One Sample: Test on a Single Proportion

Tests of hypotheses concerning proportions are required in many areas. We now consider the problem of testing the hypothesis that the proportion of successes in a binomial experiment equals some specified value. That is, we are testing the null hypothesis H_0 that $p = p_0$, where p is the parameter of the binomial distribution. The alternative hypothesis may be one of the usual one-sided or two-sided alternatives:

$$p < p_0 \quad p > p_0 \quad \text{or} \quad p \neq p_0$$

We know that if $np_0 \geq 5$ and $n(1 - p_0) \geq 5$, then the random variable $\widehat{P}$ is approximately a normal distribution with mean p_0 and standard deviation $\sigma_{\widehat{P}} = \sqrt{p_0(1 - p_0)/n}$.

The **z-value for testing** $\mathbf{p} = \mathbf{p}_0$ is given by

$$z = \frac{\hat{p} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

Hence, for a two-tailed test at the α-level of significance, the critical region is $z < -z_{\alpha/2}$ or $z > z_{\alpha/2}$. For the one-sided alternative $p < p_0$, the critical region is $z < -z_\alpha$, and for the alternative $p > p_0$, the critical region is $z > z_\alpha$.

Example

A commonly prescribed drug for relieving nervous tension is believed to be only 60% effective. Experimental results with a new drug administered to a random sample of 100 adults who were suffering from nervous tension show that 70 received relief. Is this sufficient evidence to conclude that the new drug is superior to the one commonly prescribed? Use a 0.05 level of significance.

Solution

1)

$$\begin{cases} H_0: \ p = 0.6, \\ H_1: \ p > 0.6. \end{cases}$$

2) $\alpha = 0.05$.

3) Critical region: $Z > z_\alpha$, where $z_\alpha = 1.645$. Then, the critical region: $z > 1.645$.

4) Computations: $x = 70, n = 100$, $\widehat{p} = 70/100 = 0.7$, and

$$\begin{aligned} z &= \frac{0.7 - 0.6}{\sqrt{\frac{(0.6)(0.4)}{100}}} = 2.04 \\ z &= 2.04 > 1.645 \\ P\text{-value} &= \Pr(Z > 2.04) < 0.0207. \end{aligned}$$

5) Decision: Reject H_0 and conclude that the new drug is superior.

Plan

Two Samples: Tests on Two Proportions

Situations often arise where we wish to test the hypothesis that two proportions are equal. That is, we are testing $p_1 = p_2$ against one of the alternatives $p_1 < p_2$, $p_1 > p_2$, or $p_1 = p_2$. The statistic on which we base our decision is the random variable $\widehat{P}_1 - \widehat{P}_2$. When $H_0 : p_1 = p_2 \ (= p)$ is true, we know that

$$Z = \frac{\widehat{P}_1 - \widehat{P}_2}{\sqrt{pq(1/n_1 + 1/n_2)}}$$

To compute a value of Z, however, we must estimate the parameters p and q that appear in the radical. Under H_0, both $\widehat{P}_1$ and $\widehat{P}_2$ are estimators of p.

We use the pooled estimate of the proportion p, which is

$$\widehat{p} = \frac{x_1 + x_2}{n_1 + n_2}$$

where x_1 and x_2 are the numbers of successes in each of the two samples. Substituting $\widehat{p}$ for p and $\widehat{q} = 1- \widehat{p}$ for q, the z-value for testing $p_1 = p_2$ is determined from the formula

$$z = \frac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\widehat{p}\widehat{q}(1/n_1 + 1/n_2)}}$$

The critical regions for the appropriate alternative hypotheses are set up as before, using critical points of the standard normal curve.

Example

A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. To determine if there is a significant difference in the proportions of town voters and county voters favoring the proposal, a poll is taken. If 120 of 200 town voters favor the proposal and 240 of 500 county residents favor it, would you agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters? Use an $\alpha = 0.05$ level of significance.

Solution

Let p_1 and p_2 be the true proportions of voters in the town and county, respectively, favoring the proposal.
$\widehat{p}_1 = x_1/n_1 = 120/200 = 0.6$, $\widehat{p}_2 = x_2/n_2 = 240/500 = 0.48$, and the pooled estimate
$\widehat{p} = (x_1 + x_2)/(n_1 + n_2) = (120 + 240)/(200 + 500) = 0.51.$
1)

$$\begin{cases} H_0: \; p_1 = p_2 \\ H_1: \; p_1 > p_2. \end{cases}$$

2) $\alpha = 0.05$.
3) The test statistic

$$z = \frac{\widehat{p}_1 - \widehat{p}_2}{\sqrt{\widehat{p}\widehat{q}(1/n_1 + 1/n_2)}} = \frac{0.60 - 0.48}{(0.51)(0.49)(1/200 + 1/500)} = 2.9$$

4) Critical region: $z > 1.645$. P-value $= P(Z > 2.9) = 0.0019$.
5) Decision: Reject H_0 and agree that the proportion of town voters favouring the proposal is higher than the proportion of county voters.