King Saud University

College of Science

Department of Mathematics

151 Math Exercises

# (3,3)

# Methods of Proof

## "Mathematical Induction"

## (STRONG INDUCTION)

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## Strong Induction

*STRONG INDUCTION* To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function, we complete two steps:

*BASIS STEP:* We verify that the proposition $P(1)$ is true.

*INDUCTIVE STEP:* We show that the conditional statement

$$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \to P(k+1) \text{ is true for all positive integers } k.$$

# Exercises

**1.** Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\,,\quad a_2 = 6\quad ,\qquad a_n = a_{n-1} + a_{n-2}\ \ (*) \qquad :\forall n \geq 3$$

Prove that $3|\,a_n$ for all positive integers $n$ , $\forall n \geq 1$

*Solution:* Let $P(n)$ be the proposition , $P(n)$: $3|\,a_n$ , $\Rightarrow a_n = 3c : c \in \mathbb{N}$

*BASIS STEP:* $When\ n = 1 \Rightarrow 3|\,a_1 \ :\ a_1 = 3 = 3(1) \Rightarrow\ \therefore\ P(1)\ \ is\ true$ .

$When\ n = (2)$: $3|\,a_2 \ :\ a_2 = 6 = 3(2) \Rightarrow\ \therefore\ P(2)\ \ is\ true$ .

*INDUCTIVE STEP:* Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

*Our goal is to show that* $P(\,k+1\,)$ *is also true* ?

$$a_{k+1} = 3c \;\Rightarrow\; 3|\,a_{k+1} \qquad (\text{ our goal })\ ??$$

$from\ \ (*) \ \Rightarrow \qquad a_{k+1} = a_k + a_{k-1} \qquad (**)$

$\because\ P(k)\ \&\ P(k-1)$ both are true, ( from inductive hypothesis ** ) $\Rightarrow$

from $P(k) \Rightarrow\ 3|\,a_k \ \Rightarrow a_k = 3c_1 : c_1 \in \mathbb{N}$

from $P(k-1) \Rightarrow\ 3|\,a_{k-1} \ \Rightarrow a_{k-1} = 3c_2 \ :\ c_2 \in \mathbb{N}$

$\xRightarrow{by\ subist.into(**)}$

$$a_{k+1} = a_k + a_{k-1} = 3c_1 + 3c_2 = 3(c_1 + c_2) = 3c$$

$$: c = (c_1 + c_2) \in \mathbb{N}$$

$\therefore\ a_{k+1} = 3c \ \Rightarrow\ 3|\,a_{k+1} \Rightarrow \therefore\ \ P(\,k+1\,)$ is true . #

**2.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\,,\; a_1 = 15 \;,\; a_n = \frac{a_{n-1}\,a_{n-2}}{3} + 6 \quad : \forall n \geq 2 \qquad (*)$$

Prove that $3|a_n$ <u>for all nonnegative integers</u> $n$ , $\forall n \geq 0$

*Solution:* **S(1).** Let $P(n)$ be the proposition , $P(n):\ 3|\,a_n\ ,\ \Rightarrow a_n = 3c : c \in \mathbb{N}$

**S(2).** *BASIS STEP: When* $= 0 \Rightarrow 3|\ a_0\ :\ a_0 = 9 = 3(3) \Rightarrow \therefore P(0)$ *is true .*

$When\ n = (1) \Rightarrow\ 3|\ a_1\ :\ a_1 = 15 = 3(5) \Rightarrow \therefore\ P(1)$ *is true .*

**S(3).** *INDUCTIVE STEP:* Let $k \geq 1$ and assume that

$$P(0), P(1), \ldots, P(k-2), P(k-1)\ , P(k) \qquad \text{All are true} \quad . \qquad (**)$$

Our goal is to show that $P(k+1)$ is also true .

$$3|\,a_{k+1} \;\Rightarrow\; a_{k+1} = 3c \qquad \text{(our goal)??}$$

$$from\ (*) \;\Rightarrow\; a_{k+1} = \frac{a_k\,a_{k-1}}{3} + 6 \qquad (***)$$

$\because\ P(k)\ \&\ P(k-1)$ both are true, ( from inductive hypothesis ** ) $\Rightarrow$

$$from\ \ P(k) \Rightarrow\ 3|\,a_k\ \ \Rightarrow a_k = 3c_1 :\ c_1 \in \mathbb{N}$$

$$from\ \ P(k-1) \Rightarrow\ 3|\,a_{k-1}\ \ \Rightarrow a_{k-1} = 3c_2\ :\ c_2 \in \mathbb{N}$$

$$\xRightarrow{by\ subist.into (***)}$$

$$a_{k+1} = \frac{a_k\,a_{k-1}}{3} + 6 = \frac{3c_1 .\ 3c_2}{3} + 6 = 3(c_1 . c_2) + 6$$

$$= 3(c_1 . c_2 + 2) = 3M$$

$$: c_1 . c_2 + 2 = M \in \mathbb{N}$$

$$\because\ a_{k+1} = 3M \;\Rightarrow \therefore\ 3|\,a_{k+1} \Rightarrow \therefore\ P(\,k+1\,) \quad \text{is true .}$$

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**3.** Assume $\{\mathrm{u_n}\}_{\mathrm{n=1}}^{\infty}$ is a sequence defined as:

$$u_1 = 1 \,,\; u_2 = 2 \;,\; u_n = 2u_{n-1} - u_{n-2} \quad : \forall n \geq 3 \qquad (*)$$

Prove that $u_n = n$ <u>for all positive integers</u> $n$ , $\forall n \geq 1$

*Solution:* Let $P(n)$ be the proposition , $P(n)$: $u_n = n$

*BASIS STEP:* *When* $n = 1 \Rightarrow \; u_1 = 1 \;\Rightarrow\; \therefore \; P(1)$ *is true* .

