CSC 339 – Theory of Computation
Fall 2023

9.2 Pushdown Automata (PDAs) – Part 2

Outline

- Pushing and popping strings
- Formal definition
- Instantaneous description
- Language of PDA
- PDAs accept Context-Free Languages

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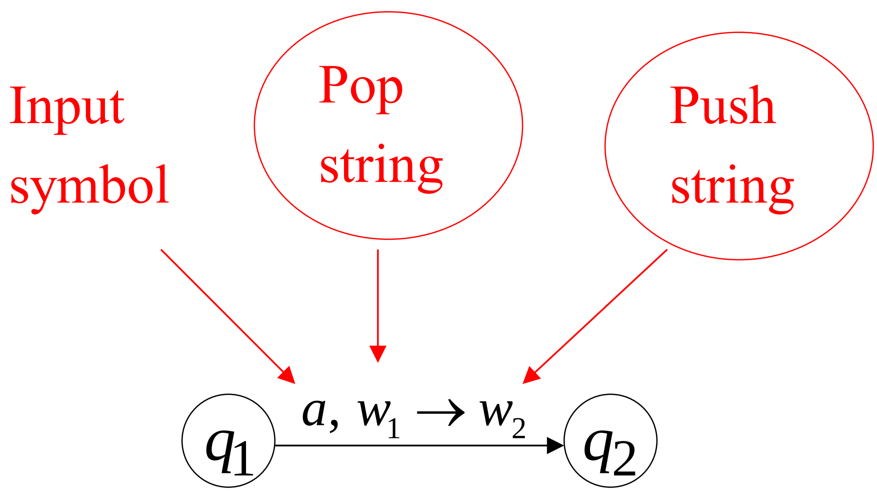
Pushing & Popping Strings



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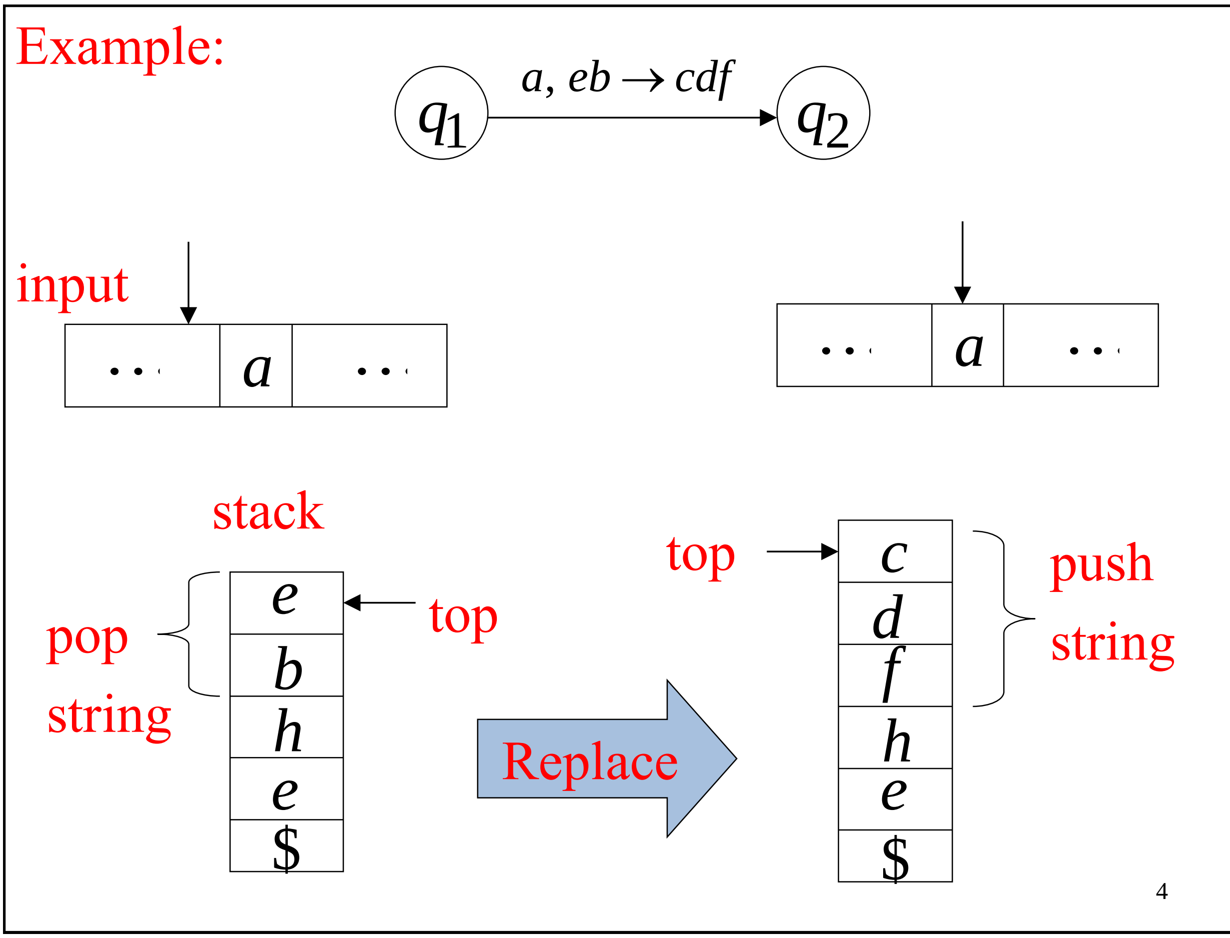
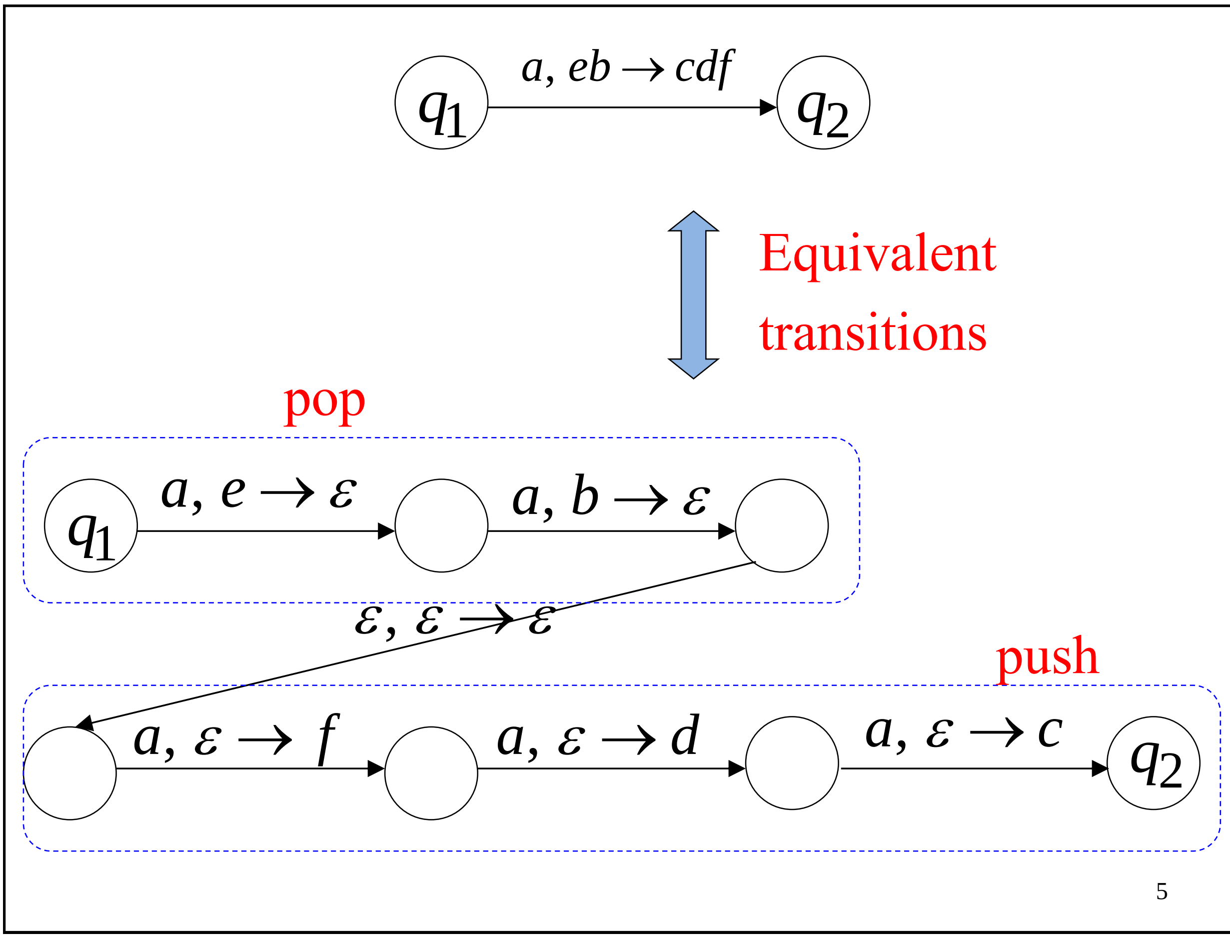

