King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(1)

Propositional Logic

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TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

Decide whether the following proposition is tautology or a contradiction or a contingency :

1) $(p \wedge q) \rightarrow (\neg p \rightarrow q)$

Solution:

p	q	$\neg p$	$p \wedge q$	$\neg p \rightarrow q$	$(p \wedge q) \rightarrow (\neg p \rightarrow q)$
T	T				
T	F				
F	T				
F	F				

2) Decide whether the following proposition is tautology or a contradiction or a contingency :

$$(p \vee \neg q) \rightarrow (p \wedge q)$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T				
T	F				
F	T				
F	F				

3) Construct the truth table of the proposition $p \land \neg[q \to (p \lor r)]$

p	q	r	$p \lor r$	$q \to (p \lor r)$	$\neg[q \to (p \lor r)]$	$p \land \neg[q \to (p \lor r)]$
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

4) Construct the truth table of the proposition : $(p \lor \neg q) \leftrightarrow (\neg r \land q)$

5) Construct the truth table of the proposition $(p \wedge q) \rightarrow (r \rightarrow q)$

6) Construct the truth table of the proposition

$$[(p \rightarrow q) \vee (q \rightarrow r)] \rightarrow (p \rightarrow \neg r)$$

7) Show that the following proposition is a tautology : (Don't use the truth tables)

$$[(p \vee q) \wedge \neg p] \rightarrow q$$

Solution:

8) Decide whether the following proposition is a tautology

$$(p \wedge q) \rightarrow [r \rightarrow (p \vee q)]$$ (Don't use the truth tables)

Solution:

9) Show that the following proposition is a contradiction : (Don't use the truth tables)

$$\neg(p \rightarrow q) \wedge (q \wedge \neg r)$$

Solution:

10) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Solution:

$$\begin{aligned}(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q)\\ &\equiv (\neg p \vee \neg q) \vee (p \vee q)\\ &\equiv (\neg p \vee p) \vee (\neg q \vee q)\\ &\equiv \mathbf{T} \vee \mathbf{T}\\ &\equiv \mathbf{T}\end{aligned}$$

11) Show that the following statement is a tautology

$$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$$

12) Prove the following proposition a tautology , (Don't use the truth table) :

$$(p \wedge q) \rightarrow [(q \vee r) \rightarrow p]$$

Proof:

$$(p \wedge q) \rightarrow [(q \vee r) \rightarrow p] \equiv \neg(p \wedge q) \vee [\neg(q \vee r) \vee p]$$

$$\equiv (\neg p \vee \neg q) \vee [(\neg q \wedge \neg r) \vee p]$$

$$\equiv (\neg p \vee \neg q) \vee [(\neg q \vee p) \wedge (\neg r \vee p)]$$

$$\equiv [(\neg p \vee \neg q) \vee (\neg q \vee p)] \wedge [(\neg p \vee \neg q) \vee (\neg r \vee p)]$$

$$\equiv [\neg p \vee \neg q \vee \neg q \vee p] \wedge [\neg p \vee \neg q \vee \neg r \vee p]$$

$$\equiv [(\neg p \vee p) \vee \neg q] \wedge [(\neg p \vee p) \vee (\neg q \vee \neg r)]$$

$$\equiv [T \vee \neg q] \wedge [T \vee (\neg q \vee \neg r)]$$

$$\equiv T \wedge T \equiv T$$

13) Show that the following proposition is a tautology :

$$[p \leftrightarrow (q \vee r)] \rightarrow [(\neg q \wedge \neg r) \vee p]$$

Solution:

$$[p \leftrightarrow (q \vee r)] \rightarrow [(\neg q \wedge \neg r) \vee p] \equiv \big[[p \rightarrow (q \vee r)] \wedge [(q \vee r) \rightarrow p]\big] \rightarrow [(\neg q \wedge \neg r) \vee p]$$

$$\equiv \big[(\neg p \vee (q \vee r)) \wedge (\neg(q \vee r) \vee p)\big] \rightarrow [(\neg q \wedge \neg r) \vee p]$$

$$\equiv \neg\big[(\neg p \vee (q \vee r)) \wedge (\neg(q \vee r) \vee p)\big] \vee [(\neg q \wedge \neg r) \vee p]$$

$$\equiv \neg(\neg p \vee (q \vee r)) \vee \neg[(\neg q \wedge \neg r) \vee p)] \vee [(\neg q \wedge \neg r) \vee p]$$

$$\equiv \neg(\neg p \vee (q \vee r)) \vee T \equiv T$$

14) Show that the following proposition is tautology or a contradiction or a contingency:

$$[(p \rightarrow q) \vee (q \rightarrow r)] \rightarrow (p \rightarrow \neg r)$$

