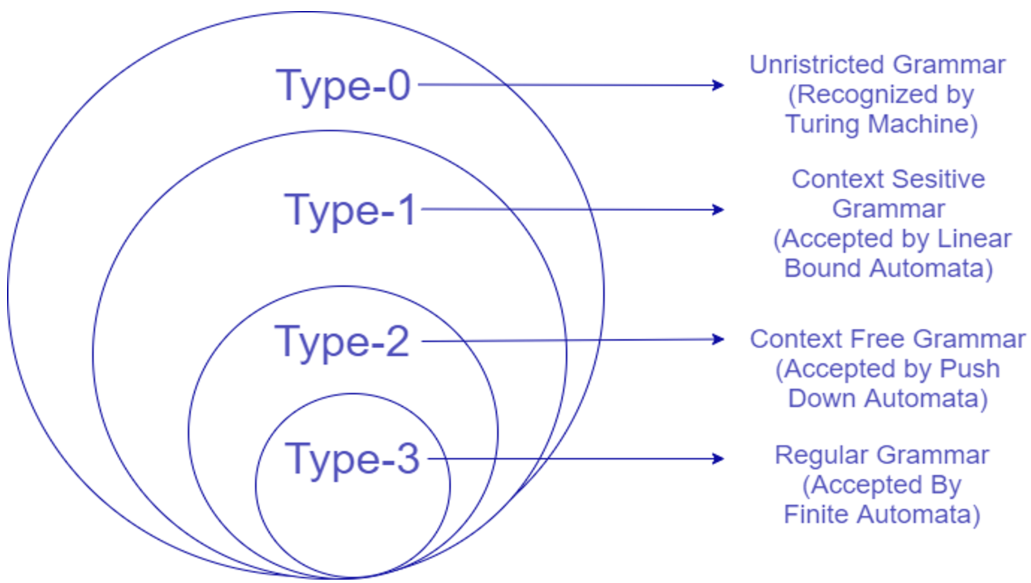
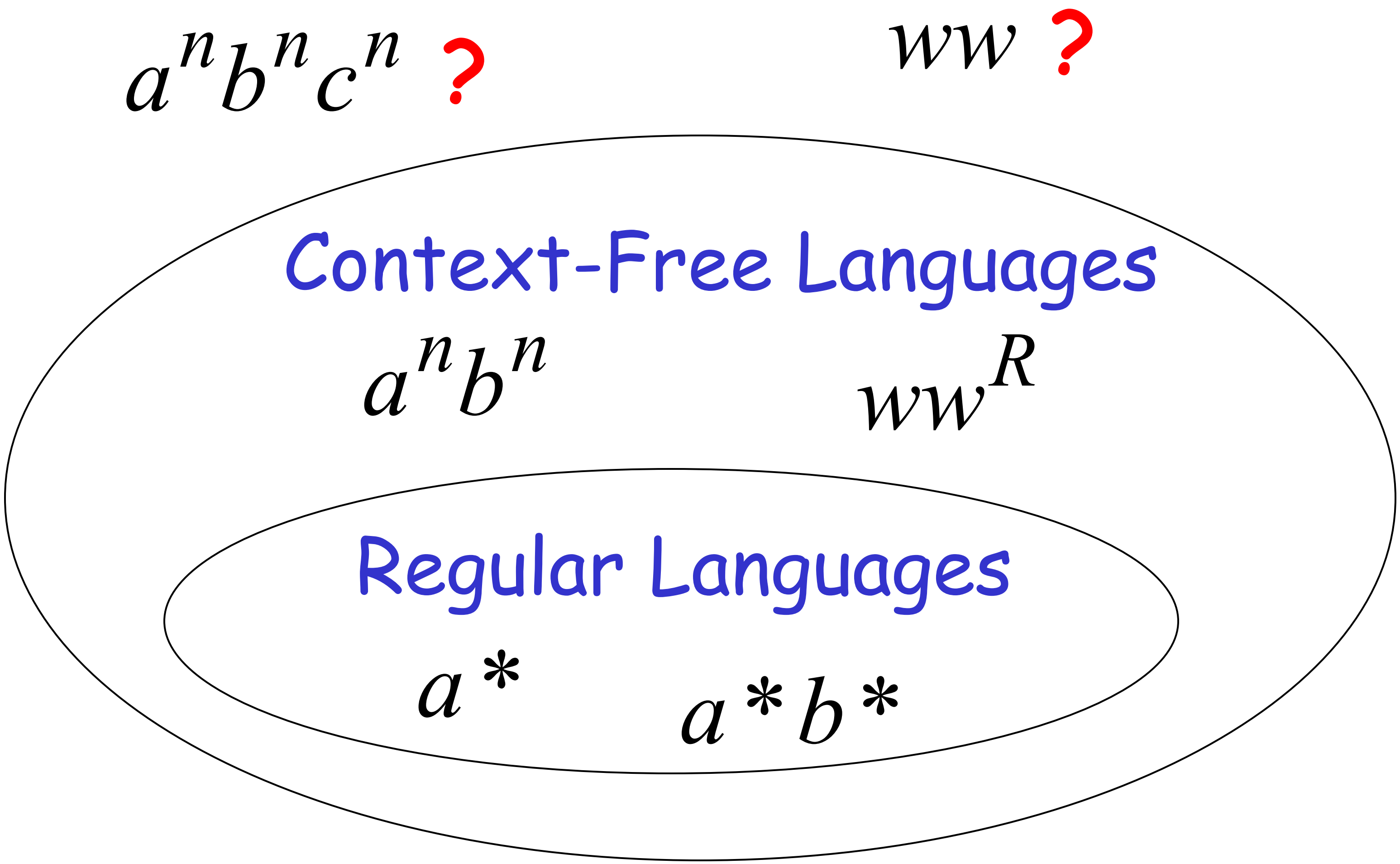
Turing Machines

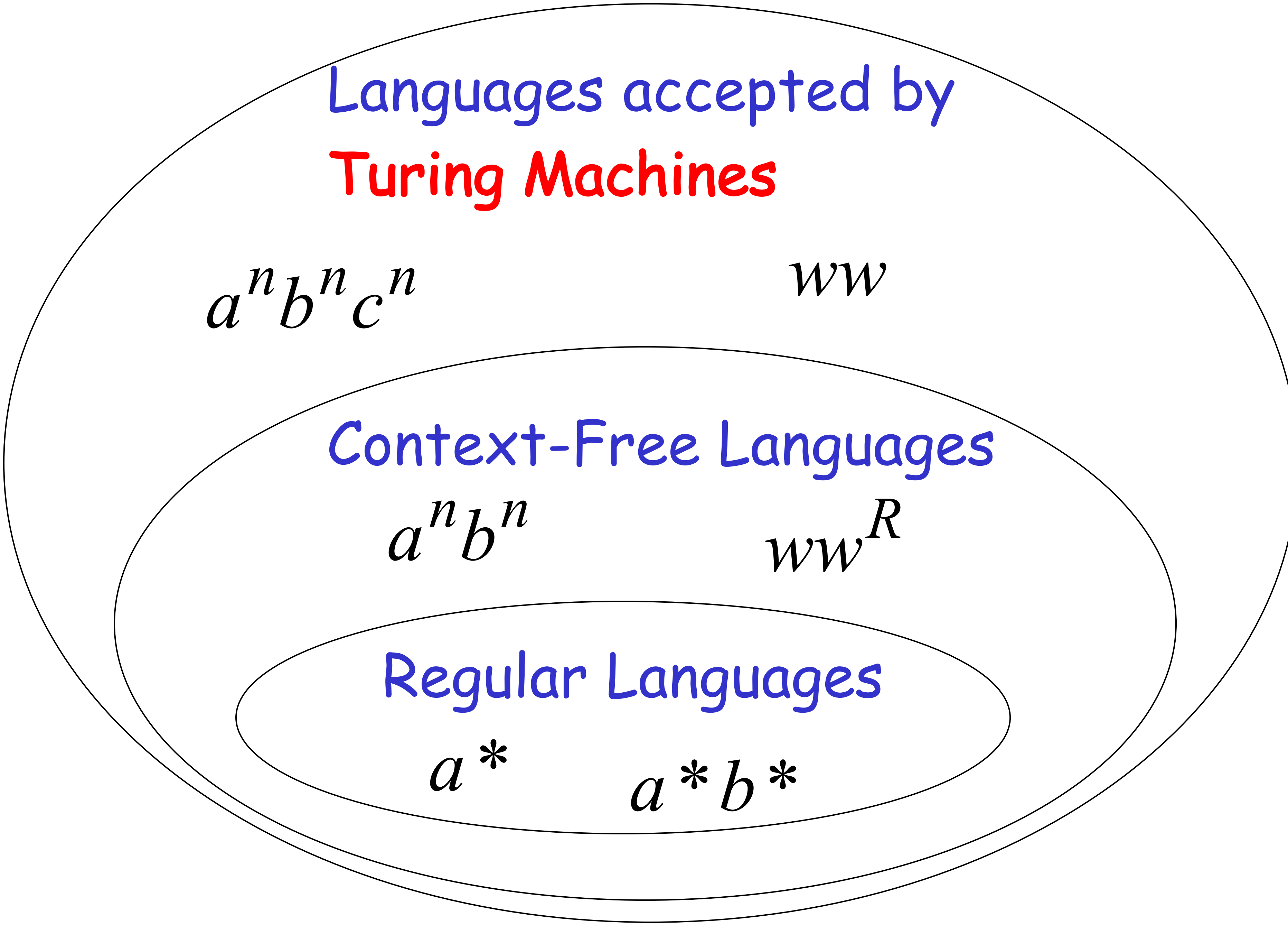
Turing Machines

A Turing Machine is an accepting device which accepts the languages (recursively enumerable set) generated by type 0 grammars. It was invented in 1936 by Alan Turing.

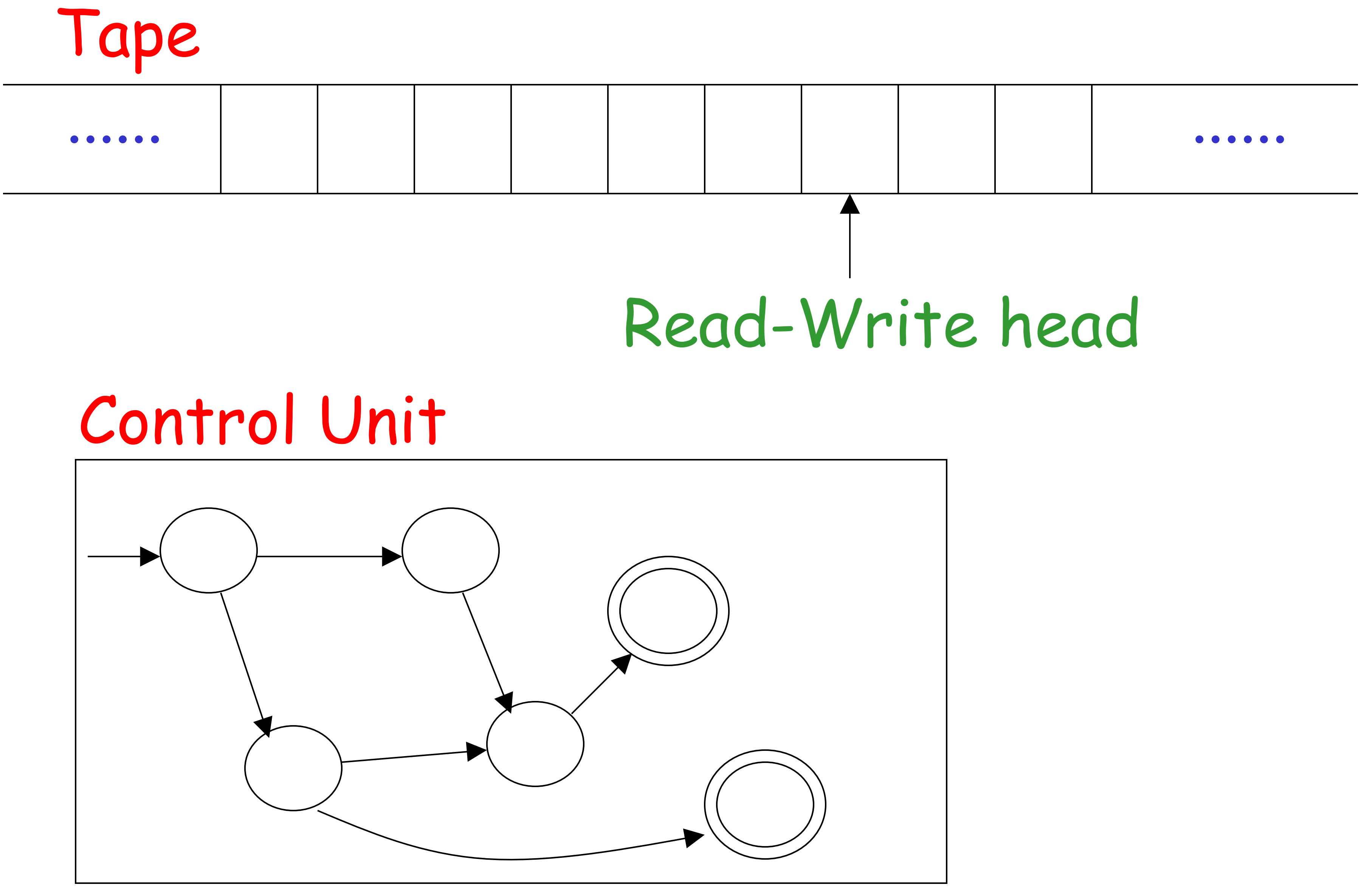


The Language Hierarchy




Languages accepted by
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
$a^n b^n$
ww^R
Regular Languages
a^*
$a^* b^*$

A Turing Machine



Definition

A Turing Machine (TM) is a mathematical model which consists of an infinite length tape divided into cells on which input is given.

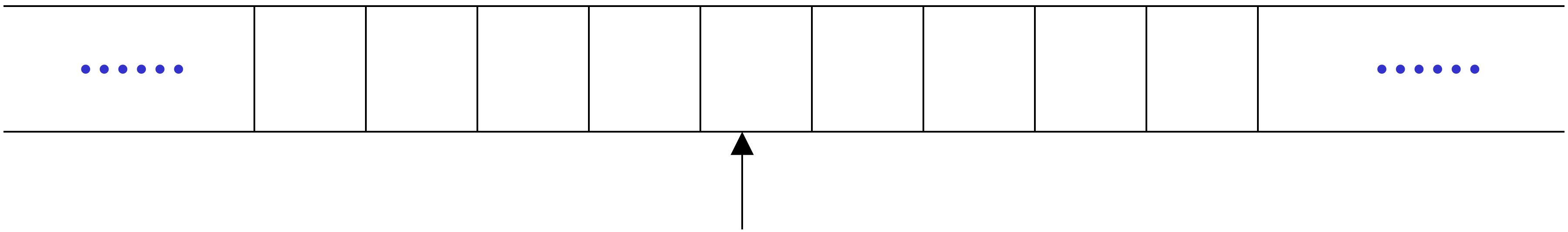
It consists of a head which reads the input tape. A state register stores the state of the Turing machine.

After reading an input symbol, it is replaced with another symbol, its internal state is changed, and it moves from one cell to the right or left.

If the TM reaches the final state, the input string is accepted, otherwise rejected.

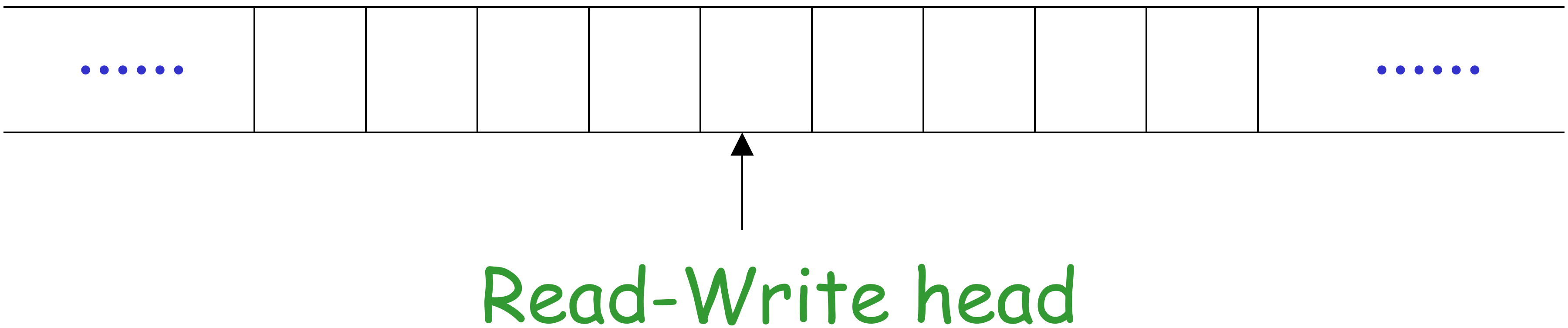
The Tape

No boundaries -- infinite length



Read-Write head

The head moves Left or Right

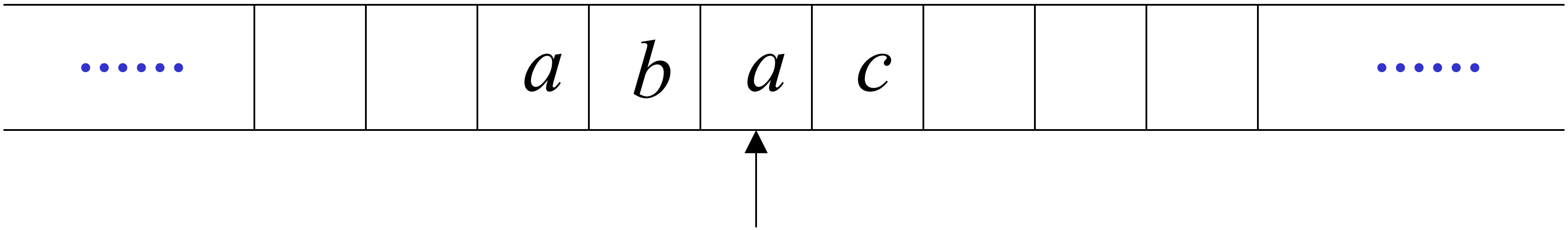


The head at each transition (time step):

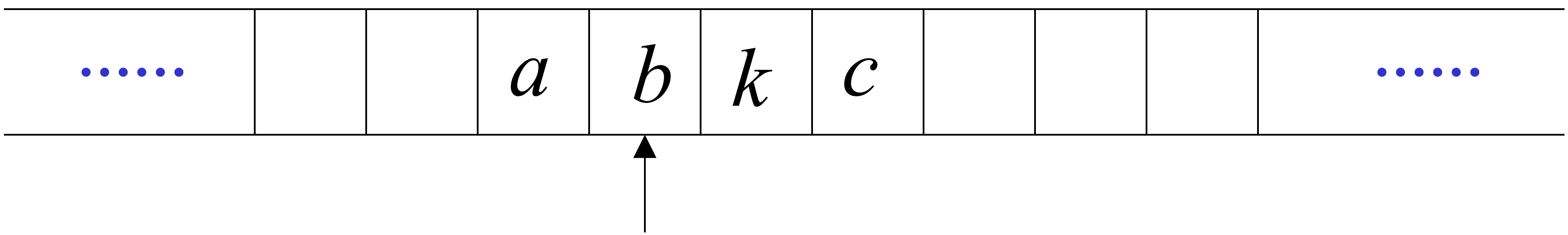
1. Reads a symbol
2. Writes a symbol
3. Moves Left or Right

Example:

Time 0



Time 1

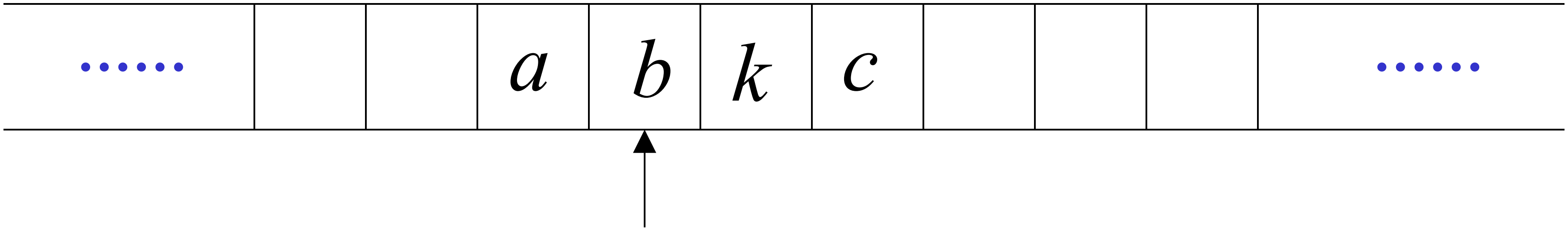


1. Reads a

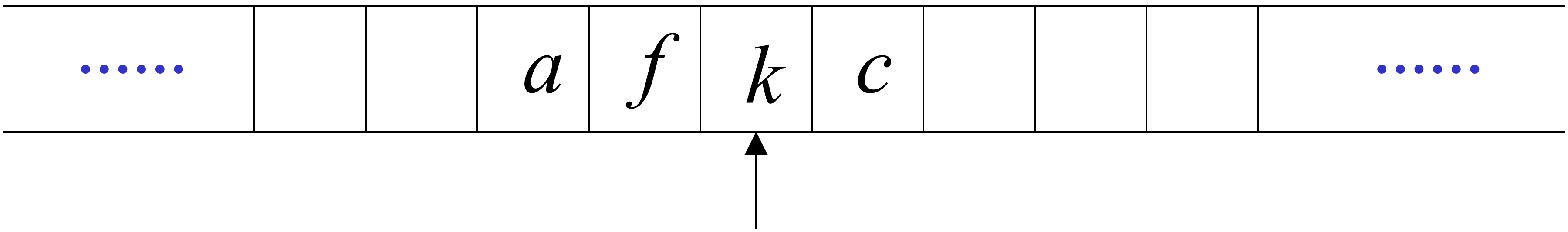
2. Writes k

3. Moves Left

Time 1

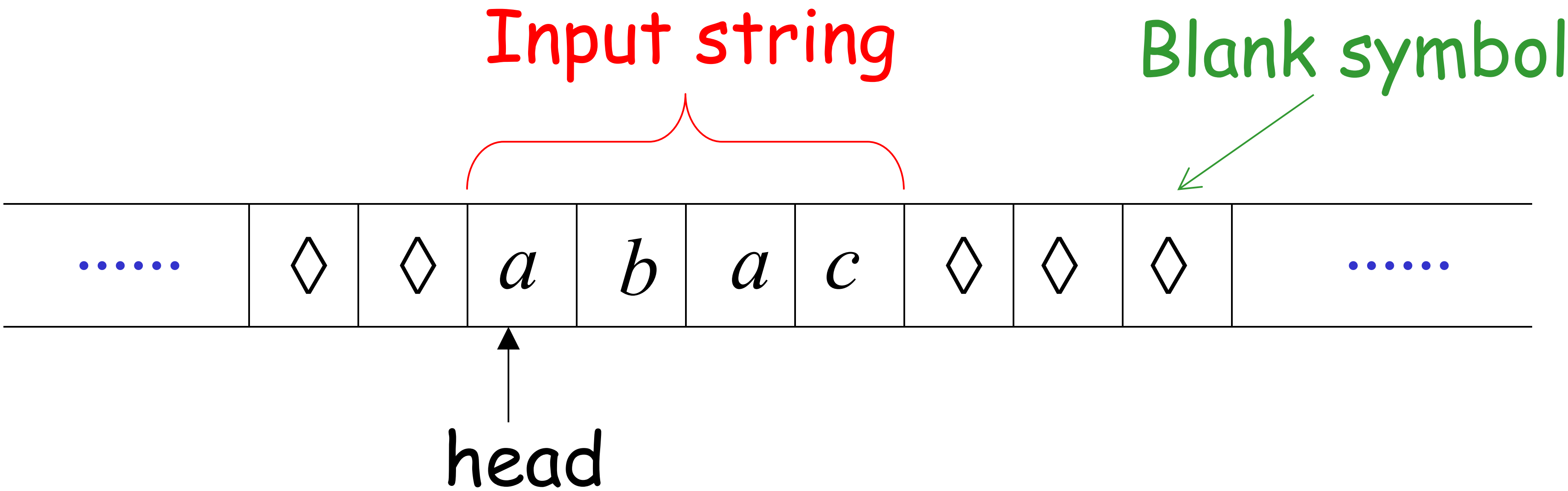


Time 2



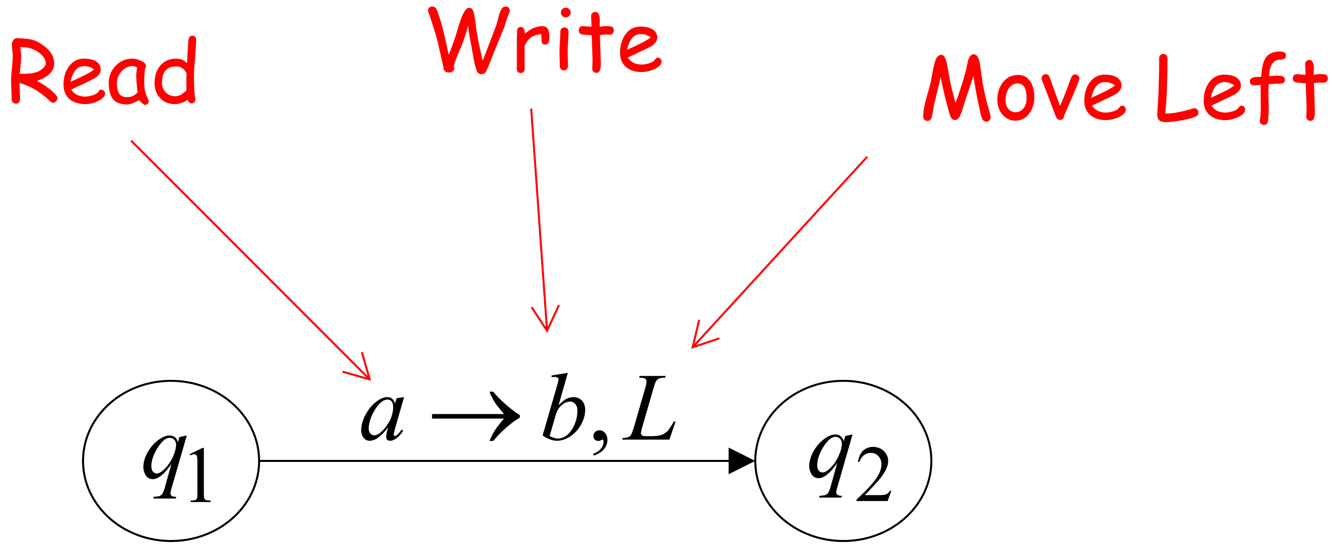
1. Reads b
2. Writes f
3. Moves Right

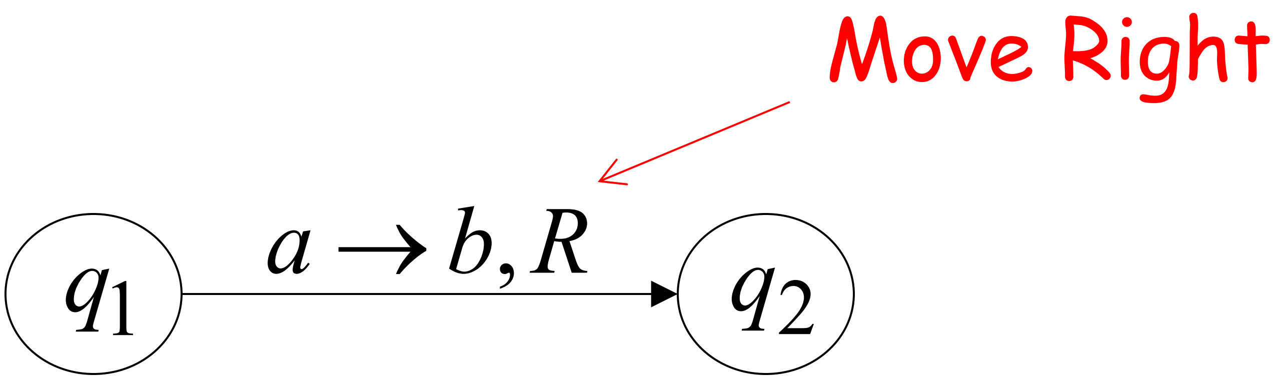
The Input String



Head starts at the leftmost position of the input string

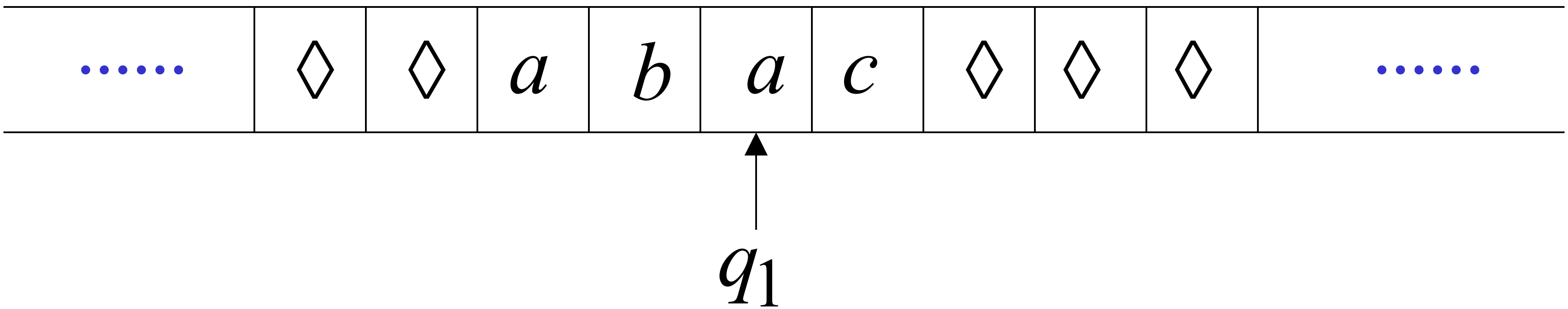
States & Transitions



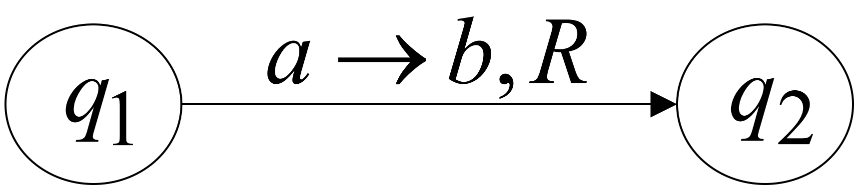


Example:

