CSC 339 – Theory of Computation
Fall 2023

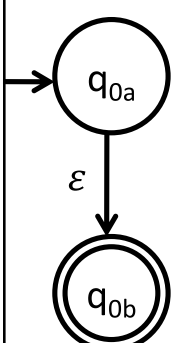
6. The Pumping Lemma

Outline

- Non-regular languages
- The pigeonhole principle
- The pumping lemma

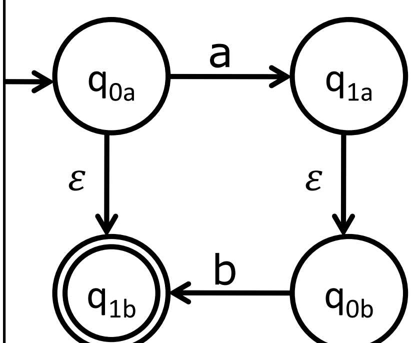
Non-Regular Languages

- $L_0 = \{a^k b^k : k \leq 0\} = \{\varepsilon\}$ is a regular language



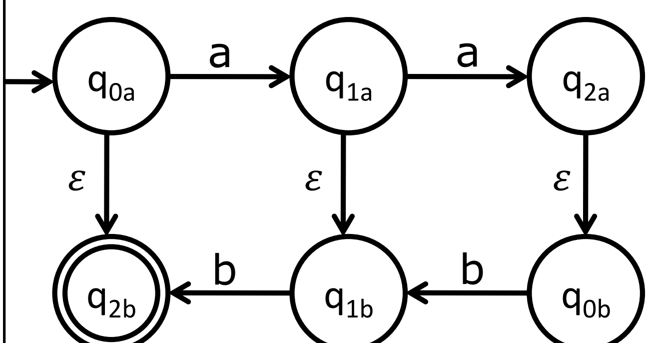
Non-Regular Languages

- $L_1 = \{a^k b^k : k \leq 1\} = \{\varepsilon, ab\}$ is regular



Non-Regular Languages

- $L_2 = \{a^k b^k : k \leq 2\} = \{\varepsilon, ab, aabb\}$ is regular



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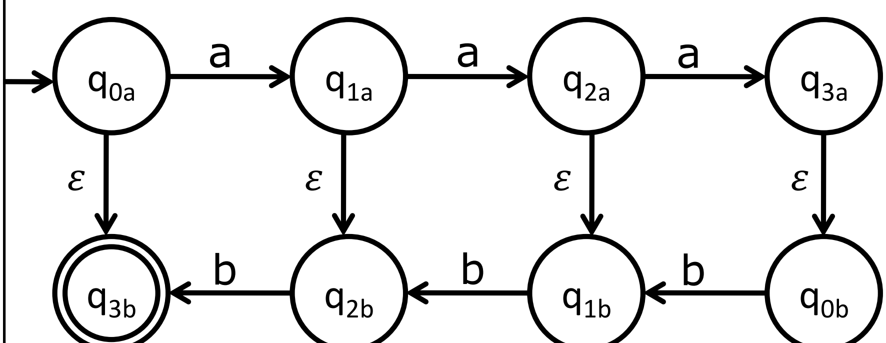
Non-Regular Languages

- $L_3 = \{a^k b^k : k \leq 3\}$ is regular



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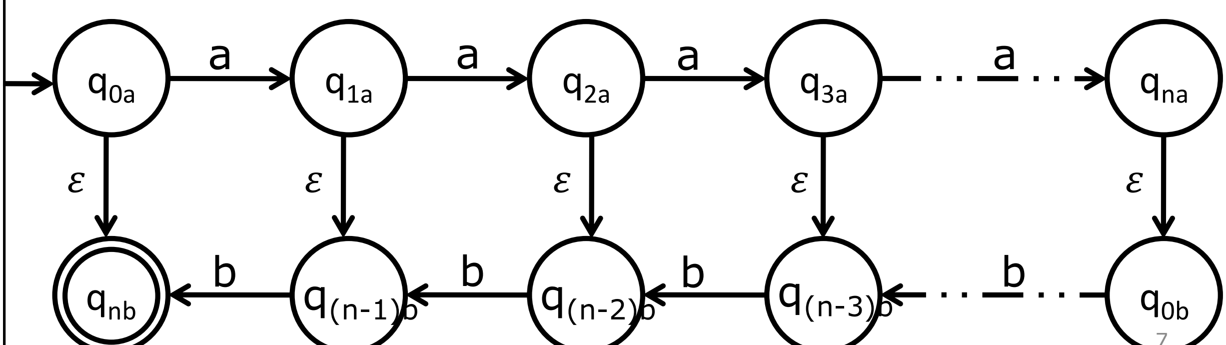
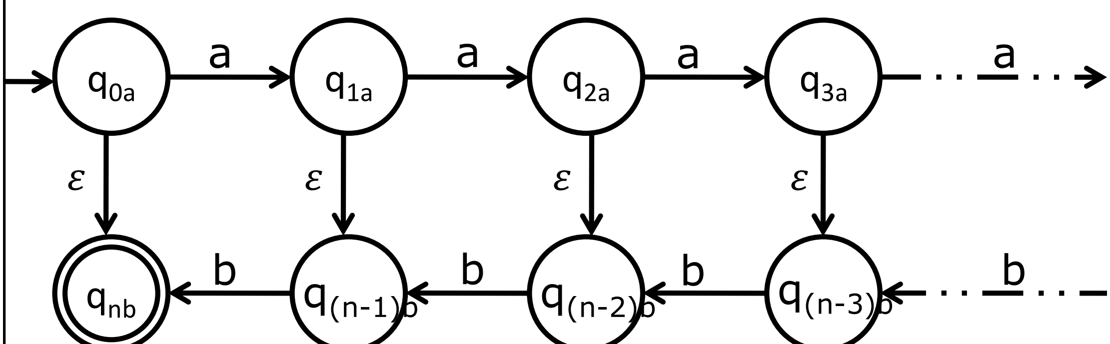
Non-Regular Languages

