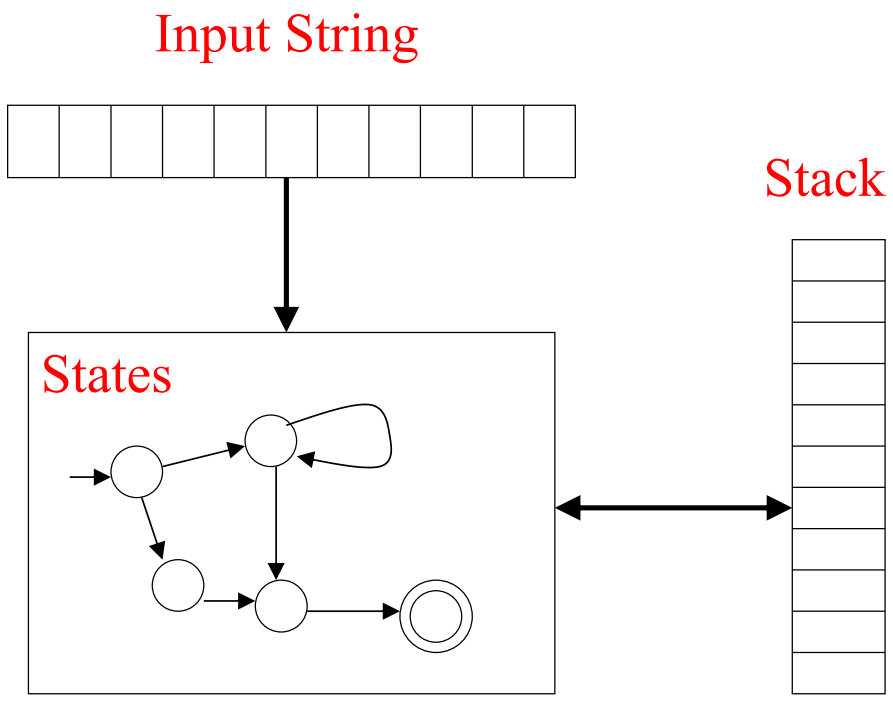
CSC 339 – Theory of Computation
Fall 2023

9.1 Pushdown Automata (PDAs) – Part 1

Outline

- Pushdown automaton (PDA)
- Pushing and popping symbols

Pushdown Automaton (PDA)



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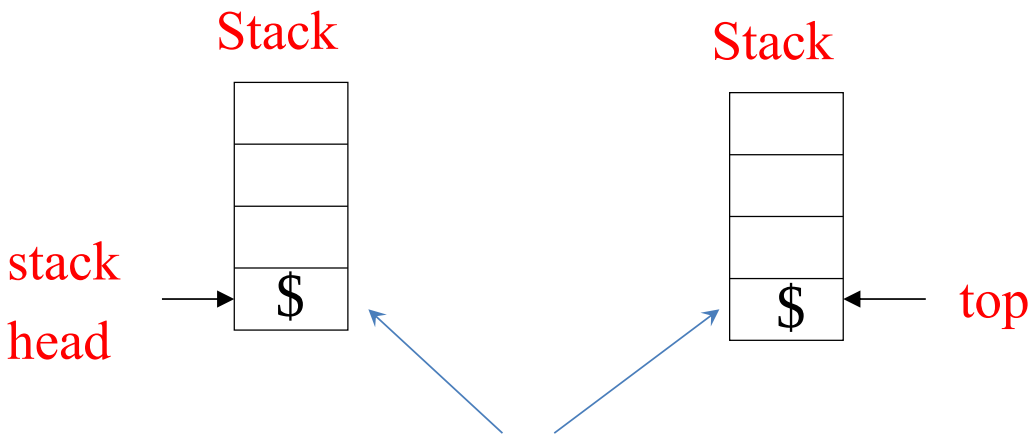
Initial Stack Symbol



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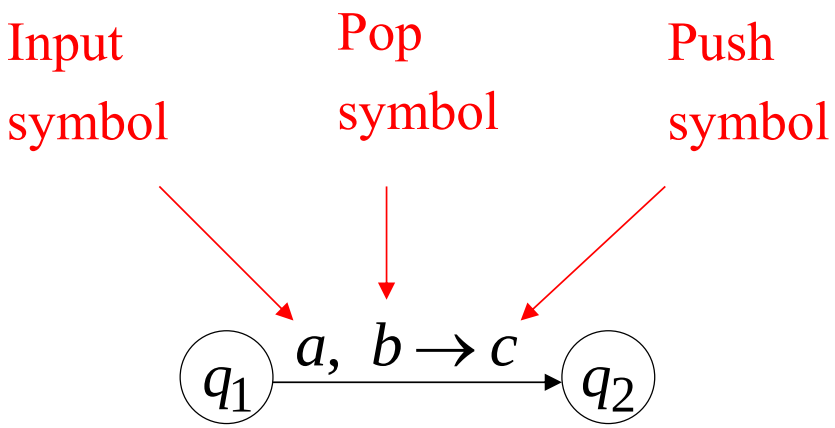
The States



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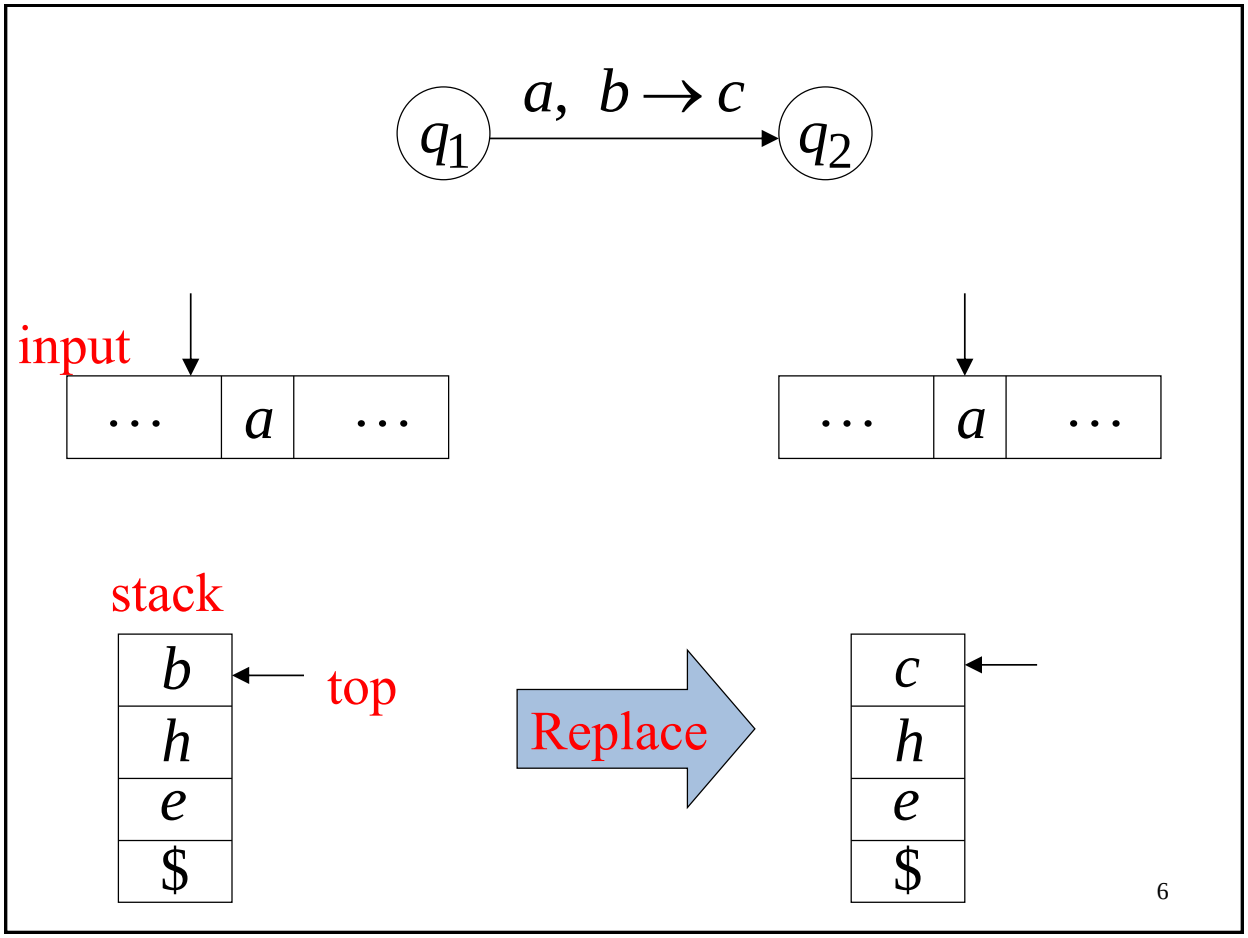
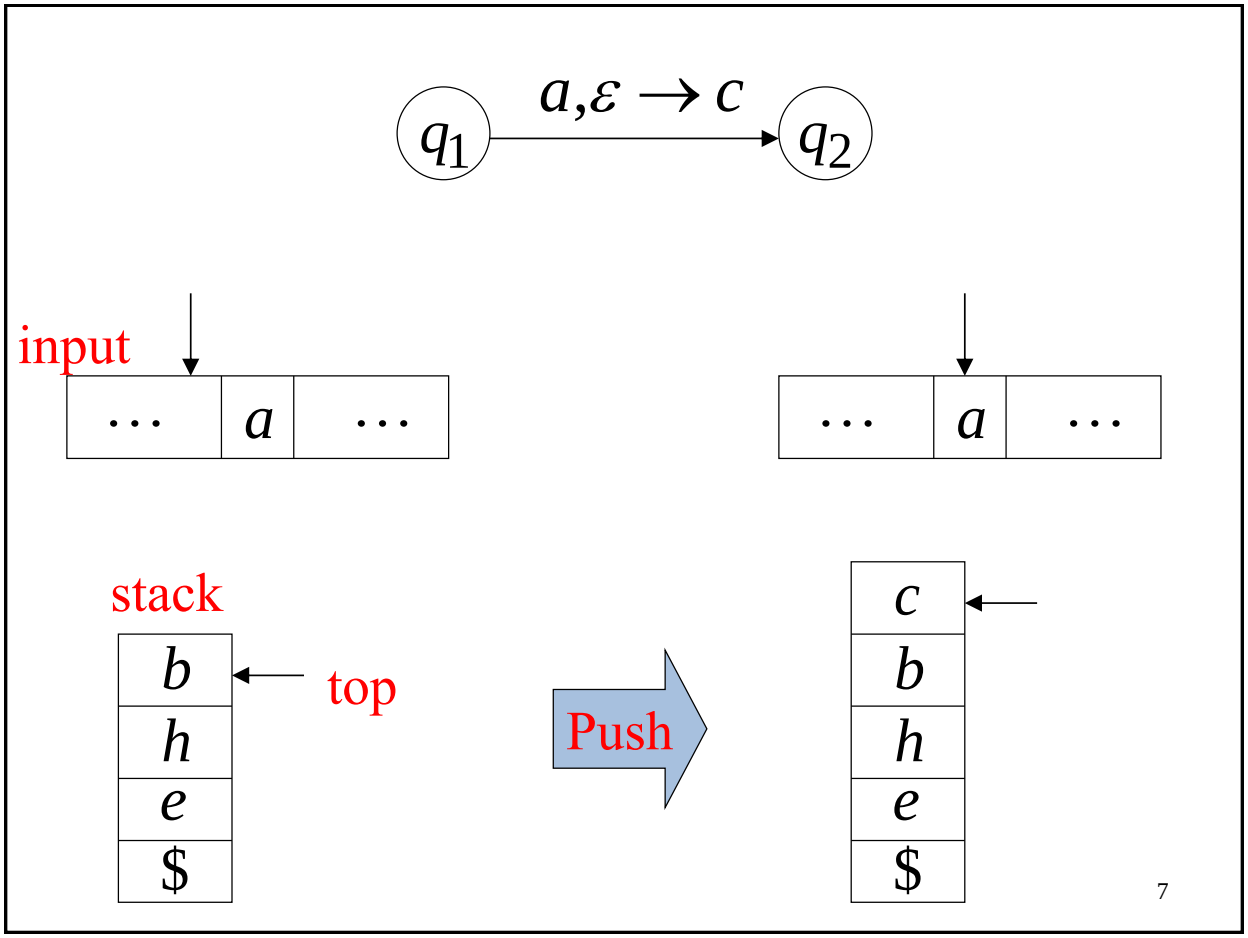
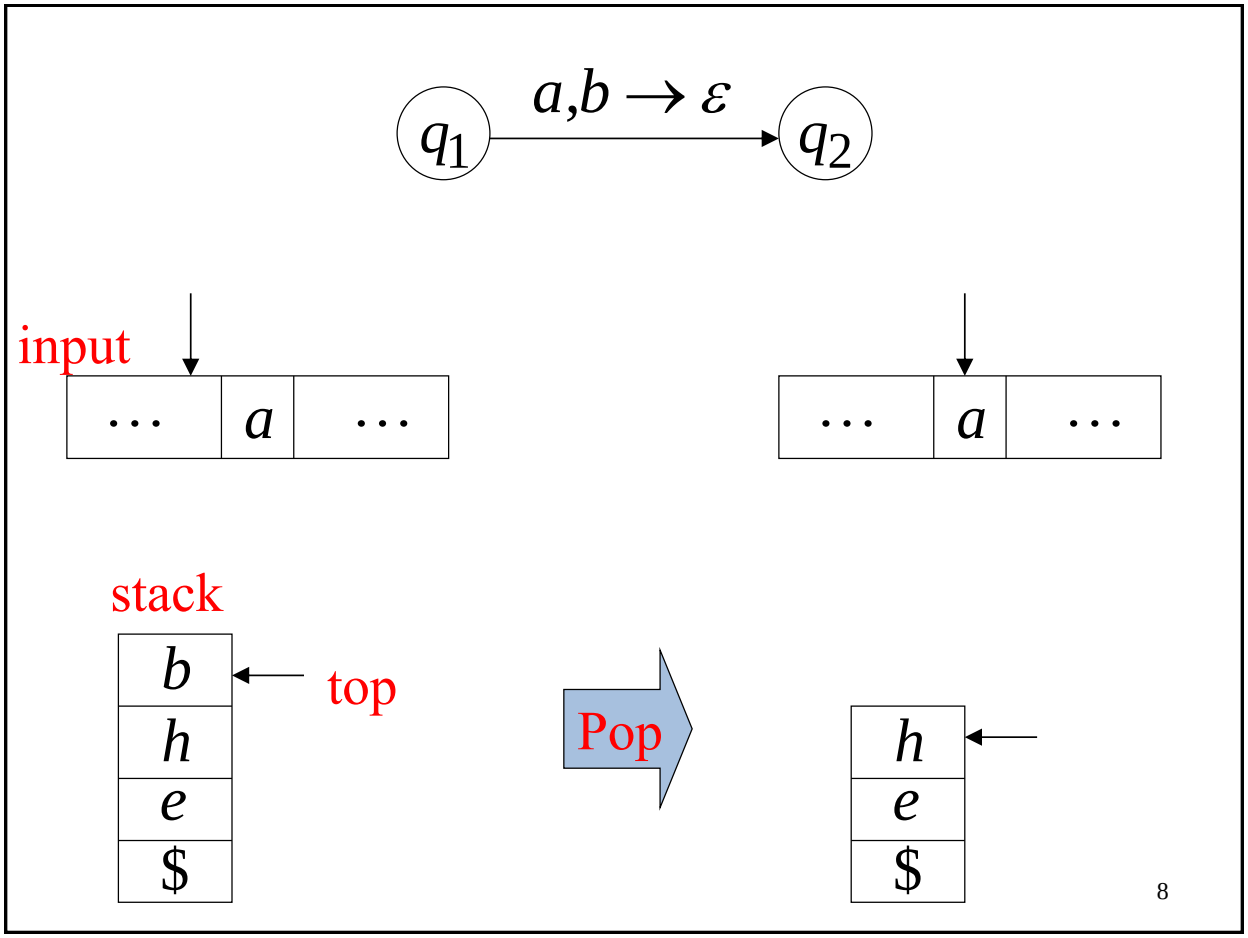
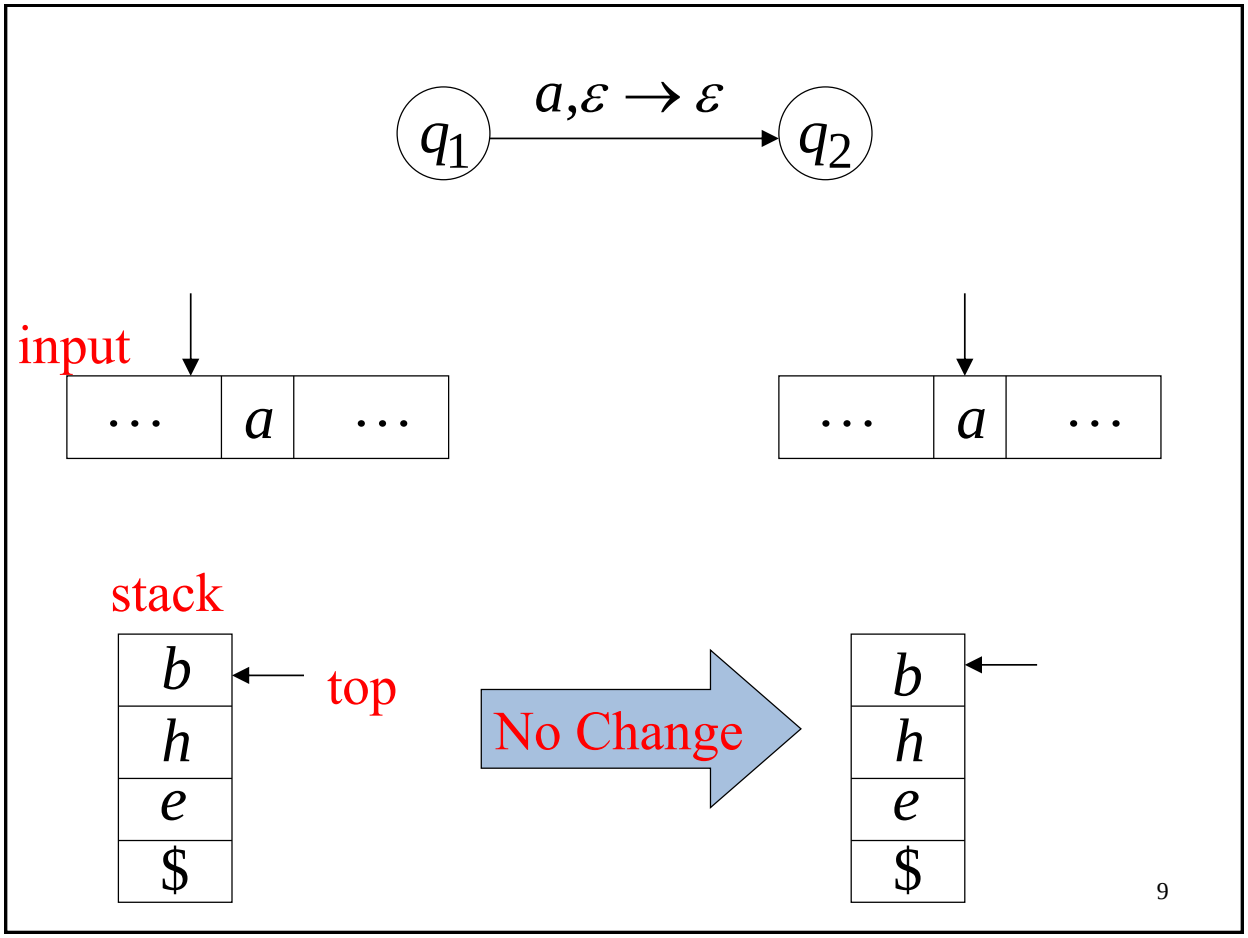
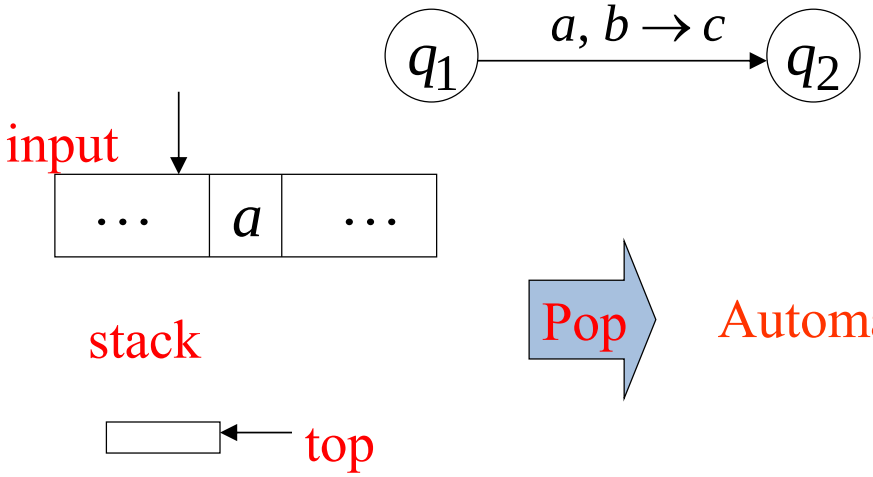