Solution:

15) Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent

Solution:

p	q	$\neg p$	$\neg q$	$\neg p \wedge q$	$p \vee (\neg p \wedge q)$	$\neg(p \vee (\neg p \wedge q))$	$\neg p \wedge \neg q$
T	T						
T	F						
F	T						
F	F						

16) Show that $(p \rightarrow q) \rightarrow q \equiv (p \vee q)$

Solution:

17) Show that $(p \rightarrow q) \rightarrow (\neg p \rightarrow r) \equiv p \vee r$

Solution:

18) Show that

$$(p \to q) \wedge (p \to r) \equiv (\neg q \vee \neg r) \to \neg p$$

Solution:

19) Show that

$$(p \to q) \vee r \equiv (p \to q) \vee (p \to r)$$

Solution:

$(p \to q) \vee (p \to r) \equiv (\neg p \vee q) \vee (\neg p \vee r)$ *(Conditional Rule)*

$\equiv \neg p \vee q \vee \neg p \vee r$

$\equiv [(\neg p \vee \neg p) \vee q] \vee r$ *Commutative and Associative Rules)*

$\equiv (\neg p \vee q) \vee r$ *(Idempotent Rule)*

$\equiv (p \to q) \vee r$ *(Conditional Rule)*

20) Show that

$$(\neg p \vee \neg r) \rightarrow (p \wedge q) \equiv p \wedge (q \vee r)$$

Solution:

21) Show that

$$(\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r} \equiv (\boldsymbol{p} \rightarrow \boldsymbol{r}) \wedge (\boldsymbol{q} \rightarrow \boldsymbol{r})$$

Solution:

$(\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r} \equiv \neg(p \vee q) \vee r$ *(Conditional Rule)*

$\equiv (\neg p \wedge \neg q) \vee$ ***(** DeMorgan's Rule*

$\equiv (\neg p \vee r) \wedge (\neg q \vee r$ *(Distributive Rule)*

$\equiv (p \rightarrow r) \wedge (q \rightarrow r$ *(Conditional Rule)*

22) Show that

$$(p \vee q) \rightarrow (\neg p \wedge r) \equiv \neg p \wedge (q \rightarrow r)$$

Solution:

$(p \vee q) \rightarrow (\neg p \wedge r) \equiv \neg(p \vee q) \vee (\neg p \wedge r)$ (*Conditional Rule*)

$\equiv (\neg p \wedge \neg q) \vee (\neg p \wedge r)$ (*DeMorgan's Rule*)

$\equiv \neg p \wedge (\neg q \vee r$ (*Distributive Rule*)

$\equiv \neg p \wedge (q \rightarrow r)$ (*Conditional Rule*)

23) Show that $p \leftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are logically equivalent

Solution: (By the truth table)

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T				
T	F				
F	T				
F	F				

24) Show that $p \leftrightarrow q$ and $\neg p \leftrightarrow \neg q$ are logically equivalent ?

Solution:

25) Show that $(p \rightarrow q) \wedge [\neg q \wedge (\neg q \vee r)] \equiv \neg(q \vee p)$

Solution:

$(p \rightarrow q) \wedge [\neg q \wedge (\neg q \vee r)] \equiv (p \rightarrow q) \wedge \neg q$ *(Absorption Rule)*

$\equiv (\neg p \vee q) \wedge \neg q$ *(Conditional Rule)*

$\equiv (\neg p \wedge \neg q) \vee (q \wedge \neg q)$ *(Distributive Rule)*

$\equiv (\neg p \wedge \neg q) \vee F$ *(Negation Rule)*

$\equiv (\neg p \wedge \neg q)$ *(Identity Rule)*

$\equiv \neg(q \vee p)$ *(DeMorgan's Rule)*

26) Show that $[p \to (q \to p)] \wedge (p \to r) \wedge (p \to \neg r) \equiv \neg p$

