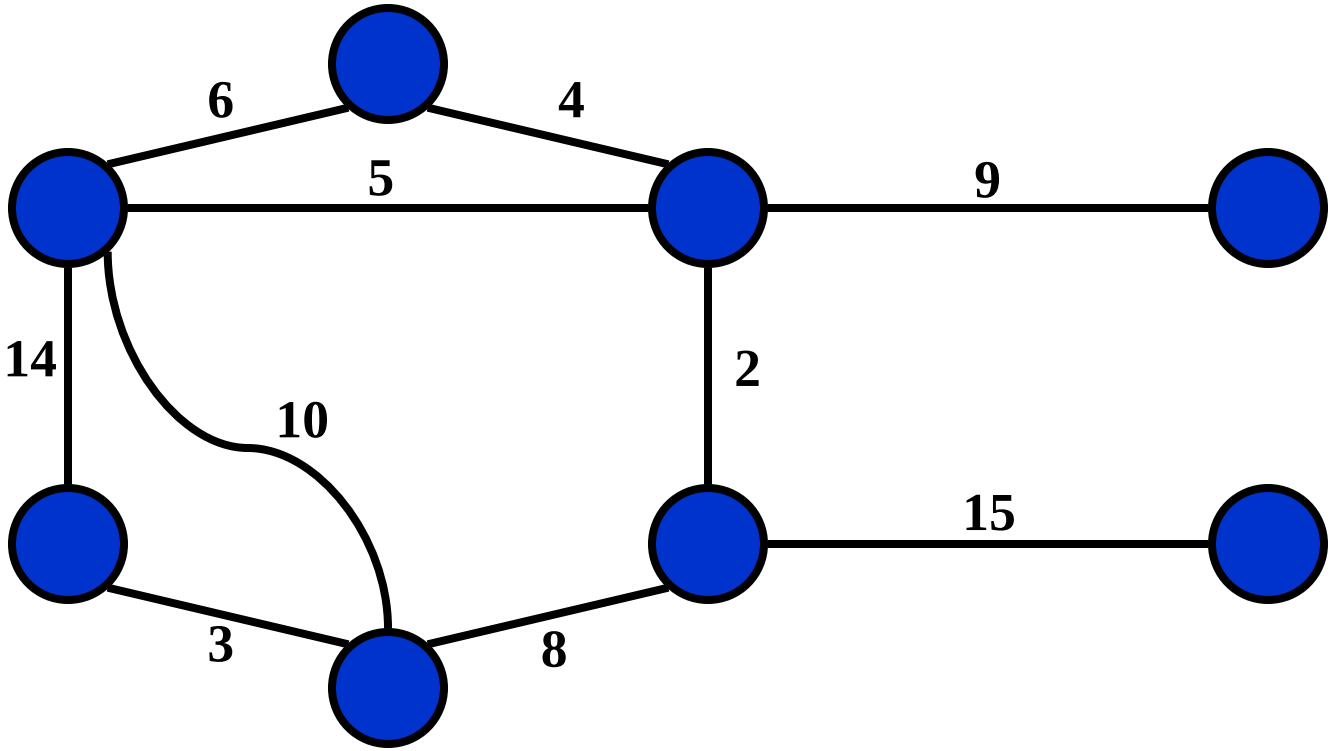
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



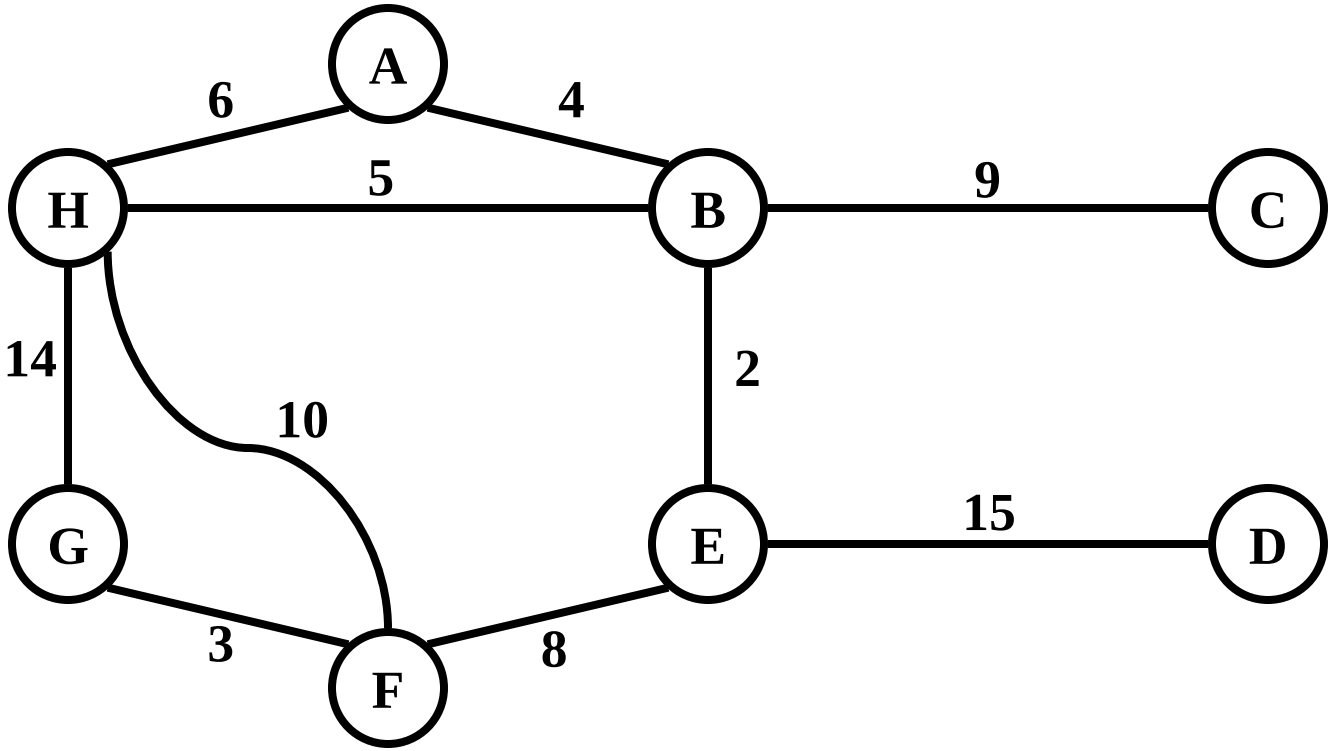
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



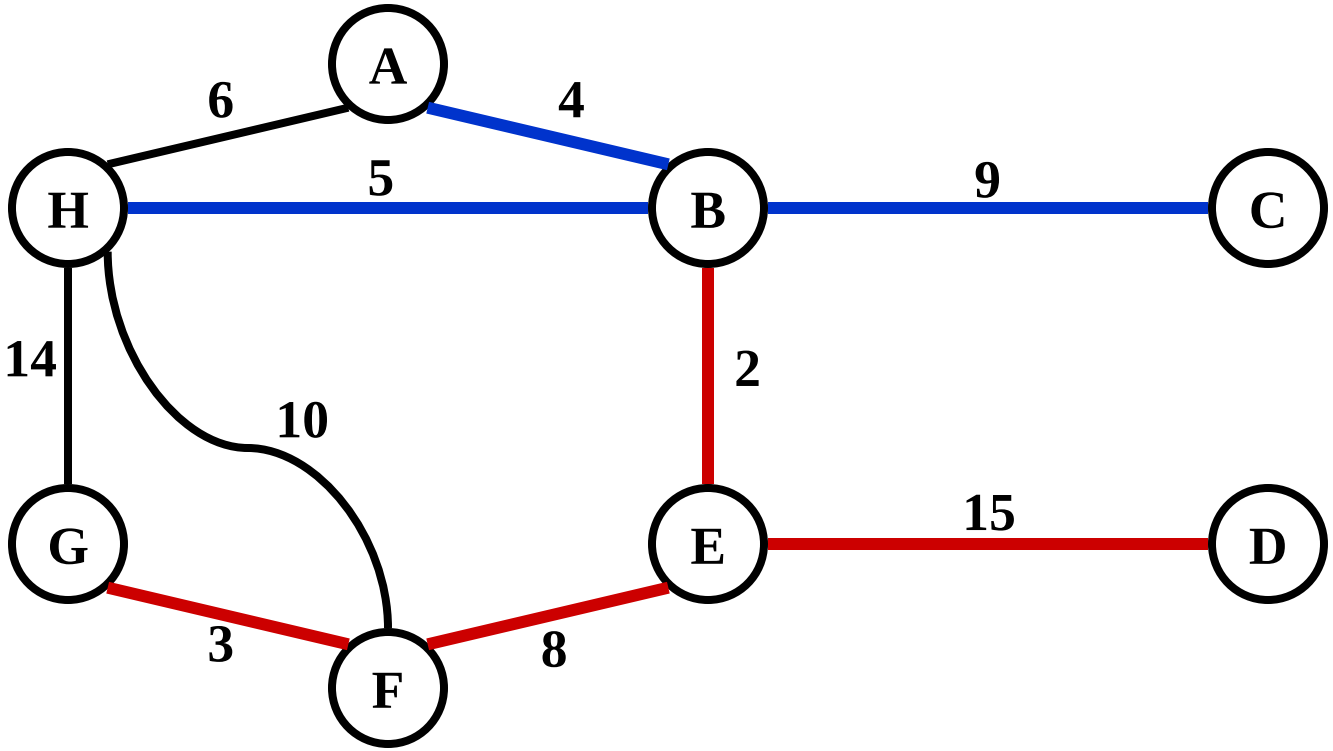
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the below graph?*



Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal tree is composed of optimal subtrees
  - Let T be an MST of G with an edge (u,v) in the middle
  - Removing (u,v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1,E_1)$, and T_2 is an MST of $G_2 = (V_2,E_2)$
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2, or T would be suboptimal)

Minimum Spanning Tree

- Thm:
  - Let T be MST of G, and let A $\subseteq$ T be subtree of T
  - Let (u,v) be min-weight edge connecting A to V-A
  - Then $(u,v) \in$ T

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Prim's Algorithm

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MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
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    while (Q not empty)
        u = ExtractMin(Q);
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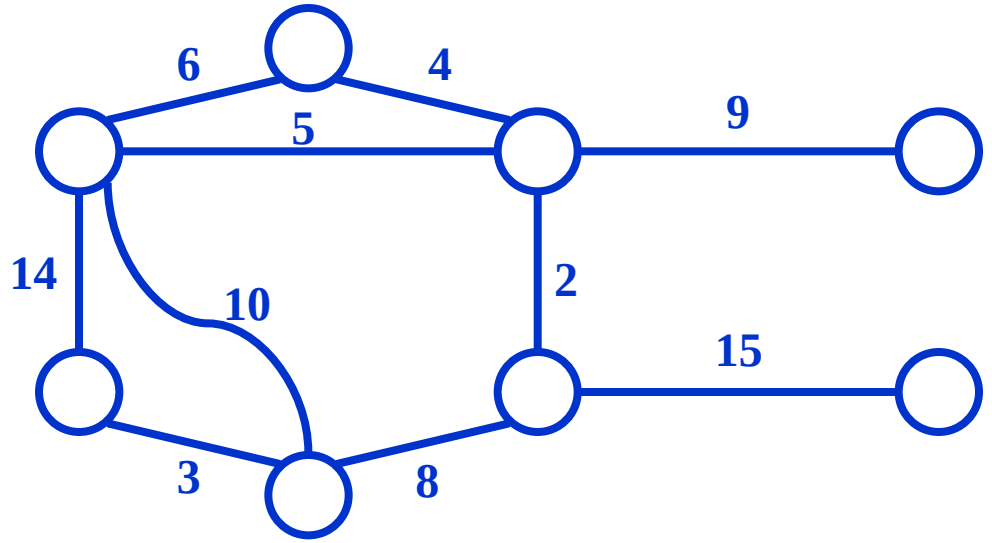

Run on example graph

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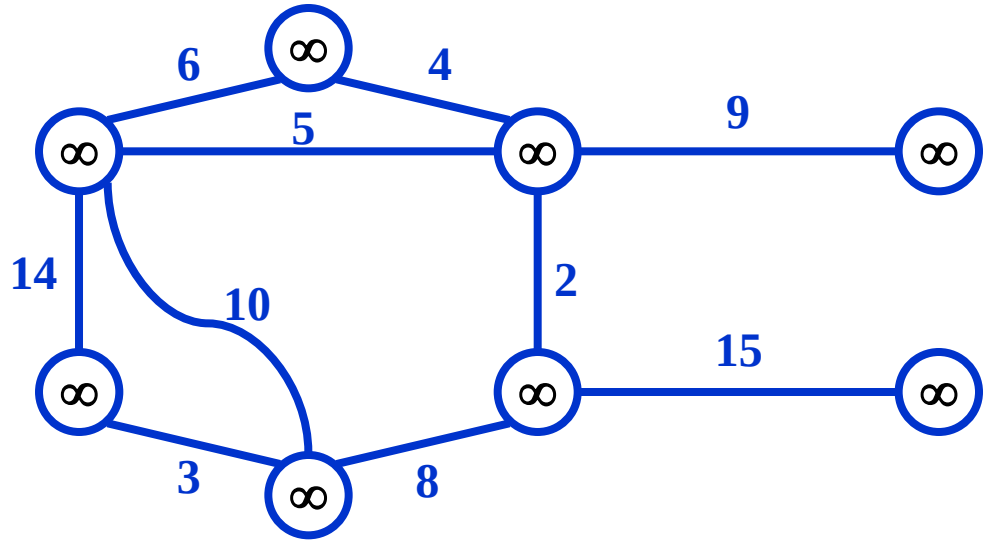

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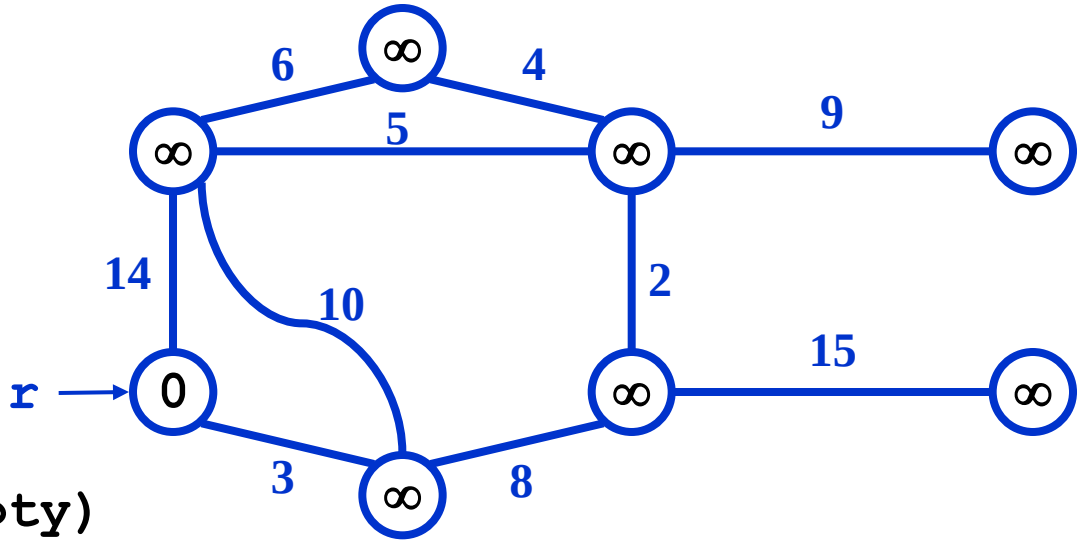

Pick a start vertex r

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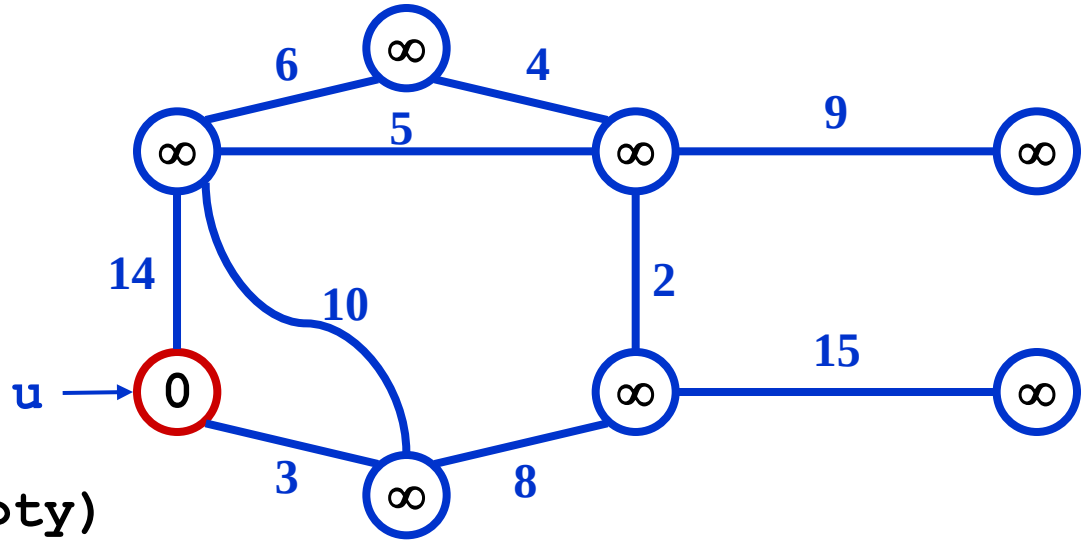

Red vertices have been removed from Q

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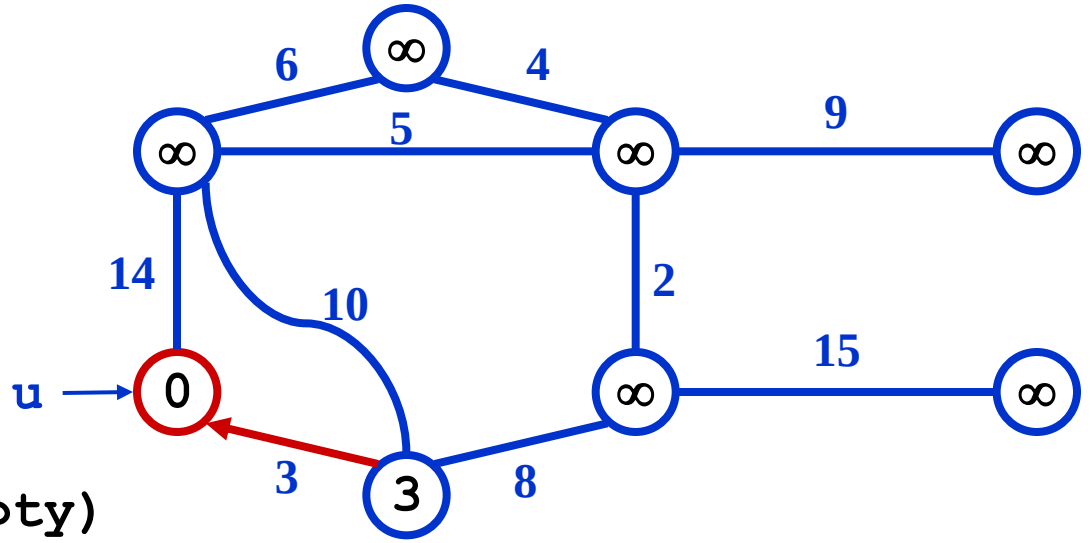

Red arrows indicate parent pointers

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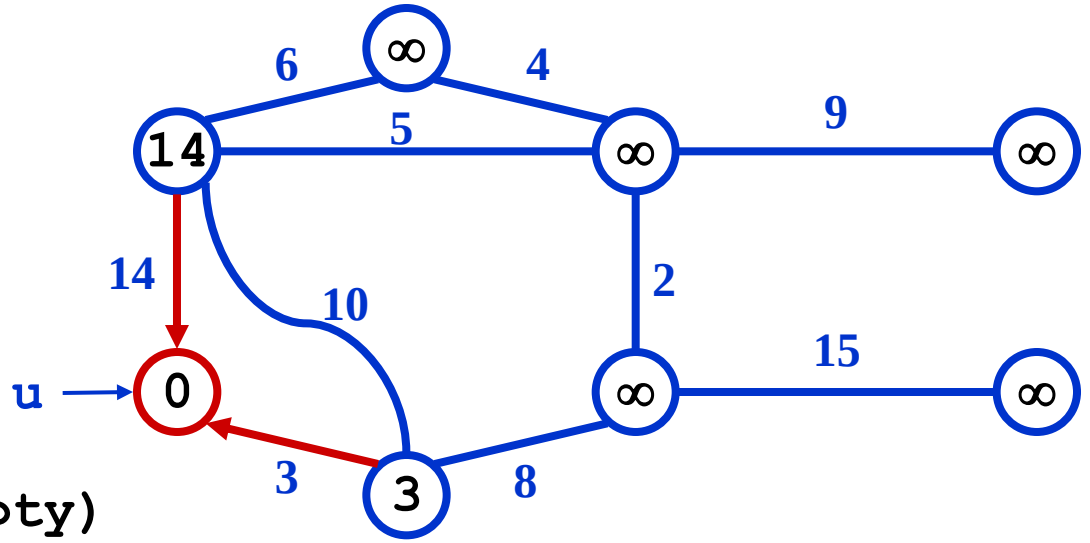

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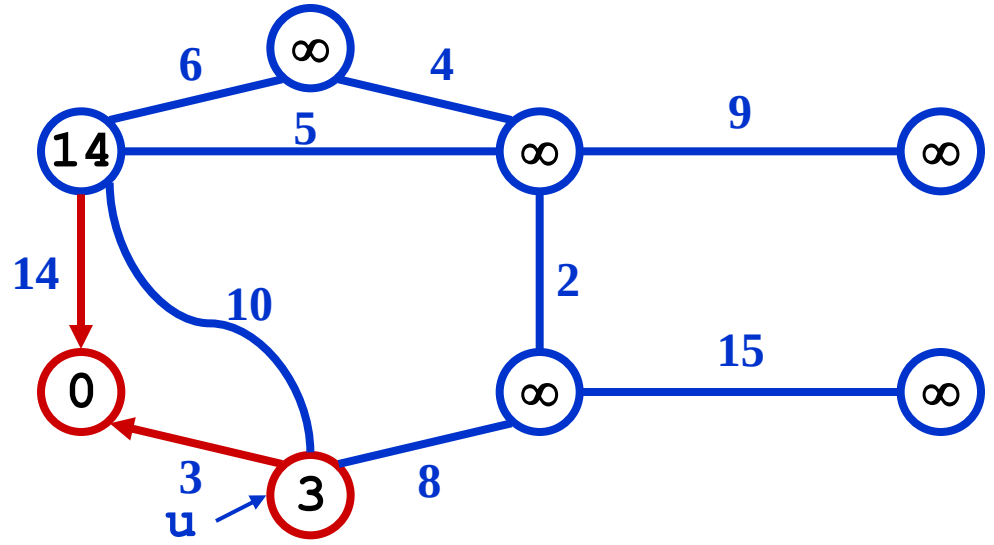

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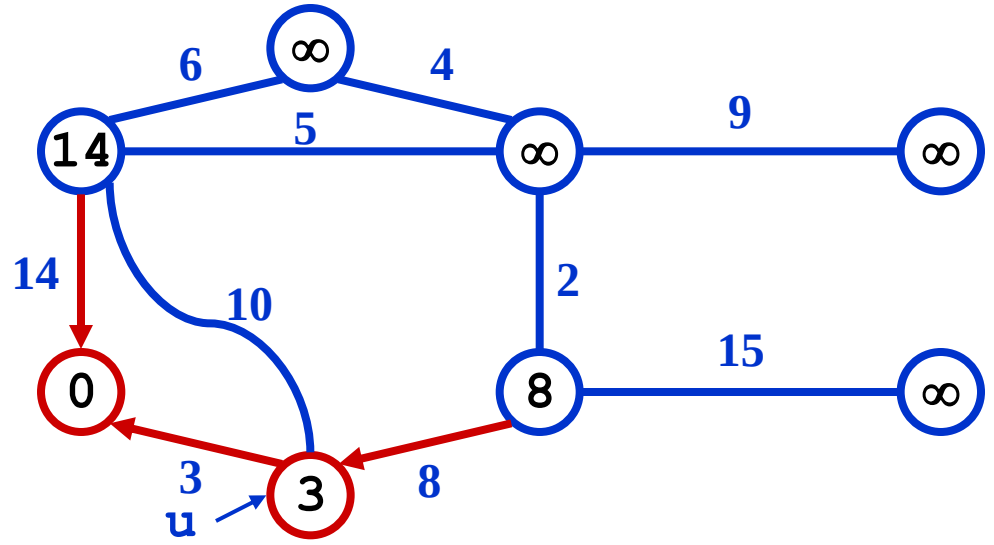