$a,\ b \rightarrow c$
q_1
q_2
input
…
a
…
…
a
…
stack
b
h
e
$
top
Replace
c
h
e
$
6


$a,\varepsilon \rightarrow c$
q_1
q_2
input
…
a
…
…
a
…
stack
b
h
e
$
top
Push
c
b
h
e
$
7


$a,b \rightarrow \varepsilon$
q_1
q_2
input
…
a
…
…
a
…
stack
b
h
e
$
top
Pop
h
e
$
8


q_1
$a,\varepsilon \rightarrow \varepsilon$
q_2
input
…
a
…
…
a
…
stack
b
h
e
$
top
No Change
b
h
e
$
9

Pop from empty stack



If the automaton attempts to pop from empty stack then it halts and rejects input.

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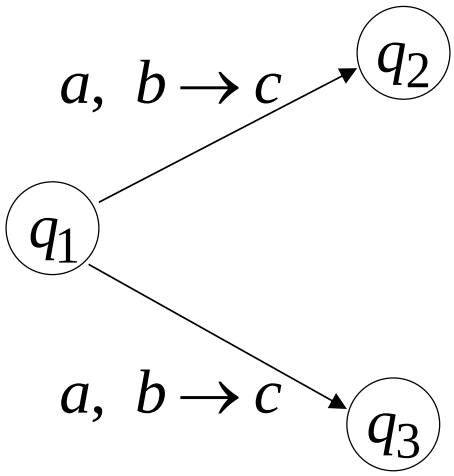
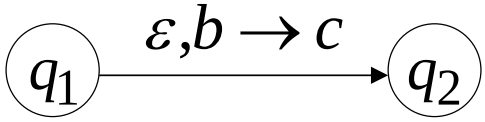
Non-Determinism

PDAs are non-deterministic

Allowed non-deterministic transitions





ε – transition

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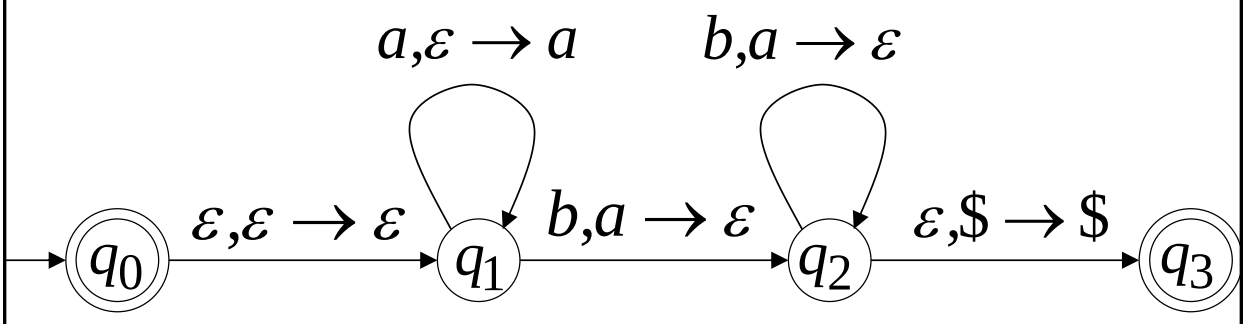
Example PDA

PDA M: $L(M) = \{a^n b^n : n \geq 0\}$



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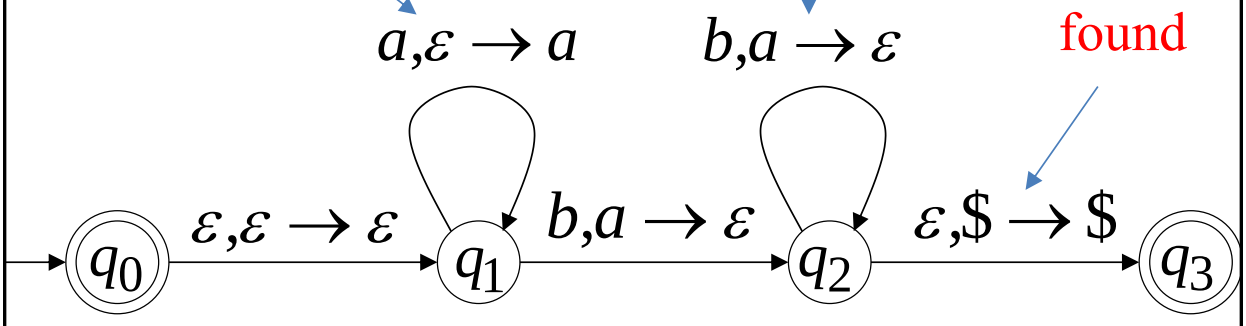
$$L(M) = \{a^n b^n : n \geq 0\}$$

Basic Idea:

1. Push the a's on the stack

2. Match the b's on input with a's on stack

3. Match found



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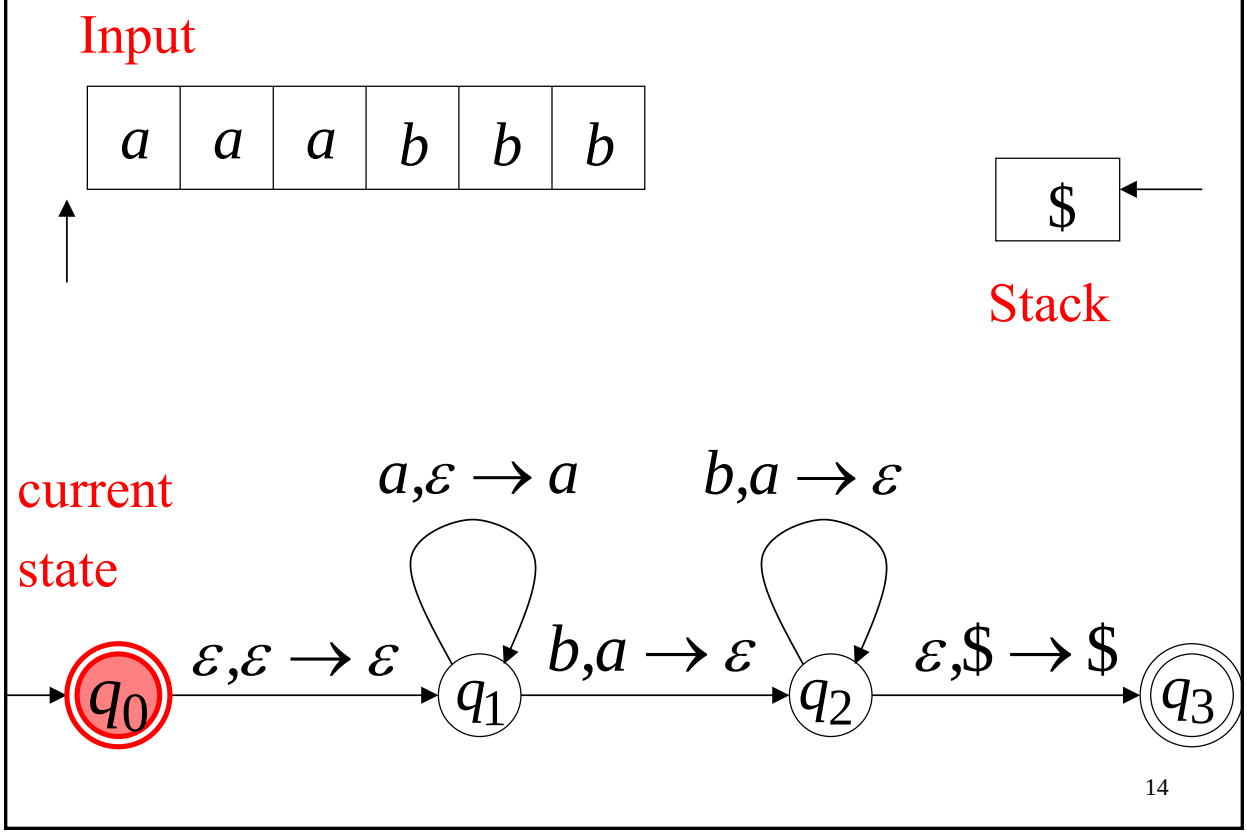
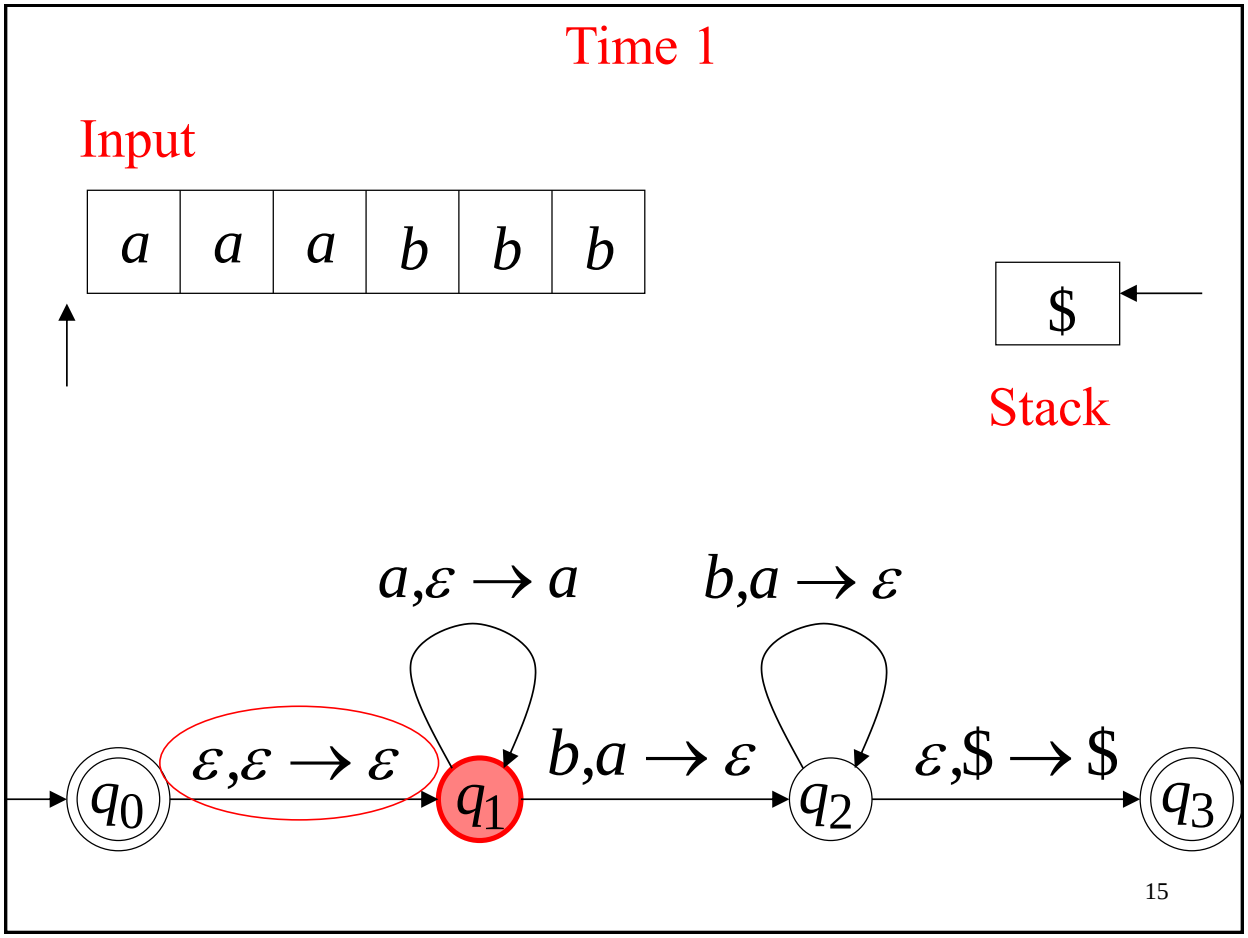
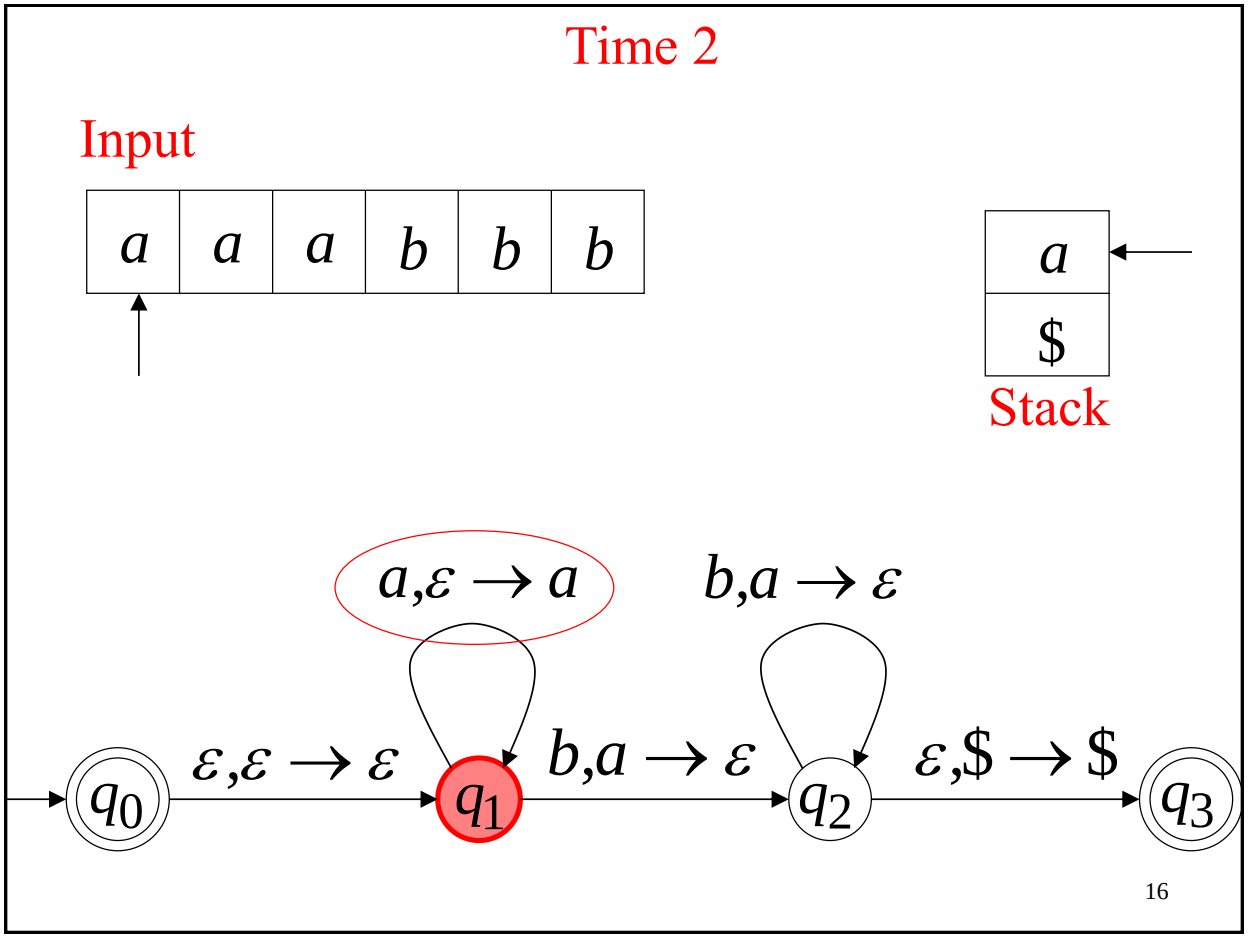
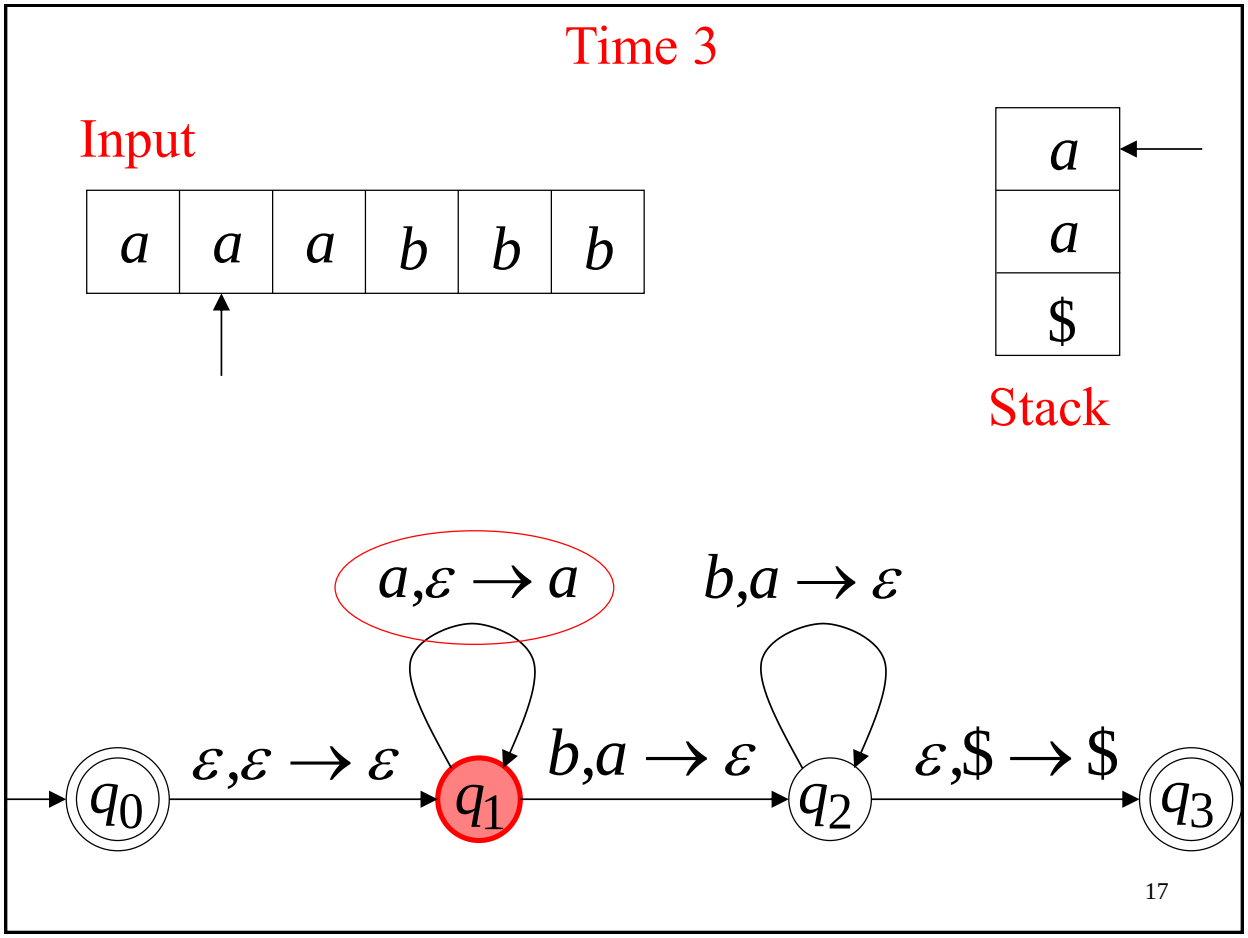
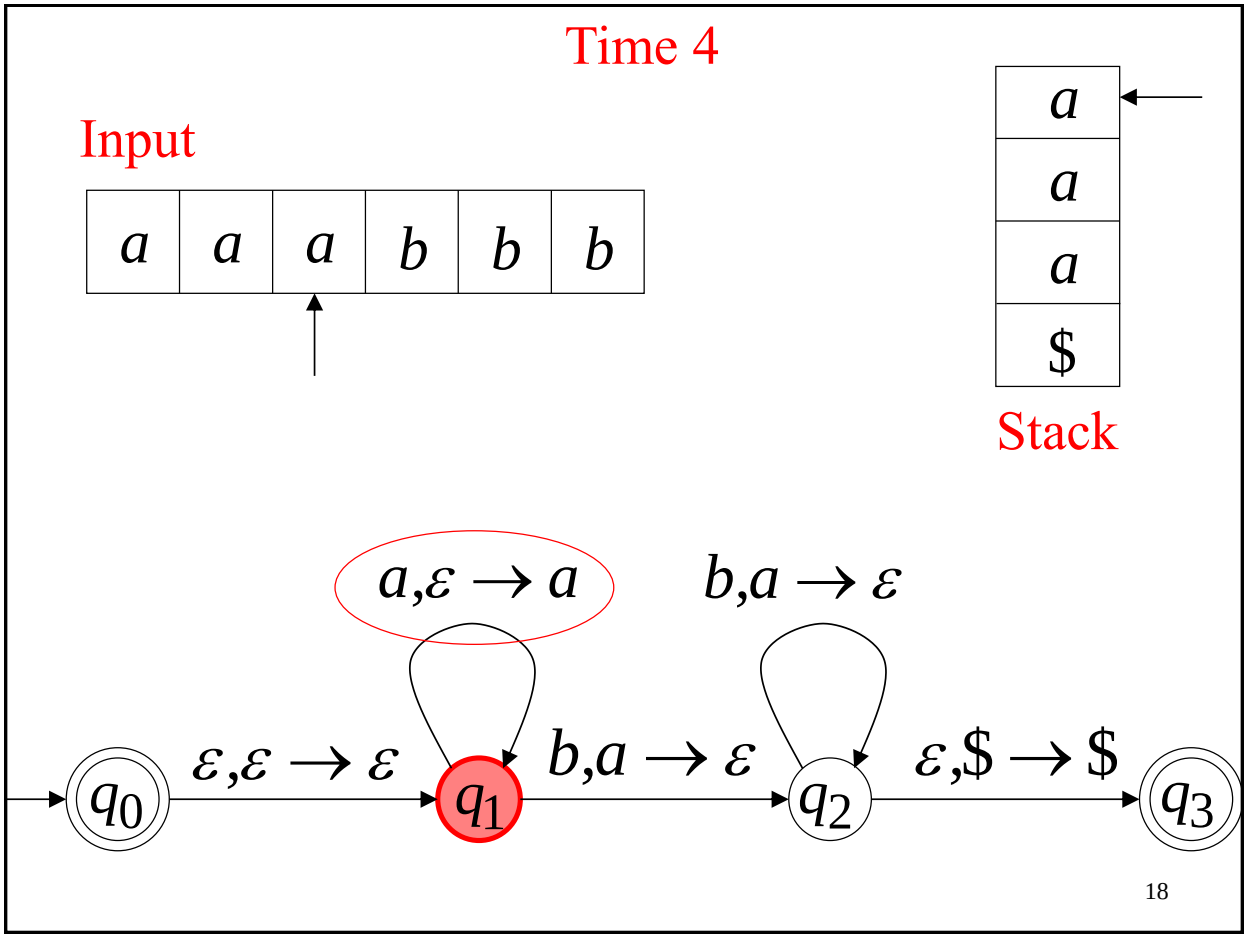
Execution Example: Time 0