- $\forall\ n \geq 0$, $L_n = \{a^k b^k : k \leq n\}$ is regular



Non-Regular Languages

- However for any $n \geq 0$, $L_n = \{a^n b^n : n \geq 0\}$
  L_n is not a regular language.
- We need an infinite number of states to build this automaton!

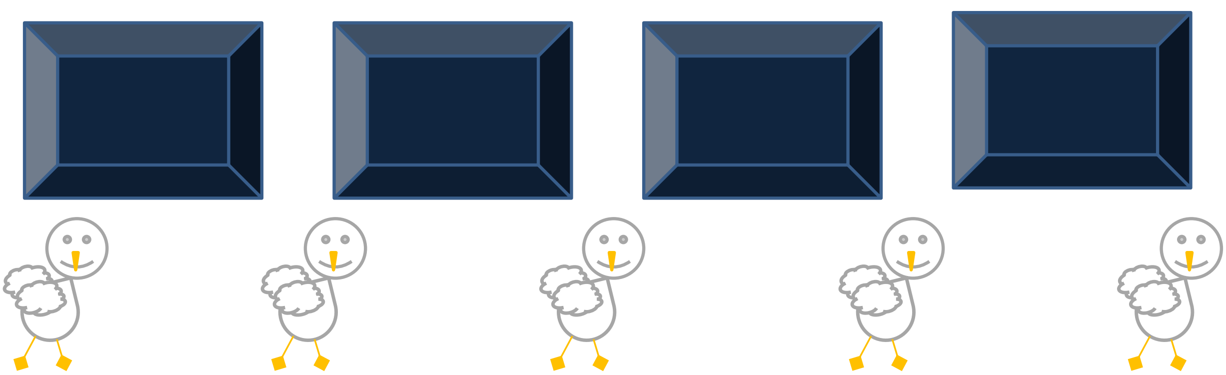


Proof

- We need mathematical proof that there is no FA that accepts L = $\{a^n b^n : n \geq 0\}$

The Pigeonhole Principle

- If we have n holes and m pigeons ($m>n$) then there is a hole with at least two pigeons.



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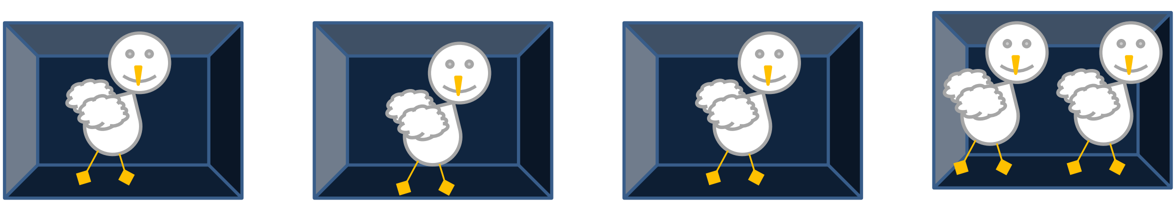
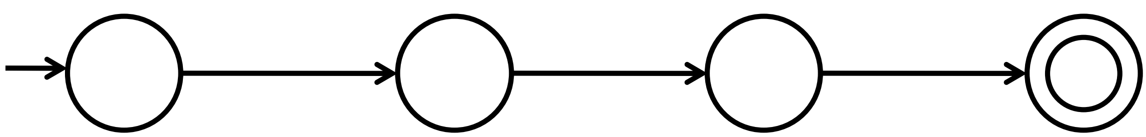
The Pigeonhole Principle (PP)

- If we have n holes and m pigeons ($m>n$) then there is a hole with at least two pigeons.