Solution:

$$[p \to (q \to p)] \wedge (p \to r) \wedge (p \to \neg r) \equiv$$

$$\equiv [p \to (q \to p)] \wedge [p \to (r \wedge \neg r)]$$

$$\equiv [p \to (q \to p)] \wedge [\neg p \vee (F)]$$

$$\equiv [p \to (q \to p)] \wedge [\neg p]$$

$$\equiv [\neg p \vee (q \to p)] \wedge \neg p$$

$$\equiv \neg p \wedge [\neg p \vee (q \to p)] \equiv \neg p$$

27) Show that

$$\neg p \vee (q \to r) \equiv (p \to r) \vee (q \to r)$$

Solution:

28) Show that

$$\neg q \lor \neg[\neg p \lor (p \land q)] \equiv \neg q$$

Solution:

29) Show that

$$p \leftrightarrow (\neg q \land \neg r) \equiv \neg p \leftrightarrow (q \lor r)$$ (Don't use the truth table)

Solution:

30) Show that the contrapositive of $(p \land q) \to r$ is logically equivalent to

$p \to (q \to r)$

Solution:

The Contrapositive

في التقرير الشرطي $p \to q$ ، يسمى التقرير p (المقدمة *Antecedent*)، بينما يسمى التقرير q (النتيجة *Consequent*).

يقترن بالتقرير الشرطي $p \to q$ تقارير شرطية أخرى هي :

العكس (*Converse*) : $q \to p$

المعكوس (*Inverse*) : $\neg p \to \neg q$

المكافئ العكسي (*Contrapositive*) : $\neg q \to \neg p$

Q - State the converse, the inverse and the contrapositive for these Propositions :

1) If mn is an odd number, then m is an odd number and also n is an odd number.

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2) If 3 divides the integers $m \; and \; n$, then 3 divides $m + n$

3) If $m \,.\, n = l$, then $m \geq 0$ or $n \geq 0$ or $l \geq 0$: $m, n, l \in \mathbb{Z}$

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4) If the integer $a + b - c$ is an even , then $a\ is\ even$ or $b\ is\ even$ or $c\ is\ even$,where $a, b, c \in \mathbb{Z}$

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5) If n is a prime number where $n \neq 2$, then n is odd .

6) If x is integer, then x is odd or x is even .

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7) If a and b are odd integers , then $a + b$ is even .

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8) If $x \geq 2$ or $y \geq 3$,then $x^2 + y^2 \geq 4$

9) I will come over whenever there is a football game on .

10) I sleep until noon, whenever I stay up late the night before .

11) If it is raining, then the home team wins .

12) If you solve all exercises then you get a good mark.

Exercises

1- Construct the truth table of the proposition $(p \to q) \vee (\neg p \to \neg q)$

2- Construct the truth table of the proposition $\neg(p \to q) \leftrightarrow (p \wedge \neg q)$

3- Construct the truth table of the proposition $(p \vee \neg q) \leftrightarrow (\neg r \wedge q)$

4- Construct the truth table of the proposition $\neg(p \wedge \neg q) \to (\neg p \vee r)$

5- Using truth table, show that $(p \to q) \to (\neg p \to r) \equiv p \vee r$

6- Using truth table, show that $[p \leftrightarrow (q \vee r)] \to [(\neg q \wedge \neg r) \vee p]$ is a tautology .

7- Without using truth tables, Show that the following proposition is a <u>contradiction</u>

$$p \wedge \neg[q \to (p \vee r)]$$

8- Without using truth tables, Show that $(p \to q) \wedge r \equiv \neg(r \to p) \vee \neg(r \to \neg q)$.

9- Without using truth tables, Show that $(u \wedge w) \to [v \to (u \wedge v \wedge w)]$ is a tautology .

10-Without using truth tables, Show that $\neg p \to [(p \wedge q) \to r]$ is a tautology .

11-Without using truth tables, Show that $p \wedge \neg[q \to (p \vee r)]$ is a <u>contradiction</u> .

12- Without using truth tables, Show that $(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \equiv p \vee q$

13- Without using truth tables, Show that $p \wedge (q \to p) \equiv (p \to q) \to p$.

14- Without using truth tables, Show that $\neg(x \to \neg y) \equiv x \wedge [x \to (y \vee \neg x)]$.

15-