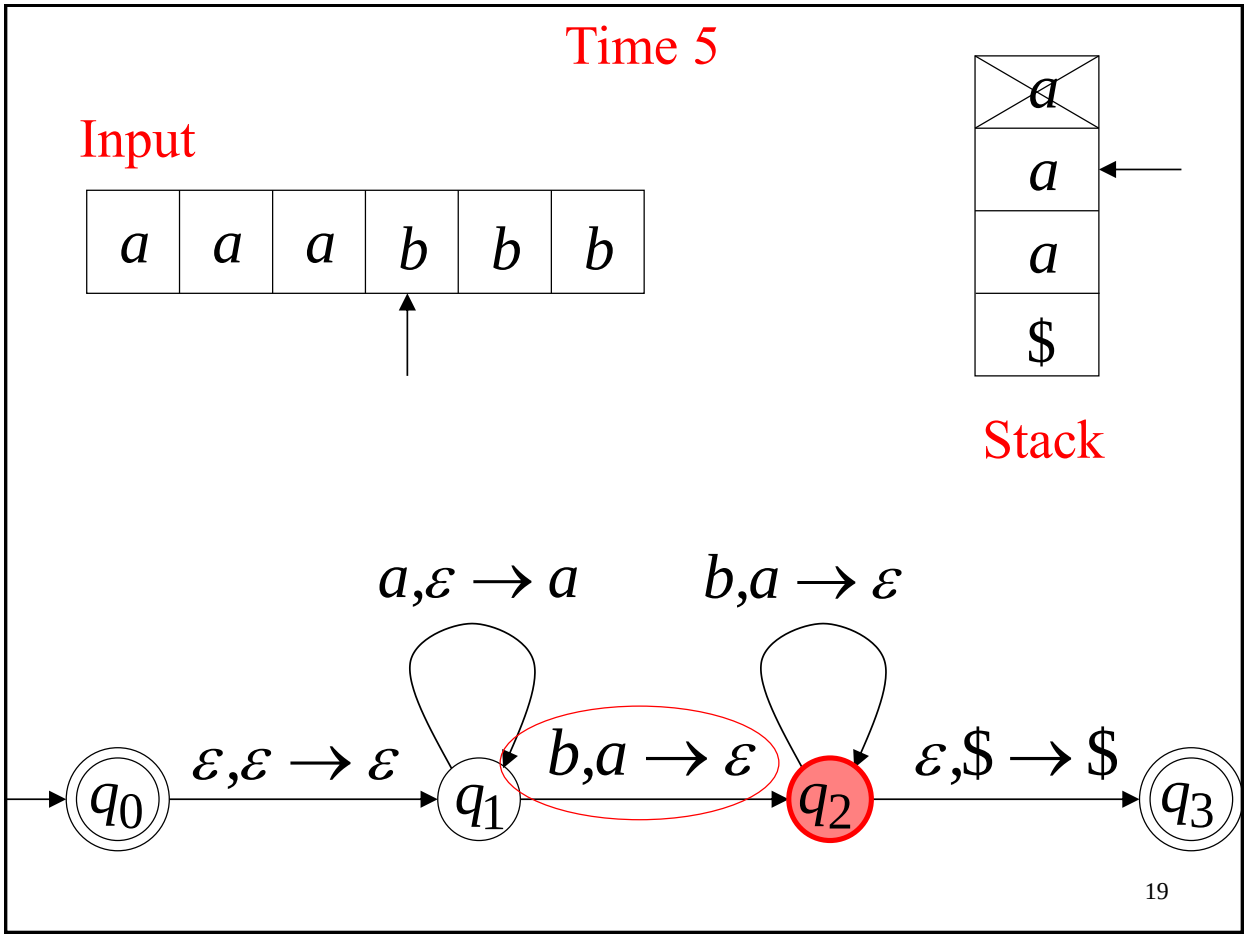


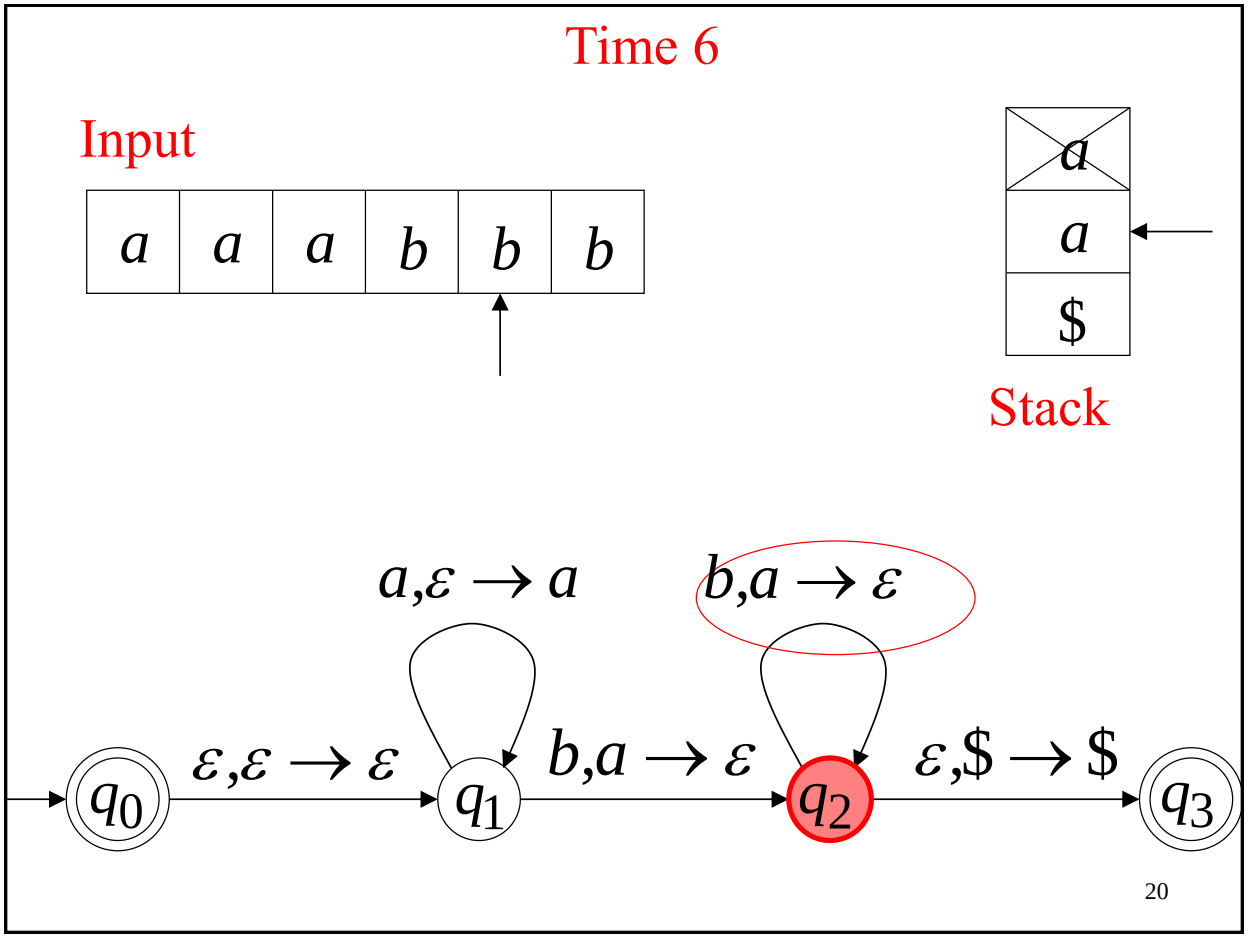

Time 1
Input
a a a b b b
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
15

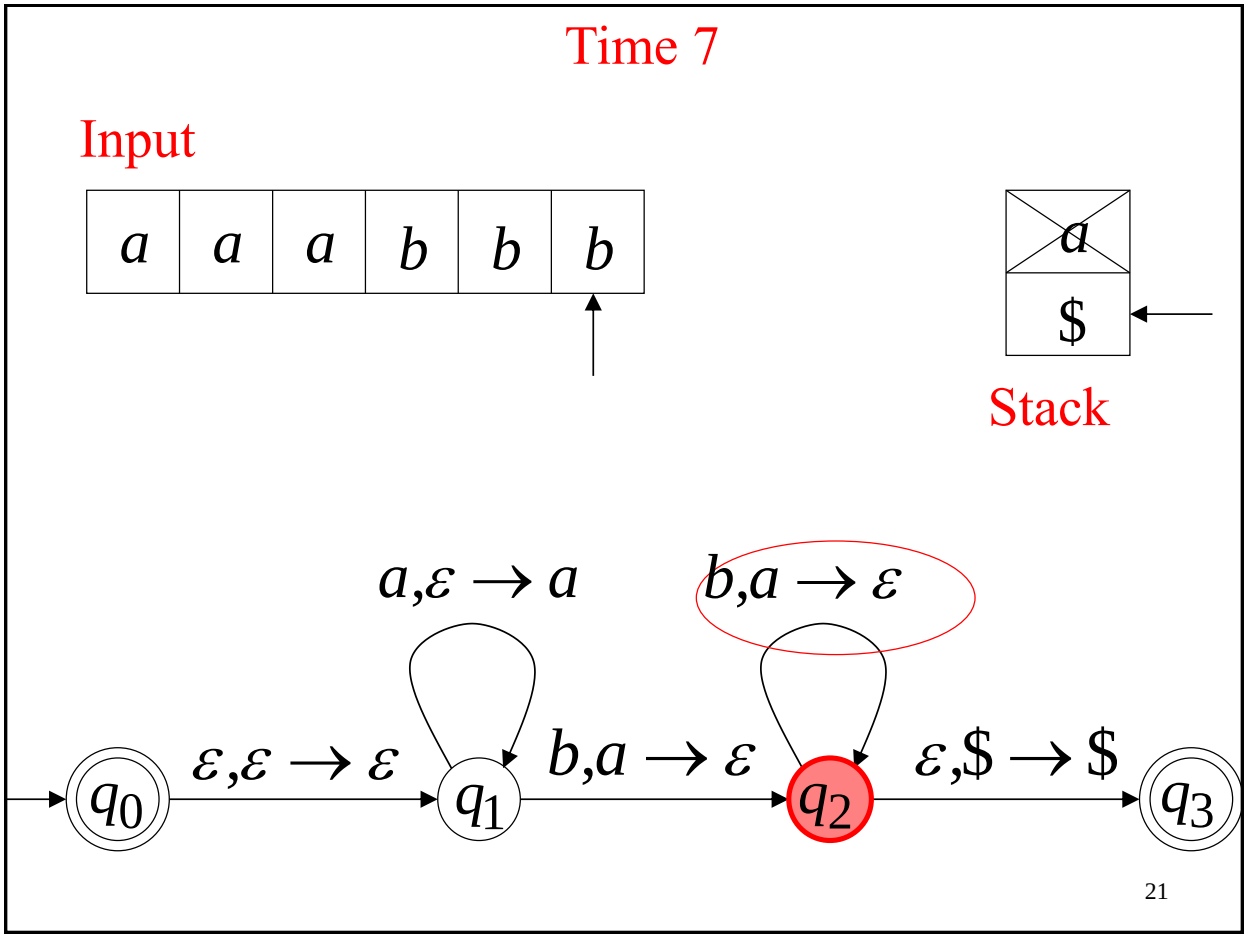

Time 2
Input
a a a b b b
a
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
16