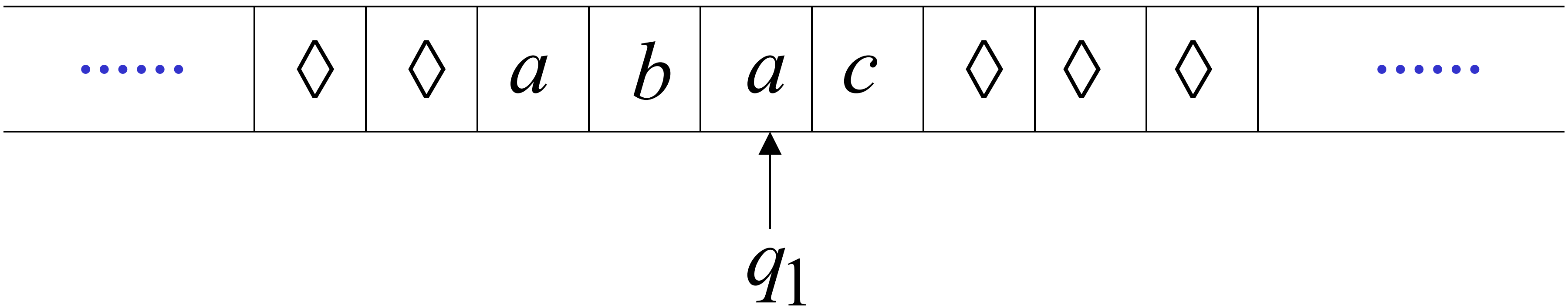
Time 1



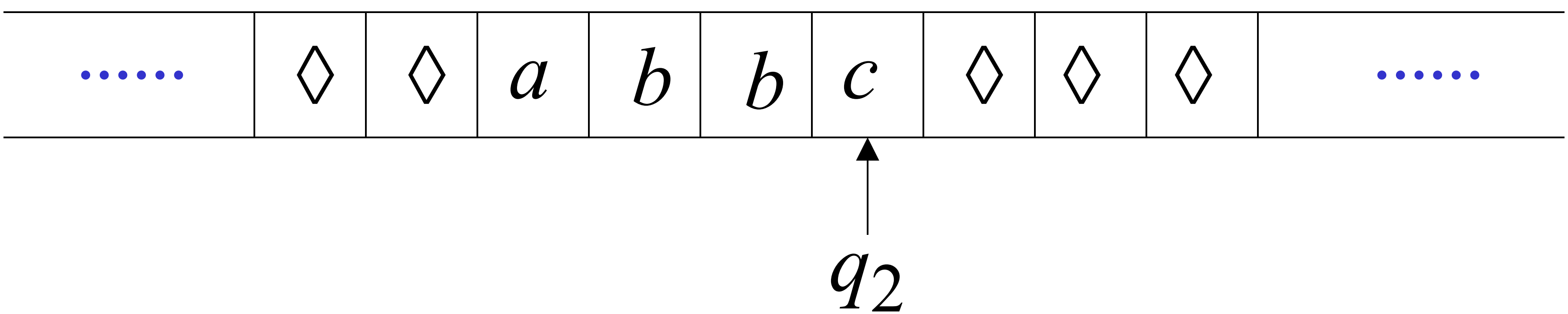
current state

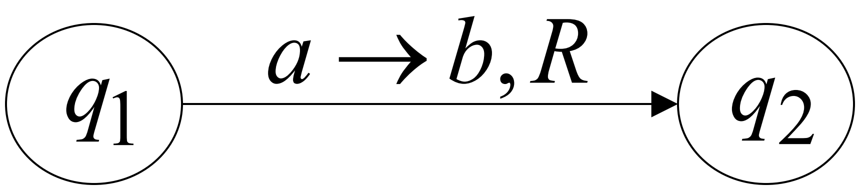


Time 1



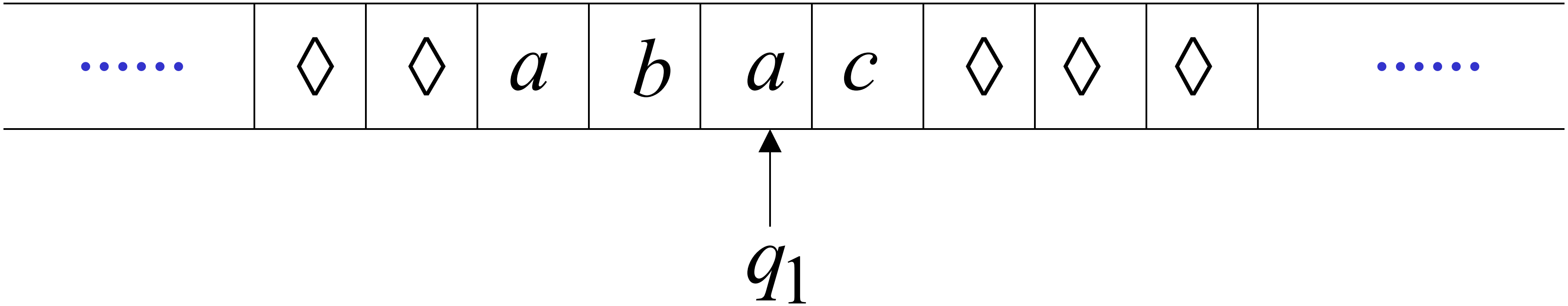
Time 2



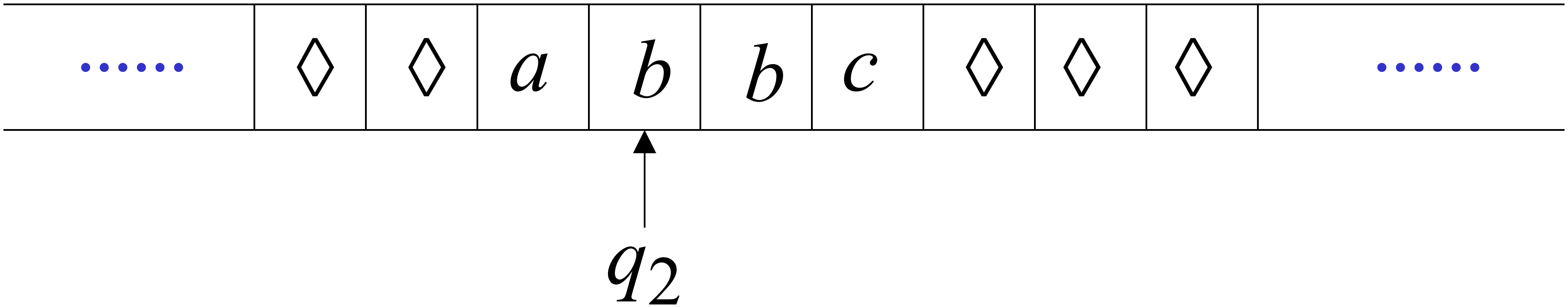


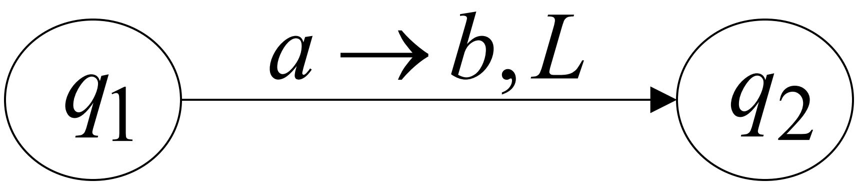
Example:

Time 1



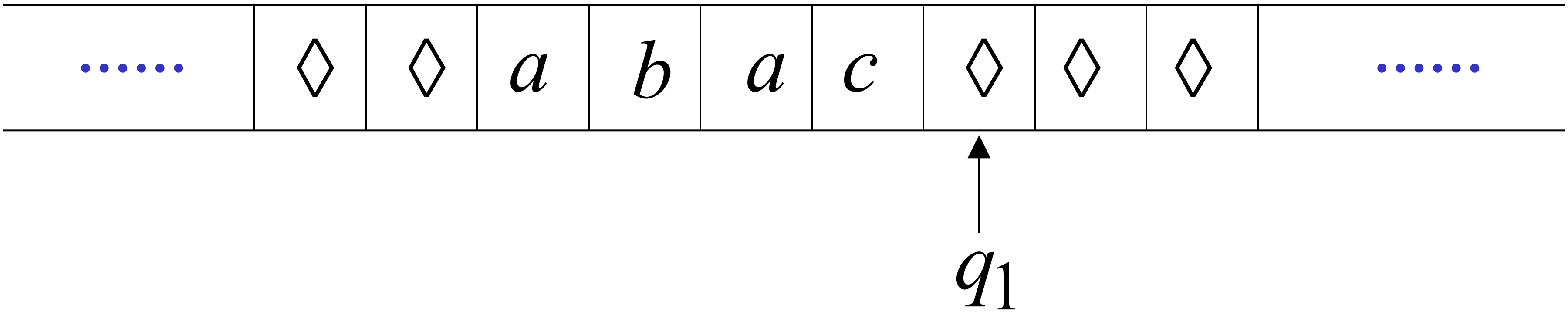
Time 2



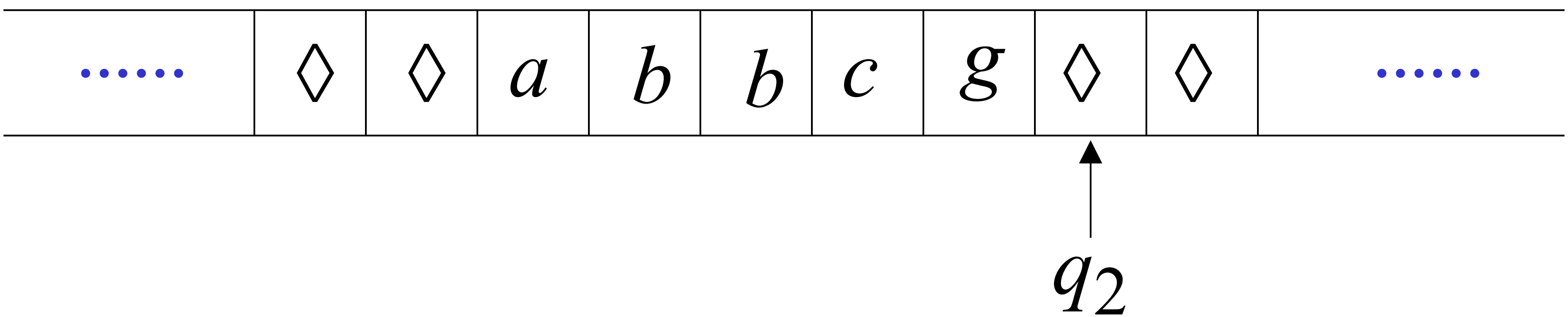


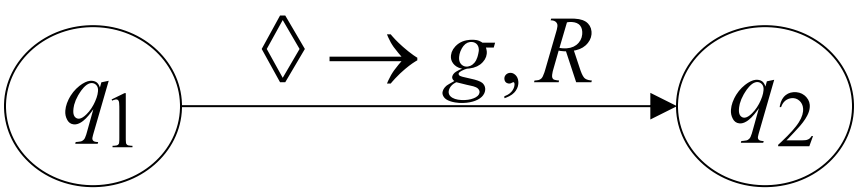
Example:

Time 1



Time 2

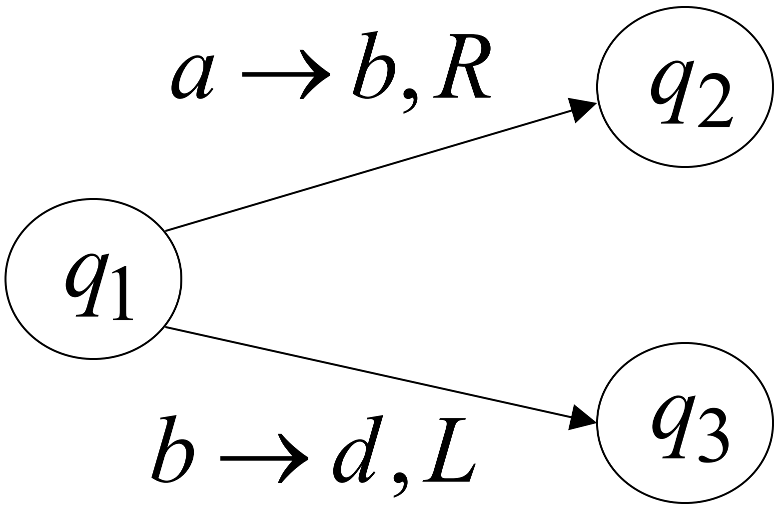




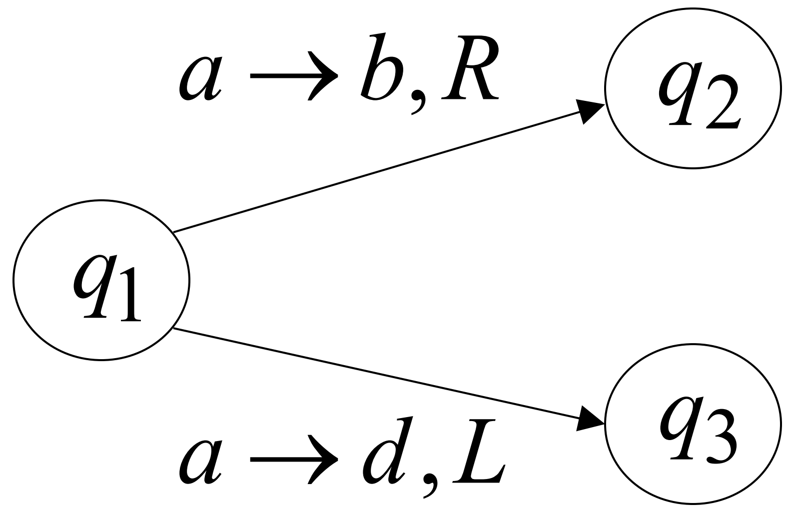
Determinism

Turing Machines are deterministic

Allowed



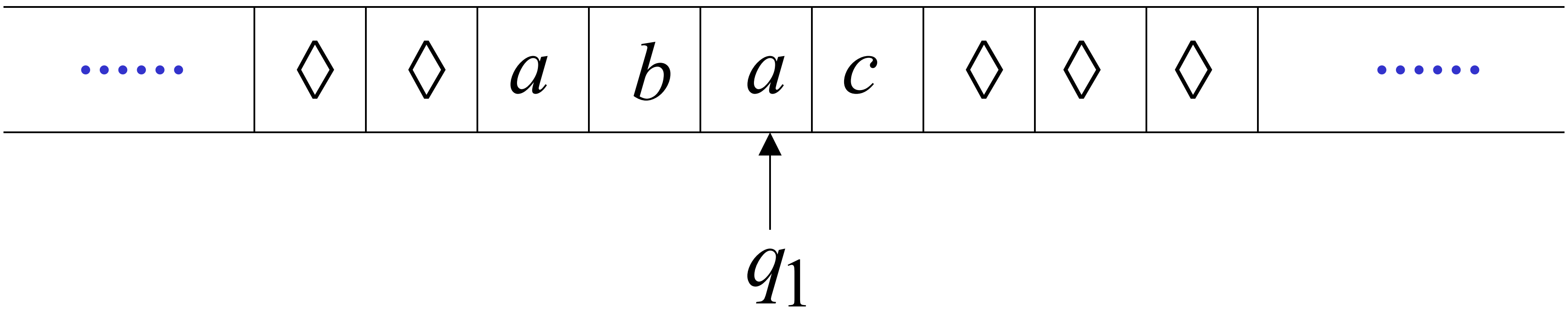
Not Allowed

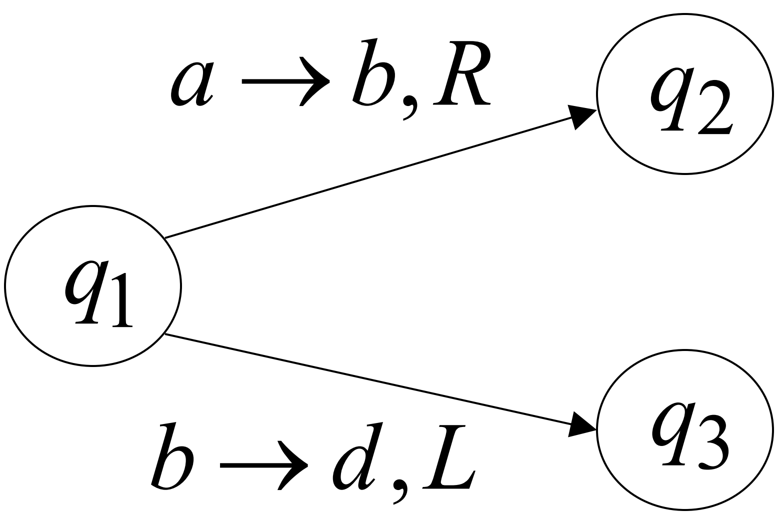


No lambda transitions allowed

Partial Transition Function

Example:





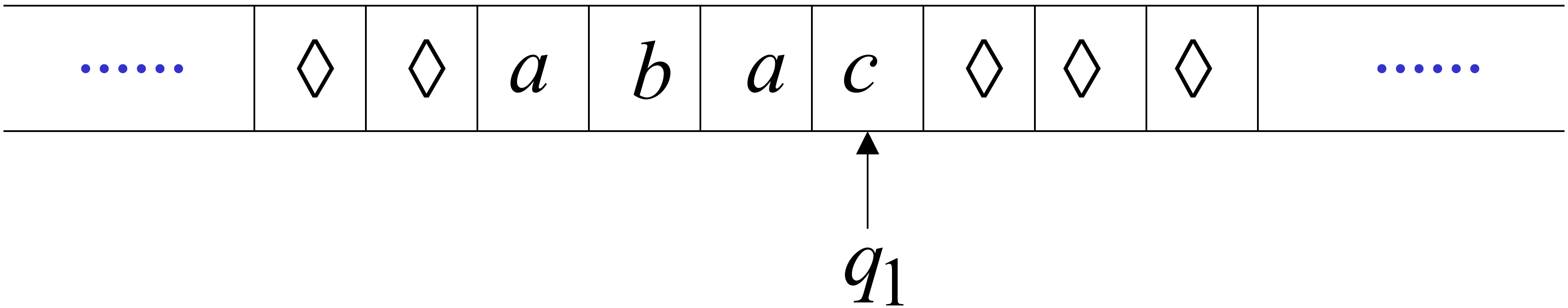
Allowed:

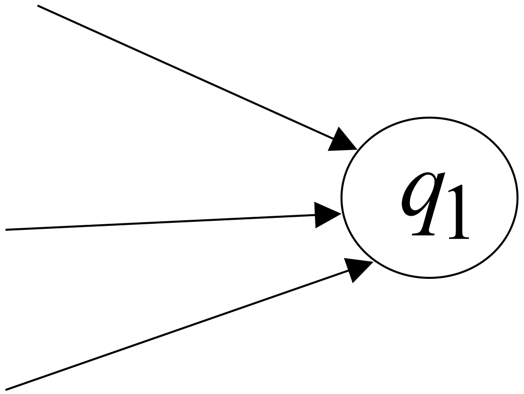
No transition for input symbol c

Halting

The machine ***halts*** in a state if there is no transition to follow

Halting Example 1:

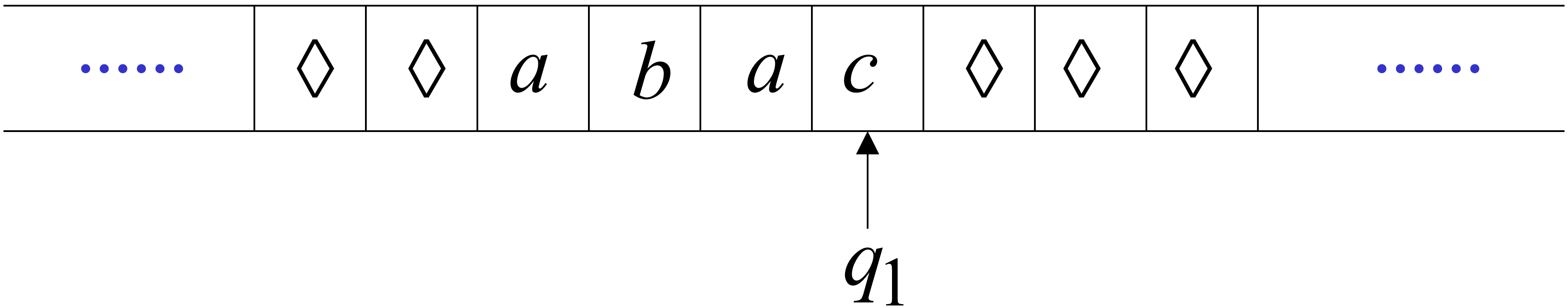


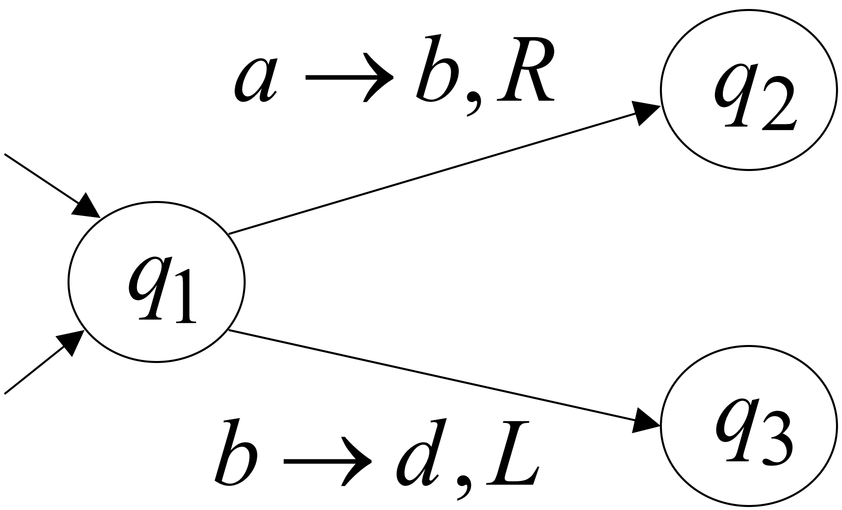


No transition from q_1

HALT!!!

Halting Example 2:

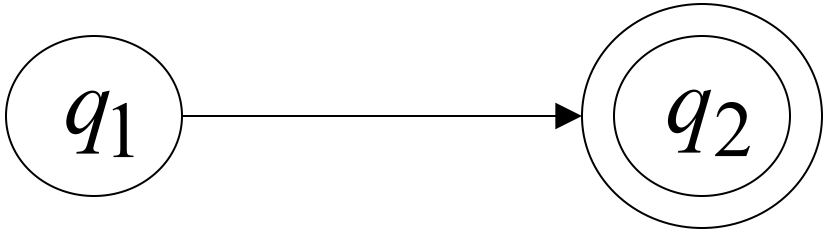




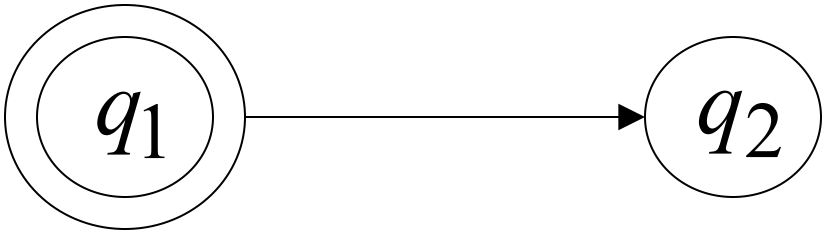
No possible transition from q_1 and symbol c

HALT!!!

Accepting States



Allowed



Not Allowed

- Accepting states have no outgoing transitions
- The machine halts and accepts

Acceptance

Accept Input string → If machine halts in an accept state

Reject Input string → If machine halts in a non-accept state

or

If machine enters an *infinite loop*

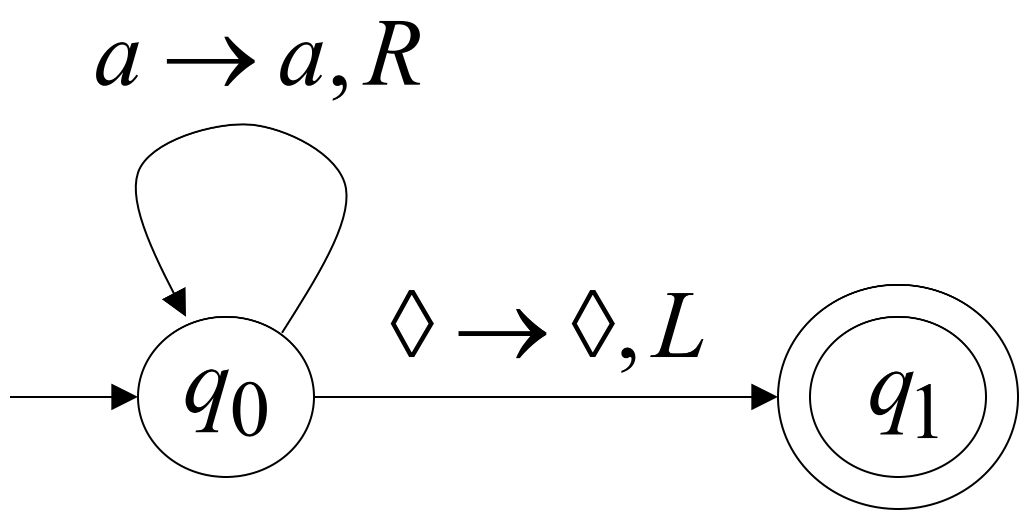
Observation:

In order to accept an input string, it is not necessary to scan all the symbols in the string

