CSC 339 – Theory of Computation
Fall 2023

12. A Universal Turing Machine

Outline

- Limitation of Turing machines
- A universal Turing machine
- Turing-acceptable languages
- Non Turing-acceptable languages

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A limitation of Turing Machines:

$$\text{Turing Machines are } \underbrace{\text{“hardwired”}}_{\text{They execute only one program.}}.$$

Real Computers are re-programmable.

Solution: Universal Turing Machine

Attributes:

- Reprogrammable machine.
- Simulates any other Turing Machine.

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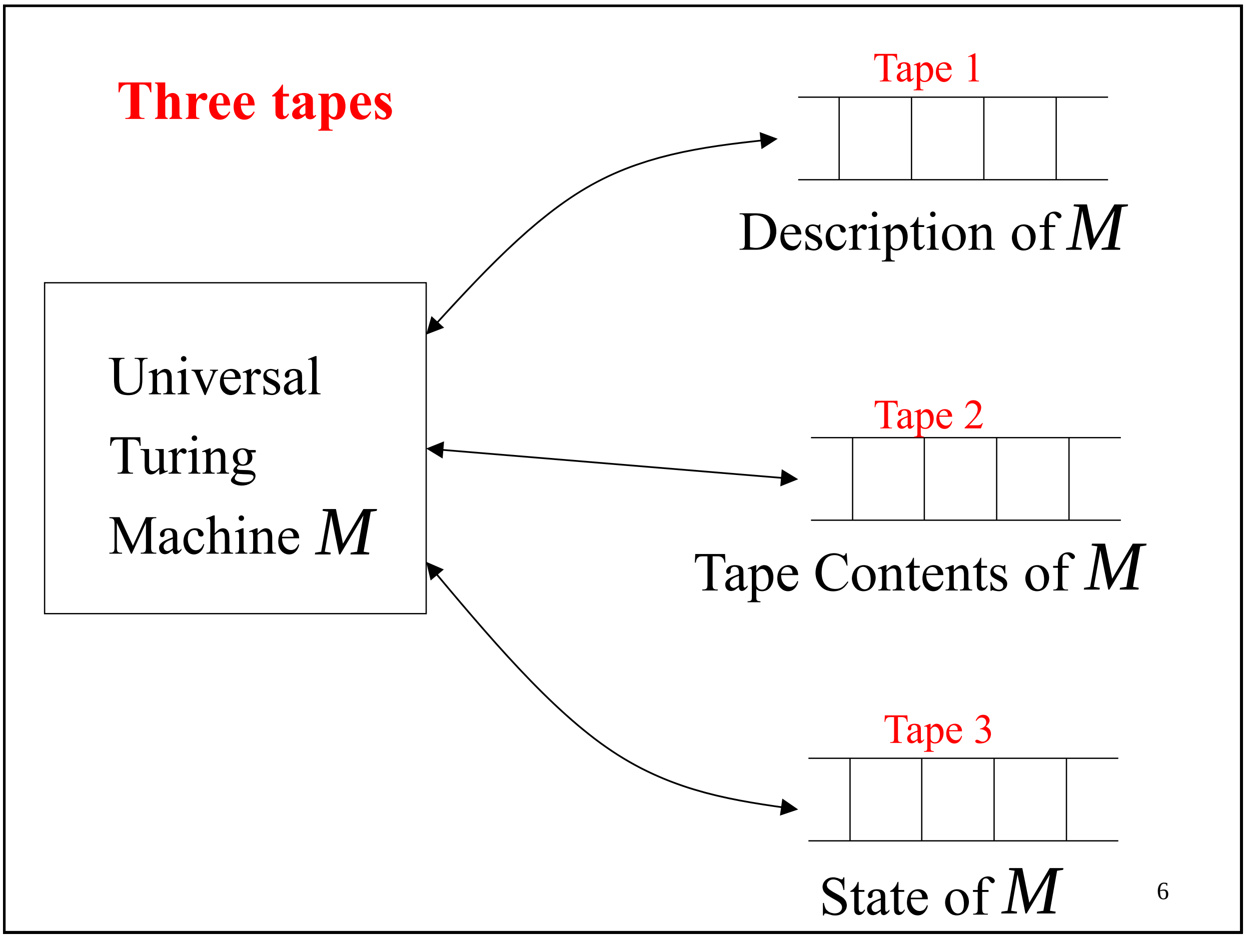
Universal Turing Machine
simulates any Turing Machine M

Input of Universal Turing Machine:

Description of transitions of M

Input string of M


Three tapes
Tape 1
Description of M
Universal Turing Machine M
Tape 2
Tape Contents of M
Tape 3
State of M
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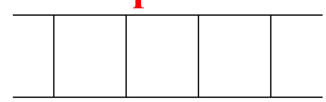
Tape 1



Description of M

We describe Turing machine M
as a string of symbols.