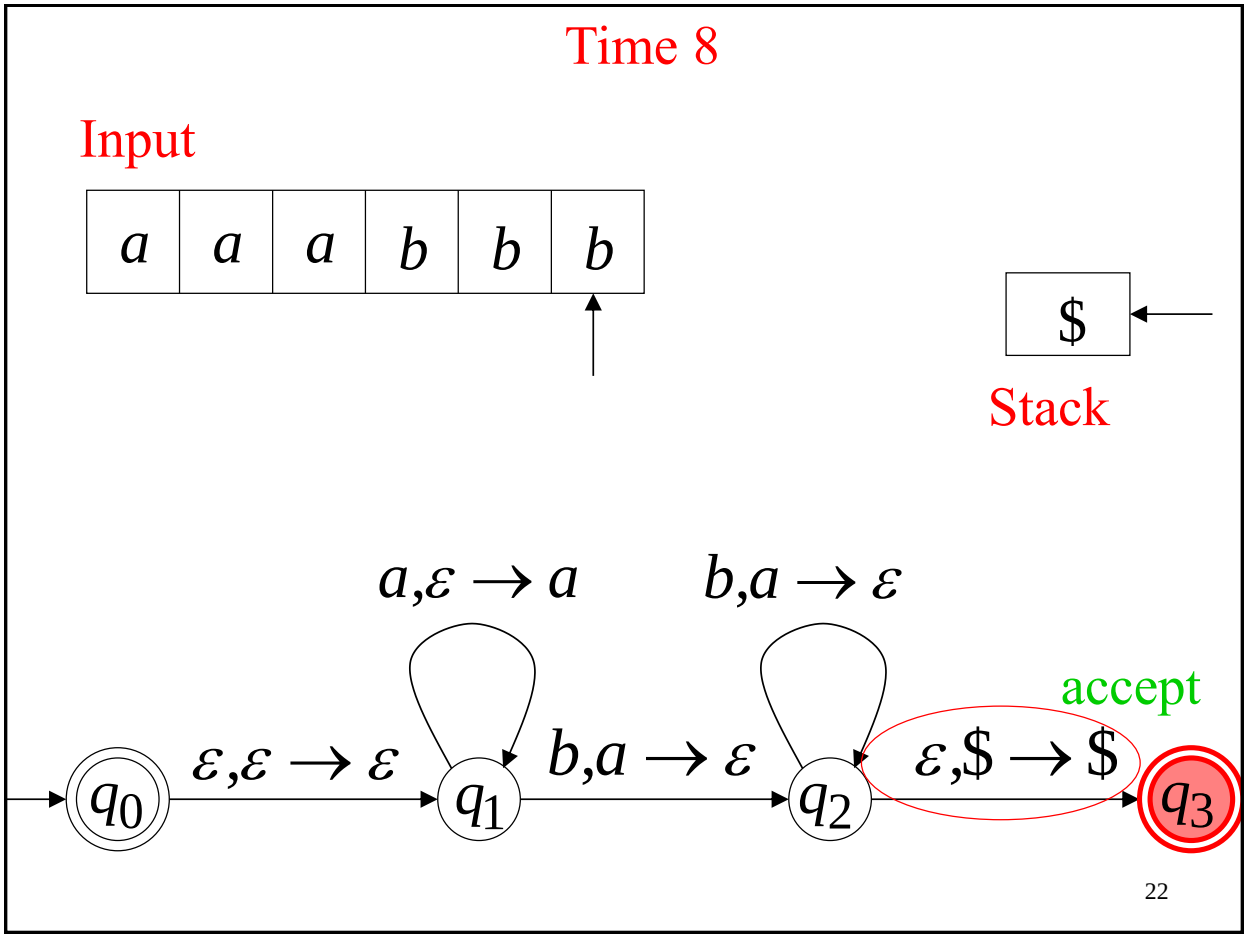

Time 3
Input
a a a b b b
a
a
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
17


Time 4
Input
a a a b b b
a
a
a
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
18


Time 5
Input
a a a b b b
a
a
a
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
19


Time 6
Input
a a a b b b
a
a
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
20


Time 7
Input
a a a b b b
a
$
Stack
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
21


Time 8
Input
a a a b b b
$
Stack
a,ε → a
b,a → ε
accept
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
22

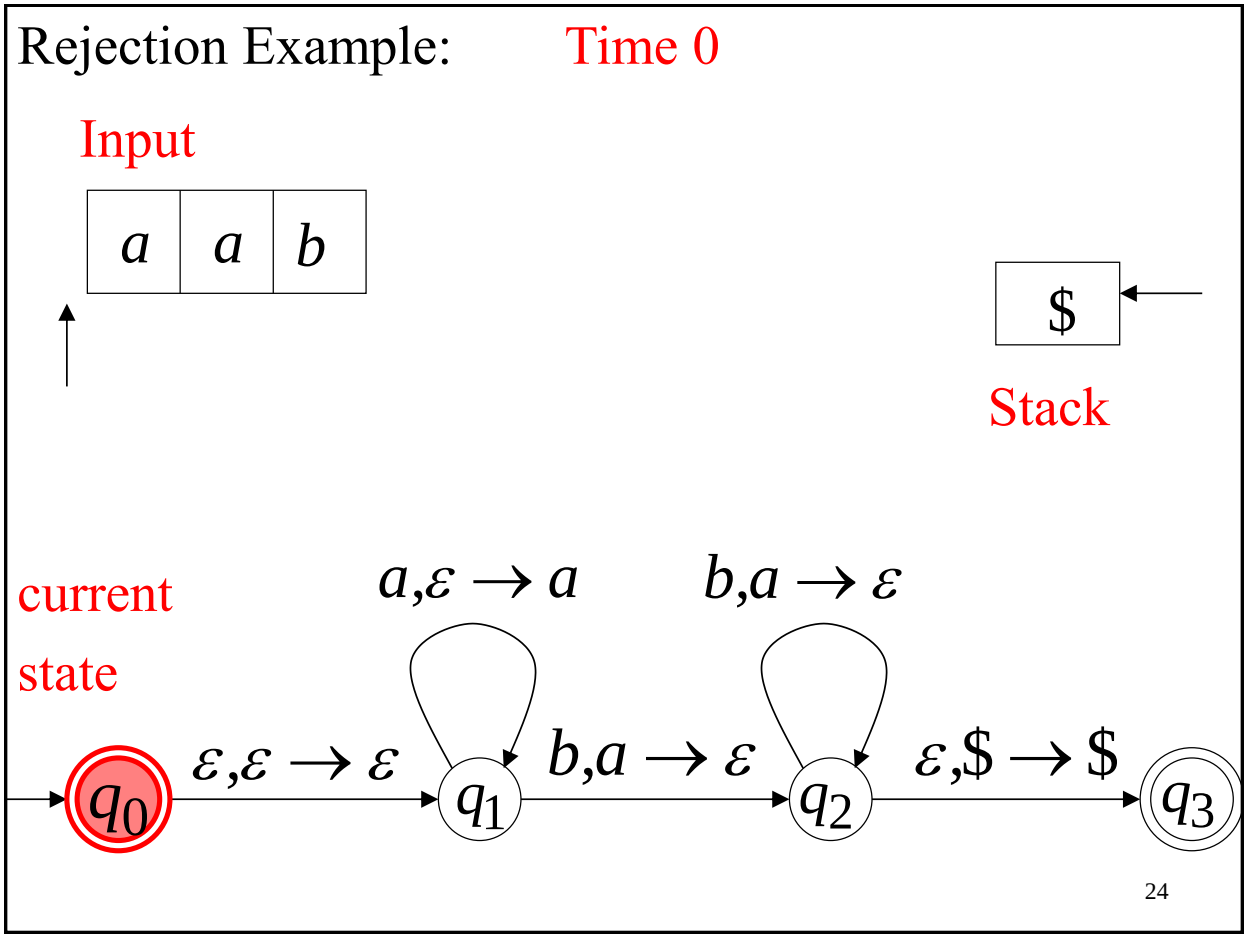
A string is accepted if there is
a computation such that:

All the input is consumed

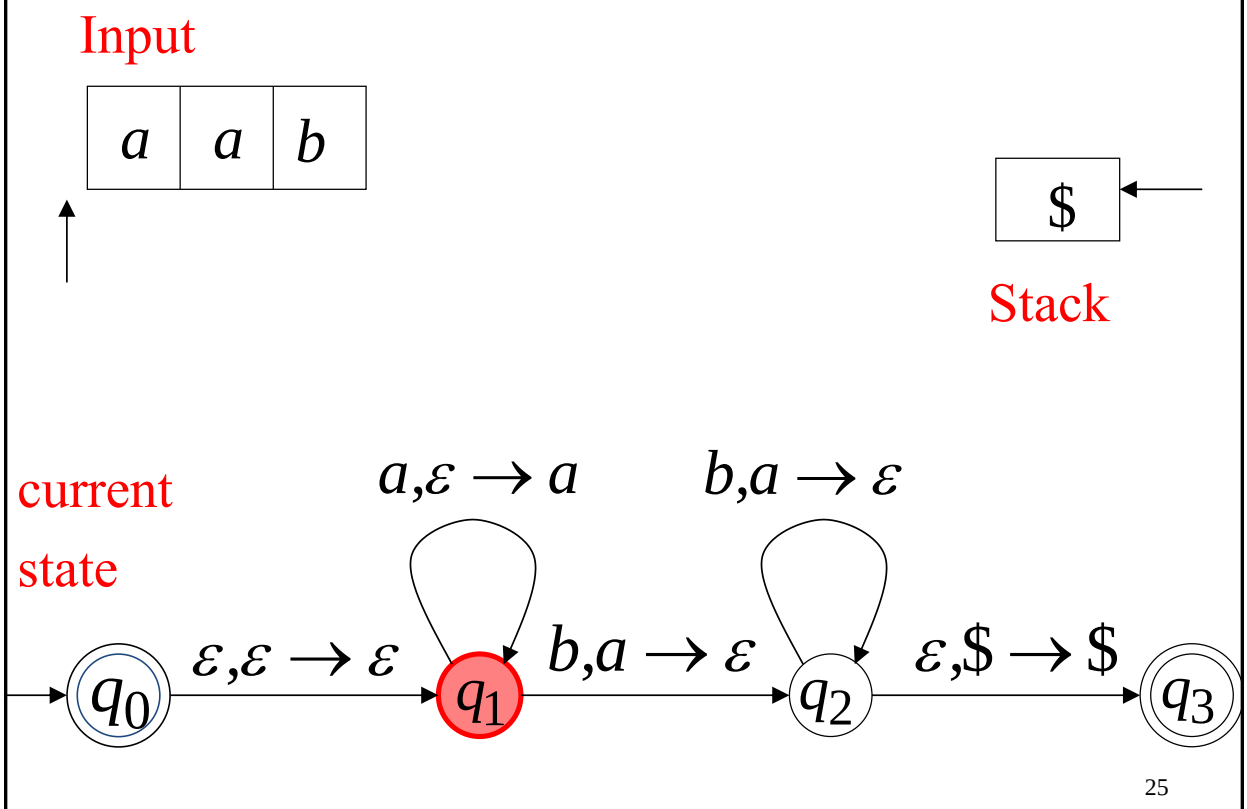
AND

The last state is an accepting state

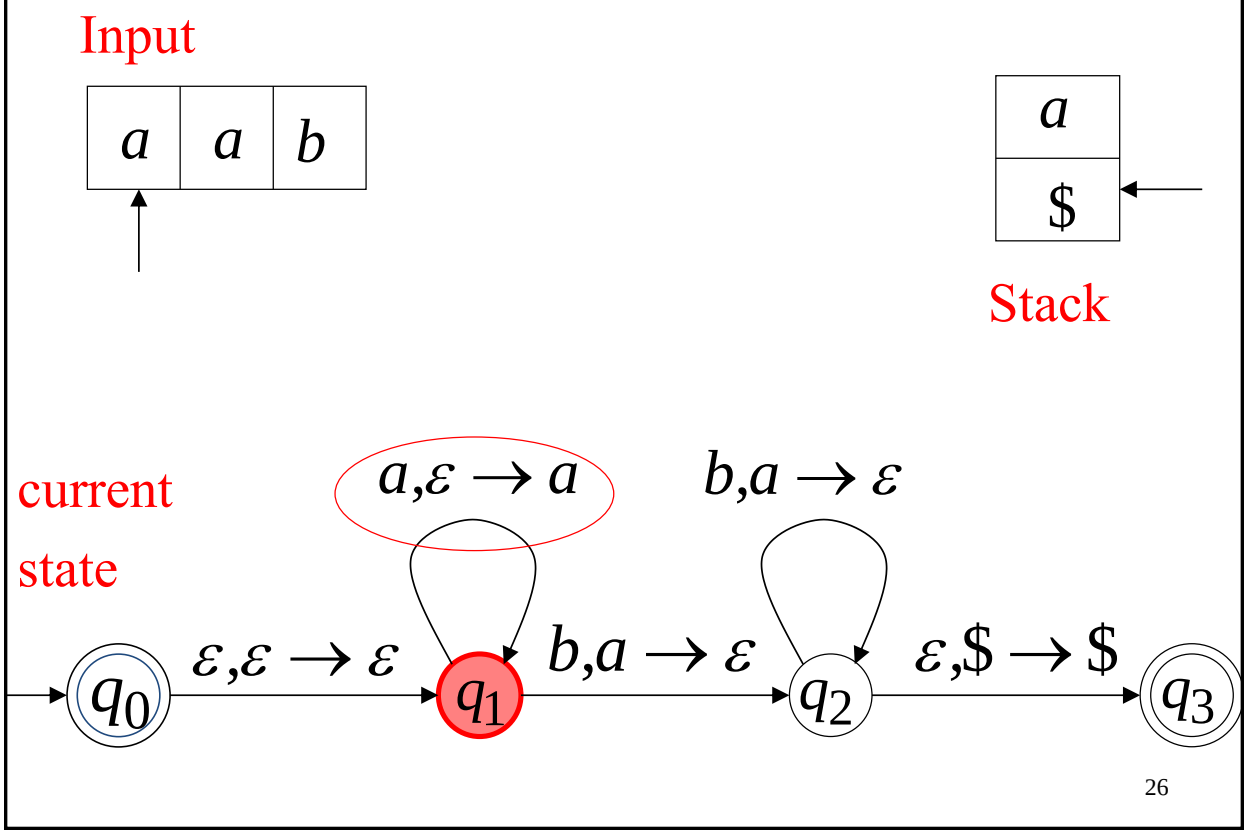
We do not care about the stack contents
at the end of the accepting computation.


Rejection Example:
Time 0
Input
a
a
b
$
Stack
current
state
a,ε → a
b,a → ε
ε,ε → ε
b,a → ε
ε,$ → $
q0
q1
q2
q3
24

