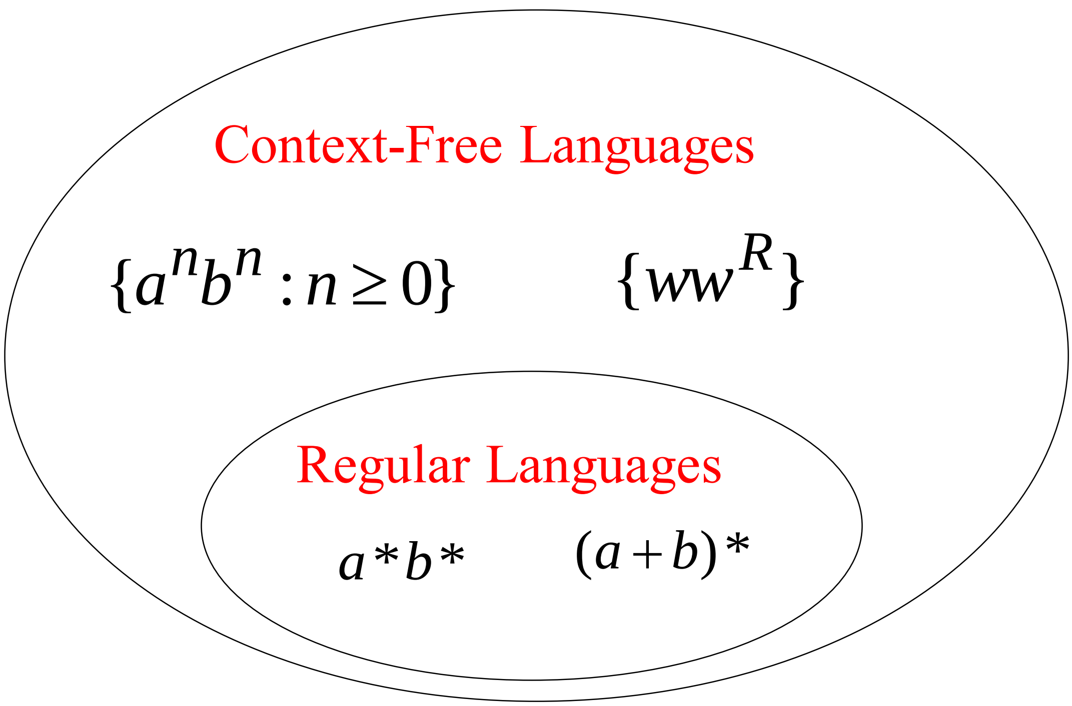
CSC 339 – Theory of Computation
Fall 2023

7. Context-Free Languages

Outline

- Context-free languages
- Grammars
- Context-free grammars
- Derivation order and derivation trees
- Ambiguity


Context-Free Languages
$\{a^n b^n : n \geq 0\}$
$\{ww^R\}$
Regular Languages
a^*b^*
$(a+b)^*$

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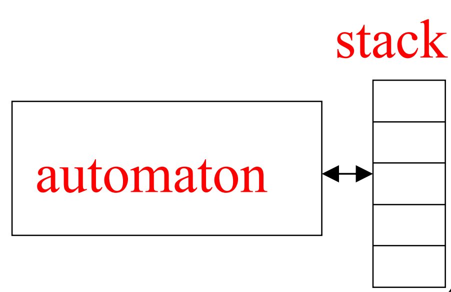
Context-Free Languages

Context-Free Grammars

Pushdown Automata



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Grammars

- Grammars express languages
- Example: the English language grammar

$$\langle sentence \rangle \rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle$$

$$\langle noun\_phrase \rangle \rightarrow \langle article \rangle \ \langle noun \rangle$$

$$\langle predicate \rangle \rightarrow \langle verb \rangle$$

$$\langle article \rangle \rightarrow a$$

$$\langle article \rangle \rightarrow the$$

$$\langle noun \rangle \rightarrow cat$$

$$\langle noun \rangle \rightarrow dog$$

$$\langle verb \rangle \rightarrow runs$$

$$\langle verb \rangle \rightarrow sleeps$$

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- Derivation of string "the dog walks":

$$\langle sentence \rangle \Rightarrow \langle noun\_phrase \rangle \; \langle predicate \rangle$$
$$\Rightarrow \langle noun\_phrase \rangle \; \langle verb \rangle$$
$$\Rightarrow \langle article \rangle \; \langle noun \rangle \; \langle verb \rangle$$
$$\Rightarrow the \; \langle noun \rangle \; \langle verb \rangle$$
$$\Rightarrow the \; dog \; \langle verb \rangle$$
$$\Rightarrow the \; dog \; sleeps$$

- Derivation of string "a cat runs":

$$
\begin{aligned}
\langle sentence \rangle &\Rightarrow \langle noun\_phrase \rangle \ \langle predicate \rangle \\
&\Rightarrow \langle noun\_phrase \rangle \ \langle verb \rangle \\
&\Rightarrow \langle article \rangle \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ \langle noun \rangle \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ \langle verb \rangle \\
&\Rightarrow a \ \ cat \ \ runs
\end{aligned}
$$

