Revision for chapter one

- Inequalities:

linear → solve for x

Absolute Value →

If $|x| < a$ then $-a < x < a$

If $|x| > a$ then $x > a$ or $x < -a$

Quadratic and Rational → factor the polynomial function then study the signs and choose the result that satisfies the inequality.

- properties of functions:

even → $f(-x) = f(x)$ → symmetric to the y-axis

odd → $f(-x) = -f(x)$ → symmetric to the origin

* all linear functions are one to one.

- Trigonometry:

pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $\tan^2\theta + 1 = \sec^2\theta$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$

Identities due to symmetry

$\sin(-\theta) = -\sin\theta$ $\quad$ $\tan(-\theta) = -\tan\theta$ $\quad$ $\cos(-\theta) = \cos\theta$

odd $\quad$ odd $\quad$ even

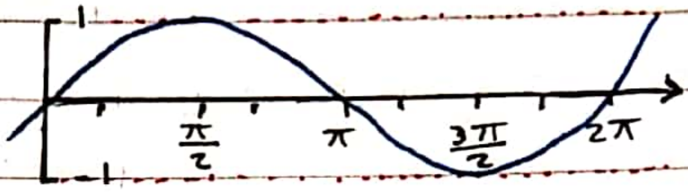
sum and difference identities

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

Graphs of trigonometric functions

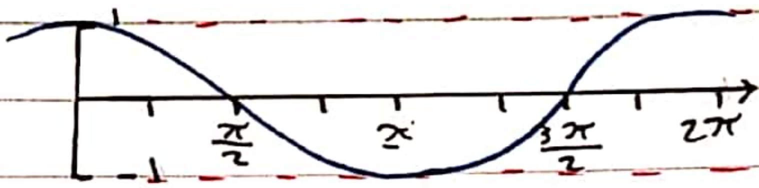
Sine:



Domain → $\mathbb{R}$ $\sin^{-1}x \rightarrow [-1, 1]$

Range → $[-1, 1]$ $\sin^{-1}x \rightarrow [\frac{-\pi}{2}, \frac{\pi}{2}]$

cosine:



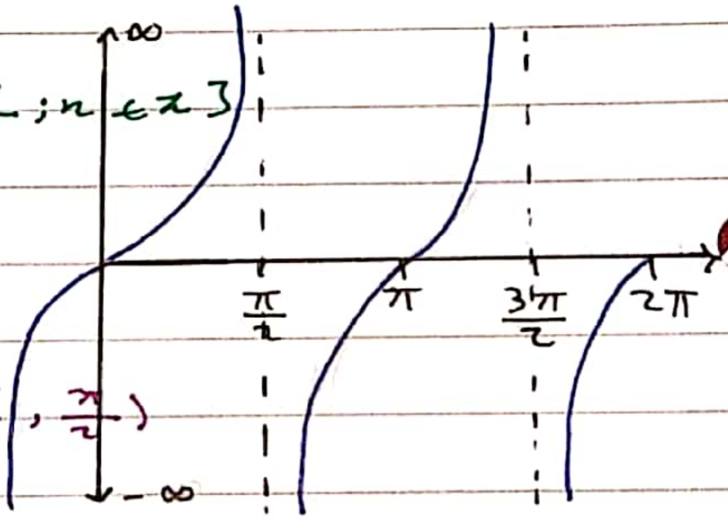
Domain → $\mathbb{R}$ $\cos^{-1}x \rightarrow [-1, 1]$

Range → $[-1, 1]$ $\cos^{-1}x \rightarrow [0, \pi]$

tangent:

Domain → $\mathbb{R} - [(2n+1)\frac{\pi}{2}; n \in \mathbb{Z}]$

Range → $\mathbb{R}$



Domain of $\tan^{-1}x \rightarrow \mathbb{R}$

Range of $\tan^{-1}x \rightarrow (\frac{-\pi}{2}, \frac{\pi}{2})$

cotangent:

Domain → $\mathbb{R} - \{n\pi; n \in \mathbb{Z}\} \rightarrow \mathbb{R}$

Range → $\mathbb{R} \rightarrow (0, \pi)$

Secant:

Domain $\rightarrow \mathbb{R} - \{(2n+1)\frac{\pi}{2} \; ; n \in \mathbb{Z}\} \rightarrow (-\infty, -1] \cup [1, \infty)$

Range $\rightarrow (-\infty, -1] \cup [1, \infty) \rightarrow [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$

Cosecant:

Domain $\rightarrow \mathbb{R} - \{n\pi \; ; n \in \mathbb{Z}\} \rightarrow (-\infty, -1] \cup [1, \infty)$

Range $\rightarrow (-\infty, -1] \cup [1, \infty) \rightarrow (0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$

chapter 2:

- section 2.1:

x if the question starts with estimate $\rightarrow$ use the table technique.