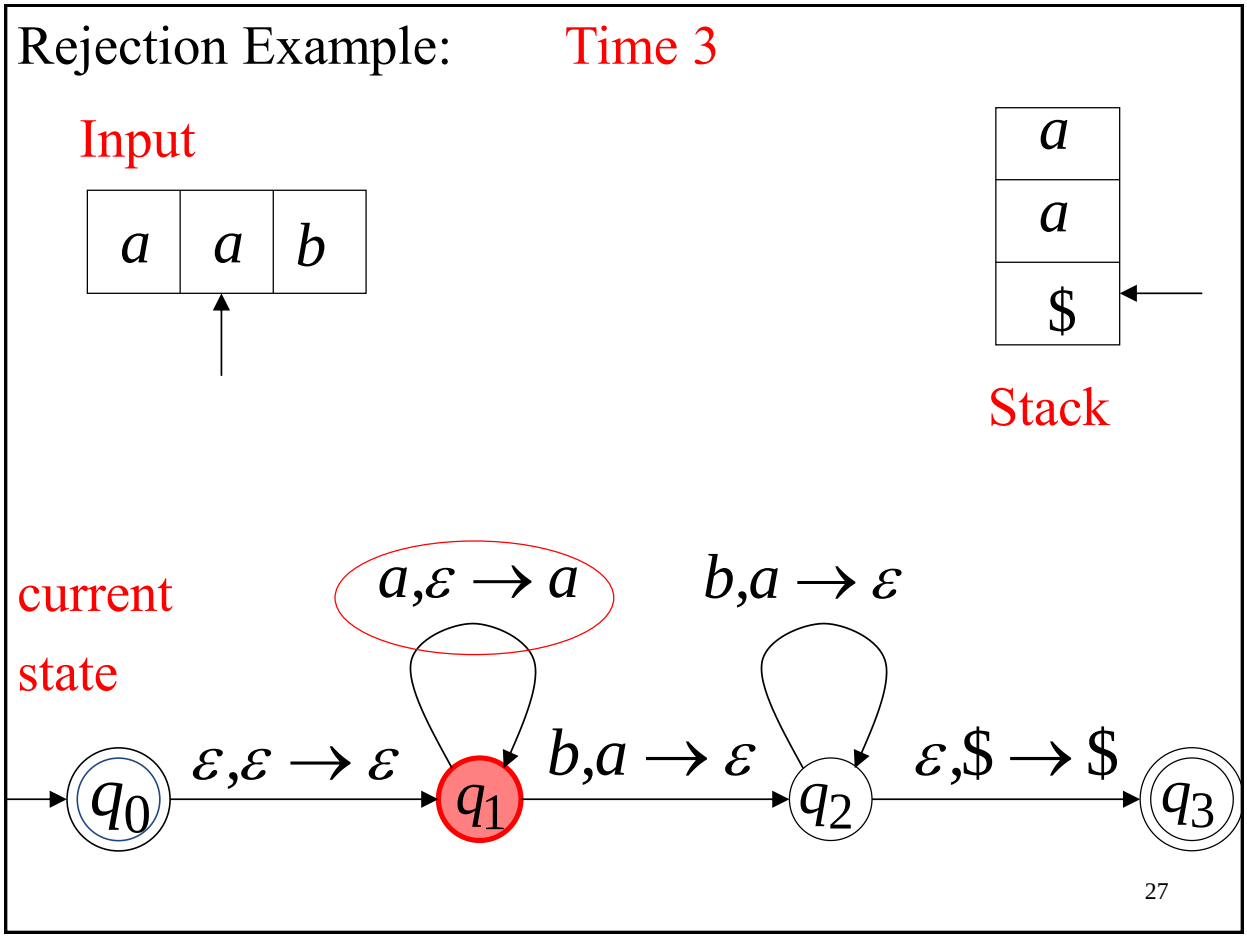
Rejection Example: Time 1



Rejection Example: Time 2

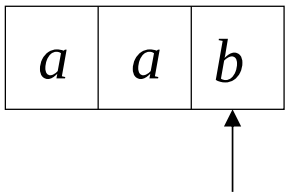


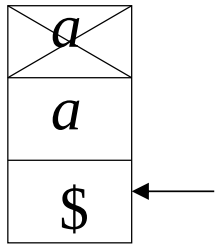
Rejection Example: Time 3



Rejection Example: Time 4

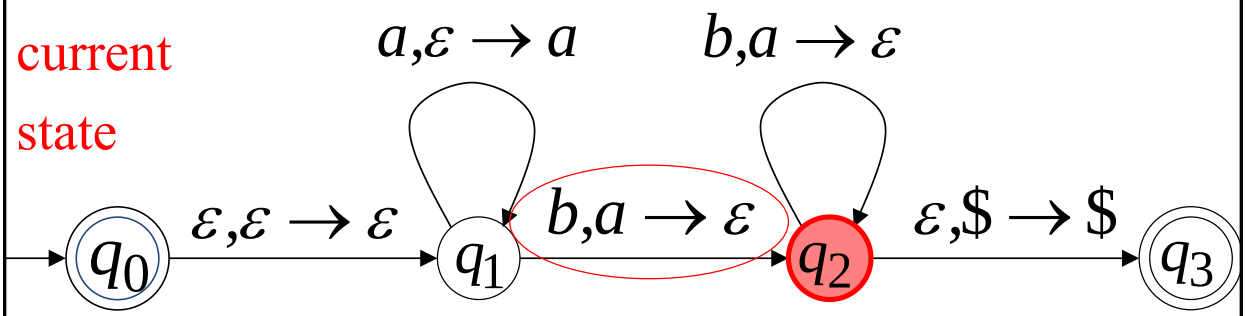
Input





Stack

current state



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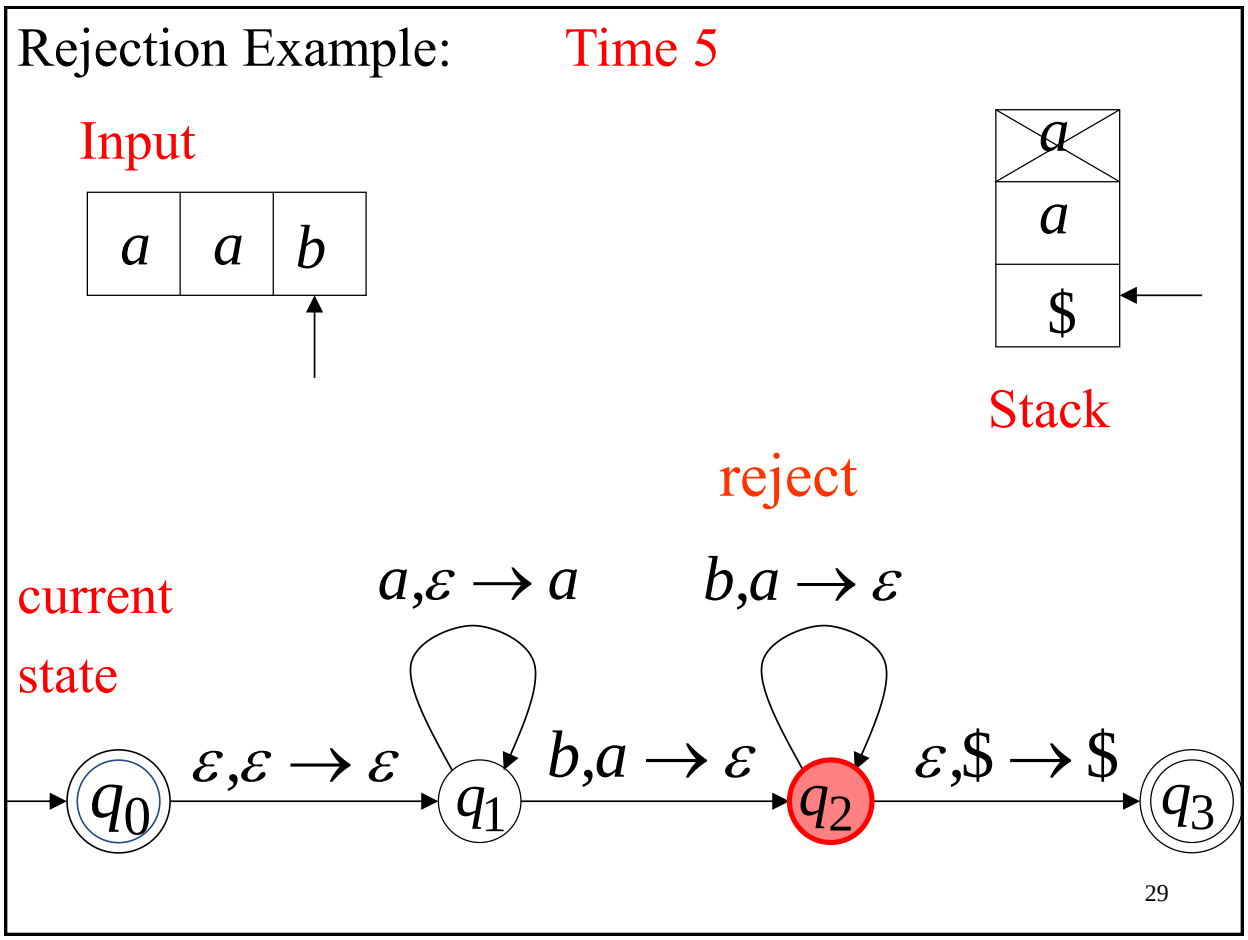
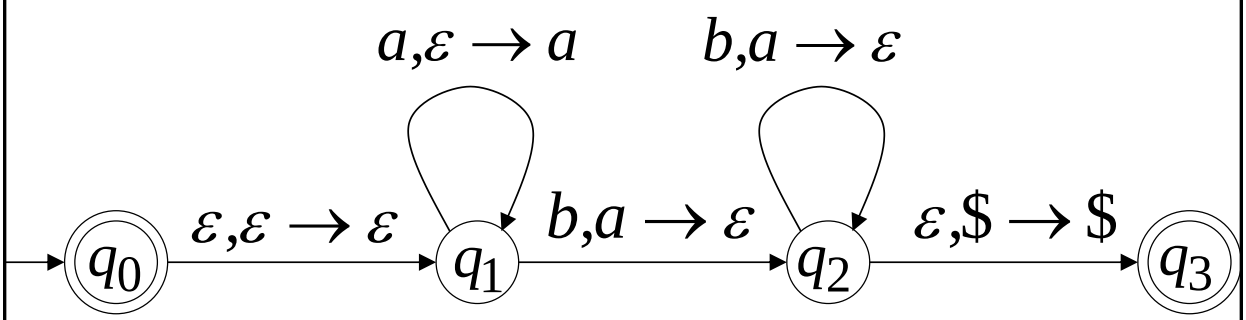
Rejection Example: Time 5



There is no accepting computation for aab.

The string aab is rejected by the PDA.



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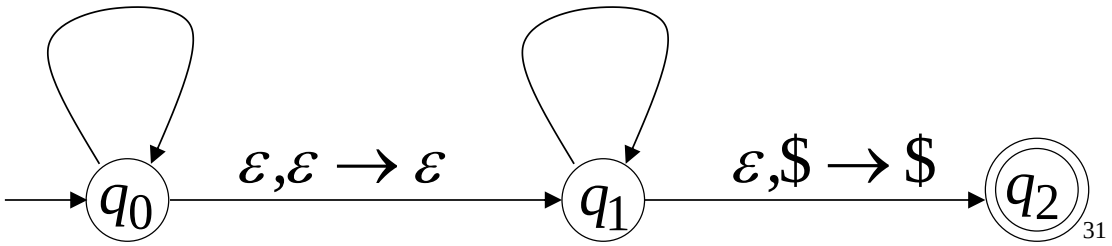
Another PDA example

PDA M: $L(M) = \{vv^R : v \in \{a,b\}^*\}$

$a, \varepsilon \to a$ $\quad$ $a, a \to \varepsilon$

$b, \varepsilon \to b$ $\quad$ $b, b \to \varepsilon$



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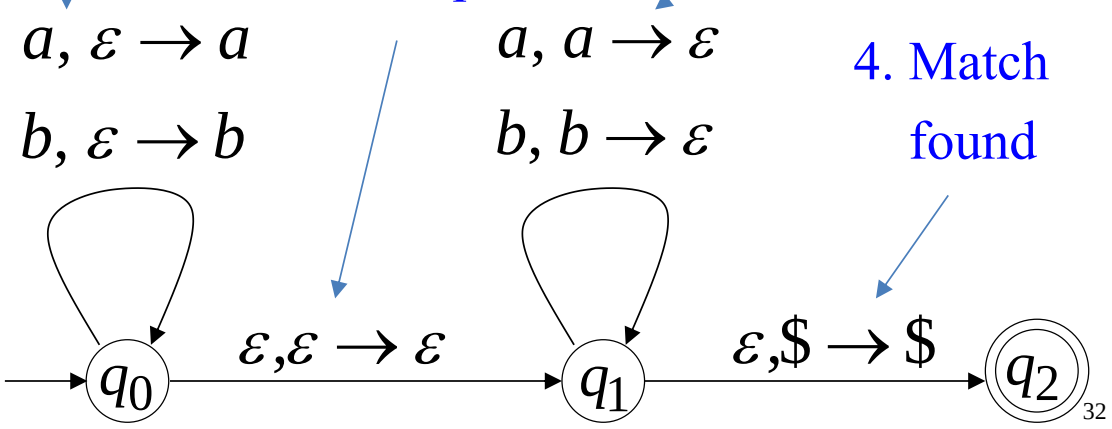
Basic Idea: $L(M) = \{vv^R : v \in \{a,b\}^*\}$

1. Push v on stack

2. Guess middle of input

3. Match v^R on input with v on the stack

4. Match found



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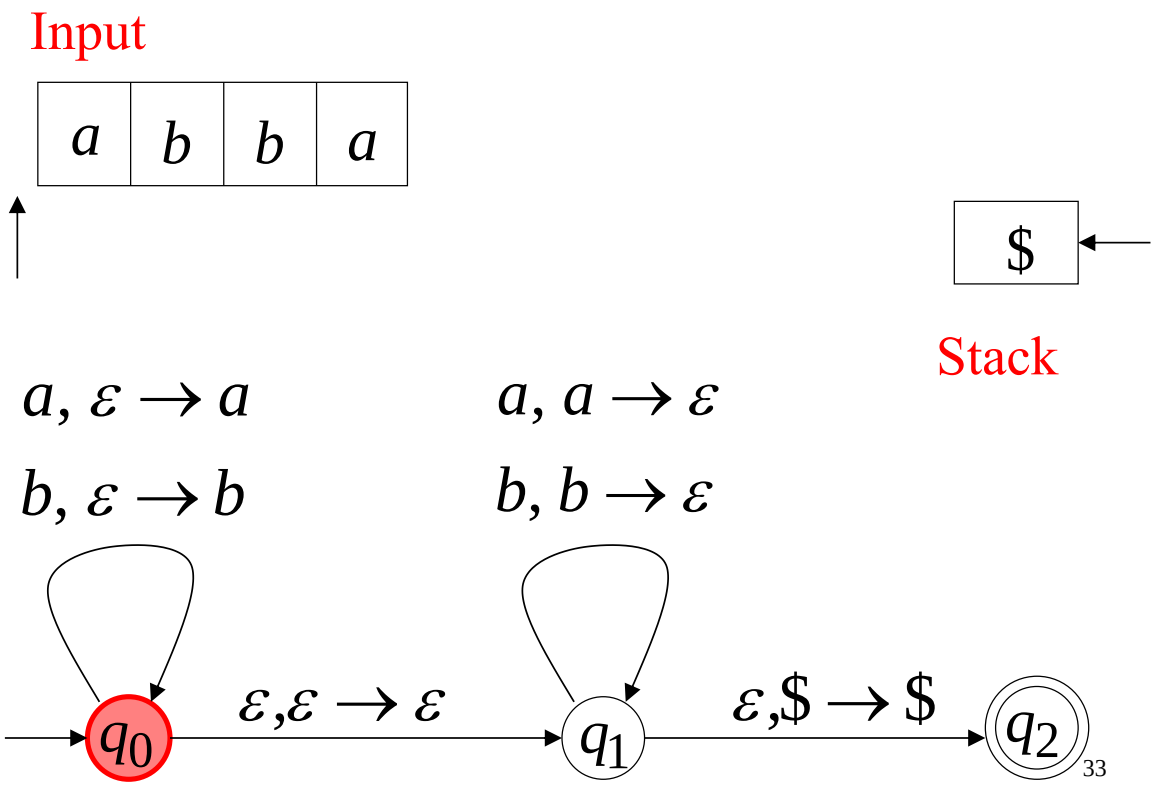
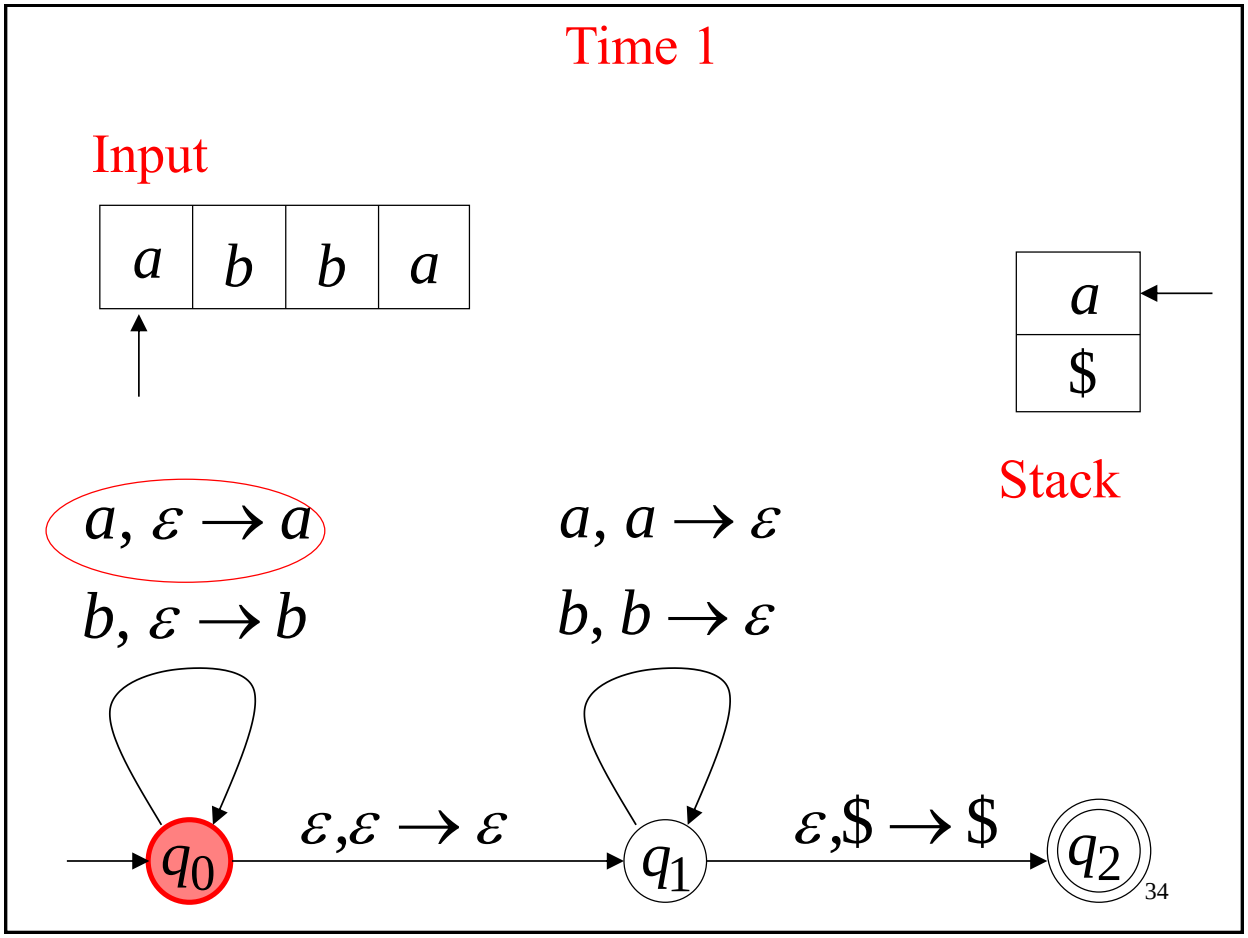
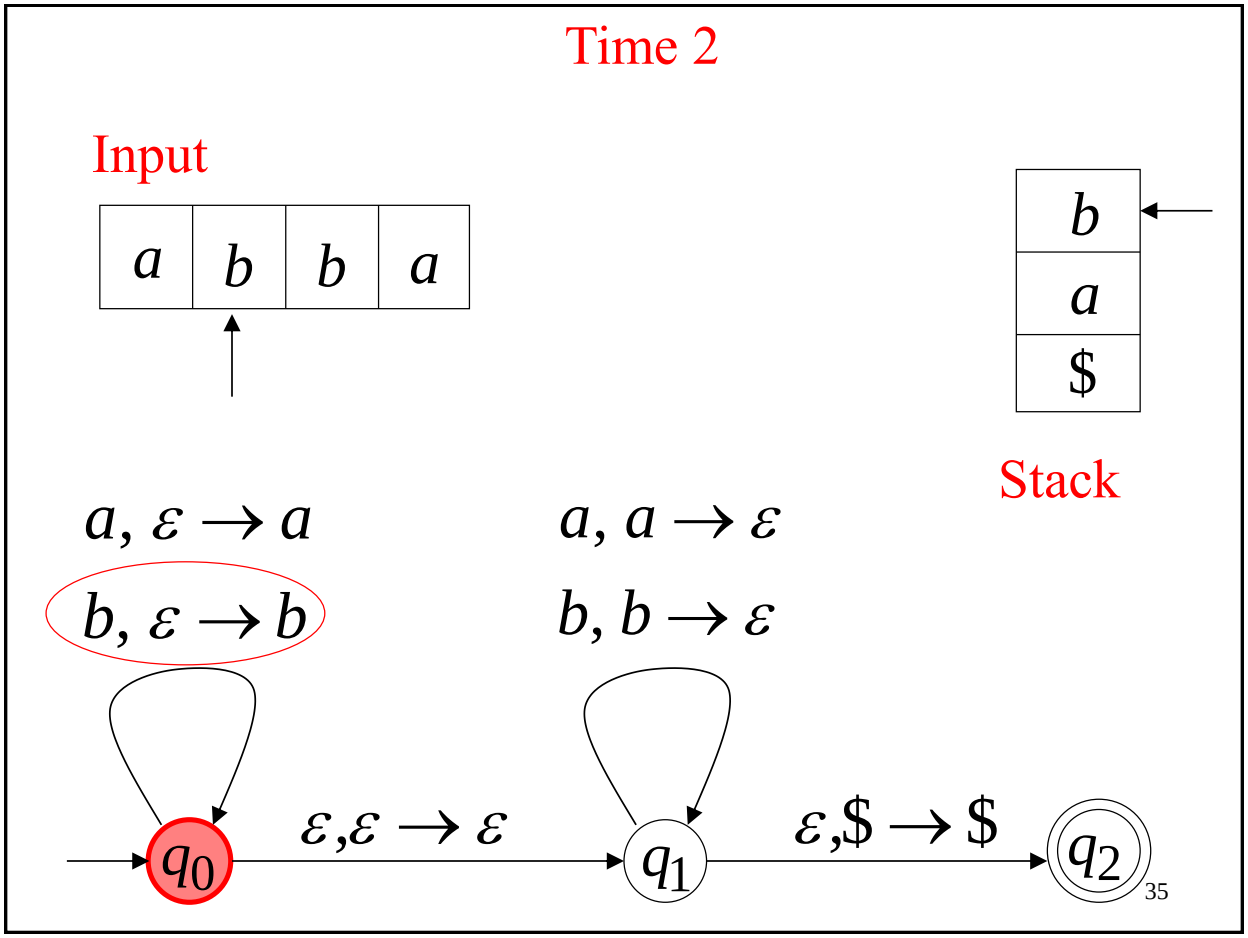
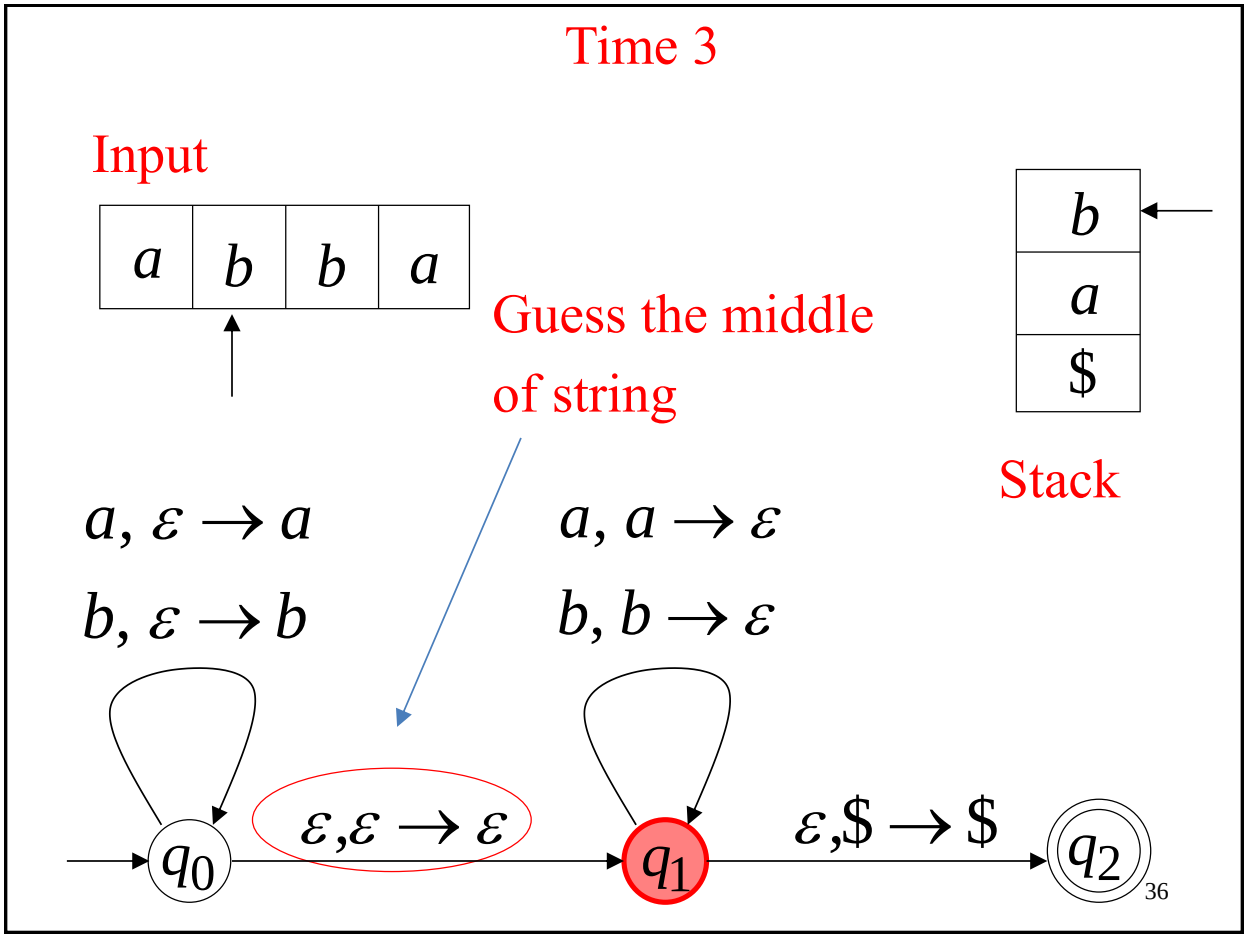
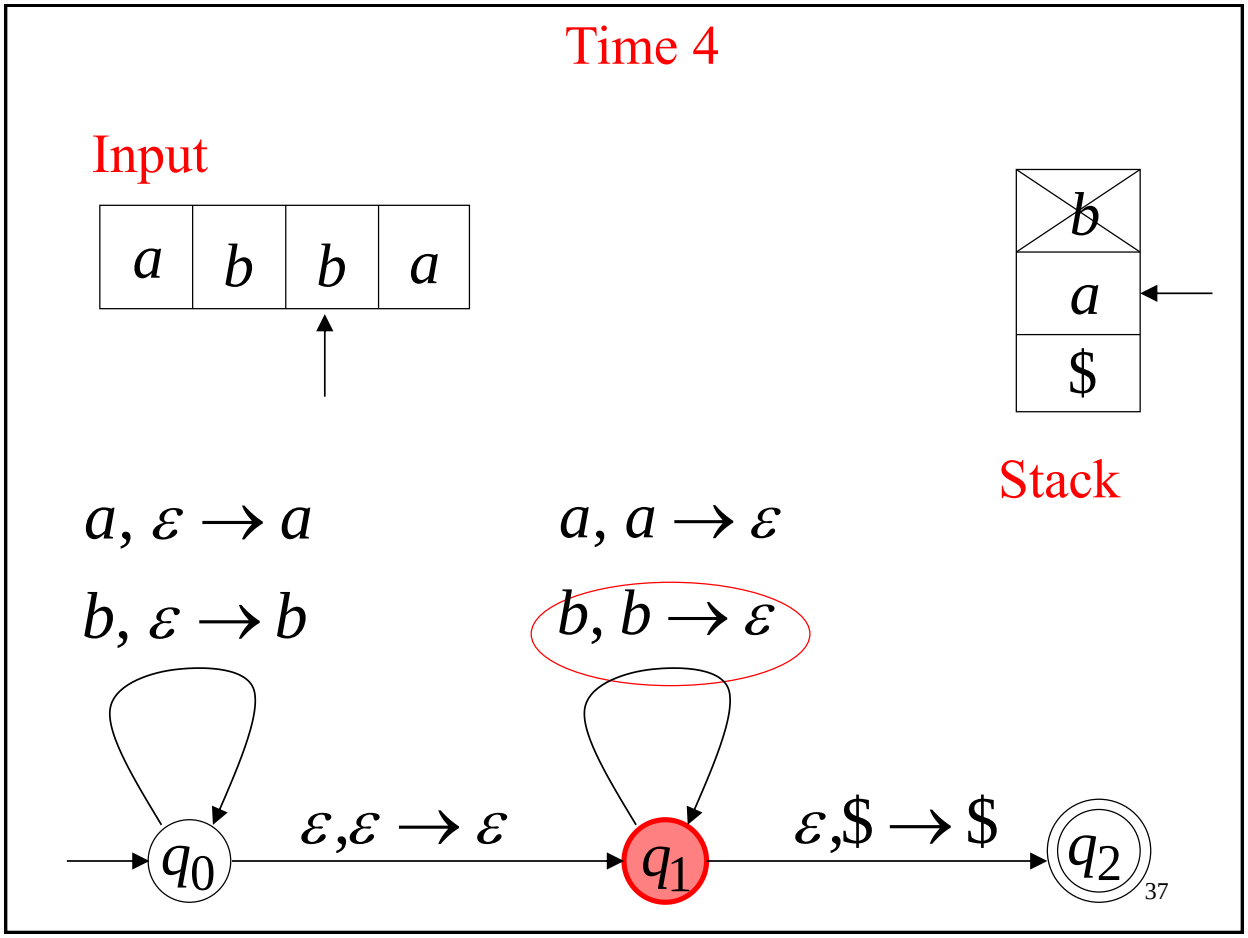
Execution Example: Time 0