- Language of the grammar:

L = { "a cat runs",
"a cat sleeps",
"the cat runs",
"the cat sleeps",
"a dog runs",
"a dog sleeps",
"the dog runs",
"the dog sleeps"… }

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Productions

$$\langle noun \rangle \rightarrow \overbrace{cat}^{\text{Sequence of Terminals (symbols)}}$$

$$\langle sentence \rangle \rightarrow \underbrace{\langle noun\_phrase \rangle \ \langle predicate \rangle}_{\text{Sequence of Variables}}$$

Variables

Another Example

Sequence of
terminals and variables

Grammar: $S \to \overbrace{aSb}$

$S \to \varepsilon$

Variable

The right side
may be ε

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- Grammar:

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

- Derivation of string ab:

$$S \Rightarrow aSb \Rightarrow ab$$

$$S \rightarrow aSb \qquad\qquad S \rightarrow \varepsilon$$

- Grammar:

$$S \to aSb$$

$$S \to \varepsilon$$

- Derivation of string *aabb*:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$$S \to aSb \qquad\qquad S \to \varepsilon$$

Grammar: $S \to aSb$

$S \to \varepsilon$

Other derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb$$
$$\Rightarrow aaaaSbbbb \Rightarrow aaaabbbb$$

Grammar: $S \rightarrow aSb$

$S \rightarrow \varepsilon$

Language of the grammar:

$$L = \{a^n b^n : n \geq 0\}$$

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A Convenient Notation

- We write:

$$S \overset{*}{\Rightarrow} aaabbb$$

for zero or more derivation steps

- Instead of:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

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In general we write: $w_1 \overset{*}{\Rightarrow} w_n$

If: $w_1 \Rightarrow w_2 \Rightarrow w_3 \Rightarrow \cdots \Rightarrow w_n$

in zero or more derivation steps

Trivially: $w \overset{*}{\Rightarrow} w$

Example Grammar

$$S \rightarrow aSb$$

$$S \rightarrow \varepsilon$$

Possible Derivations

$$S \overset{*}{\Rightarrow} \varepsilon$$

$$S \overset{*}{\Rightarrow} ab$$

$$S \overset{*}{\Rightarrow} aaabbb$$

$$S \overset{*}{\Rightarrow} aaSbb \overset{*}{\Rightarrow} aaaaaSbbbbb$$

Another convenient notation:

$$S \to aSb$$
$$S \to \varepsilon$$

$$\Longrightarrow \quad S \to aSb \mid \varepsilon$$

Example:

$$\langle article \rangle \to a$$
$$\langle article \rangle \to the$$

$$\Longrightarrow \quad \langle article \rangle \to a \mid the$$

Formal Definitions

Grammar: $G = (V, T, S, P)$

Set of variables

Set of terminal symbols

Start variable

Set of productions

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Context-Free Grammar: $G = (V, T, S, P)$

All productions in P are of the form:

$$A \rightarrow s$$

Variable

String of
variables and
terminals

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Example of Context-Free Grammar

$$\boxed{S \to aSb \mid \varepsilon}$$

productions

$$P = \{S \to aSb,\ \ S \to \varepsilon\}$$

$$G = (V, T, S, P)$$

$$V = \{S\}$$

variables

$$T = \{a, b\}$$

terminals

start variable

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Language of a Grammar:

For a grammar G with start variable S

$$L(G) = \{w : S \overset{*}{\Rightarrow} w, \quad w \in T^*\}$$

String of terminals or ε

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Example:

Context-free grammar: G $\boxed{S \to aSb \mid \varepsilon}$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Since, there is derivation

$$S \overset{*}{\Rightarrow} a^n b^n \text{ for any } n \geq 0$$

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Context-Free Language:

- A language L is context-free
  - if there is a context-free grammar G
  - with $L = L(G)$

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Example 1:

$$L = \{a^n b^n : n \geq 0\}$$

is a context-free language

since context-free grammar: G $\boxed{S \rightarrow aSb \mid \varepsilon}$

that generates $L(G) = L$

Example 2:

Context-free grammar: G

$$S \to aSa \mid bSb \mid \varepsilon$$

Example derivations:

$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$

$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$

$$L(G) = \{ww^R : \quad w \in \{a,b\}^*\}$$

Palindromes of even length

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Example 3:

Context-free grammar: G

$$S \to aSb \mid SS \mid \varepsilon$$

Example derivations:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$L(G) = \{w : n_a(w) = n_b(w),$$

and $n_a(v) \geq n_b(v)$

in any prefix $v\}$

Describes matched parentheses: () ((( ))) (( )) $a = ($; $b =)$

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Derivation Order
and
Derivation Trees

Derivation Order

Consider the following example grammar with 5 productions:

1. $S \to AB$	2. $A \to aaA$	4. $B \to Bb$
	3. $A \to \varepsilon$	5. $B \to \varepsilon$

1. $S \rightarrow AB$	2. $A \rightarrow aaA$	4. $B \rightarrow Bb$
	3. $A \rightarrow \varepsilon$	5. $B \rightarrow \varepsilon$

Leftmost derivation order of string aab:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