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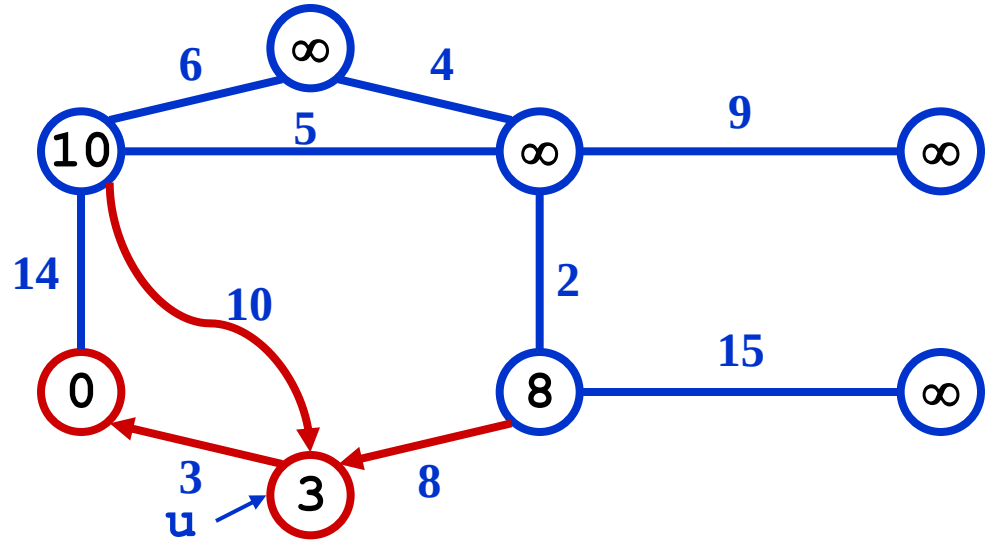

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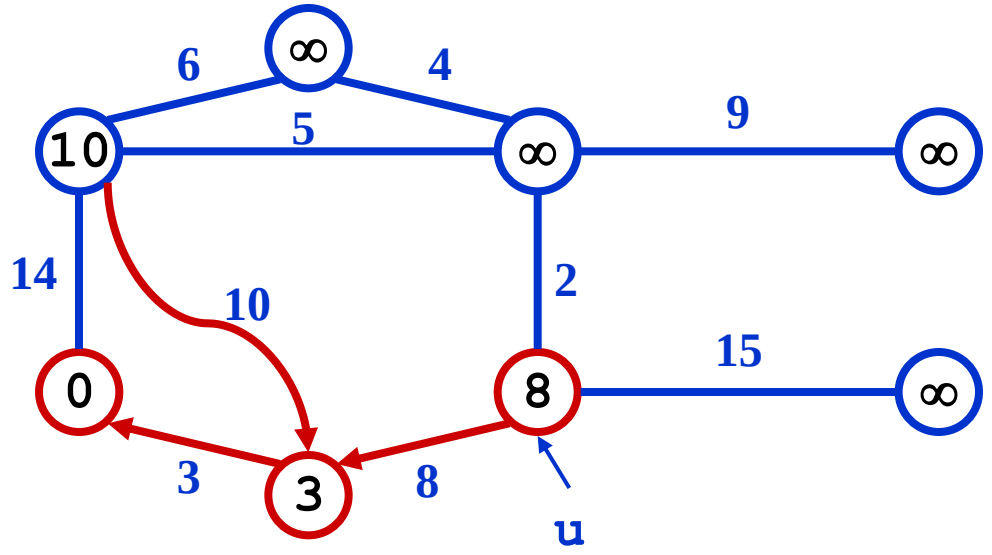

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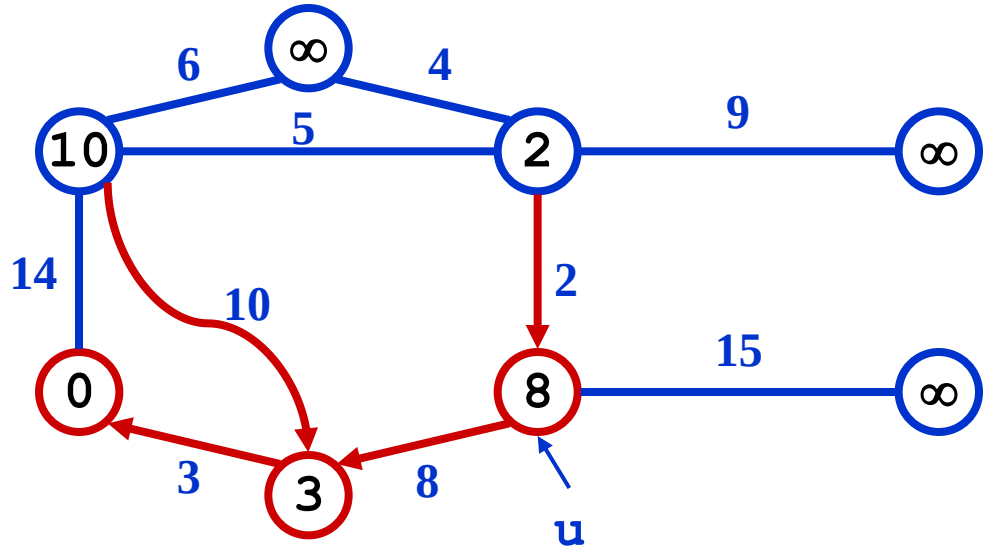

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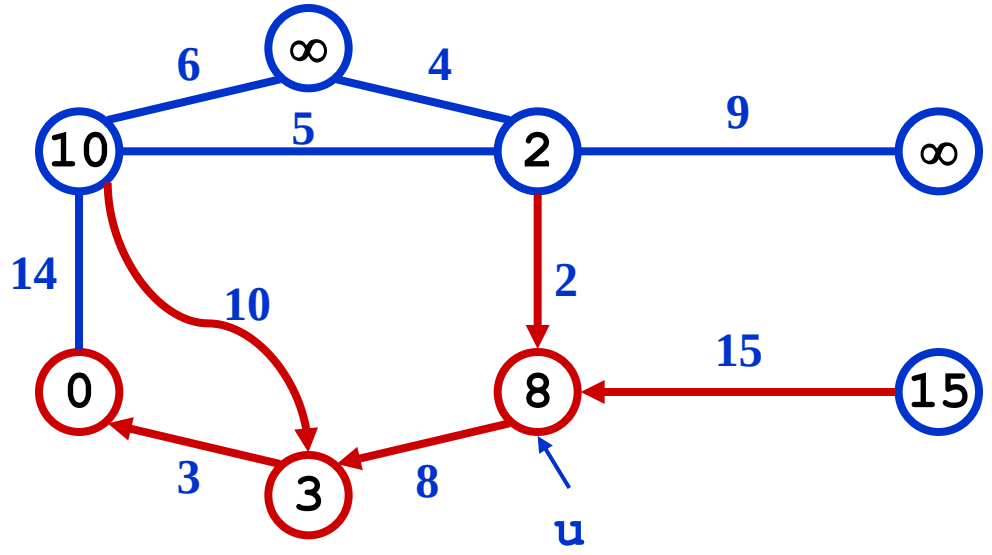

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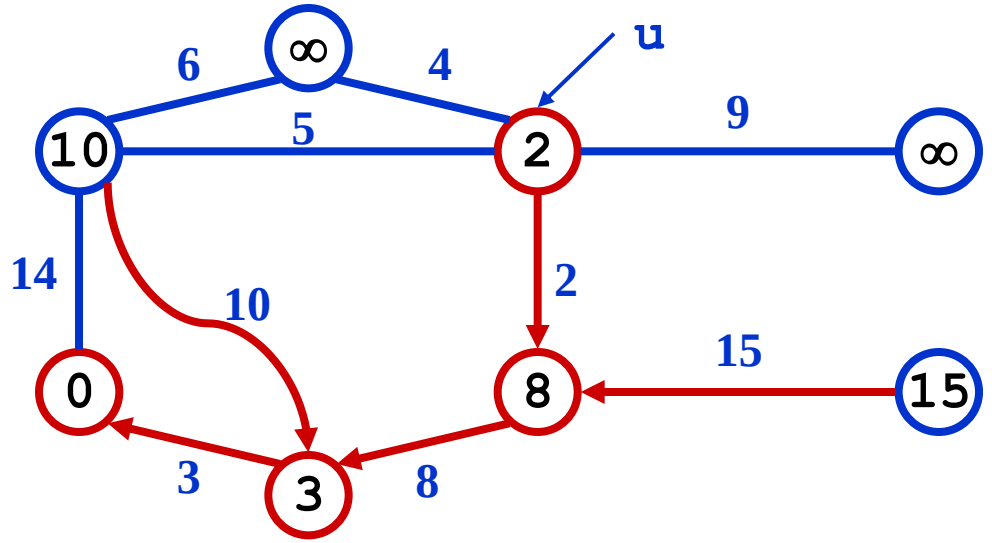

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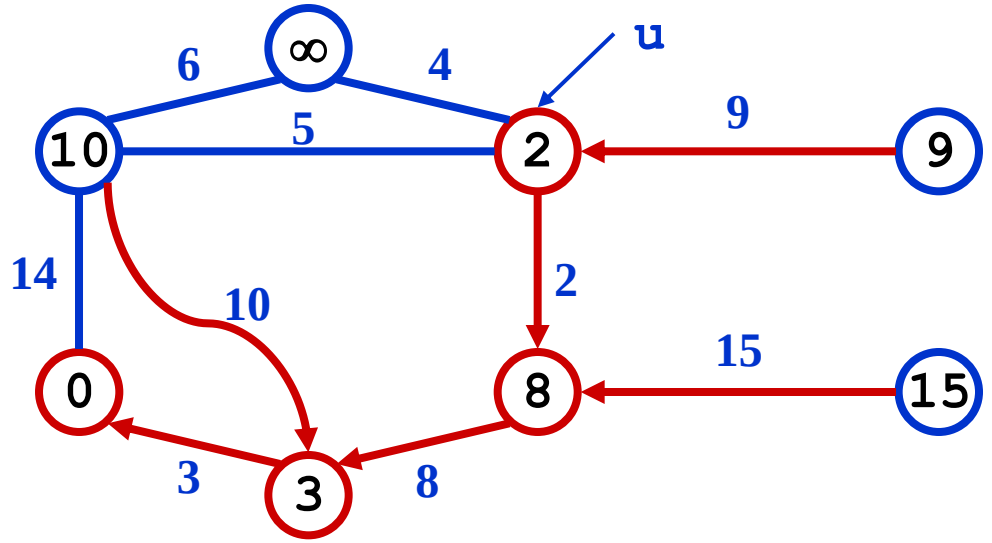

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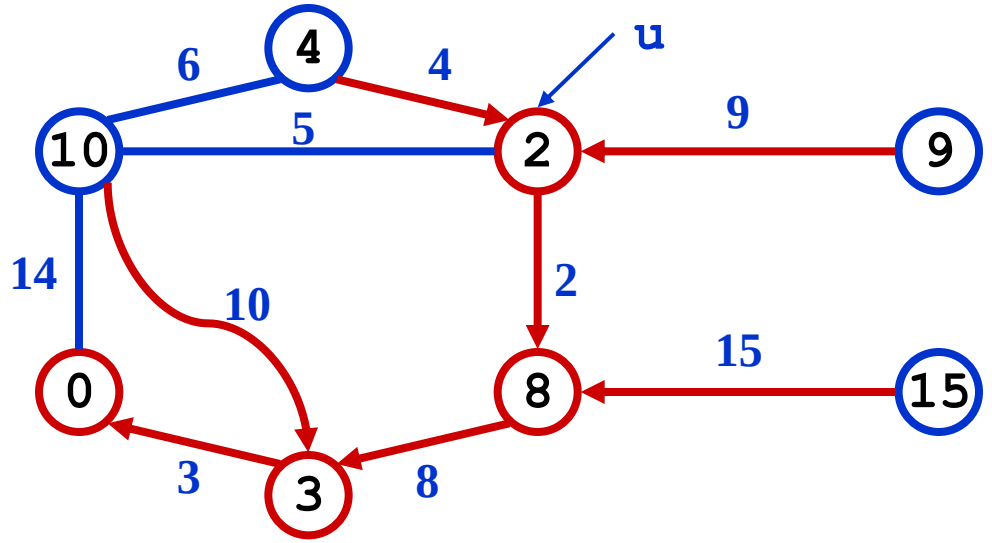

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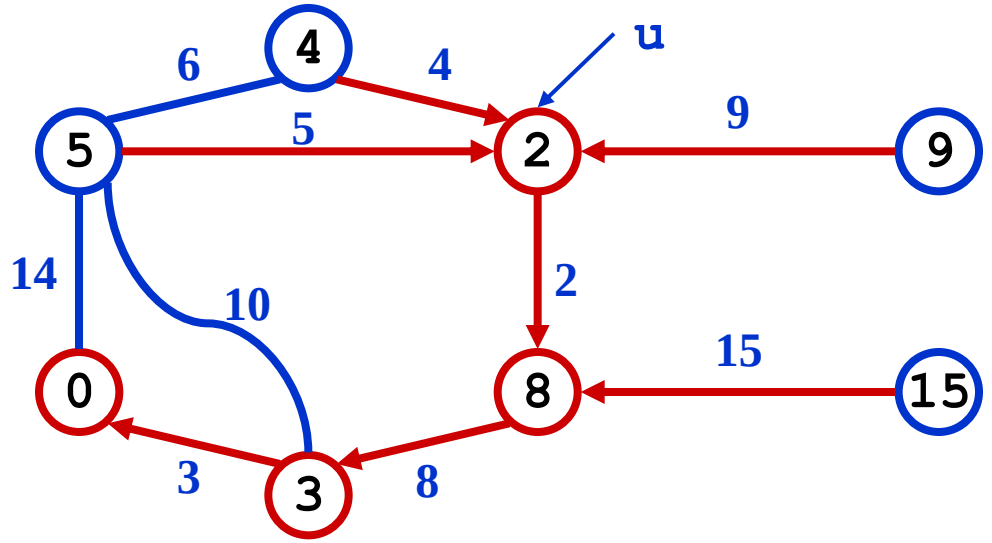

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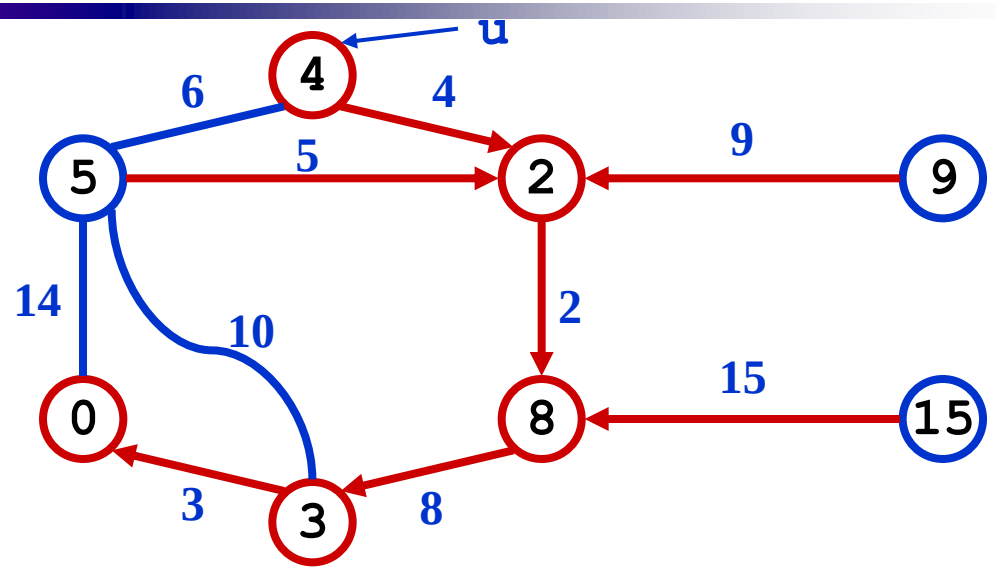
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
                key[v] = w(u,v);
```



Prim's Algorithm

```
MST-Prim(G, w, r)
    Q = V[G];
    for each u ∈ Q
        key[u] = ∞;
    key[r] = 0;
    p[r] = NULL;
    while (Q not empty)
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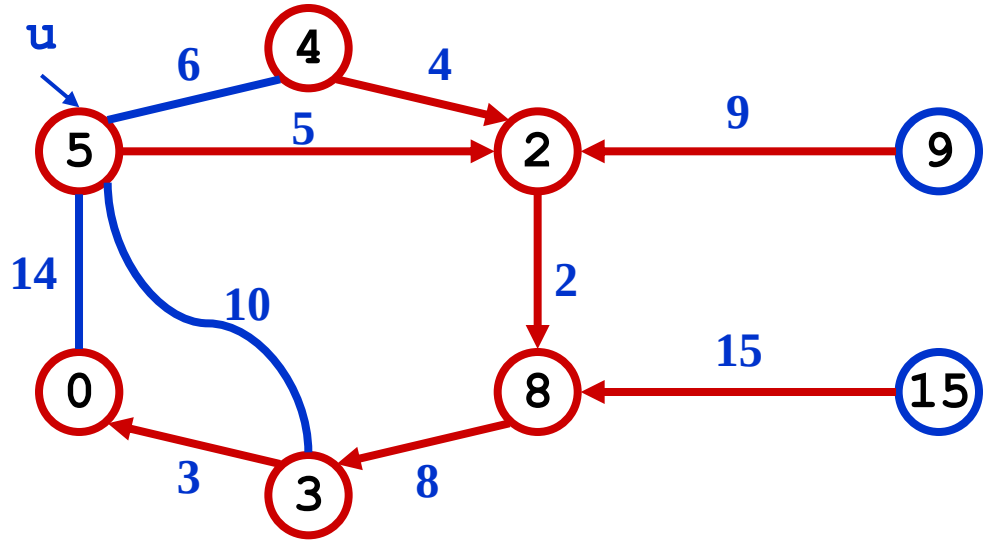
```



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            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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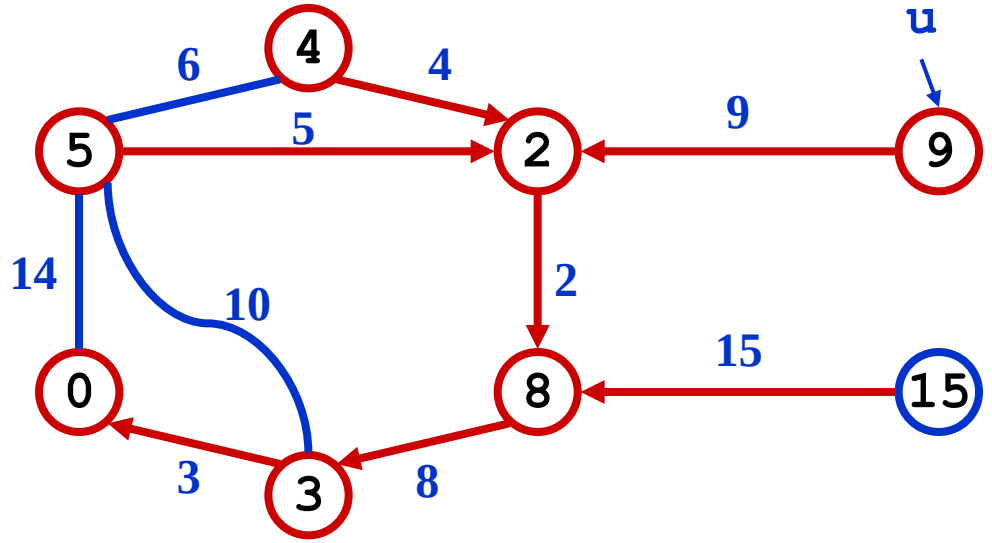
```



Prim's Algorithm

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        u = ExtractMin(Q);
        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
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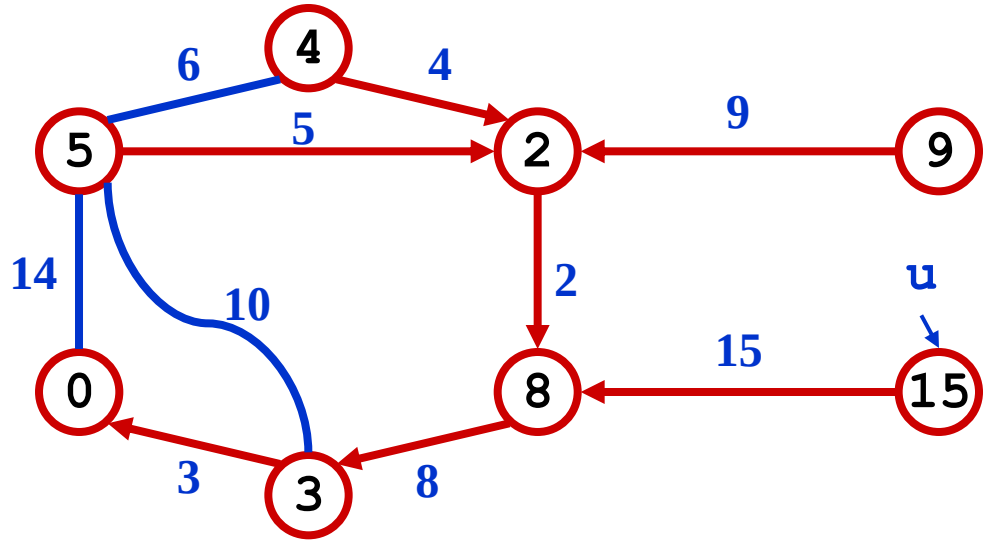
```



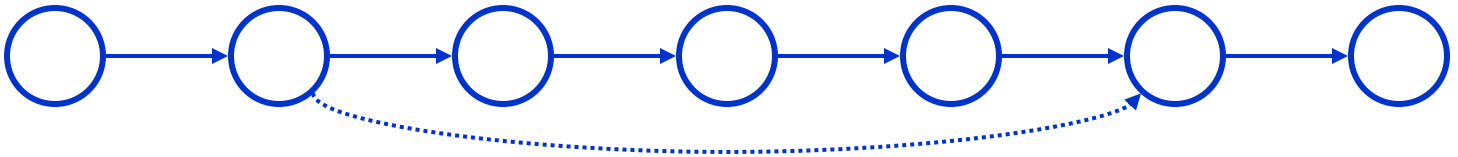
Single-Source Shortest Path

Single-Source Shortest Path

- Problem: given a weighted directed graph G, find the minimum-weight path from a given source vertex s to another vertex v
  - "Shortest-path" = minimum weight
  - Weight of path is sum of edges
  - E.g., a road map: what is the shortest path from Louisville to Memphis ?

Shortest Path Properties

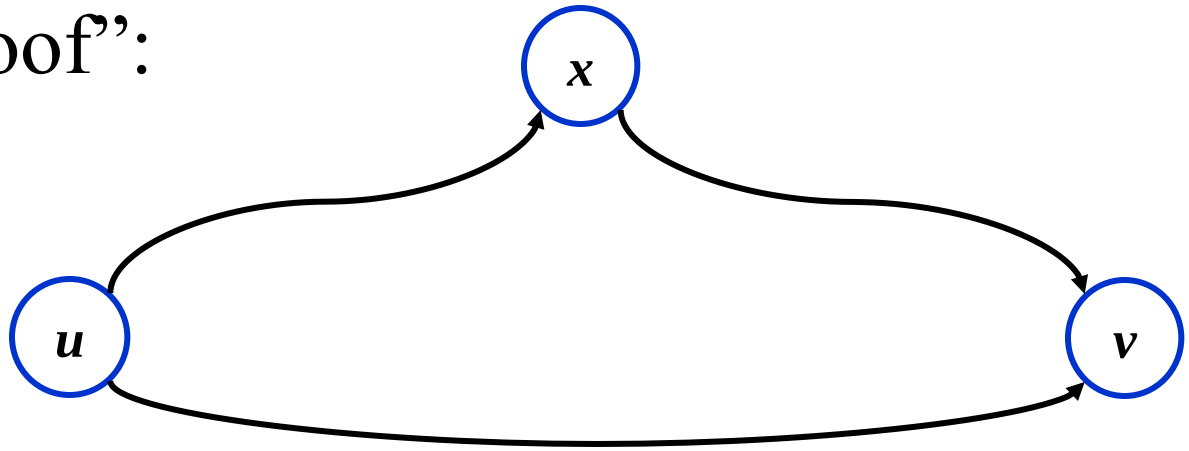
- Again, we have *optimal substructure*: the shortest path consists of shortest subpaths:



  - Proof: suppose some subpath is not a shortest path
    - There must then exist a shorter subpath
    - Could substitute the shorter subpath for a shorter path
    - But then overall path is not shortest path. Contradiction

Shortest Path Properties

- Define $\delta(u,v)$ to be the weight of the shortest path from u to v
- Shortest paths satisfy the *triangle inequality*: $\delta(u,v) \leq \delta(u,x) + \delta(x,v)$
- "Proof":

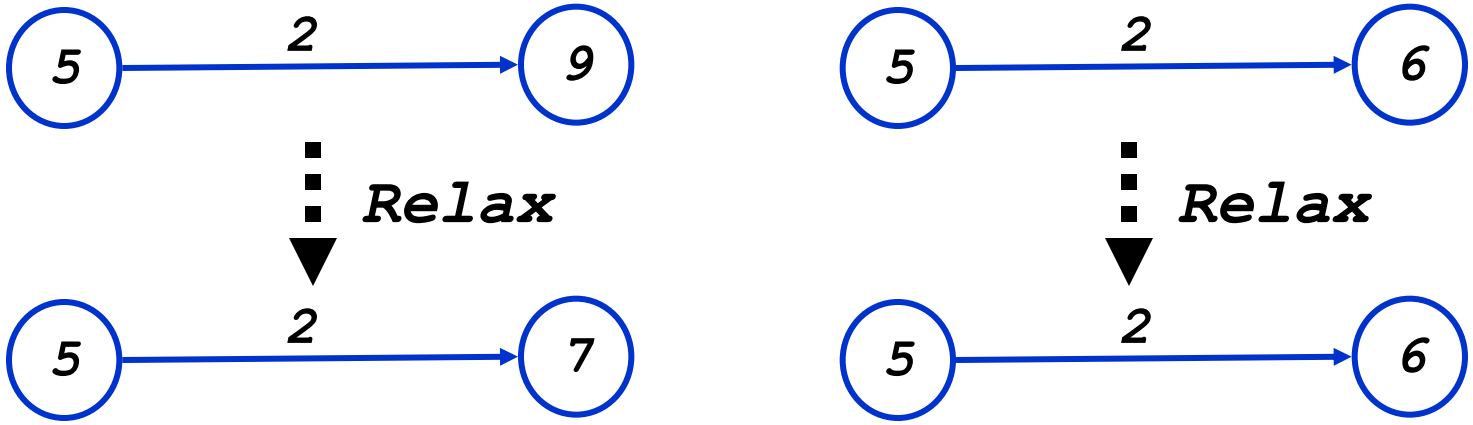


This path is no longer than any other path

Relaxation

- A key technique in shortest path algorithms is *relaxation*
  - Idea: for all *v*, maintain upper bound d[*v*] on δ(s,v)

```
Relax(u,v,w) {
     if (d[v] > d[u]+w) then d[v]=d[u]+w;
}
```



Bellman-Ford Algorithm

```
BellmanFord()
    for each v ∈ V
        d[v] = ∞;
    d[s] = 0;
    for i=1 to |V|-1
        for each edge (u,v) ∈ E
            Relax(u,v, w(u,v));
    for each edge (u,v) ∈ E
        if (d[v] > d[u] + w(u,v))
            return "no solution";
```

Initialize d[], which will converge to shortest-path value δ

Relaxation: Make |V|-1 passes, relaxing each edge

Test for solution
Under what condition do we get a solution?