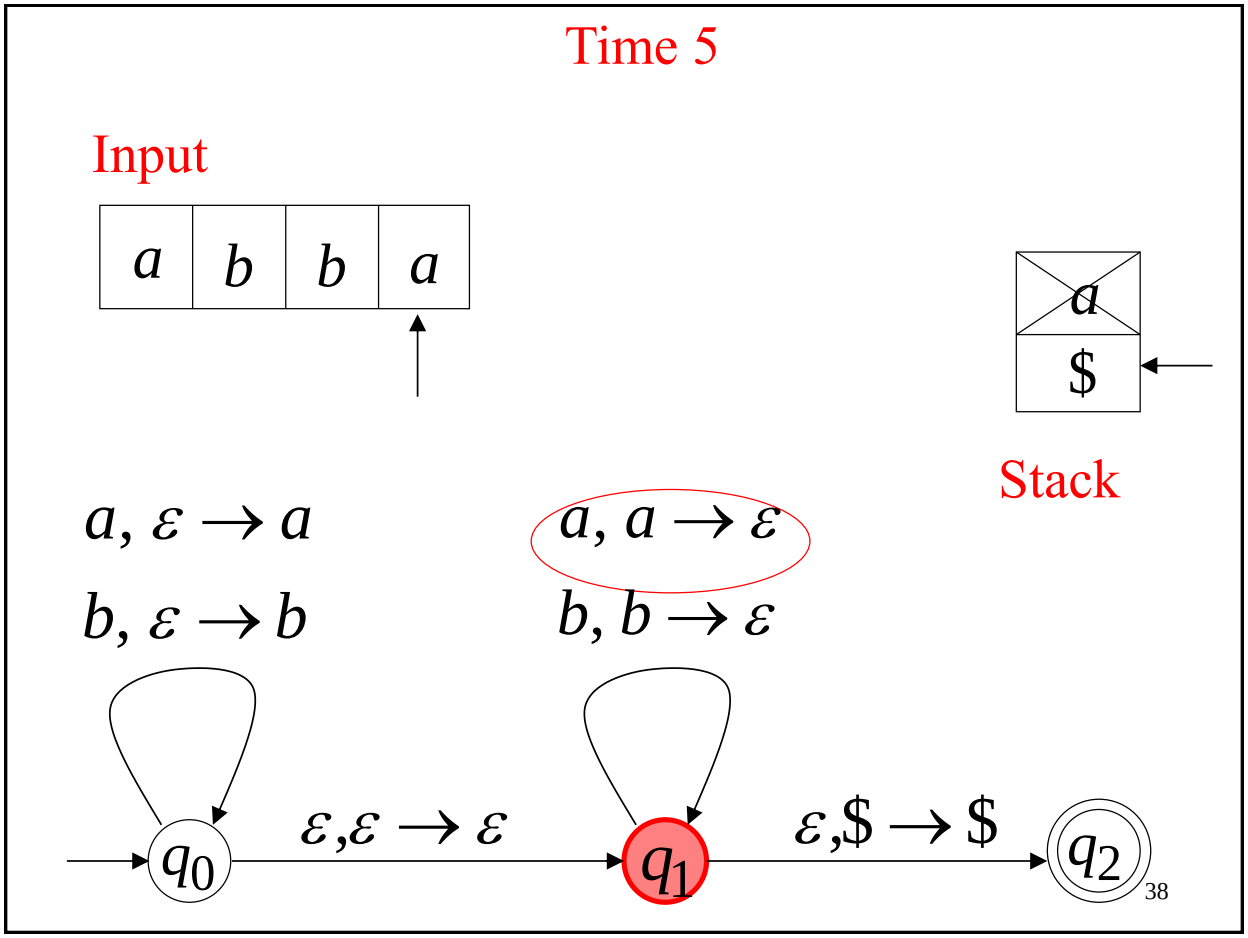


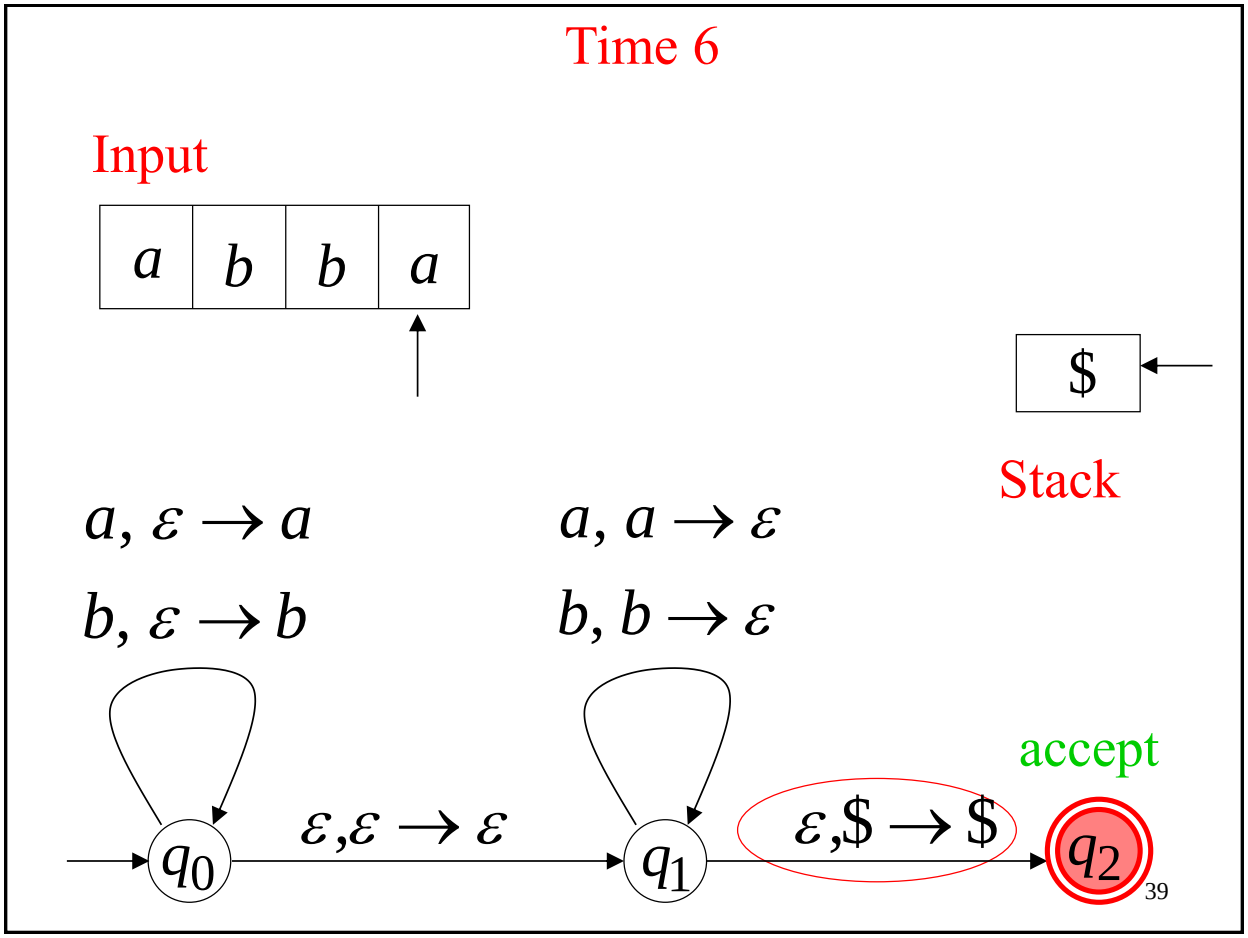

Time 1
Input
a
b
b
a
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
34


Time 2
Input
a
b
b
a
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
35

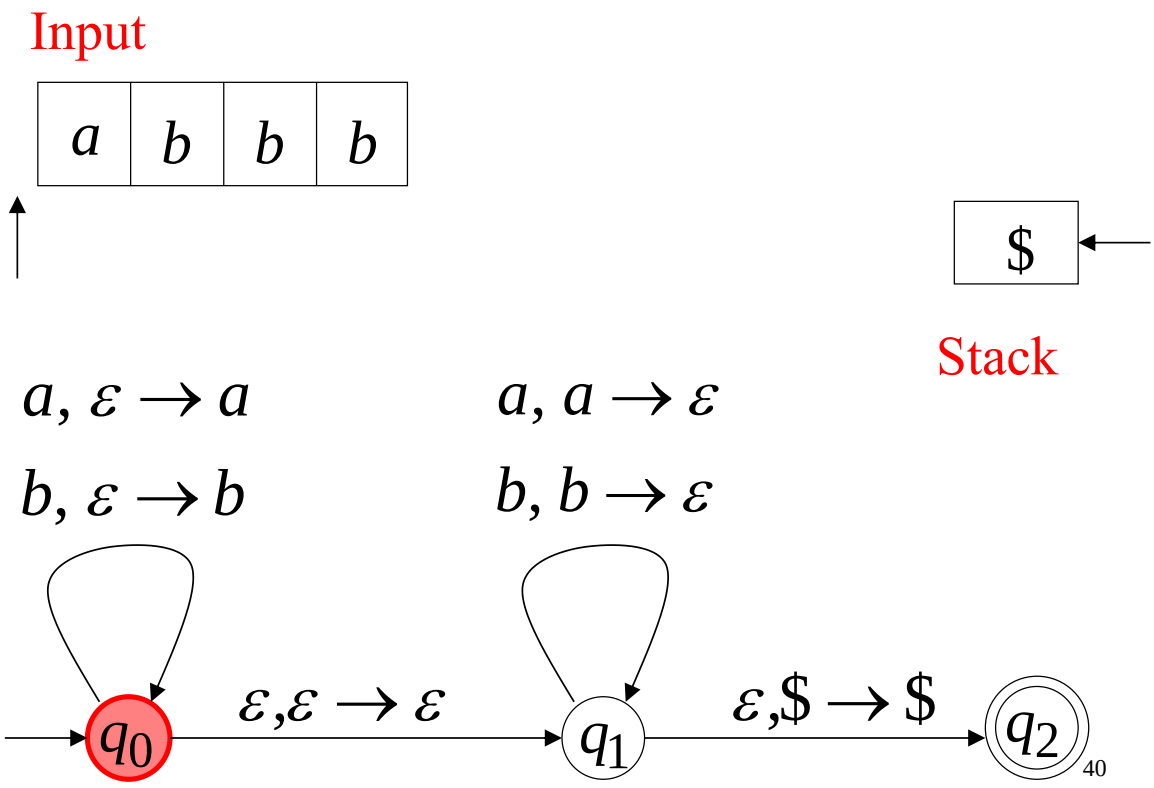

Time 3
Input
a
b
b
a
Guess the middle
of string
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
36