Example:
$a,\ eb \to cdf$
q_1
q_2
input
a
a
stack
pop
string
top
e
b
h
e
$\$$
Replace
top
push
string
c
d
f
h
e
$\$$
4



$a,\ eb \rightarrow cdf$
q_1
q_2
Equivalent transitions
pop
$a,\ e \rightarrow \varepsilon$
$a,\ b \rightarrow \varepsilon$
$\varepsilon,\ \varepsilon \rightarrow \varepsilon$
push
$a,\ \varepsilon \rightarrow f$
$a,\ \varepsilon \rightarrow d$
$a,\ \varepsilon \rightarrow c$
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Another PDA example

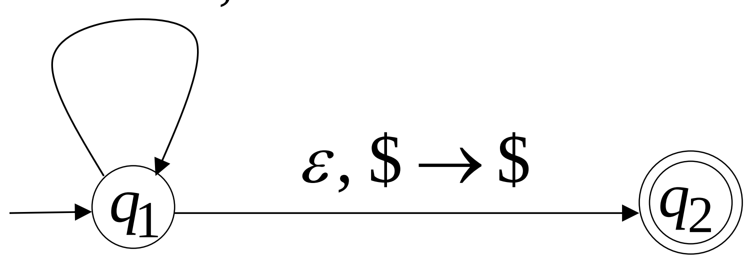
$$L(M) = \{w \in \{a,b\}^* : n_a(w) = n_b(w)\}$$

PDA M

$a, \$ \to 0\$$ $\quad$ $b, \$ \to 1\$$

$a, 0 \to 00$ $\quad$ $b, 1 \to 11$

$a, 1 \to \varepsilon$ $\quad$ $b, 0 \to \varepsilon$



Execution Example: Time 0

Input

a	b	b	b	a	a

$\$$

Stack

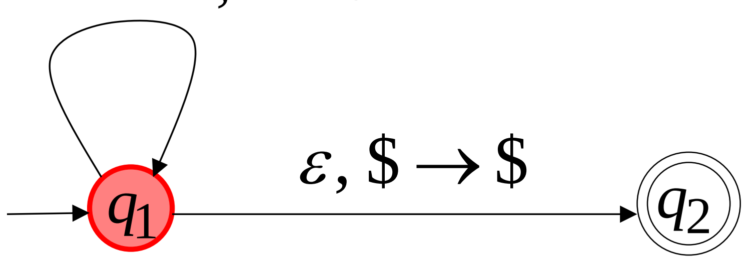
$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $\quad$ $b, 0 \rightarrow \varepsilon$

current

state



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Time 1

Input

a	b	b	b	a	a

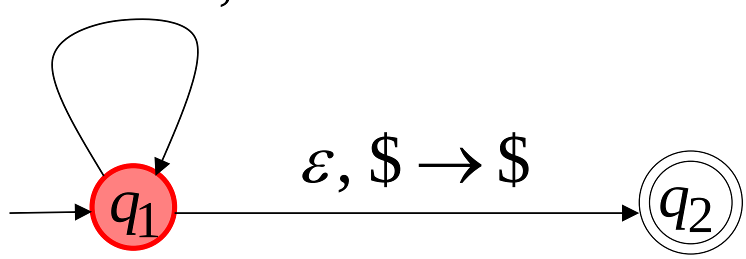
0
$

Stack

$a, \$ \rightarrow 0\$$ $\quad b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $\quad b, 0 \rightarrow \varepsilon$



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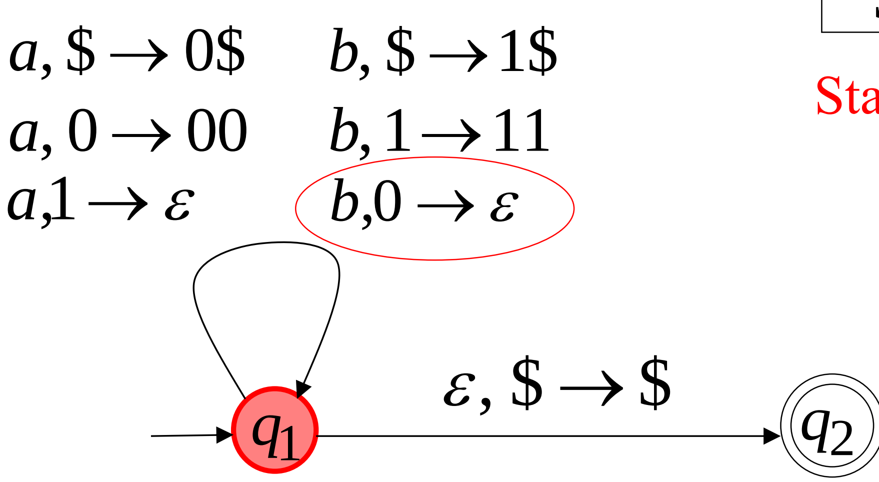
Time 2

Input

a	b	b	b	a	a

$\cancel{0}$
$\$$

Stack



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Time 3

Input

a	b	b	b	a	a

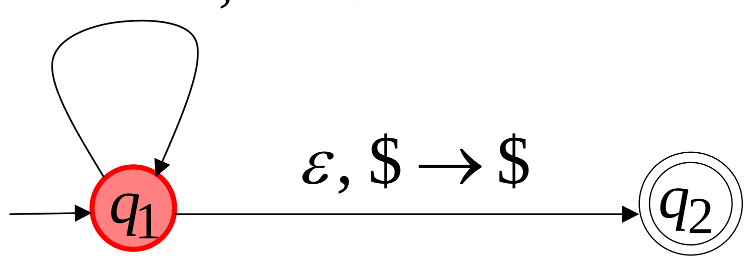
Stack

1
\$

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $\quad$ $b, 0 \rightarrow \varepsilon$