```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Bellman-Ford Algorithm

```
BellmanFord()
    for each v ∈ V
        d[v] = ∞;
    d[s] = 0;
    for i=1 to |V|-1
        for each edge (u,v) ∈ E
            Relax(u,v, w(u,v));
    for each edge (u,v) ∈ E
        if (d[v] > d[u] + w(u,v))
            return "no solution";
```

What will be the running time?

```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Bellman-Ford Algorithm

```
BellmanFord()
    for each v ∈ V
        d[v] = ∞;
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        for each edge (u,v) ∈ E
            Relax(u,v, w(u,v));
    for each edge (u,v) ∈ E
        if (d[v] > d[u] + w(u,v))
            return "no solution";
```

What will be the running time?

A: O(VE)

```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Bellman-Ford Algorithm

```
BellmanFord()
    for each v ∈ V
        d[v] = ∞;
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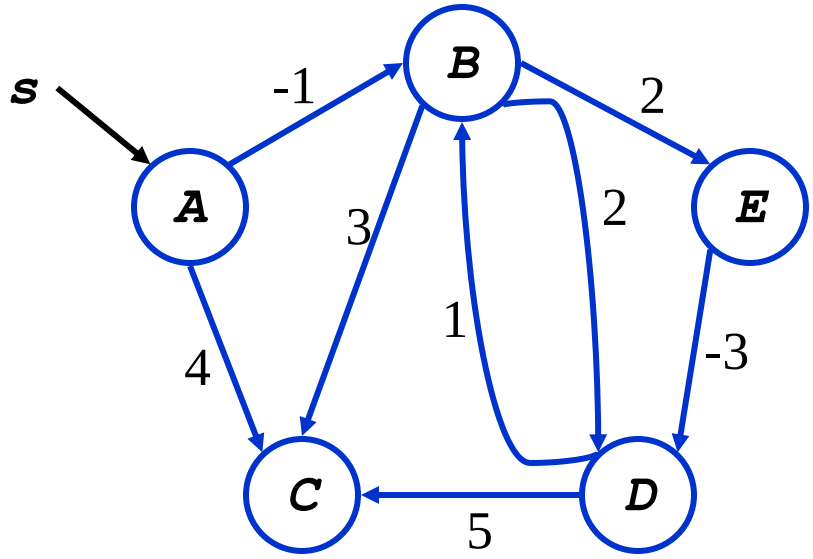
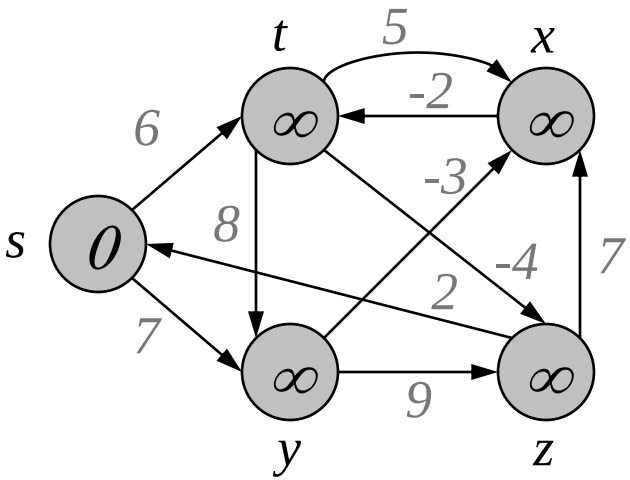
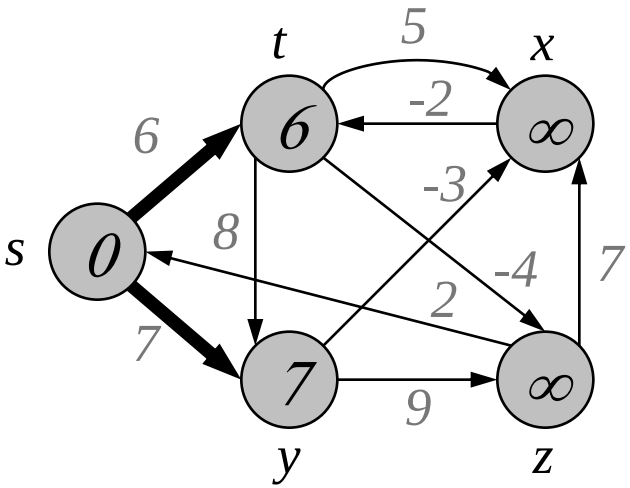
    for i=1 to |V|-1
        for each edge (u,v) ∈ E
            Relax(u,v, w(u,v));
    for each edge (u,v) ∈ E
        if (d[v] > d[u] + w(u,v))
            return "no solution";
```

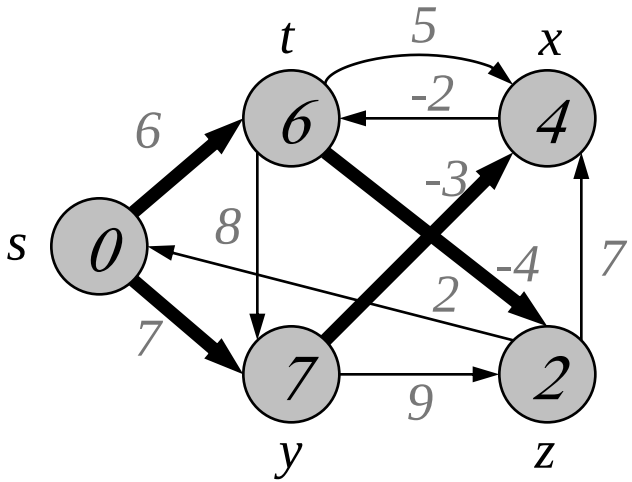


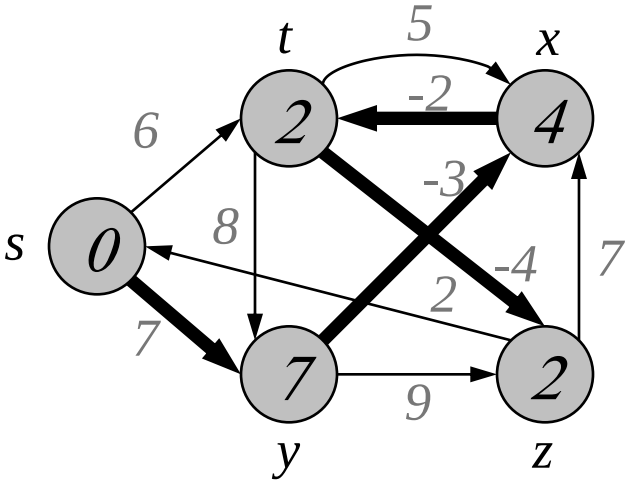

```
Relax(u,v,w): if (d[v] > d[u]+w) then d[v]=d[u]+w
```

Bellman-Ford Example

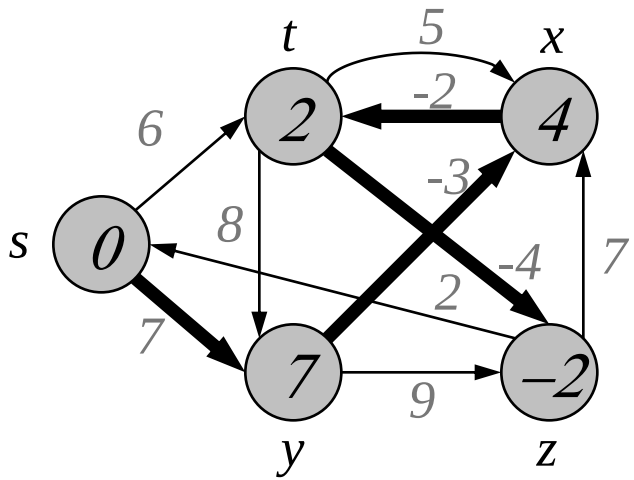






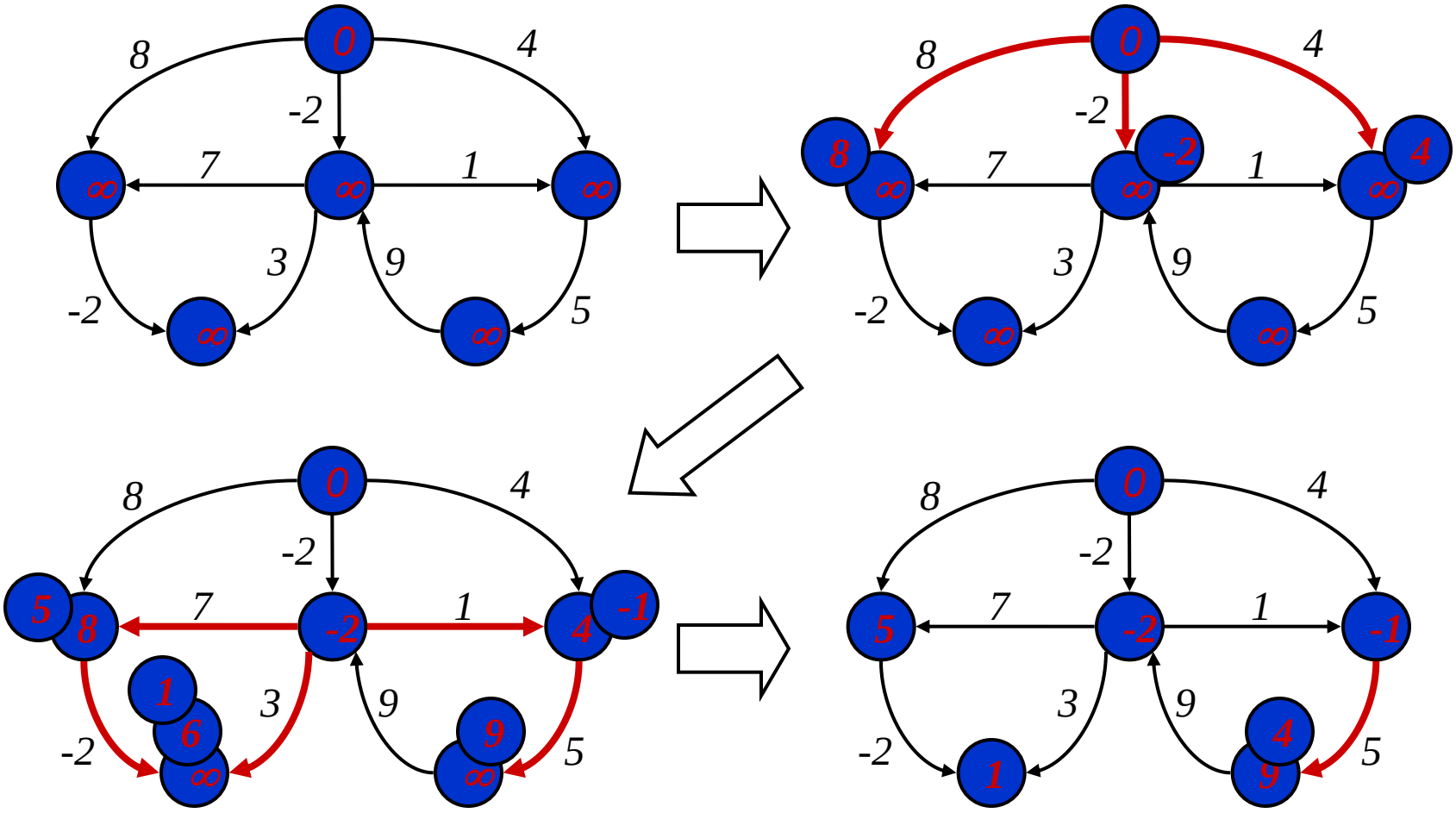


Bellman-Ford Example



- Bellman-Ford running time:
  - $(|V|-1)|E| + |E| = \Theta(VE)$

Bellman-Ford Example



Dijkstra's algorithm

Dijkstra's algorithm - is a solution to the single-source shortest path problem in graph theory.

Works on both directed and undirected graphs. However, all edges must have nonnegative weights.

Approach: Greedy

Input: Weighted graph G={E,V} and source vertex $v \in V$, such that all edge weights are nonnegative

Output: Lengths of shortest paths (or the shortest paths themselves) from a given source vertex $v \in V$ to all other vertices

Dijkstra's algorithm - Pseudocode

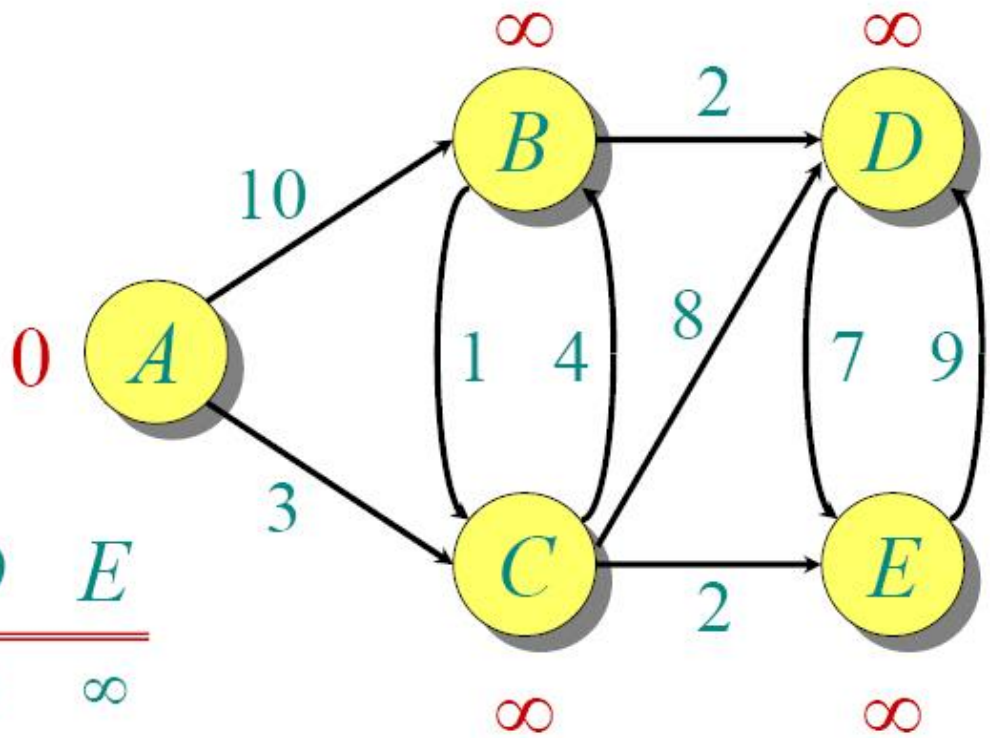
```
dist[s] ←o                                  (distance to source vertex is zero)
for  all v ∈ V–{s}
      do  dist[v] ←∞                        (set all other distances to infinity)
S←∅                                         (S, the set of visited vertices is initially empty)
Q←V                                 (Q, the queue initially contains all vertices)
while Q ≠∅                                  (while the queue is not empty)
do   u ← mindistance(Q,dist)                (select the element of Q with the min. distance)
     S←S∪{u}                                (add u to list of visited vertices)
      for all v ∈ neighbors[u]
            do  if  dist[v] > dist[u] + w(u, v)                (if new shortest path found)
                  then     d[v] ←d[u] + w(u, v)        (set new value of shortest path)
                        (if desired, add traceback code)
return dist
```

Dijkstra's Algorithm

- If no negative edge weights, we can beat BF
- Similar to breadth-first search
  - Grow a tree gradually, advancing from vertices taken from a queue
- Also similar to Prim's algorithm for MST
  - Use a priority queue keyed on d[v]

Dijkstra Animated Example

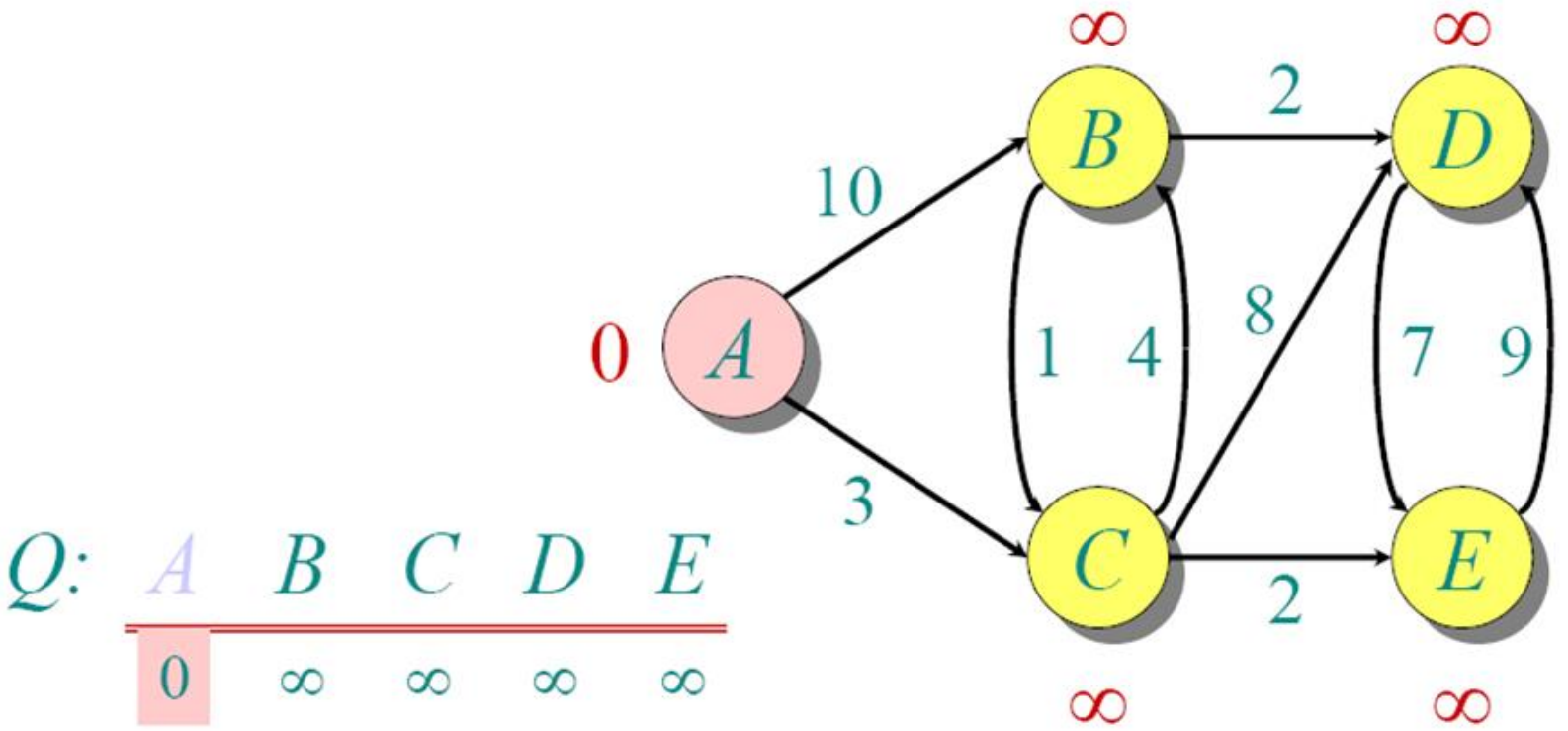
Initialize:



Q:	A	B	C	D	E
	0	∞	∞	∞	∞

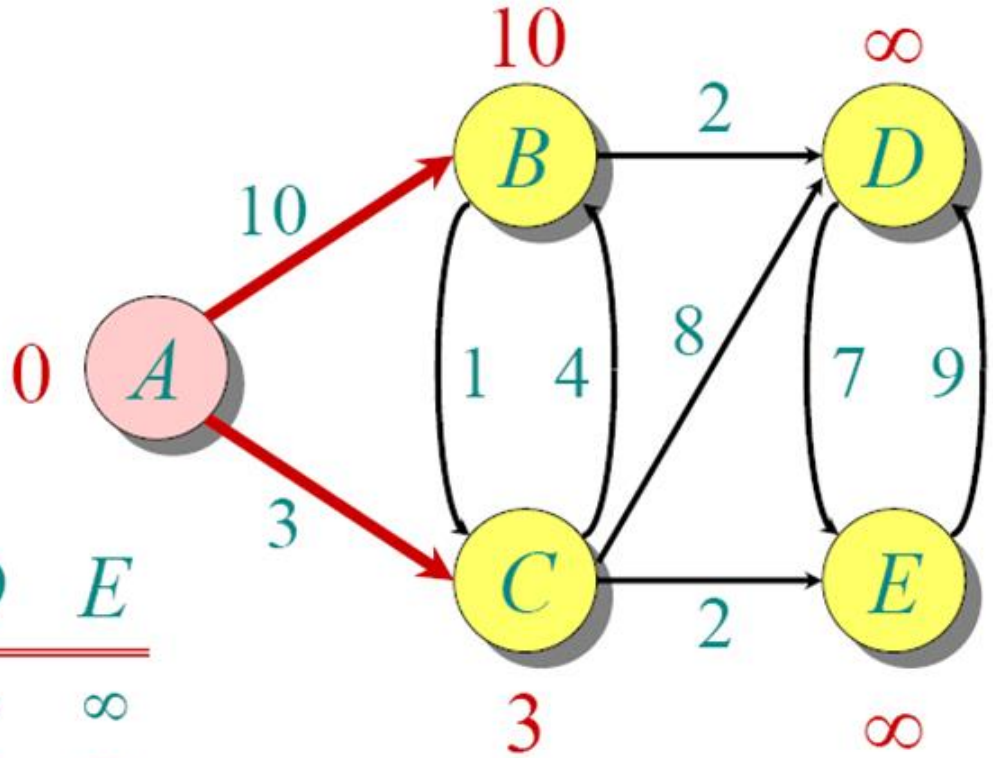
S: {}

Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞

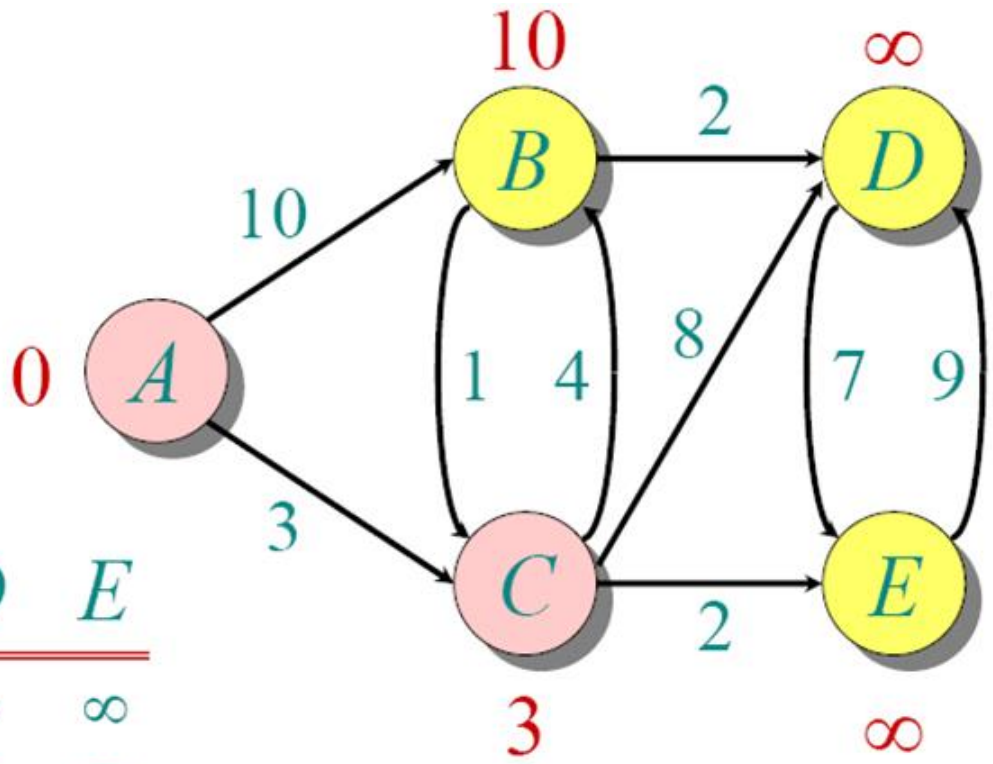
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

