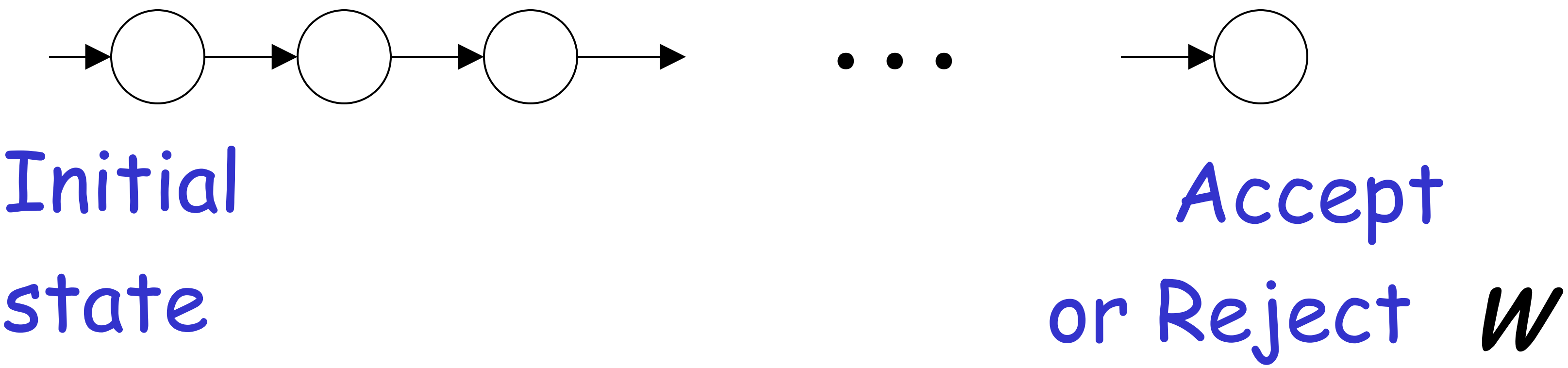
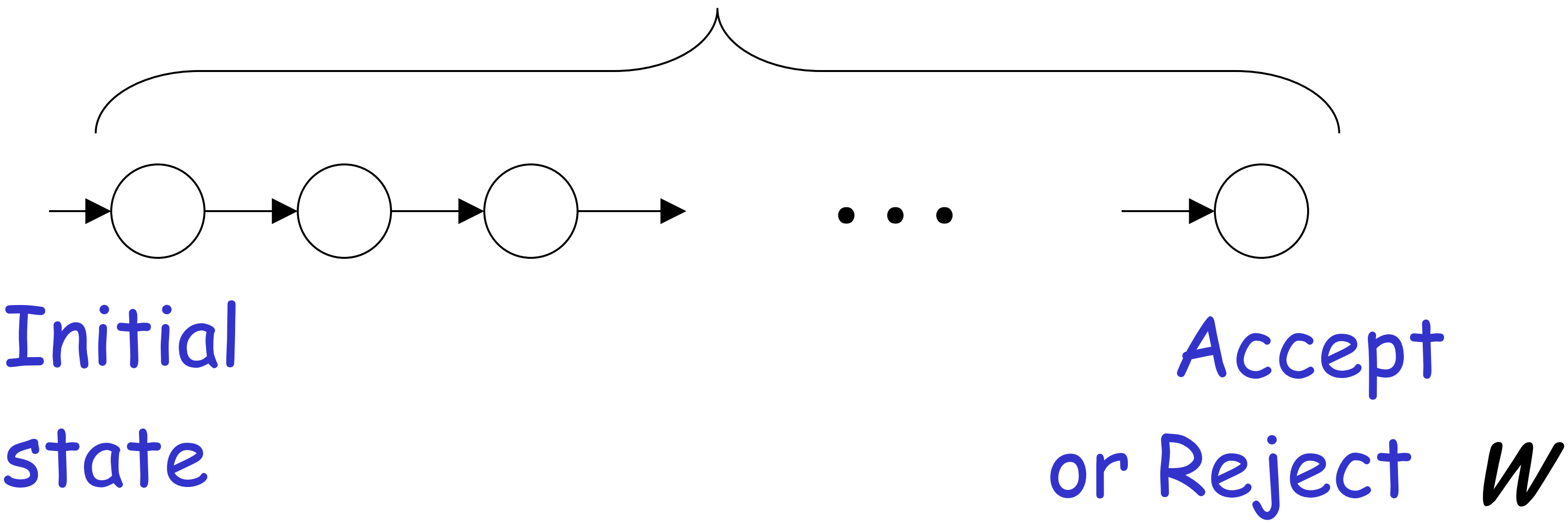
Time Complexity

Consider a deterministic Turing Machine M which decides a language L

For any string w the computation of M terminates in a finite amount of transitions

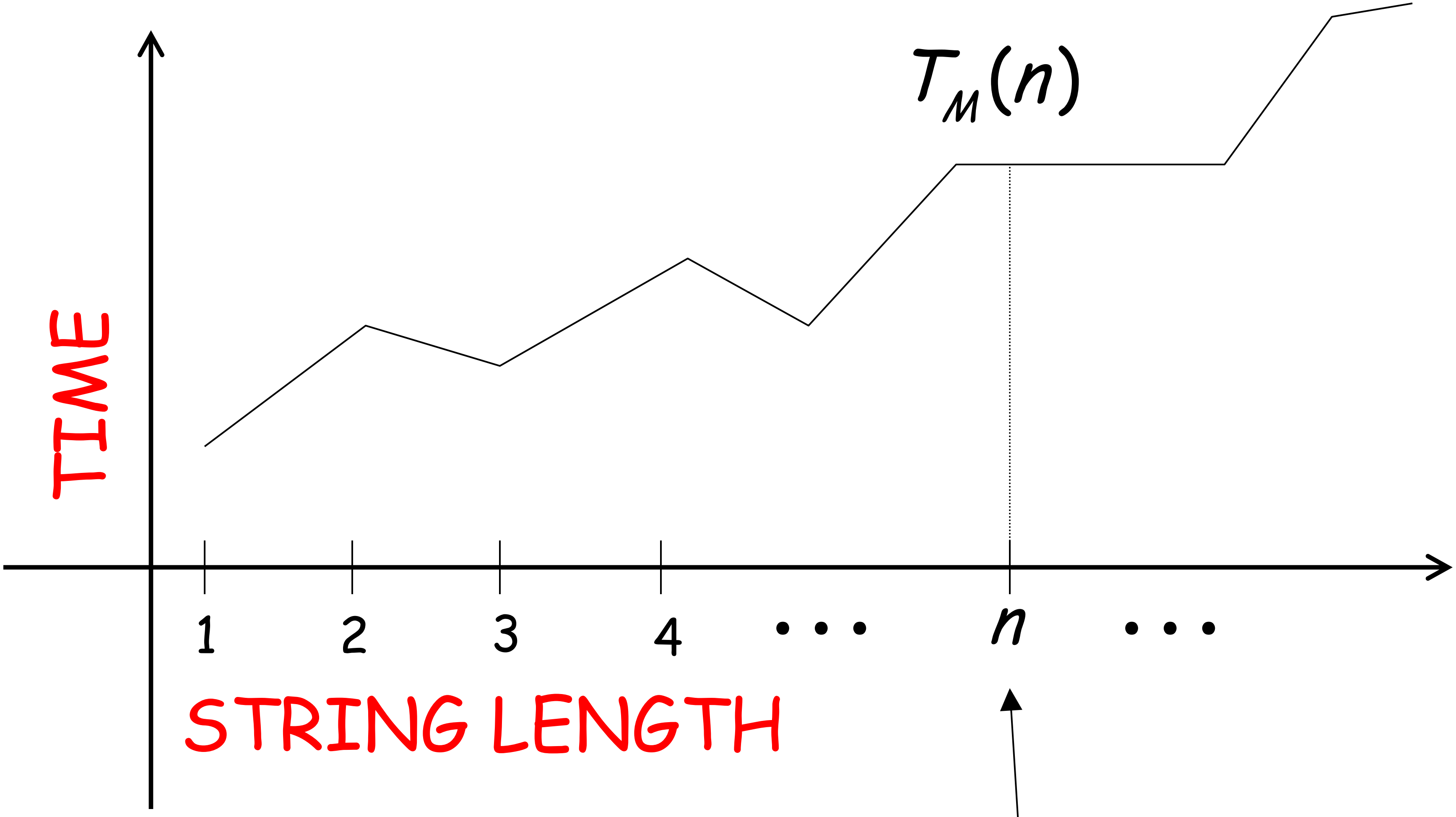


Decision Time = #transitions



Consider now all strings of length n

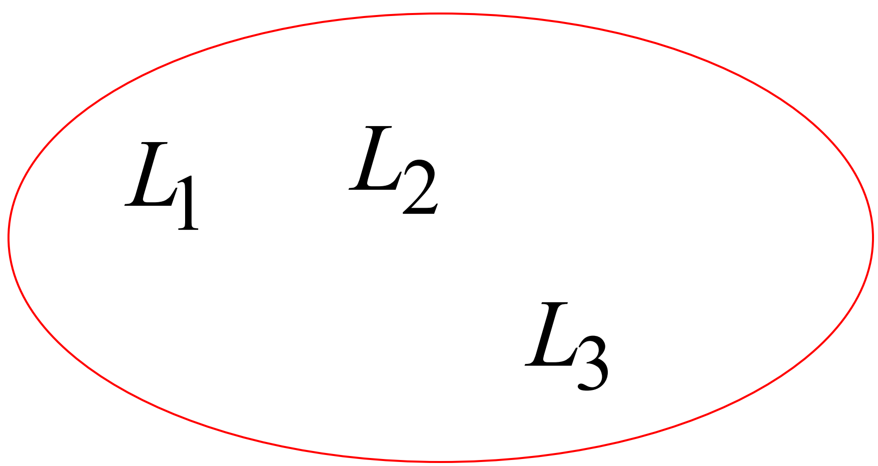
$T_M(n)$ = maximum time required to decide any string of length n



Max time to accept a string of length n

Time Complexity Class: $TIME(T(n))$

All Languages decidable by a deterministic Turing Machine in time $O(T(n))$



Example: $L_1 = \{a^n b : n \geq 0\}$

This can be decided in $O(n)$ time

$TIME(n)$

$L_1 = \{a^n b : n \geq 0\}$

Other example problems in the same class

$$TIME(n)$$

$$L_1 = \{a^n b : \ n \geq 0\}$$

$$\{ab^n aba : \ n,k \geq 0\}$$

$$\{b^n : \ n \text{ is even}\}$$

$$\{b^n : \ n = 3k\}$$

Examples in class:

$$TIME(n^2)$$

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}\}$$

$$\{ww : w \in \{a,b\}\}$$

Turing machine for the language $\{a^n b^n\}$

$n \geq 1$

Basic Idea:

Match a's with b's:

Repeat:

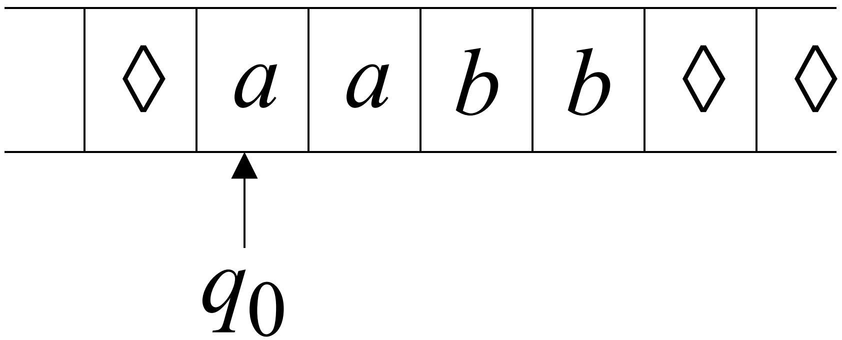
replace leftmost a with x

find leftmost b and replace it with y

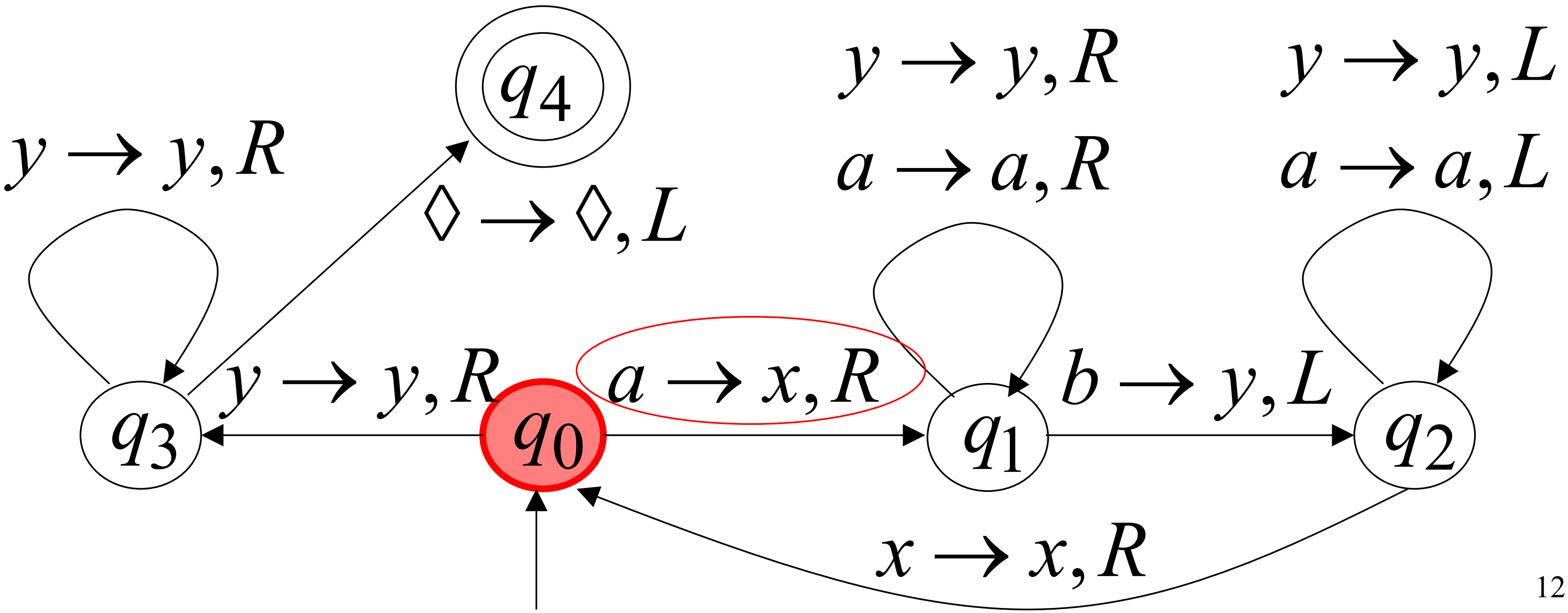
Until there are no more a's or b's

If there is a remaining a or b reject

Time 0

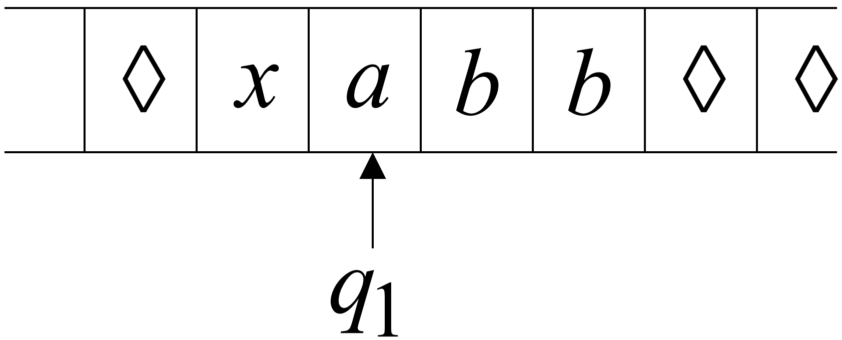


◊
a
a
b
b
◊
◊
q_0

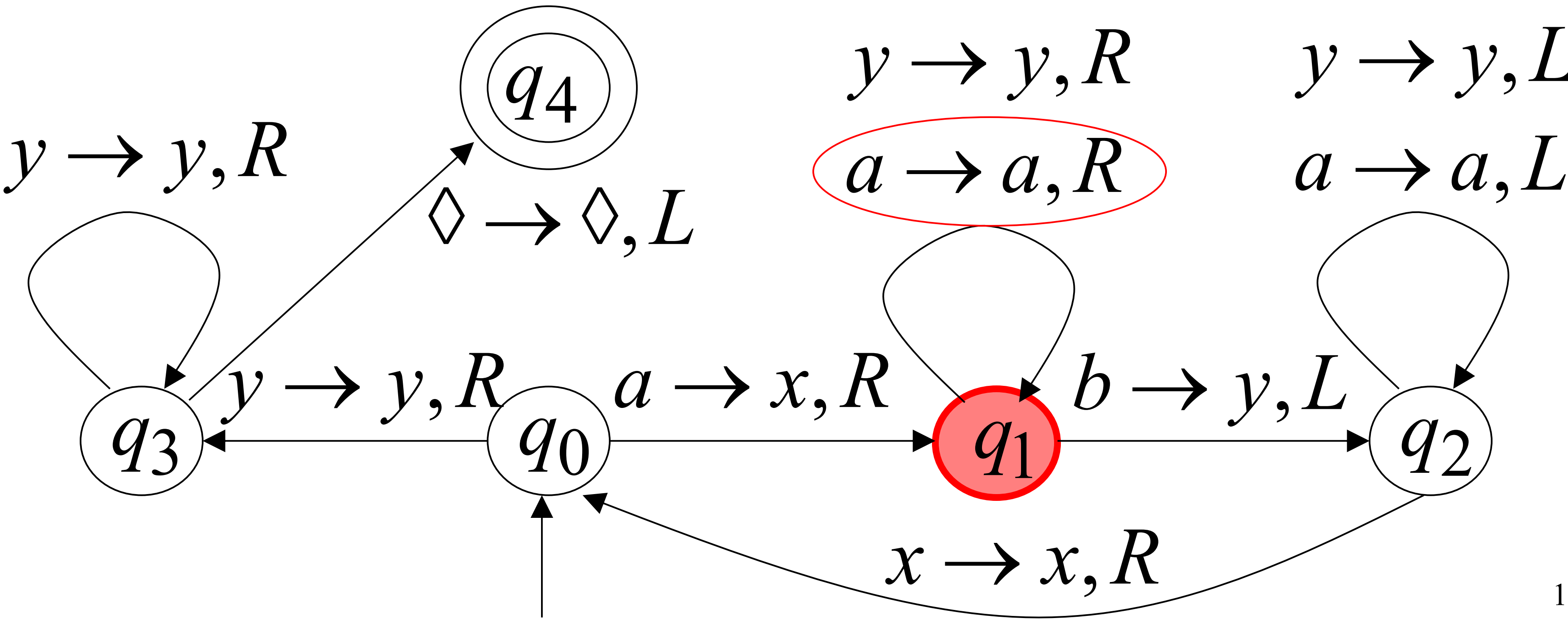


q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 1

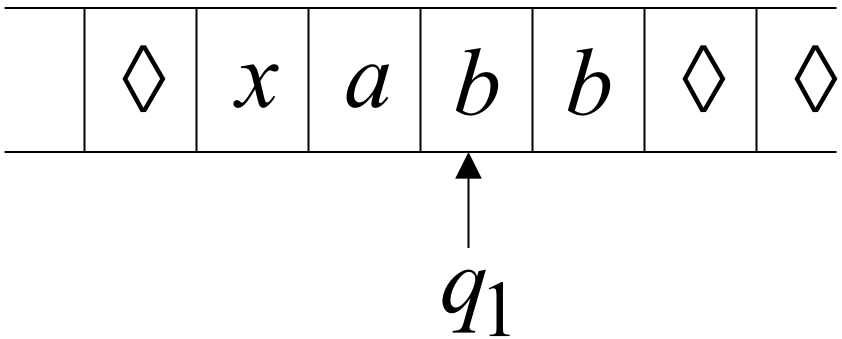


◊
x
a
b
b
◊
◊
q_1