At each step, we substitute the leftmost variable

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1. $S \to AB$
2. $A \to aaA$
3. $A \to \varepsilon$
4. $B \to Bb$
5. $B \to \varepsilon$

Rightmost derivation order of string aab :

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

At each step, we substitute the rightmost variable

1. $S \to AB$	2. $A \to aaA$	4. $B \to Bb$
	3. $A \to \varepsilon$	5. $B \to \varepsilon$

Leftmost derivation of aab:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

Rightmost derivation of aab:

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

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Derivation Trees

Consider the same example grammar:

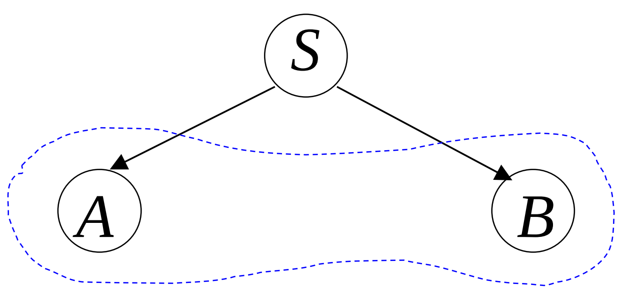
$$S \to AB \qquad A \to aaA \mid \varepsilon \qquad B \to Bb \mid \varepsilon$$

And a derivation of aab:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

$$S \rightarrow AB \qquad A \rightarrow aaA \mid \varepsilon \qquad B \rightarrow Bb \mid \varepsilon$$