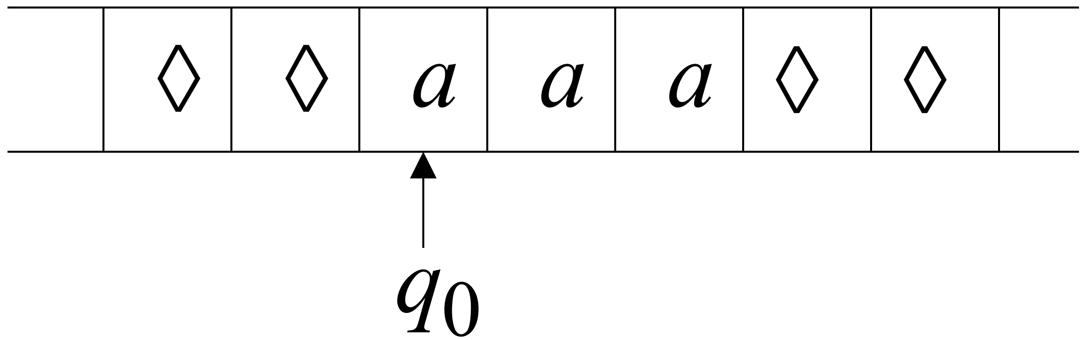
Turing Machine Example

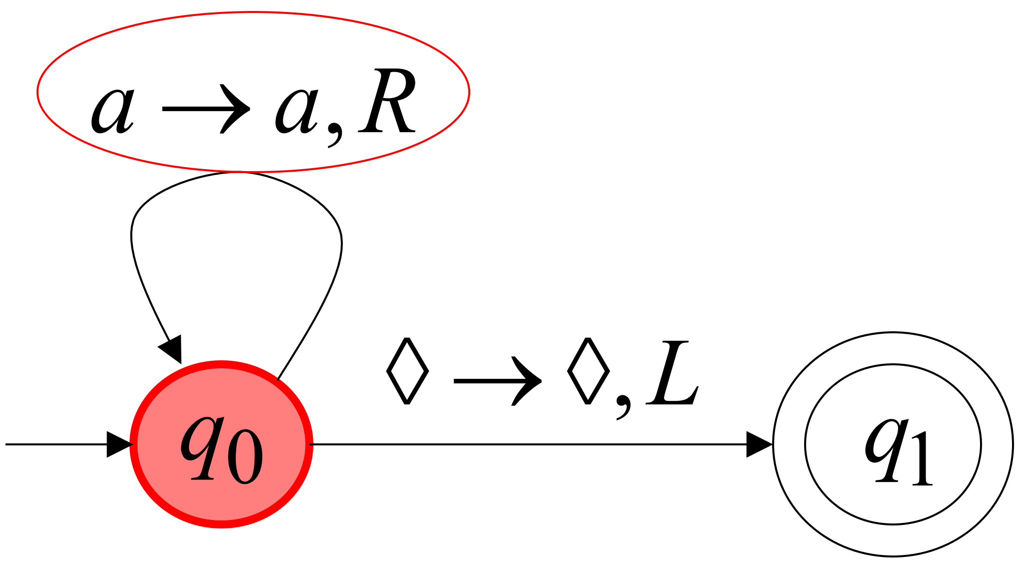
Input alphabet $\Sigma = \{a, b\}$

Accepts the language: a^*

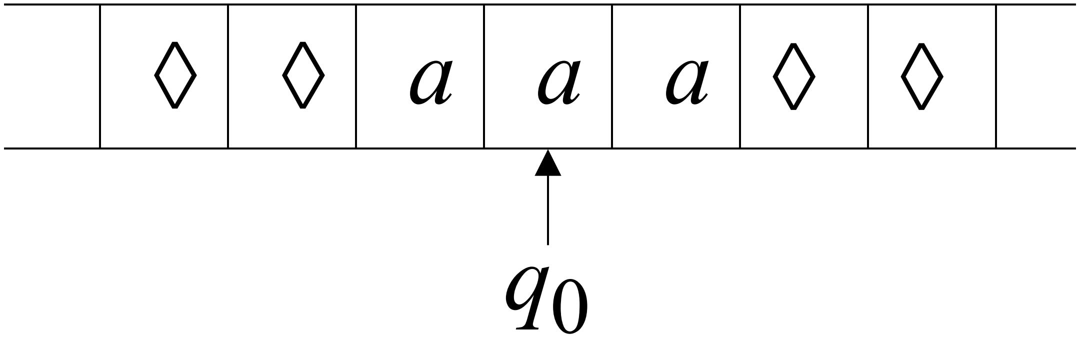


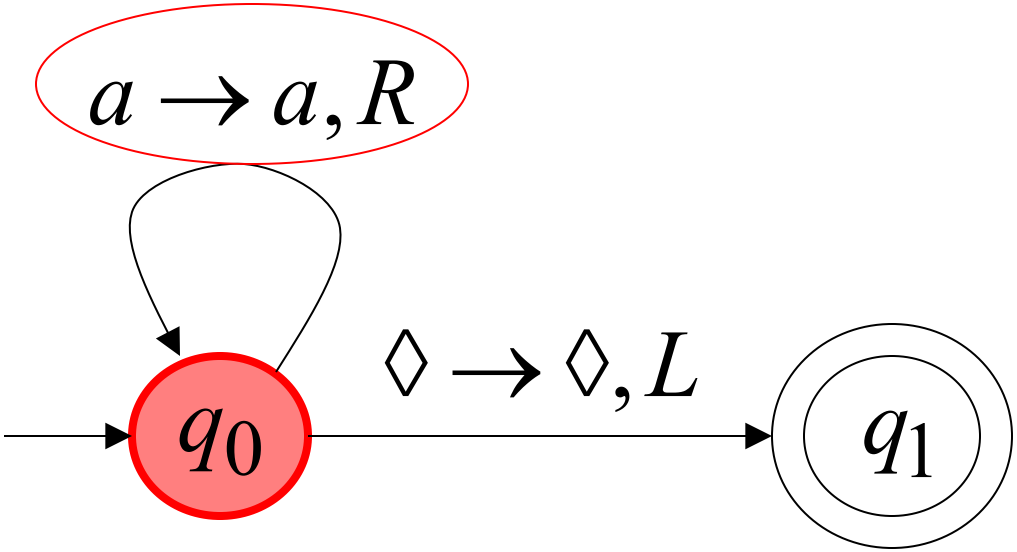
Time 0



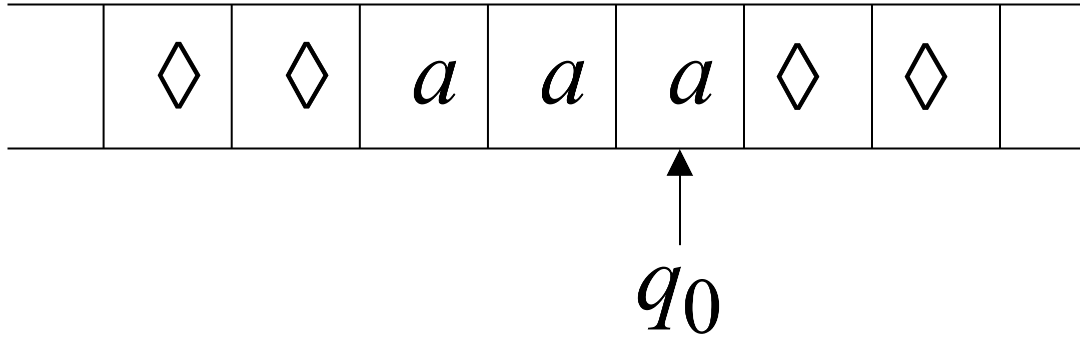


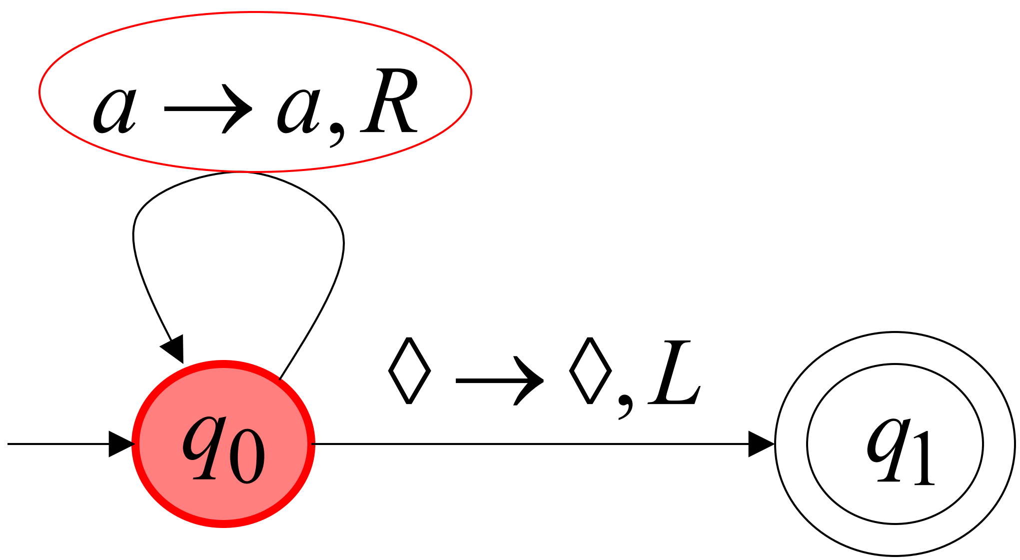
Time 1



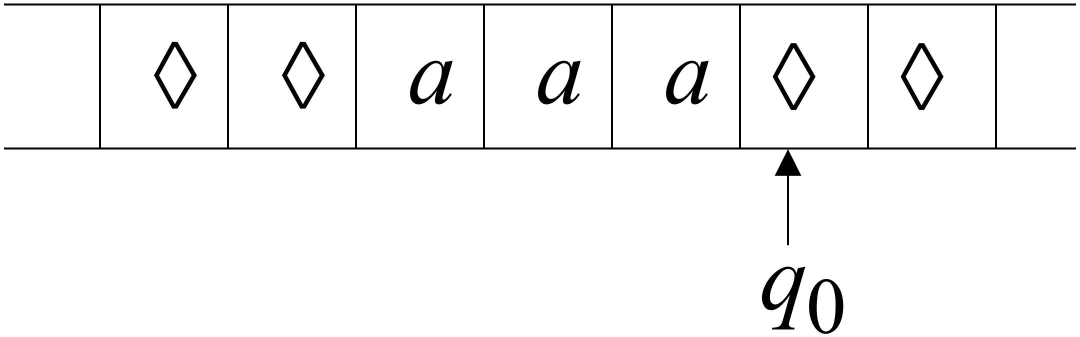


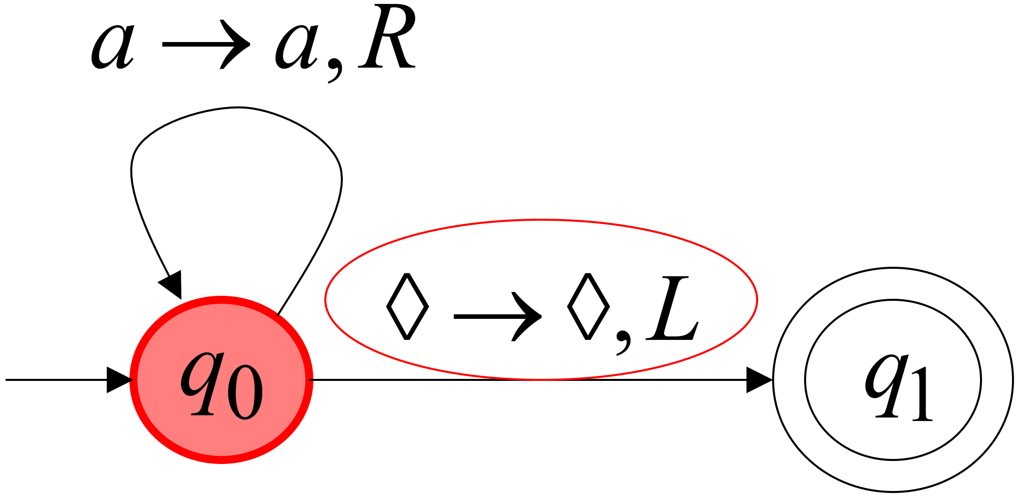
Time 2



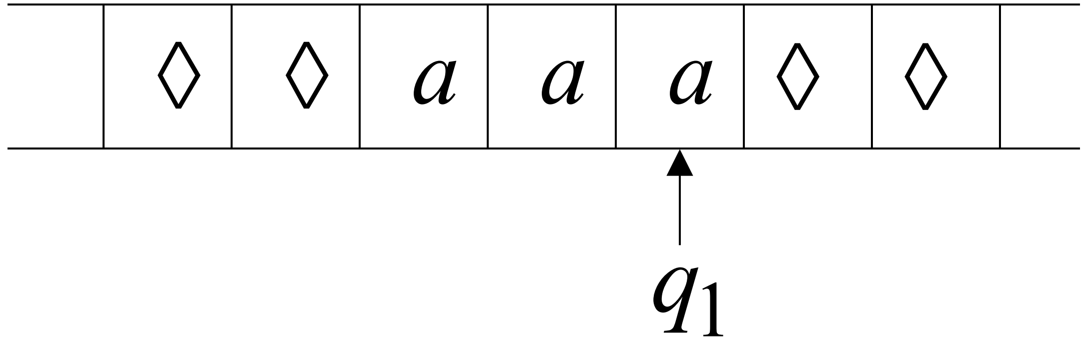


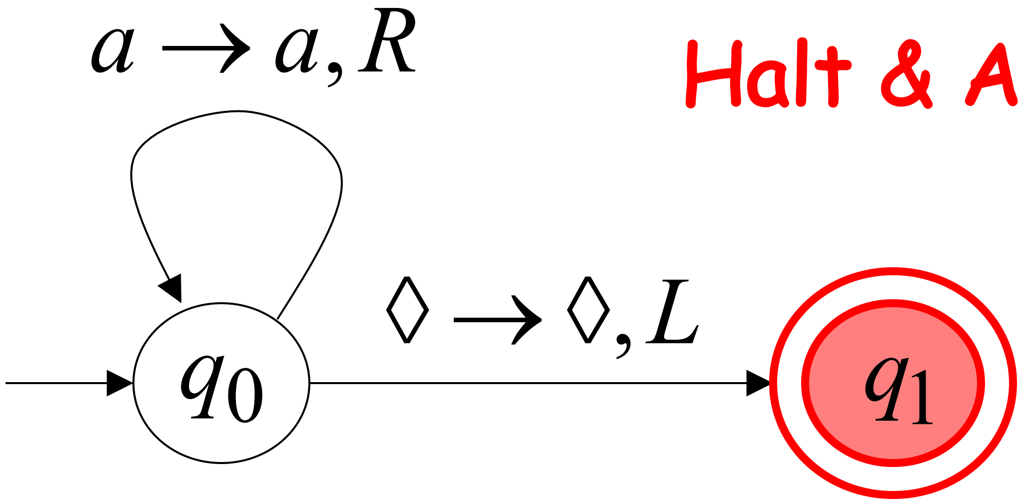
Time 3





Time 4

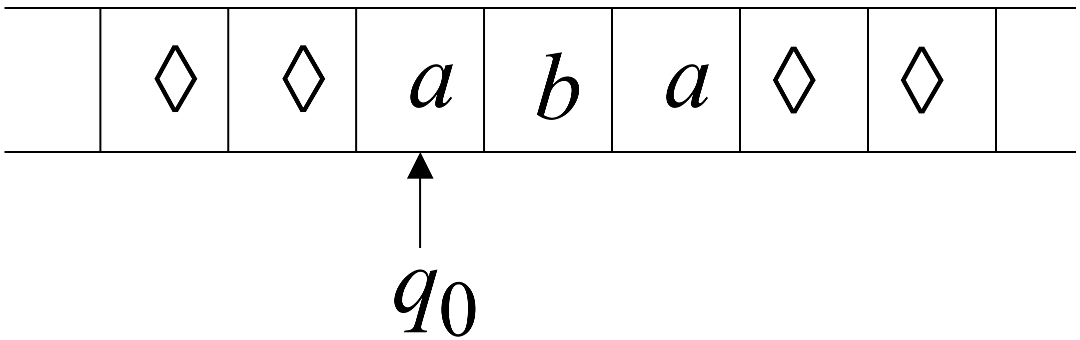


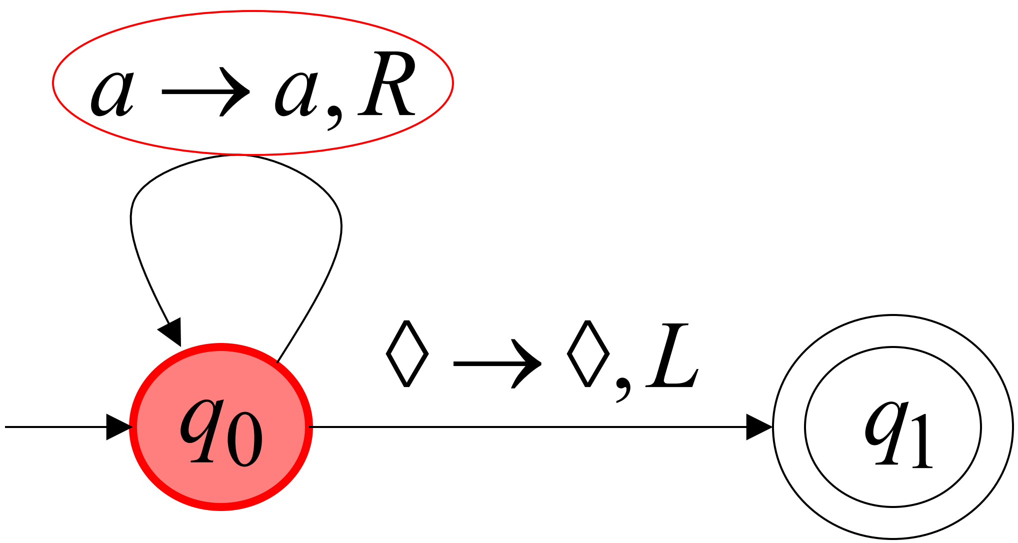


Halt & Accept

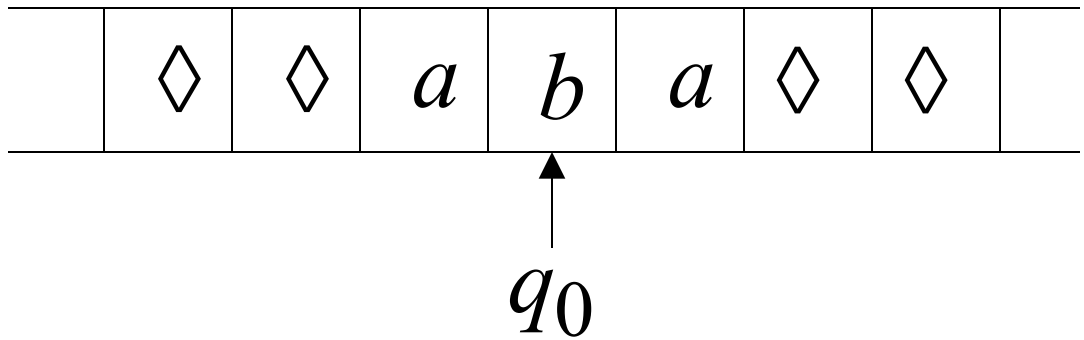
Rejection Example

Time 0



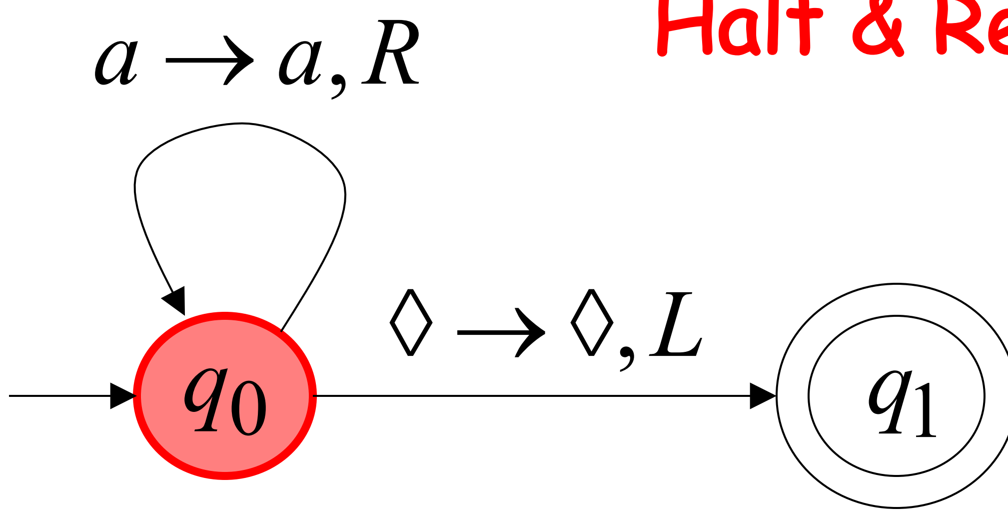


Time 1



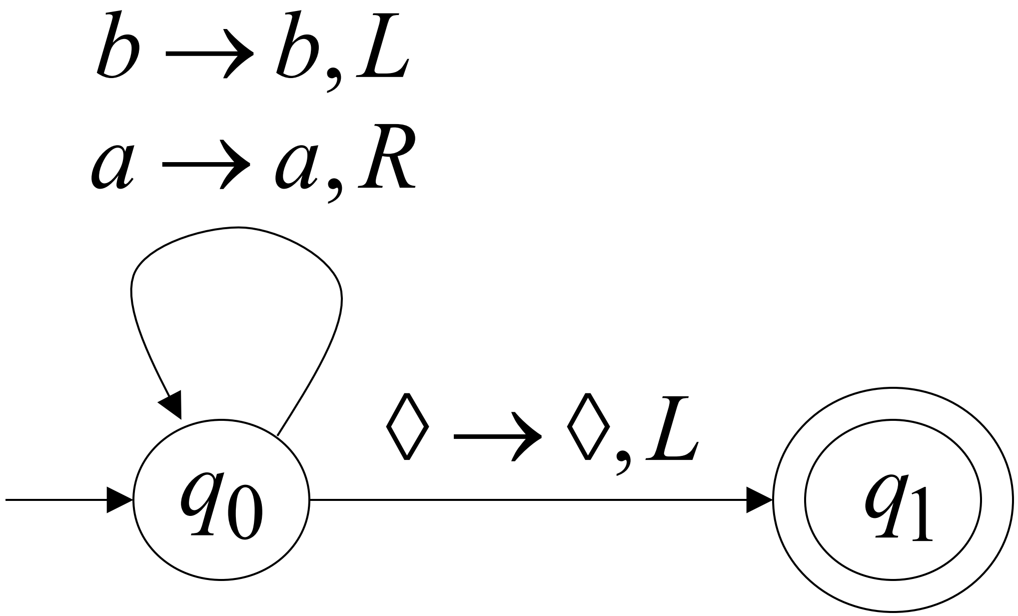
No possible Transition

Halt & Reject

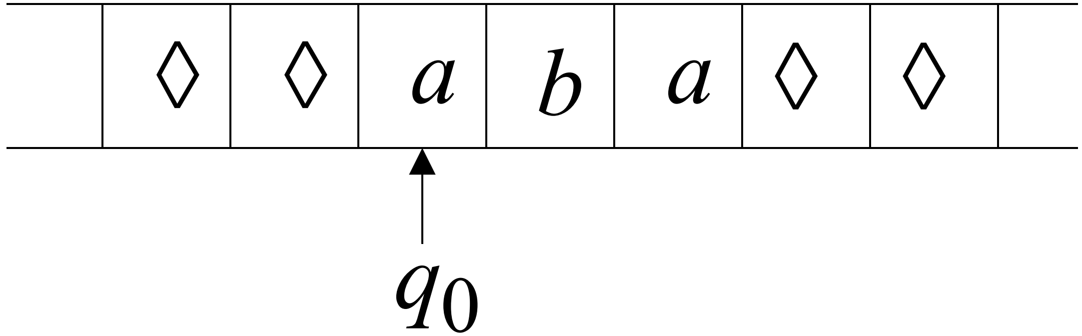


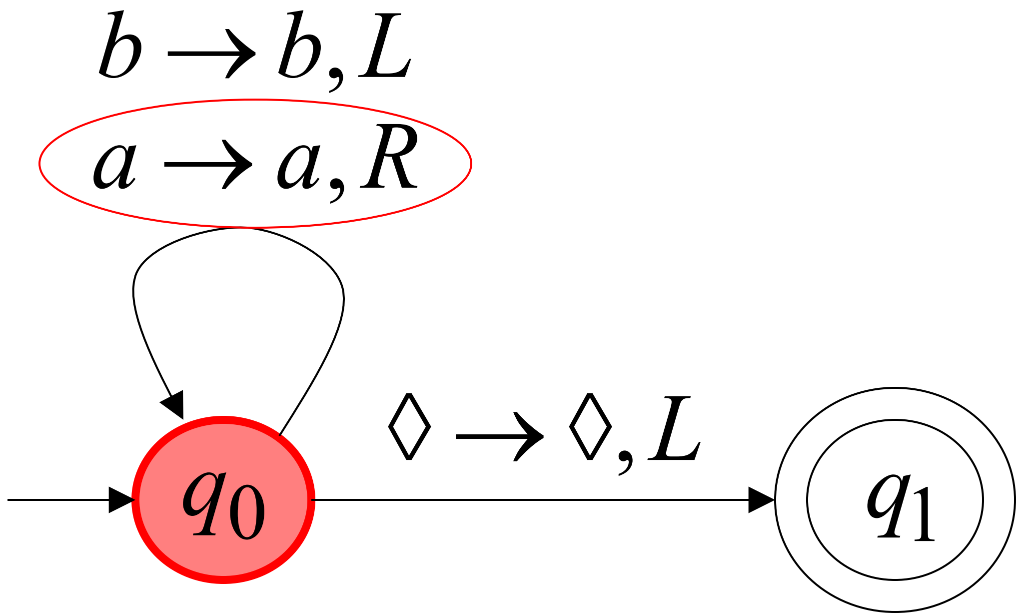
Infinite Loop Example

A Turing machine
for language $a^* + b(a+b)^*$

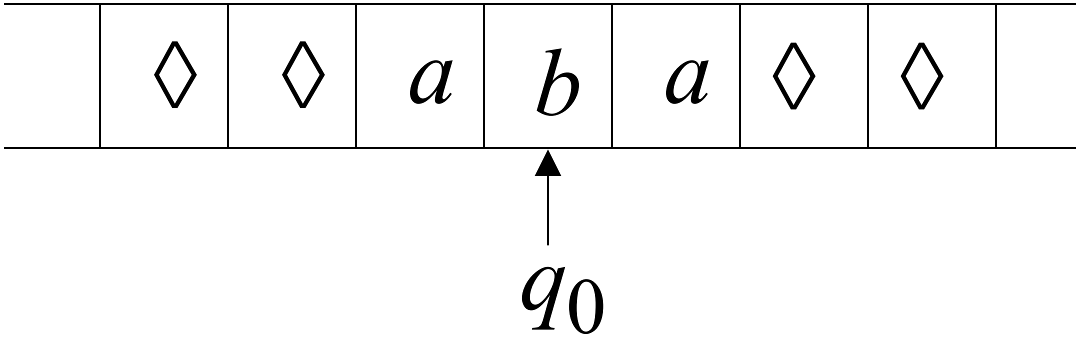


Time 0

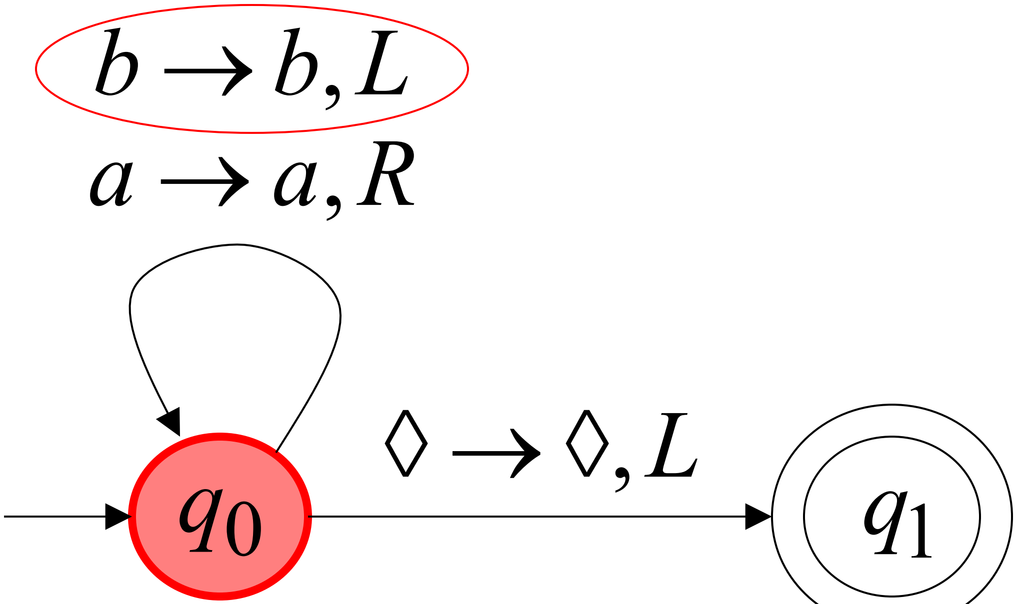

◊ ◊ a b a ◊ ◊
q_0


$b \rightarrow b, L$
$a \rightarrow a, R$
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Time 1

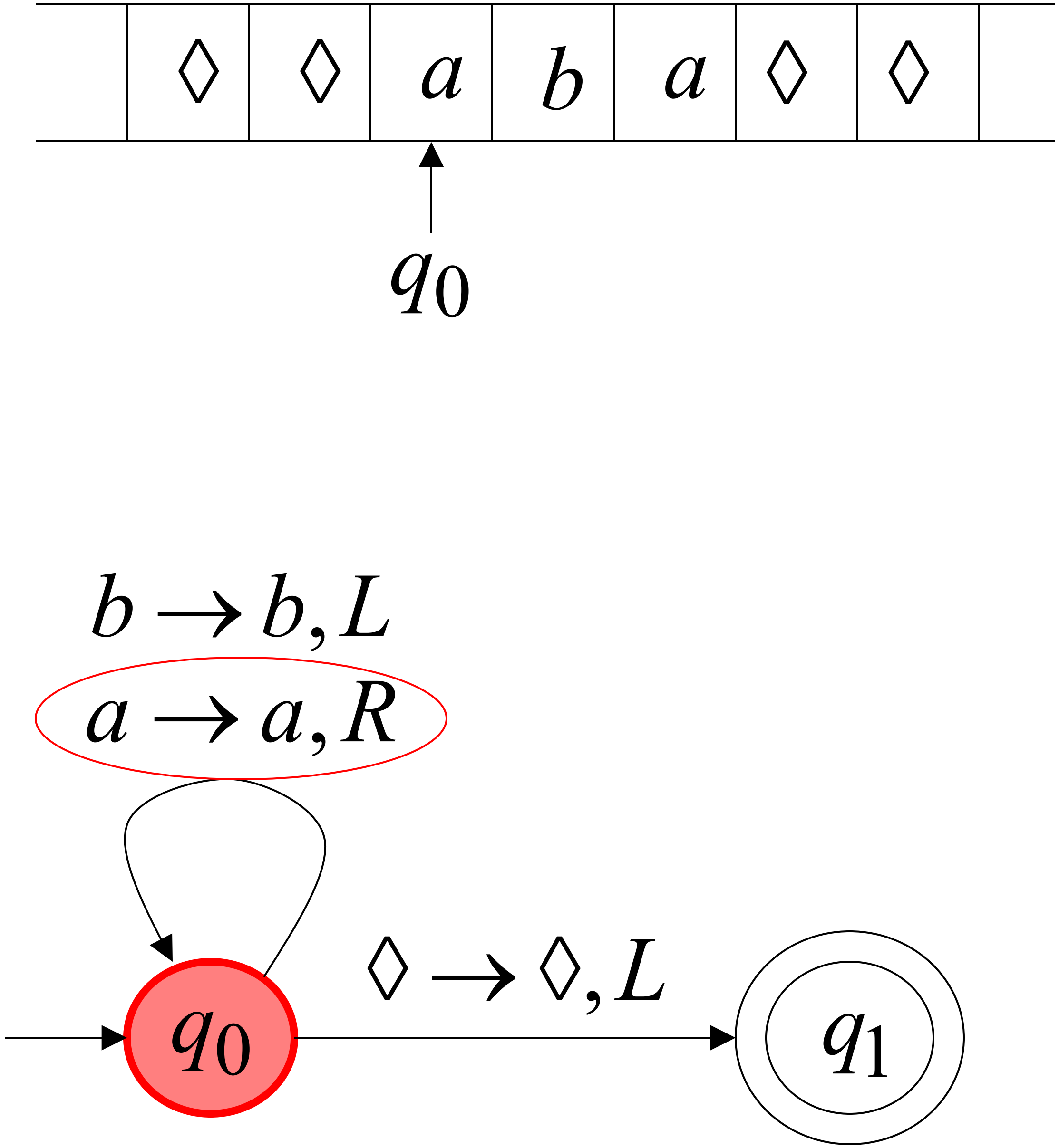


◊
◊
a
b
a
◊
◊
q_0

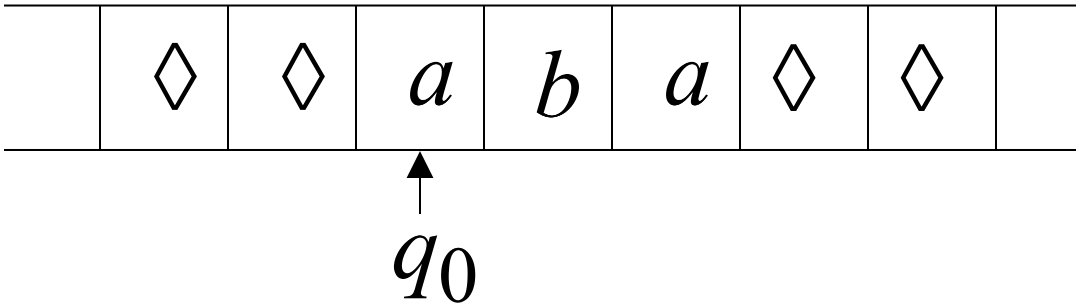


$b \rightarrow b, L$
$a \rightarrow a, R$
$\lozenge \rightarrow \lozenge, L$
q_0
q_1

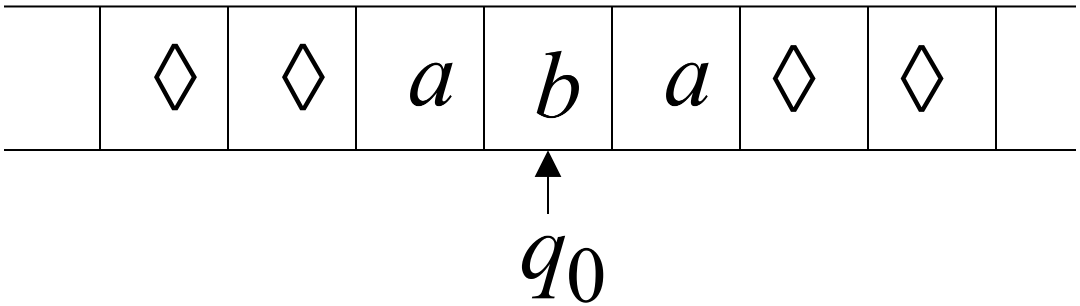
Time 2



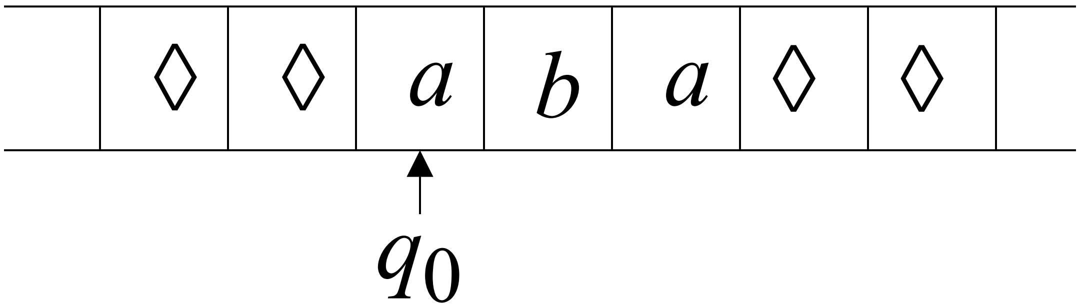
Time 2



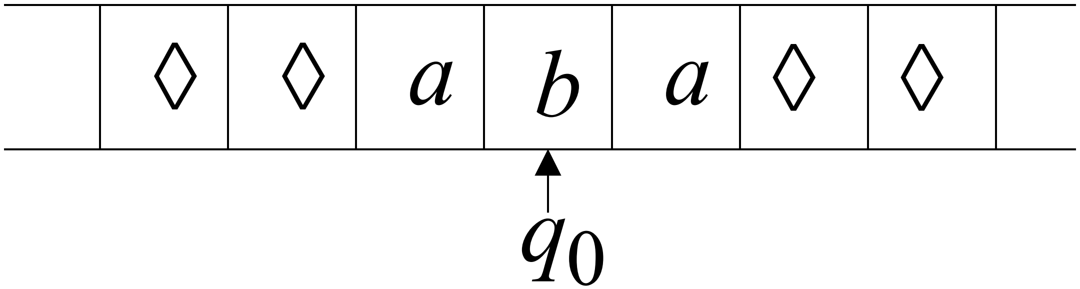
Time 3



Time 4



Time 5



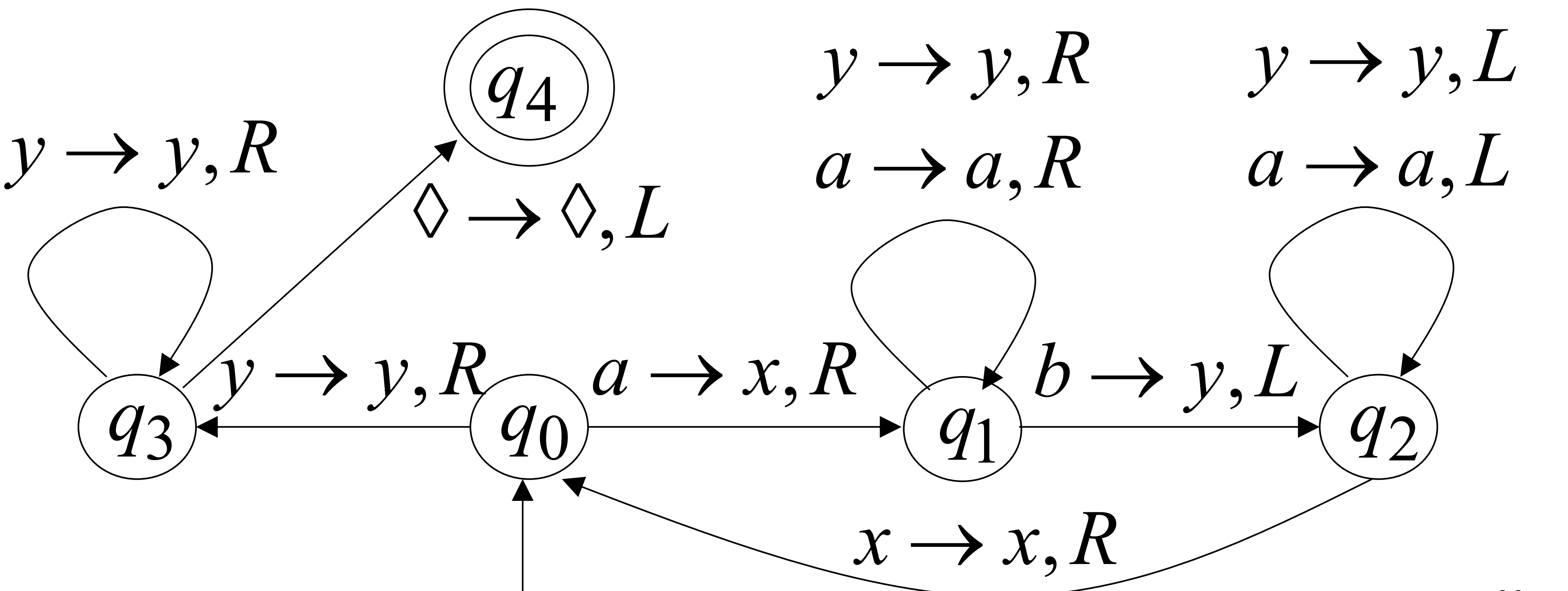
Infinite loop

Because of the infinite loop:

- The accepting state cannot be reached
- The machine never halts
- The input string is rejected

Another Turing Machine Example

Turing machine for the language $\{a^n b^n\}$
$n \geq 1$



Basic Idea:

Match a's with b's:

Repeat:

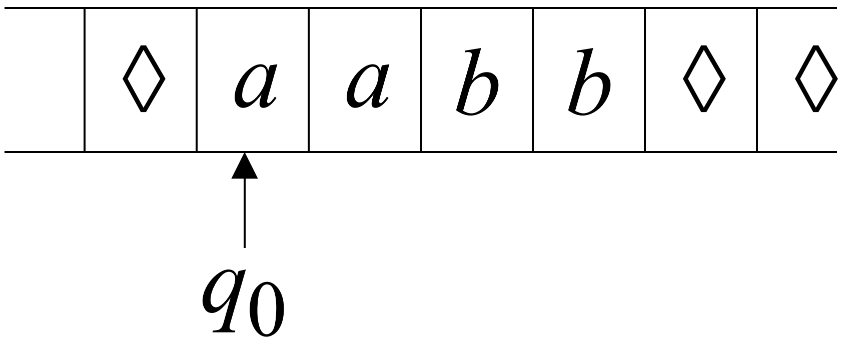
replace leftmost a with x

find leftmost b and replace it with y

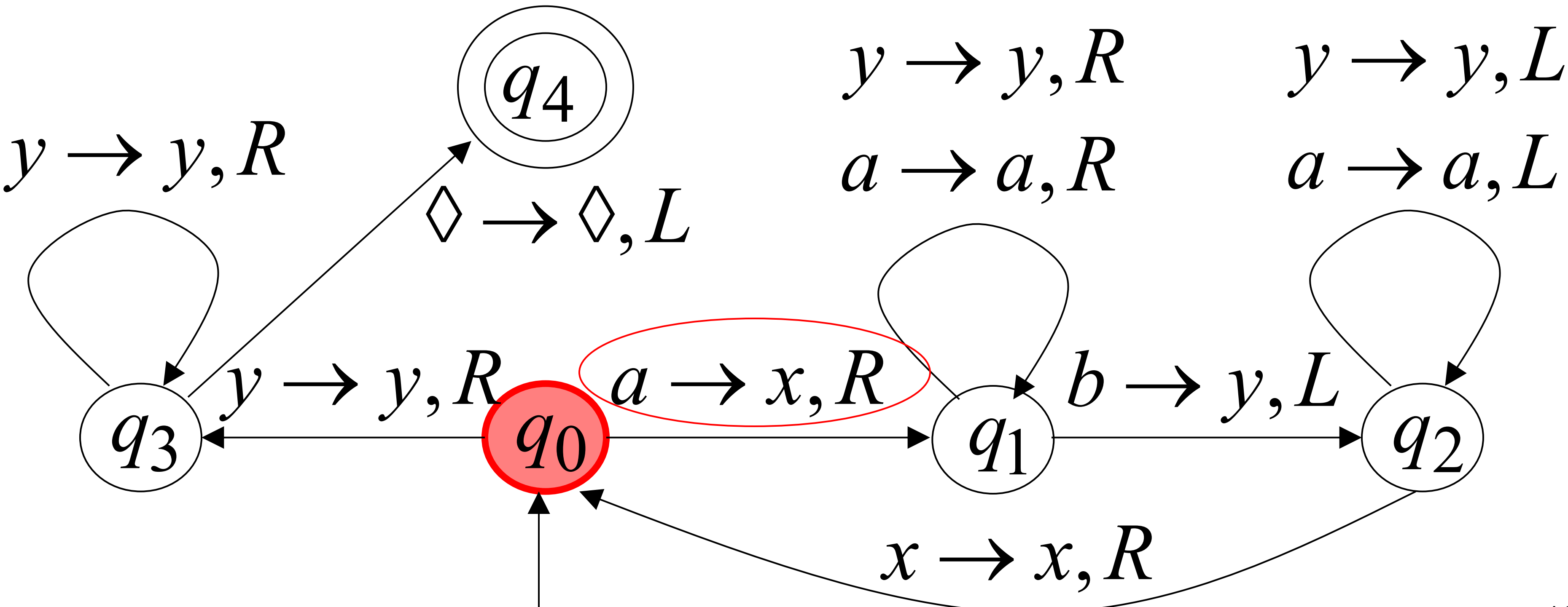
Until there are no more a's or b's

If there is a remaining a or b reject

Time 0

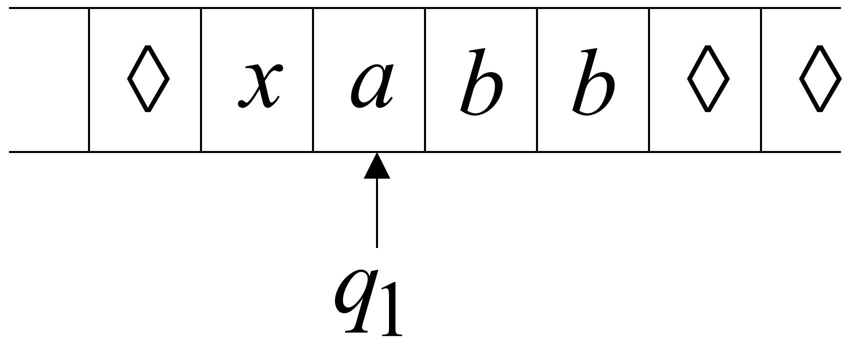


◊
a
a
b
b
◊
◊
q_0

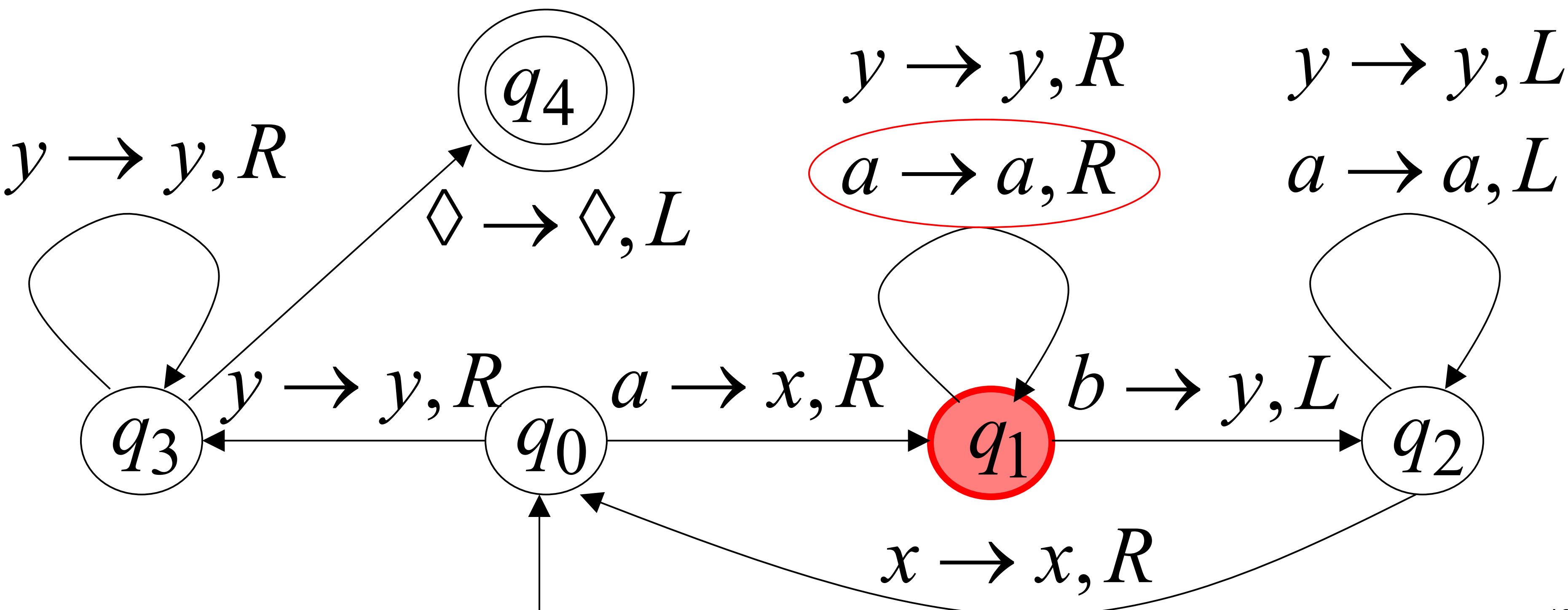


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 1

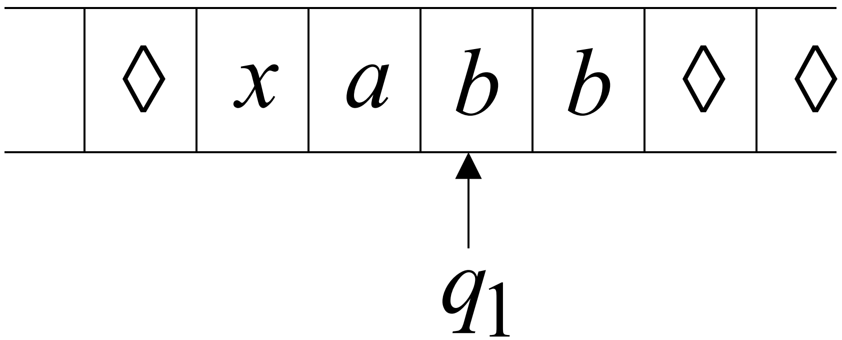


◊ x a b b ◊ ◊
q_1

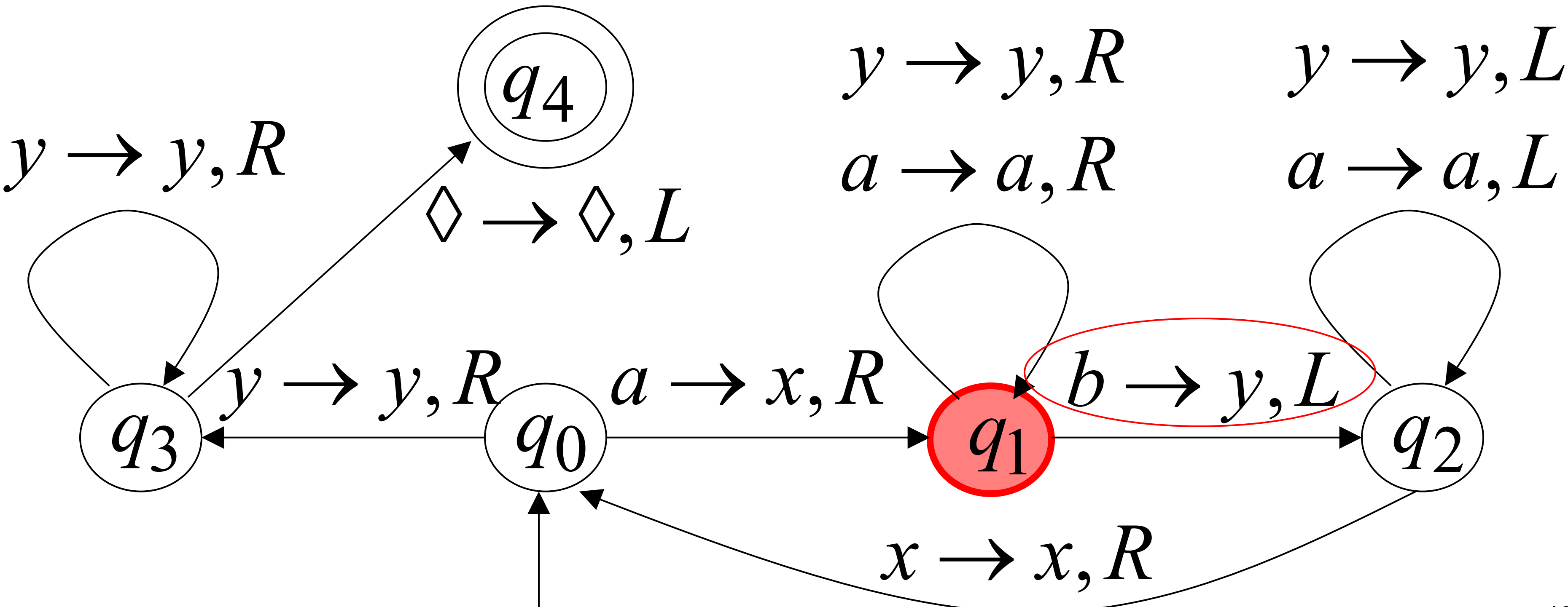


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 2

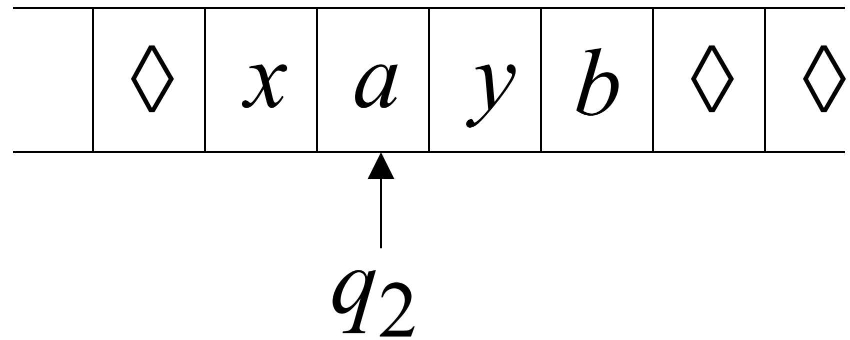


◊
x
a
b
b
◊
◊
q_1

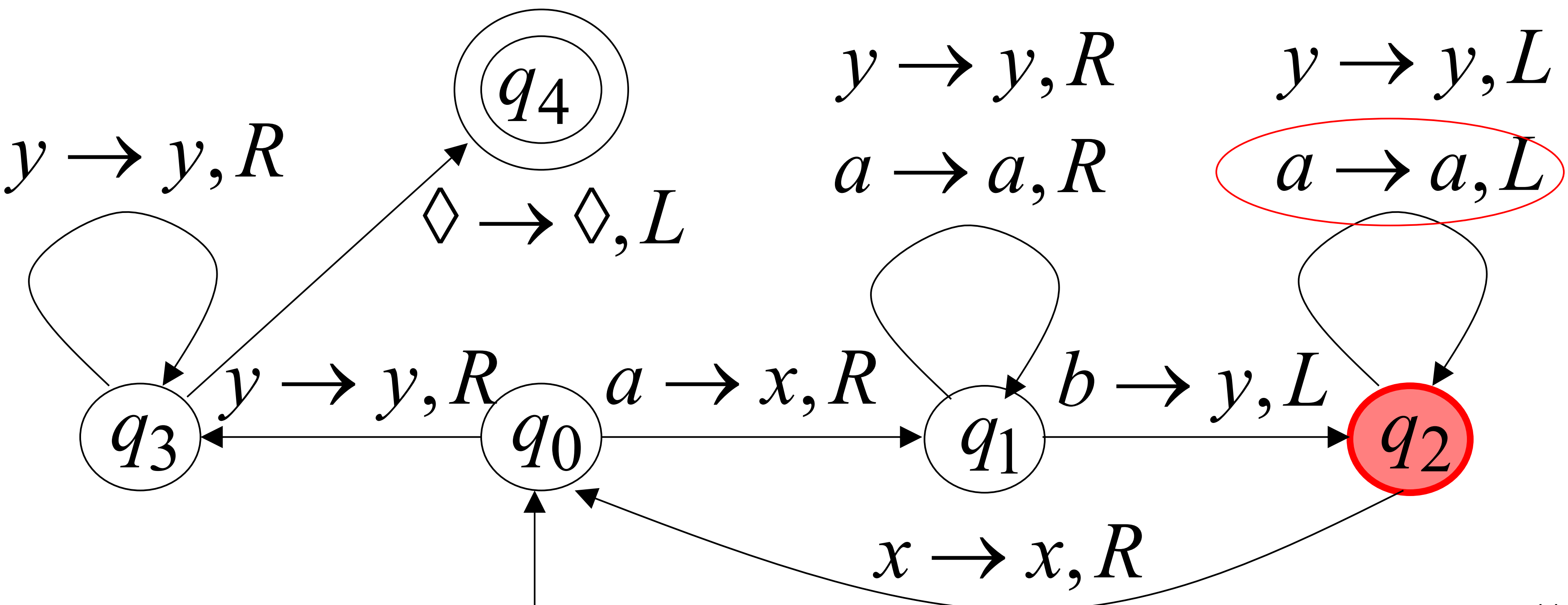


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 3

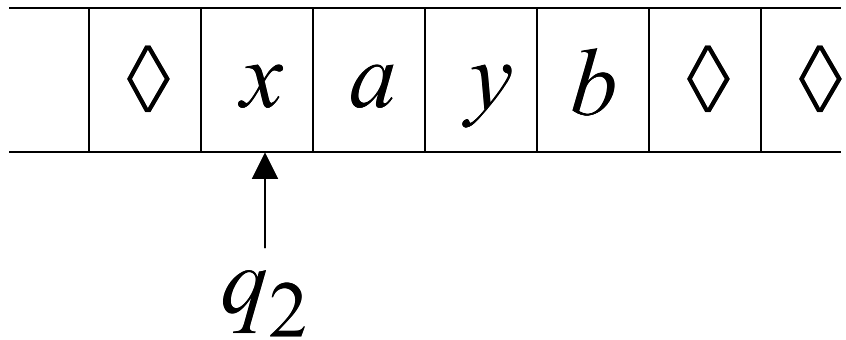


◊ x a y b ◊ ◊
q_2

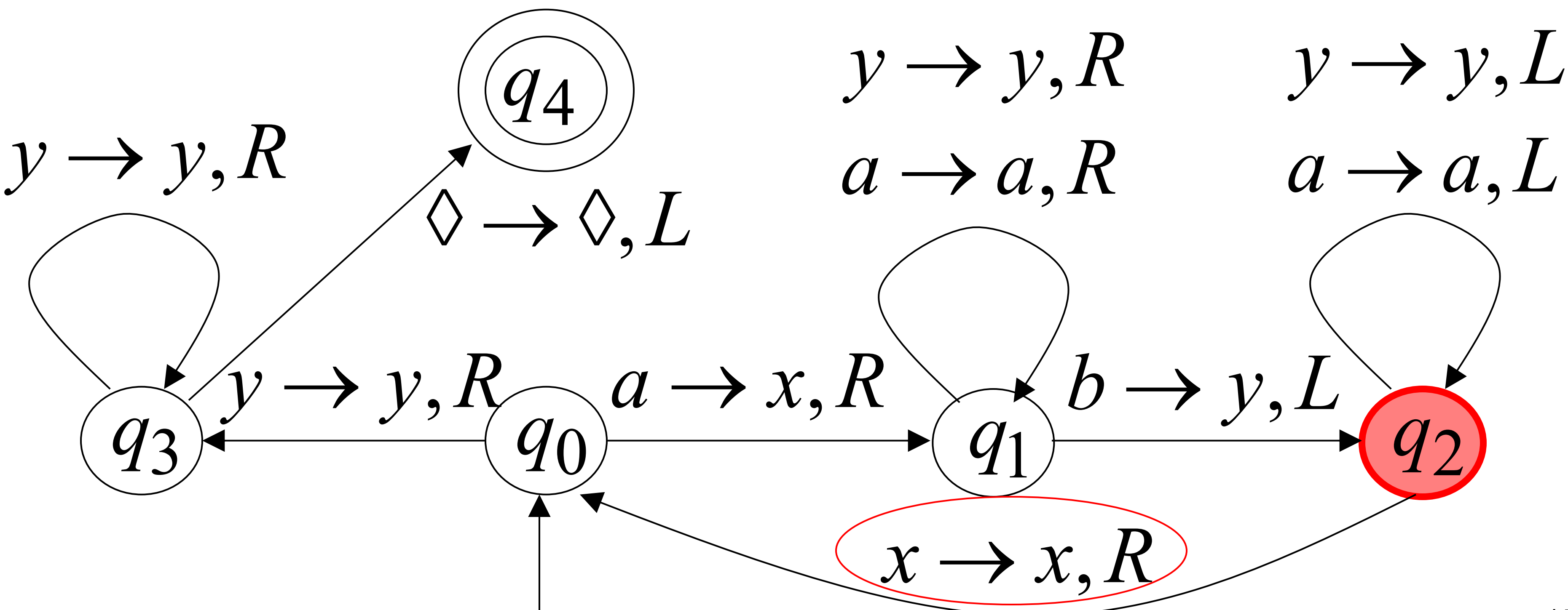


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 4

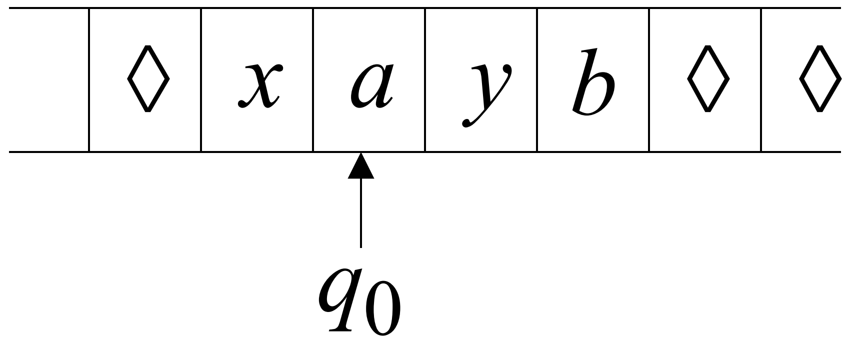


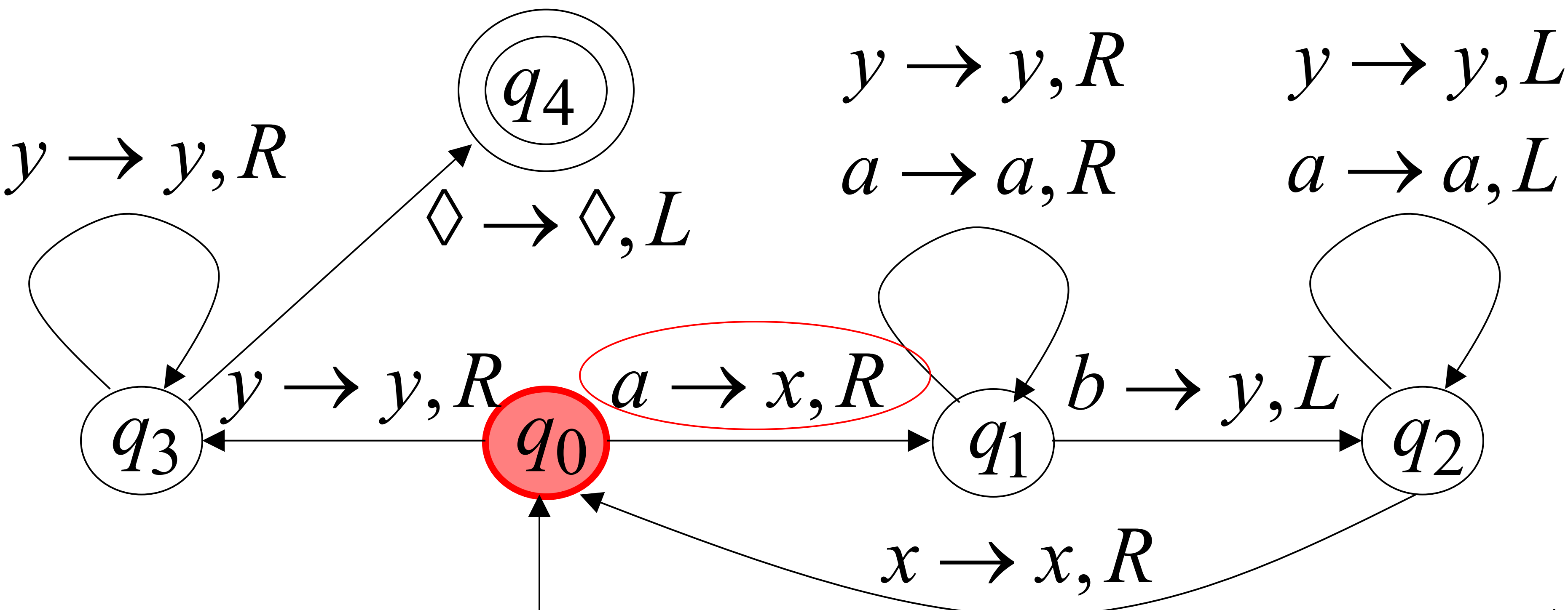
◊
x
a
y
b
◊
◊
q_2



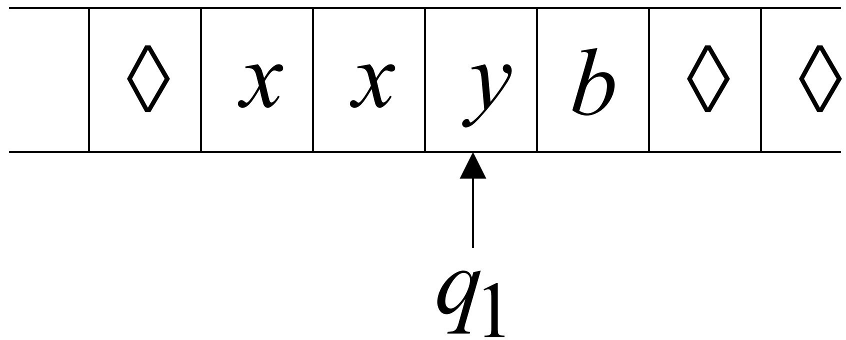
q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 5