PP and Finite Automata

- If an automaton with n states accepts a string with length m ($m \geq n$), there should be at least one repeating state for every accepting path.



$s = aaba$

PP and Finite Automata

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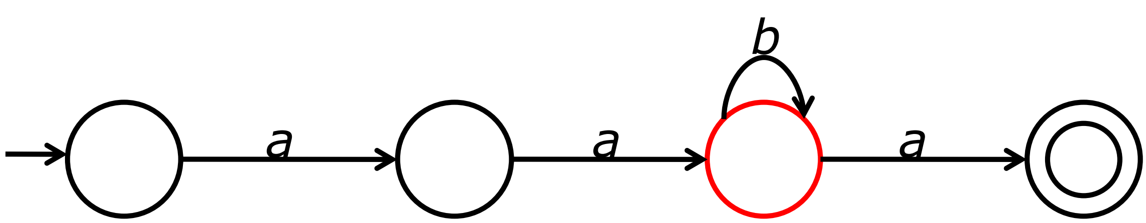

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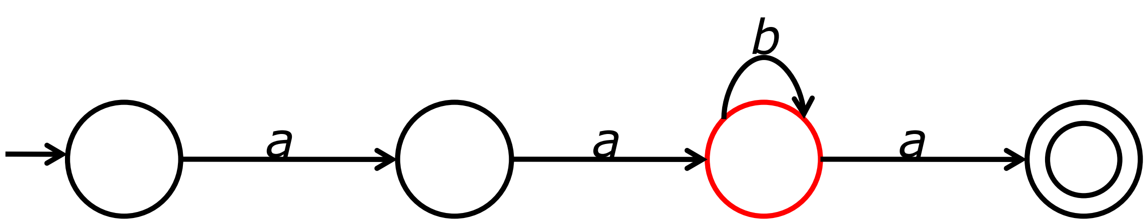
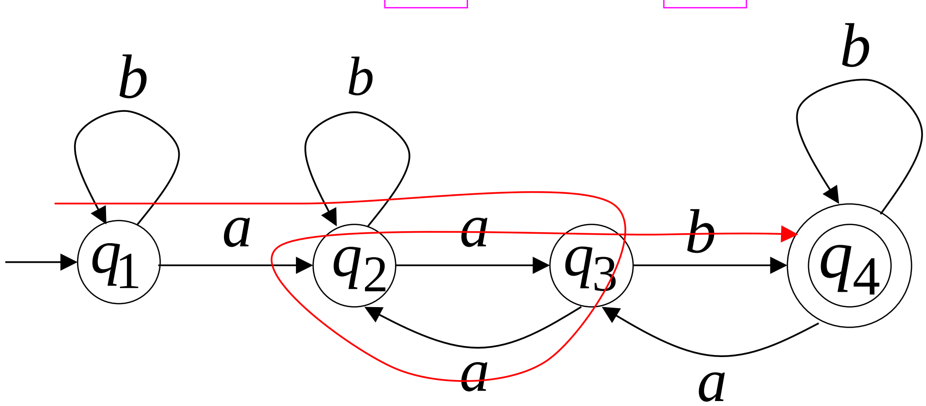


Any string of the form *aab*a* should be accepted!

Consider the walk of a "long" string: *aaaab*
(length at least 4)

A state is repeated in the walk of *aaaab*

$$q_1 \xrightarrow{a} q_2 \xrightarrow{a} \boxed{q_3} \xrightarrow{a} q_2 \xrightarrow{a} \boxed{q_3} \xrightarrow{b} q_4$$



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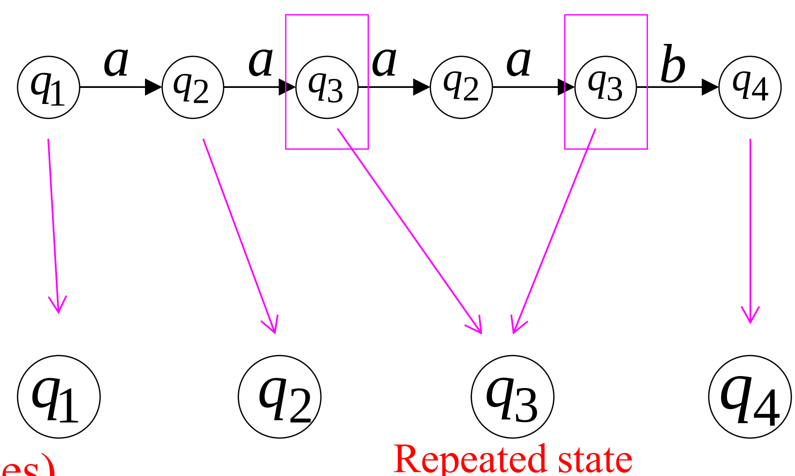
The state is repeated as a result of the pigeonhole principle