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Time 4

Input

a	b	b	b	a	a

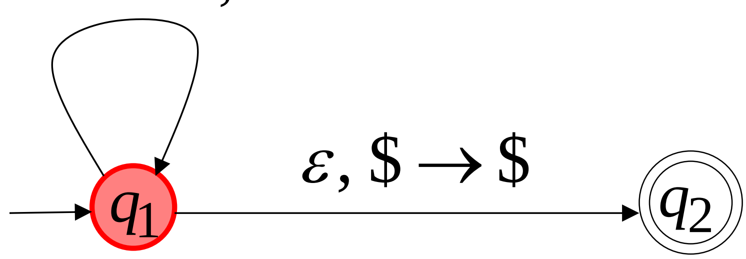
Stack

1
1
$

$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$

$$a, 0 \to 00 \qquad b, 1 \to 11$$

$$a, 1 \to \varepsilon \qquad b, 0 \to \varepsilon$$



Time 5

Input

a	b	b	b	a	a

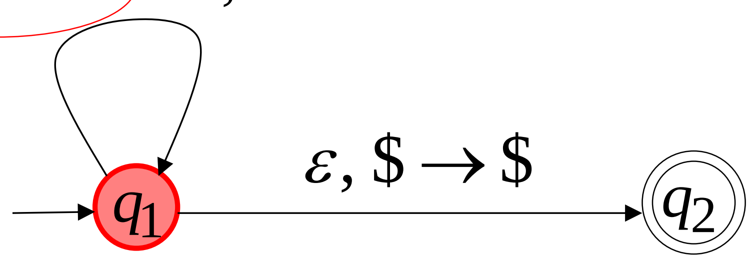
~~1~~
1
$

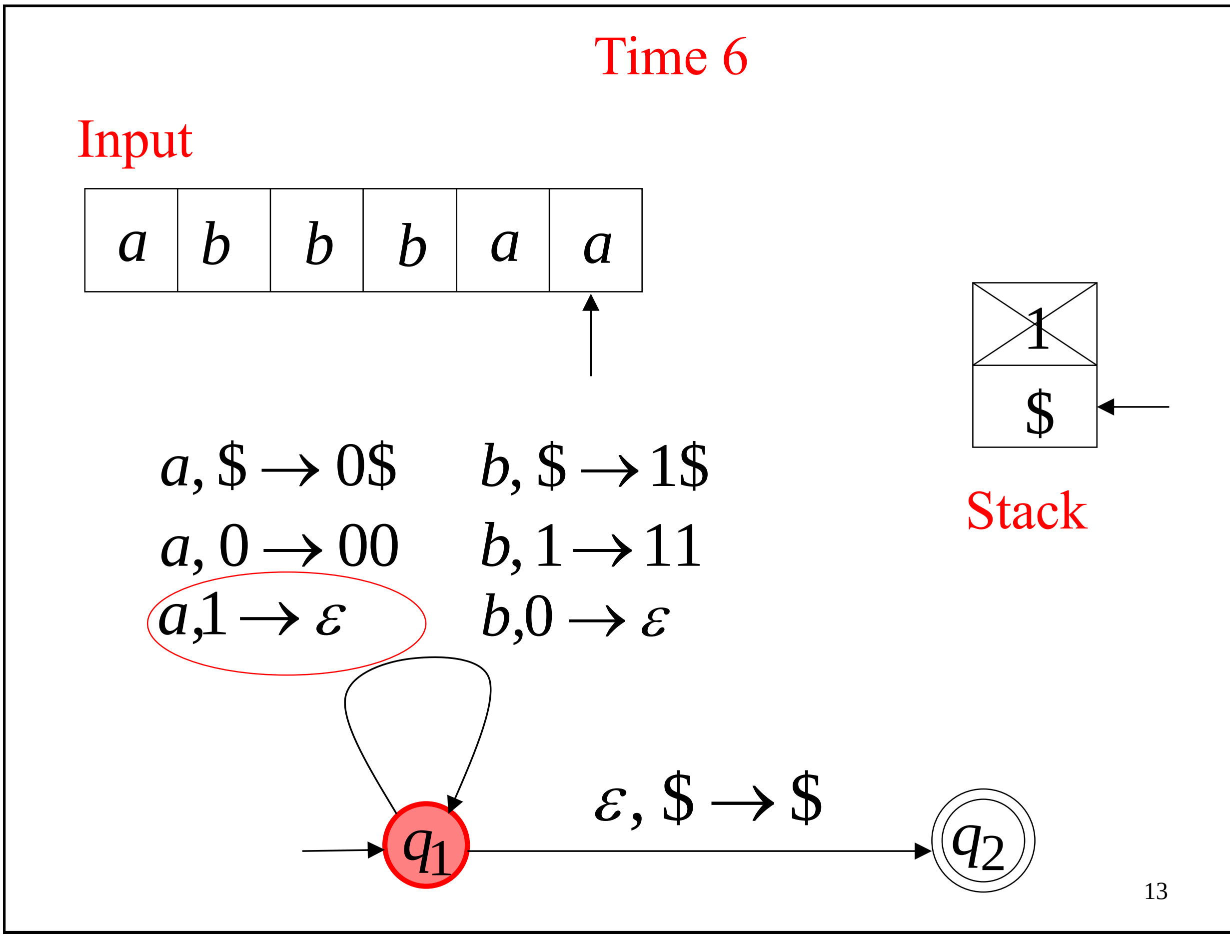
Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \varepsilon$ $\quad$ $b, 0 \rightarrow \varepsilon$




Time 6
Input
a b b b a a
1
$
Stack
a, $ → 0$
b, $ → 1$
a, 0 → 00
b, 1 → 11
a,1 → ε
b,0 → ε
ε, $ → $
q1
q2
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Time 7

Input

a	b	b	b	a	a

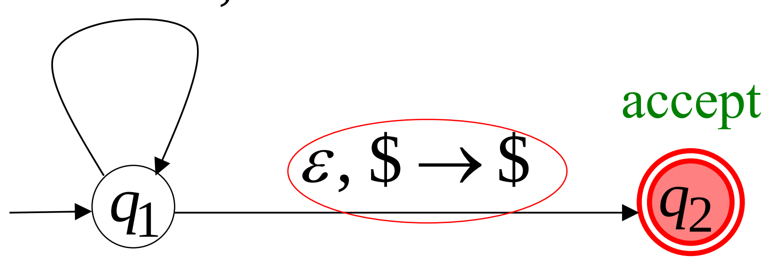
$

Stack

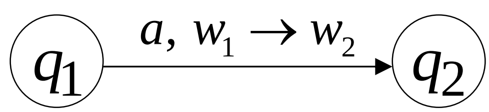
$$a, \$ \to 0\$ \qquad b, \$ \to 1\$$$

$$a, 0 \to 00 \qquad b, 1 \to 11$$

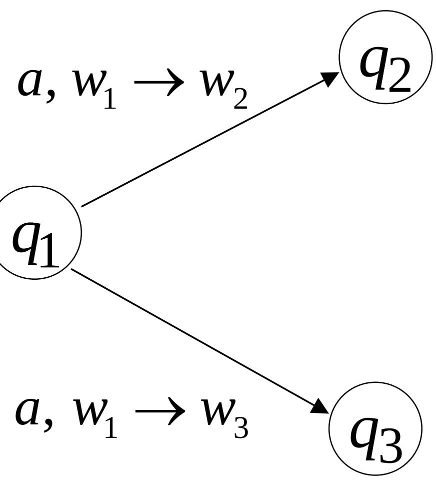
$$a, 1 \to \varepsilon \qquad b, 0 \to \varepsilon$$