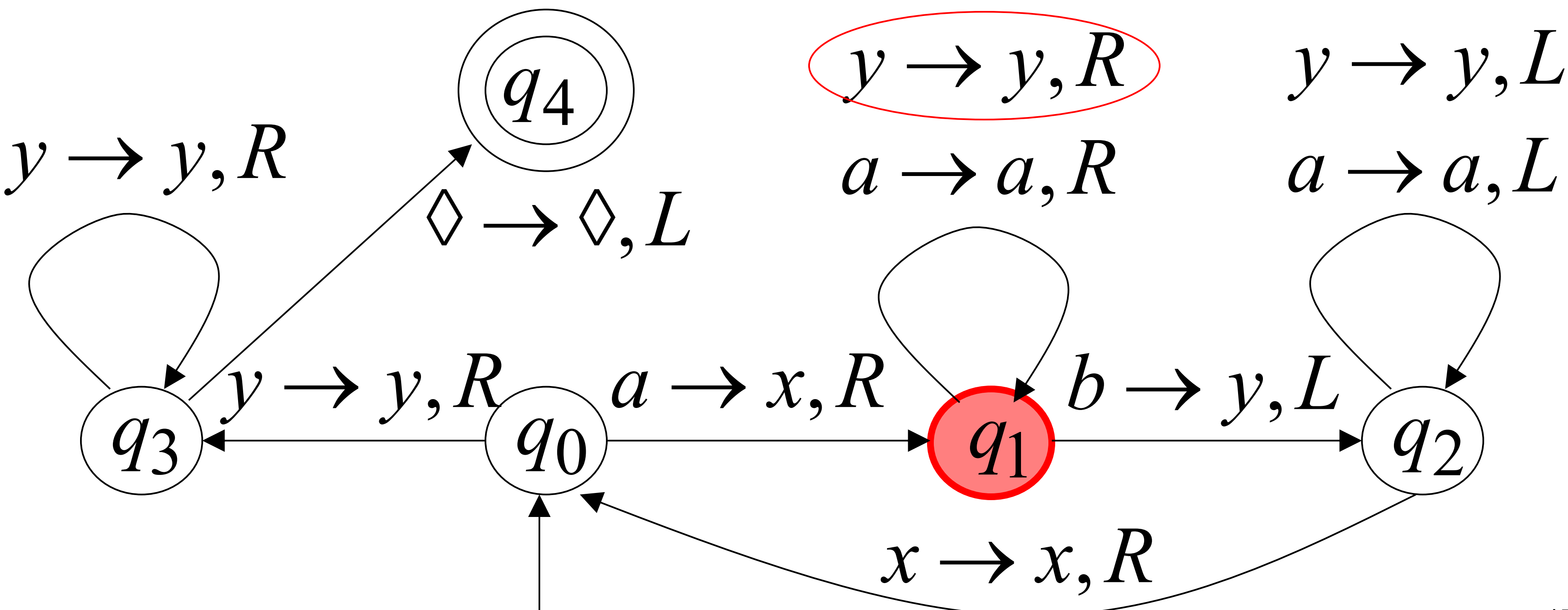

◊
x
a
y
b
◊
◊
q_0


q_4
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 6

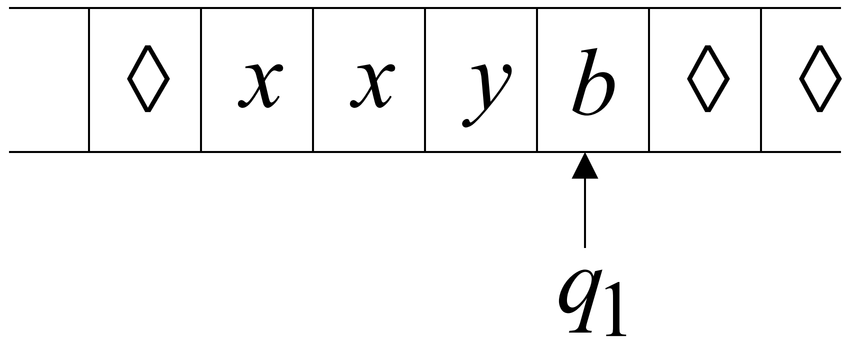


◊
x
x
y
b
◊
◊
q_1

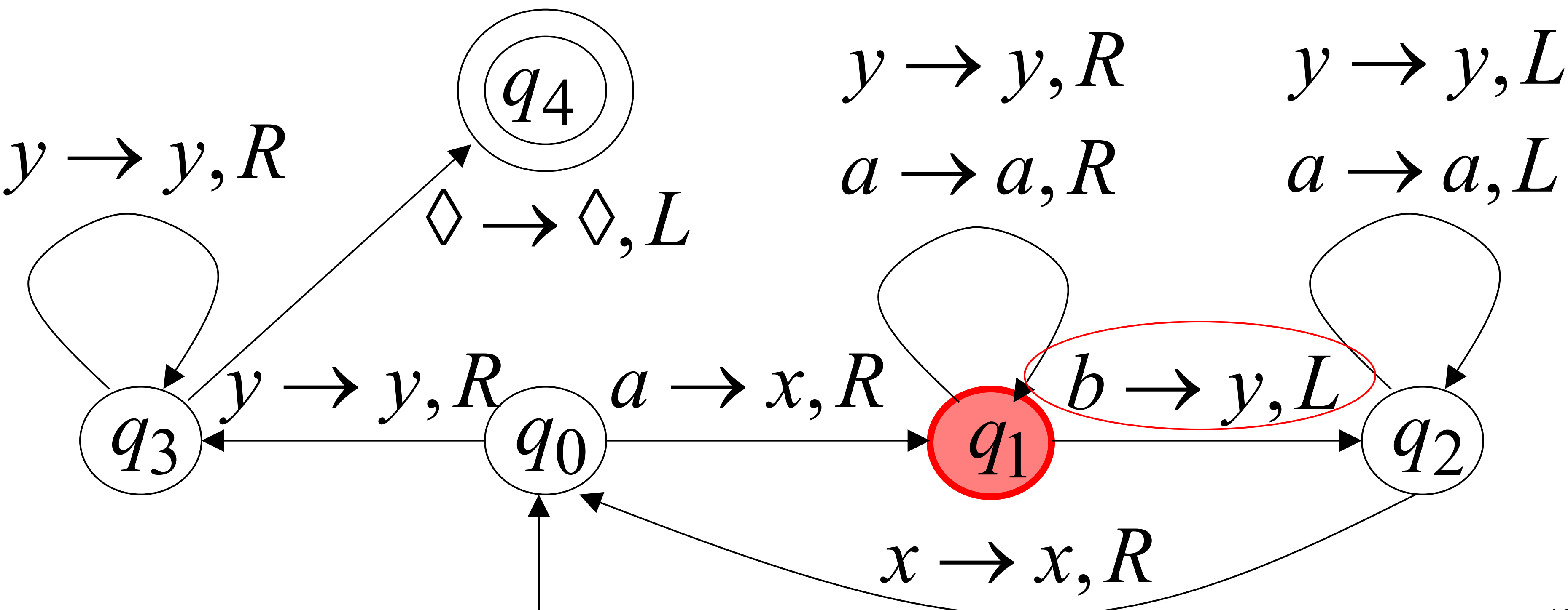


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 7

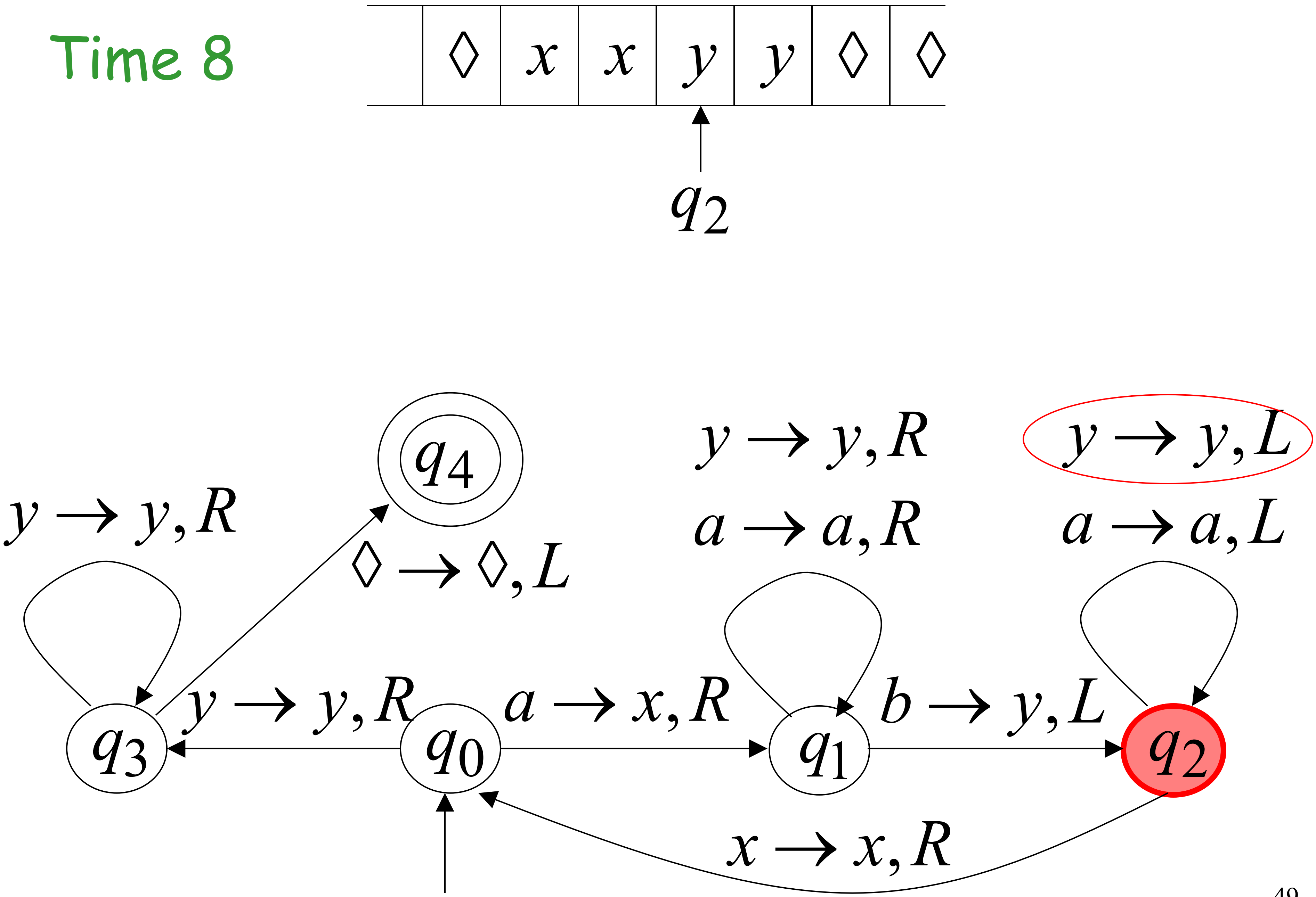


◊ x x y b ◊ ◊
q_1

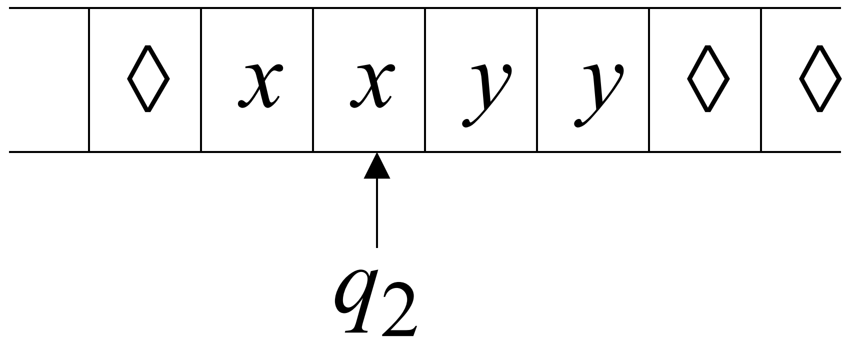


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

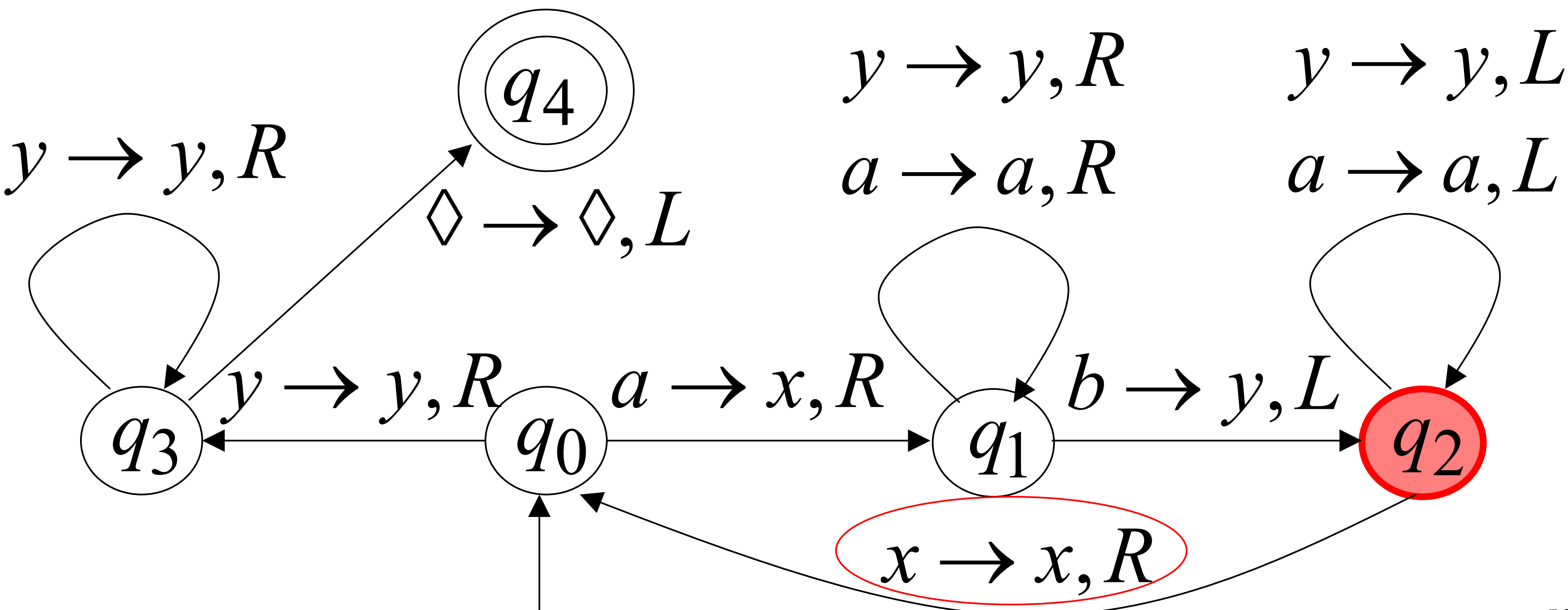
Time 8


◊
x
x
y
y
◊
◊
q_2
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 9

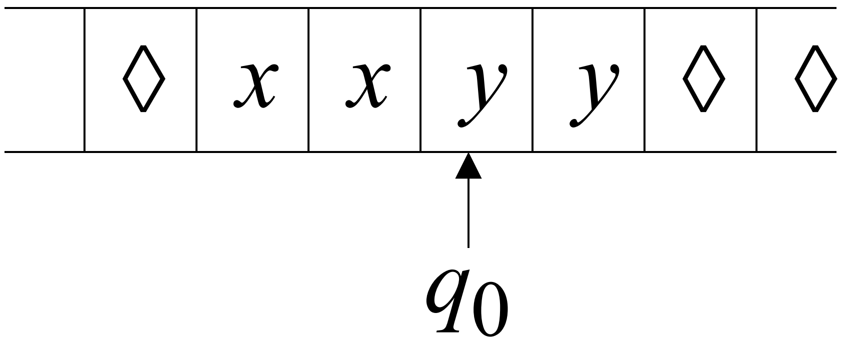


◊
x
x
y
y
◊
◊
q_2

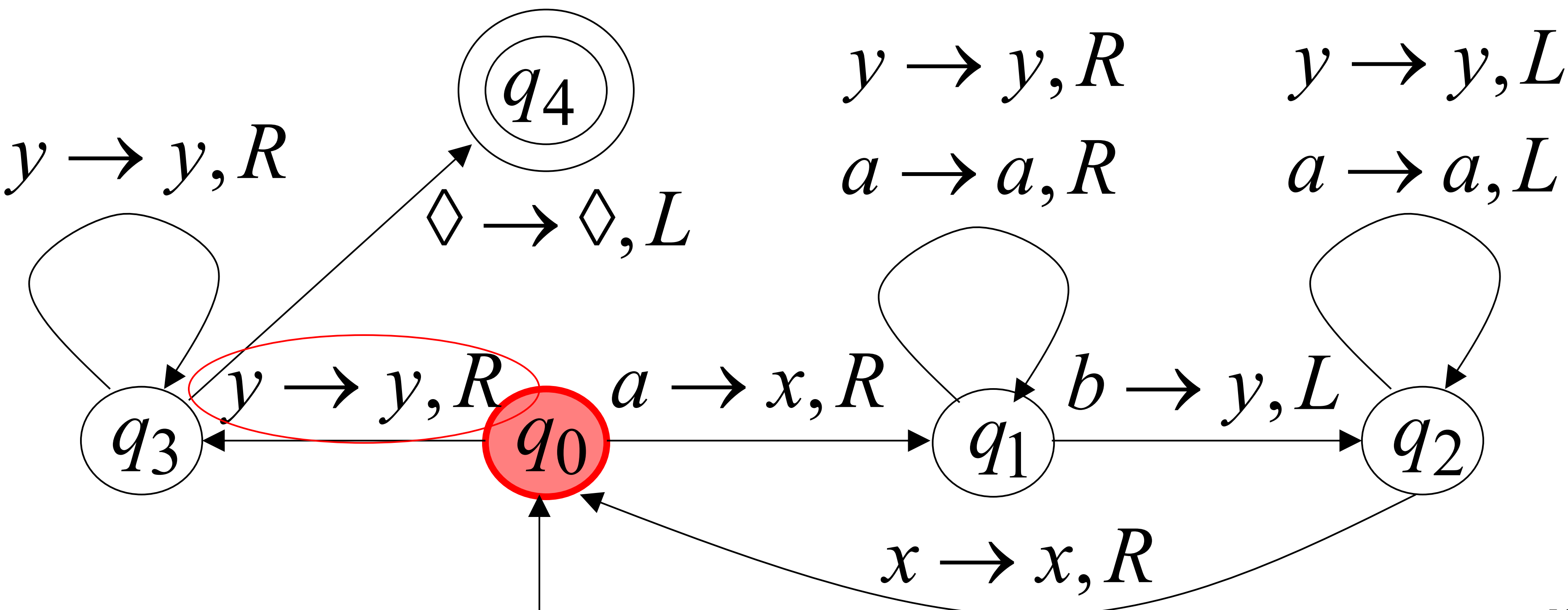


q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 10

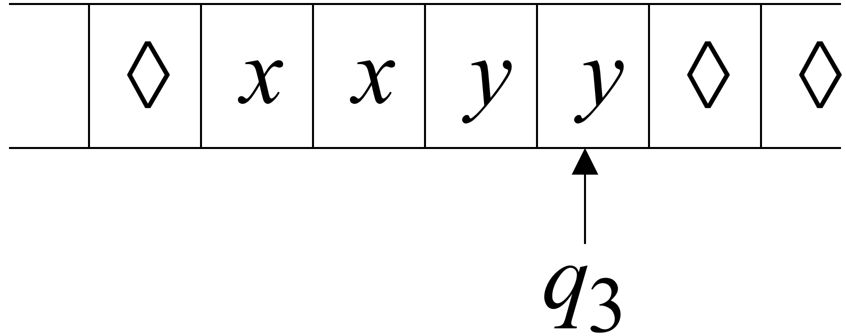


$\Diamond$
x
x
y
y
$\Diamond$
$\Diamond$
q_0

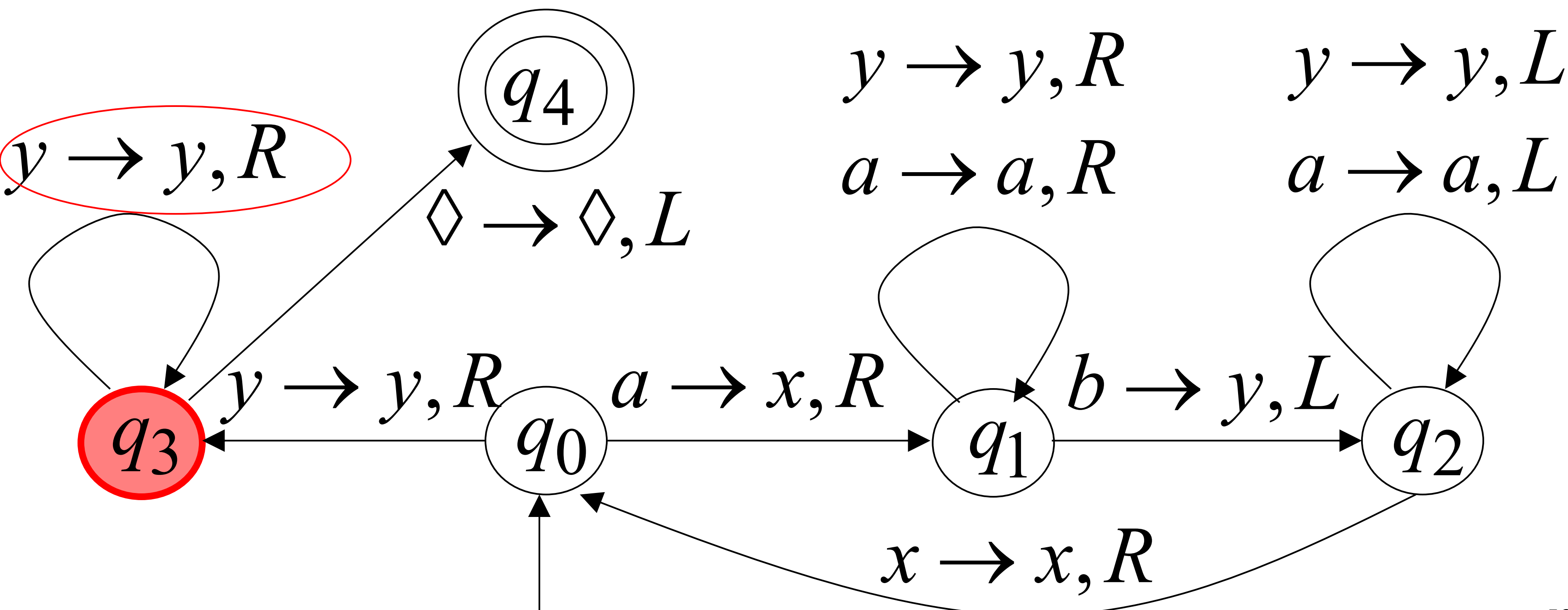


q_4
$y \rightarrow y, R$
$y \rightarrow y, L$
$a \rightarrow a, R$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 11

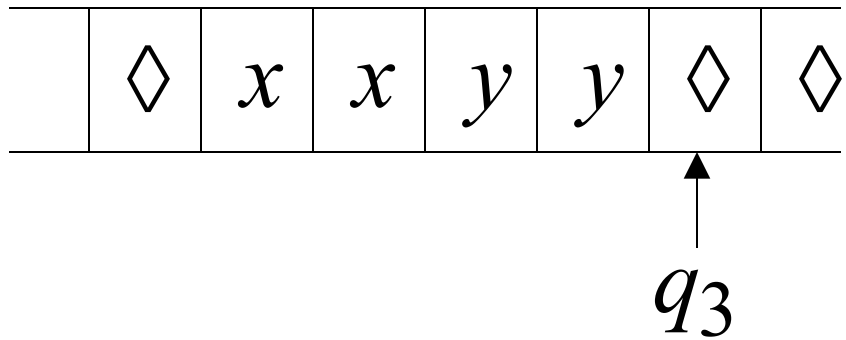


$\Diamond$ x x y y $\Diamond$ $\Diamond$
q_3

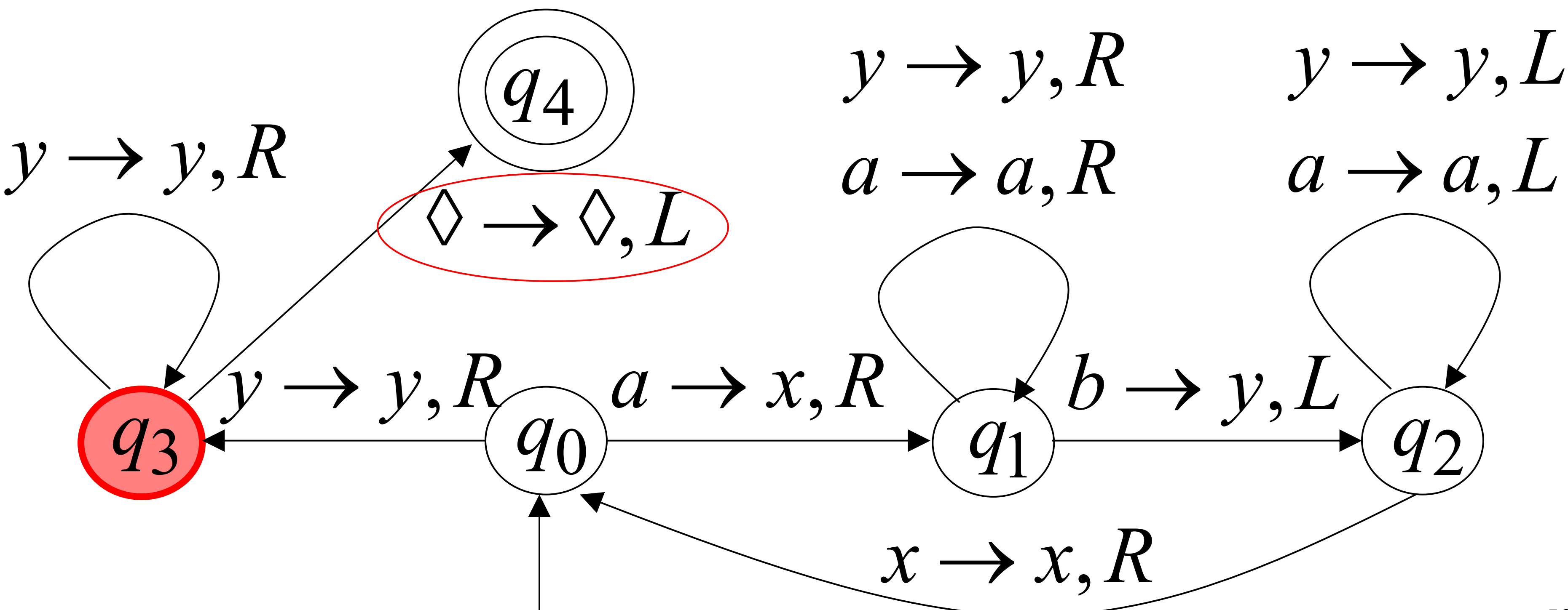


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 12

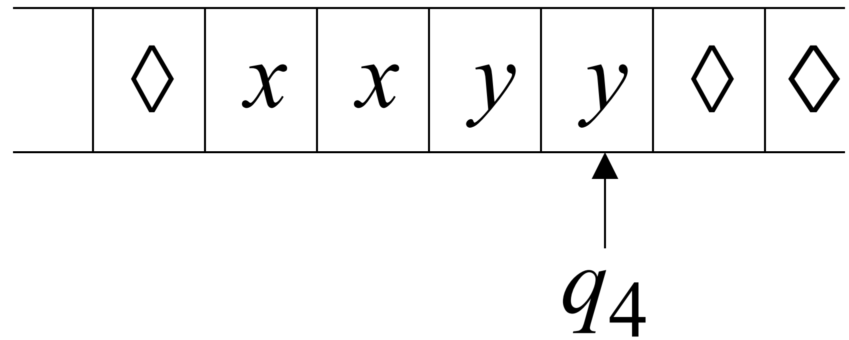


◊ x x y y ◊ ◊
q_3

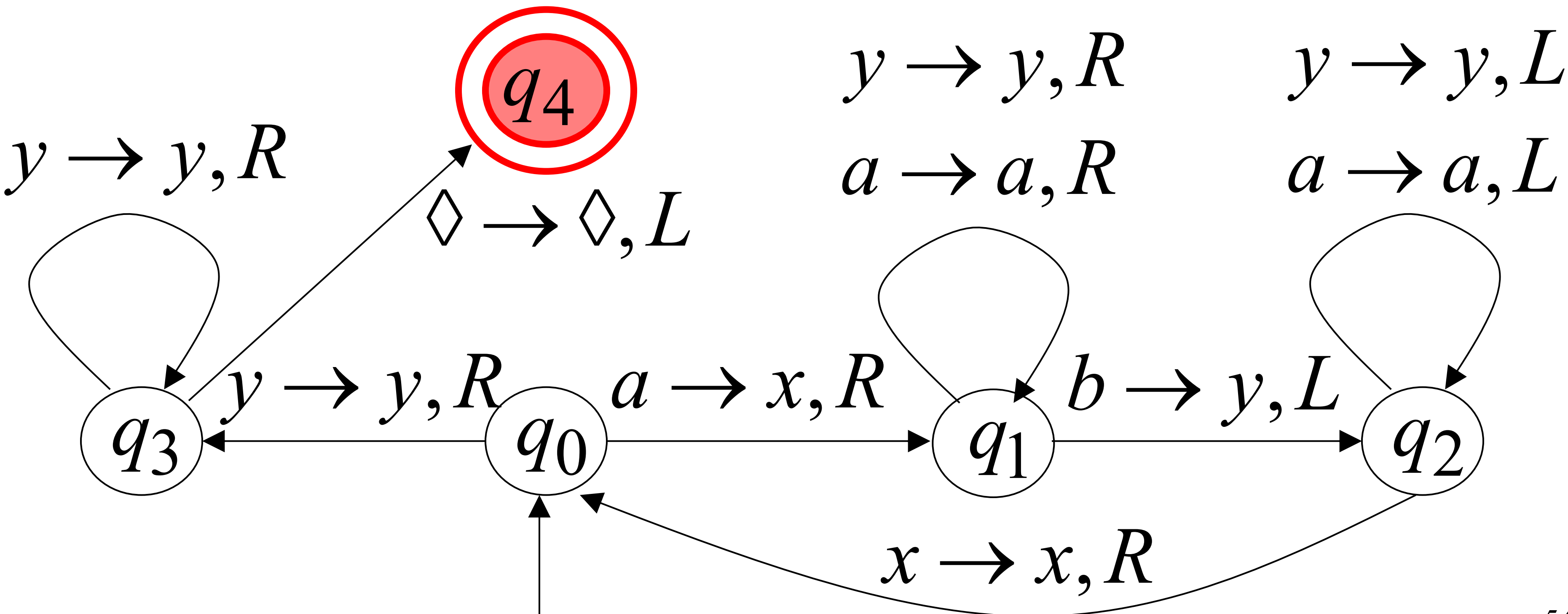


q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 13



Halt & Accept



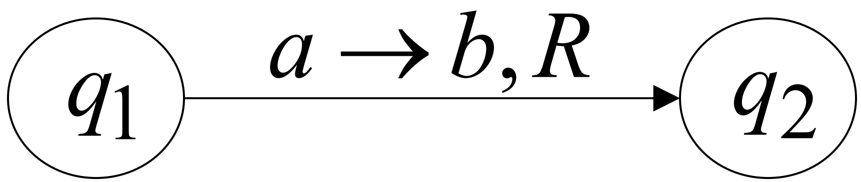
Observation:

If we modify the
machine for the language $\{a^n b^n\}$

we can easily construct
a machine for the language $\{a^n b^n c^n\}$

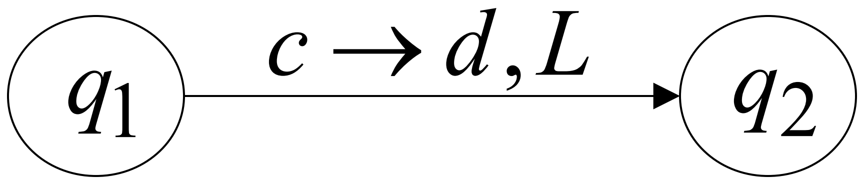
Formal Definitions for Turing Machines

Transition Function



$$\delta(q_1, a) = (q_2, b, R)$$

Transition Function



$$\delta(q_1, c) = (q_2, d, L)$$

Turing Machine:

States

Input alphabet

Tape alphabet

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

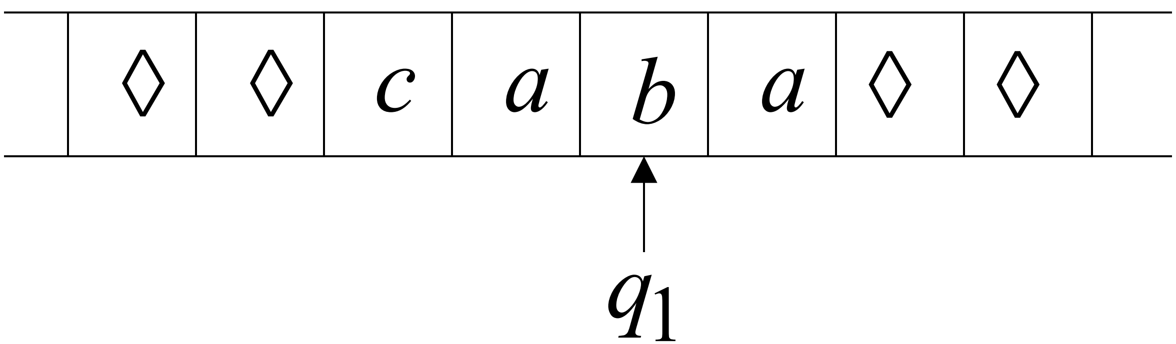
Transition function

Initial state

blank

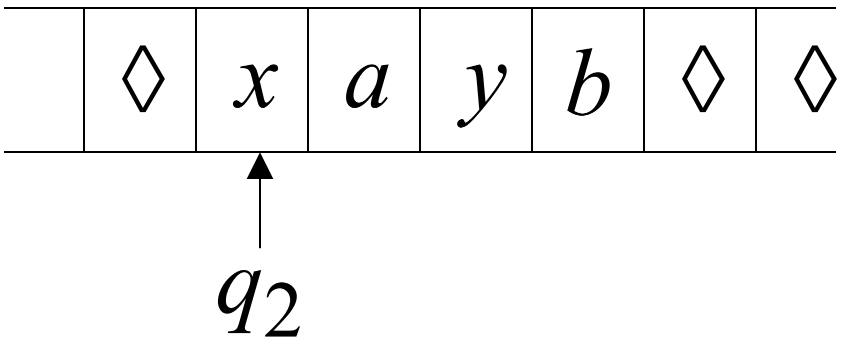
Accept states

Configuration

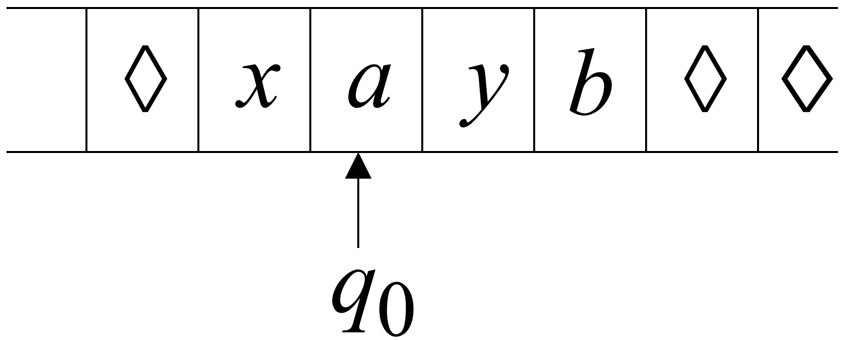


Instantaneous description: $ca\ q_1\ ba$

Time 4



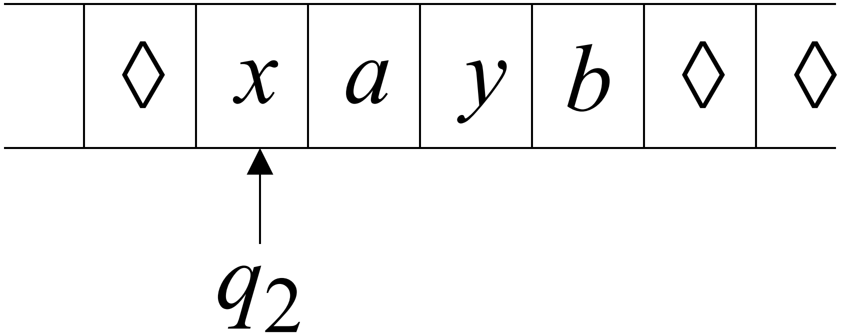
Time 5



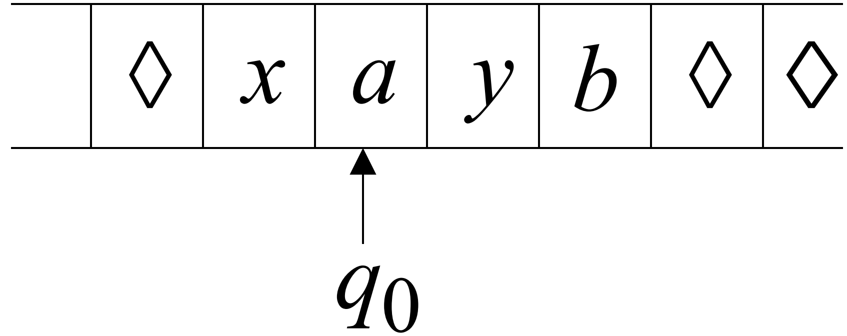
A Move: $q_2\ xayb \succ x\ q_0\ ayb$

(yields in one move)