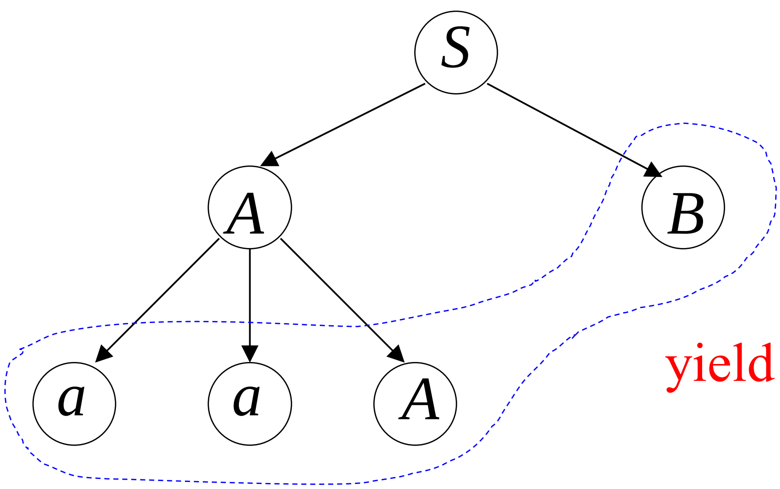
$$S \Rightarrow AB$$



yield AB

$$S \to AB \qquad A \to aaA \mid \varepsilon \qquad B \to Bb \mid \varepsilon$$

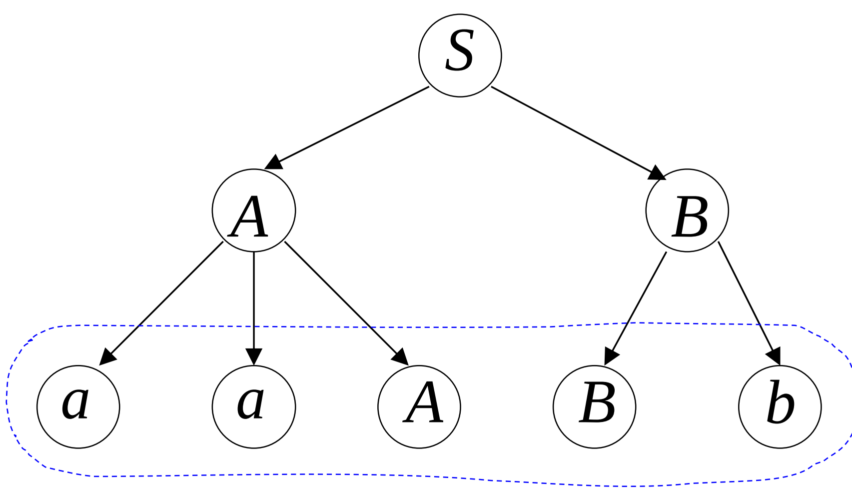
$$S \Rightarrow AB \Rightarrow aaAB$$



yield $aaAB$

$$S \to AB \qquad A \to aaA \mid \varepsilon \qquad B \to Bb \mid \varepsilon$$

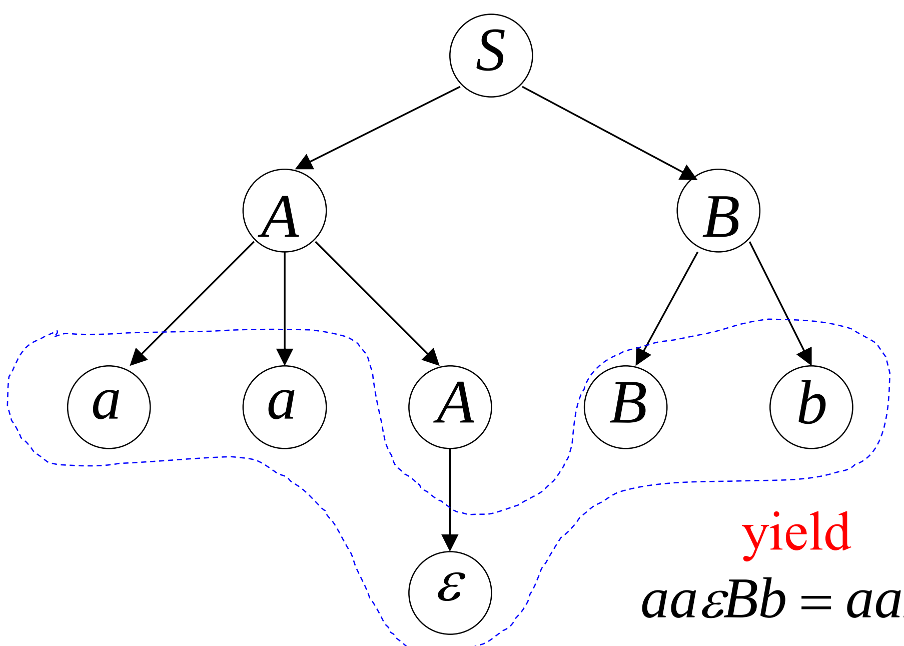
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$



yield $aaABb$

$$S \to AB \qquad A \to aaA \mid \varepsilon \qquad B \to Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb$$



yield

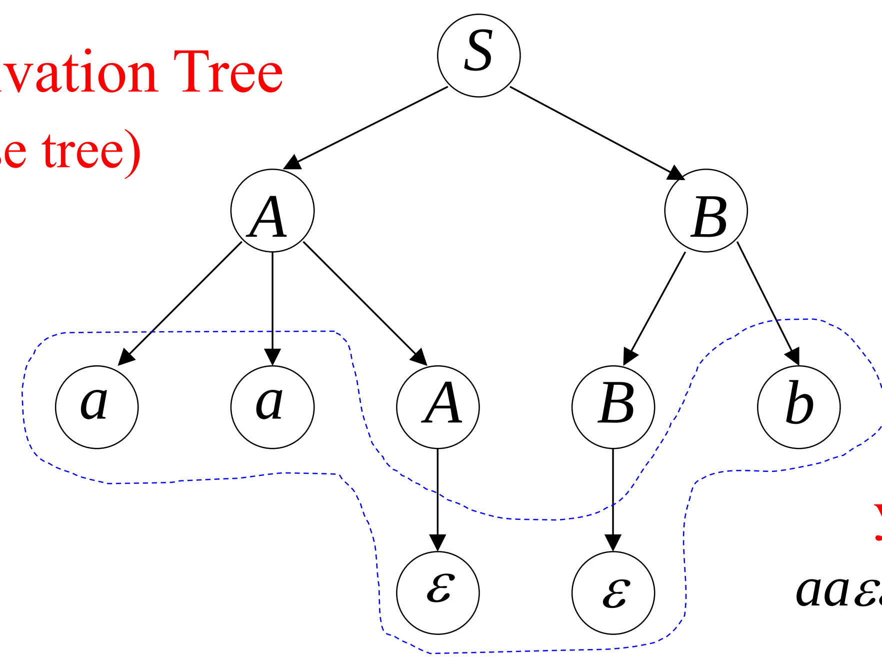
$$aa\varepsilon Bb = aaBb$$

$$S \rightarrow AB \qquad A \rightarrow aaA \mid \varepsilon \qquad B \rightarrow Bb \mid \varepsilon$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree

(parse tree)



yield

$aa\varepsilon\varepsilon b = aab$

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Sometimes, derivation order doesn't matter

Leftmost derivation:

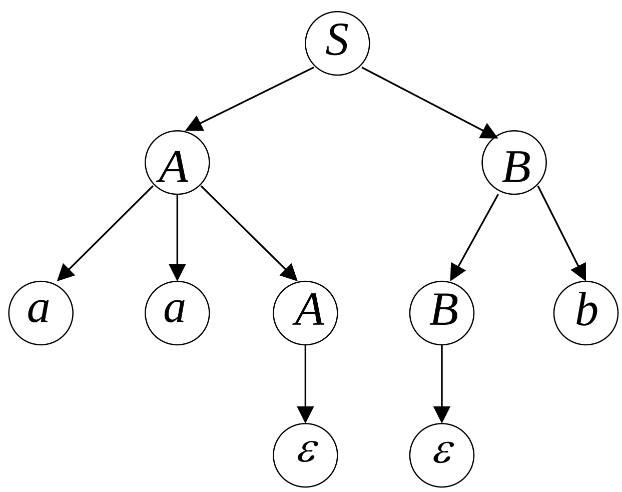
$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost derivation:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