Walk of *aaaab*

Pigeons:
(walk states)

Are more than

Nests:
(Automaton states)



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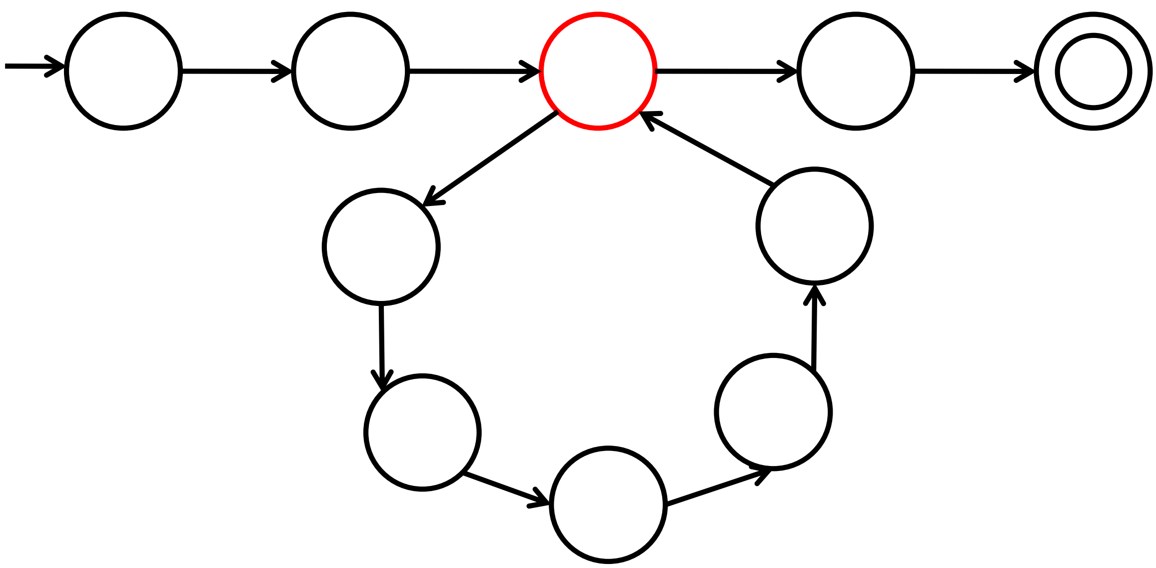
PP and Finite Automata



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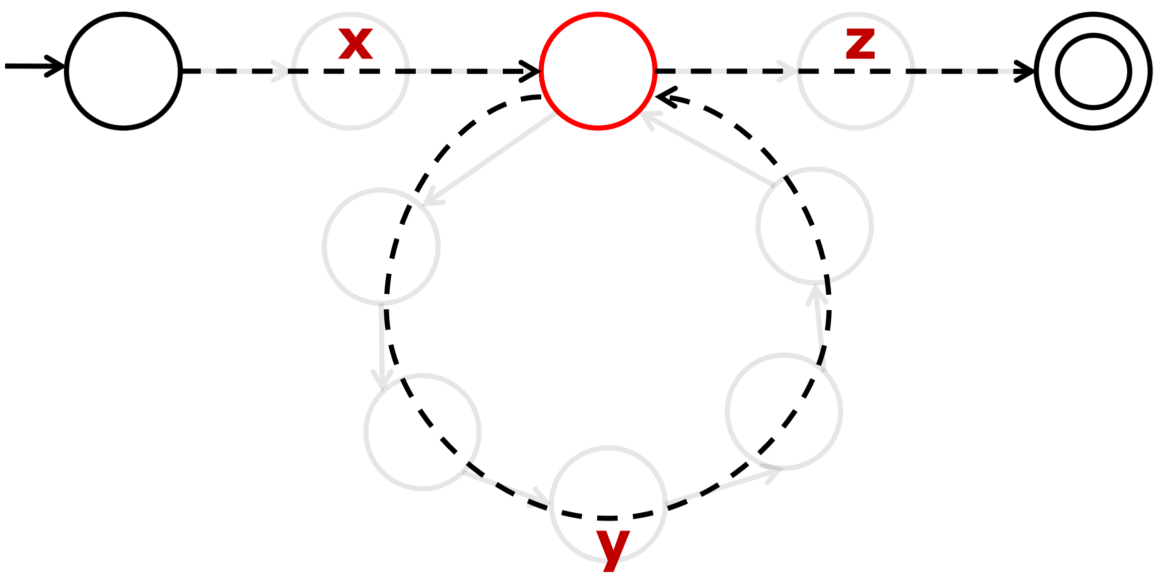
PP and Finite Automata