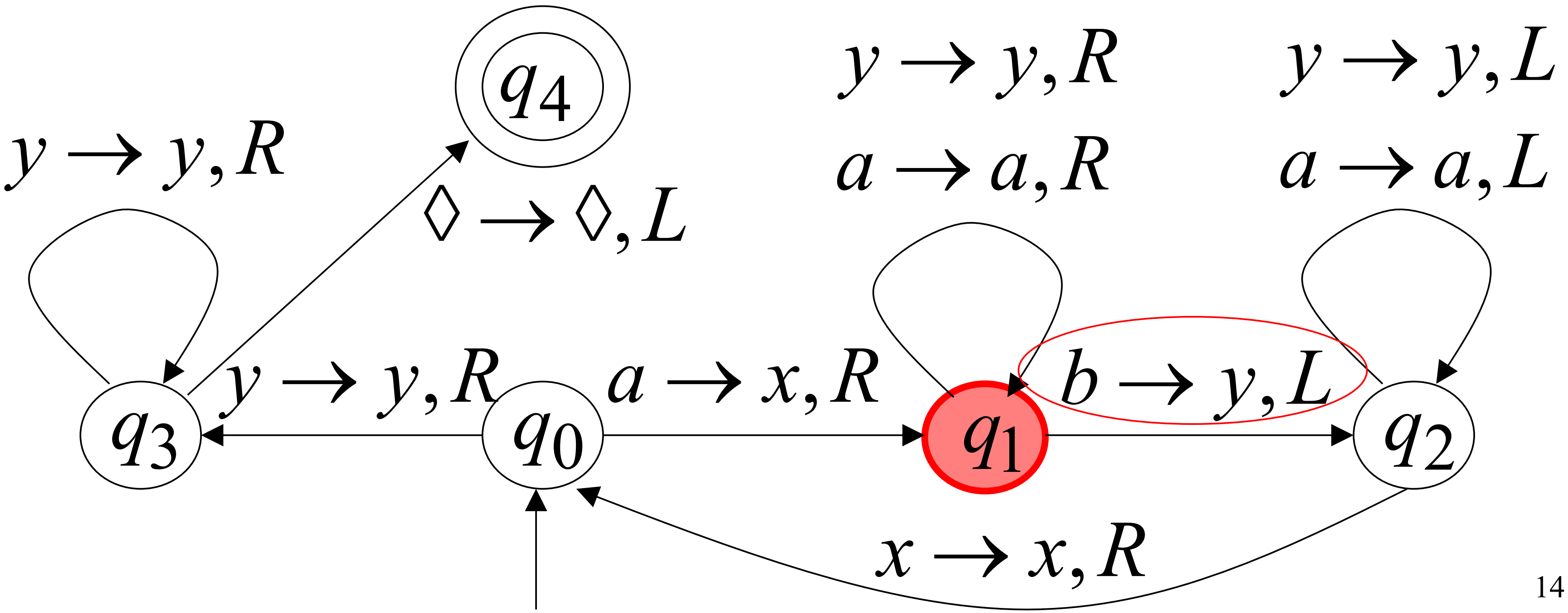


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 2

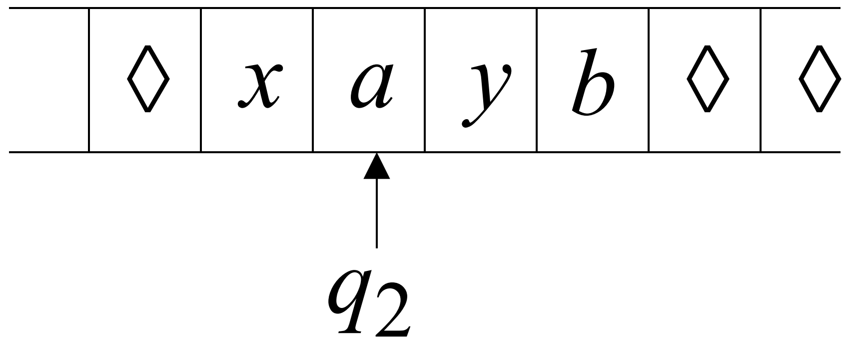


$\Diamond$ x a b b $\Diamond$ $\Diamond$
q_1

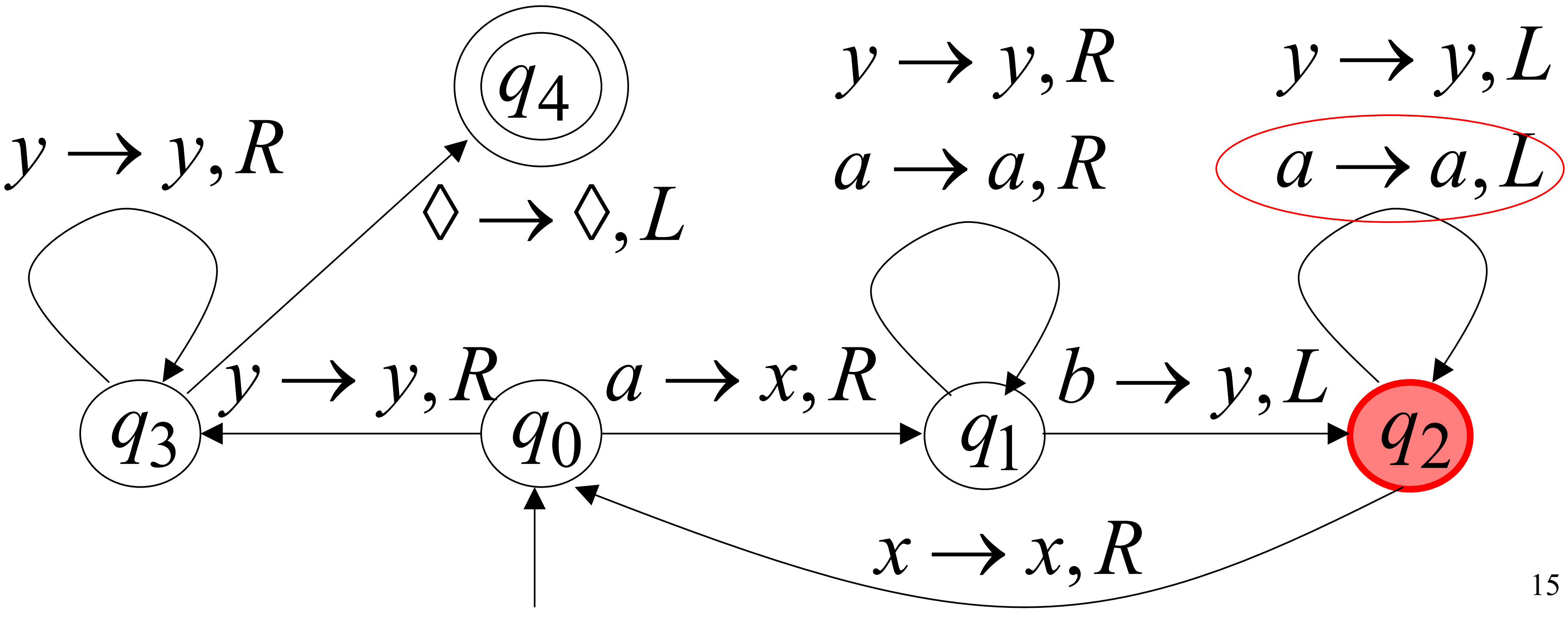


q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 3

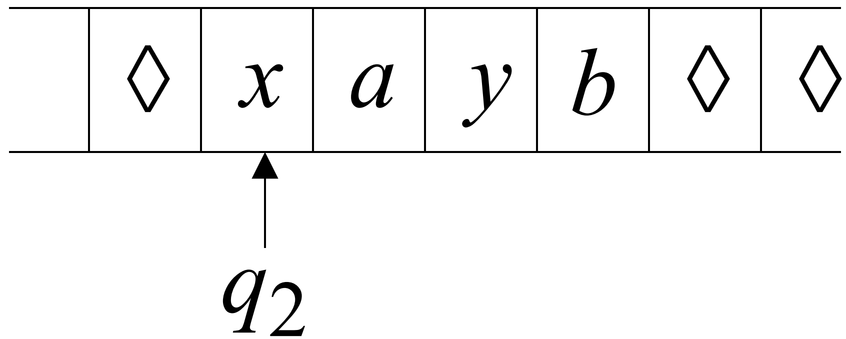


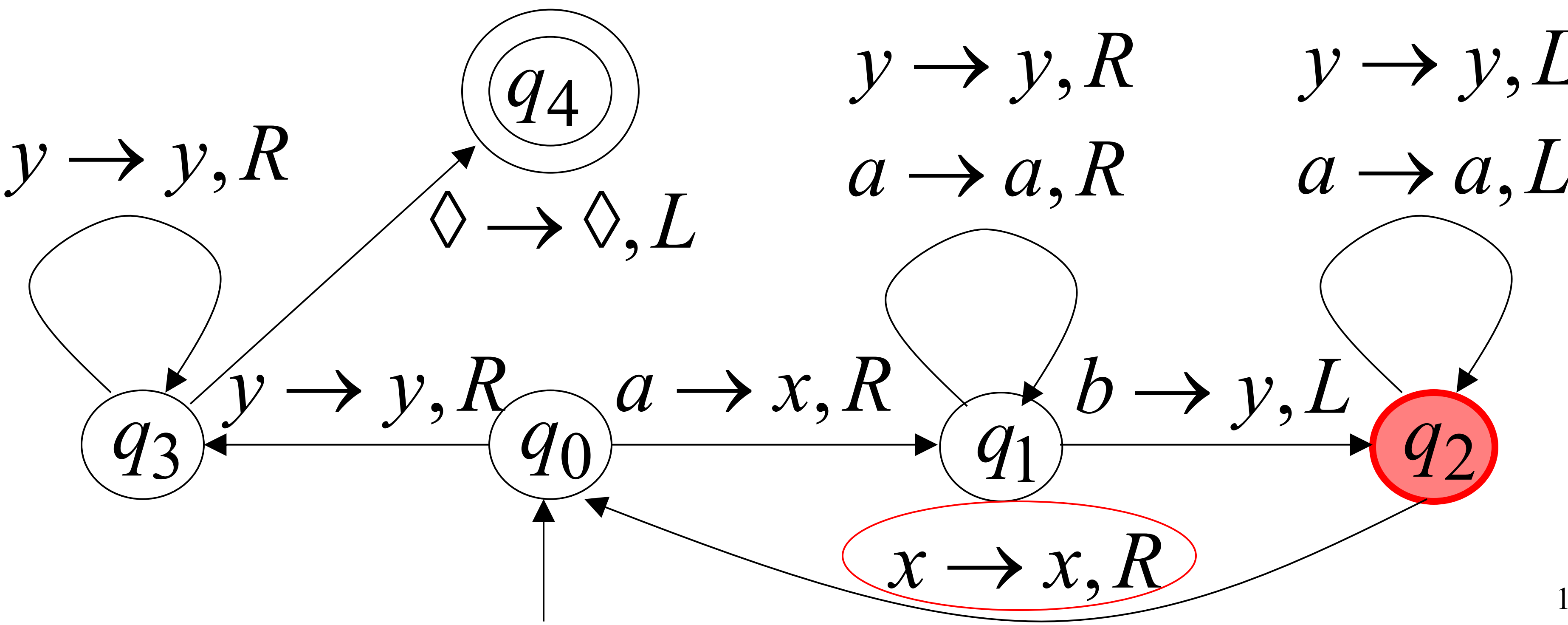
◊
x
a
y
b
◊
◊
q_2



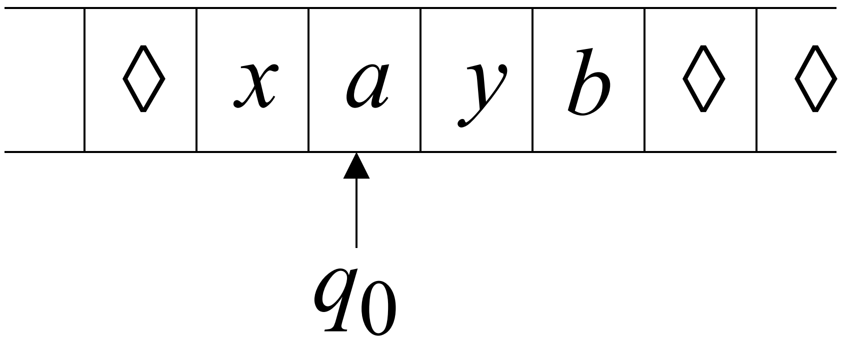
q_4
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