- formal definition of limits $\rightarrow$ epsilon and delta

- section 2.2:

- If a limit has an odd root $\rightarrow \sqrt{\lim_{x \to a} a}$
- If a limit has an even root. first, test if $a > 0 \rightarrow \sqrt{\lim_{x \to a} a}$

If $a < 0$. Then the limit does not exist.

- squeeze theorem is used when $\rightarrow \lim_{x \to a} \sin / \cos \left.\frac{x}{0}\right\} \rightarrow \infty$

zeroes of sin $\longrightarrow$ $n\pi$; $n \in \mathbb{Z}$

zeroes of cos $\longrightarrow$ $(2n+1)\frac{\pi}{2}$; $n \in \mathbb{Z}$

Trigonometric theorems :

$$\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b} \qquad \lim_{x \to 0} \frac{1-\cos x}{x} = 0$$

$$\lim_{x \to 0} \frac{x}{\sin x} = 1 \qquad \lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b}$$

factoring rules/formulas:

$(x+y)^2 = x^2 + 2xy + y^2$ $\qquad x^2 - y^2 = (x+y)(x-y)$

$(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$ $\qquad x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

• section 2.3:

Vertical asymptote:

- find the zeroes of denominator.
- simplify the equation if necessary
- check the behavior of f as it approaches to the zeroes
- if you get infinity then vertical asymptote at x = the zeroes.

Horizontal asymptote:

• $\lim_{x \to \pm\infty}$ → $\pm\infty$ → no Horizontal asymptote.

→ n → Horizontal asymptote at $y = n$

Limits at infinity for polynomials:

- if the degree of denominator is greater than the numerator: $\lim_{x \to \pm\infty} f(x) = 0$
- if the degree of both denominator and numerator are equal:

$$\lim_{x \to \pm\infty} f(x) = \frac{\text{coefficient of } x^n}{\text{coefficient of } x^n}$$

- if the degree of numerator is greater than the denominator:

$$\lim_{x \to +\infty} f(x) = \frac{\text{sign of coefficient}}{\text{sign of coefficient}} \cdot \infty$$

$$\lim_{x \to -\infty} f(x) = \frac{\text{sign of coefficient}}{\text{sign of coefficient}} \cdot (-1)^{n-d} \cdot \infty$$

• cases of infinit limits:

• $\frac{\infty}{n}$ ∞ • $\frac{n}{\infty}$ 0 • $\frac{\infty}{\infty}$ Indet. • $\infty - \infty$ Indet. • $\infty \cdot 0$ Indet

• section 2.4:

A function is continuous when:

- $f(c)$ is defined
- $\lim f(x)$ exists
- $f(c) = \lim f(x)$

* a polynomial function is continuous on $\mathbb{R}$
* a rational function is continuous on $\mathbb{R} - \{\text{zeroes of denom}\}$

Therefore, a function is always continuous on it's domain.

Intermidiate value theorem:

- f should be continuous on a closed interval $[a, b]$
- k is any number between $f(a)$ and $f(b)$
- Then there's atleast one number c in $[a, b]$; $f(c) = k$

chapter 3:

- section 3.1:

The formal definition of derivatives:

- $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ • $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ • $f'(a) = \lim_{x \to a} \frac{f(x) -}{x -}$

The equation of the tangent line:

- find the point $(a, f(a))$
- find the slope $\rightarrow$ m or $f'(a)$
- $y = m(x-a) + f(a)$

A function fails to have derivative when:

- f is discontinuous
- f has a corner or cusp
- f has a vertical tangent i.e. $m = \frac{n}{0}$ undefined

Instantaneous velocity:

$v(t) \rightarrow s'(t)$ $\quad$ $a(t) \rightarrow v'(t)$

- Section 3.2:

The product rule: $(fg)' = fg' + gf'$

The quotient rule: $\frac{gf' - fg'}{(g)^2} = \left(\frac{f}{g}\right)'$

• section 3.3:

Derivative of trigonometric functions:

$\frac{d}{dx} \sin x \rightarrow \cos x$ $\frac{d}{dx} \cos x \rightarrow -\sin x$

$\frac{d}{dx} \tan x \rightarrow \sec^2 x$ $\frac{d}{dx} \cot x \rightarrow -\csc^2 x$

$\frac{d}{dx} \sec x \rightarrow \sec x \tan x$ $\frac{d}{dx} \csc x \rightarrow -\csc x \cot x$

• section 3.4:

The general power rule:

$\frac{d}{dx}((g(x))^r) \rightarrow r(g(x))^{r-1} \cdot \frac{d}{dx} g(x)$

• This can be applied to trigonometric functions

• section 3.5:

Implicit differentiation:

• differentiate both sides of the equation → with respect to x

• seperate y' to one side and solve for it

• section 3.7:

* if $f'(x) \neq 0$ Then f^{-1} has a derivative.