*When* $n = (2) \Rightarrow \; u_2 = 2 \;\Rightarrow\; \therefore \; P(2)$ *is true* .

*INDUCTIVE STEP:* Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,, P(k)$ All are true (**)

Our goal is to show that $P(k+1)$ is also true .

$$u_{k+1} = k + 1 \qquad \text{( our goal ) ??}$$

*from* (*) $\Rightarrow \qquad u_{k+1} = 2u_k - u_{k-1} \qquad$ (***)

$\because \; P(k)$ & $P(k-1)$ both are true, ( from inductive hypothesis ** ) $\Rightarrow$

$from \;\; P(k) \;\Rightarrow \qquad u_k = k$

$from \;\; P(k-1) \Rightarrow \;\; u_{k-1} = k - 1$

$\xRightarrow{by\ subist.into(***)}$

$$u_{k+1} = 2u_k - u_{k-1} = 2k - (k-1) = 2k - k + 1 = k + 1$$

$\because \; u_{k+1} = k + 1 \;\Rightarrow\; \therefore \quad P(\,k+1\,)$ is true .

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# Exercises

**4.** Assume $\{a_n\}_{\text{n}=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\,,\ \ a_2 = 6\ \ ,\ a_n = a_{n-1} + a_{n-2}\ :\forall n \geq 3$$

Prove that $3|\,a_n$ for all integers $n \geq 1$.

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**5.** Assume $\{a_n\}_{\text{n}=1}^{\infty}$ is a sequence defined as:

$$a_1 = 8\,,\ a_2 = 4\ \ ,\ a_n = a_{n-1} + a_{n-2}\ :\forall n \geq 3$$

Prove that $a_n$ is even for all integers $n \geq 1$.

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**6.** Assume $\{a_n\}_{\text{n}=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\,,\ a_1 = 15\ \ ,\ \ a_n = \frac{a_{n-1}\,a_{n-2}}{3} + 6\ \ :\forall n \geq 2$$

Prove that $3|a_n$ for all integers $n \geq 0$

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**7.** Assume $\{\text{u}_\text{n}\}_{\text{n}=1}^{\infty}$ is a sequence defined as:

$$u_1 = 1\,,\ u_2 = 2\ \ ,\ \ u_n = 2u_{n-1} - u_{n-2}\ \ :\forall n \geq 3$$

Prove that $u_n = n$ for all positive integers *n*.

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**8.** Assume $\{a_n\}_{\text{n}=1}^{\infty}$ is a sequence defined as:

$$a_1 = -1\,,\ \ a_2 = -\frac{1}{2}\,,\ a_3 = -\sqrt{10}\ \ ,\ \ a_{n+1} = a_n.a_{n-1}.a_{n-2}\ \ :\forall n \geq 3$$

Prove that $a_n \leq 0$ for all positive integers *n*.

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**9.** Assume $\{a_n\}_{\text{n}=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 5\ \ ,\ \ a_{n+1} = 2a_n + 3\,a_{n-1}\ \ :\forall n \geq 2$$

Prove that $3^n \leq a_{n+1} \leq 2\,.\,3^n$ for all positive integers *n*.

**10.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \quad a_n = 2a_{n-1} + 1 \quad : \forall n \geq 1$$

Prove that $a_n = 2^{n+1} - 1$ for all integers $n \geq 0$

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**11.** Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3 \ , \quad a_2 = 9 \ , \quad a_3 = 15 \ , \quad a_{n+1} = a_n + a_{n-1} + a_{n-2} \quad : \forall n \geq 3$$

Show that $a_n$ is an integer divisible by 3, for all integers $n \geq 1$

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**12.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \quad a_1 = 1 \ , \quad a_{n+1} = 2a_n + a_{n-1} \quad : \forall n \geq 1$$

Show that $a_n$ is odd for all integers $n \geq 0$

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**13.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 0 \ , \quad a_1 = 4 \ , \quad a_{n+1} = -2a_n + 3\,a_{n-1} \quad : \forall n \geq 1$$

Show that $a_n = 1 - (-3)^n$ for all integers $n \geq 0$

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**14.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 0 \ , \quad a_1 = 2 \ , \quad a_{n+1} = 4a_n - 3a_{n-1} \quad : \forall n \geq 1$$

Show that $a_n = 3^n - 1$ for all integers $n \geq 0$

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**15.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9 \ , \quad a_1 = 15 \ , \quad a_2 = 3 \qquad a_n = \frac{a_{n-1}\,a_{n-2}\,a_{n-3}}{9} + 6 \quad : \forall n \geq 3$$

Show that $3 | a_n$ for all integers $n \geq 0$

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**16.** Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 1 \ , u_2 = 2 \ , u_3 = 3 \ , \ u_n = 3u_{n-1} - u_{n-2} - u_{n-3} - 2 : \forall n \geq 4$$

Show that $u_n = n$ for all integers $n \geq 1$.