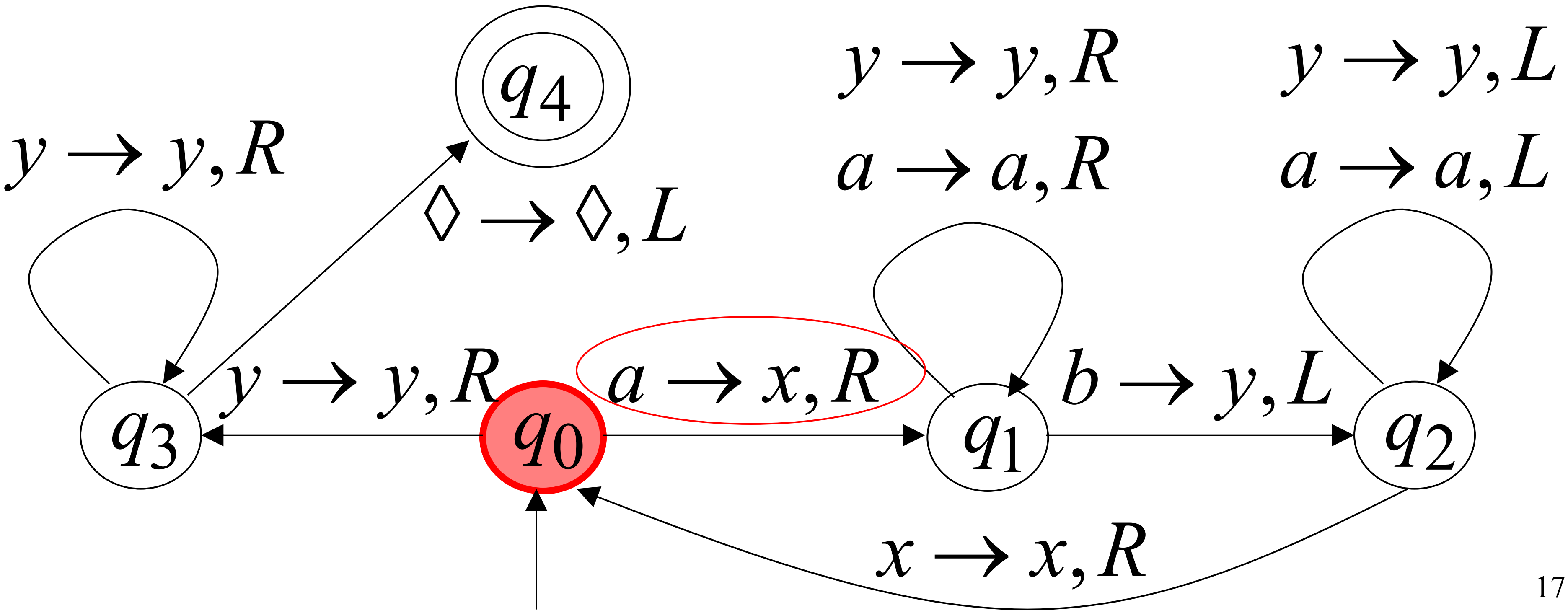
Time 4


◊ x a y b ◊ ◊
q_2

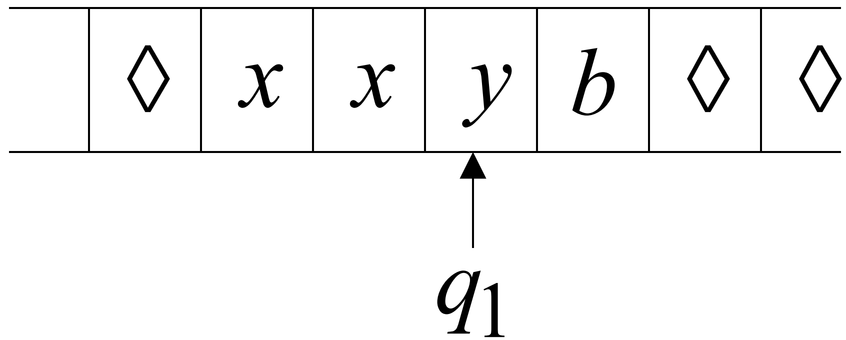

q_4
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

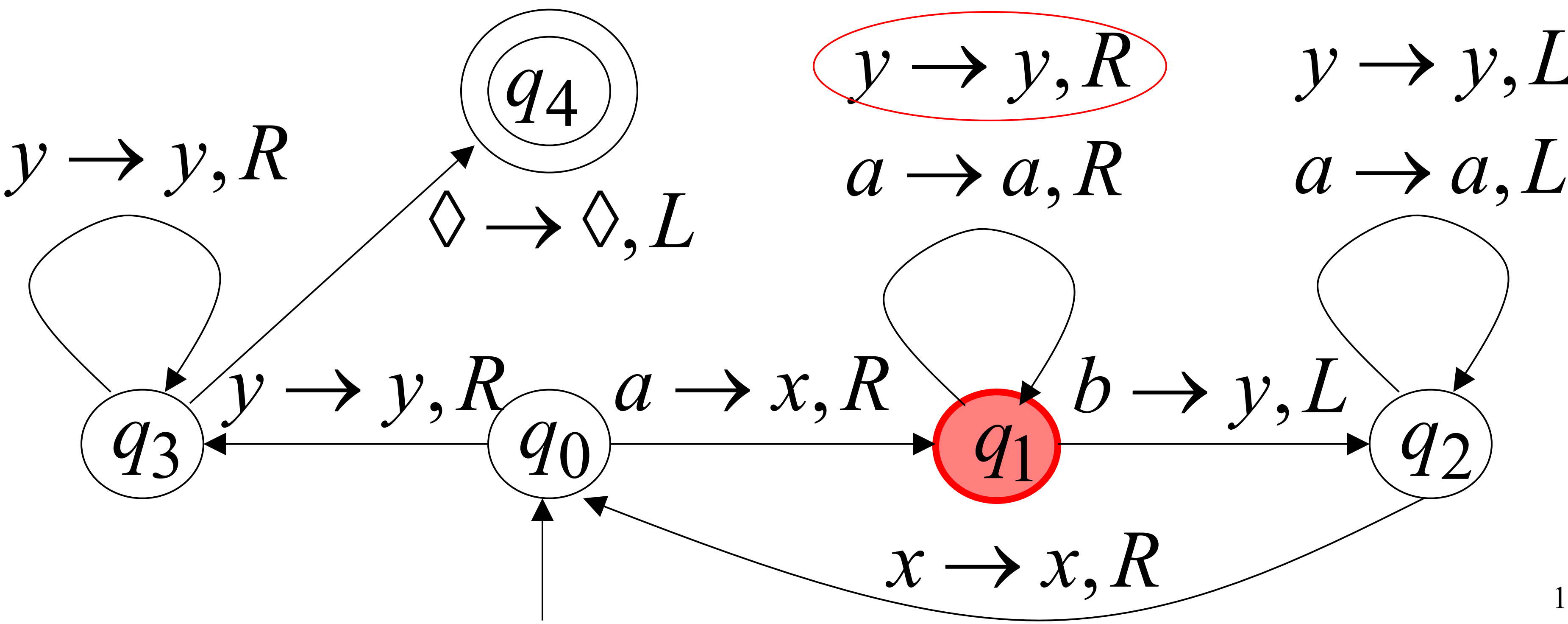
Time 5


◊
x
a
y
b
◊
◊
q_0

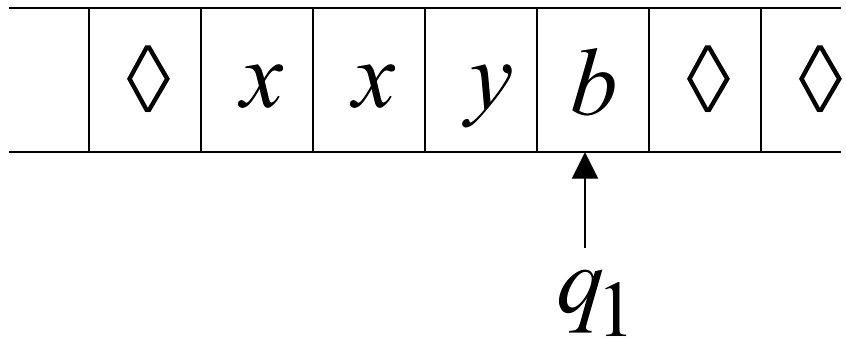

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 6

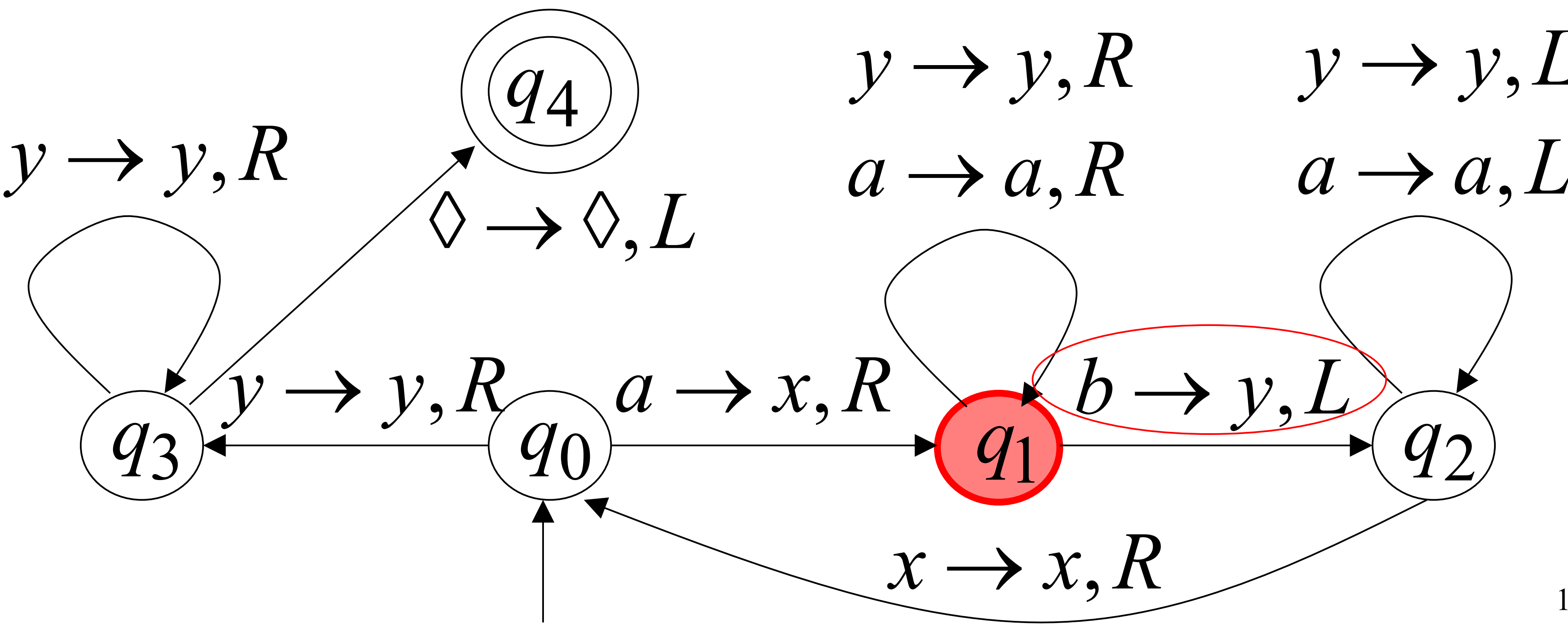

◊ x x y b ◊ ◊
q_1


q_4
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

Time 7

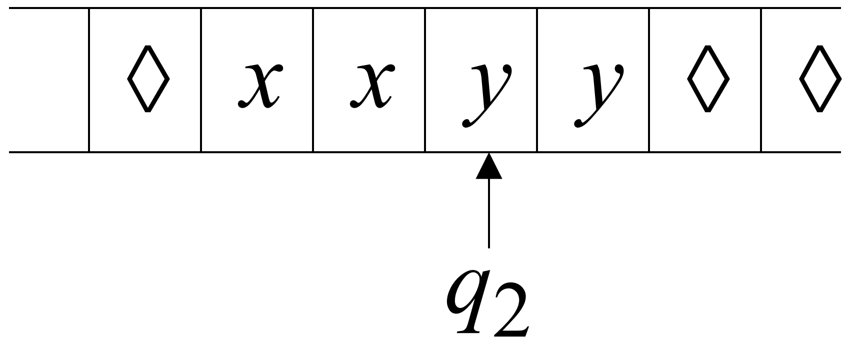