The second formula to inverse derivative:

$\frac{dy}{dx} \longrightarrow \frac{1}{dx/dy}$

Generalized derivatives of inverse trigonometric functions:

$$\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \qquad \frac{d}{dx}(\cos^{-1}u) = -\frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2} \cdot \frac{du}{dx} \qquad \frac{d}{dx}(\cot^{-1}u) = -\frac{1}{1+u^2} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}(\sec^{-1}u) = \frac{1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx} \qquad \frac{d}{dx}(\csc^{-1}u) = -\frac{1}{u\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

Chapter 4:

- Section 4.1:

To find the critical number:

- find the domain of f
- find the derivative: $f' = 0$ or f' DNE
- find c and if $c \in D_f$ then c is a critical number

Absolute extremas:

* if a function is continuous on a closed interval $\longrightarrow$ There has to be
* if a function is continuous on an open interval $\longrightarrow$ Doesn't have to be

Guideline for absolute extremas:

- find the critical numbers of f in (a,b)
- evaluate $f(c)$ and the end points $f(a)$ and $f(b)$
- The greatest value is an absolute max and the least is min

- Section 4.2:

Rolle's theorem:

- f should be continuous on a closed interval $[a,b]$
- f is differentiable on an open interval (a,b)
- $f(a) = f(b)$

Thus, f satisfies the conditions of Rolle's theorem, $\exists c \in (a,b); f'(c) =$

Mean value theorem:

- f is continuous on a closed interval $[a,b]$
- f is differentiable on an open interval (a,b)

Thus, f satisfies the conditions of Mean value theorem.

$\exists\ c \in (a,b);\ f'(c) = \dfrac{f(b) - f(a)}{b - a}$

• section 4.3:

Increasing and decreasing intervals:

- find the domain of f
- find the critical numbers
- draw the domain on a number line with the critical number
- evaluate the signs of $f'(x)$ in interval notation

Local extremas: "first derivative test."

- if f has a critical number then there is an extrema
- determining the local extrema is the same procedure as the Increasing and decreasing except you write it as a point.

• section 4.4

Determining the concavity intervals:

- find the domain of f
- find both $f'(x)$ and $f''(x)$
- find the values of which $f''(x)=0$ or $f''(x)$ DNE

Points of inflection:

Let f be a continuous function on (a,b) and $c \in (a,b)$. If the graph of f concaves upwards on (a,c) then downwards on (c,b) or vice versa. Then, the point $(c, f(c))$ is an inflection point.

Local extremas: "Second derivative test."

- find the domain of f
- find the critical numbers
- find $f''(x)$: if $f''(a) > 0 \longrightarrow$ local min at $(a, f(a))$
  if $f''(b) < 0 \longrightarrow$ local max at $(b, f(b))$
  if $f''(c) = 0 \longrightarrow$ test fails $\rightarrow$ do the 1st derivati

Identifying graphs:

$f(x)$	$f'(x)$	$f''(x)$
- critical numbers: • where the function is discontinuous. • corners • peaks and troughs	- critical numbers: any value/point that crosses the x-axis.	—
- Increasing and decreasing: The old fashioned way ;)	- Increasing and decreasing: "Interval notation" Inc: above the x-axis Dec: under the x-axis	—
- local extrema: Peak → maxima trough → minima	- local extrema: Pinpoint the critical values on a number line then evaluate the extremas	—
- concavity: upwards → ∪ downwards → ∩	- concavity: upwards → where f is inc downwards → where f is dec	- concavity: upwards: above the x-axis downwards: under the x-axis
- Inflection points: The point where the concavity changes.	- Inflection points: The point where the concavity changes. i.e ∩ ∪ or ∪ ∩	- Infliction points: The point where the function crosses the x-axis $f''=0$

• Section 4.5:

Curve sketching:

1- find the domain of f
2- find the x-y intercepts.
3- find the critical number and local extrema → using $f'(x)$
4- find the concavity and inflection points → using $f''(x)$
5- find the Horizontal and vertical asymptote
6- plot the key points and sketch.