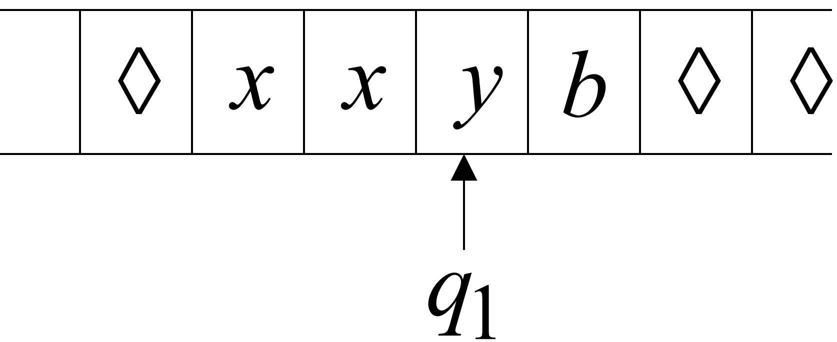
Time 4



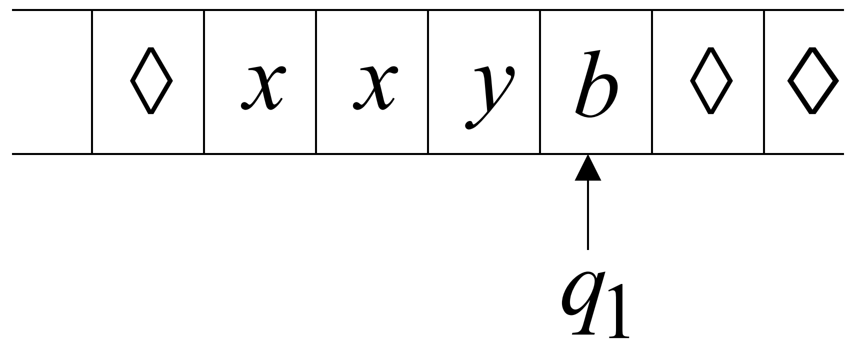
Time 5



Time 6



Time 7



A computation

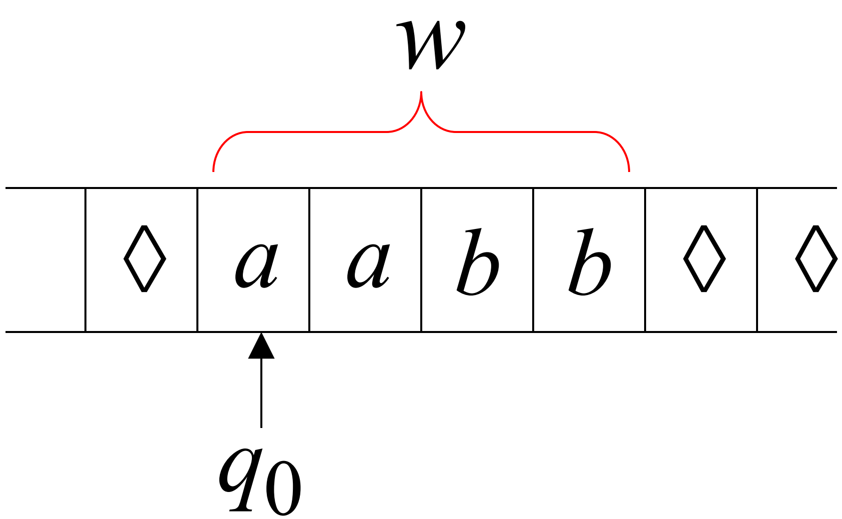
$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

$$q_2\ xayb \succ x\ q_0\ ayb \succ xx\ q_1\ yb \succ xxy\ q_1\ b$$

Equivalent notation: $q_2\ xayb \overset{*}{\succ} xxy\ q_1\ b$

Initial configuration: $q_0 \, w$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{w : \quad q_0\, w \overset{*}{\succ} x_1\, q_f\, x_2\}$$

Initial state

Accept state

If a language L is accepted by a Turing machine M then we say that L is:

- Turing Recognizable

Other names used:

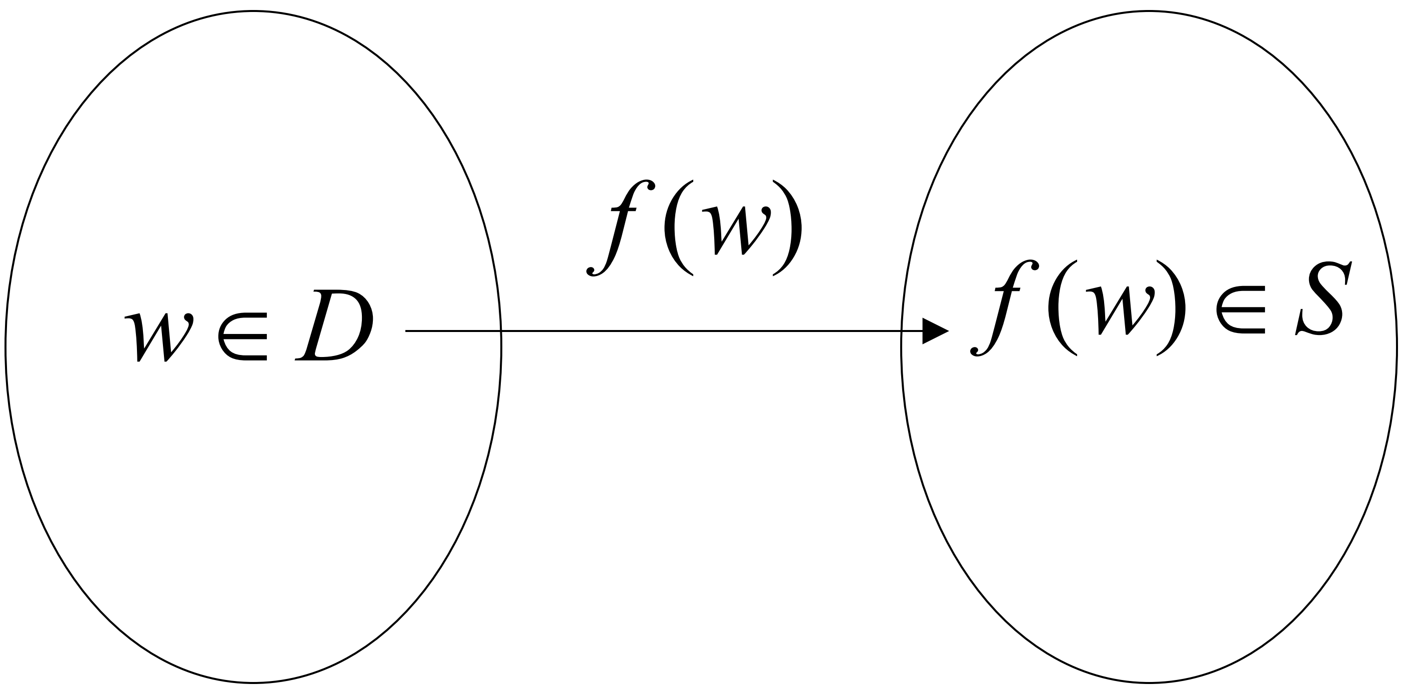
- Turing Acceptable
- Recursively Enumerable

Computing Functions with Turing Machines

A function $f(w)$ has:

Domain: D

Result Region: S



A function may have many parameters:

Example: Addition function

$$f(x, y) = x + y$$

Integer Domain

Decimal: 5

Binary: 101

Unary: 11111

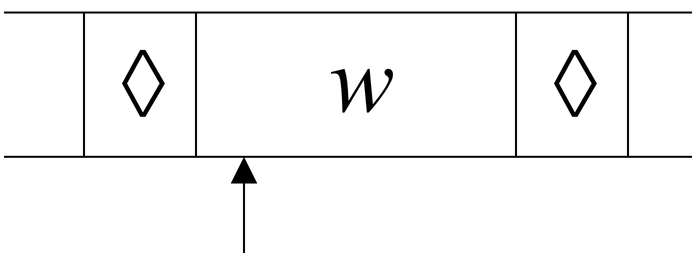
We prefer **unary** representation:

easier to manipulate with Turing machines

Definition:

A function f is computable if
there is a Turing Machine M such that:

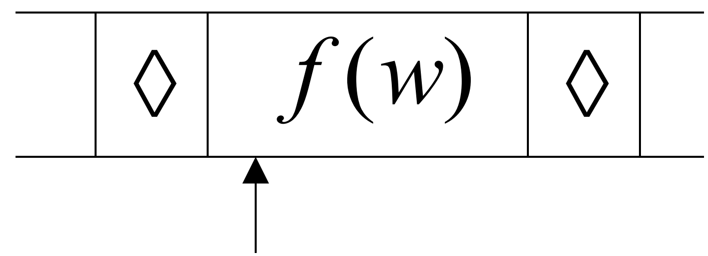
Initial configuration



q_0

initial state

Final configuration



q_f

accept state

For all $w \in D$ Domain

In other words:

A function f is computable if
there is a Turing Machine M such that:

$$q_0\, w \overset{*}{\succ} q_f\, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

Example

The function $f(x, y) = x + y$ is computable

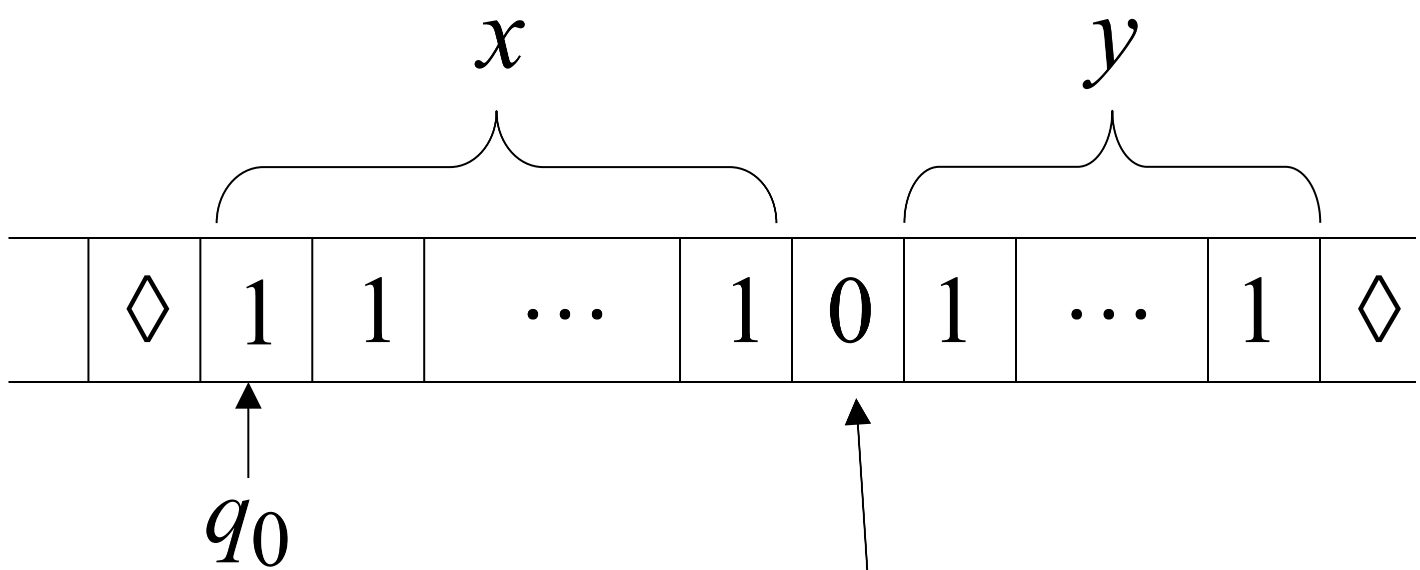
x, y are integers

Turing Machine:

Input string: $x0y$ unary

Output string: $xy0$ unary

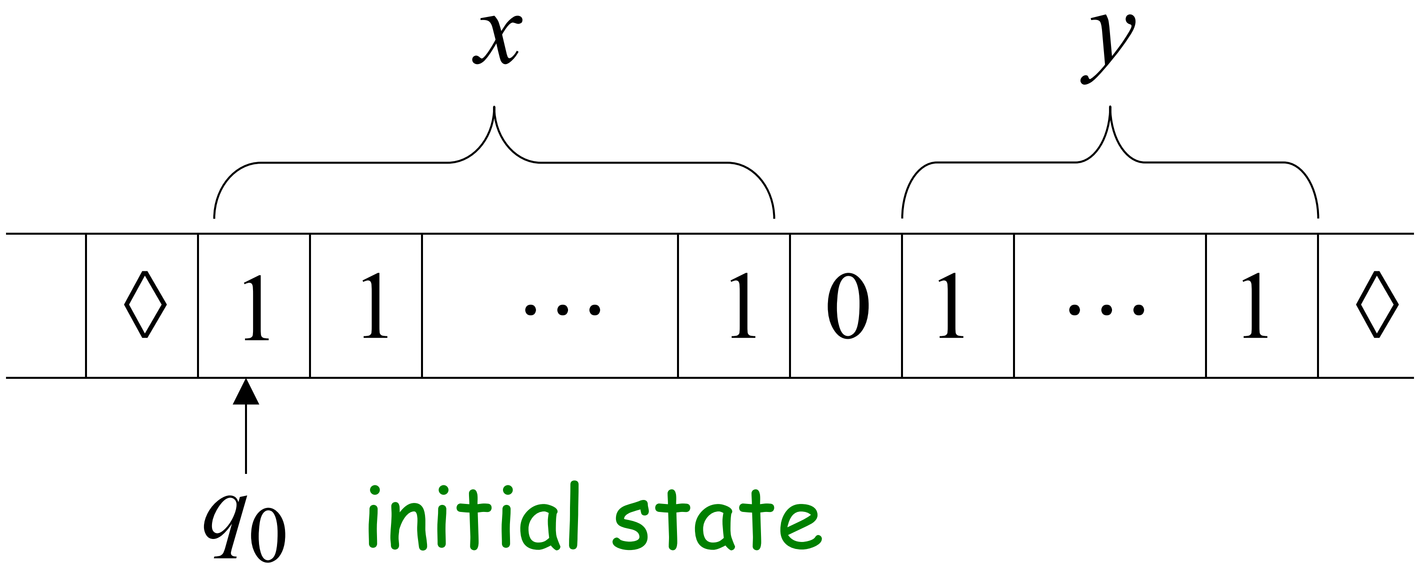
Start



initial state

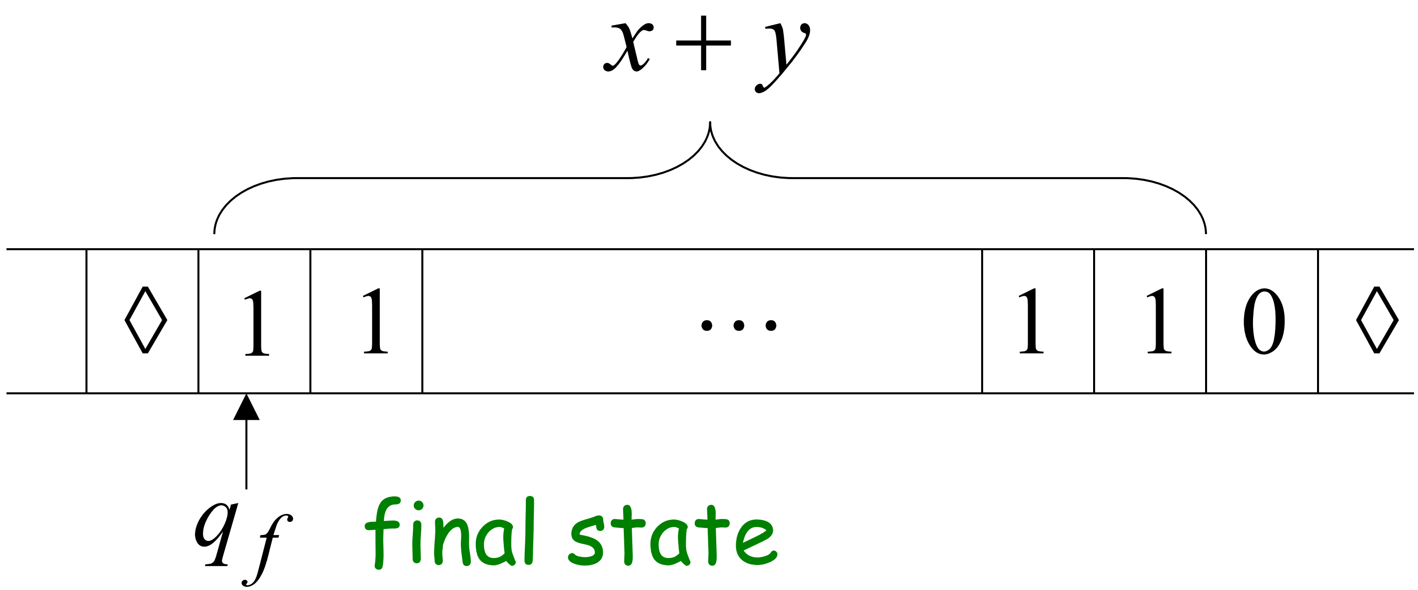
The 0 is the delimiter that separates the two numbers

Start



initial state

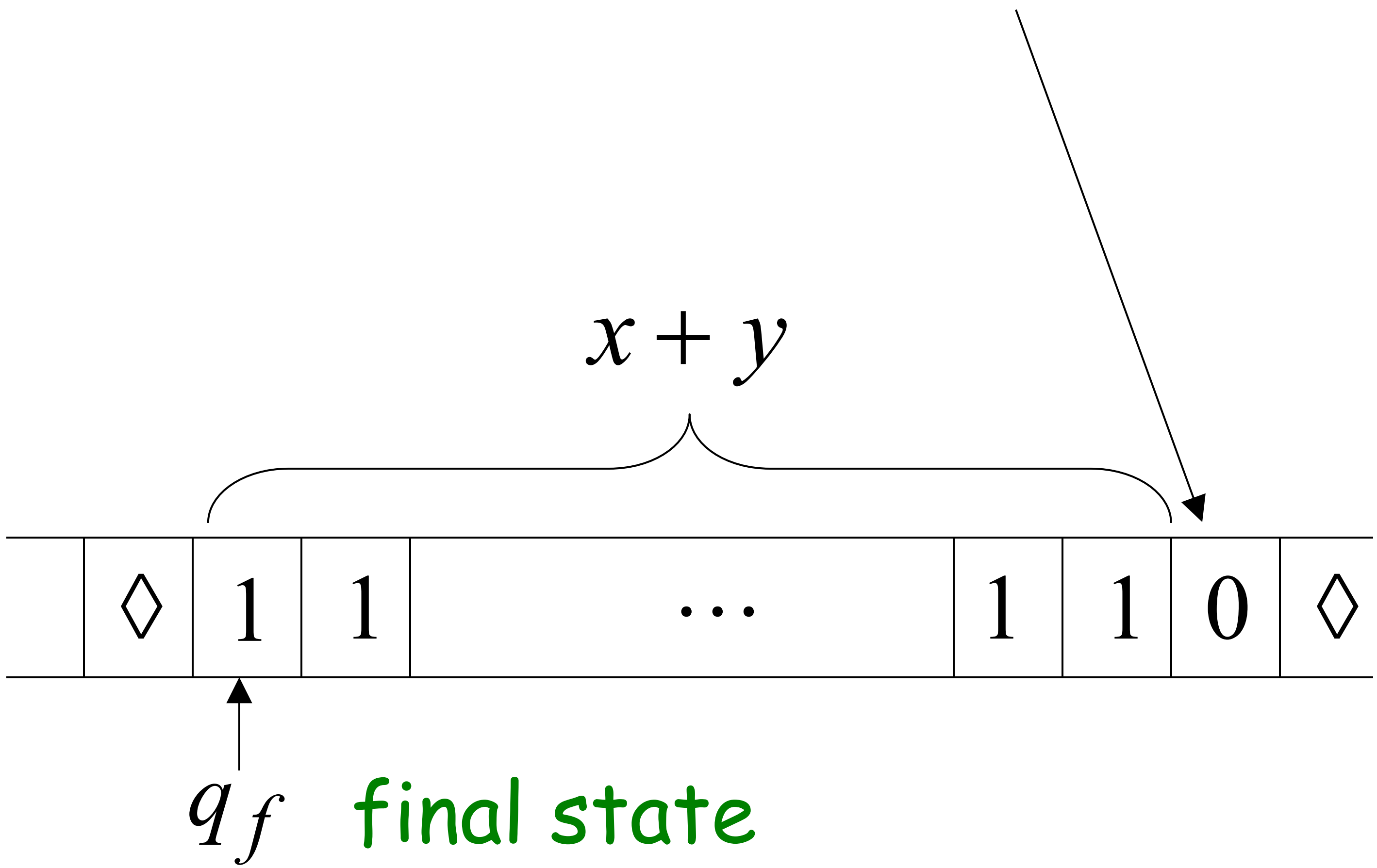
Finish



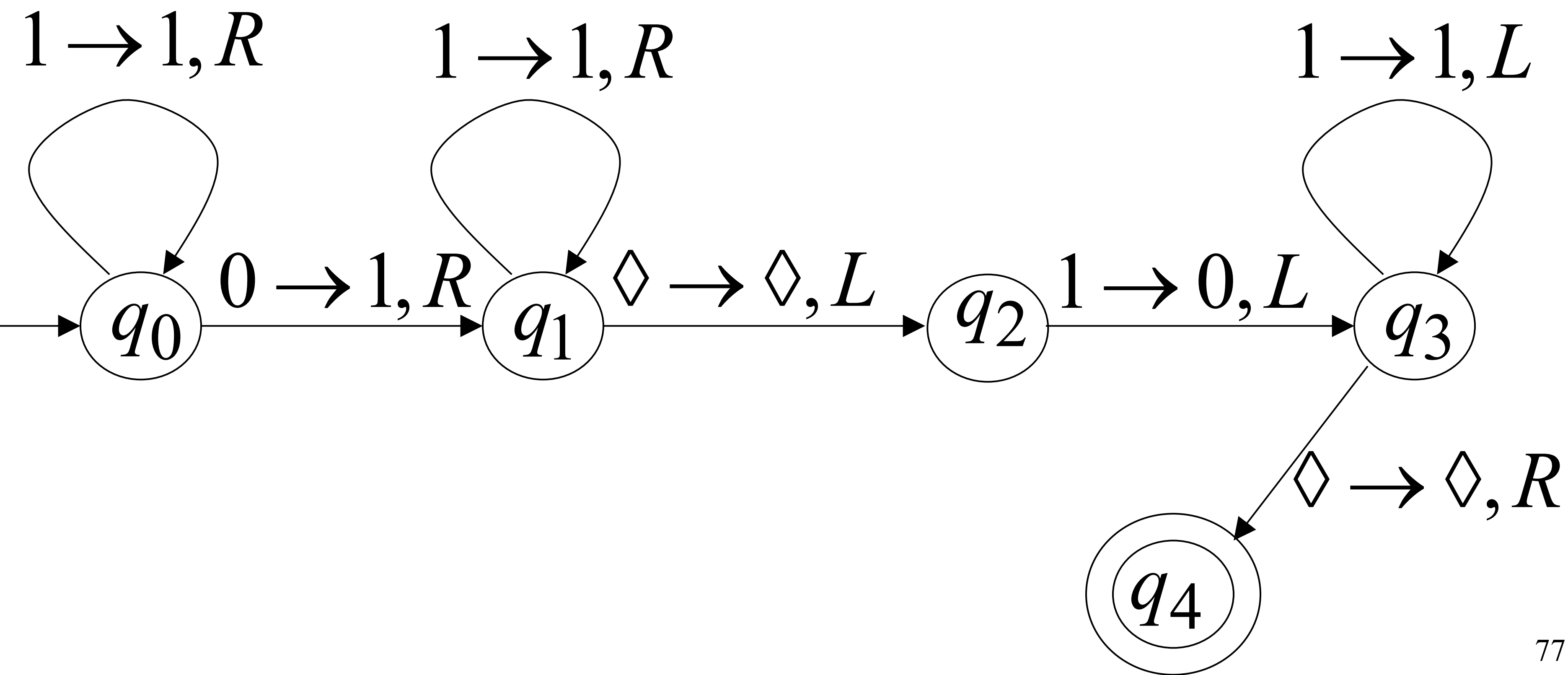
final state

The 0 here helps when we use the result for other operations

Finish



Turing machine for function $f(x, y) = x + y$

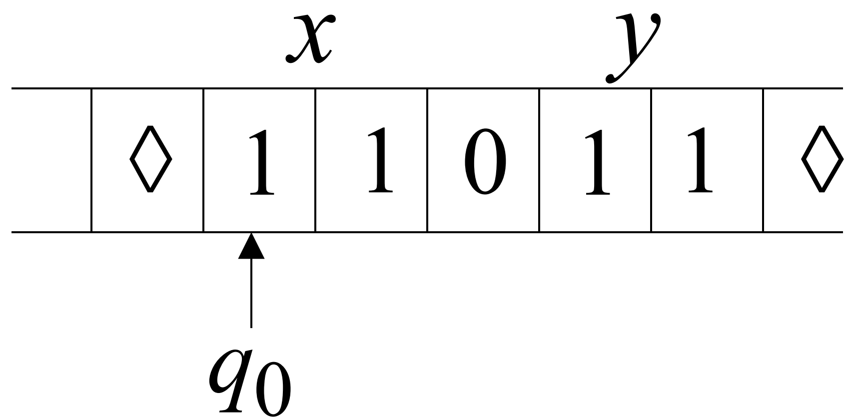


Execution Example:

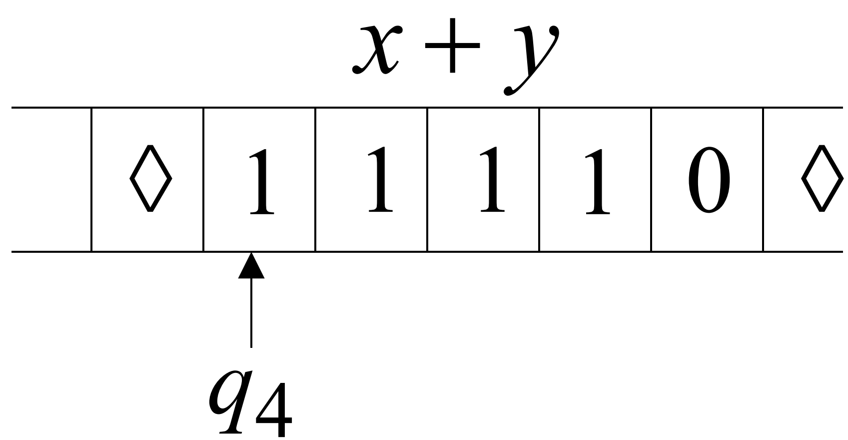
$x = 11$ (=2)

$y = 11$ (=2)