$\Diamond$ x x y b $\Diamond$ $\Diamond$
q_1

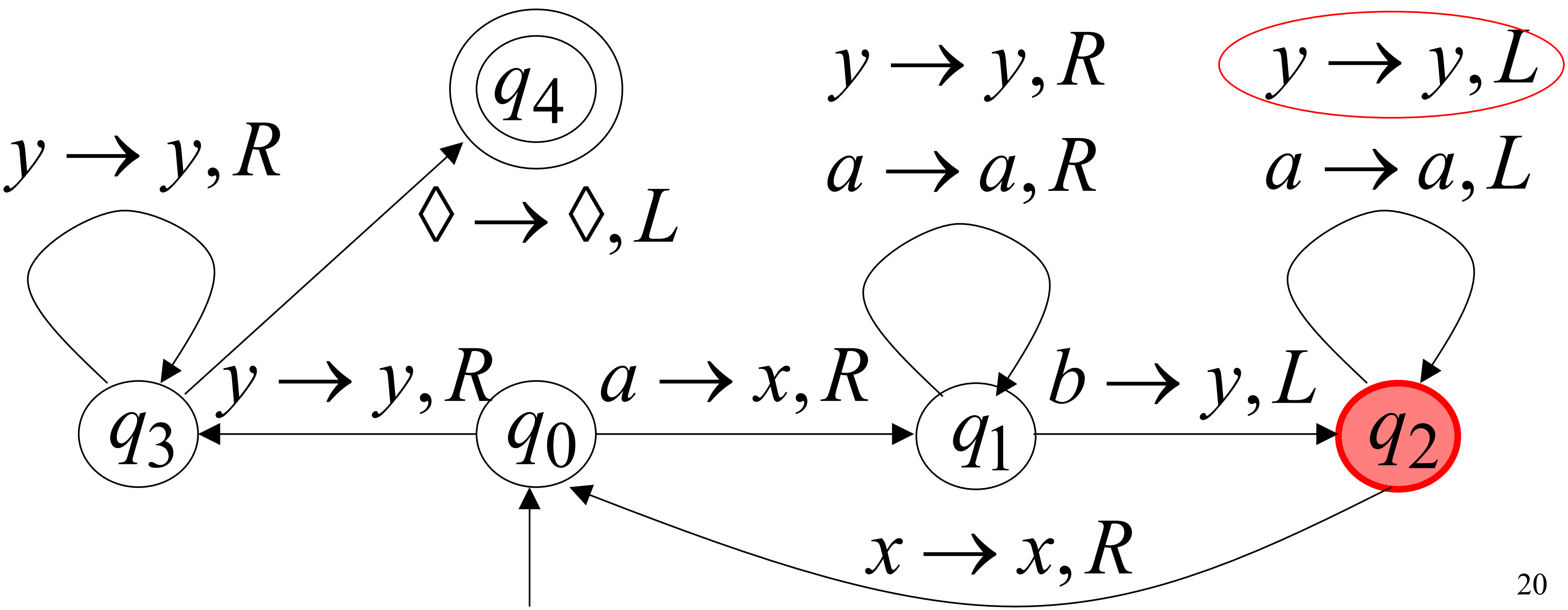


q_0
q_1
q_2
q_3
q_4
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
$x \rightarrow x, R$

Time 8

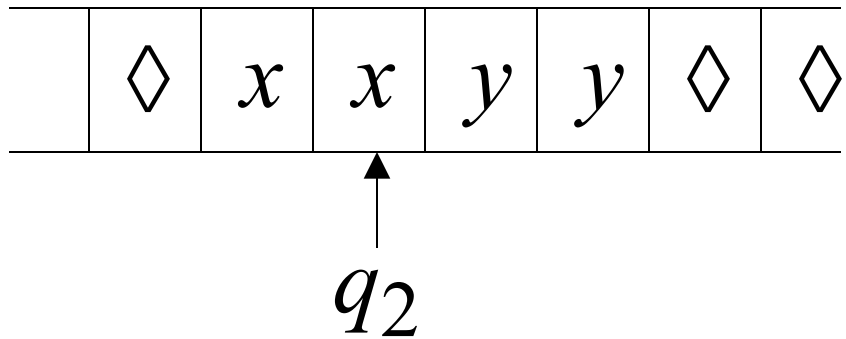


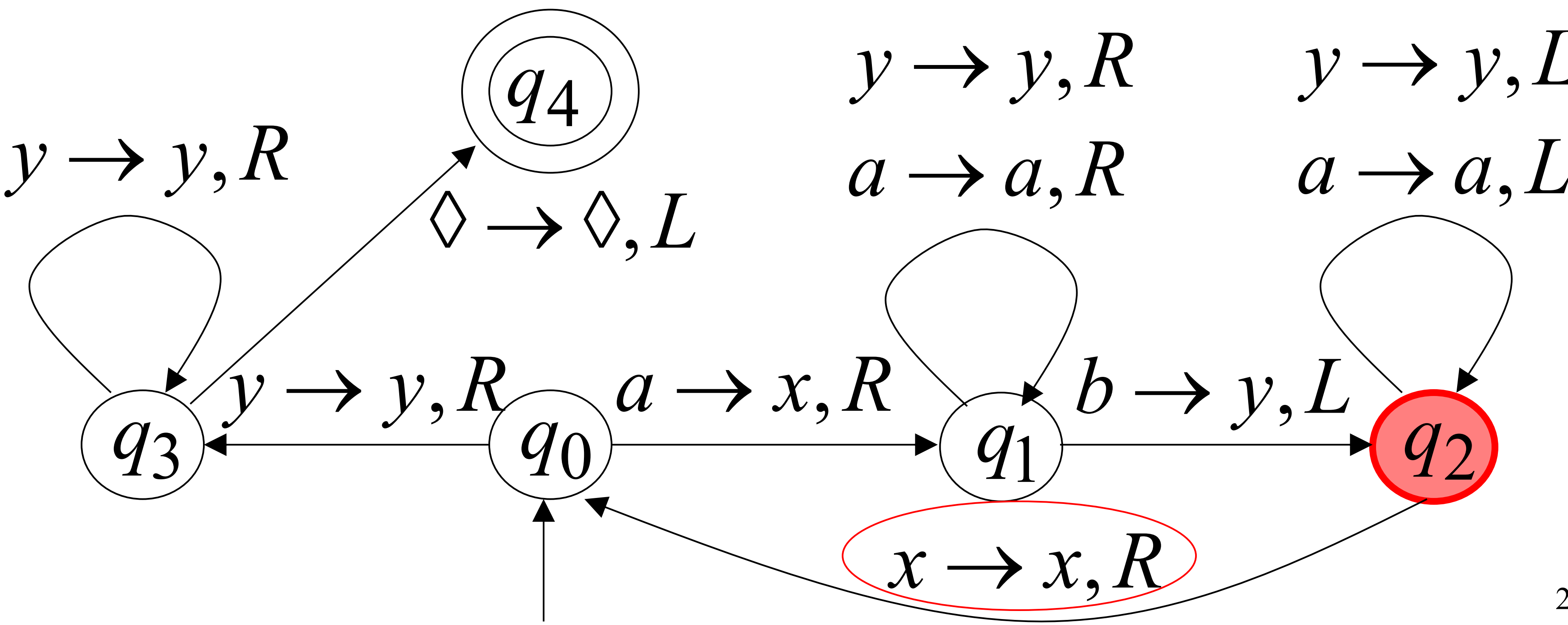
◊
x
x
y
y
◊
◊
q_2



q_4
$\Diamond \rightarrow \Diamond, L$
$y \rightarrow y, R$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$y \rightarrow y, R$
$a \rightarrow a, R$
$b \rightarrow y, L$
q_2
$y \rightarrow y, L$
$a \rightarrow a, L$
$x \rightarrow x, R$