Formal Definition



Transition function: $\delta(q_1, a, w_1) = \{(q_2, w_2)\}$



Transition function: $\delta(q_1, a, w_1) = \{(q_2, w_2),\ (q_3, w_3)\}$

Formal Definition

Pushdown Automaton (PDA)

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

Stack alphabet

Transition function

Initial state

Stack start symbol

Accept states

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Instantaneous Description

$$(q, u, s)$$

Current state

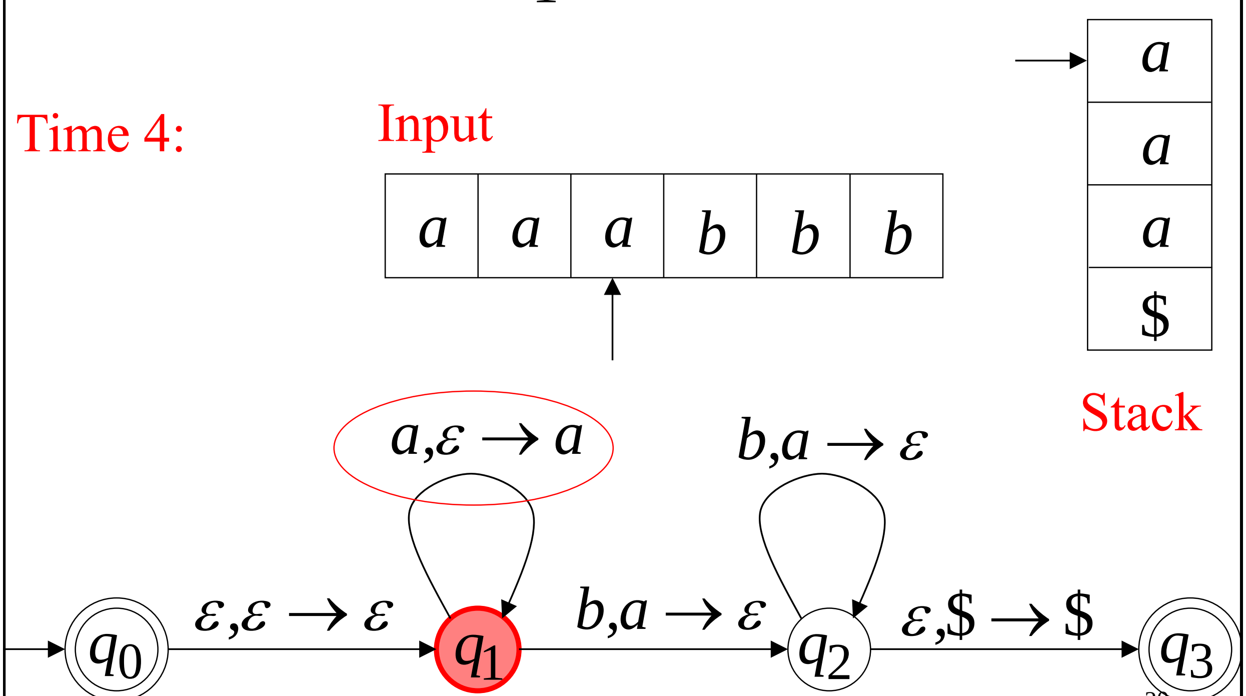
Remaining input

Current stack contents

Example: **Instantaneous Description**

$$(q_1, bbb, aaa\$)$$

Time 4:



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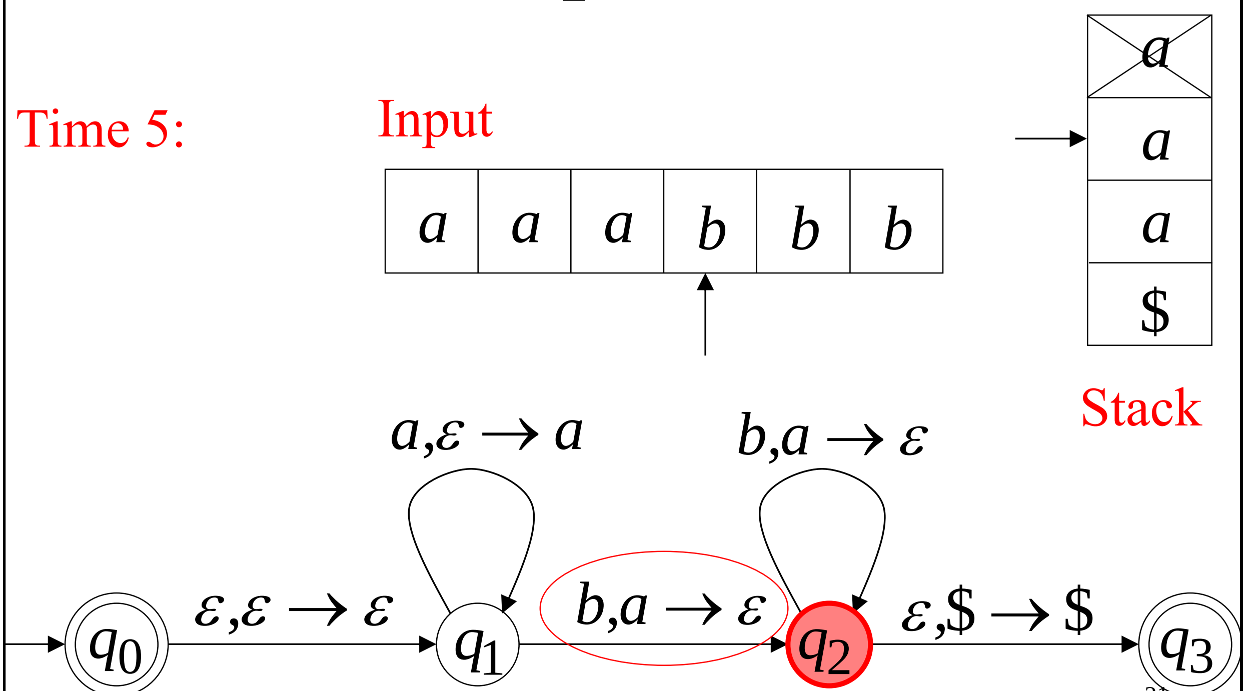
Example: **Instantaneous Description**

$$(q_2, bb, aa\$)$$

Time 5:



We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4 Time 5

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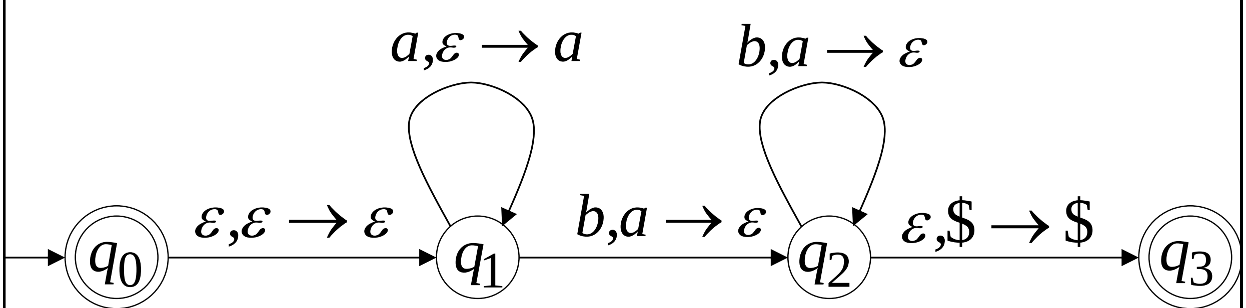
A computation:

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$



$$(q_0, aaabbb,\$) \succ (q_1, aaabbb,\$) \succ$$
$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$
$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda,\$) \succ (q_3, \lambda,\$)$$

For convenience we write:

$$(q_0, aaabbb,\$) \overset{*}{\succ} (q_3, \lambda,\$)$$

Language of PDA

Language $L(M)$ accepted by PDA M:

$$L(M) = \{w : (q_0, w, z) \overset{*}{\succ} (q_f, \lambda, s)\}$$

Initial state

Accept state

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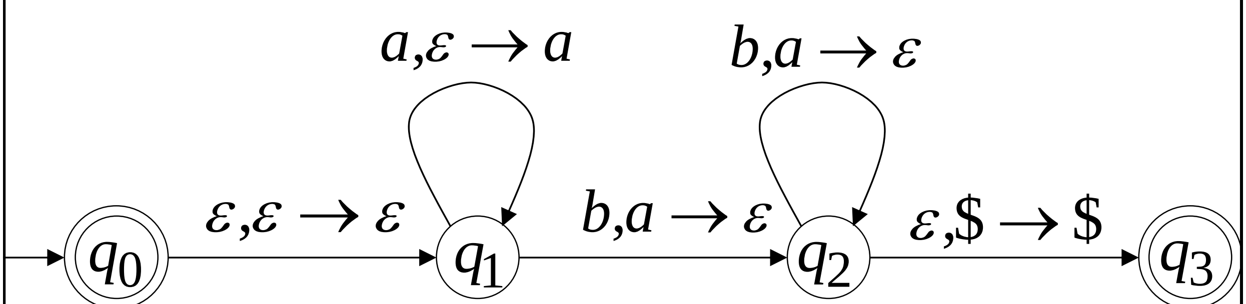
Example:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$aaabbb \in L(M)$$

PDA M:



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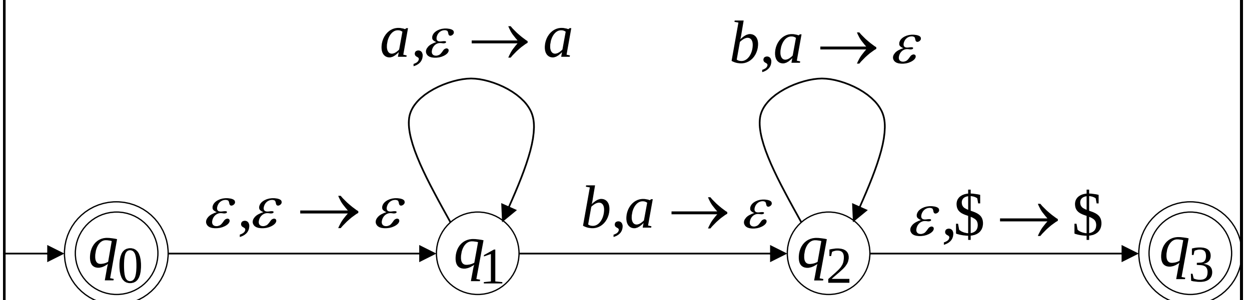
$$(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

$$\Downarrow$$

$$a^n b^n \in L(M)$$

PDA M:



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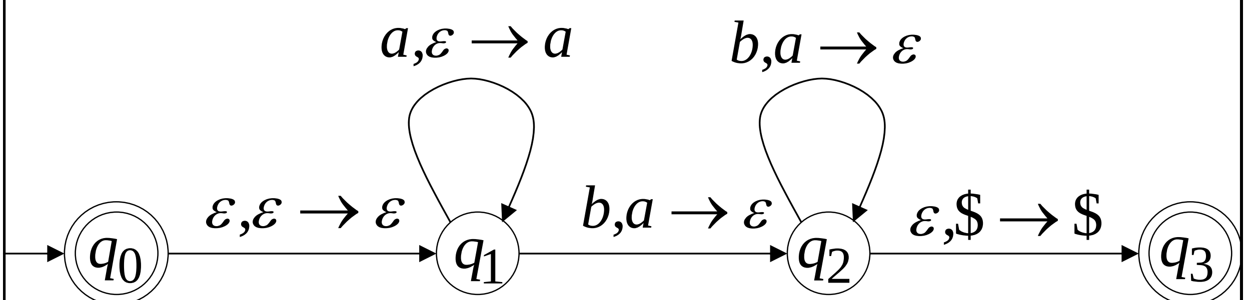
Therefore: $L(M) = \{a^n b^n : n \geq 0\}$

PDA M:



PDAs Accept Context-Free Languages

Theorem:

$$\left\{\begin{array}{c}\text{Context-Free}\\\text{Languages}\\\text{(Grammars)}\end{array}\right\} = \left\{\begin{array}{c}\text{Languages}\\\text{Accepted by}\\\text{PDAs}\end{array}\right\}$$

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Proof - Step 1:

$$\left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{(Grammars)}\end{array}\right\} \subseteq \left\{\begin{array}{c}\text{Languages}\\ \text{Accepted by}\\ \text{PDAs}\end{array}\right\}$$

Convert any context-free grammar G
to a PDA M with: $L(G) = L(M)$

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Proof - Step 2:

$$\left\{\begin{array}{c}\text{Context-Free}\\ \text{Languages}\\ \text{(Grammars)}\end{array}\right\} \supseteq \left\{\begin{array}{c}\text{Languages}\\ \text{Accepted by}\\ \text{PDAs}\end{array}\right\}$$

Convert any PDA M to a context-free grammar G with: $L(G) = L(M)$

Proof - step 1

Convert Context-Free Grammars to PDAs

Take an arbitrary context-free grammar G

We will convert G to a PDA M such that:

$$L(G) = L(M)$$

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Conversion Procedure:

For each production in G

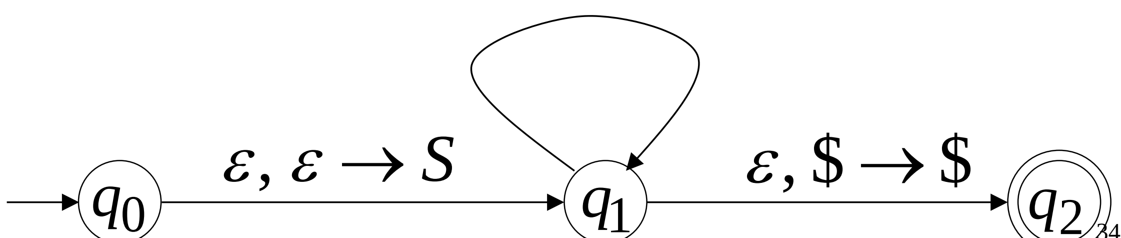
$A \to w$

For each terminal in G

a

Add transitions

$\varepsilon,\ A \to w \qquad a,\ a \to \varepsilon$



Example

Grammar

$S \to aSTb$

$S \to b$

$T \to Ta$

$T \to \varepsilon$

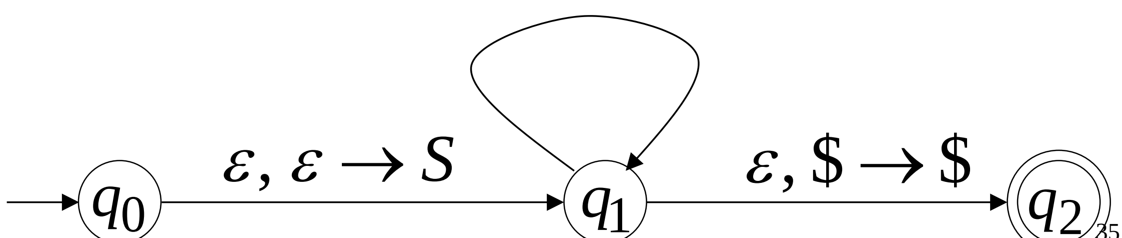
PDA

$\varepsilon, S \to aSTb$

$\varepsilon, S \to b$

$\varepsilon, T \to Ta$ $\quad a, a \to \varepsilon$

$\varepsilon, T \to \varepsilon$ $\quad b, b \to \varepsilon$



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PDA simulates leftmost derivations

Grammar
Leftmost Derivation

$$\begin{aligned}&S\\ &\Rightarrow \cdots\\ &\Rightarrow \boxed{\sigma_1 \cdots \sigma_k}\,\boxed{X_1 \cdots X_m}\\ &\Rightarrow \cdots\\ &\Rightarrow \sigma_1 \cdots \sigma_k \sigma_{k+1} \cdots \sigma_n\end{aligned}$$

PDA Computation

$$\begin{aligned}&(q_0, \sigma_1 \cdots \sigma_k \sigma_{k+1} \cdots \sigma_n, \$)\\ &\succ (q_1, \boxed{\sigma_1 \cdots \sigma_k}\,\sigma_{k+1} \cdots \sigma_n, S\$)\\ &\succ \cdots\\ &\succ (q_1, \sigma_{k+1} \cdots \sigma_n, \boxed{X_1 \cdots X_m}\,\$)\\ &\succ \cdots\\ &\succ (q_2, \varepsilon, \$)\end{aligned}$$

Scanned symbols

Stack contents

Grammar

Leftmost Derivation

Terminals

Leftmost variable

Variables or terminals

$$\Rightarrow \cdots$$

$$\Rightarrow xAy$$

$$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$$

Production applied

$$A \rightarrow \underbrace{\sigma_i \cdots \sigma_j}_{\text{Terminals}} Bz$$

Variables or terminals

Variable

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Grammar

Leftmost Derivation

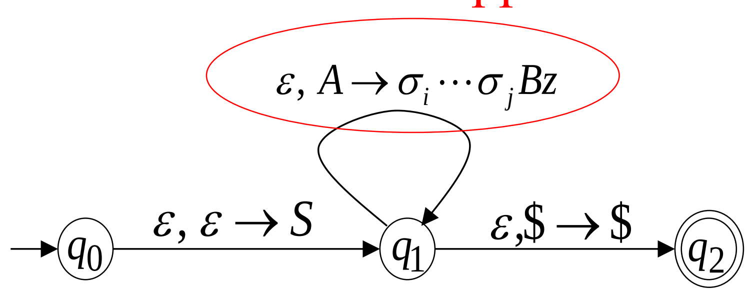
PDA Computation

$\Rightarrow \cdots$	$\succ \cdots$
$\Rightarrow xAy$	$\succ (q_1, \sigma_i \cdots \sigma_n, Ay\$)$
$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$	$\succ (q_1, \sigma_i \cdots \sigma_n, \sigma_i \cdots \sigma_j Bzy\$)$

Production applied

$$A \rightarrow \sigma_i \cdots \sigma_j Bz$$

Transition applied



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Grammar

Leftmost Derivation

PDA Computation

$\Rightarrow \cdots$

$\Rightarrow xAy$

$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$

$\succ \cdots$

$\succ (q_1, \sigma_i \cdots \sigma_n, Ay\$)$

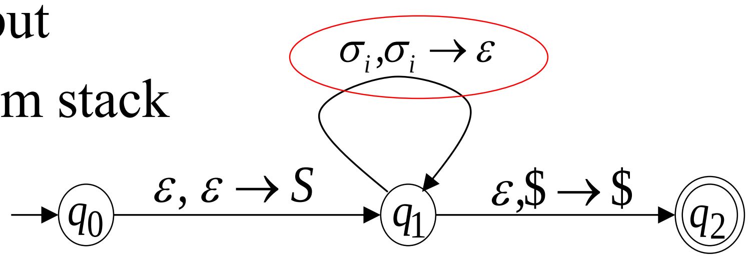
$\succ (q_1, \sigma_i \cdots \sigma_n, \sigma_i \cdots \sigma_j Bzy\$)$

$\succ (q_1, \sigma_{i+1} \cdots \sigma_n, \sigma_{i+1} \cdots \sigma_j Bzy\$)$

Transition applied

Read σ_i from input

and remove it from stack



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Grammar

Leftmost Derivation

$\Rightarrow \cdots$

$\Rightarrow xAy$

$\Rightarrow x\sigma_i \cdots \sigma_j Bzy$

PDA Computation

$\succ \cdots$

$\succ (q_1, \sigma_i \cdots \sigma_n, Ay\$)$

$\succ (q_1, \sigma_i \cdots \sigma_n, \sigma_i \cdots \sigma_j Bzy\$)$

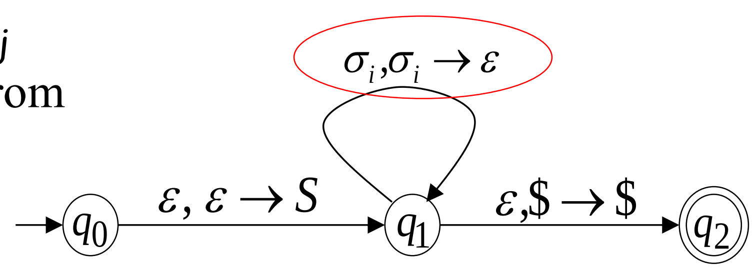
$\succ (q_1, \sigma_{i+1} \cdots \sigma_n, \sigma_{i+1} \cdots \sigma_j Bzy\$)$

$\succ \cdots$

$\succ (q_1, \sigma_{j+1} \cdots \sigma_n, Bzy\$)$

Last transition applied

All symbols $\sigma_i \cdots \sigma_j$ have been removed from top of stack



The process repeats with the next leftmost variable

$$
\begin{aligned}
&\Rightarrow \cdots && \succ \cdots \\
&\Rightarrow xAy && \succ (q_1, \sigma_{j+1} \cdots \sigma_n, Bzy\$) \\
&\Rightarrow x\sigma_i \cdots \sigma_j Bzy && \succ (q_1, \sigma_{j+1} \cdots \sigma_n, \sigma_{j+1} \cdots \sigma_k Cpzy\$) \\
&\Rightarrow x\sigma_i \cdots \sigma_j \sigma_{j+1} \cdots \sigma_k Cpzy && \succ \cdots \\
& && \succ (q_1, \sigma_{k+1} \cdots \sigma_n, Cpzy\$)
\end{aligned}
$$