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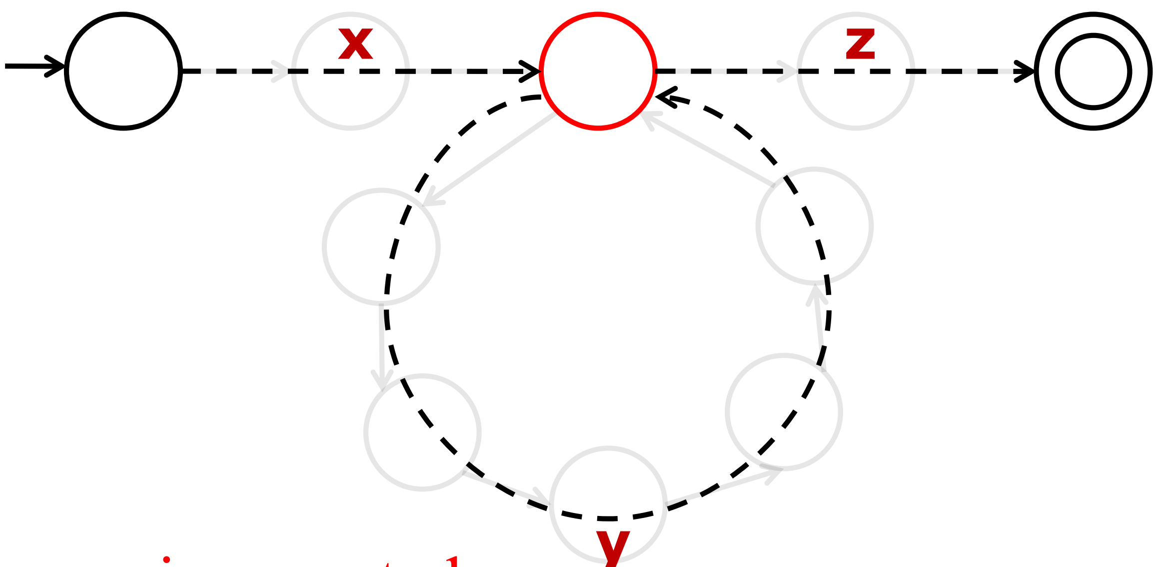
PP and Finite Automata



xyz is accepted

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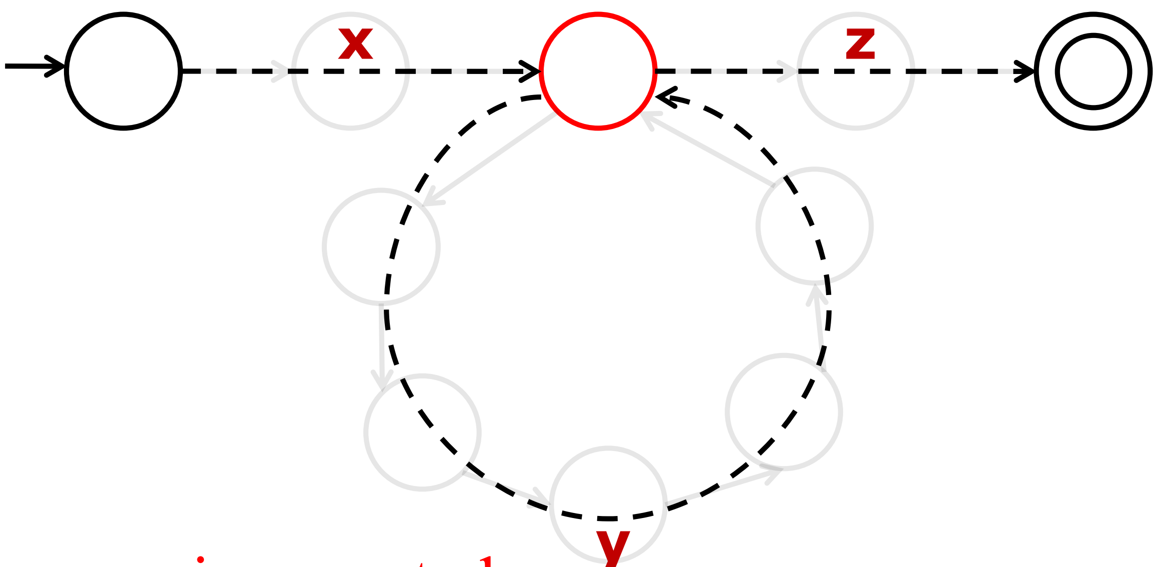
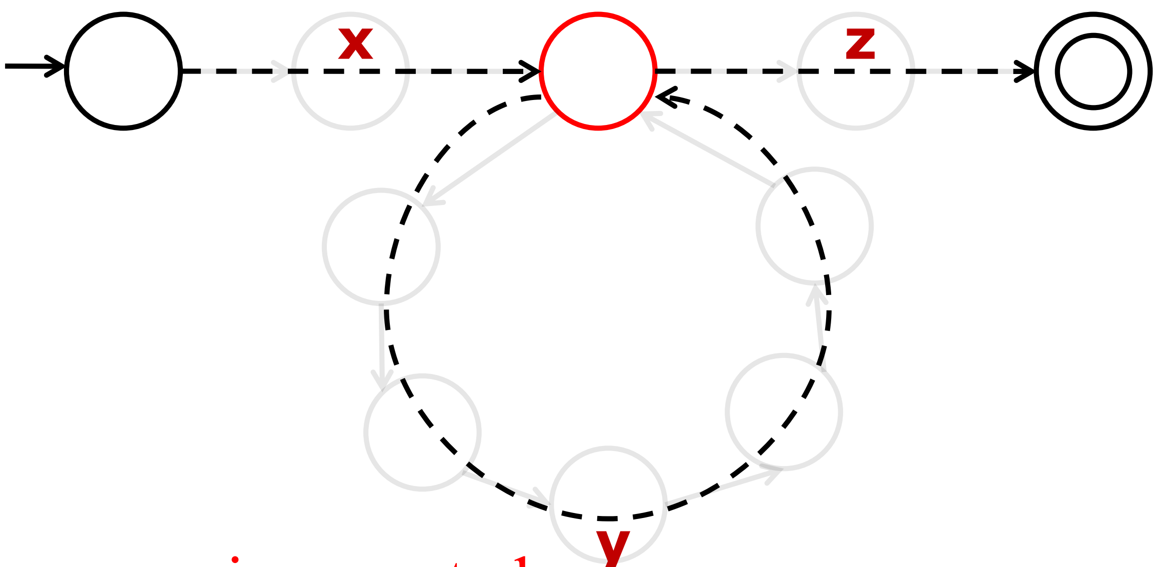
PP and Finite Automata