Time 9


◊
x
x
y
y
◊
◊
q_2


q_4
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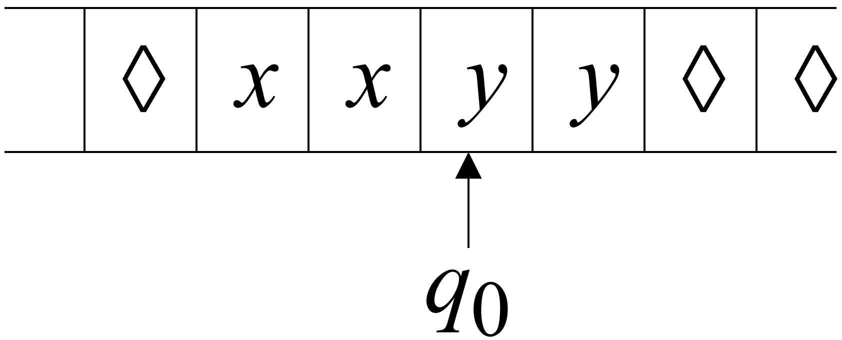
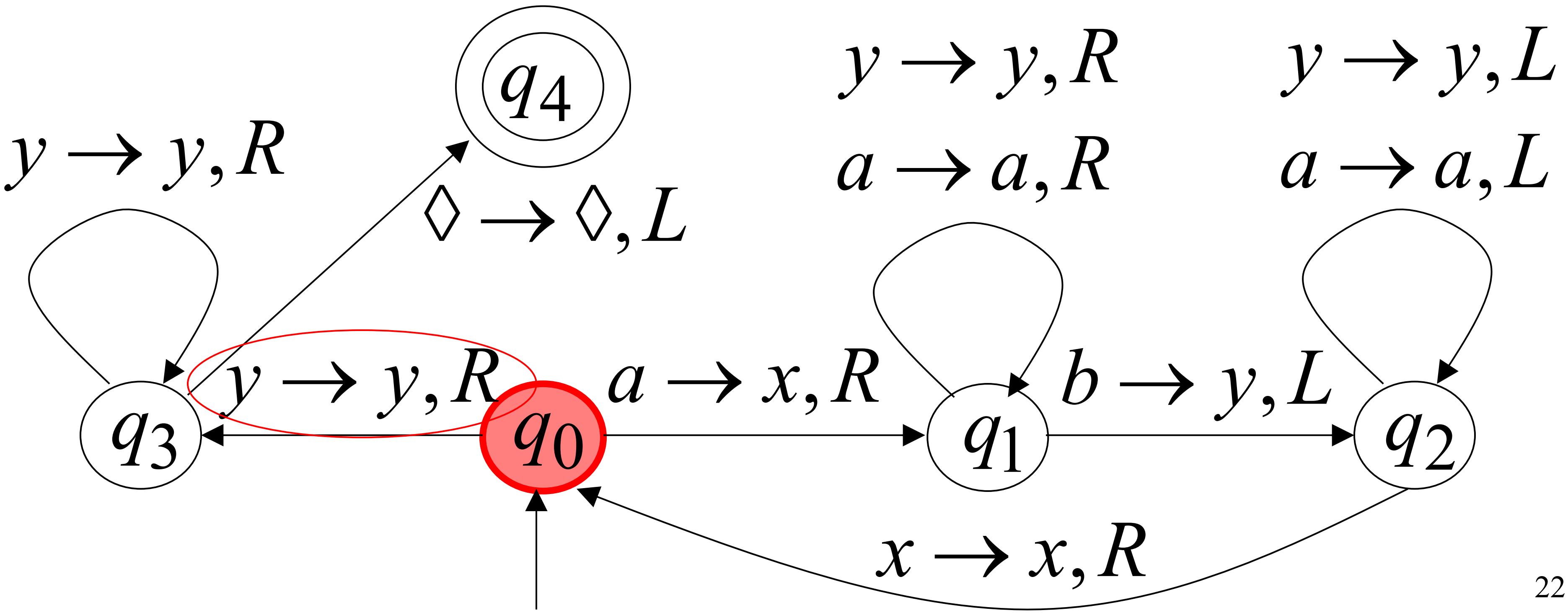
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 10



◊
x
x
y
y
◊
◊
q_0



q_4
$y \rightarrow y, R$
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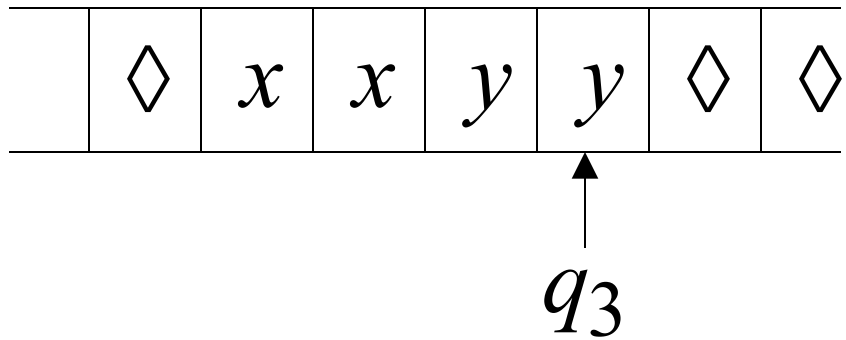
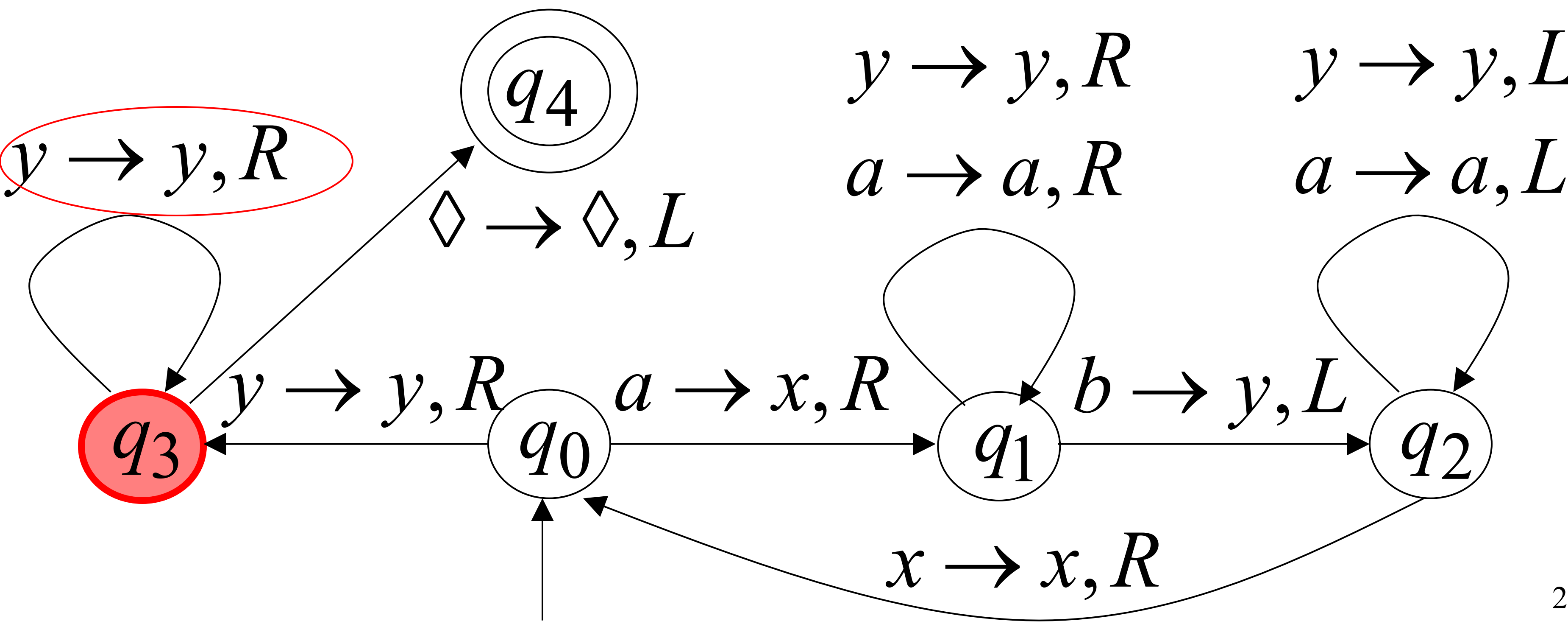
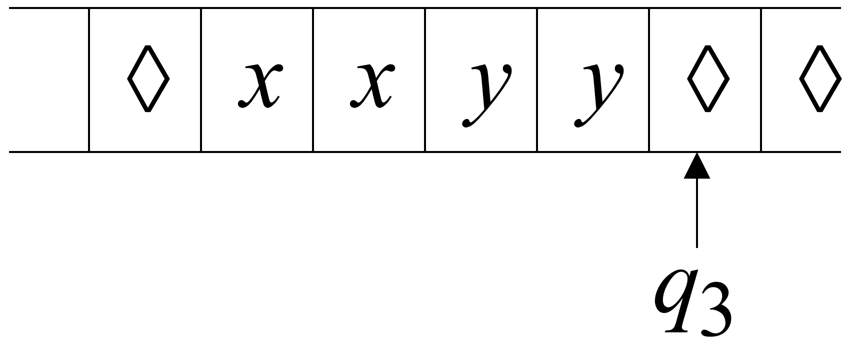
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
$y \rightarrow y, R$
$a \rightarrow x, R$
$b \rightarrow y, L$
q_3
q_0
q_1
q_2
$x \rightarrow x, R$