$S: \{ A \}$

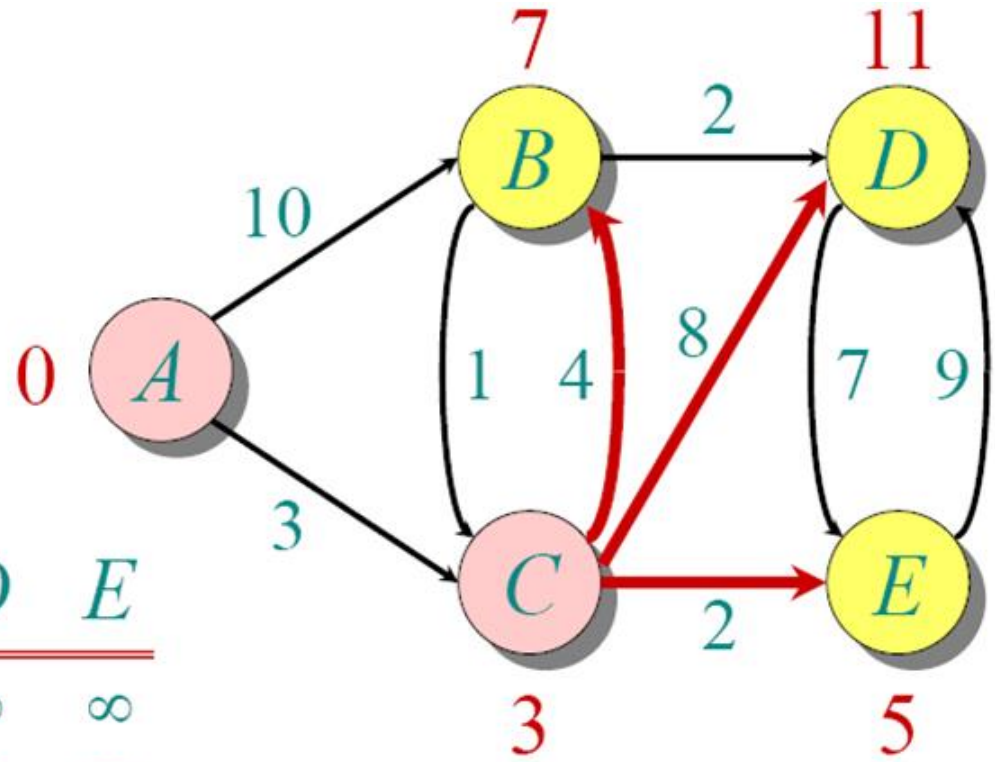
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞

$S: \{ A, C \}$

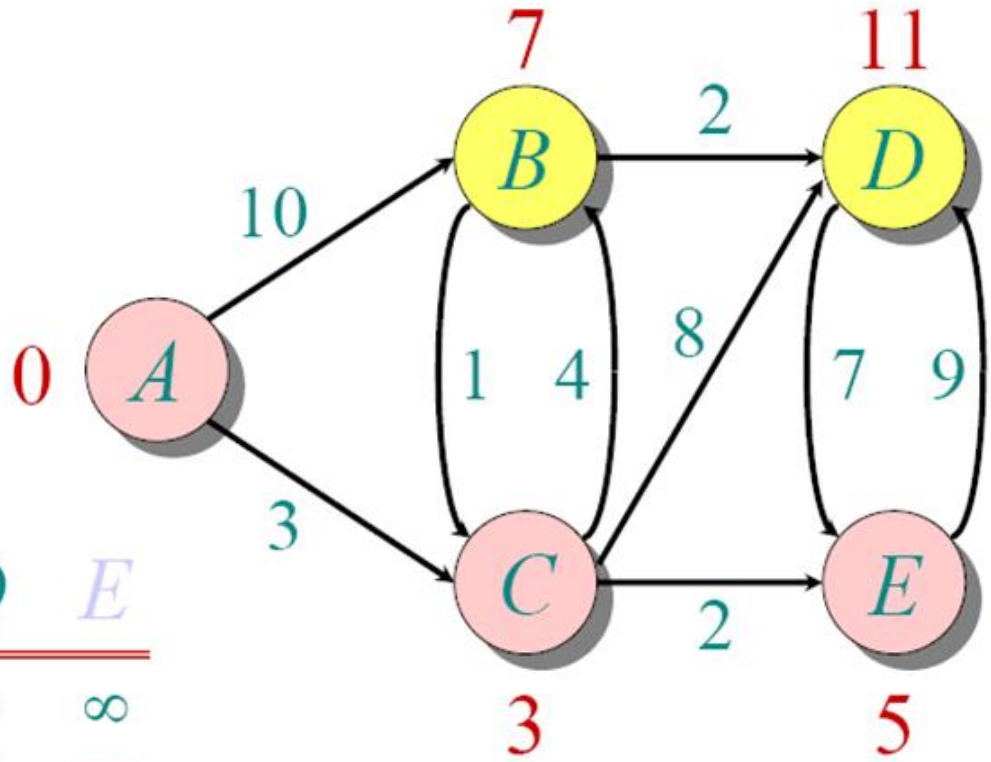
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

$S: \{ A, C \}$

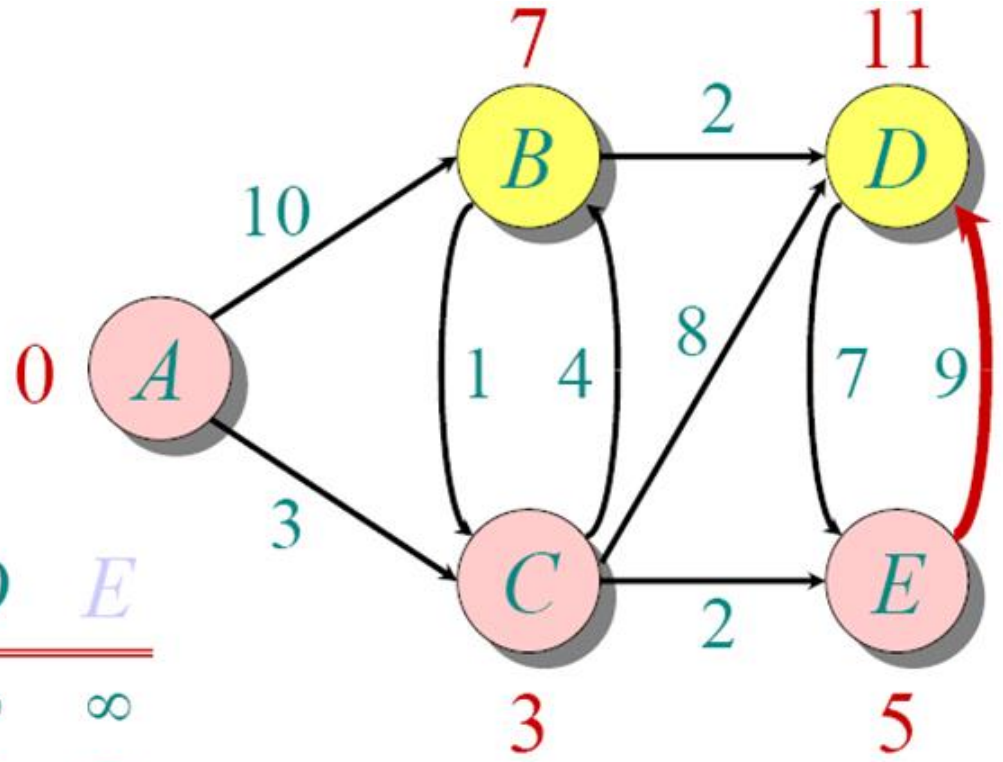
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

$S: \{ A, C, E \}$

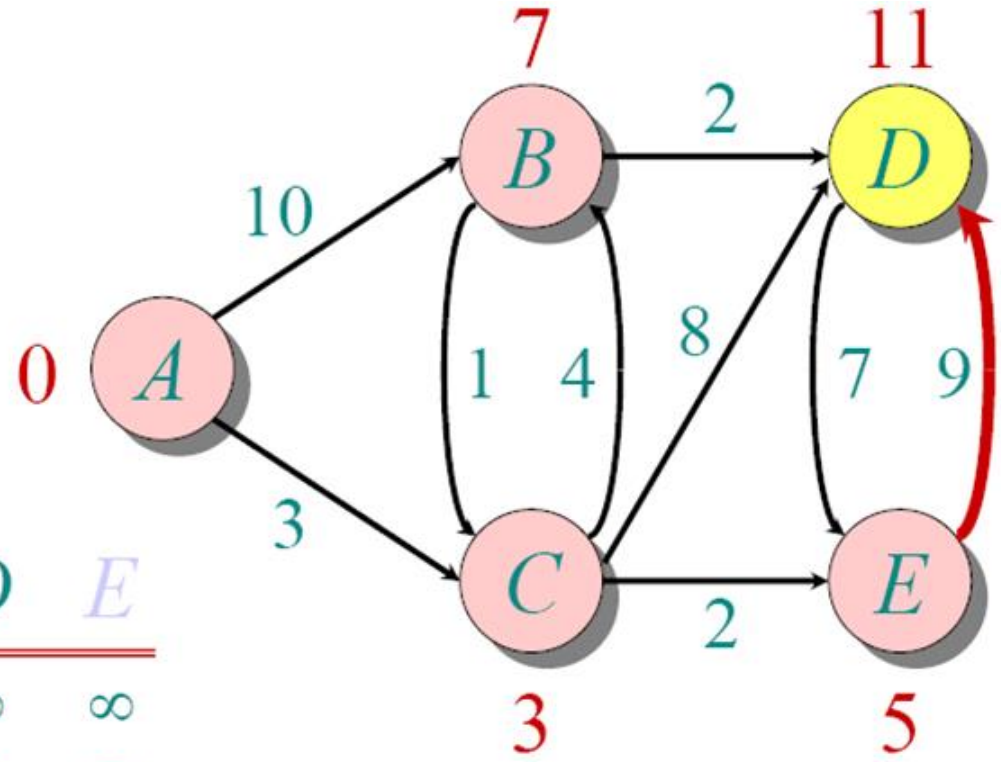
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

$S: \{ A, C, E \}$

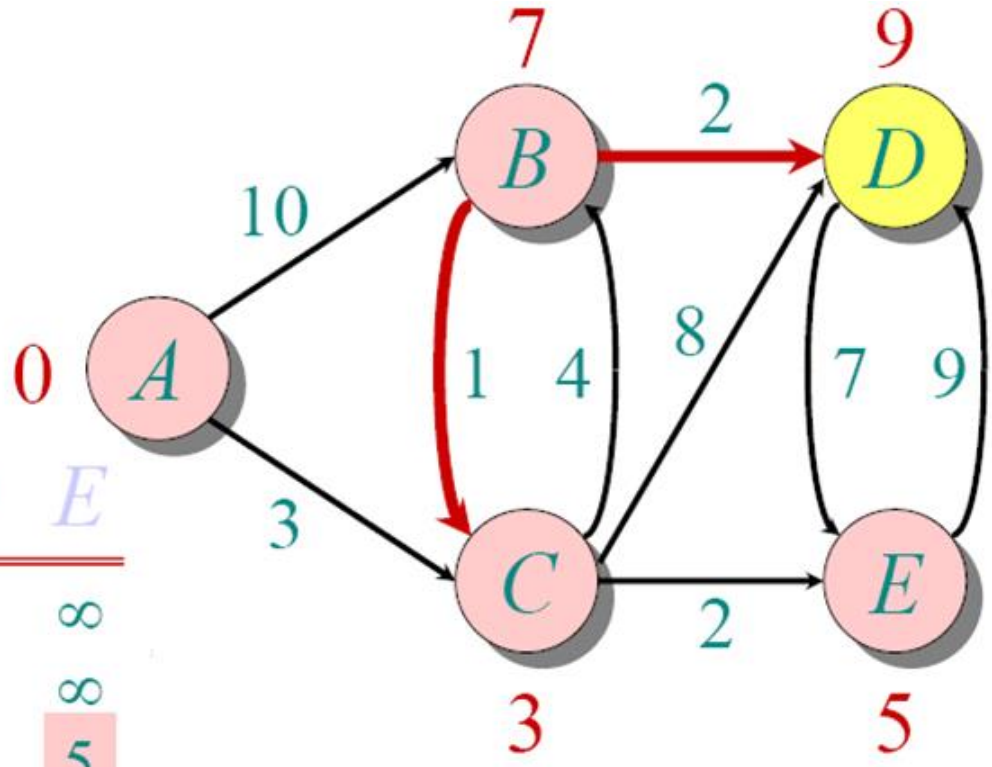
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

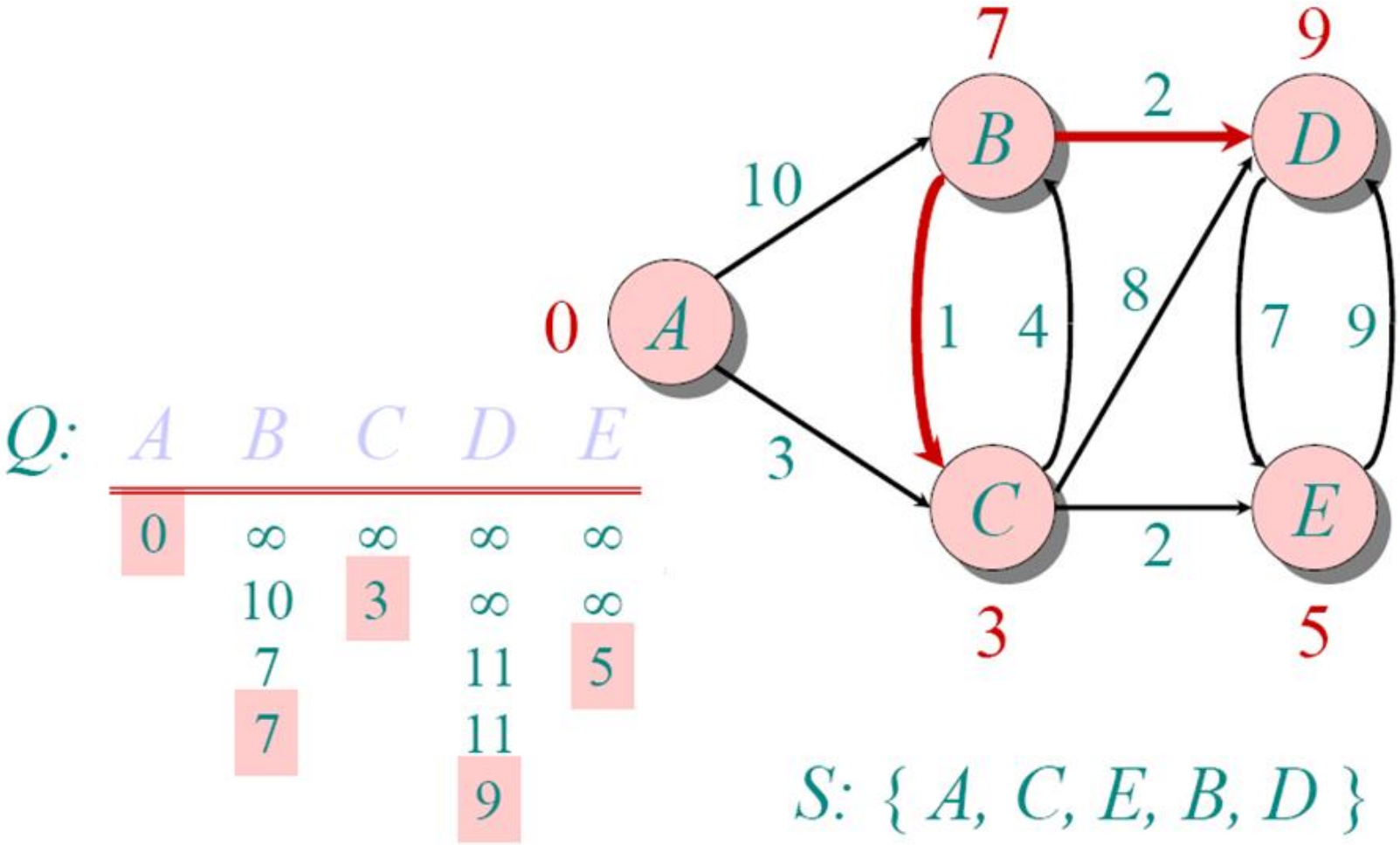
Dijkstra Animated Example



Q:	A	B	C	D	E
	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { A, C, E, B }

Dijkstra Animated Example



Implementations and Running Times

For sparse graphs, or graphs with very few edges and many nodes, it can be implemented more efficiently storing the graph in an adjacency list using a binary heap or priority queue. This will produce a running time of

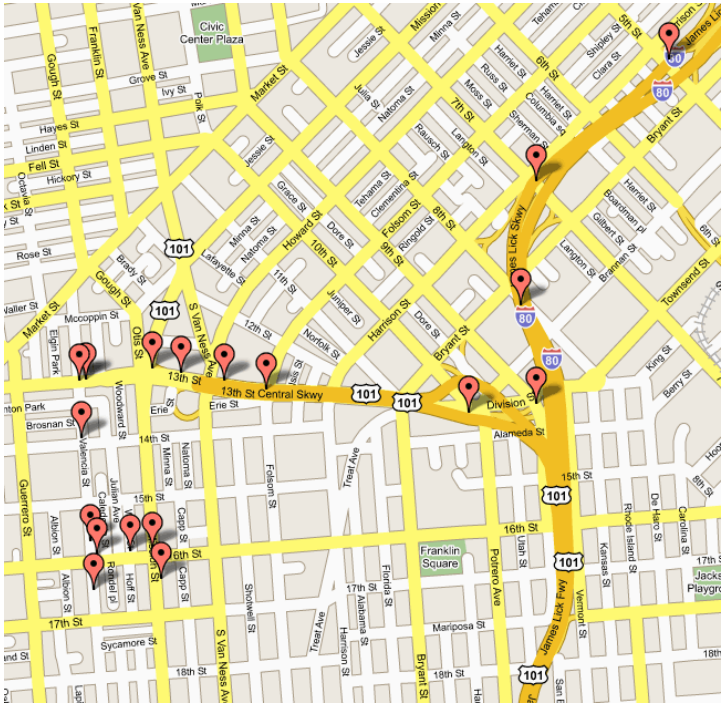
$O((|E|+|V|) \log |V|)$

Dijkstra's Algorithm - Why use it?

- As mentioned, Dijkstra's algorithm calculates the shortest path to every vertex.
- However, it is about as computationally expensive to calculate the shortest path from vertex *u* to every vertex using Dijkstra's as it is to calculate the shortest path to some particular vertex *v*.
- Therefore, anytime we want to know the optimal path to some other vertex from a determined origin, we can use Dijkstra's algorithm.

Applications of Dijkstra's Algorithm

- Traffic Information Systems are most prominent use
- Mapping (Map Quest, Google Maps)
- Routing Systems



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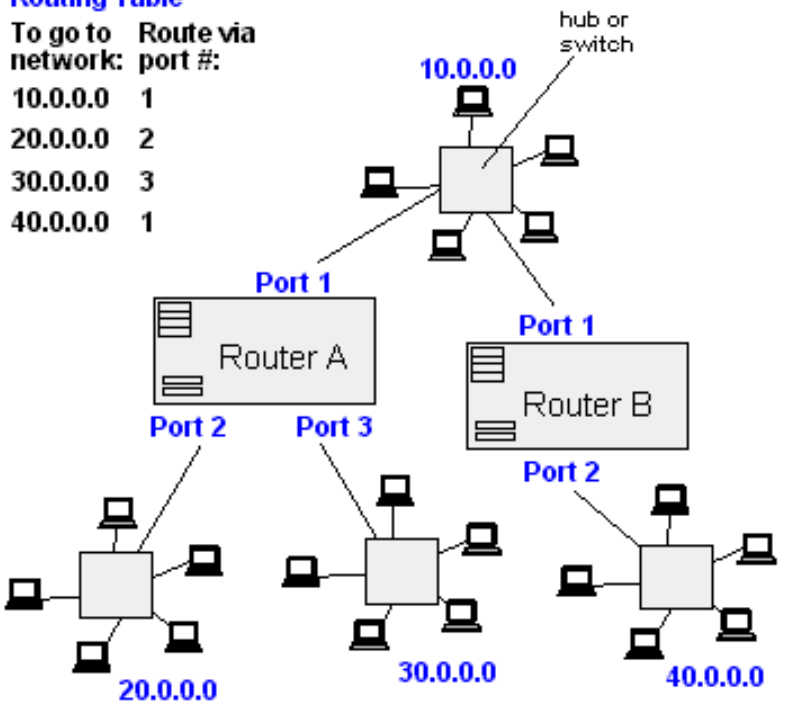
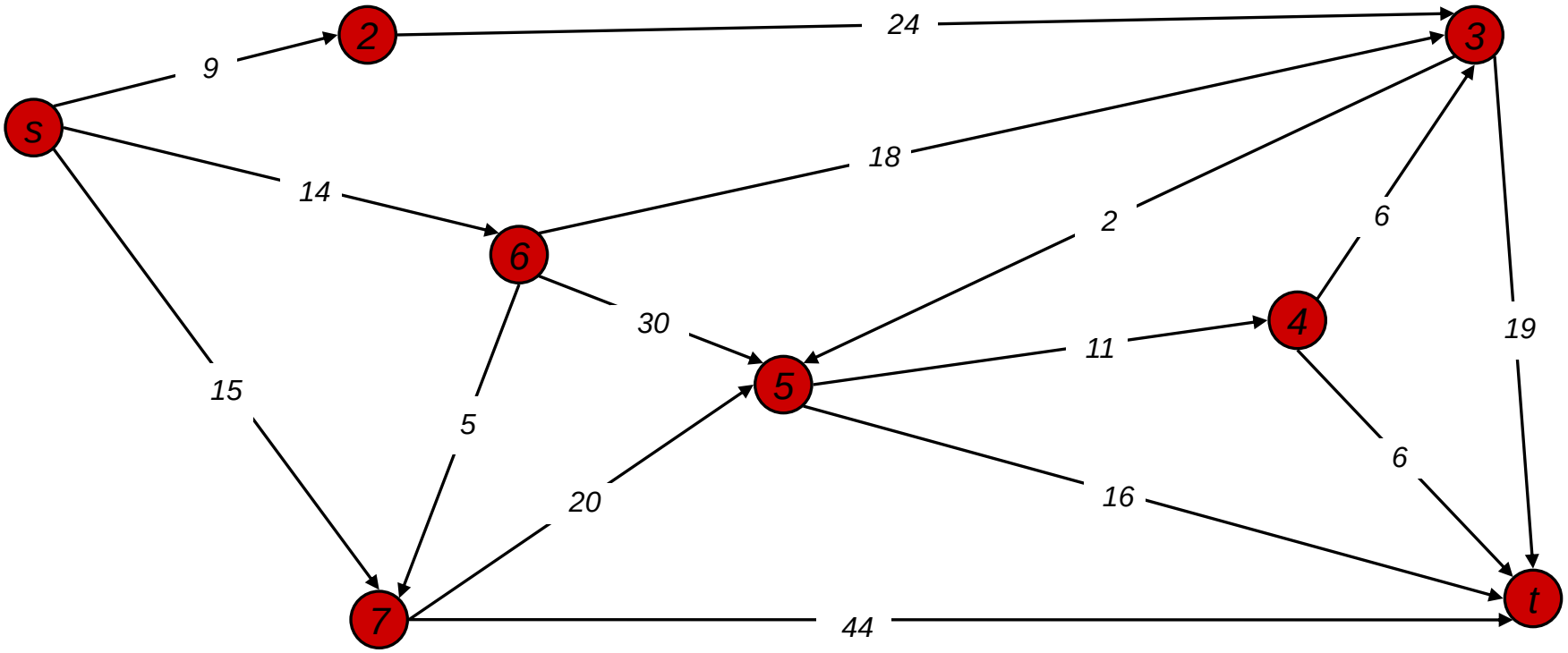
Router A
Routing Table

To go to network:	Route via port #:
10.0.0.0	1
20.0.0.0	2
30.0.0.0	3
40.0.0.0	1



Dijkstra's Shortest Path Algorithm

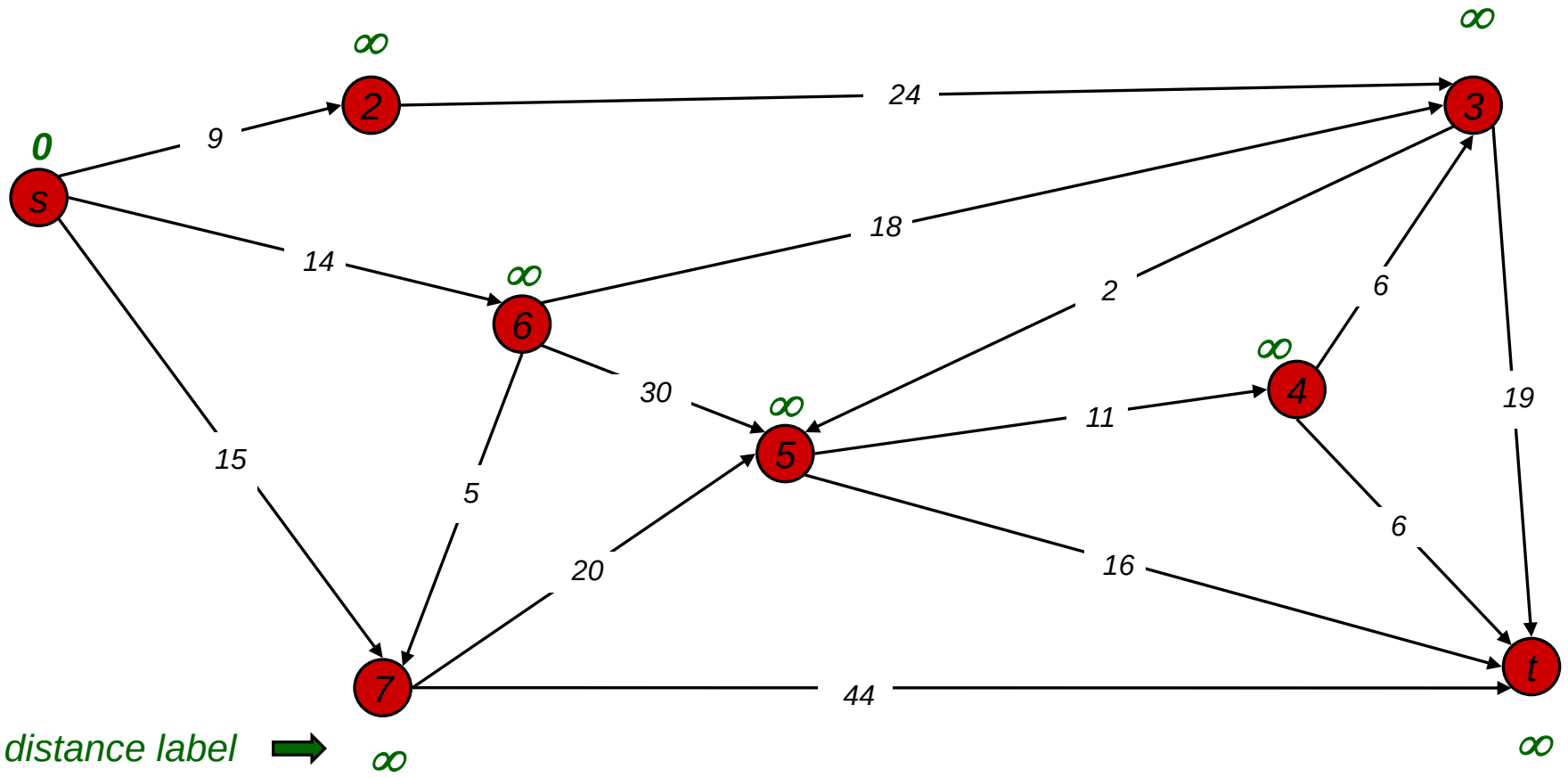
- Find shortest path from s to t.