Give same

derivation tree



Ambiguity

Grammar for mathematical expressions

$$\boxed{E \to E + E \ \mid E * E \mid (E) \mid a}$$

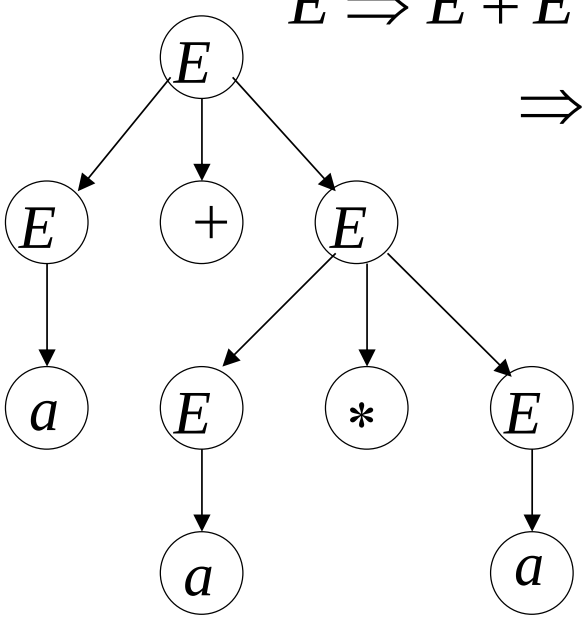
Example strings:

$$(a + a) * a + (a + a * (a + a))$$

Denotes any number

$$\boxed{E \to E + E \;\mid\; E * E \mid (E) \mid a}$$

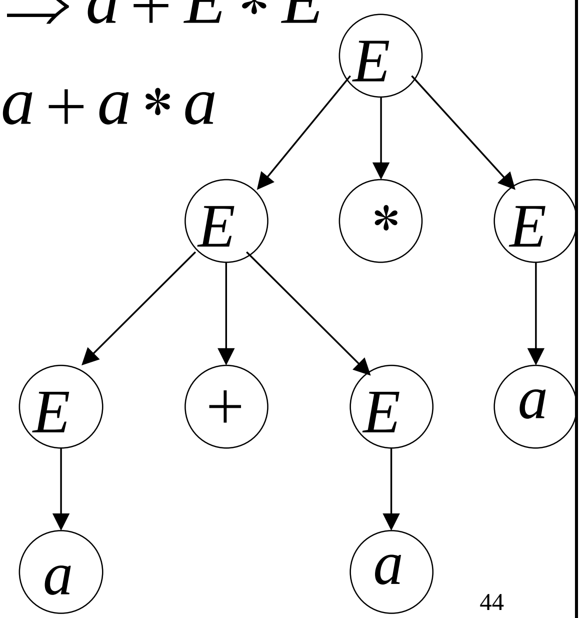
$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$



A leftmost derivation for $a + a * a$

$$E \to E + E \;\mid\; E * E \mid (E) \mid a$$

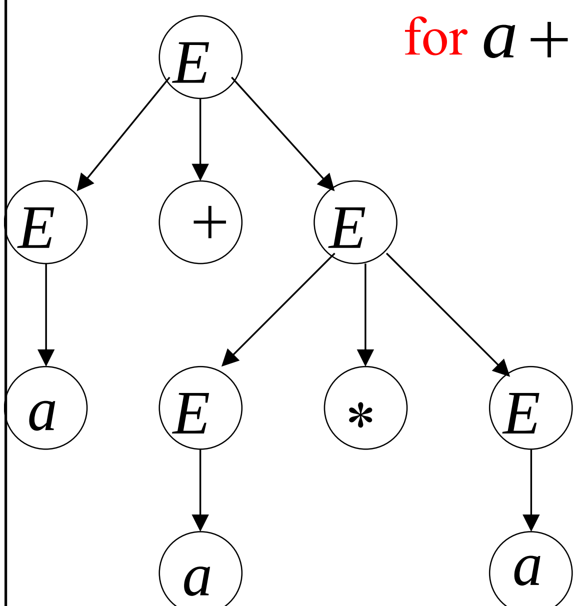
$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