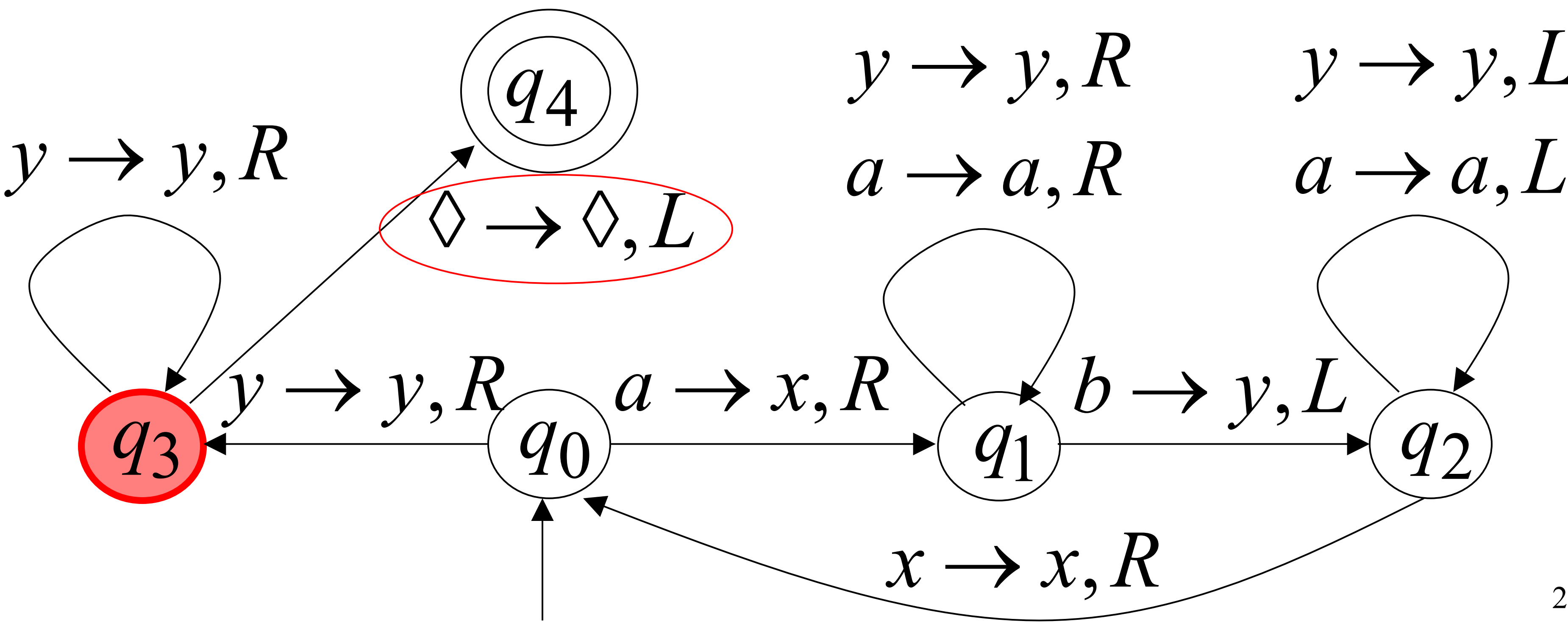
Time 11


◊ x x y y ◊ ◊
q_3

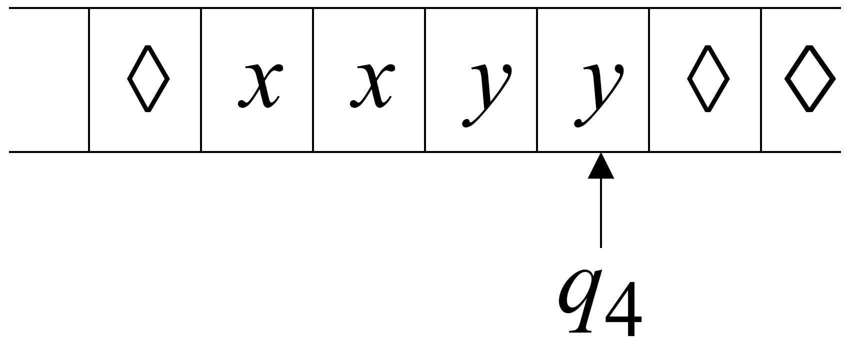

$y \rightarrow y, R$
q_4
$\lozenge \rightarrow \lozenge, L$
$y \rightarrow y, R$
$a \rightarrow a, R$
$y \rightarrow y, L$
$a \rightarrow a, L$
q_3
$y \rightarrow y, R$
q_0
$a \rightarrow x, R$
q_1
$b \rightarrow y, L$
q_2
$x \rightarrow x, R$

Time 12

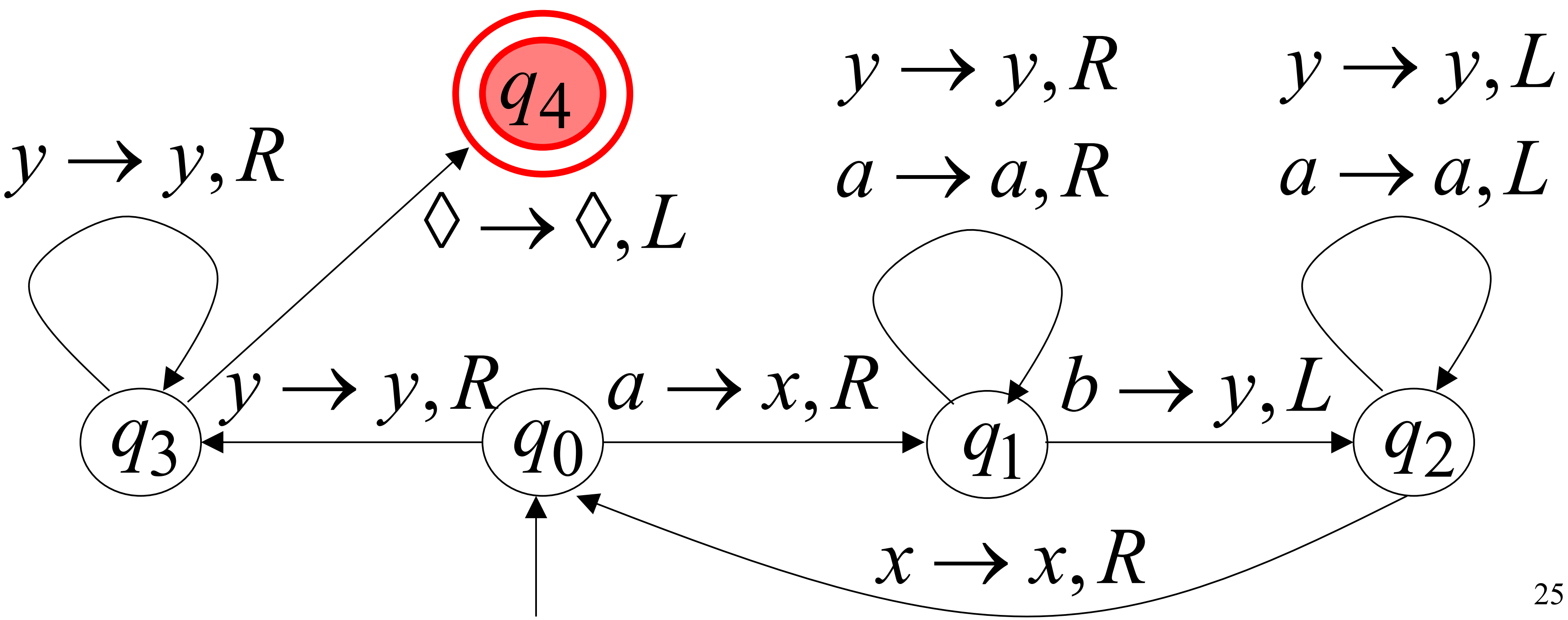




Time 13



Halt & Accept



Polynomial time algorithms: $TIME(n^k)$

constant $k > 0$

Represents tractable algorithms:

for small k we can decide the result fast

The Time Complexity Class P

$$P = \bigcup_{k>0} TIME(n^k)$$

Represents:

- polynomial time algorithms

- "tractable" problems

Class P

$\{a^n b\}$

$\{a^n b^n\}$

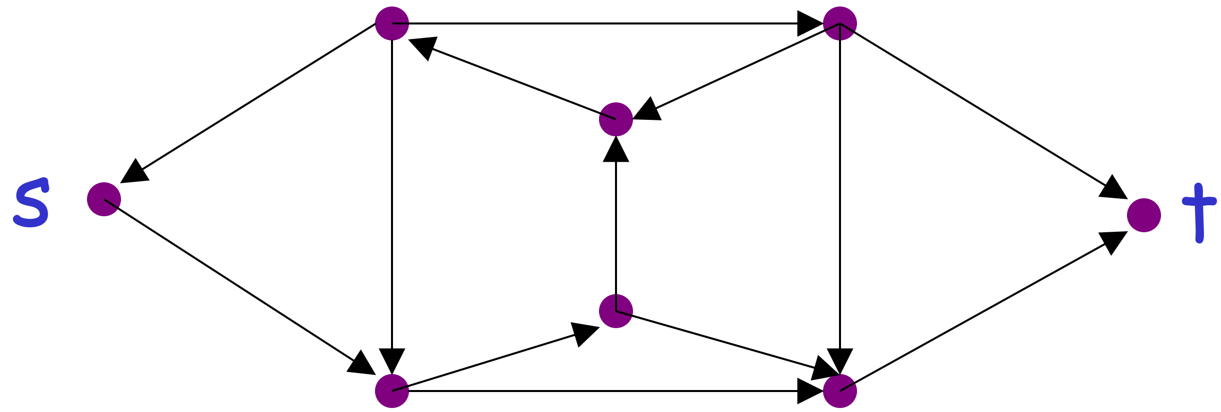
$\{ww\}$

Exponential time algorithms: $TIME(2^{n^k})$

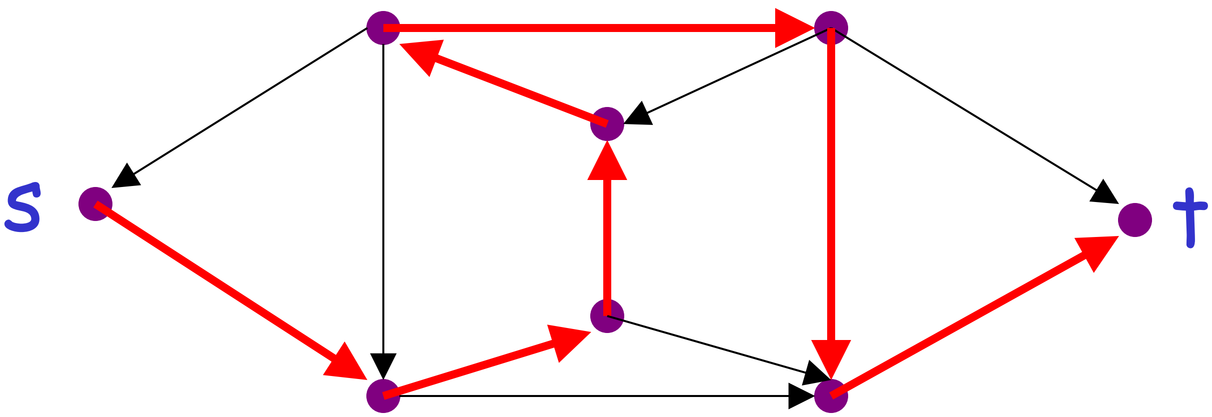
Represent intractable algorithms:

Some problem instances
may take centuries to solve

Example: the Hamiltonian Path Problem



Question: is there a Hamiltonian path from s to t?