We encode M as a string of symbols.

Alphabet Encoding

Symbols:	a	b	c	d	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Encoding:	1	11	111	1111	

State Encoding

States:	q_1	q_2	q_3	q_4	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
Encoding:	1	11	111	1111	

Head Move Encoding

Move:	L	R
	$\downarrow$	$\downarrow$
Encoding:	1	11

Transition Encoding

Transition: $\delta(q_1, a) = (q_2, b, L)$

Encoding: $1\,0\,1\,0\,1\,1\,0\,1\,1\,0\,1$

Separator

Turing Machine Encoding

Transitions:

$\delta(q_1, a) = (q_2, b, L)$ $\delta(q_2, b) = (q_3, c, R)$

Encoding:

1 0 1 0 11 0 11 0 1 00 11 0 11 0 111 0 111 0 11

Separator

Tape 1 Description of Universal Turing Machine:

Binary encoding of the simulated machine M

Tape 1

1 0 1 0 11 0 11 0 10011 0 11 0 111 0 111 0 1100…

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A Turing Machine is described with a binary string of 0's and 1's.

Therefore, the set of Turing machines forms a language:

Each string of this language is the binary encoding of a Turing Machine.

Language of Turing Machines:

L = { 010100101, (Turing Machine 1)

00100100101111, (Turing Machine 2)

111010011110010101, (Turing Machine 2)

…… } …

Theorem:
The set of all Turing Machines is countable.

Proof:

Any Turing Machine can be encoded with a binary string of 0's and 1's.

Find an enumeration procedure for the set of Turing Machine strings.

Enumerator:

Repeat

1. Generate the next binary string of 0's and 1's in proper order.
2. Check if the string describes a Turing Machine

   if YES: print string on output tape.

   if NO: ignore string.

Binary strings | **Turing Machines**

0

1

00

01

⋮

10101101100

10101101101 $\xrightarrow{s_1}$ 10101101101

⋮

101101010010101101 $\xrightarrow{s_2}$ 101101010010101101

⋮

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Theorem:

If S is an infinite countable set, then the powerset 2^S of S is uncountable.

(the powerset 2^S is the set whose elements are all possible sets made from the elements of S)

Application to Languages

Consider Alphabet : $A = \{a, b\}$

The set S of all strings over A is infinite and countable:

$$S = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

(we can enumerate the strings in proper order)

Any language L over A is a subset of S.

Example:

$$L = \{aa, ab, aab\}$$

Application to Languages

Consider Alphabet : $A = \{a, b\}$

The set S of all strings over A is infinite and countable:

$$S = \{a, b\}^* = \{\varepsilon, a, b, aa, ab, ba, bb, aaa, aab, \ldots\}$$

(we can enumerate the strings in proper order)

The powerset of S contains all languages:

$$2^S = \{\emptyset, \{\varepsilon\}, \{a\}, \{a, b\}, \{aa, b\}, \ldots, \{aa, ab, aab\}, \ldots\}$$

The powerset of S is uncountable.

Consider Alphabet : $A = \{a, b\}$

countable

Turing machines: $M_1 \quad M_2 \quad M_3 \quad \cdots$

accepts

$$\begin{array}{cccc} M_1 & M_2 & M_3 & \cdots \\ \downarrow & \downarrow & \downarrow & \\ L_1 & L_2 & L_3 & \cdots \end{array}$$

Languages accepted
By Turing Machines: $L_1 \quad L_2 \quad L_3 \quad \cdots$

countable

Consider Alphabet : $A = \{a, b\}$

countable

Turing machines: $M_1 \quad M_2 \quad M_3 \quad \cdots$

accepts

Languages accepted
By Turing Machines: $L_1 \quad L_2 \quad L_3 \quad \cdots$

countable

Denote: $X = \{L_1, L_2, L_3, \ldots\}$ countable

Note: $X \subseteq 2^S \quad \left(S = \{a, b\}^*\right)$

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Languages accepted by Turing machines: X countable