Dijkstra's Shortest Path Algorithm

$S = \{ \}$

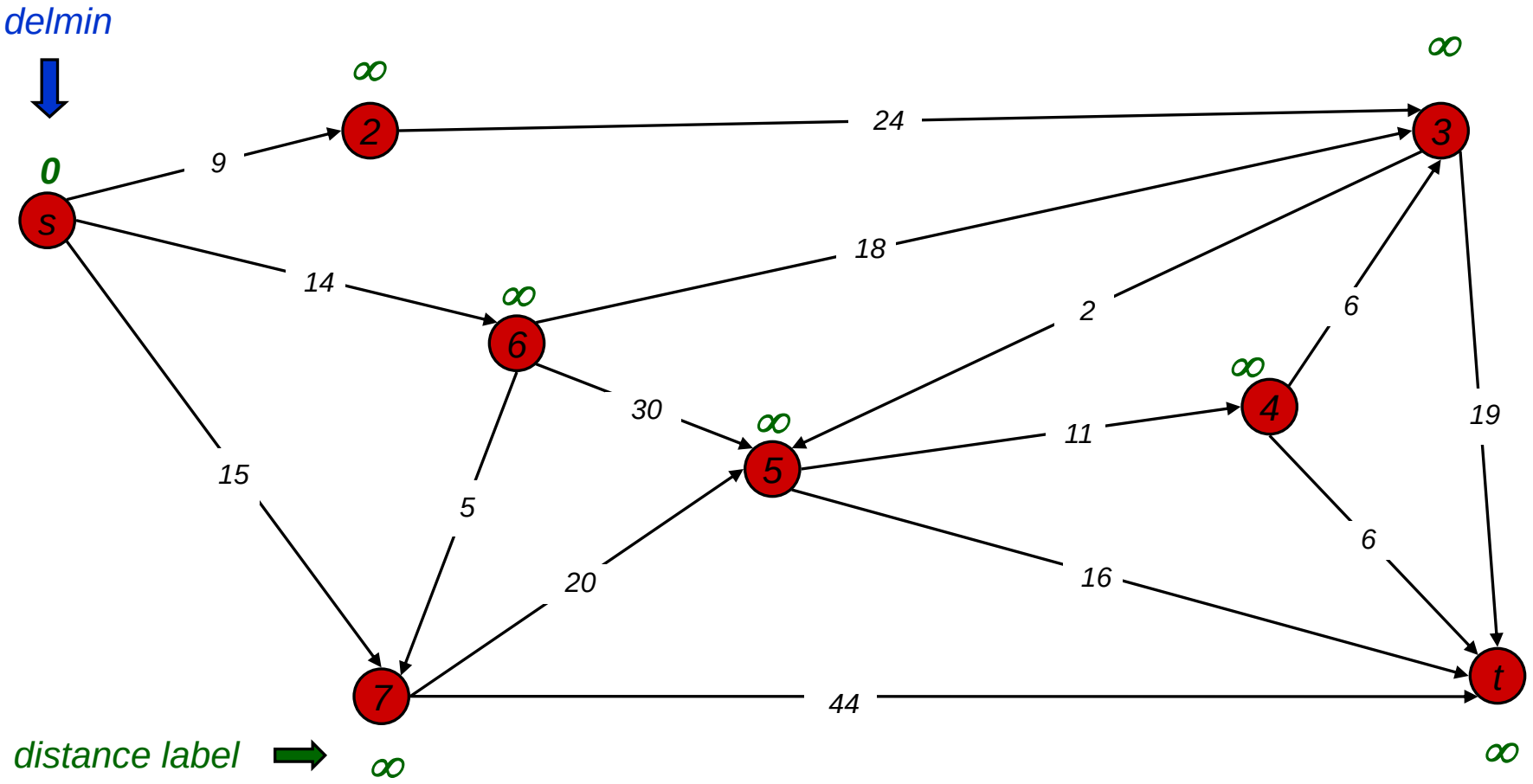
$PQ = \{ s, 2, 3, 4, 5, 6, 7, t \}$



Dijkstra's Shortest Path Algorithm

$S = \{\ \}$

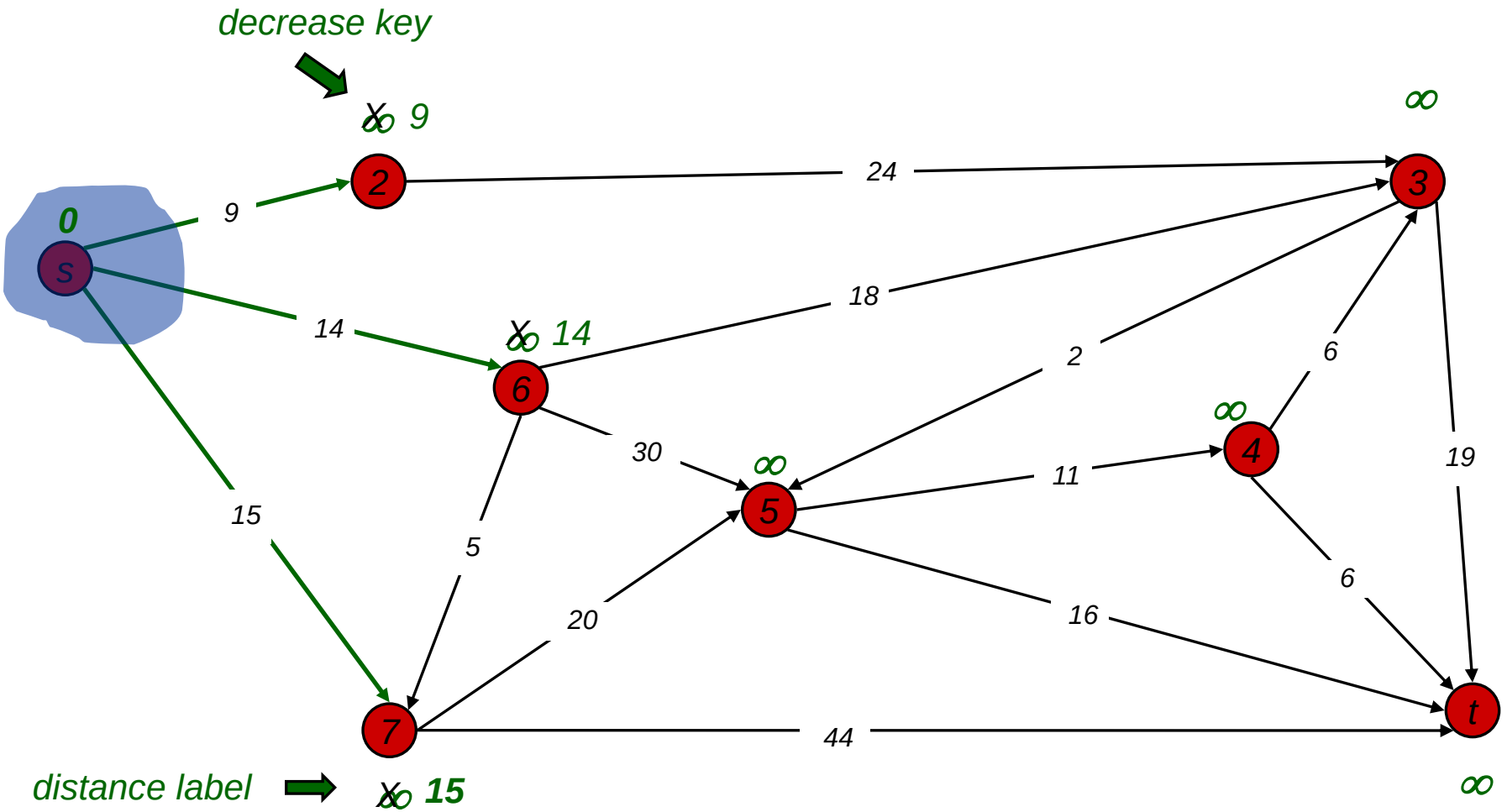
$PQ = \{ s, 2, 3, 4, 5, 6, 7, t \}$



Dijkstra's Shortest Path Algorithm

$S = \{ s \}$

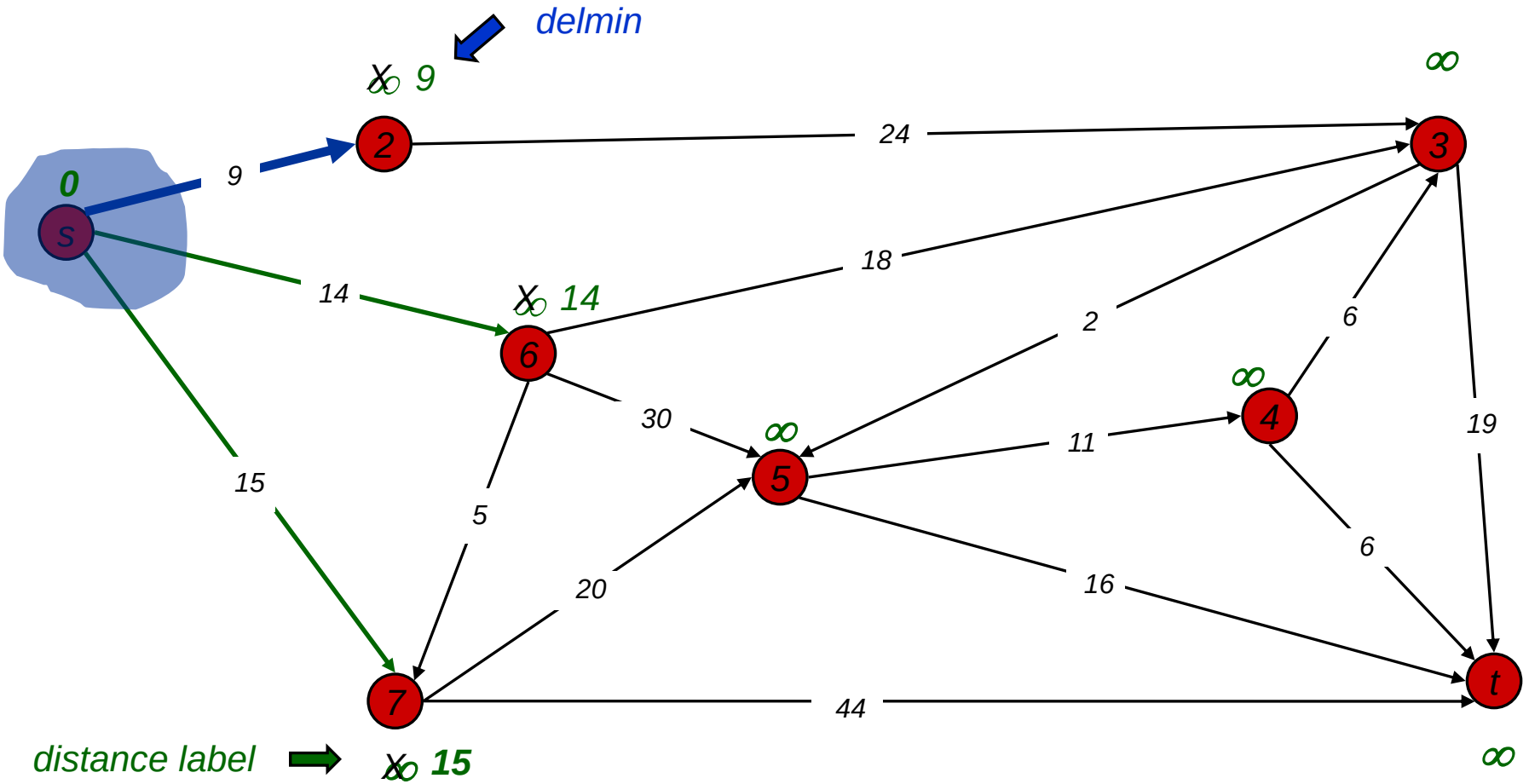
$PQ = \{ 2, 3, 4, 5, 6, 7, t \}$



Dijkstra's Shortest Path Algorithm

$S = \{ s \}$

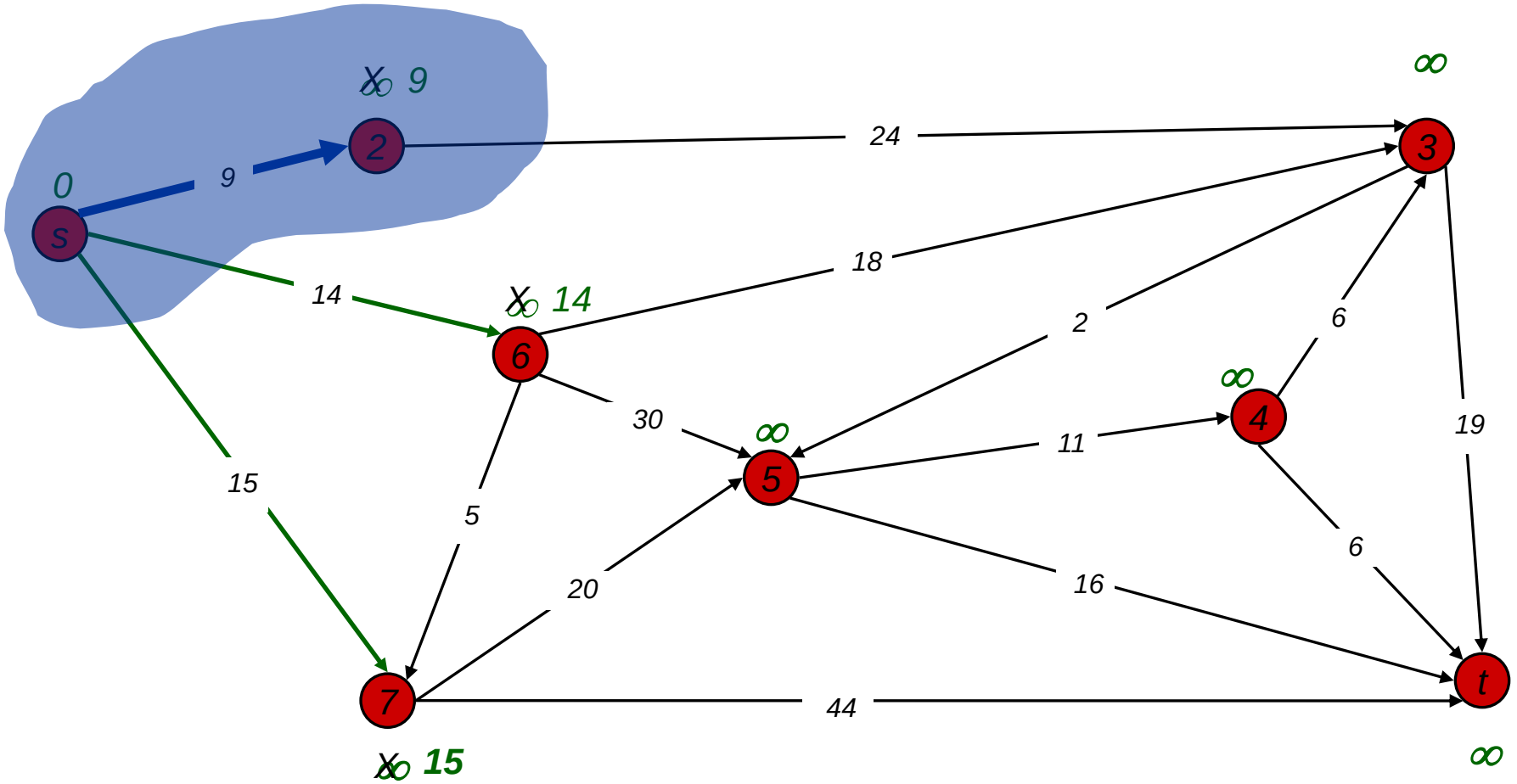
$PQ = \{ 2, 3, 4, 5, 6, 7, t \}$



Dijkstra's Shortest Path Algorithm

S = { s, 2 }

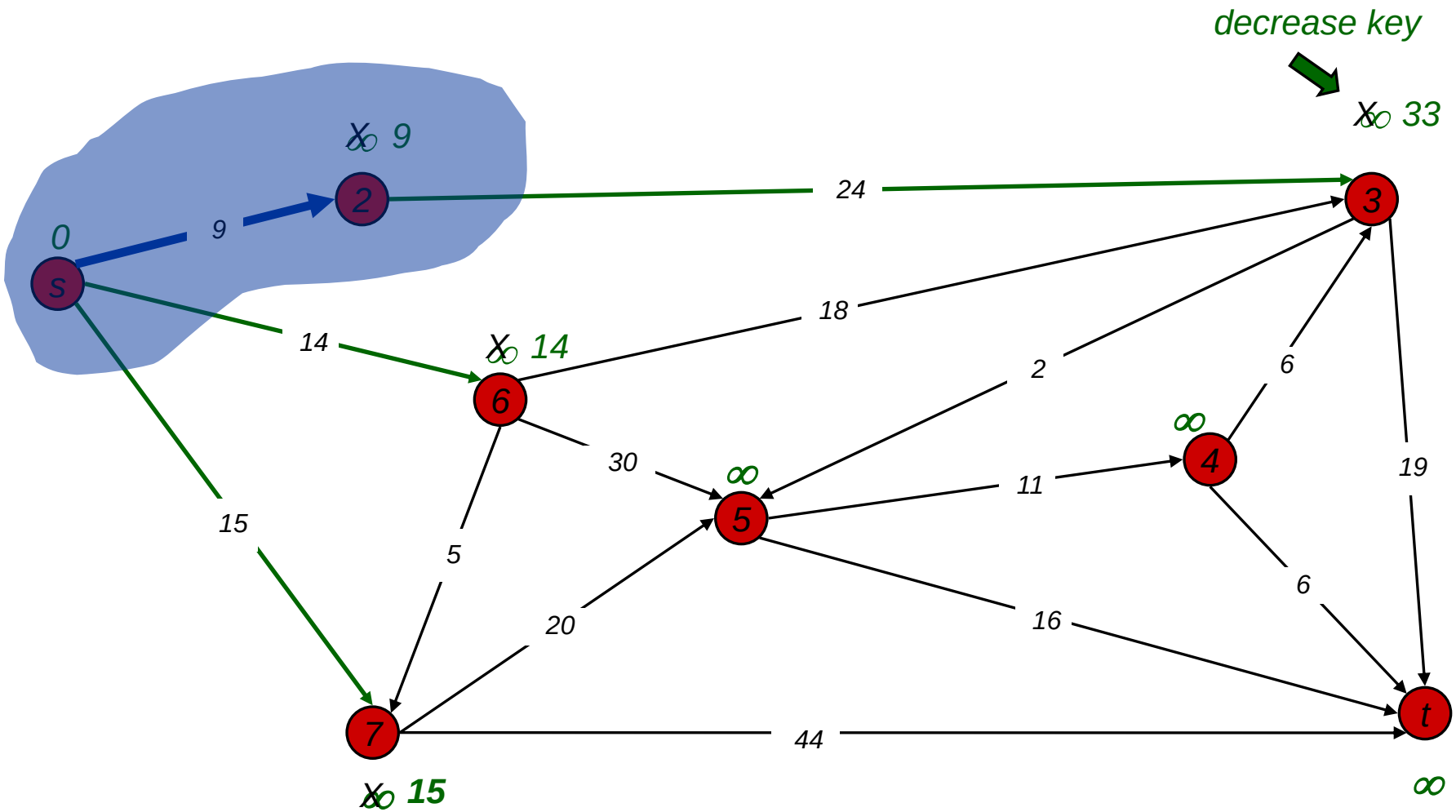
PQ = { 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