YES!

A solution: search exhaustively all paths

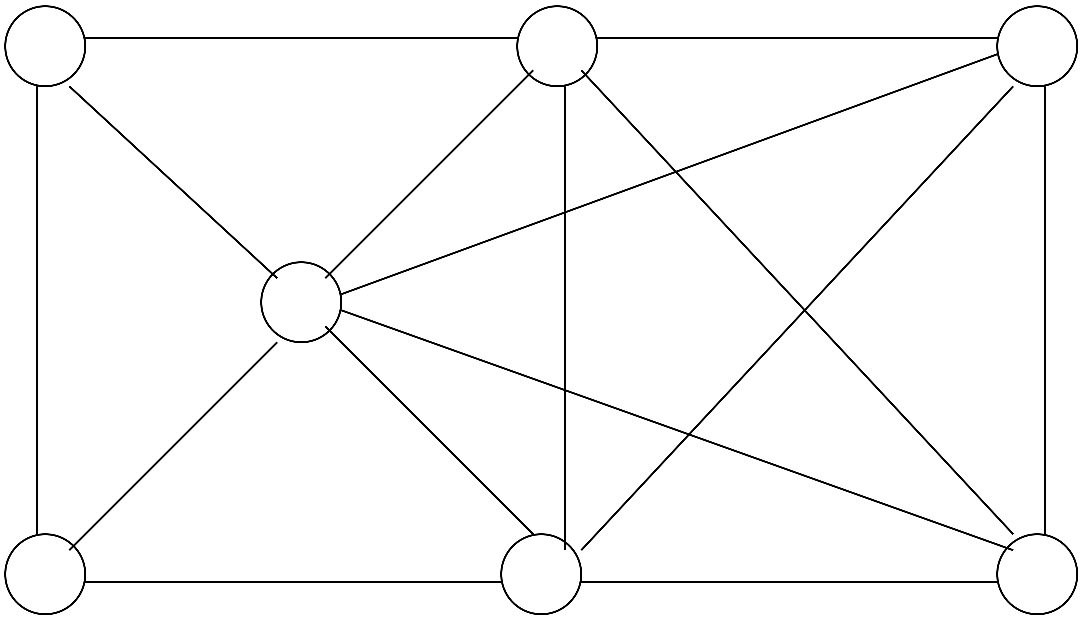
L = {<G,s,t>: there is a Hamiltonian path in G from s to t}

$$L \in TIME(n!) \approx TIME(2^{n^k})$$

Exponential time

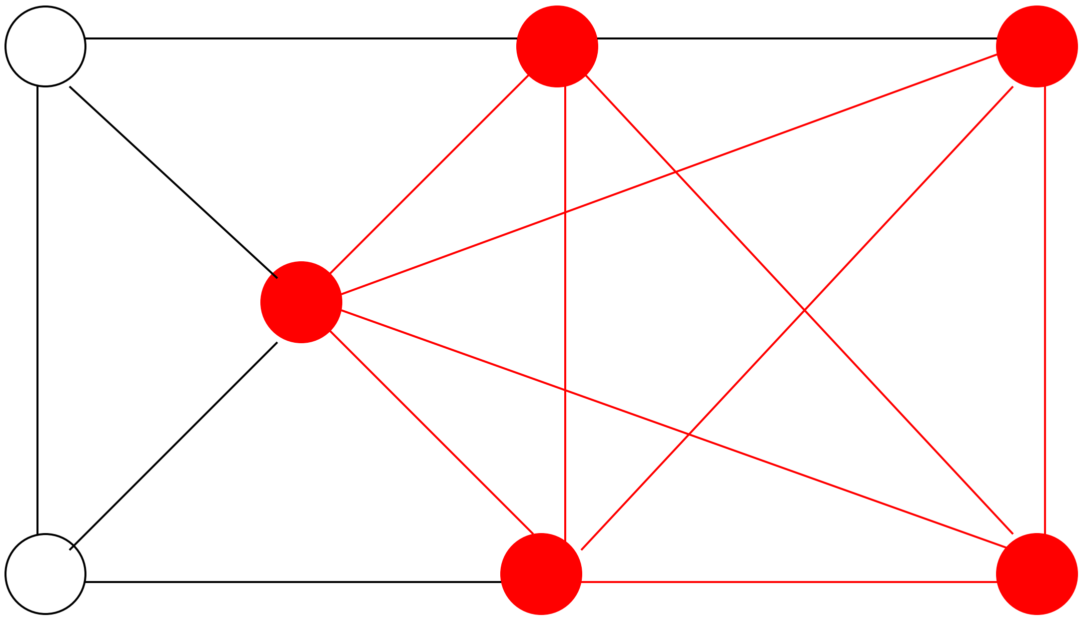
Intractable problem

The clique problem



Does there exist a clique of size 5?

The clique problem



Does there exist a clique of size 5?

Example: The Satisfiability Problem

Boolean expressions in
Conjunctive Normal Form:

$$t_1 \wedge t_2 \wedge t_3 \wedge \cdots \wedge t_k \quad \text{clauses}$$

$$t_i = x_1 \vee \bar{x}_2 \vee x_3 \vee \cdots \vee \bar{x}_p$$

Variables

Question: is the expression satisfiable?

Example: $(\overline{x}_1 \vee x_2) \wedge (x_1 \vee x_3)$

Satisfiable: $x_1 = 0,\ x_2 = 1,\ x_3 = 1$

$$(\overline{x}_1 \vee x_2) \wedge (x_1 \vee x_3) = 1$$

Example: $(x_1 \vee x_2) \wedge \overline{x}_1 \wedge \overline{x}_2$

Not satisfiable

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

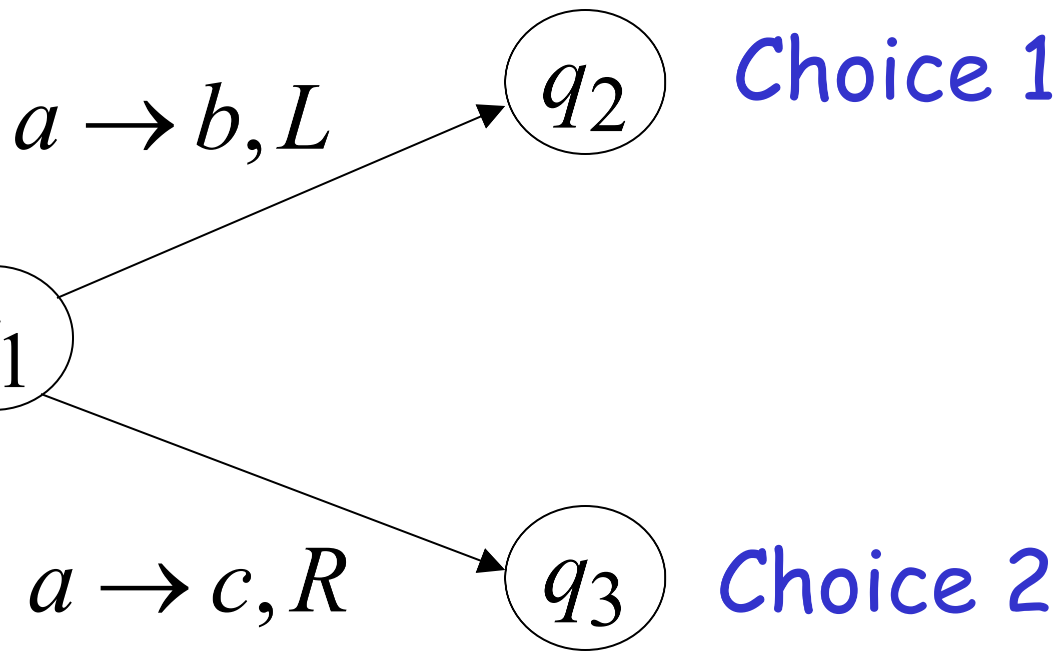
$$L \in TIME(2^{n^k})$$

exponential

Algorithm:

search exhaustively all the possible binary values of the variables

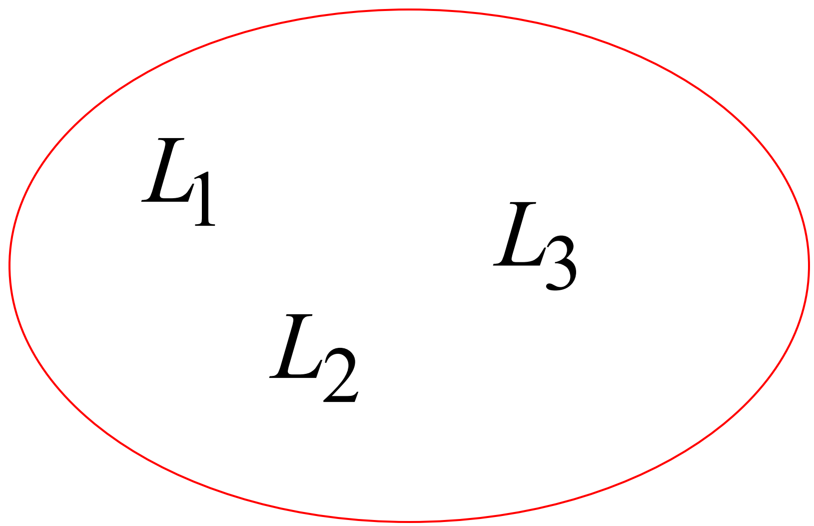
Variation of Turing Machine: Nondeterministic Turing Machines



Allows Non Deterministic Choices

Non-Determinism

Language class: $NTIME(T(n))$



A Non-Deterministic Turing Machine decides each string of length n in time $O(T(n))$

Non-Deterministic Polynomial time algorithms:

$$L \in NTIME(n^k)$$

The class NP

$$NP = \bigcup_{k>0} NTIME(n^k)$$

Non-Deterministic Polynomial time

Example: The satisfiability problem

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

Non-Deterministic algorithm:

- Guess an assignment of the variables
- Check if this is a satisfying assignment

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

Time for n variables:

- Guess an assignment of the variables $O(n)$

- Check if this is a satisfying assignment $O(n)$

Total time: $O(n)$

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

$$L \in NP$$

The satisfiability problem is an NP - Problem

Observation:

$$P \subseteq NP$$

Deterministic Polynomial

Non-Deterministic Polynomial

Open Problem: $P = NP\ ?$

WE DO NOT KNOW THE ANSWER

Open Problem: $P = NP\,?$

Example: Does the Satisfiability problem have a polynomial time deterministic algorithm?

WE DO NOT KNOW THE ANSWER