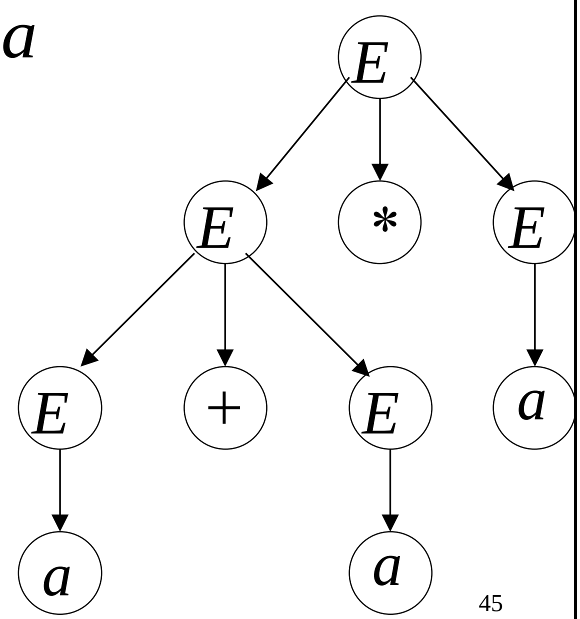


Another
leftmost derivation
for $a + a * a$

$$E \to E + E \;\mid\; E * E \;\mid\; (E) \;\mid\; a$$

Two derivation trees for $a + a * a$





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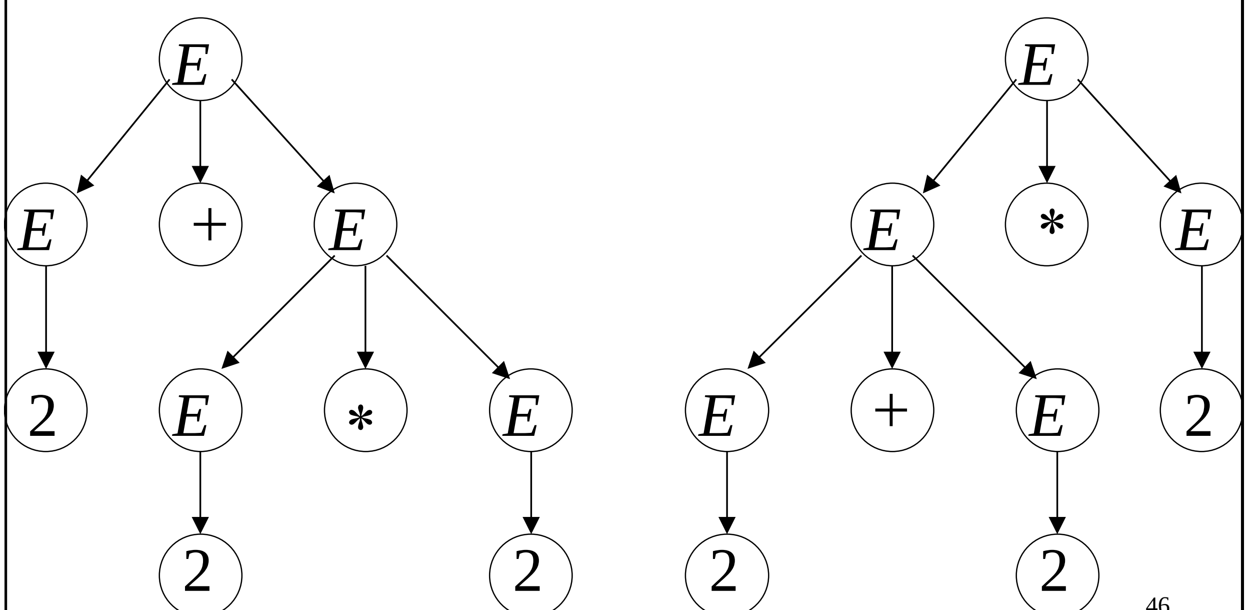
take $a = 2$

$$a + a * a = 2 + 2 * 2$$



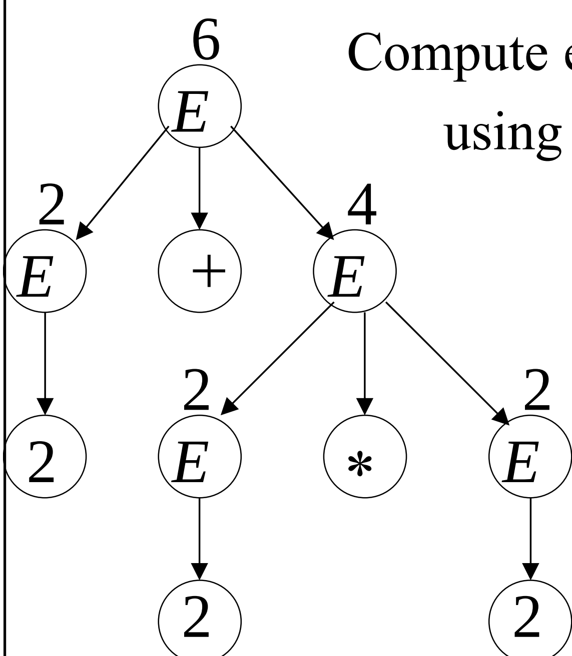
Good Tree

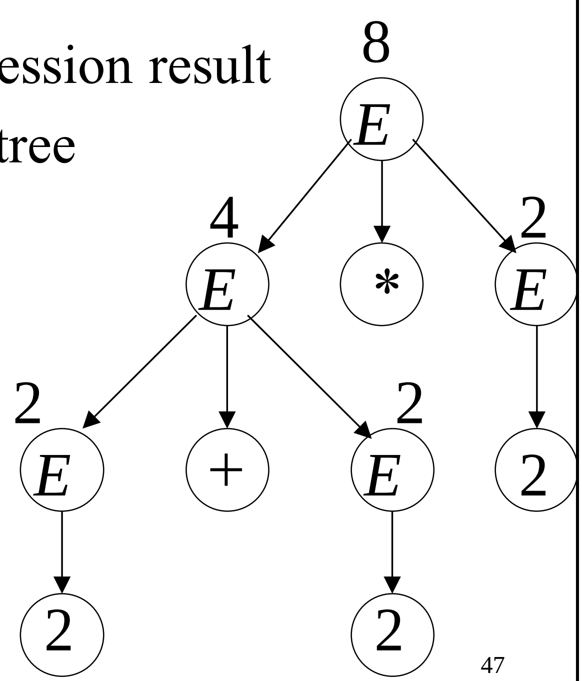
$$2+2*2=6$$

Bad Tree

$$2+2*2=8$$

Compute expression result using the tree





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Two different derivation trees
may cause problems in applications which
use the derivation trees:

- Evaluating expressions
- In general, in compilers
  for programming languages

Ambiguous Grammar:

A context-free grammar G is ambiguous if there is a string $w \in L(G)$ which has:

Two different derivation trees

or

Two leftmost derivations

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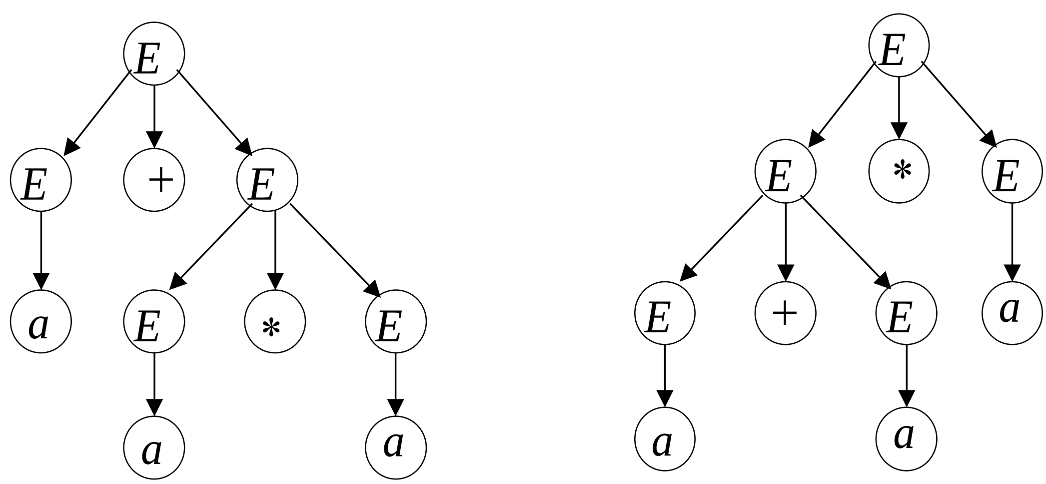
Example: $E \to E + E \;\; | \; E * E \; | \; (E) \; | \; a$

This grammar is ambiguous since

string $a + a * a$ has two derivation trees



$$\boxed{E \to E + E \;\mid\; E * E \mid (E) \mid a}$$

This grammar is ambiguous also because string $a + a * a$ has two leftmost derivations

$$E \Rightarrow E + E \Rightarrow a + E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

Another ambiguous grammar:

IF_STMT $\rightarrow$ if EXPR then STMT

| if EXPR then STMT else STMT

Variables Terminals

Common piece of grammar
in programming languages

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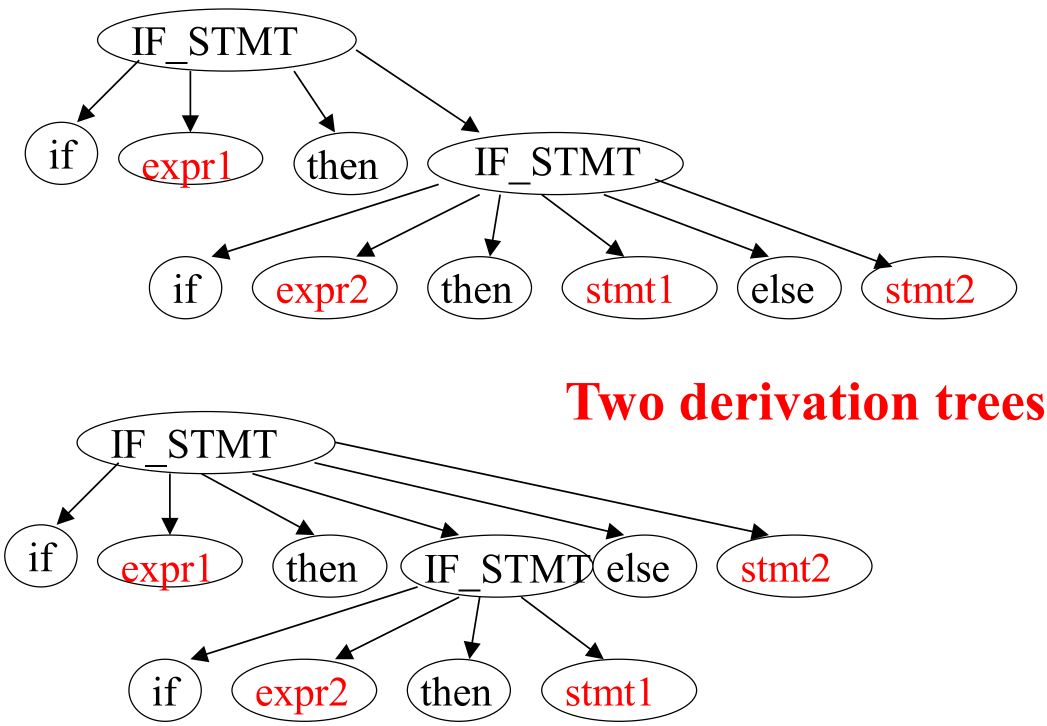
If expr1 then if expr2 then stmt1 else stmt2



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In general, ambiguity represents an
issue and we want to remove it.

