4.1 Mean of a Random Variable

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Definition

Let X be a random variable with probability distribution $f(x)$.

The **mean** or **expected value** of the random variable X is

$$\mu = E[X] = \sum xf(x) \qquad \text{If } X \text{ is discrete, and}$$

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)dx \qquad \text{If } X \text{ is continuous.}$$

Example

The random variable X = *Number of heads in a single toss of a fair coin* has the possible values $X = 0$ and $X = 1$ with probabilities

$$P(X = 0) = \frac{1}{2} \text{ and } P(X = 1) = \frac{1}{2}.$$

the mean $\mu = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}.$

Example

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components.

A sample of 3 is taken by the inspector.

Find the expected value of the number of good components in this sample.

Solution

Let X represent the number of good components in the sample.

The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \quad x = 0, 1, 2, 3.$$

Simple calculations lead to $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, $f(3) = 4/35$. Therefore,

Example

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components.

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Solution

Simple calculations lead to $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, $f(3) = 4/35$. Therefore,

$$\mu = E(X) = 0 \cdot \frac{1}{35} + 1 \cdot \frac{12}{35} + 2 \cdot \frac{18}{35} + 3 \cdot \frac{4}{35} = \frac{12}{7} = 1.7.$$

It means that if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it would contain, on average, 1.7 good components.

Example

In a gambling game a man is paid \$5 if he gets all heads or all tails when three coins are tossed, and he will pay out \$3 if either one or two heads show. What is his expected gain?

Solution

The sample space for the possible outcomes when three coins are tossed simultaneously, or equivalently if 1 coin is tossed 3 times, is

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

Each of these possibilities is equally likely and occurs with probability equal to 1/8.

The random variable of interest is Y, the amount the gambler can win; and the possible values of Y are \$5 if event $E_1 = \{HHH, TTT\}$ occurs and -\$3 if event $E_2 = \{HHT, HTH, THH, HTT, THT, TTH\}$ occurs.

Since E_1 and E_2 occur with probabilities 1/4 and 3/4, respectively, it follows that

$$\mu = E(Y) = 5 \cdot \frac{1}{4} + (-3) \cdot \frac{3}{4} = -1.$$

The gambler will, on average, lose \$1 per toss of the 3 coins.

In the previous two examples, the random variables are discrete.

The next example is about a continuous random variable where an engineer is interested in the *mean life* of a certain type of electronic device.

Example

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} 20{,}000/x^3, & x > 100 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected life of this type of device.

Solution

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20{,}000}{x^3} dx = \int_{100}^{\infty} \frac{20{,}000}{x^2} dx = 200.$$

We expect this type of device to last, *on average*, 200 hours.

Now, we consider a new random variable $g(X)$, which depends on X; that is each value of $g(X)$ is determined by knowing the values of X.

For example, $g(X)$ might X^2 or $3X - 1$, so that whenever X assumes the value 2, $g(X)$ assumes the value $g(2)$.

Theorem

Let X be a random variable with probability distribution $f(X)$.

The mean or expected value of the random variable $g(X)$ is

$$\mu_{g(x)} = E[g(x)] = \sum g(x) f(x)$$ If X is discrete, and

$$\mu_{g(x)} = E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$ If X is continuous.

Example

Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	1/12	1/12	1/4	1/4	1/6	1/6

Let $g(X) = 2X - 1$ represent the amount of money in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution By the previous Theorem, the attendant expects to receive

$$E[g(x)] = E(2X - 1) = \sum_{x=4}^{9} (2x - 1) f(x)$$

$$= 7 \cdot 1/12 + 9 \cdot 1/12 + 11 \cdot 1/4 + 13 \cdot 1/4 + 15 \cdot 1/6 + 17 \cdot 1/6$$

$$= 12.67 \text{ dollars.}$$

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Example

Let X be a random variable with the density function

$$f(x) = \begin{cases} x^2/3, & -1 < x < 2 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Solution

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3}\,dx = \frac{1}{3}\int_{-1}^{2} (4x^3+3x^2)\,dx = 8$$

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Assignment: 1, 2, 4, 5, 6, 7, 8, 12, 13, 14, 15, 17, 22.