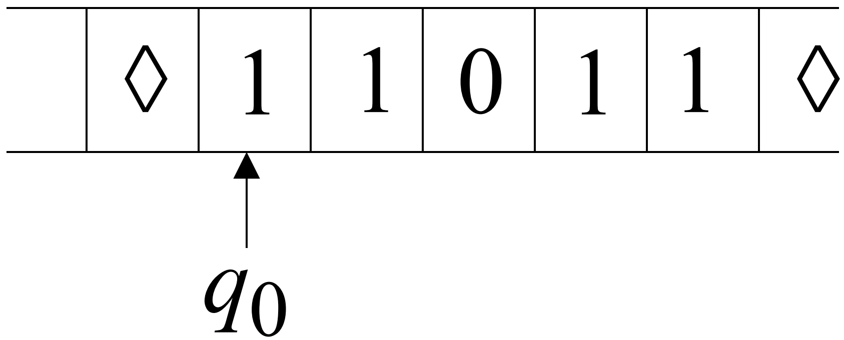
Time 0

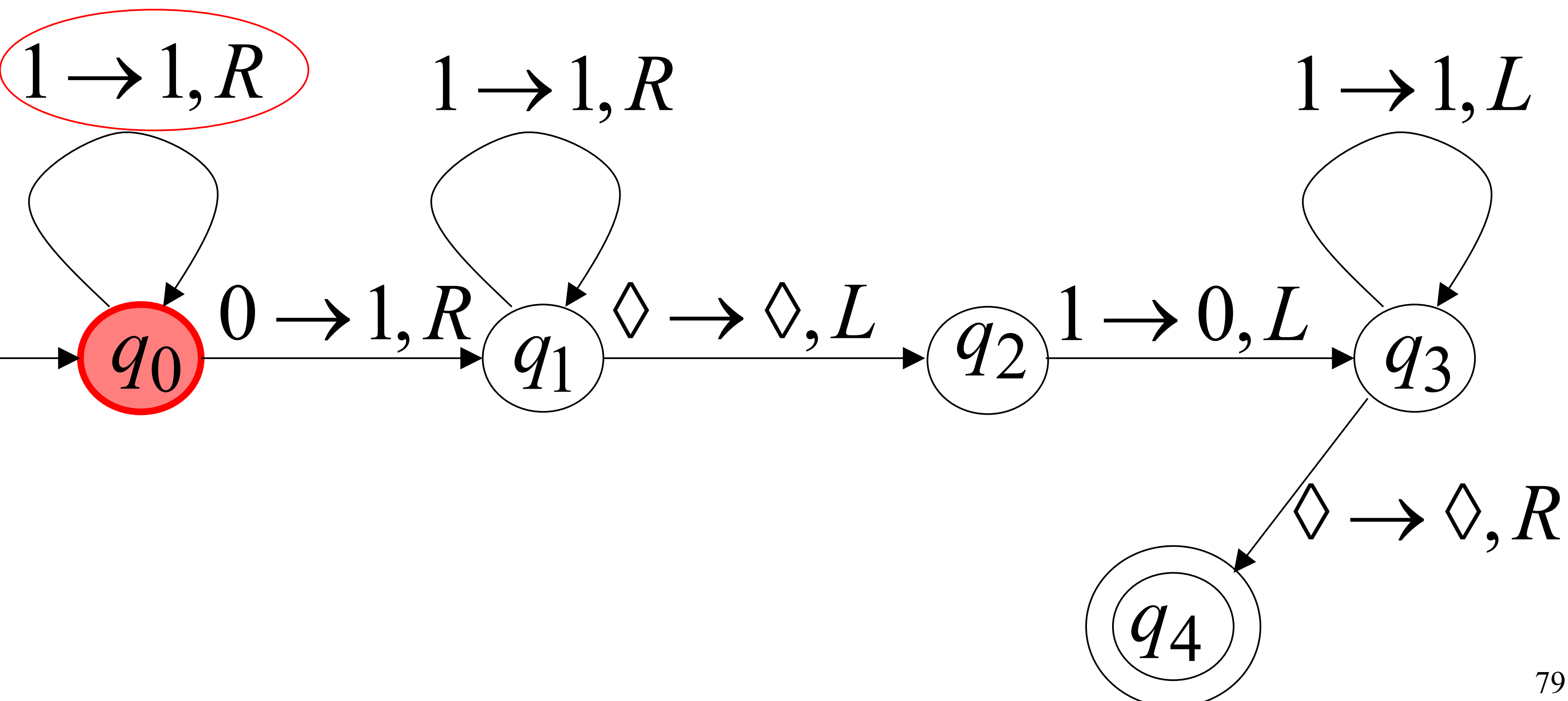


Final Result

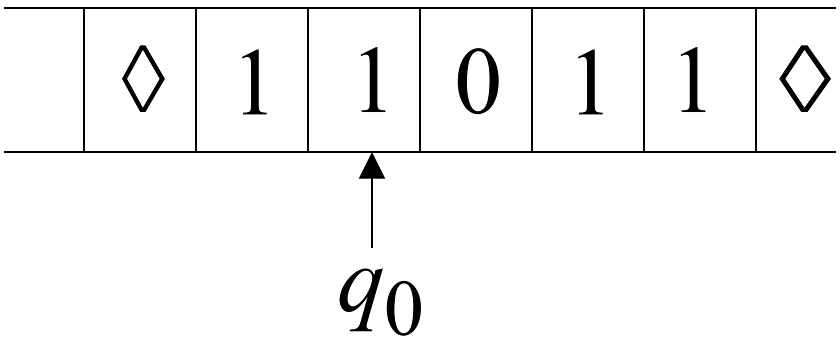


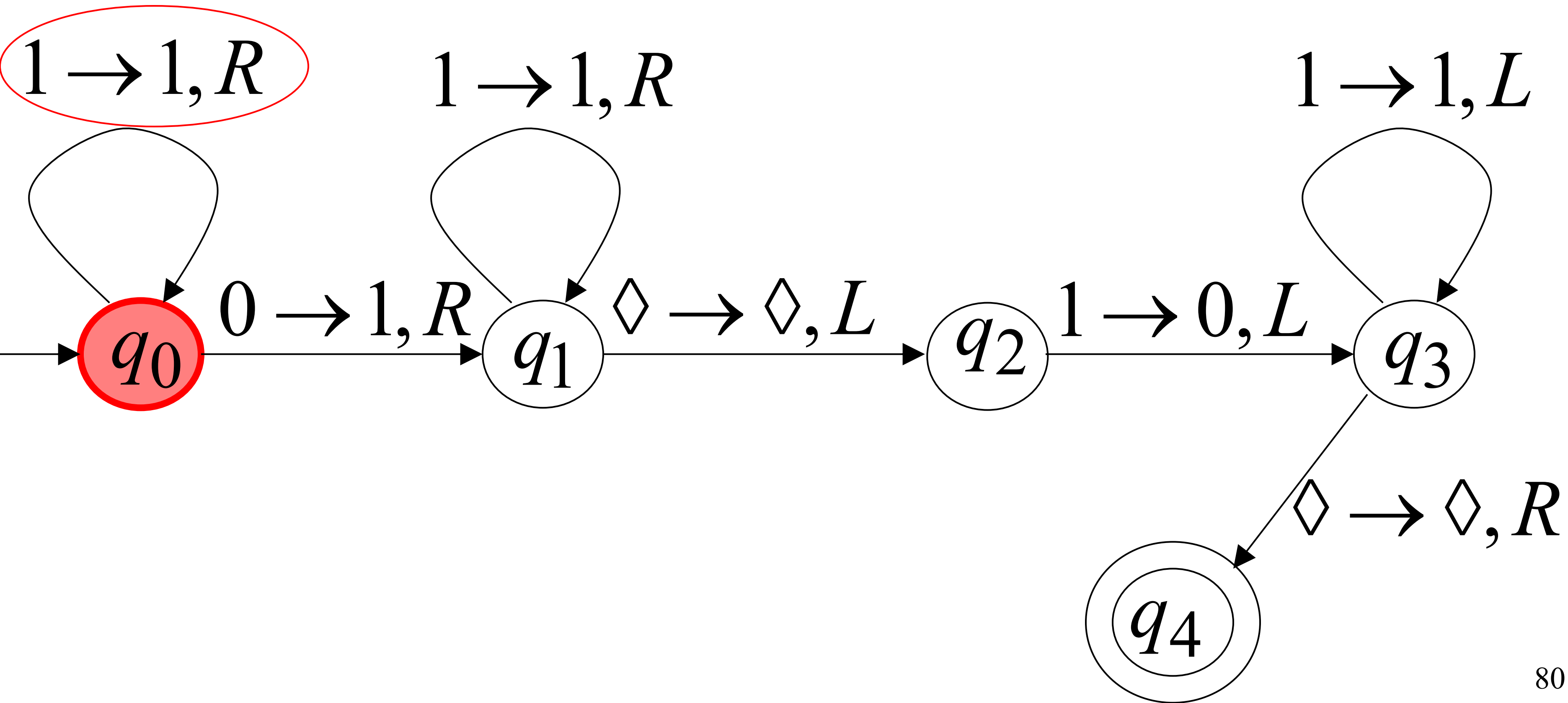
Time 0


◊ 1 1 0 1 1 ◊
q_0

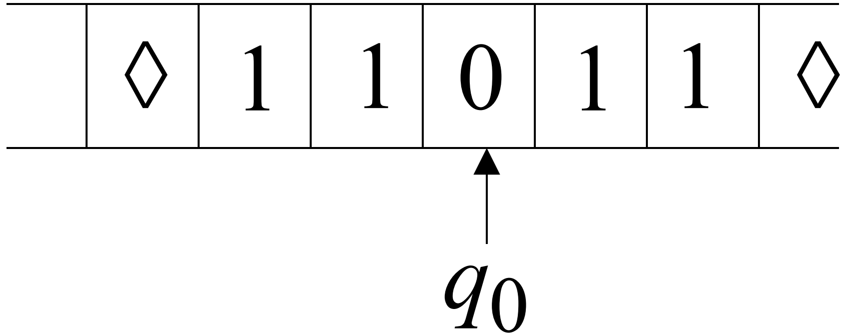

$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

Time 1

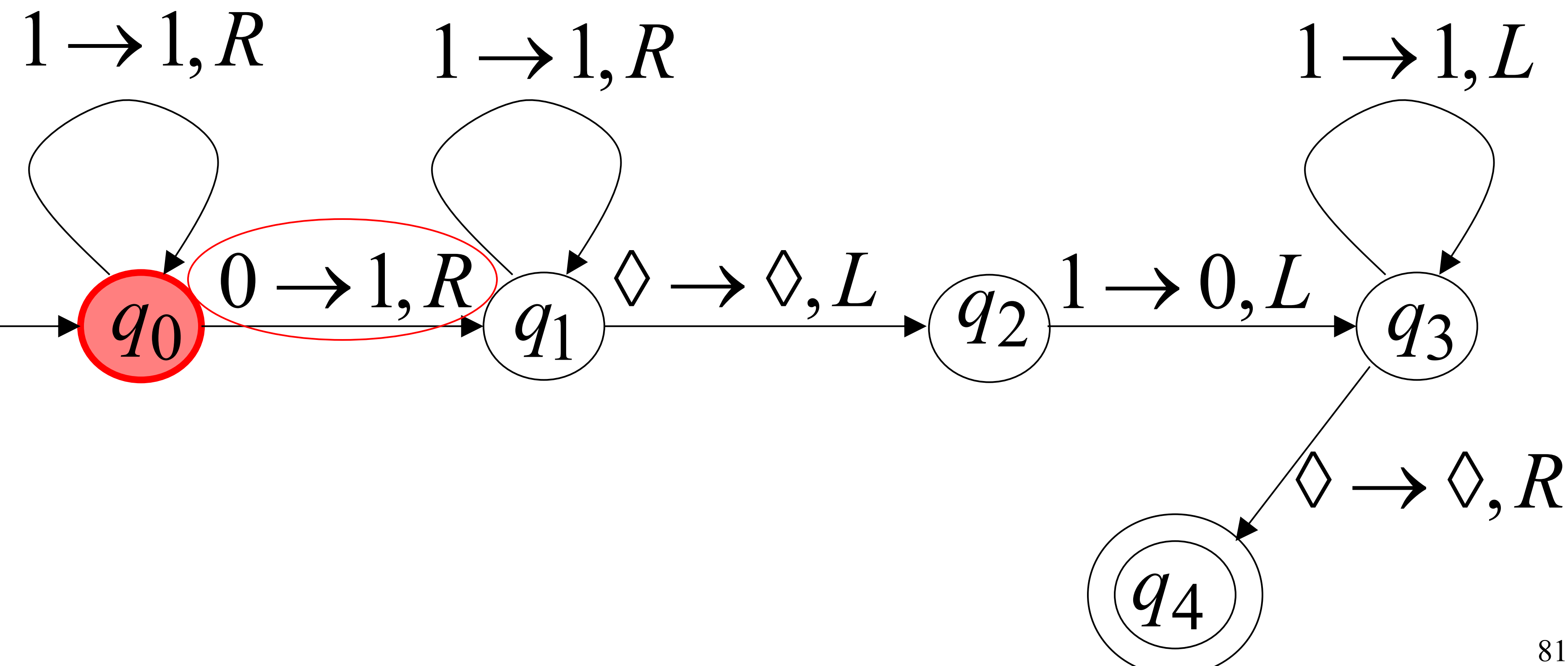

◊
1
1
0
1
1
◊
q_0


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\lozenge \rightarrow \lozenge, L$
$1 \rightarrow 0, L$
$\lozenge \rightarrow \lozenge, R$
q_0
q_1
q_2
q_3
q_4

Time 2

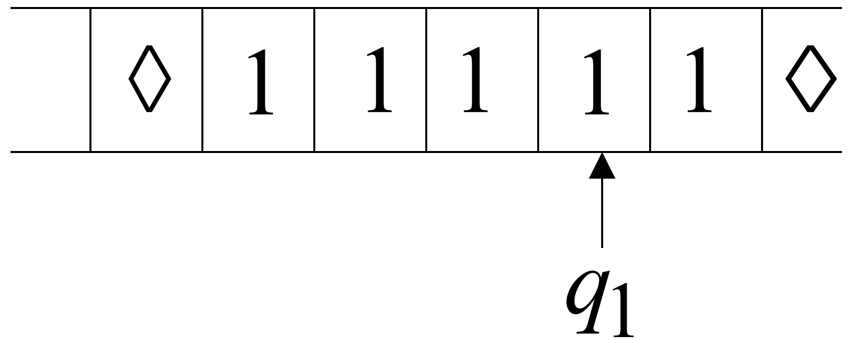


◊ 1 1 0 1 1 ◊
q_0

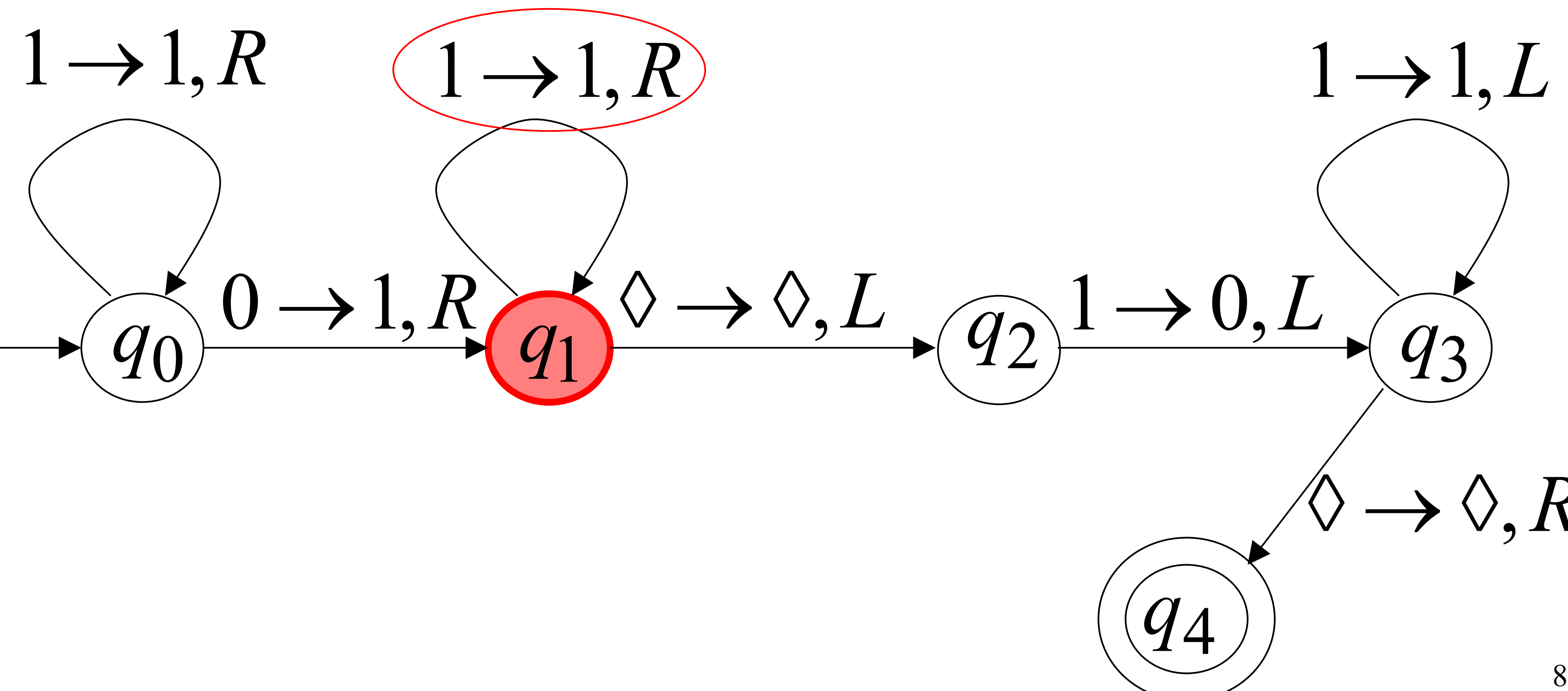


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 3

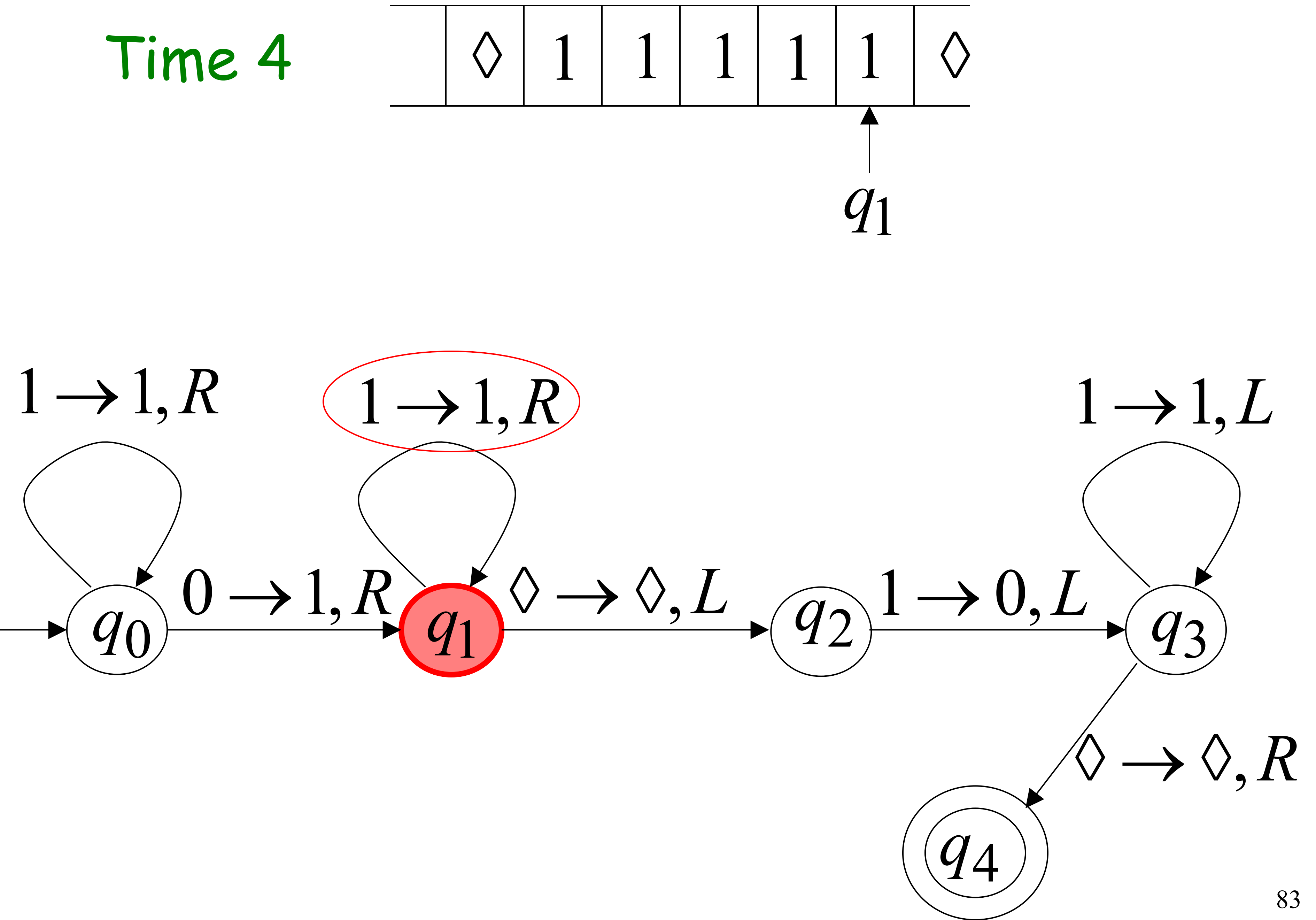


◊ 1 1 1 1 1 ◊
q_1

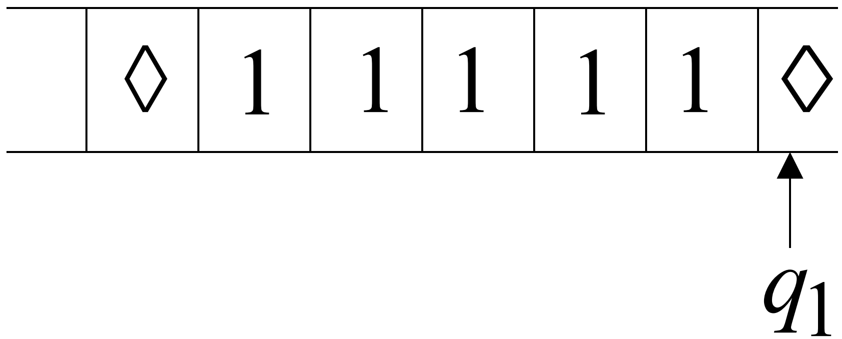


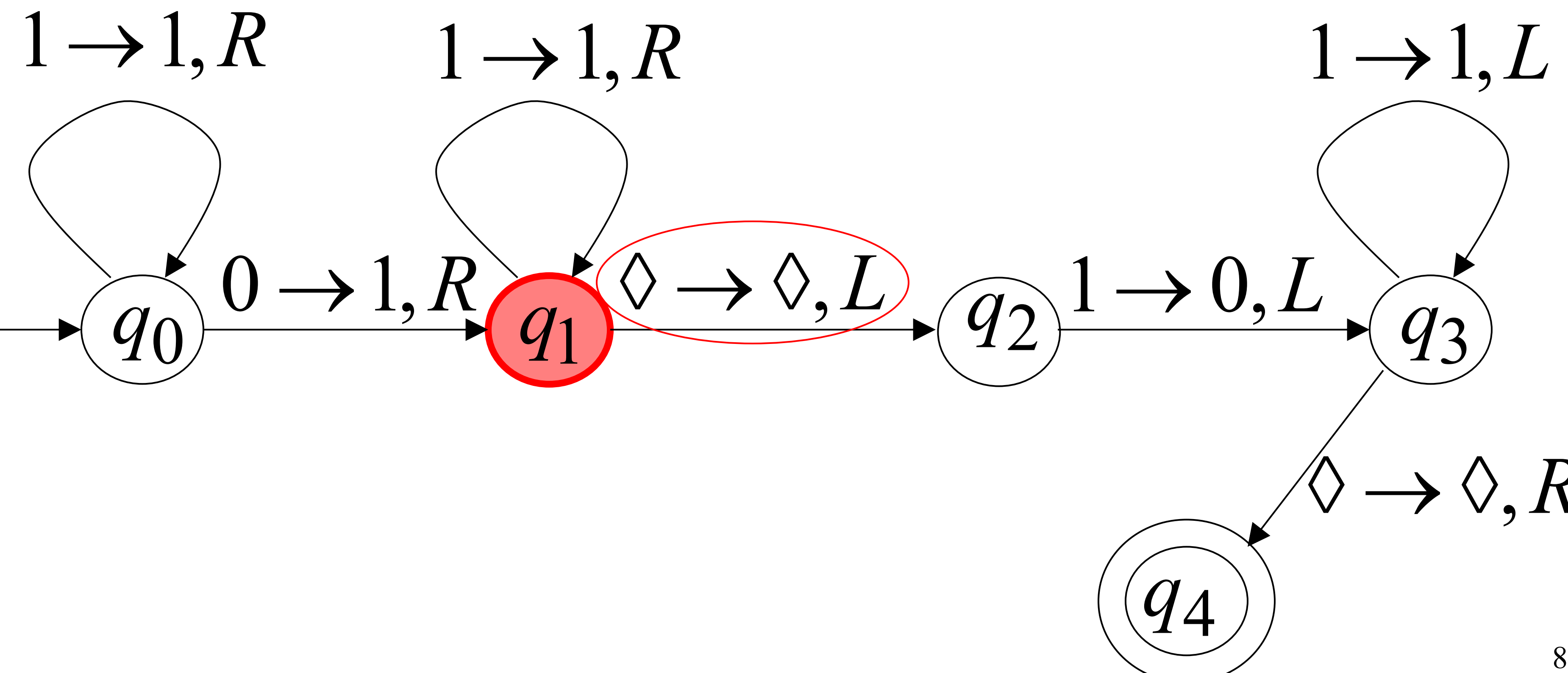
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 4

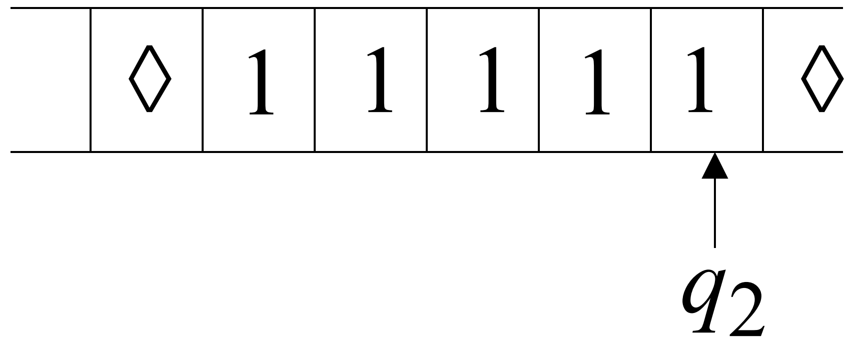


Time 5

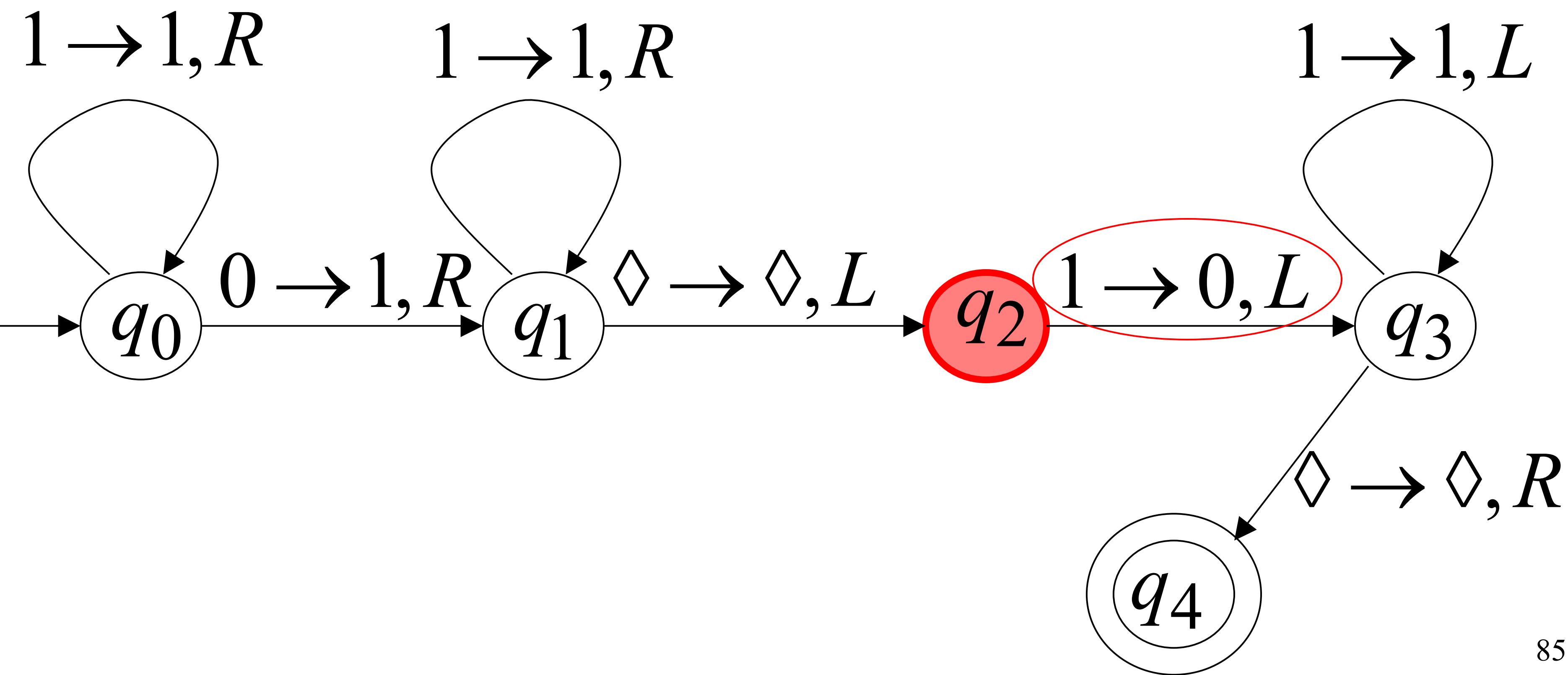

◊
1
1
1
1
1
◊
q_1


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 6

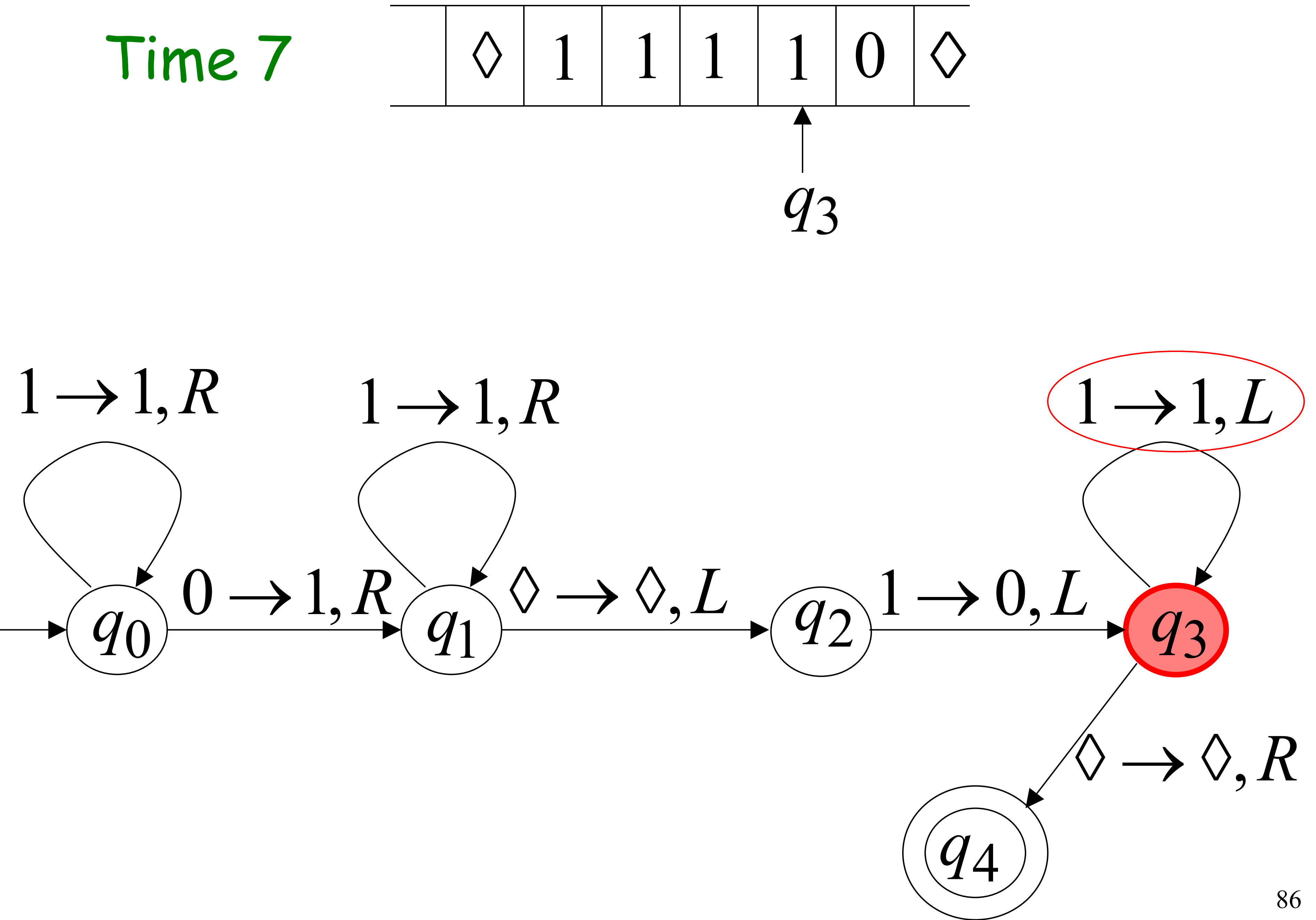


◊ 1 1 1 1 1 ◊
q_2

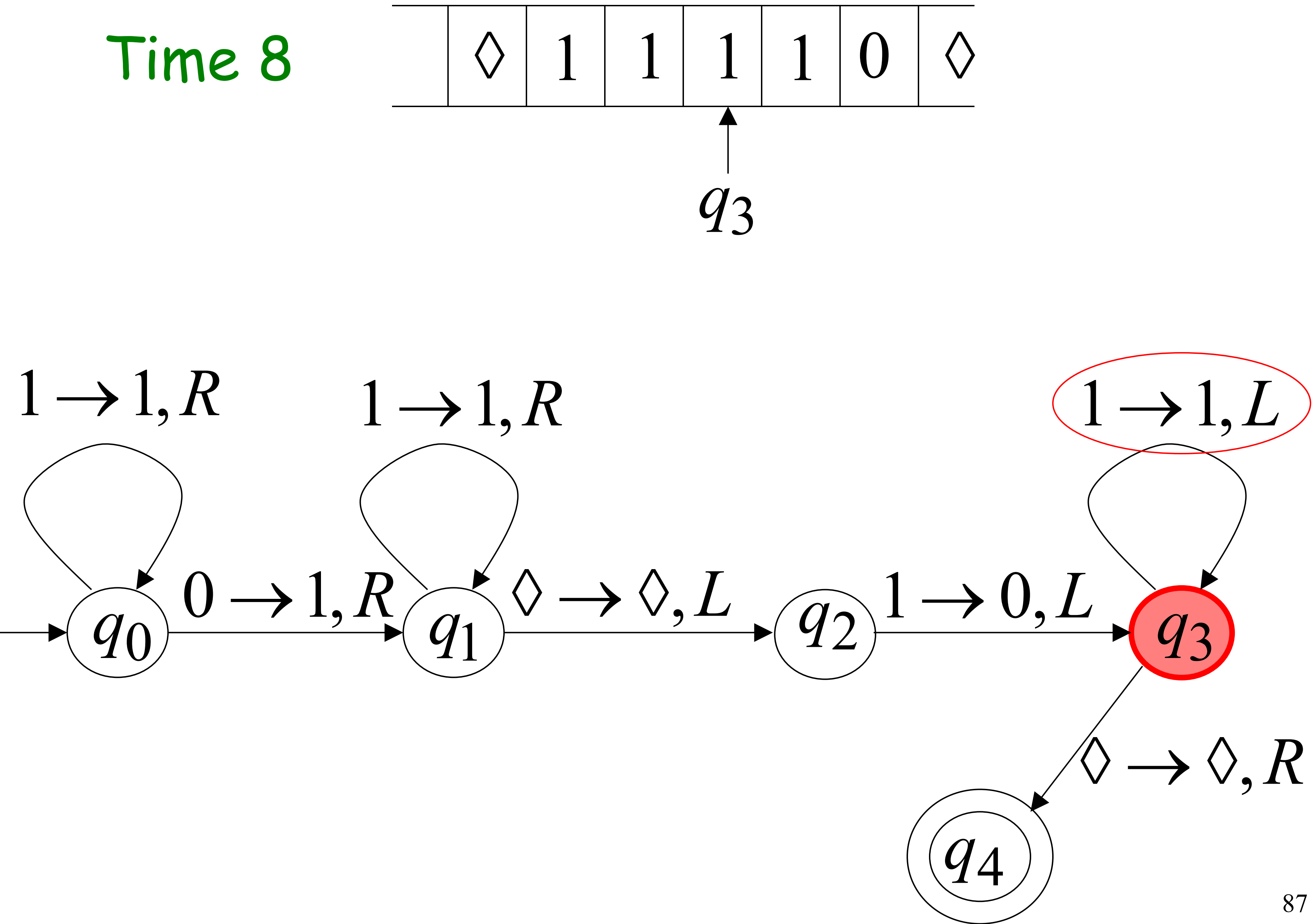


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
q_4

Time 7

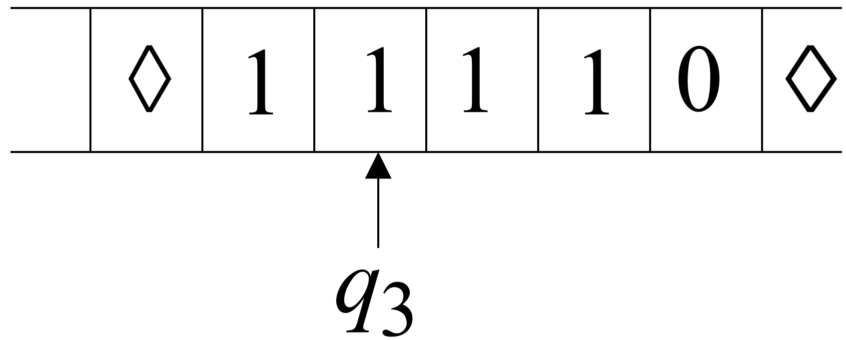

◊
1
1
1
1
0
◊
q_3
$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\lozenge \to \lozenge, L$
$1 \to 0, L$
$\lozenge \to \lozenge, R$
q_0
q_1
q_2
q_3
q_4