Production applied

$$B \rightarrow \sigma_{j+1} \cdots \sigma_k Cp$$

And so on……

Example:

Input

a	b	a	b

Time 0

$\varepsilon, S \to aSTb$

$\varepsilon, S \to b$

$\varepsilon, T \to Ta$

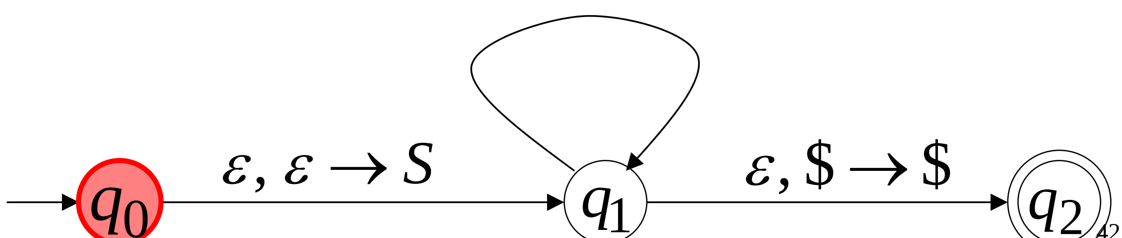
$\varepsilon, T \to \varepsilon$

$a, a \to \varepsilon$

$b, b \to \varepsilon$

$

Stack



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Derivation: S

Input

a	b	a	b

Time 1

$\varepsilon, S \rightarrow aSTb$

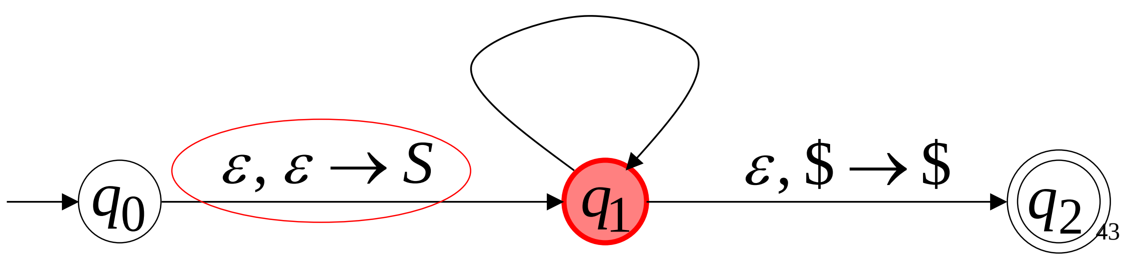
$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$ $\quad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon$ $\quad b, b \rightarrow \varepsilon$

Stack
S
$\$$

Stack



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Derivation: $S \Rightarrow aSTb$

Input

a	b	a	b

Time 2

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$

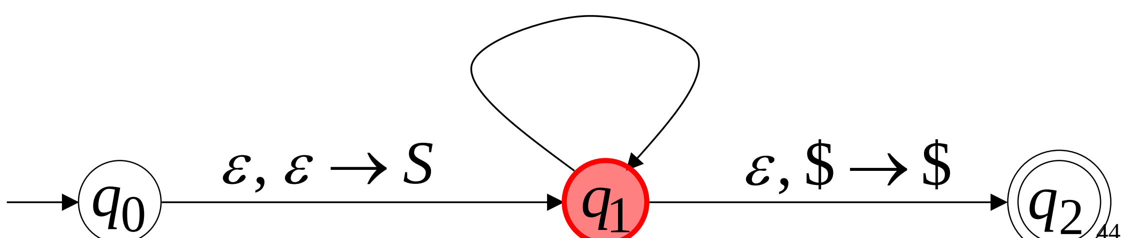
$\varepsilon, T \rightarrow \varepsilon$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

a
S
T
b
$\$$

Stack



Derivation: $S \Rightarrow aSTb$

Input

a	b	a	b

Time 3

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$

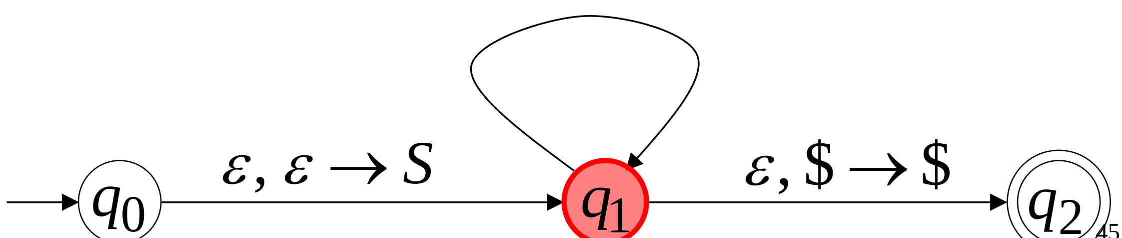
$\varepsilon, T \rightarrow \varepsilon$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

Stack

$\cancel{a}$
S
T
b
$\$$



Derivation: $S \Rightarrow aSTb \Rightarrow abTb$

Input

a	b	a	b

Time 4

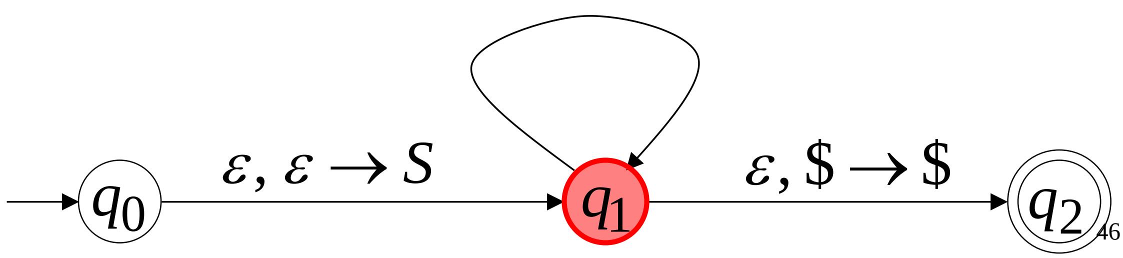
$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$ $\qquad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon$ $\qquad b, b \rightarrow \varepsilon$

Stack
b
T
b
$\$$



Derivation: $S \Rightarrow aSTb \Rightarrow abTb$

Input

a	b	a	b

Time 5

Stack

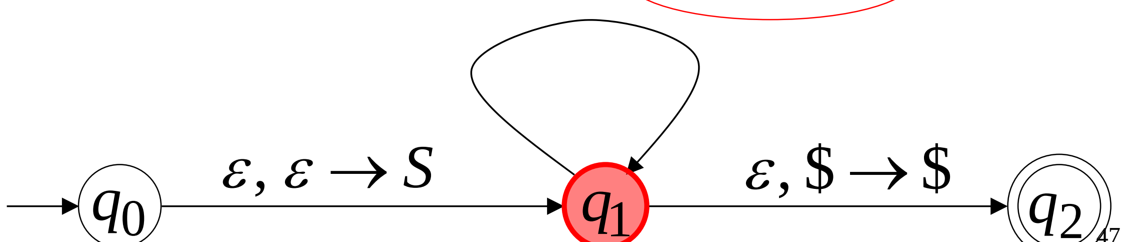
$\cancel{b}$
T
b
$\$$

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$ $\quad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon$ $\quad b, b \rightarrow \varepsilon$