xyyz is accepted

PP and Finite Automata



xyyyz is accepted

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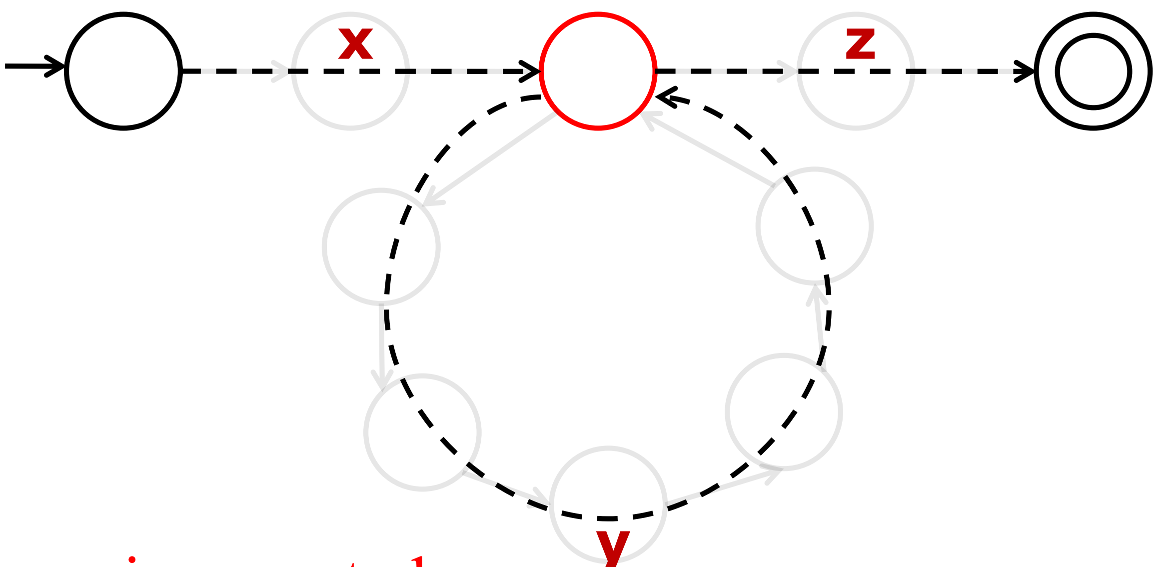
PP and Finite Automata