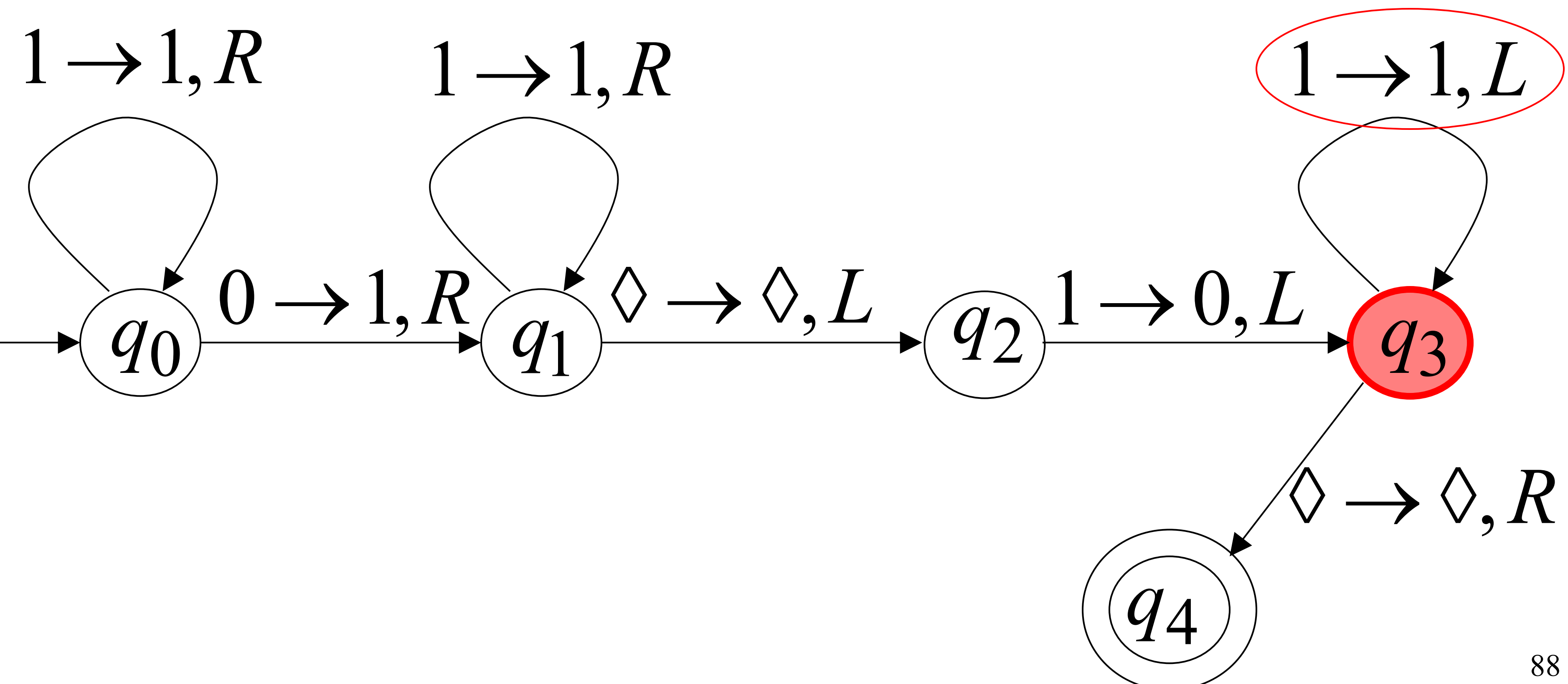
Time 8



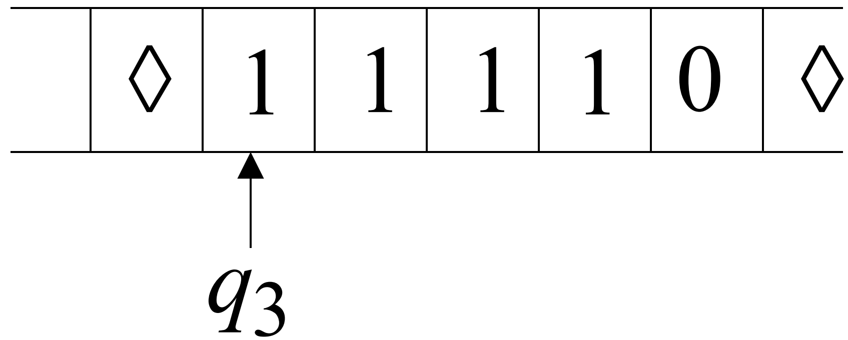
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 9

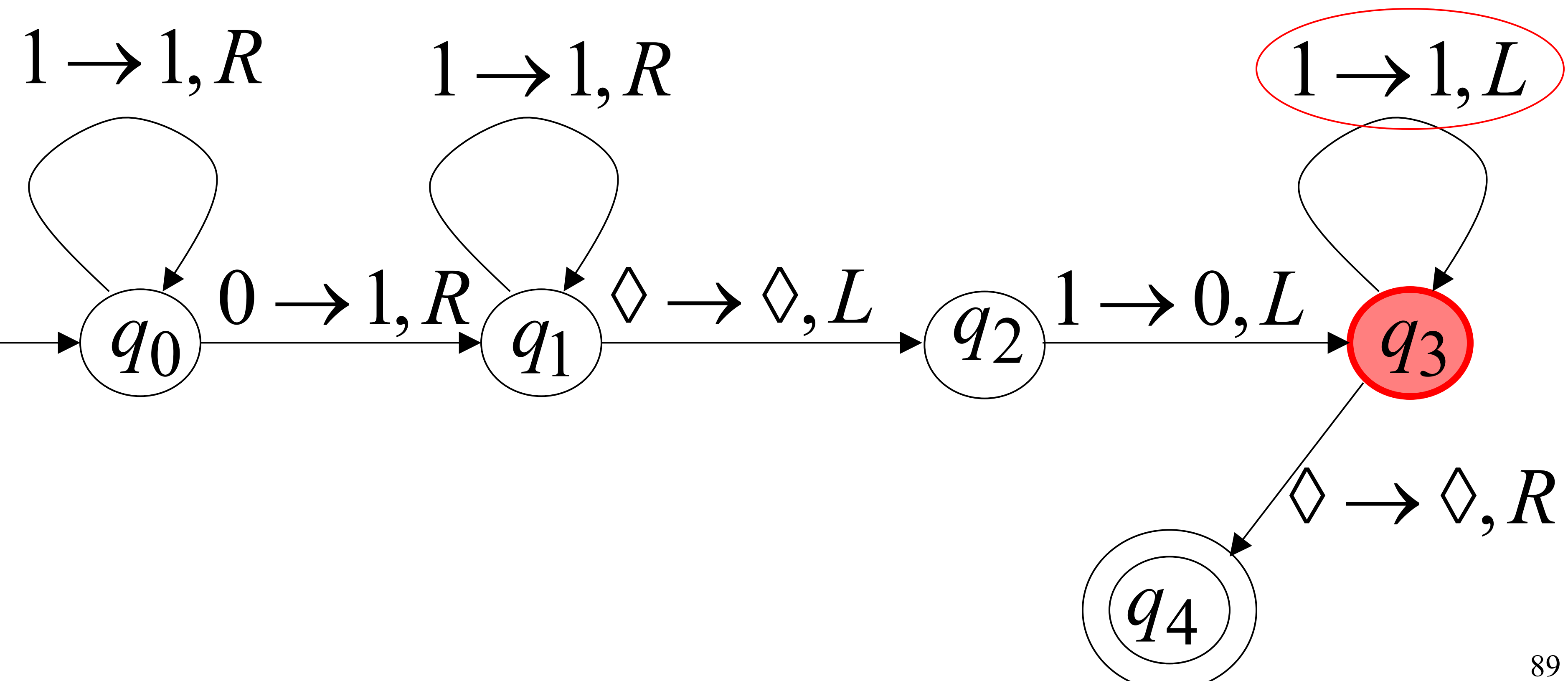

◊ 1 1 1 1 0 ◊
q_3


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\lozenge \rightarrow \lozenge, L$
q_2
$1 \rightarrow 0, L$
q_3
$\lozenge \rightarrow \lozenge, R$
q_4

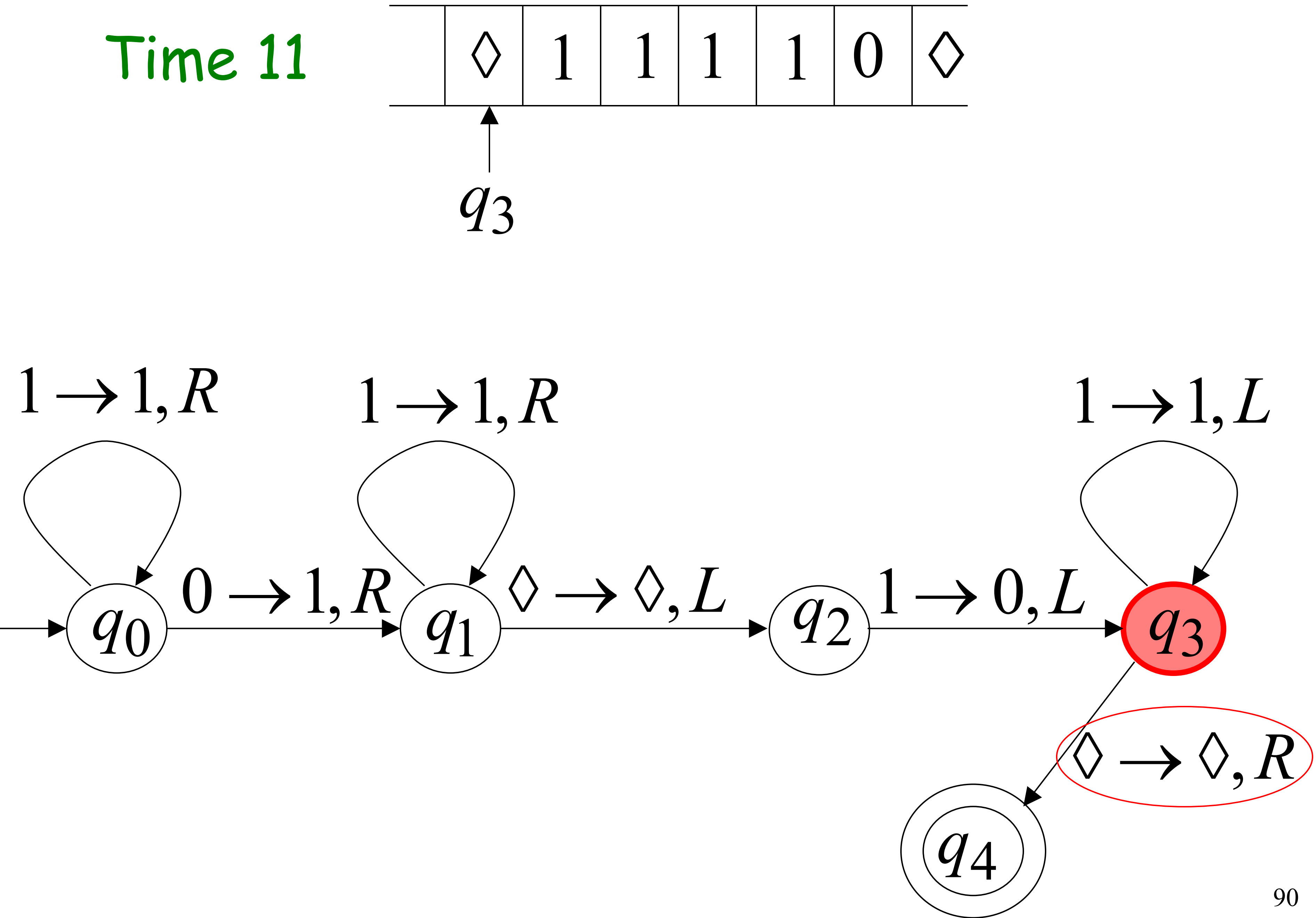
Time 10



◊
1
1
1
1
0
◊
q_3

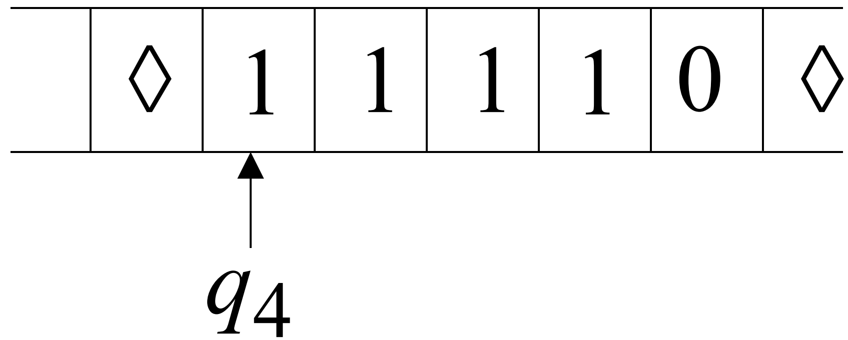


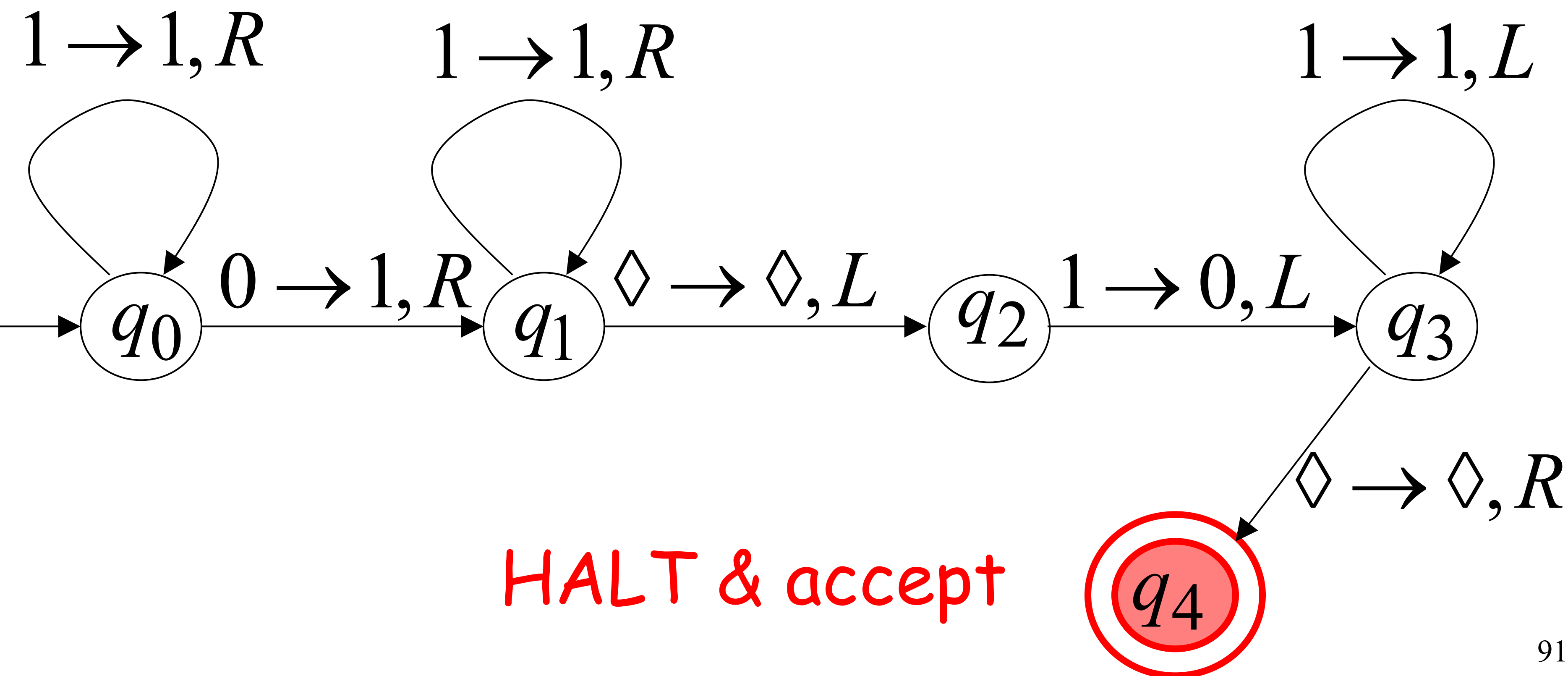
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$\Diamond \rightarrow \Diamond, L$
$1 \rightarrow 0, L$
$\Diamond \rightarrow \Diamond, R$
q_0
q_1
q_2
q_3
q_4



Time 11
◊ 1 1 1 1 0 ◊
q_3
$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
$0 \rightarrow 1, R$
$◊ \rightarrow ◊, L$
$1 \rightarrow 0, L$
$◊ \rightarrow ◊, R$
q_0
q_1
q_2
q_3
q_4

Time 12


◊ 1 1 1 1 0 ◊
q_4


$1 \rightarrow 1, R$
$1 \rightarrow 1, R$
$1 \rightarrow 1, L$
q_0
$0 \rightarrow 1, R$
q_1
$\Diamond \rightarrow \Diamond, L$
q_2
$1 \rightarrow 0, L$
q_3
$\Diamond \rightarrow \Diamond, R$
HALT & accept
q_4

Another Example

The function $f(x) = 2x$ is computable

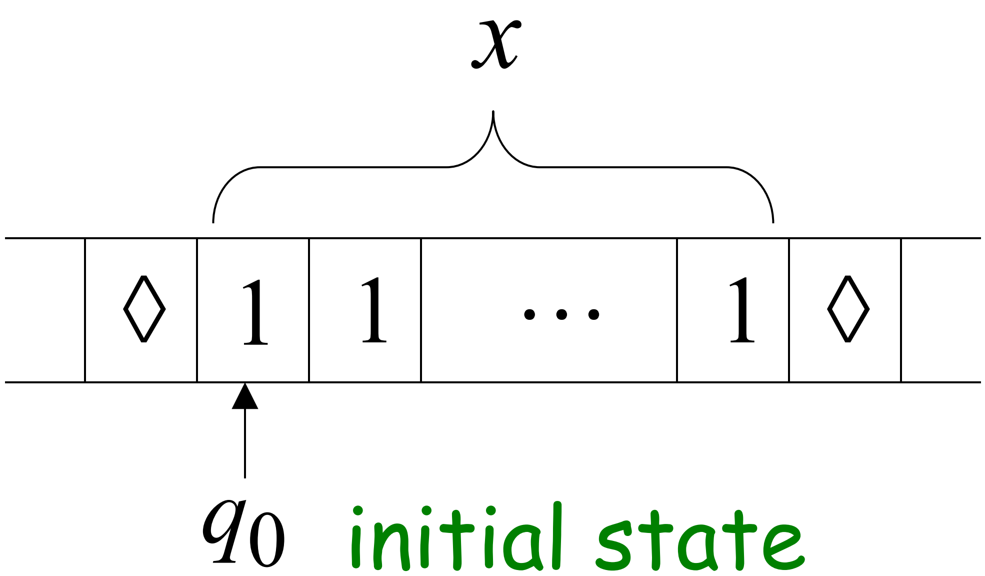
x is integer

Turing Machine:

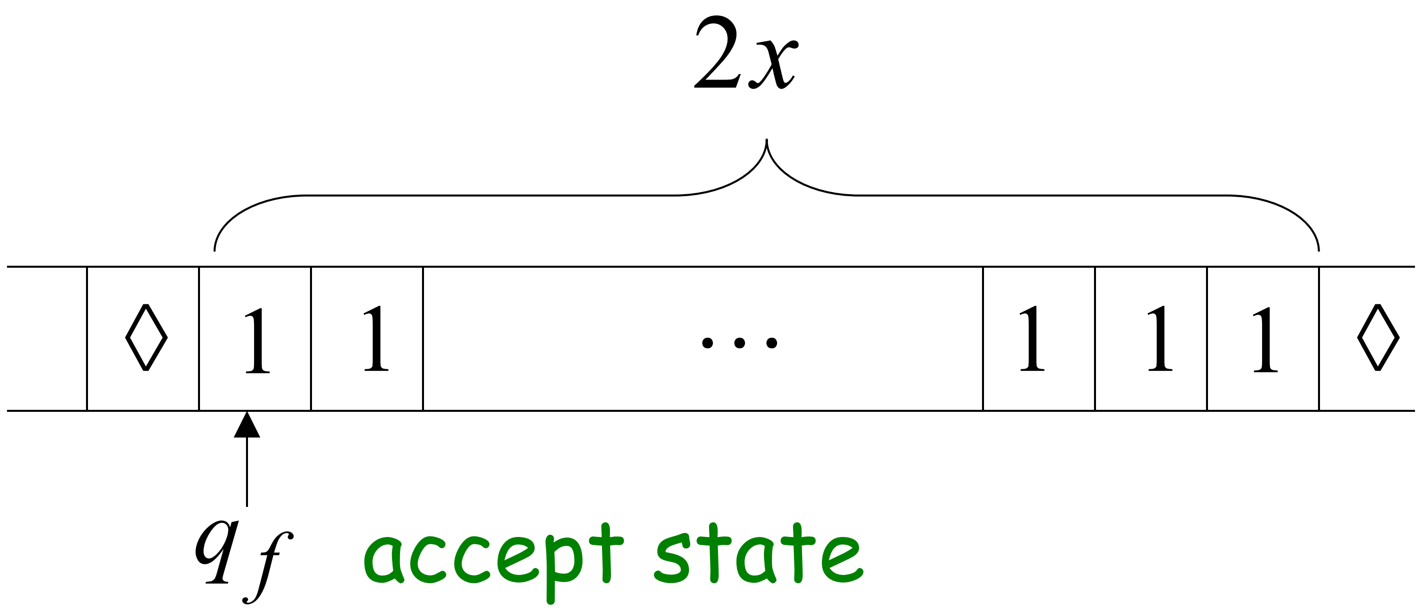
Input string: x unary

Output string: xx unary

Start



Finish

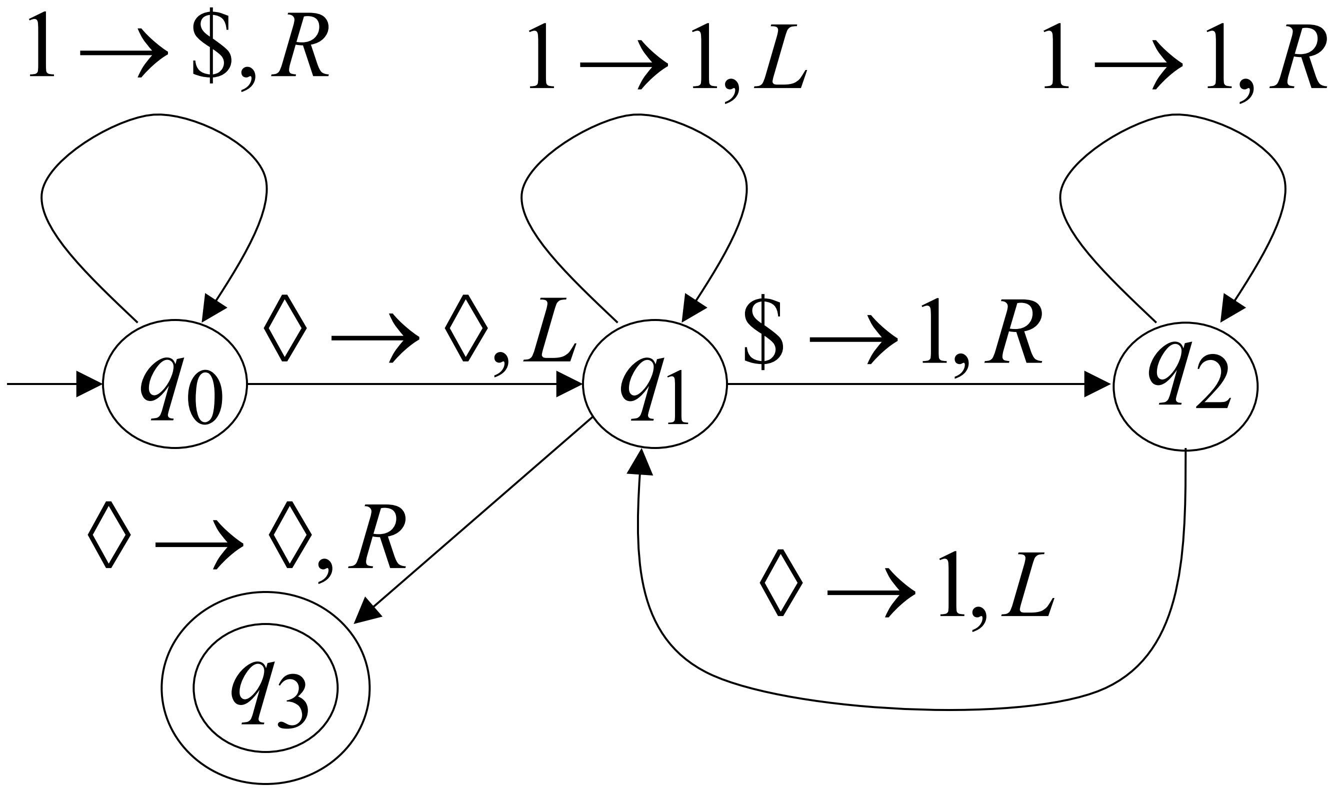


Turing Machine Pseudocode for $f(x) = 2x$

- Replace every 1 with \$
- Repeat:
  - Find rightmost \$, replace it with 1
  - Go to right end, insert 1

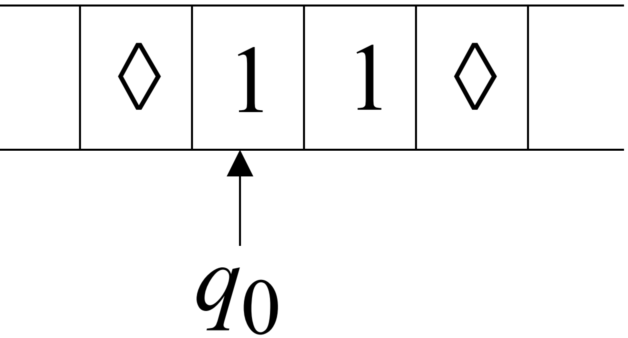
Until no more \$ remain

Turing Machine for $f(x) = 2x$

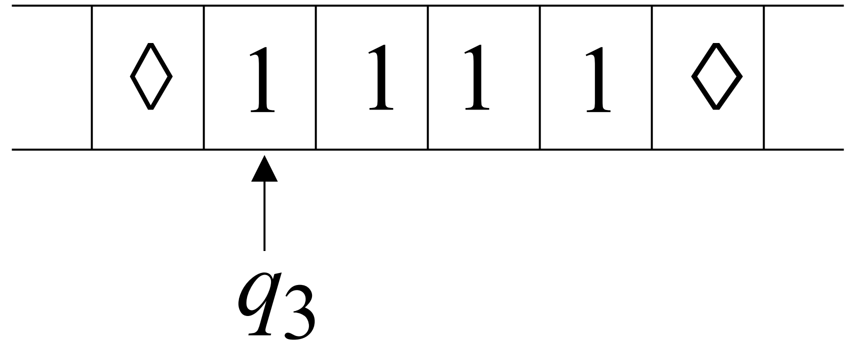


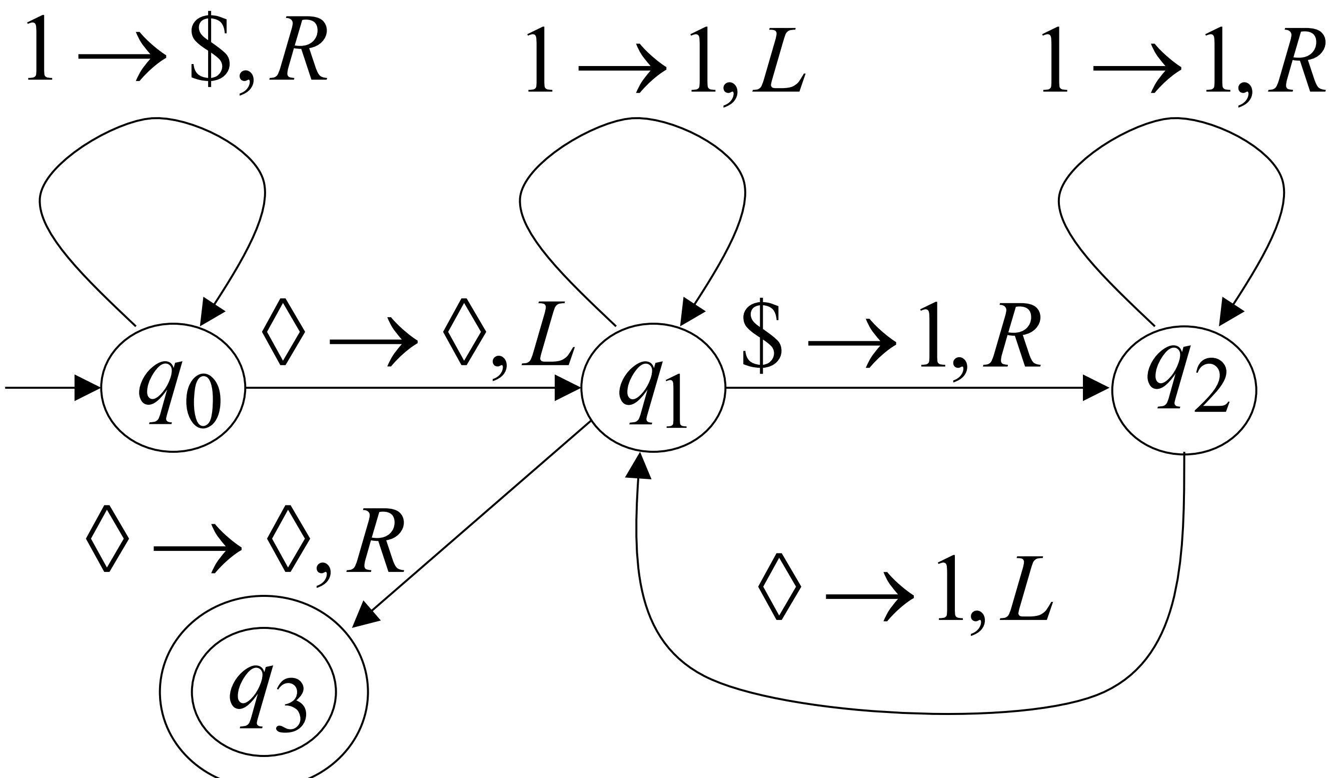
Example

Start



Finish





Another Example

The function is computable

$$f(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \le y \end{cases}$$

Input: $x0y$

Output: 1 or 0

Turing Machine Pseudocode:

- Repeat

  Match a 1 from x with a 1 from y

  Until all of x or y is matched

- If a 1 from x is not matched

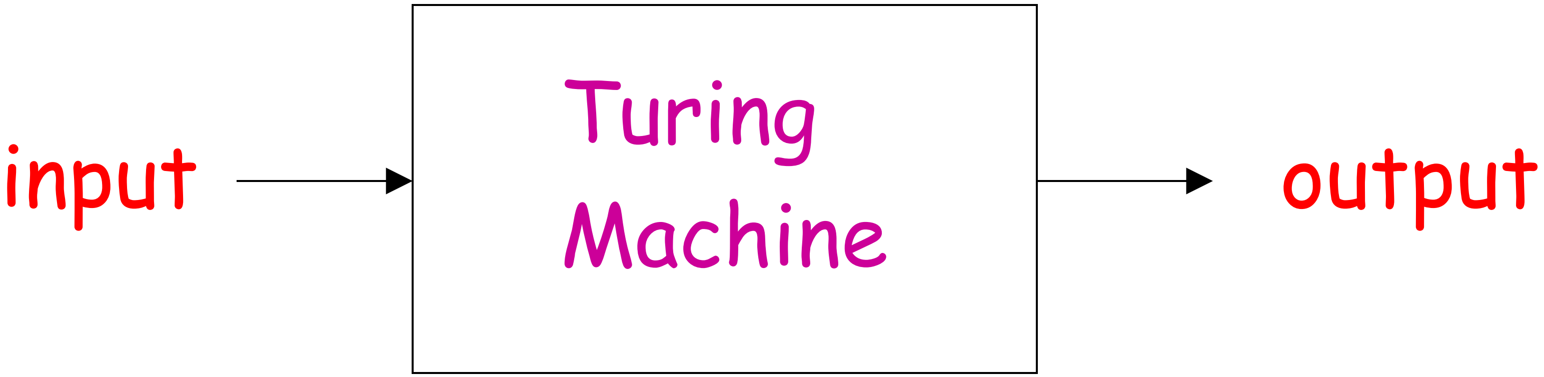
  erase tape, write 1 $(x > y)$

  else

  erase tape, write 0 $(x \leq y)$

Combining Turing Machines

Block Diagram



Example:

$$f(x,y) = \begin{cases} x+y & \text{if} \quad x > y \\ 0 & \text{if} \quad x \le y \end{cases}$$