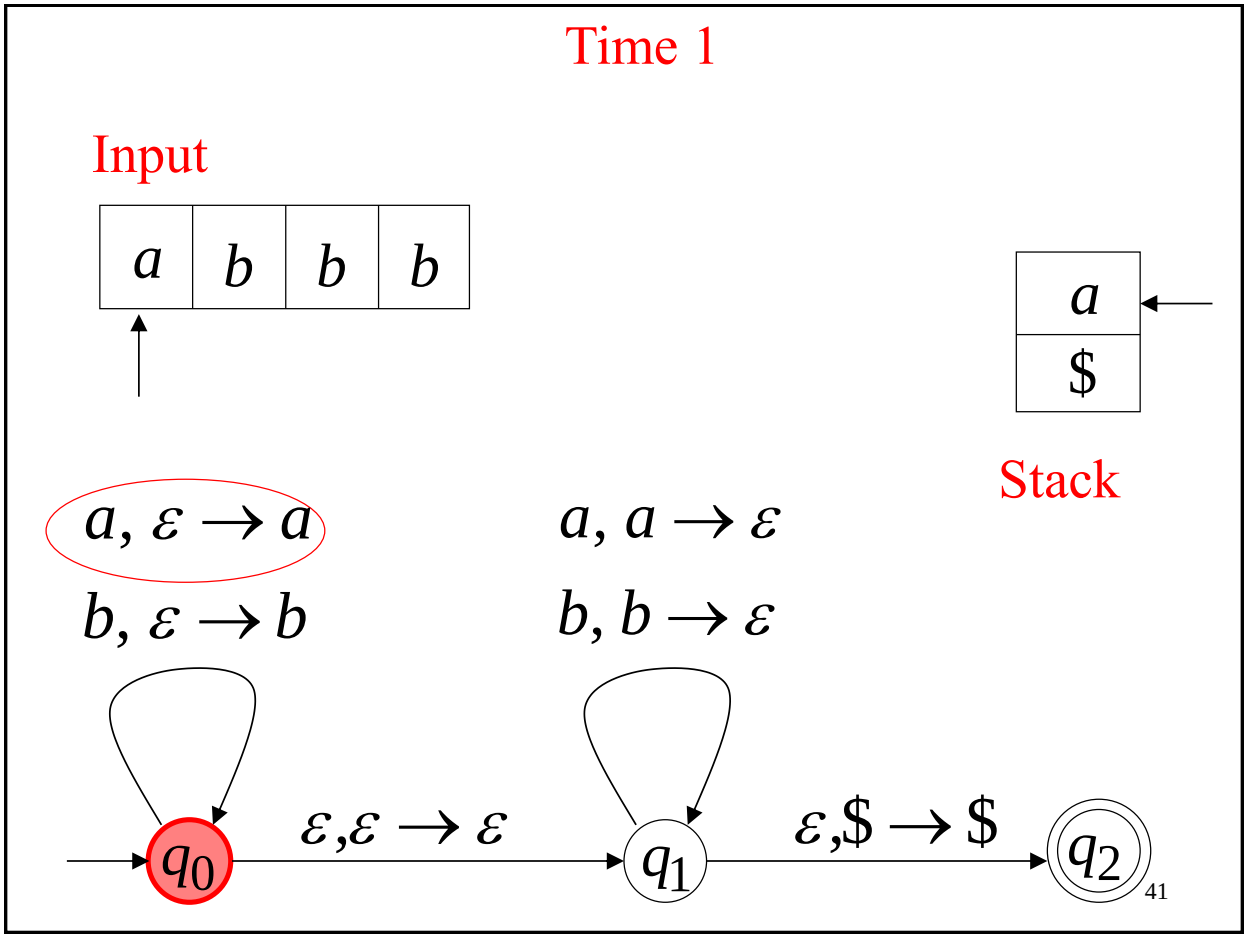

Time 4
Input
a
b
b
a
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
37

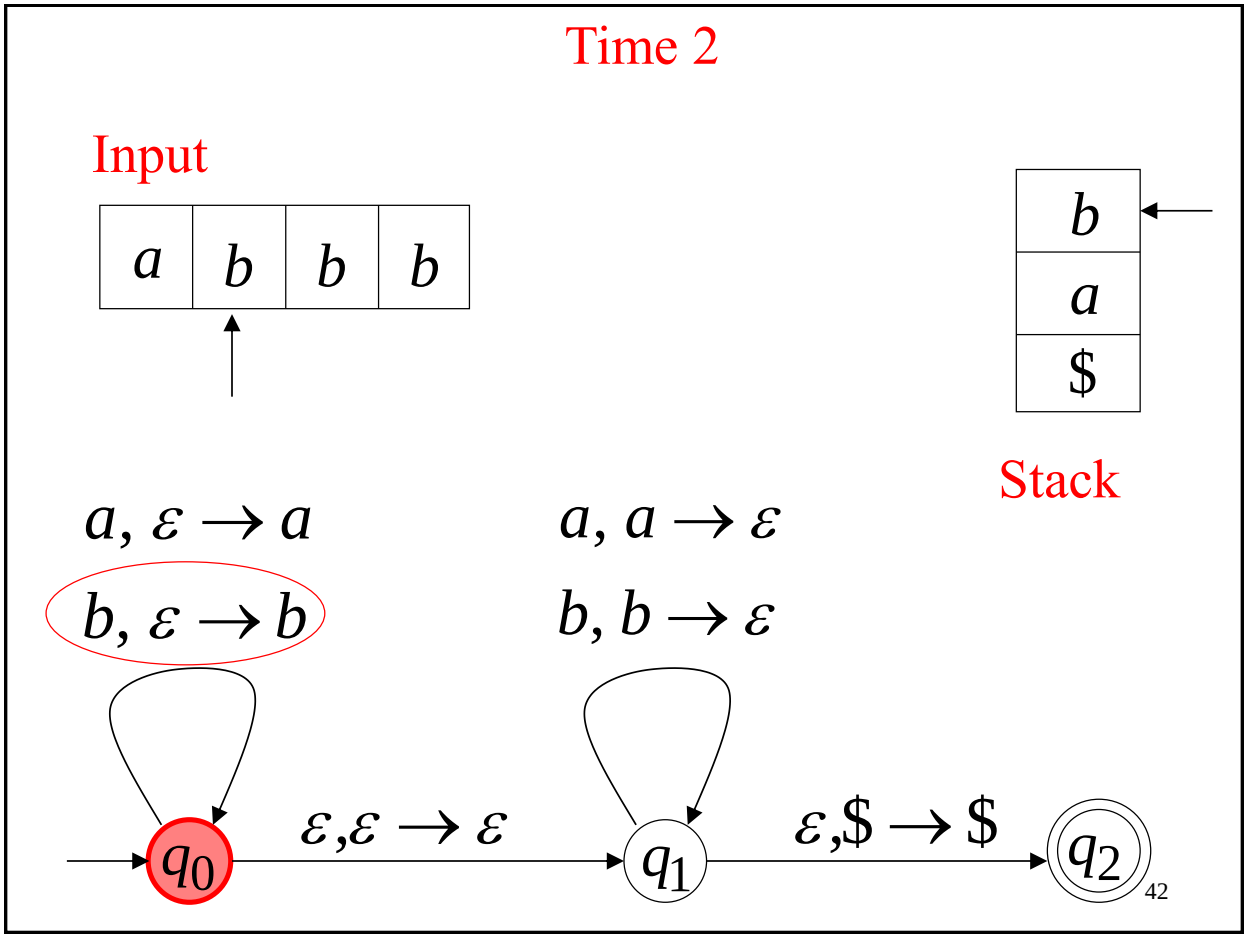

Time 5
Input
a
b
b
a
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
38

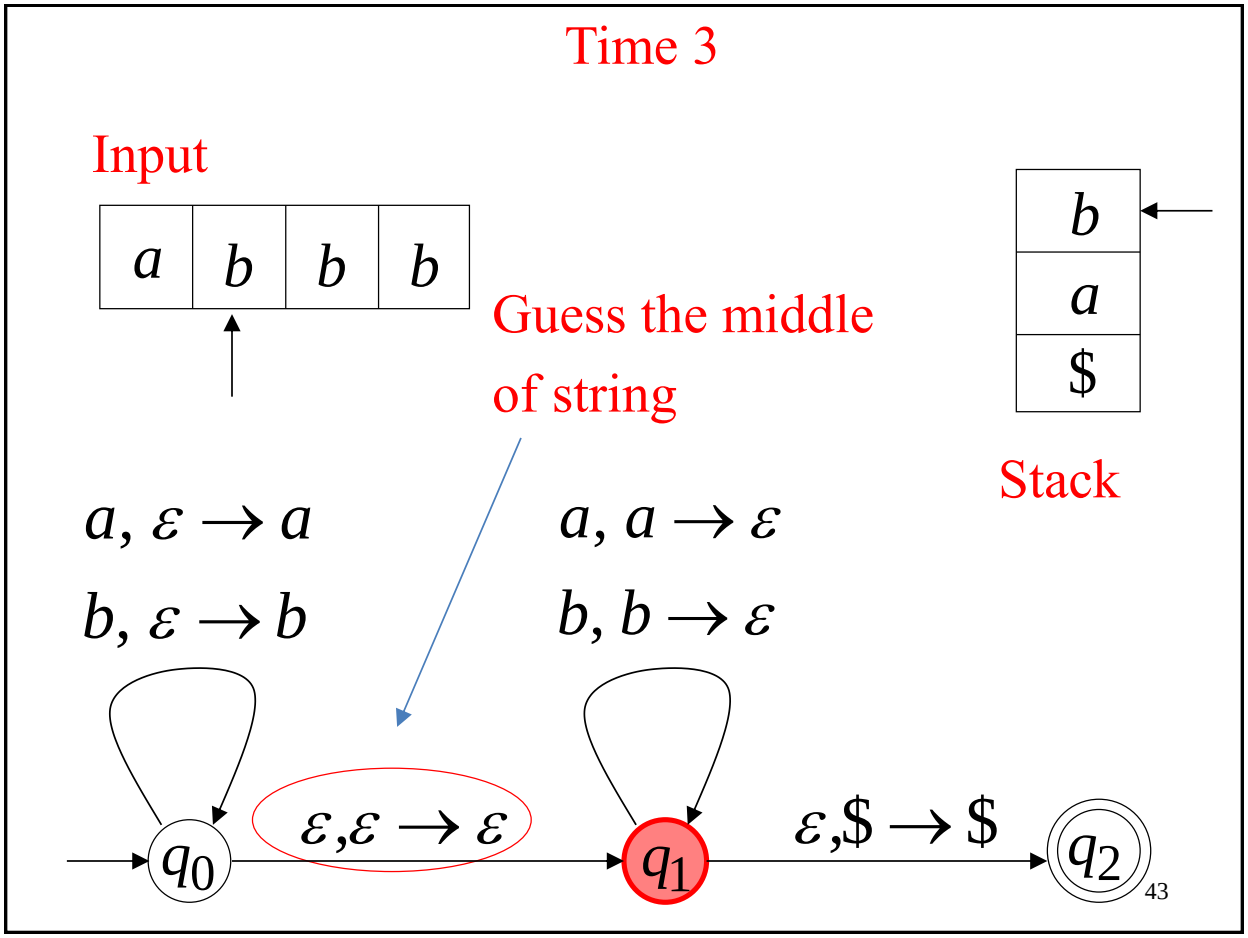

Time 6
Input
a
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
accept
ε,ε → ε
ε,$ → $
q0
q1
q2

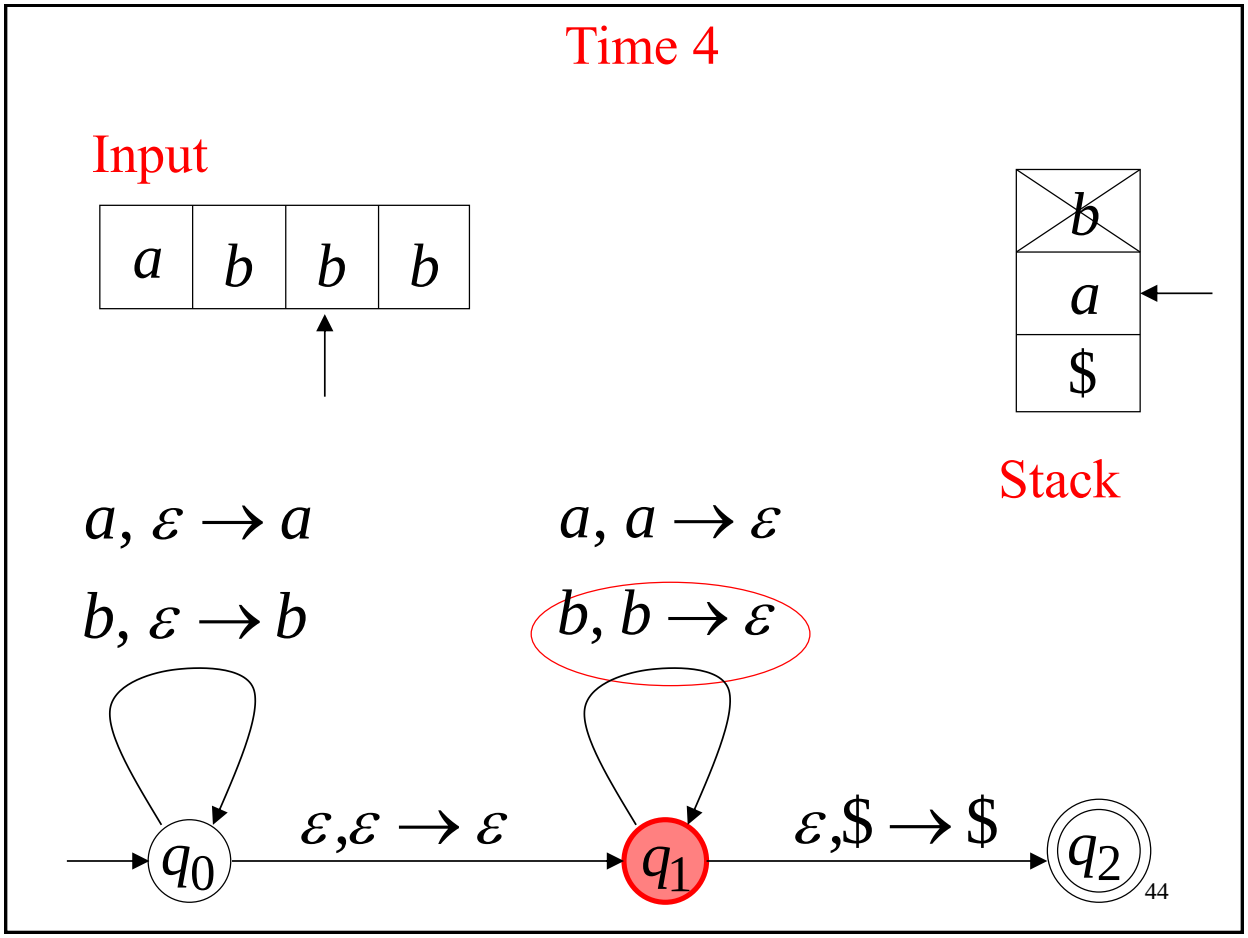
Rejection Example: Time 0




Time 1
Input
a
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
41


Time 2
Input
a
b
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
42

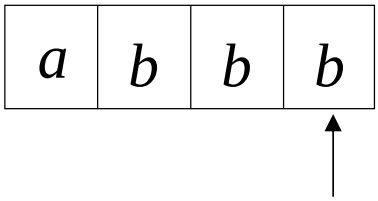

Time 3
Input
a
b
b
b
Guess the middle
of string
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
43


Time 4
Input
a
b
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
44

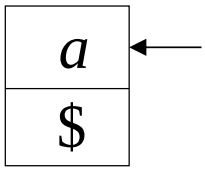
Time 5

There is no possible transition.