All possible languages: 2^S uncountable

Therefore: $X \neq 2^S$

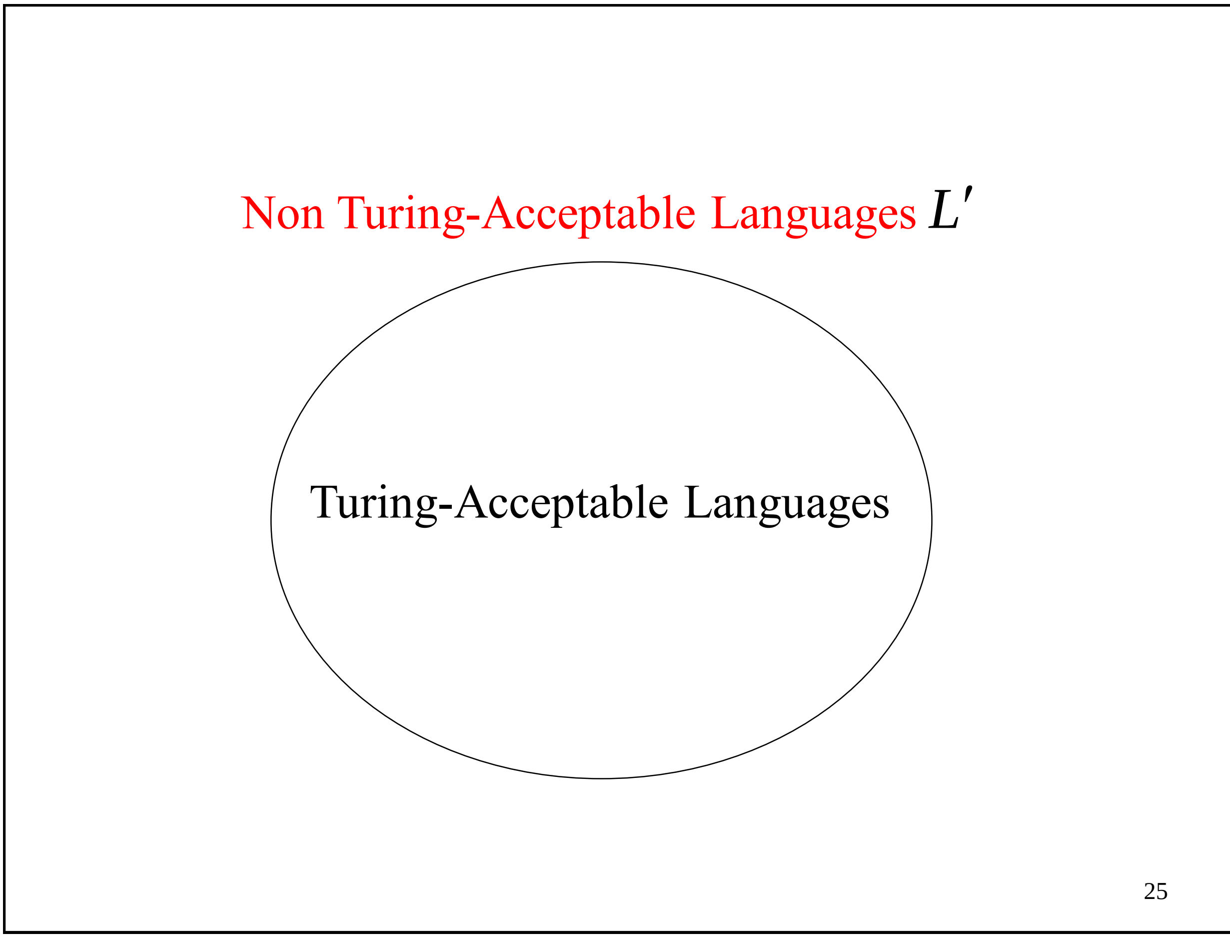
since $X \subseteq 2^S$, we have $X \subset 2^S$

Conclusion:

There is a language L' not accepted by any Turing Machine.

$$X \subset 2^S \Longrightarrow \exists L' \in 2^S \text{ and } L' \notin X$$

The language L' cannot be described by any algorithm.


Non Turing-Acceptable Languages L'
Turing-Acceptable Languages

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