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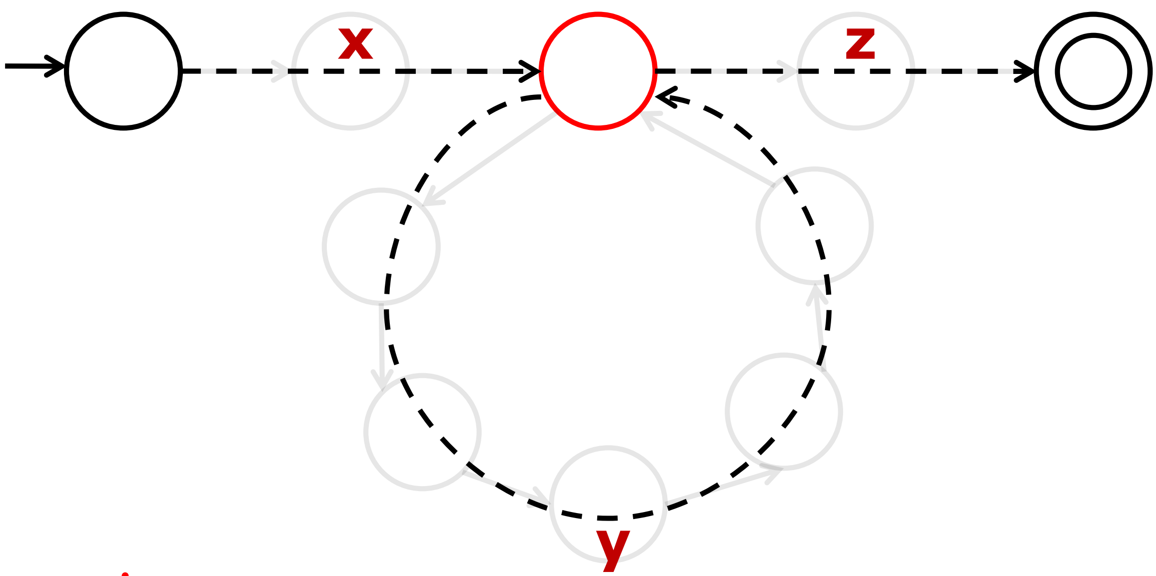
PP and Finite Automata



xy^iz, for $i \geq 0$ is accepted

The pumping lemma

For every infinite regular language L
there exists a pumping length $k > 0$ such that
for any string s in L with length $|s| \geq k$
we can write $s = xyz$
with $|xy| \leq k$ and $|y| \geq 1$
such that xy^iz *in L* for every $i \geq 0$.

Proof

- If L is regular then there exists a DFA M which accepts L. Set k to be the number of states of M.
- L is infinite, there exists a string s with a length greater than k.
- The number of states of M is k.
- The string is of length at least k, there is a part in the path that is repeated.

Proof

- Split s into 3 parts x, y, z with y being the first repeated part.
- Since we have k states the first repetition should take place ($|y| \geq 1$) in at most k transitions ($|xy| \leq k$).
- Since the path under y is a loop we can follow it as many times as we want (maybe none).
- Thus xy^iz for any $i \geq 0$ should lead us to the same accepting state as xyz.

Prove that L is not regular

- Given is an infinite language L.

Prove that L is not regular

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- If L is regular

Prove that L is not regular

- Given L an infinite language
- If L is regular
- Then the Pumping lemma holds:
  - *There exists* a pumping length k such that
  - *for all* strings s ($|s| \geq k$) in L
  - *there is* a splitting of s in x, y, and z ($|xy| \leq k$ and $|y| \geq 1$) such that *for all* i, xy^iz is in L.

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Prove that L is not regular

- Given is an infinite language L
- If L is regular the pumping lemma holds.
- The negation of the pumping lemma:
  - *For all* pumping lengths k
  - *there exist* a string s ($|s| \geq k$) in L such that
  - *for every possible* splitting of s in x, y, and z ($|xy| \leq k$ and $|y| \geq 1$) *there is* an i for which xy^iz is *not* in L.

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
  - *Fix* an *arbitrary* pumping length k.
  - *Specify* a string s ($|s| \geq k$) in L.
  - Show that *for every possible splitting* of s in x, y, and z ($|xy| \leq k$ and $|y| \geq 1$)
  - *there is* an i for which xy^iz is *not* in L.
- Contradiction! L is then not regular.

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Assume that L is regular. The pumping lemma holds!

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Fix an arbitrary pumping length k for L.

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

The string $s = a^k b^k$ should be in the language.

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Example

- L = $\{a^n b^n: n \geq 0\}$ is not regular.