Input



Input is not consumed



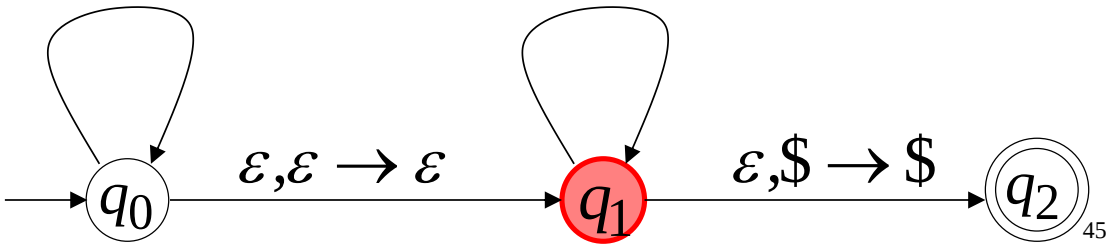
Stack

$a, \varepsilon \rightarrow a$

$b, \varepsilon \rightarrow b$

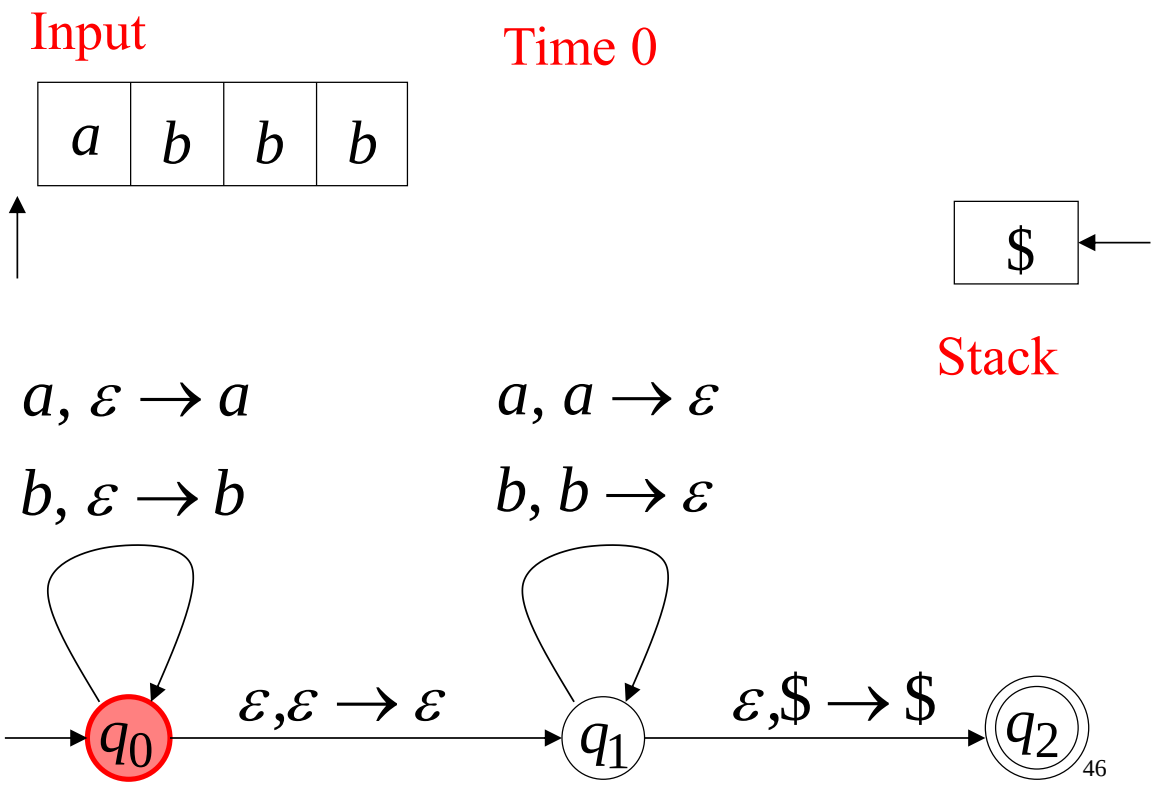
$a, a \rightarrow \varepsilon$

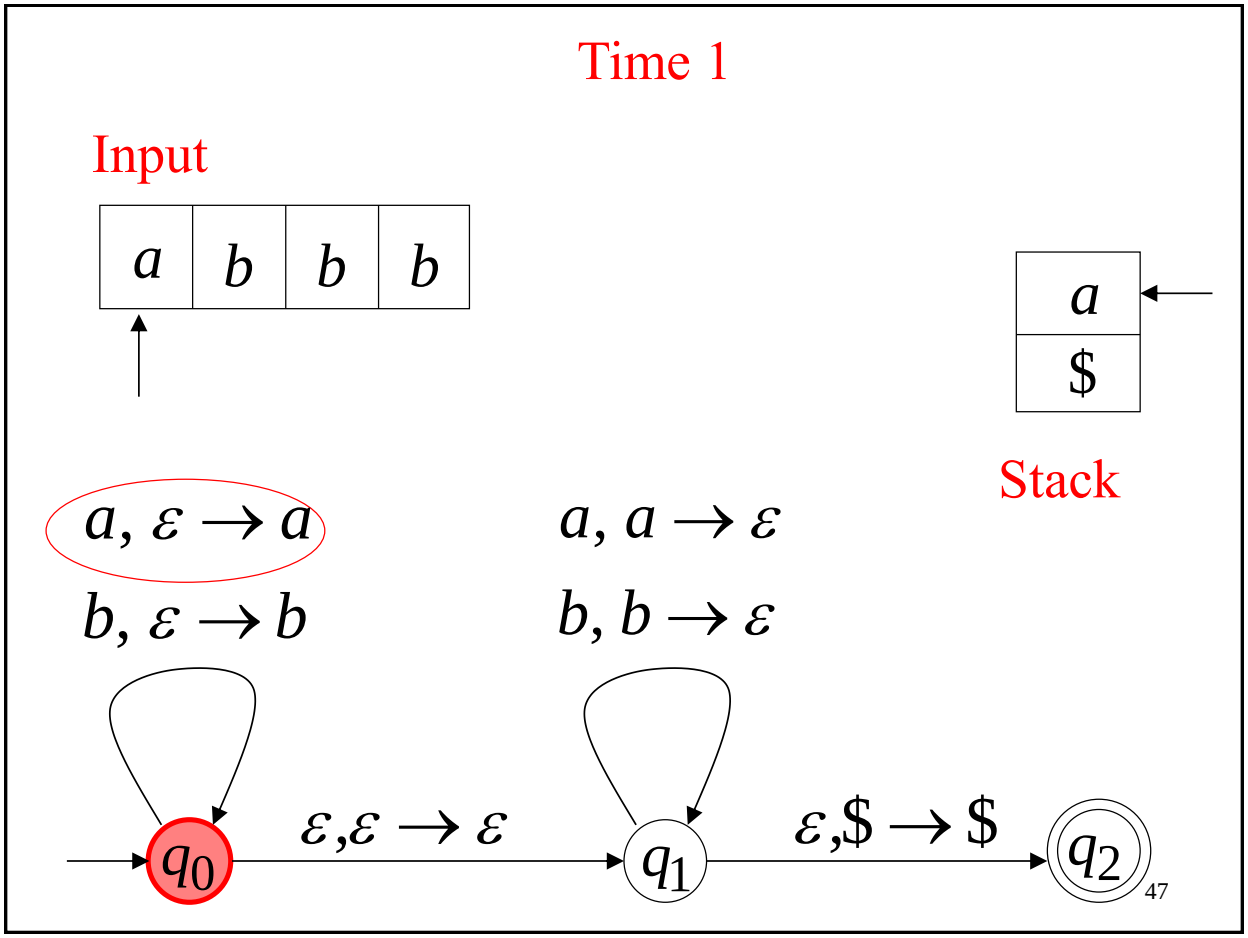
$b, b \rightarrow \varepsilon$

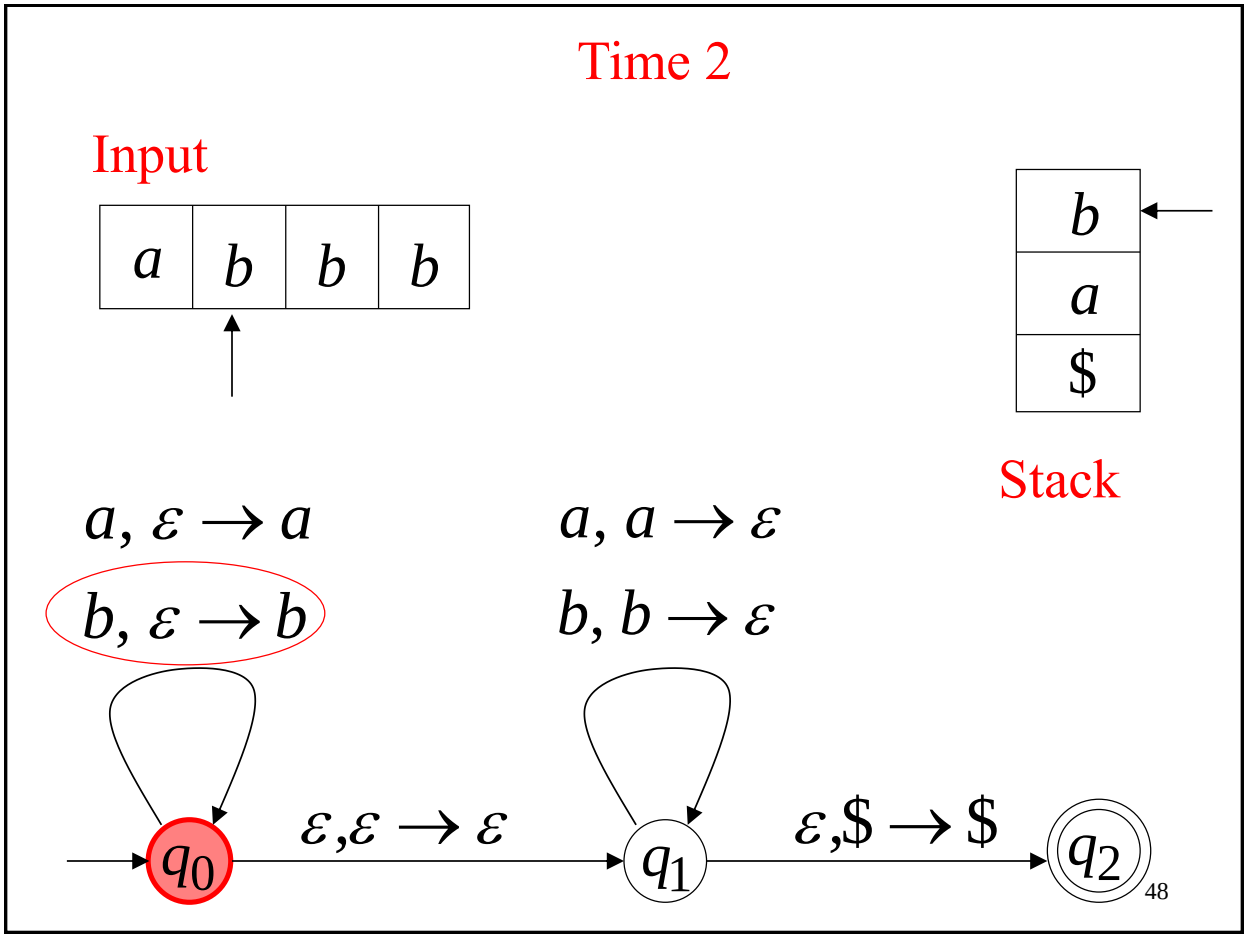


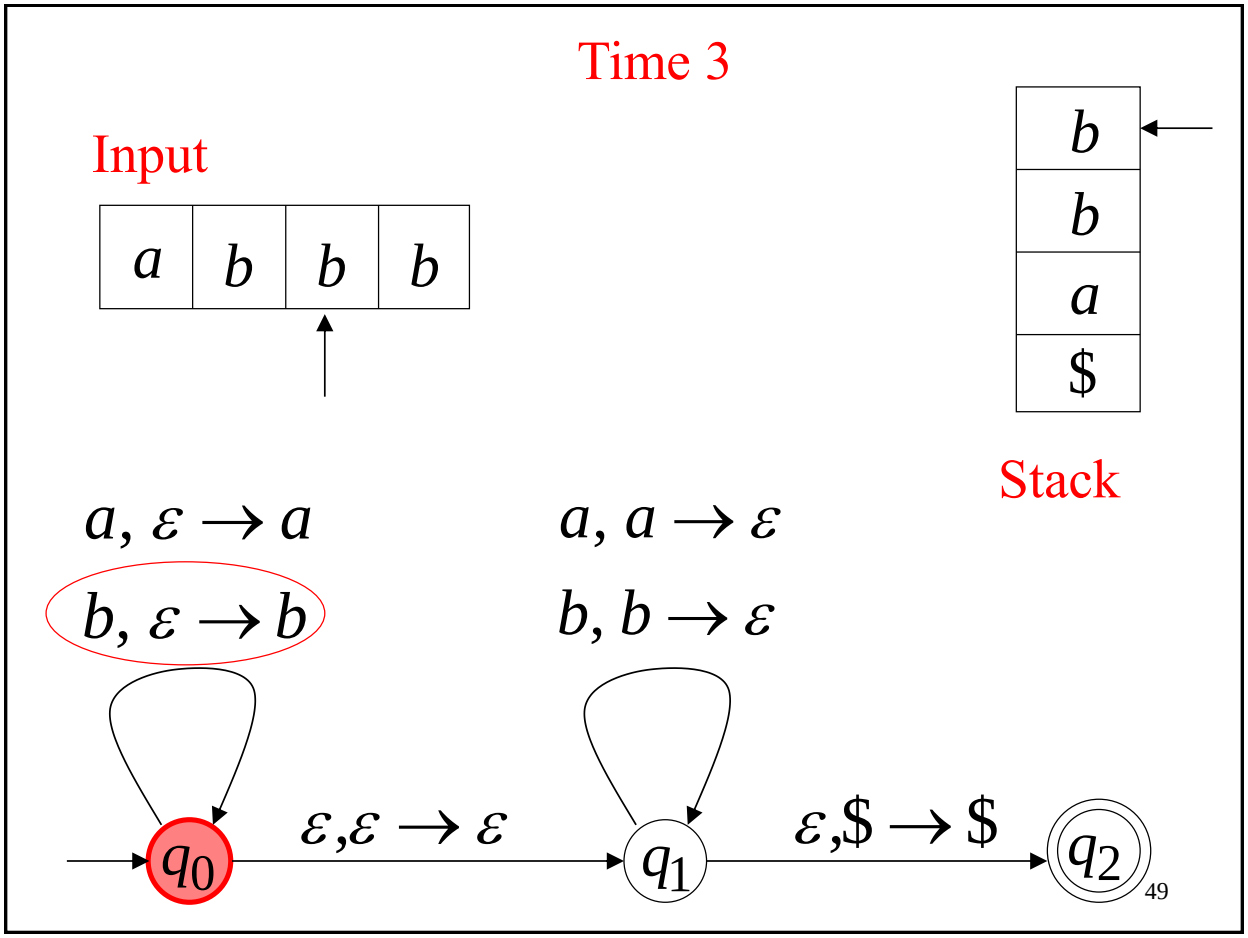
45

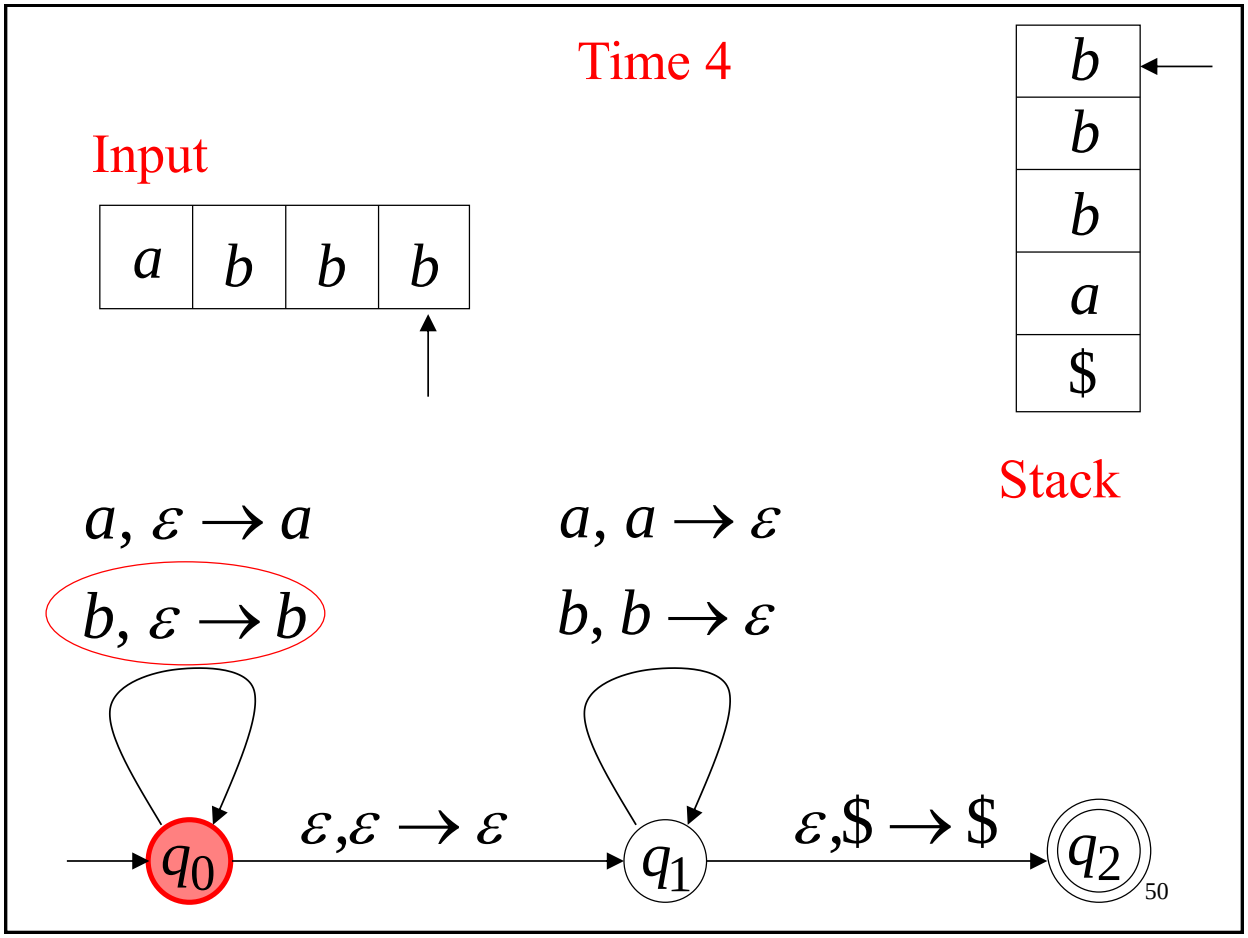
Another computation on the same string:

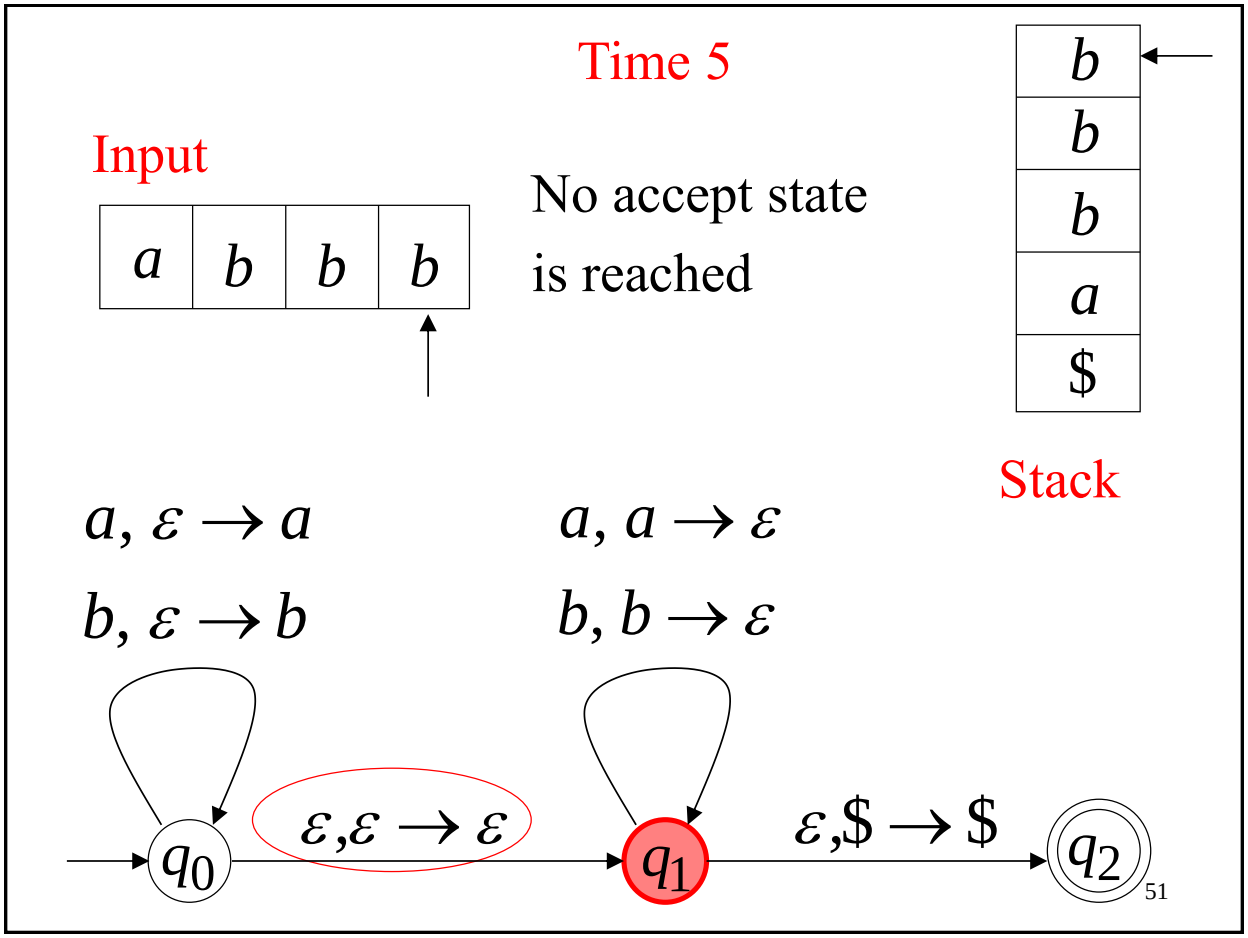



Time 1
Input
a b b b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
47


Time 2
Input
a
b
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
48


Time 3
Input
a
b
b
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
49


Time 4
Input
a
b
b
b
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
50


Time 5
Input
a b b b
No accept state
is reached
b
b
b
a
$
Stack
a, ε → a
b, ε → b
a, a → ε
b, b → ε
ε,ε → ε
ε,$ → $
q0
q1
q2
51

There is no computation
that accepts string $abbb$

$$abbb \notin L(M)$$