$S = \{ s, 2 \}$

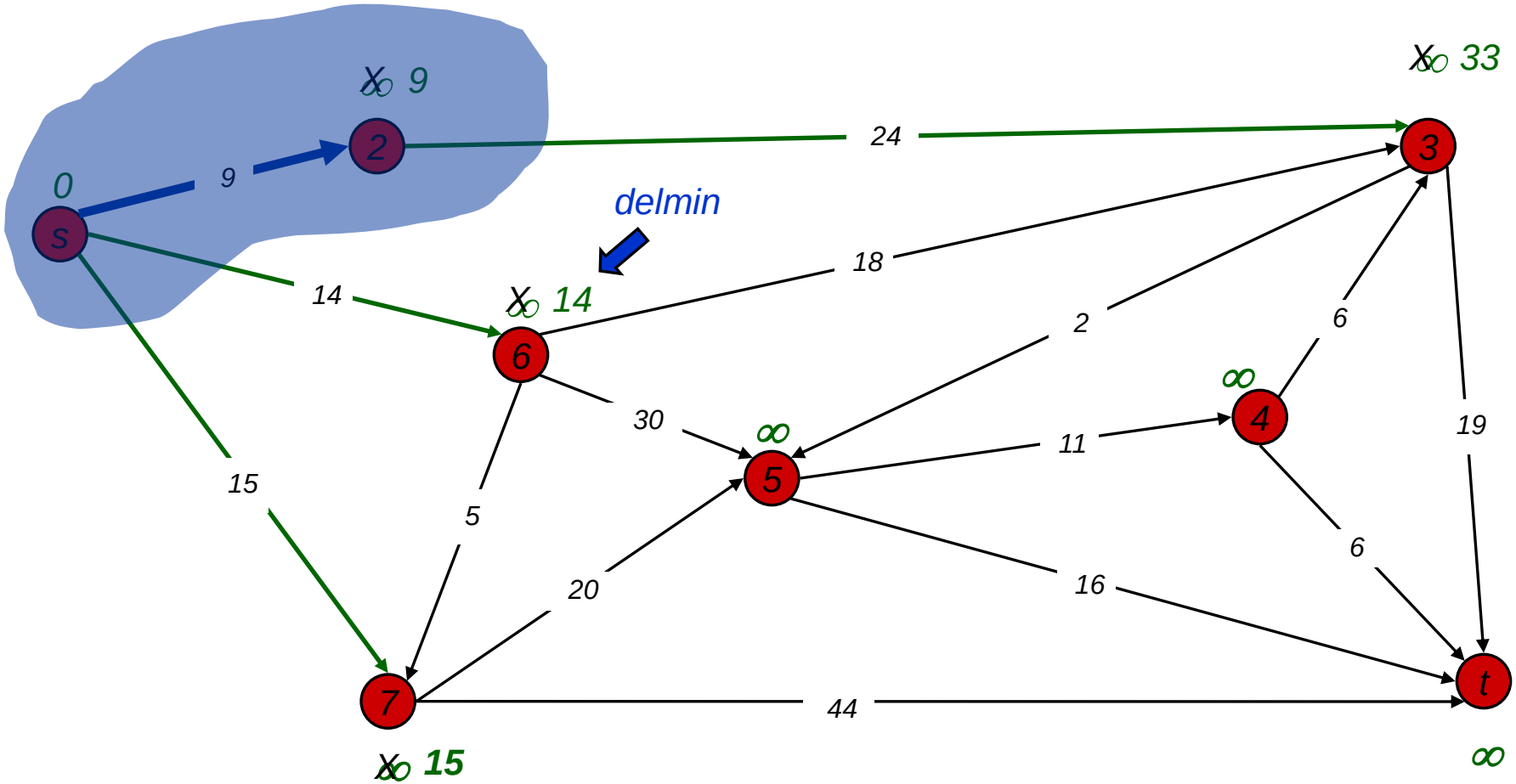
$PQ = \{ 3, 4, 5, 6, 7, t \}$



Dijkstra's Shortest Path Algorithm

S = { s, 2 }

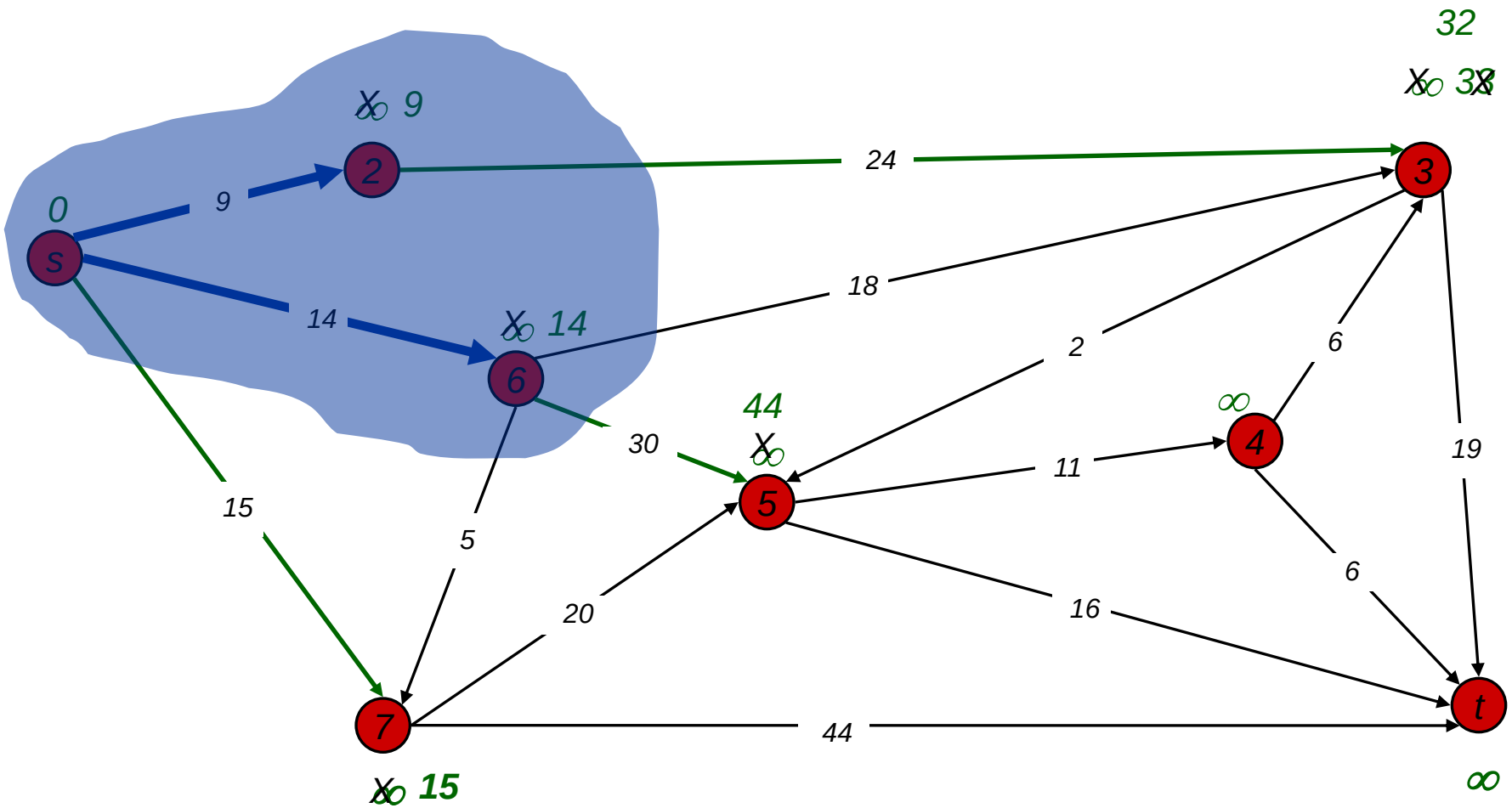
PQ = { 3, 4, 5, 6, 7, t }



Dijkstra's Shortest Path Algorithm

$S = \{ s, 2, 6 \}$

$PQ = \{ 3, 4, 5, 7, t \}$



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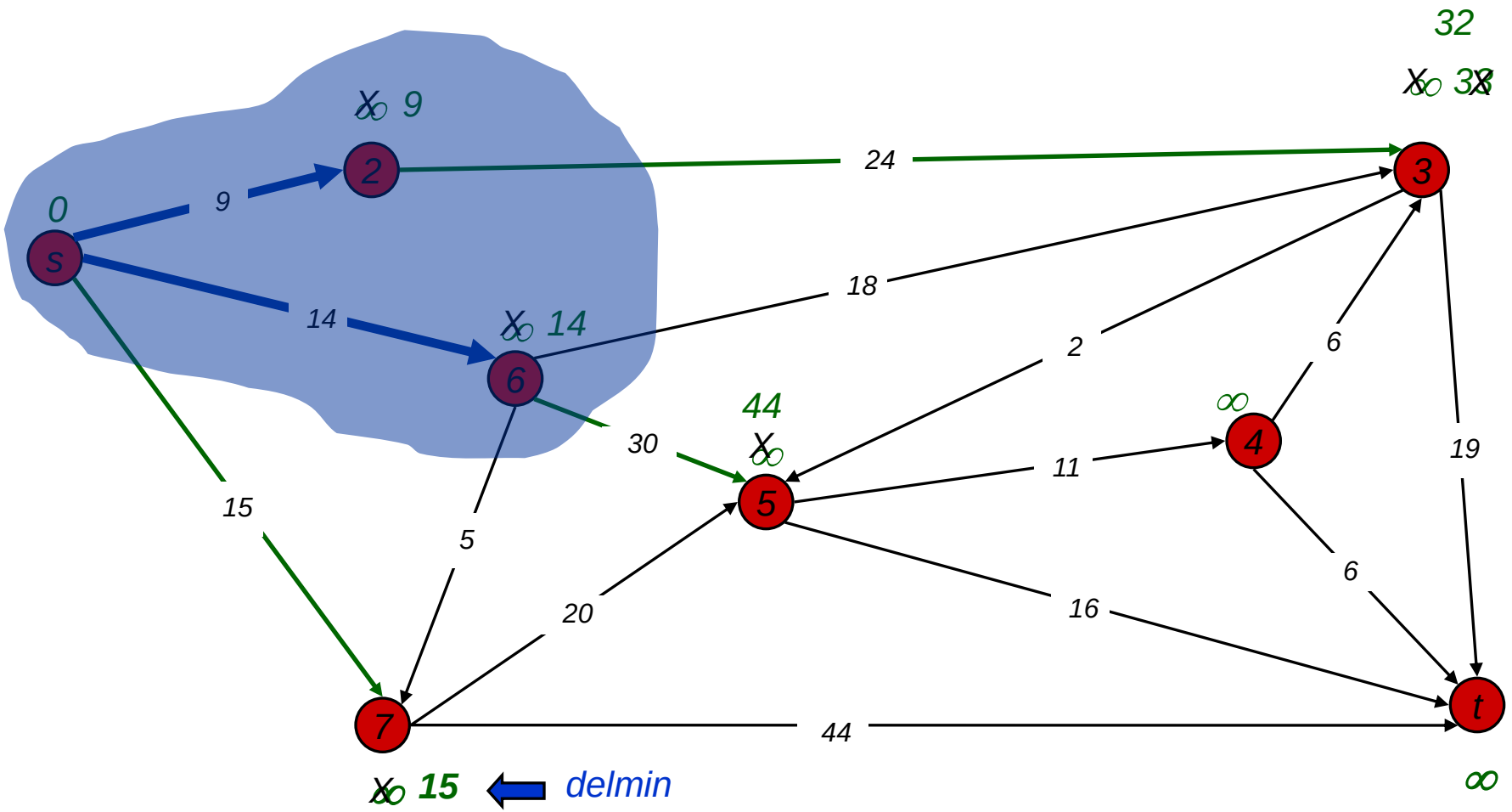

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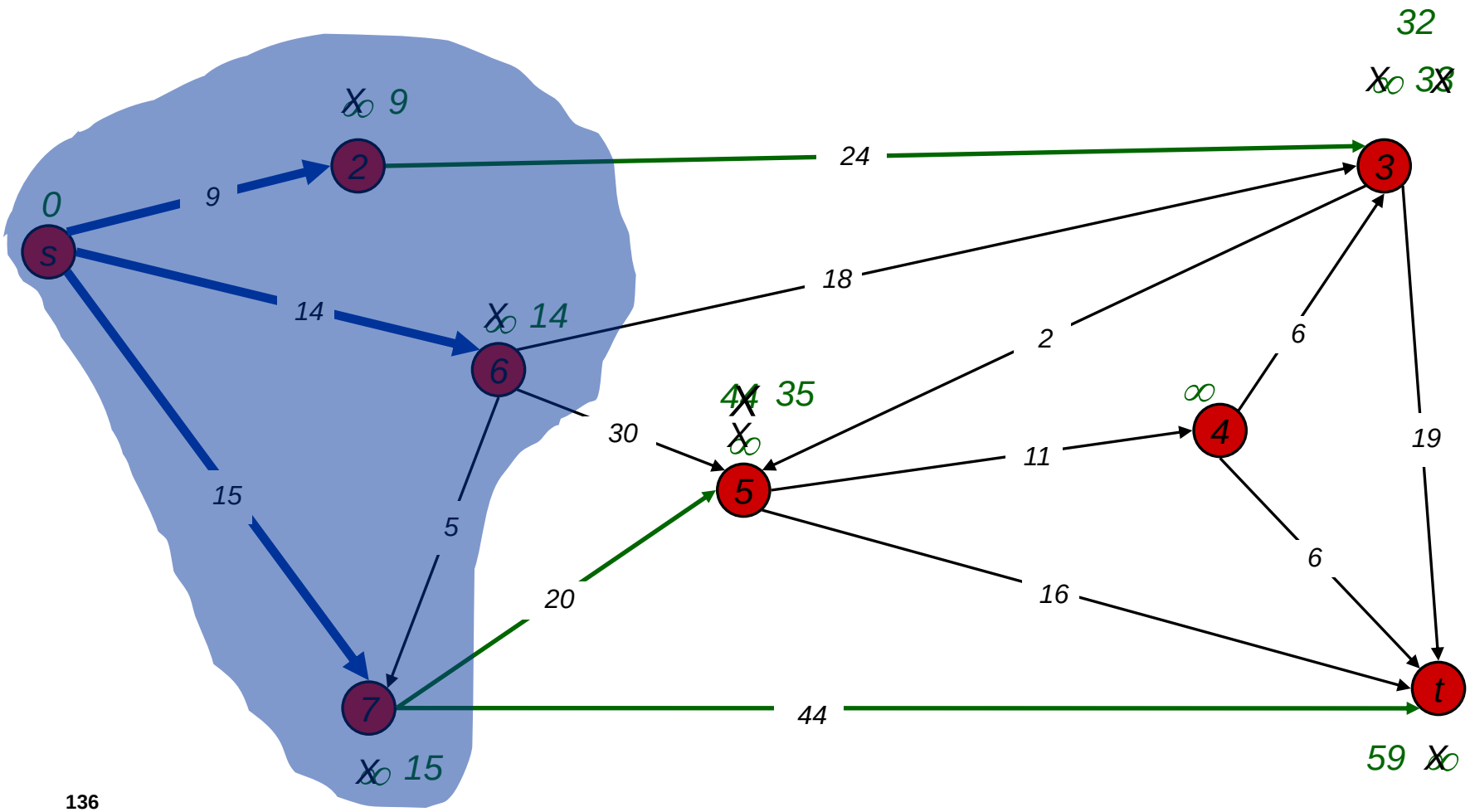

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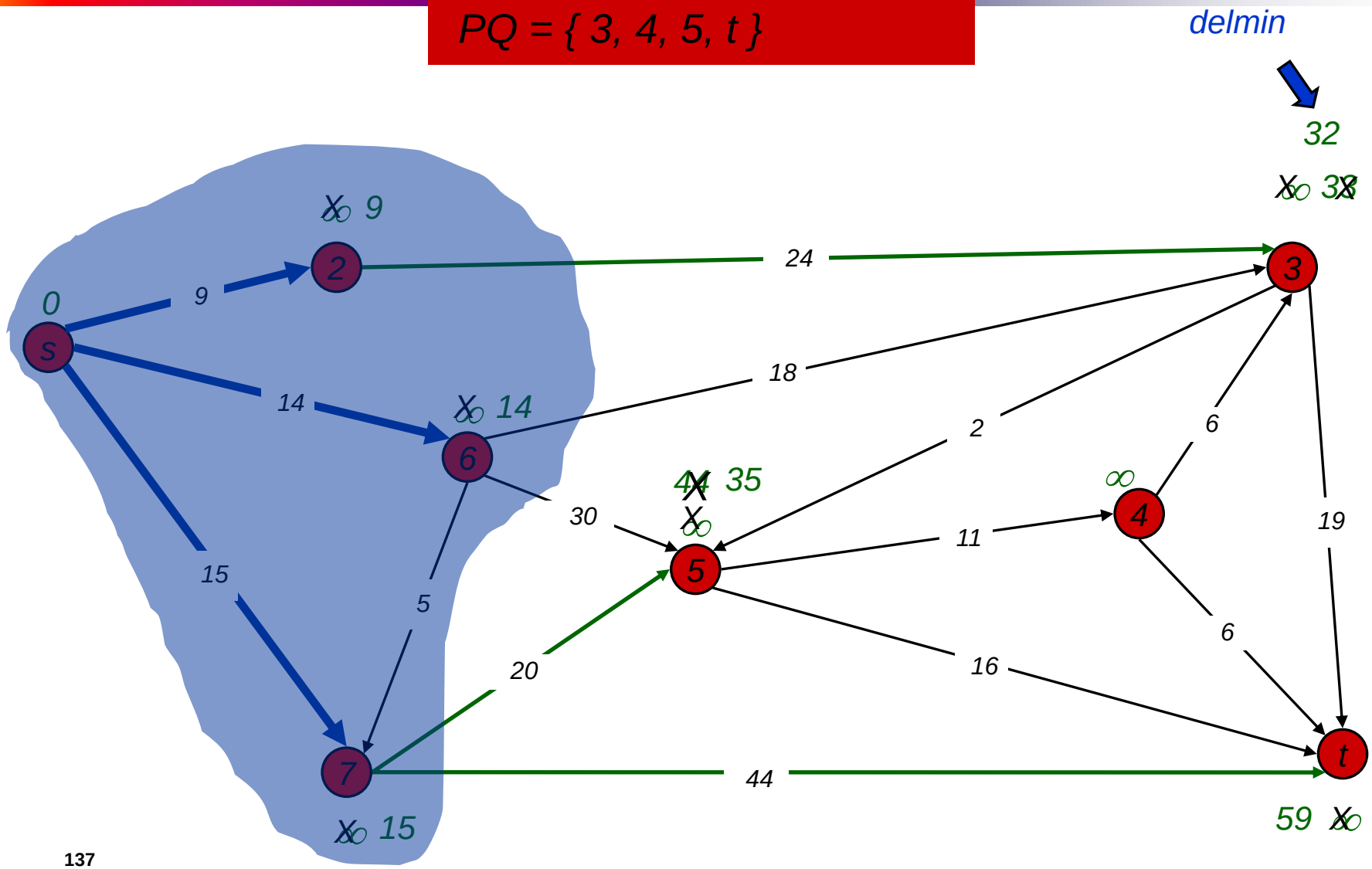

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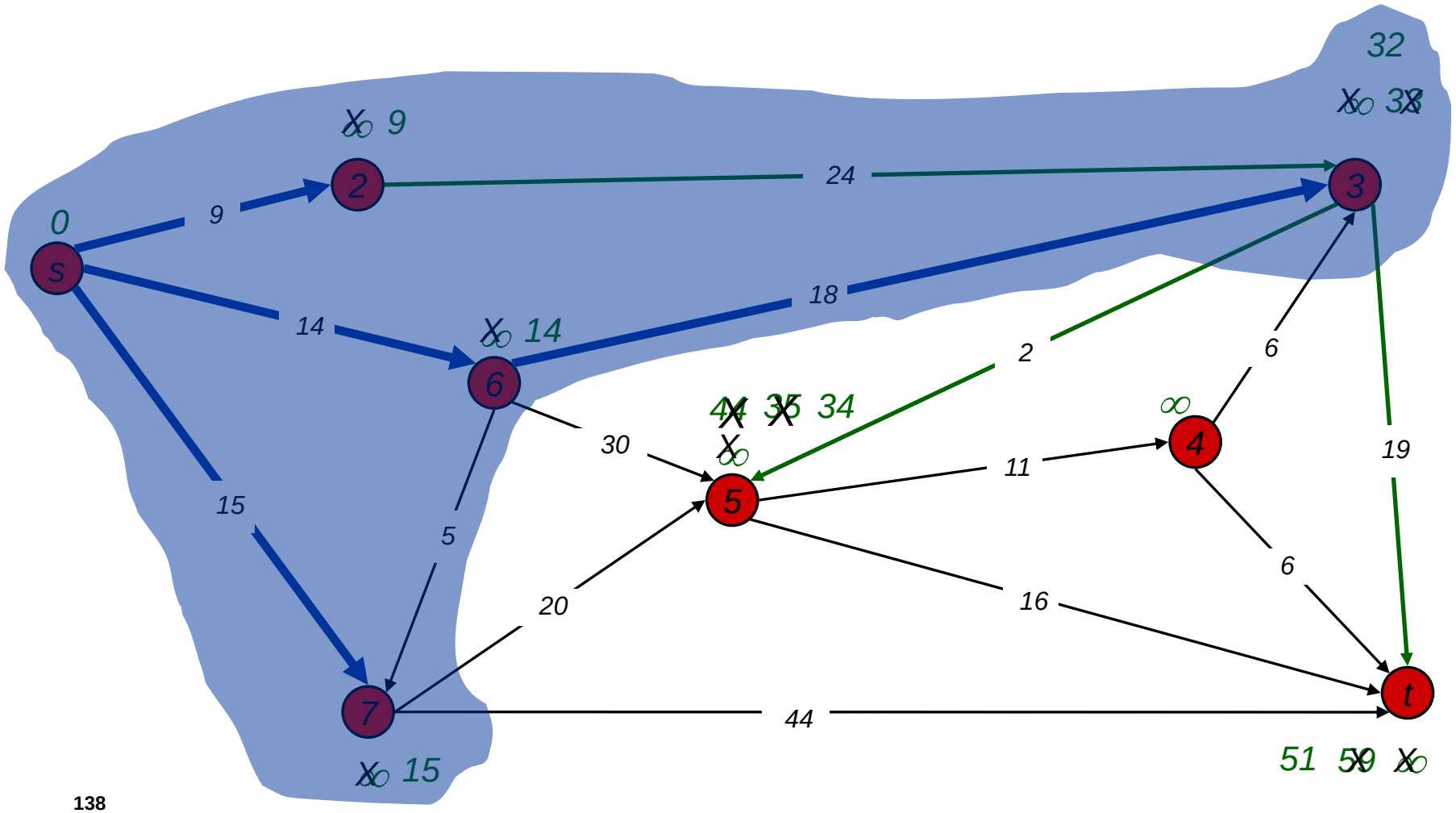

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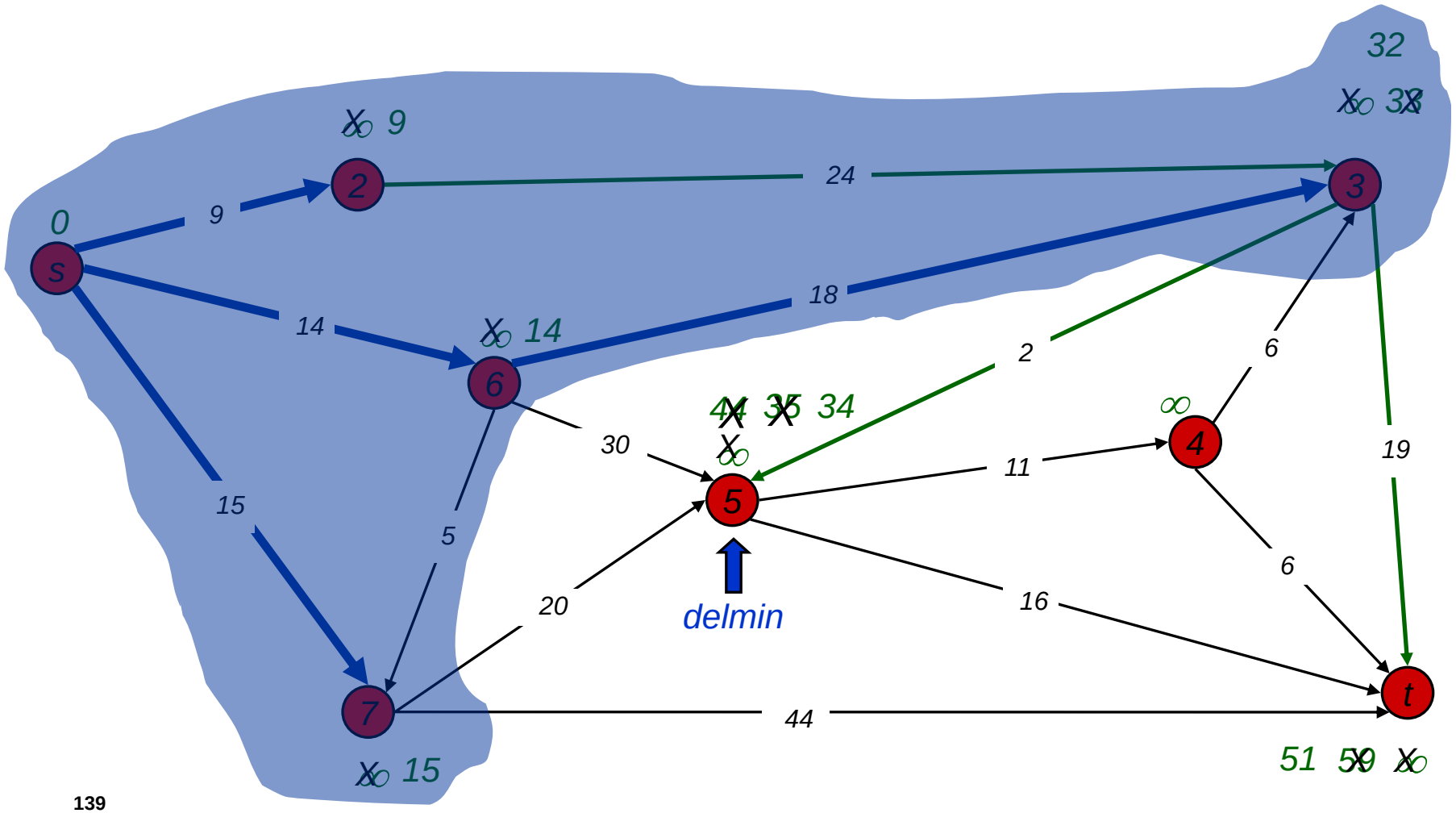