Sometimes it is possible to find
a non-ambiguous grammar for a language.

But, in general, we cannot do so.

A successful example:

Ambiguous Grammar

$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$
$$E \to a$$

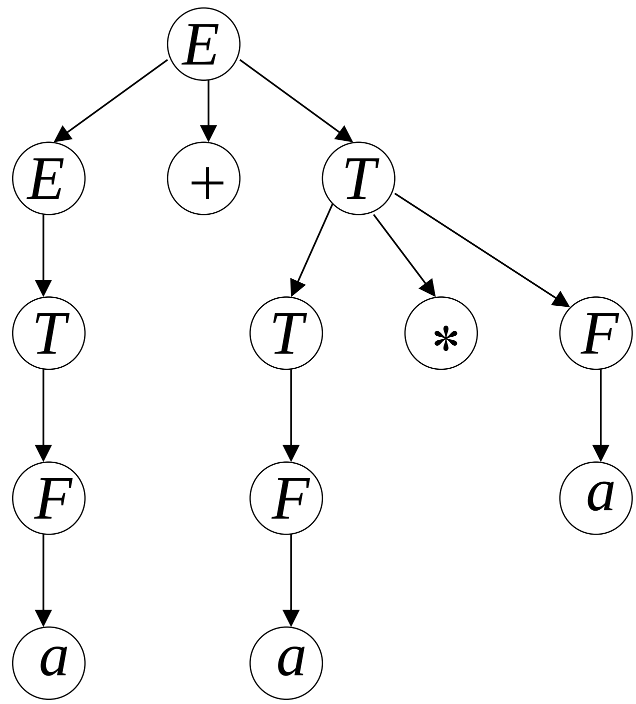
Equivalent Non-Ambiguous Grammar

$$E \to E + T \mid T$$
$$T \to T * F \mid F$$
$$F \to (E) \mid a$$

It generates the same language

$$E \Rightarrow E+T \Rightarrow T+T \Rightarrow F+T \Rightarrow a+T \Rightarrow a+T*F$$
$$\Rightarrow a+F*F \Rightarrow a+a*F \Rightarrow a+a*a$$

$$E \to E+T \mid T$$
$$T \to T*F \mid F$$
$$F \to (E) \mid a$$



Unique
derivation tree
for $a+a*a$

An un-successful example:

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\} \quad n, m \geq 0$$

L is inherently ambiguous:

Every grammar that generates this language is ambiguous.

Example (ambiguous) grammar for L :

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

$$S \to S_1 \mid S_2$$

$$S_1 \to S_1 c \mid A$$
$$A \to aAb \mid \varepsilon$$

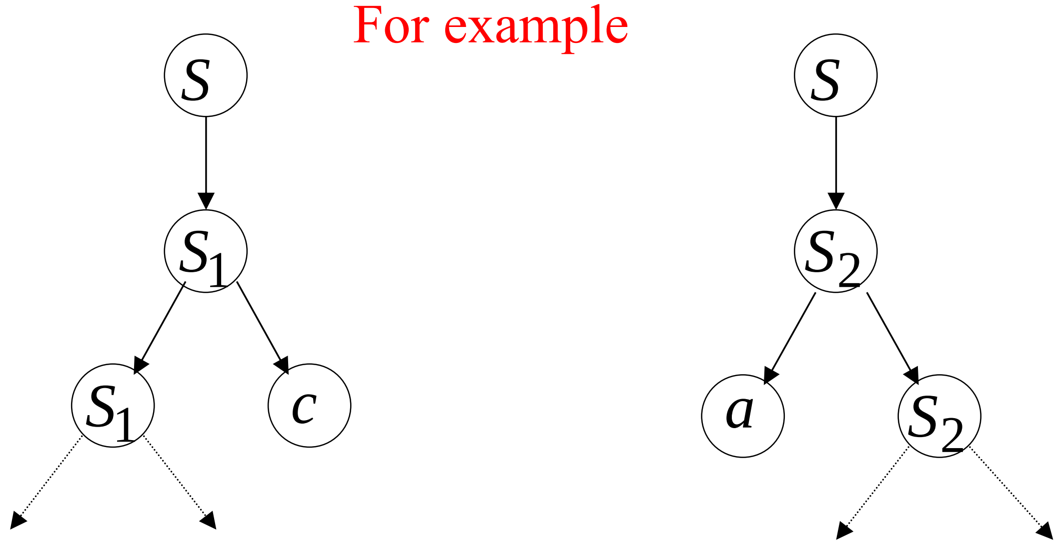
$$S_2 \to aS_2 \mid B$$
$$B \to bBc \mid \varepsilon$$

The string $a^n b^n c^n \in L$

has always two different derivation trees

(for any grammar)

For example



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