Proof of theorems:

- Theorem 2.2.1

a. $\lim_{x \to a} c = c$

we want to prove that, $\forall \varepsilon > 0, \exists \delta > 0$; if

$$0 < |x-a| < \delta, \text{ then } |f(x)-c| < \varepsilon.$$

But $\forall \varepsilon > 0$ if

$$0 < |x-a| < \delta, \text{ then } |c-c| = 0 < \varepsilon$$

Thus, we can choose any $\delta > 0$ because it doesn't depend on ε

$$\lim_{x \to a} f(x) = \lim_{x \to a} c = c$$

b. $\lim_{x \to a} x = a$

we want to probe that, $\forall \varepsilon > 0, \exists \delta > 0$; if

$$0 < |x-a| < \delta, \text{ then } |x-a| < \varepsilon.$$

If we choose $\delta = \varepsilon$, then the inequality $0 < |x-a| < \delta$ implies the inequality $|x-a| < \varepsilon$.

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

- Theorem 3.1.1

If a function f is differentiable at a, then f is continuous at a

If $x \in D_f$ and $x \neq a$ then $f(x)$ may be written as follows:

$$f(x) = f(a) + \frac{f(x)-f(a)}{x-a}(x-a)$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} f(a) + \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a}(x-a)$$

$$= f(a) + f'(a) \cdot 0 = f(a)$$

Thus, by definition of continuity at a point, f is continuous at a

- Theorem 3.2.1

If $f(x) = mx + c$ where m and c are real numbers, then

$$f'(x) = m$$

From the derivative definition, we have

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(m(x+h)+c) - (mx+c)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{mx} + m\cancel{h} + \cancel{c} - \cancel{mx} - \cancel{c}}{\cancel{h}}$$

$$= \lim_{h\to 0} m = m.$$

- Theorem 3.3.1

a. $\frac{d}{dx}(\sin x) = \cos x$

$$\frac{d}{dx} \sin x = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

Recall the identity $\rightarrow$ $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$$\frac{d}{dx} \sin x = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

Factoring out $\sin x$ $\longrightarrow$ $= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$

$$= \sin x \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$= \sin x (0) \rightarrow$ by theorem $+ \cos x (1) \rightarrow$ by theorem

Thus, $\frac{d}{dx} \sin x = \cos x$

b. $\frac{d}{dx} \cos x = -\sin x$

$$\frac{d}{dx} \cos x = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

Recall The identity $\rightarrow \cos(a+b) = \cos a \cos b - \sin a \sin b$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \cos x \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \lim_{h\to 0} \frac{\sin h}{h}$$

$= \cos x (0) \rightarrow$ theorem $- \sin x (1) \rightarrow$ theorem

Thus, $\frac{d}{dx} \cos x = -\sin x$

• Theorem 3.7.2

a. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

let $y = \sin^{-1} x$

$\sin y = x$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\frac{d}{dy} \sin y = \frac{d}{dy} x$ → differentiating with respect to y

$\cos y = \frac{dx}{dy}$ → $\sin^2 y + \cos^2 y = 1$ → $\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1-x^2} = \frac{dx}{dy}$ → $\cos^2 y = 1 - x^2$

2nd formula ← $\frac{dy}{dx} = \frac{1}{dx/dy}$ → $\cos y = \sqrt{1-x^2}$

Thus, $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

c. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

let $y = \tan^{-1} x$

$\tan y = x$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$\frac{d}{dy} \tan y = \frac{d}{dy} x$

$\sec^2 y = \frac{dx}{dy}$ → $1 + \tan^2 y = \sec^2 y$ → $\sec^2 y = 1 + x^2$

$\sec^2 y = 1 + x^2 = \frac{dx}{dy}$

$\frac{dy}{dx} = \frac{1}{dx/dy}$

Thus, $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$

e. $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \quad |x| > 1$

let $y = \sec^{-1}x$

$\sec y = x \quad$ for $\quad 0 < y < \frac{\pi}{2}$

$\frac{d}{dy}\sec y = \frac{d}{dy}x$ → differentiating both sides with respect to y

$\sec y \tan y = \frac{dx}{dy}$ → $\sec^2 y = 1 + \tan^2 y$

$\frac{dy}{dx} = \frac{1}{dx/dy}$ $\qquad \tan^2 y = \sec^2 y - 1$

$\frac{dx}{dy} = x\sqrt{x^2-1}$ $\qquad \tan y = \sqrt{x^2-1}$

Thus, $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \ |x| > 1$ $\qquad \sec y = x \rightarrow \sec^2 y = x^2$