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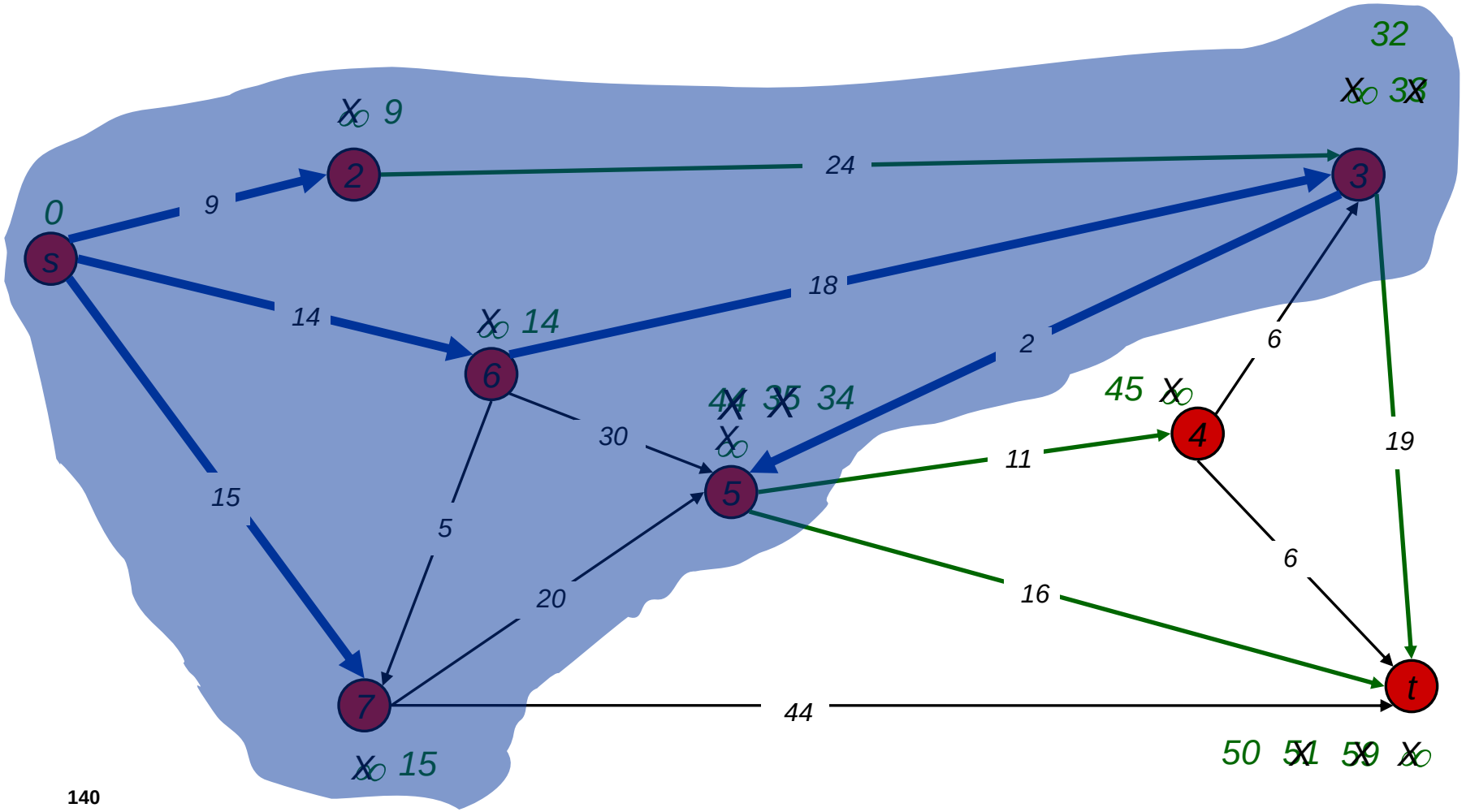

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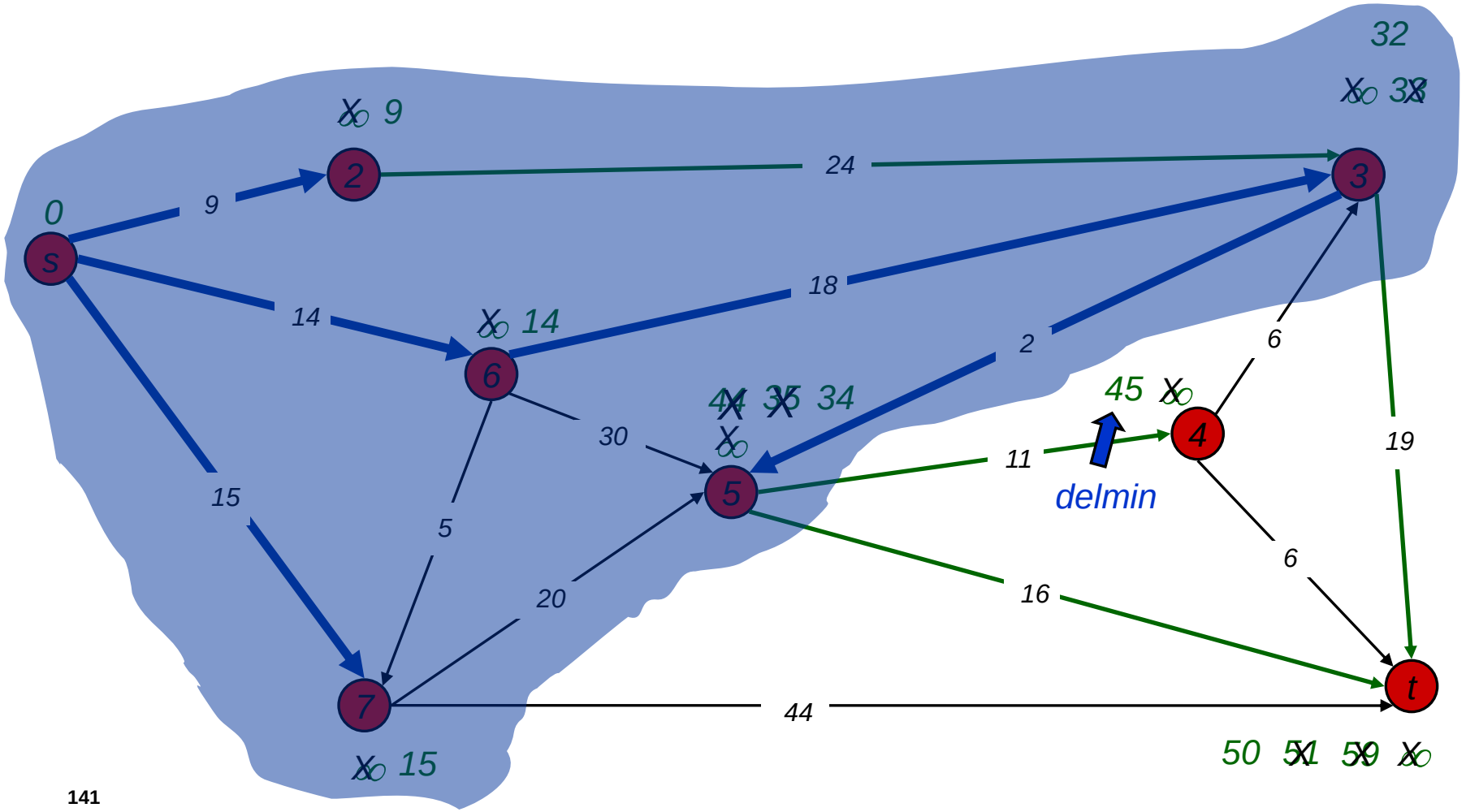

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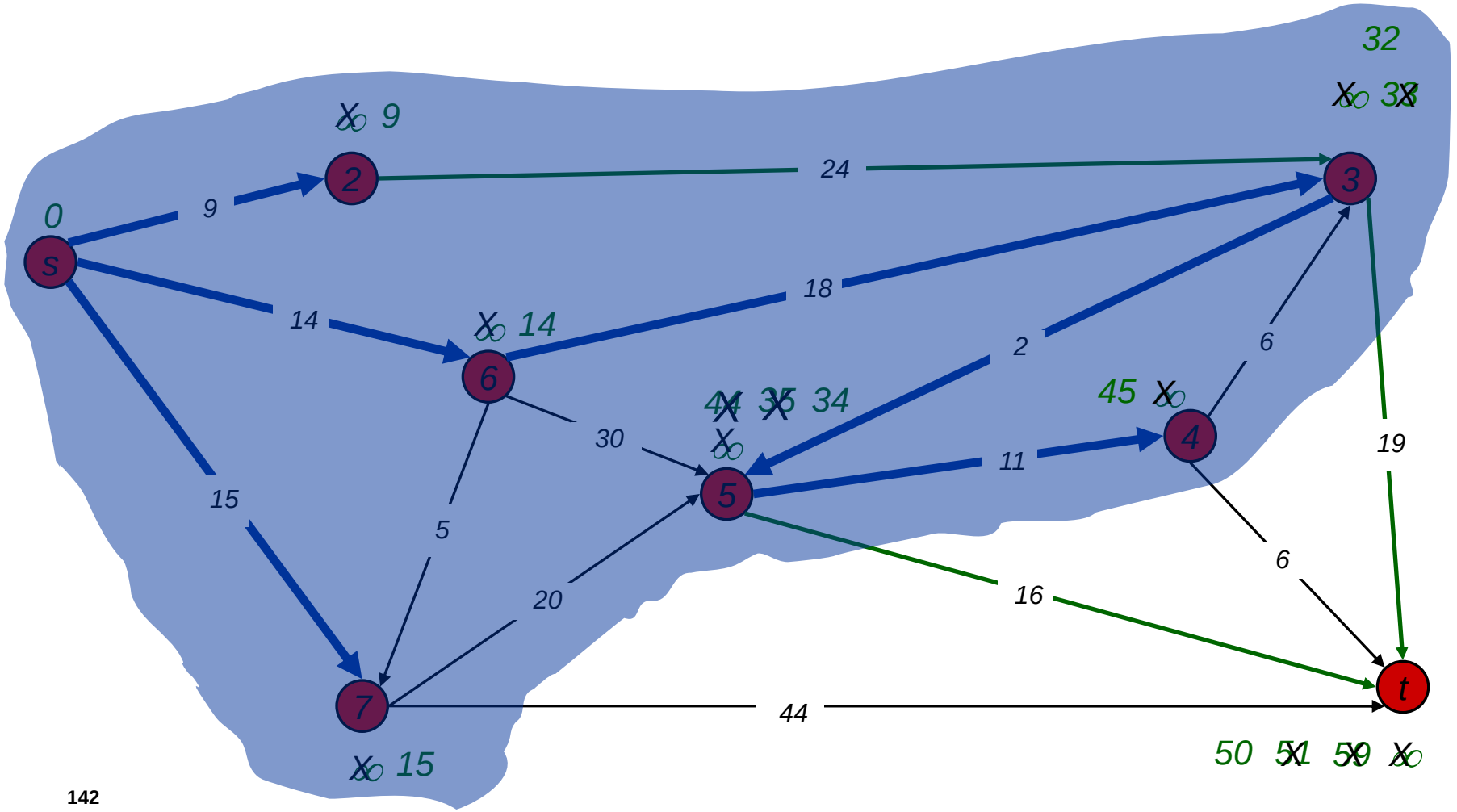

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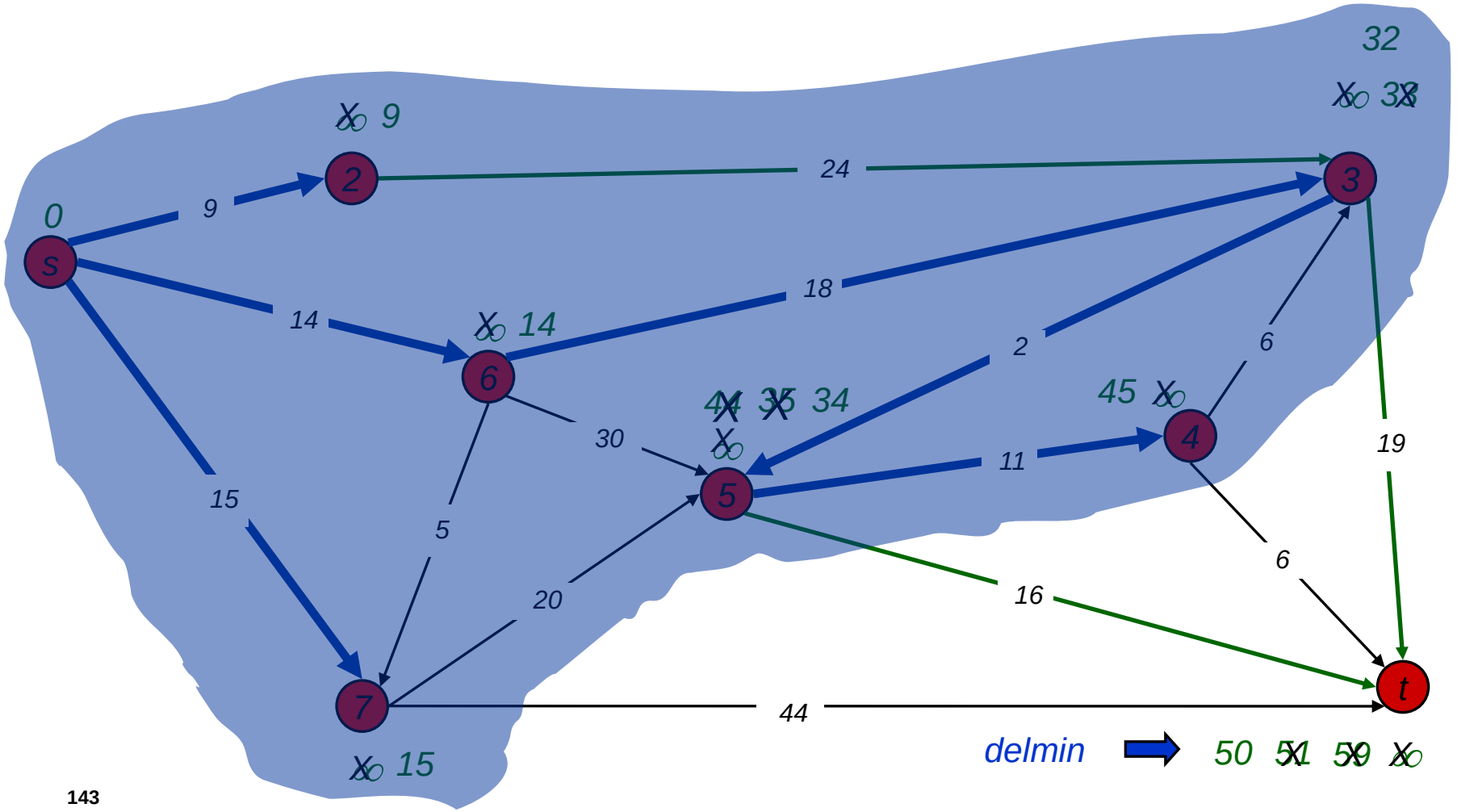

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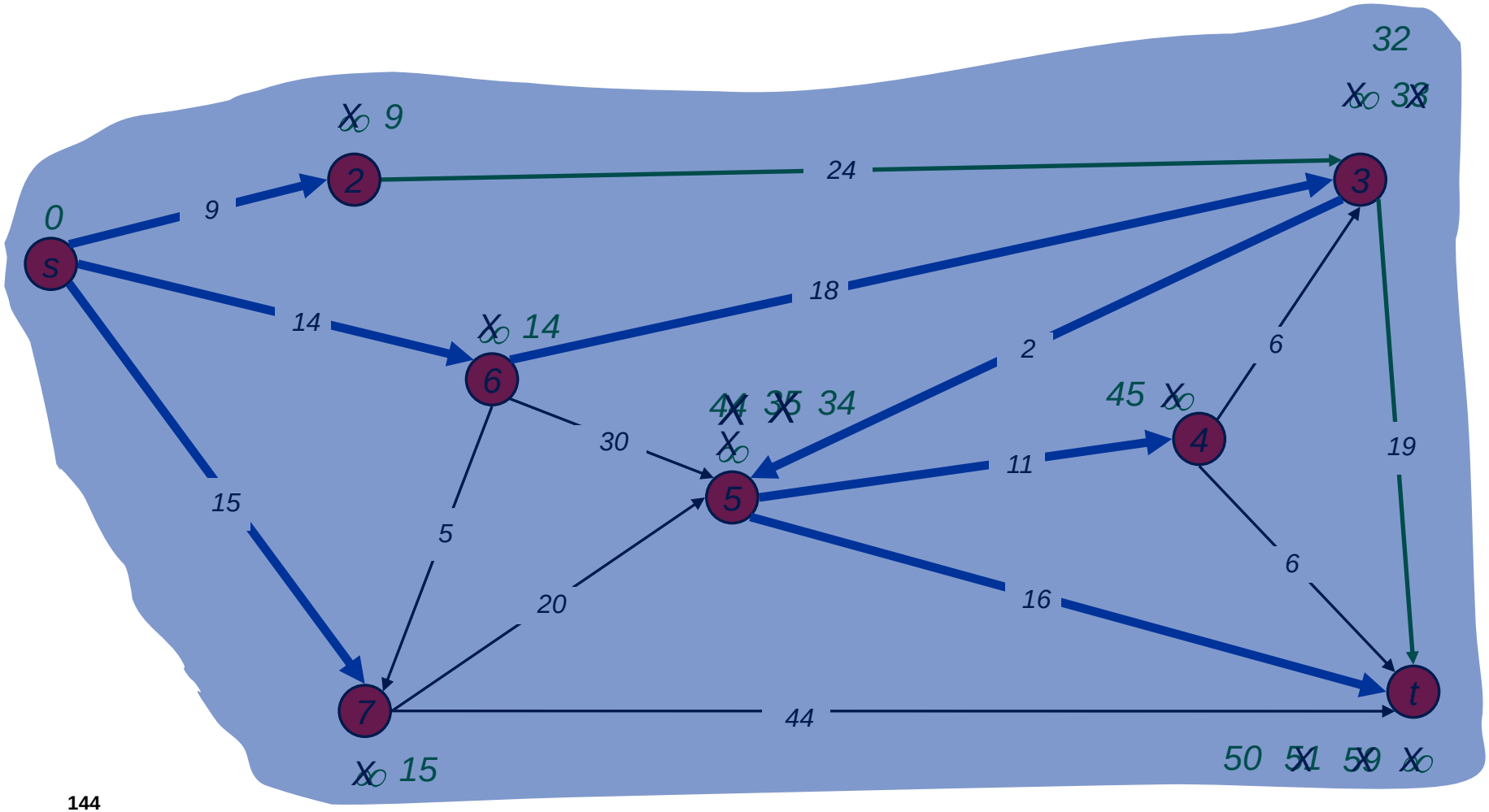

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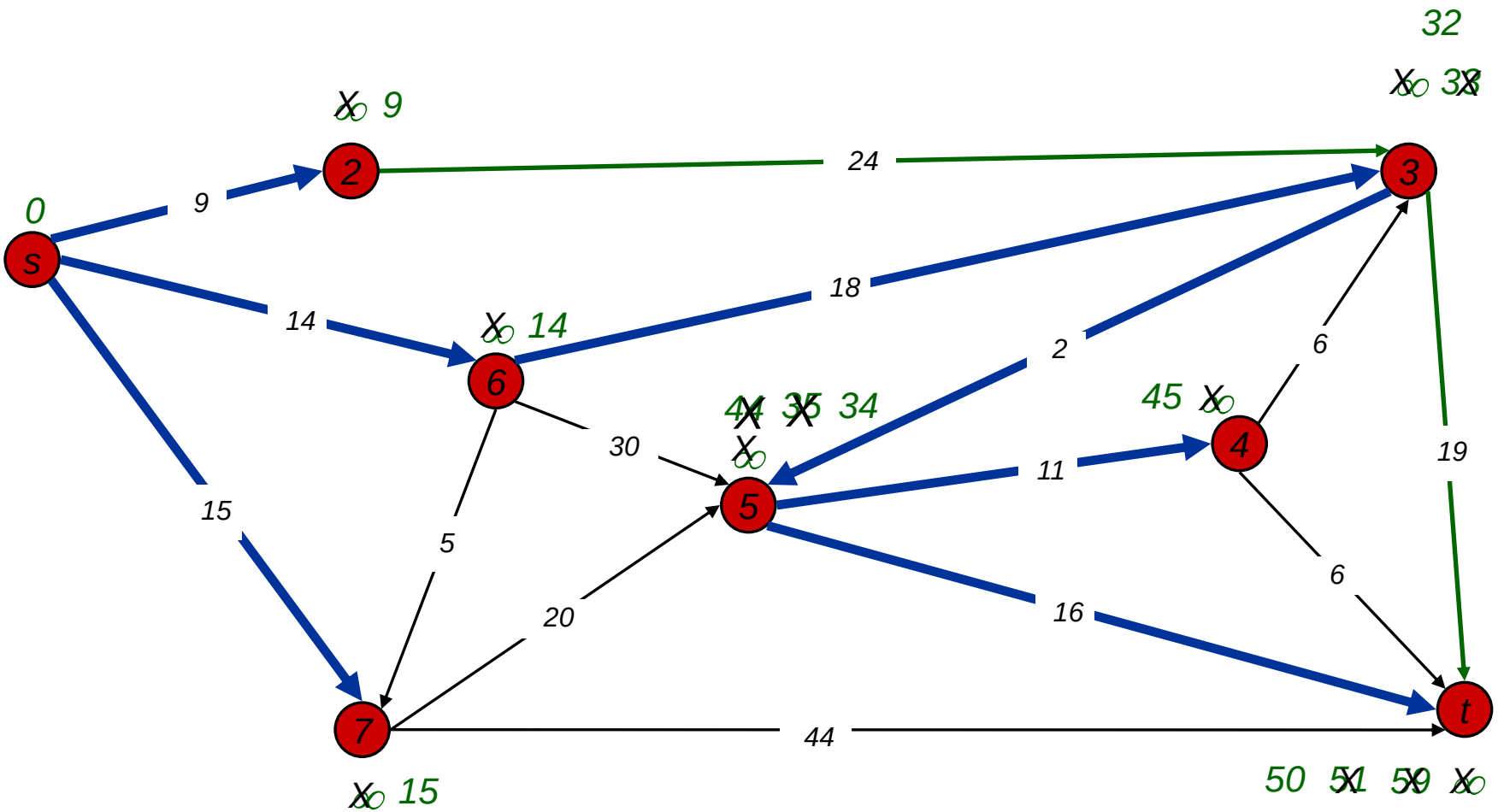
PQ = { }



Kruskal's Algorithm

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Kruskal()
{
    T = ∅;
    for each v ∈ V
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    sort E by increasing edge weight w
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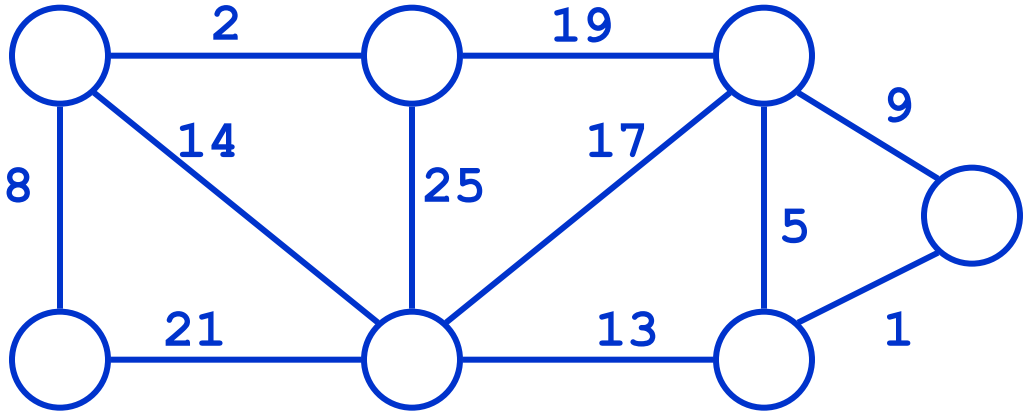

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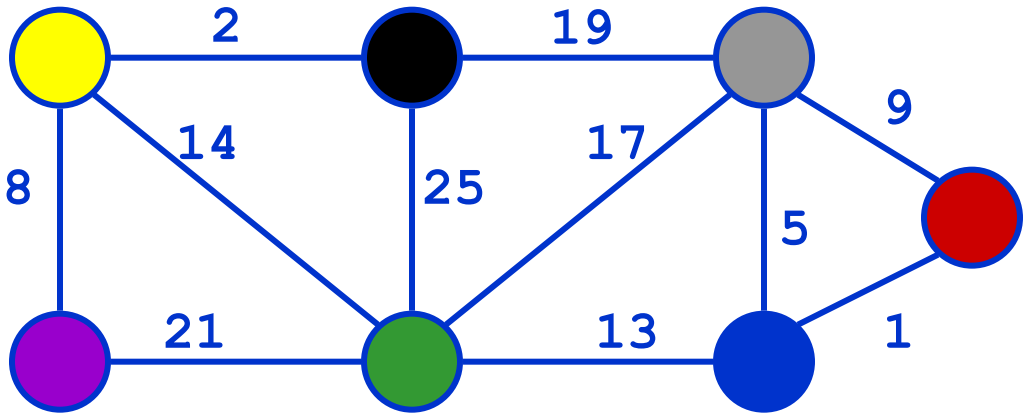

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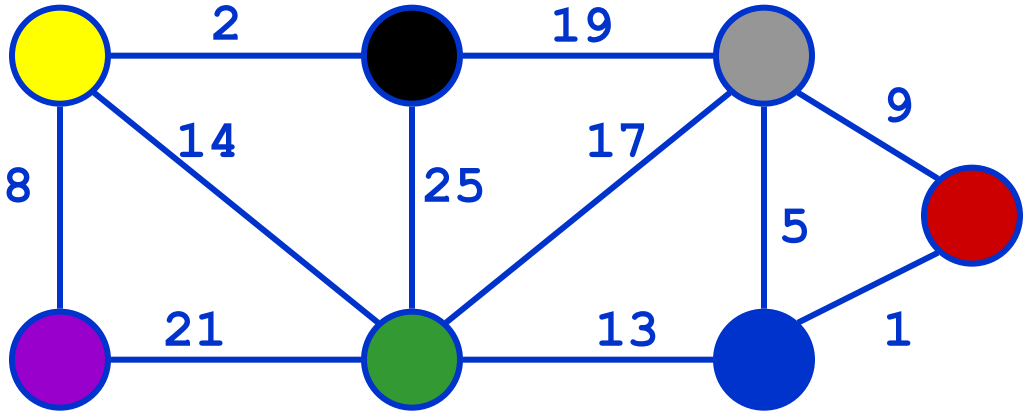