Proof:

$|s| \geq k$: $|s| = 2k$ is greater than k

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Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with:

- $|xy| \leq k$ and
- $|y| \geq 1$

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$$\overbrace{a\ a\ a\ a\ a\ a\ a\ a\ a}^{k}\ \overbrace{b\ b\ b\ b\ b\ b\ b\ b\ b}^{k}$$

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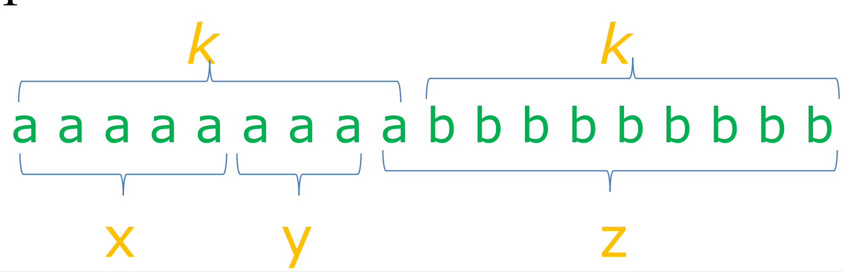

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with ($|xy| \leq k$ and $|y| \geq 1$)

- $y = a^m$, for $1 \leq m \leq k$

$$\overbrace{a\ a\ a\ a\ a\ a\ a\ a\ a}^{k}\ \overbrace{b\ b\ b\ b\ b\ b\ b\ b\ b}^{k}$$

$$\underbrace{a\ a\ a\ a\ a}_{x}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ b\ b\ b\ b\ b\ b\ b\ b\ b}_{z}$$

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Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with ($|xy| \leq k$ and $|y| \geq 1$)

- $y = a^m$, for $1 \leq m \leq k$
- for i =2, $xy^2z = a^{k+m}b^k$ is not in L!

$$\underbrace{a\ a\ a\ a\ a}_{x}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ b\ b\ b\ b\ b\ b\ b\ b\ b}_{z}$$

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Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with ($|xy| \leq k$ and $|y| \geq 1$)

- xy^2z is not in L!

CONTRADICTION!

$$\underbrace{a\;a\;a\;a\;a}_{x}\;\underbrace{a\;a\;a}_{y}\;\underbrace{a\;a\;a}_{y}\;\underbrace{a\;b\;b\;b\;b\;b\;b\;b\;b\;b}_{z}$$

How to use the pumping lemma

- The pumping lemma mentions that if L is a regular language then it can be pumped.
- The contrapositive is true: If L cannot be pumped then it shouldn't be regular!

How not to use the pumping lemma

- The pumping lemma mentions that if L is a regular language then it can be pumped.
- The converse is not true: If a language can be pumped this doesn't mean that it is regular!

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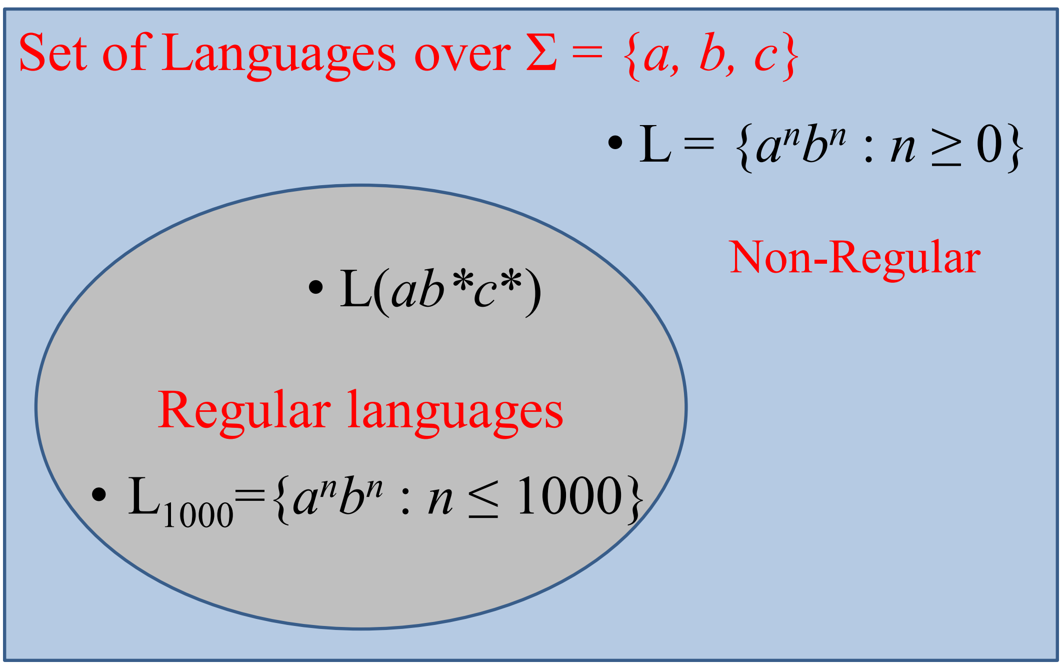
Language universe



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Summary

- Every language of finite size has to be regular
  - We can easily construct an NFA that accepts every string in the language.
- Therefore, every non-regular language has to be of infinite size.

Summary

To prove that an infinite language L is not regular:

1. Assume the opposite: L is regular.
2. The pumping lemma should hold for L.
3. Use the pumping lemma to obtain a contradiction.
4. Therefore, L is not regular.