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Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab$

Input

a	b	a	b

Time 6

$\varepsilon, S \rightarrow aSTb$

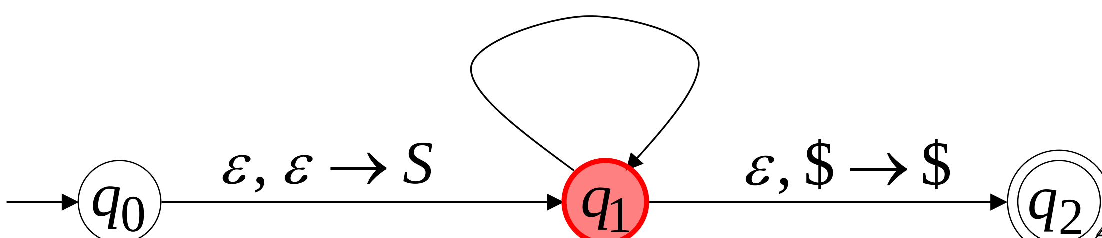
$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$ $\quad a, a \rightarrow \varepsilon$

$\varepsilon, T \rightarrow \varepsilon$ $\quad b, b \rightarrow \varepsilon$

T
a
b
$\$$

Stack



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Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Time 7

Stack

$\cancel{T}$
a
b
$\$$

$\varepsilon, S \rightarrow aSTb$

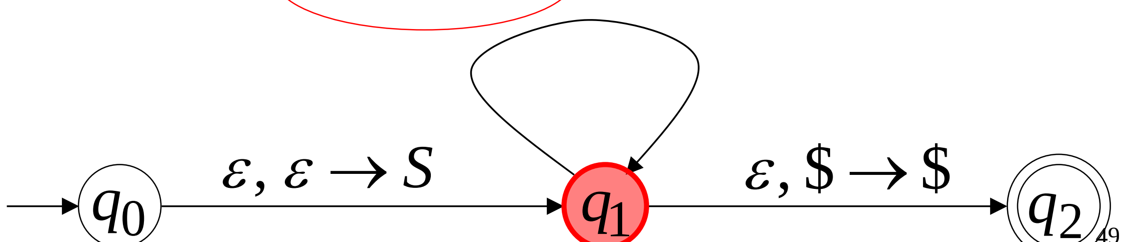
$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$

$\varepsilon, T \rightarrow \varepsilon$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$



Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Time 8

$\varepsilon, S \rightarrow aSTb$

$\varepsilon, S \rightarrow b$

$\varepsilon, T \rightarrow Ta$

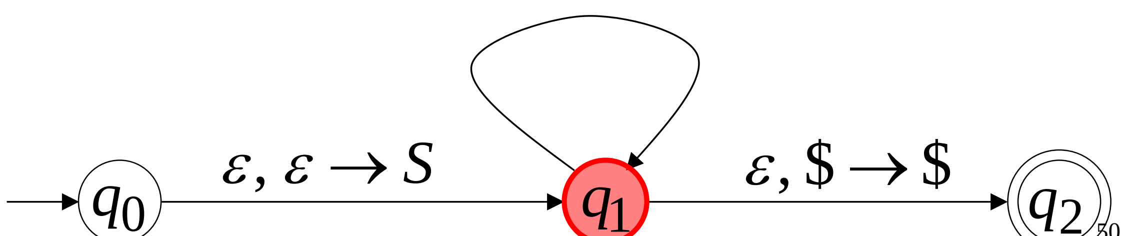
$\varepsilon, T \rightarrow \varepsilon$

$a, a \rightarrow \varepsilon$

$b, b \rightarrow \varepsilon$

$\cancel{a}$
b
$\$$

Stack



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Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Time 9

$\varepsilon, S \to aSTb$

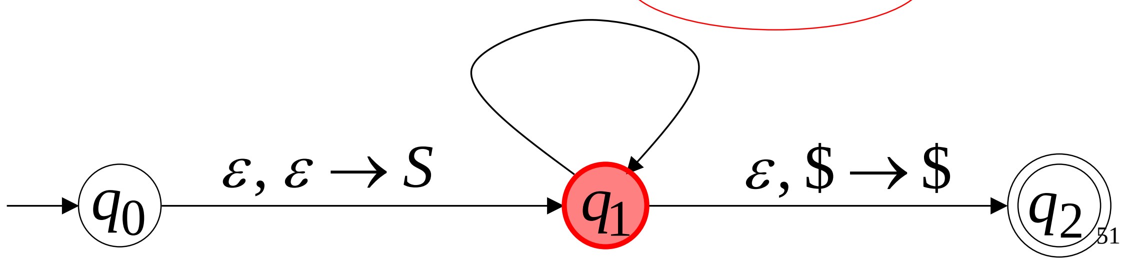
$\varepsilon, S \to b$

$\varepsilon, T \to Ta$ $\quad a, a \to \varepsilon$

$\varepsilon, T \to \varepsilon$ $\quad b, b \to \varepsilon$

$\cancel{b}$
$\$$

Stack



Derivation: $S \Rightarrow aSTb \Rightarrow abTb \Rightarrow abTab \Rightarrow abab$

Input

a	b	a	b

Time 10

$\varepsilon, S \to aSTb$

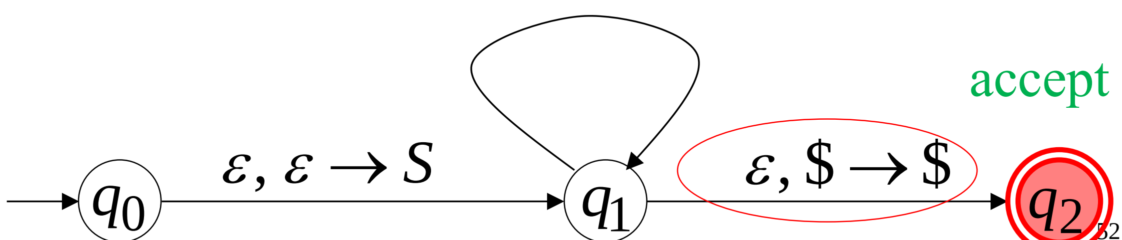
$\varepsilon, S \to b$

$\varepsilon, T \to Ta$ $\quad a, a \to \varepsilon$

$\varepsilon, T \to \varepsilon$ $\quad b, b \to \varepsilon$

$

Stack



Grammar
Leftmost Derivation

$$S$$
$$\Rightarrow aSTb$$
$$\Rightarrow abTb$$
$$\Rightarrow abTab$$
$$\Rightarrow abab$$

PDA Computation

$$(q_0, abab, \$)$$
$$\succ (q_1, abab, S\$)$$
$$\succ (q_1, bab, STb\$)$$
$$\succ (q_1, bab, bTb\$)$$
$$\succ (q_1, ab, Tb\$)$$
$$\succ (q_1, ab, Tab\$)$$
$$\succ (q_1, ab, ab\$)$$
$$\succ (q_1, b, b\$)$$
$$\succ (q_1, \varepsilon, \$)$$
$$\succ (q_2, \varepsilon, \$)$$

In general, it can be shown that:

Grammar G
generates
string w

$$S \overset{*}{\Rightarrow} w$$

If and
Only if

PDA M
accepts w

$$(q_0, w, \$) \succ (q_2, \lambda, \$)$$

Therefore $L(G) = L(M)$