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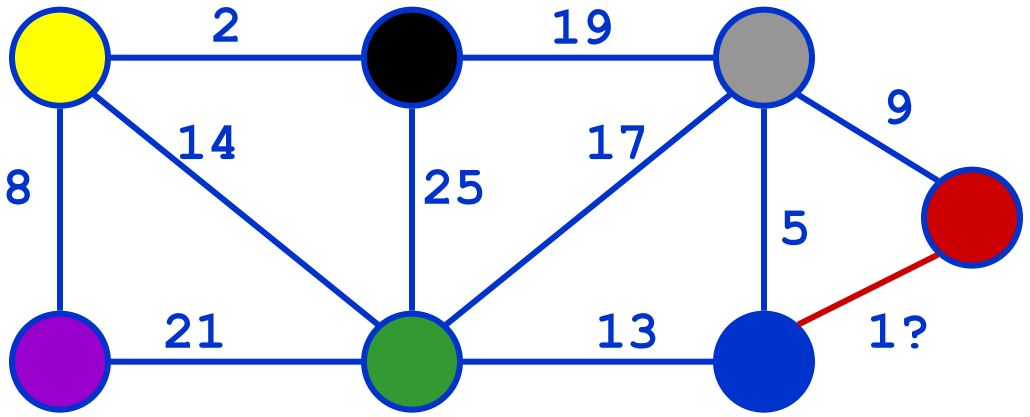

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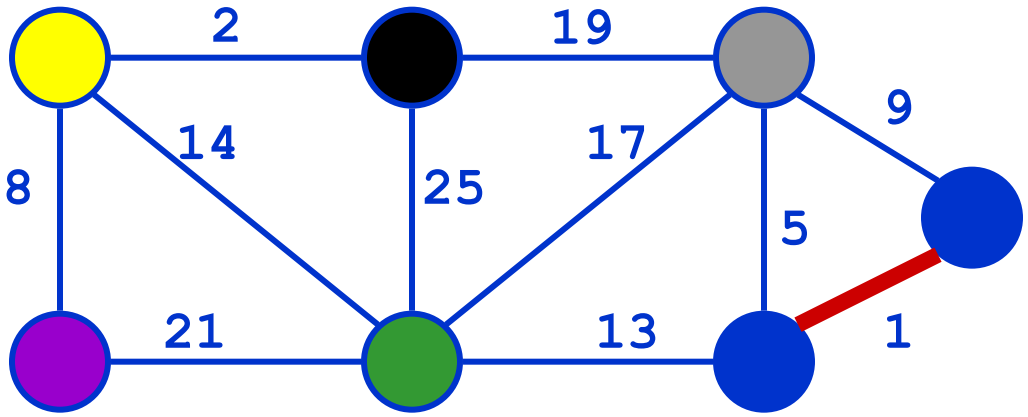

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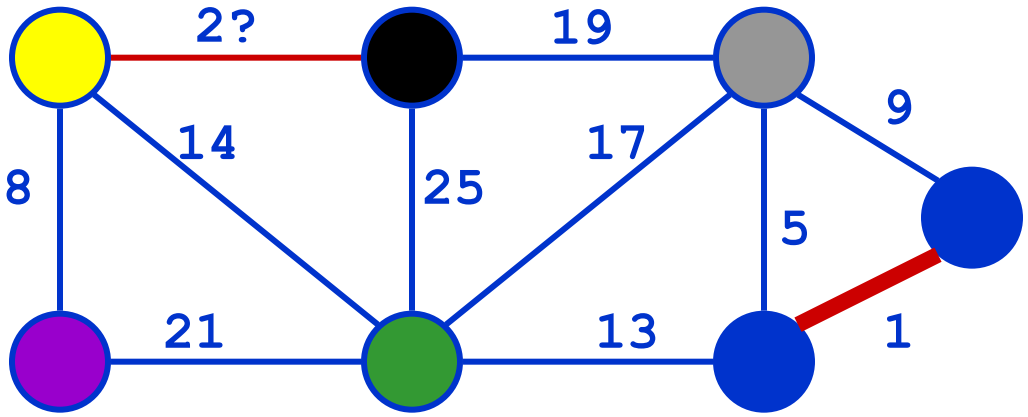

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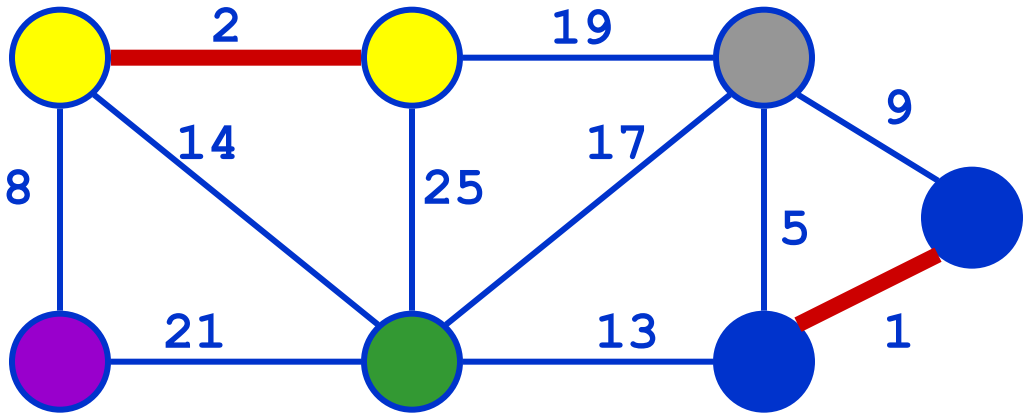

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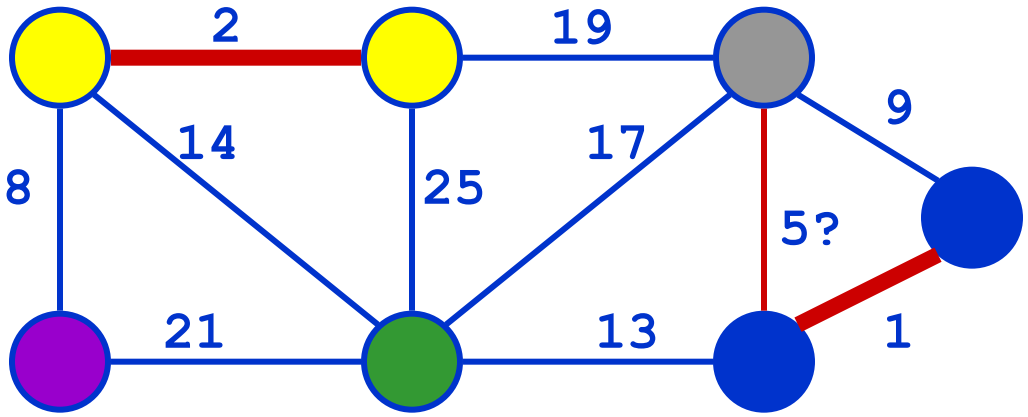

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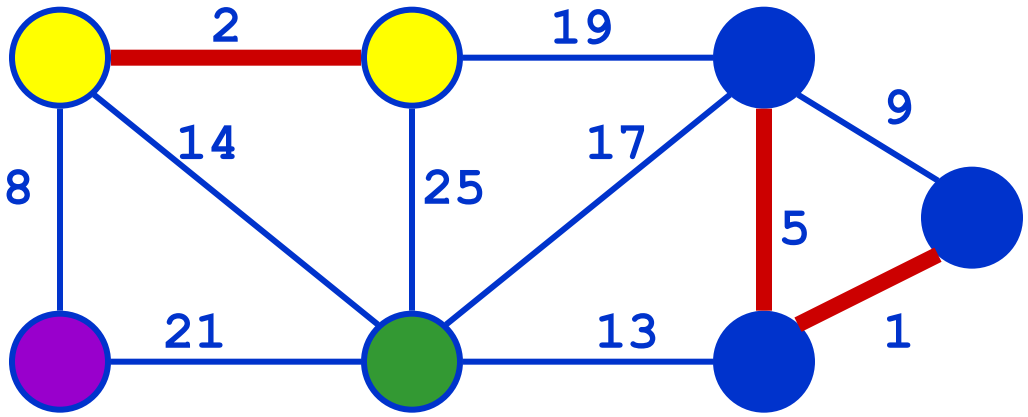

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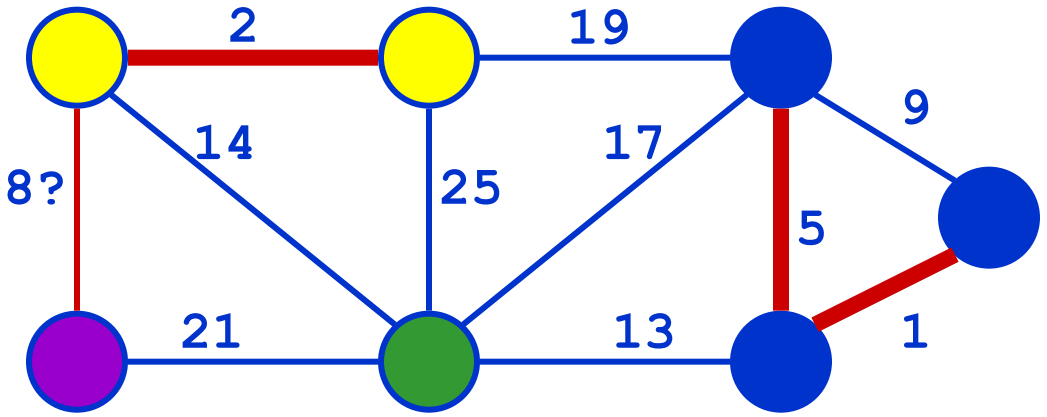

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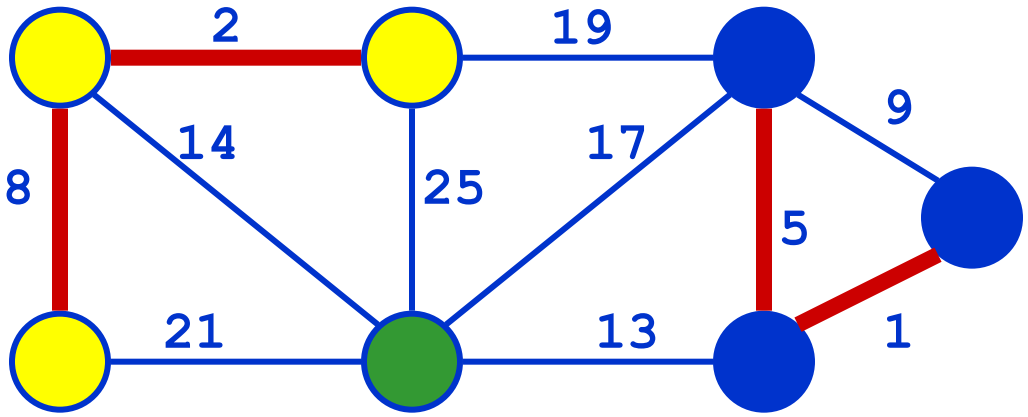

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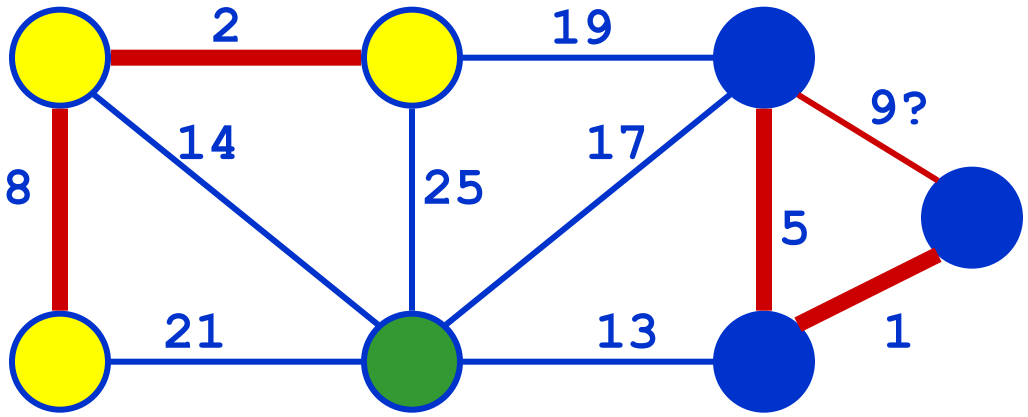

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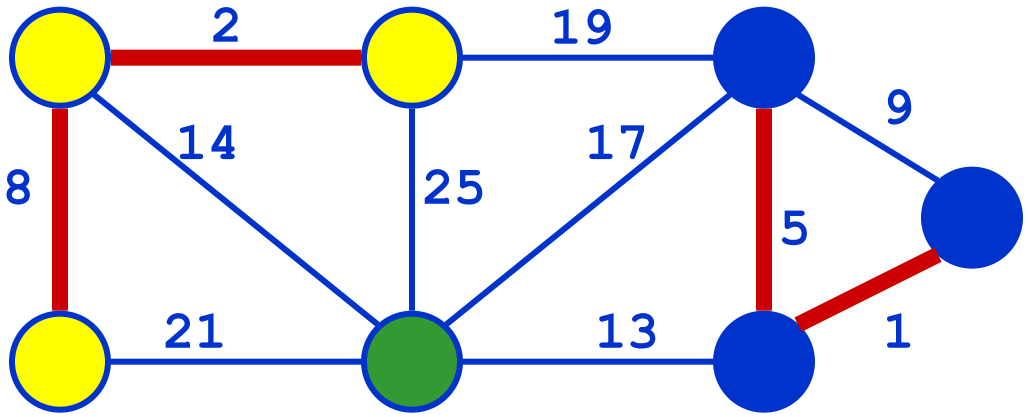

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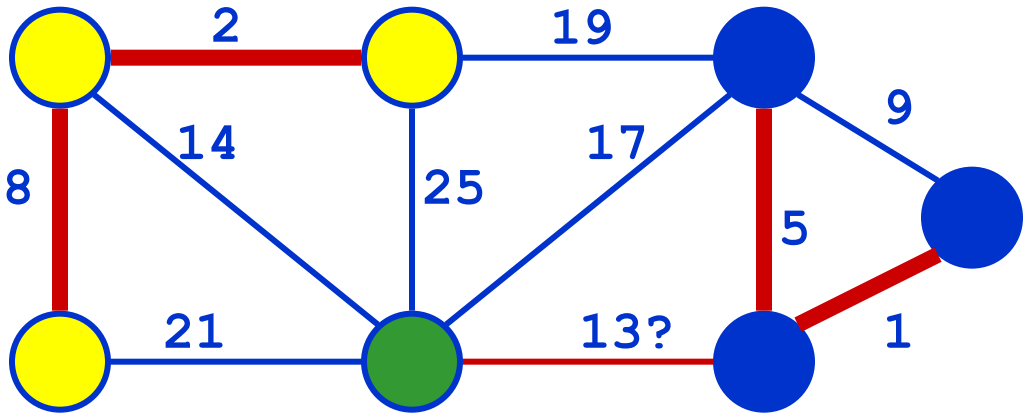

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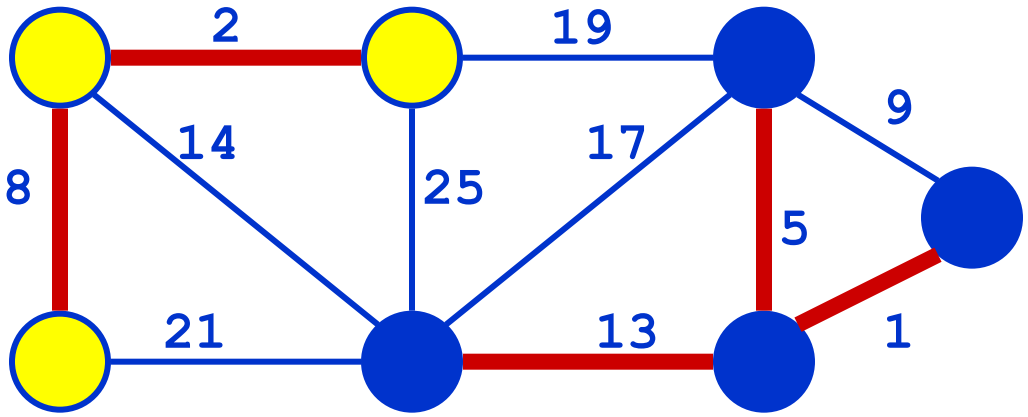

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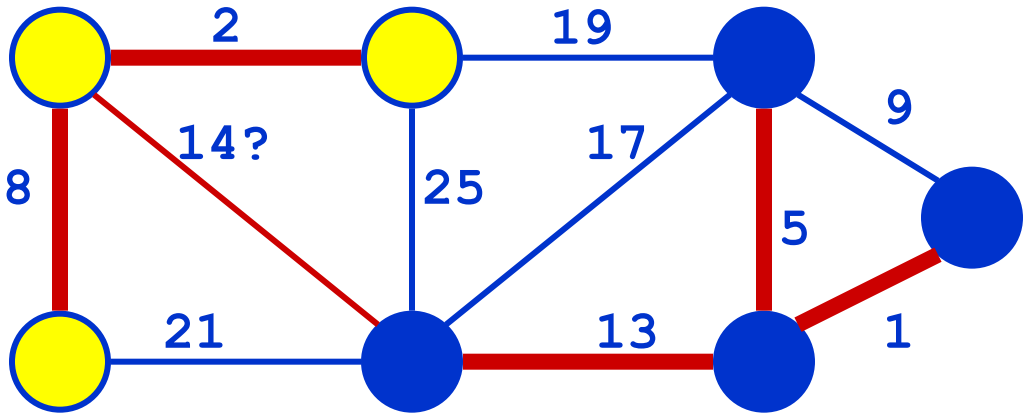

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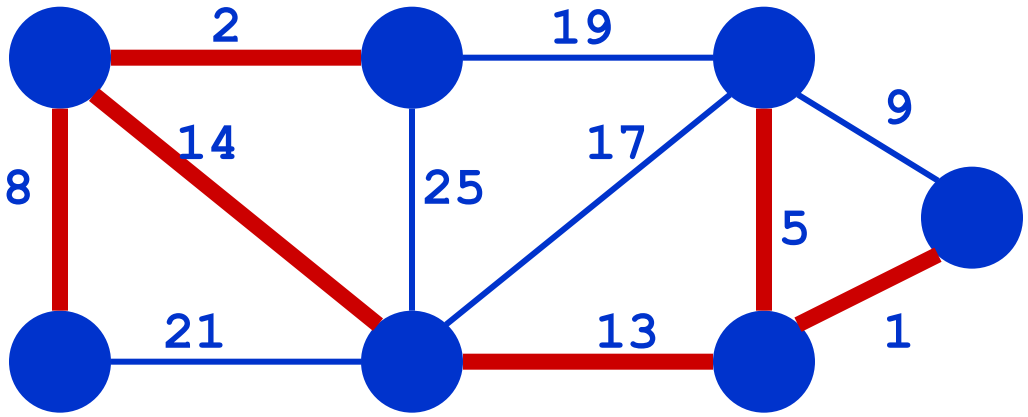

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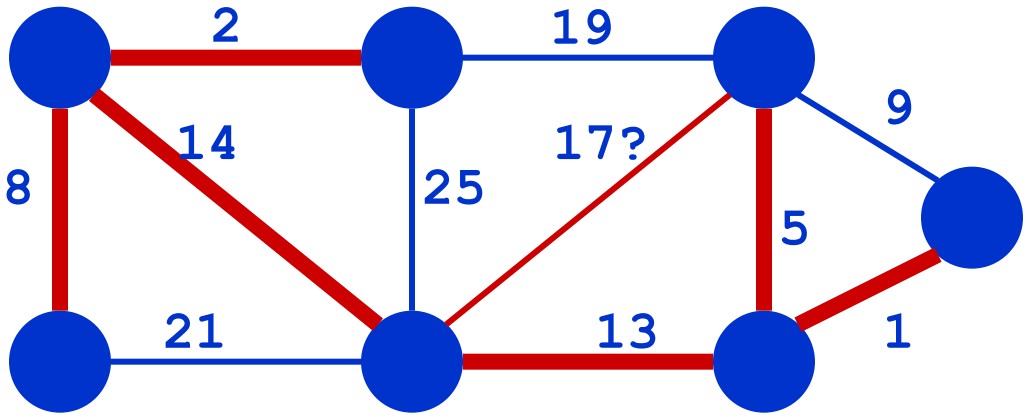

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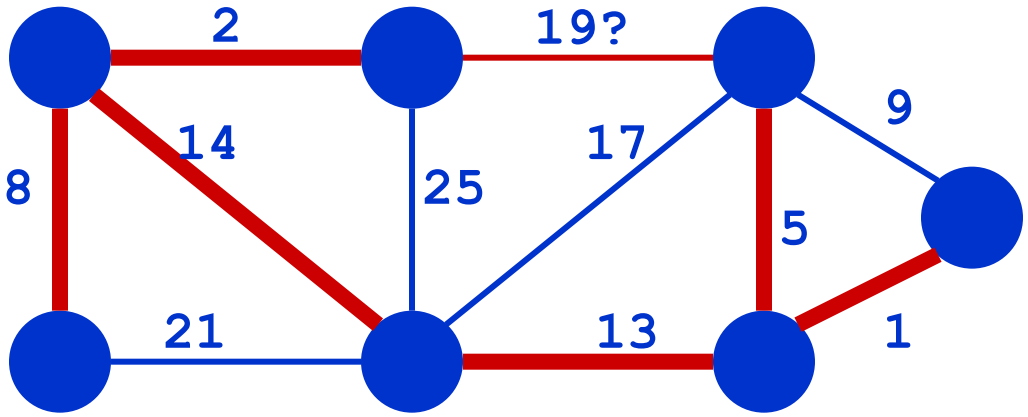

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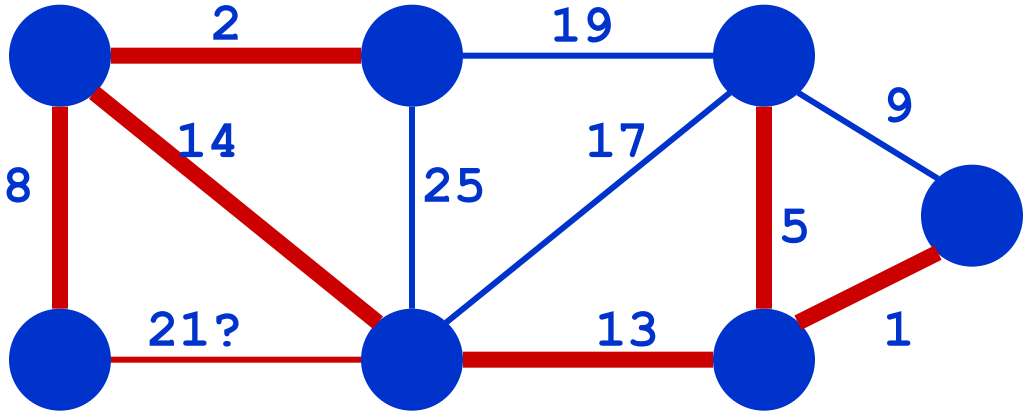
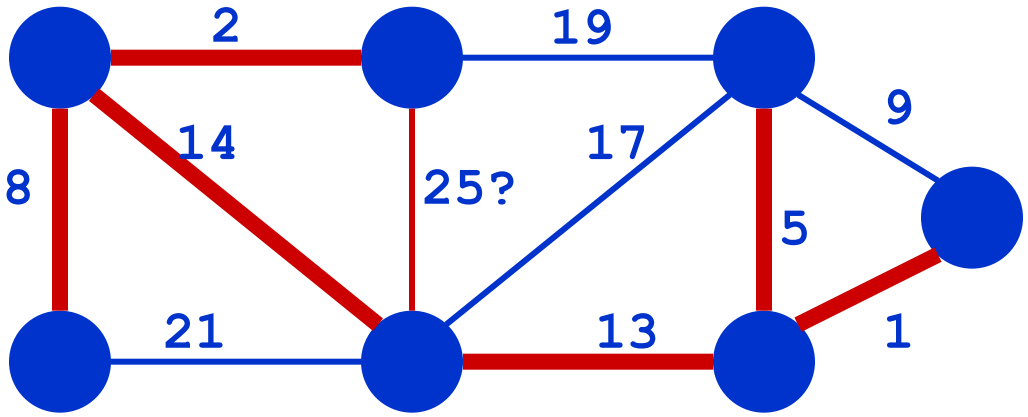


Kruskal's Algorithm

```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            T = T U {{u,v}};
            Union(FindSet(u), FindSet(v));
}
```

Run the algorithm:



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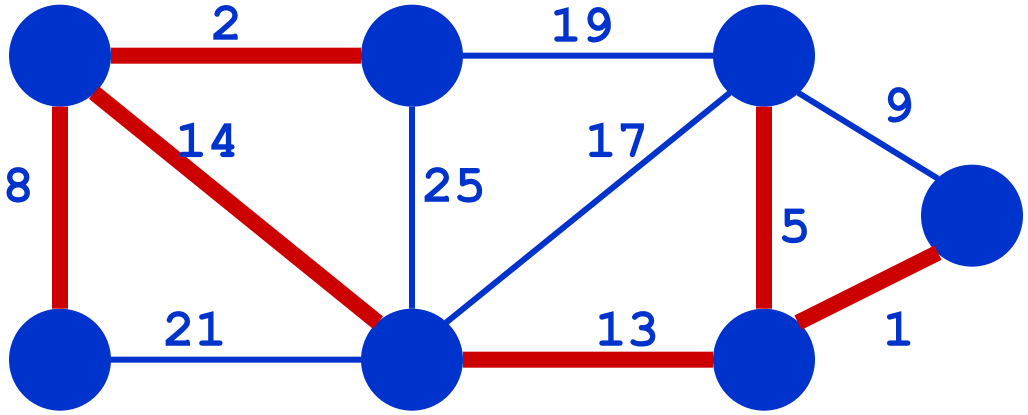

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