**17.** Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 2\,, a_3 = 3\ ,\quad a_n = \frac{a_{n-1} + a_{n-2} + a_{n-3}}{3}\quad : \forall n \geq 4$$

Show that $1 \leq a_n \leq 3$ for all integers $n \geq 1$

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**18.** Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = \frac{3}{4}\,,\ u_2 = \frac{8}{13}\quad ,\quad u_n = \frac{3\,u_{n-1} + 2\,u_{n-2} - 3}{3}\quad : \forall n \geq 3$$

Show that $u_n < 1$ for all integers $n \geq 1$

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**19.** Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 2\,,\ u_2 = 4\quad ,\quad u_n = \frac{2u_{n-1} + u_{n-2} + 8}{3}\quad : \forall n \geq 3$$

Show that $u_n = 2n$ for all positive integers $n$.

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**20.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\quad a_1 = 2\ ,\ a_2 = 3\ ,\quad a_n = a_{n-1} + a_{n-2} + 2a_{n-3}\ : \forall n \geq 3$$

Show that $a_n \leq 3^n$ for all integers $n \geq 0$

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**21.**Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\quad a_1 = 2\quad ,\quad a_n = 4a_{n-1} - 4a_{n-2}\quad : \forall n \geq 2$$

Prove that $a_n = 2^n$ for all nonnegative integers $n$.

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**22.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\quad a_1 = 1\quad ,\quad a_n = 4a_{n-1} - 4a_{n-2}\quad : \forall n \geq 2$$

Show that $a_n = 2^n - n2^{n-1}$ for all integers $n \geq 0$

**23.** Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 3 \quad , \quad a_n = a_{n-1} + a_{n-2} \qquad : \forall n \geq 3$$

Prove that $a_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$ for all positive integers *n*.

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**24.** Assume $\{a_n\}_{n=1}^{\infty}$ is a "Fibonacci" sequence defined as:

$$a_1 = 1\,,\ a_2 = 1 \quad , \quad a_n = a_{n-1} + a_{n-2} \qquad : \forall n \geq 3$$

Prove that $a_n \leq (\frac{1+\sqrt{5}}{2})^n$ for all positive integers *n*.

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**25.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\ a_1 = 1\ , a_2 = 3\ , \quad a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad : \forall n \geq 3$$

Show that $a_n < 3^n$ for all integers $n \geq 0$

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**26.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 2\,,\ a_1 = 4\ , a_2 = 6\ , \quad a_n = 5a_{n-3} \quad : n = 3,4,5,\ldots$$

Show that $2|a_n$ for all integers $n \geq 0$

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**27.** Assume $\{a_n\}_{n=1}^{\infty}$ is a "Fibonacci" sequence defined as:

$$a_1 = 1\,,\ a_2 = 2 \quad , \quad a_n = 2a_{n-1} + a_{n-2} \qquad : \forall n \geq 2$$

Prove that $a_n \leq (\frac{5}{2})^n$ for all positive integers *n*.

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**28.** Assume $\{u_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$u_0 = 2\,,\ u_1 = 3 \quad , \quad u_{n+1} = 3u_n - 2u_{n-1} - 1 \ : \forall n \geq 1$$

Show that $u_n = n + 2$ for all integers $n \geq 0$

**29.** Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$u_1 = 0$ , $u_2 = 1$ , $u_{n+1} = 3u_n - 2u_{n-1} - 1$ $for$ $n = 2,3,4,\ldots$

Show that $u_n = n - 1$ for all integers $n \geq 1$

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**30.** Assume $\{u_n\}_{n=0}^{\infty}$ is a sequence defined as:

$u_0 = 12$ , $u_1 = 21$ , $u_{n+1} = \frac{(u_n)^2 \; u_{n-1}}{9}$ $for$ $n = 1,2,3,\ldots$

Show that $u_n$ is an integer divisible by 3, for all integers $n \geq 0$

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**31.** Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$u_1 = 2$ , $u_2 = 5$ , $u_{n+1} = 2u_n - u_{n-1} + 2$ $for$ $n = 2,3,4,\ldots$

Show that $u_n = n^2 + 1$ for all integers $n \geq 1$

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**32.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$a_0 = 2$ , $a_1 = 4$ , $a_{n+1} = 4a_n - 3a_{n-1}$ : $\forall n \geq 1$

Show that $a_n = 1 + 3^n$ for all integers $n \geq 0$

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**33.** Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$a_0 = 2$ , $a_1 = 5$ , $a_{n+1} = 5a_n - 6a_{n-1}$ : $\forall n \geq 1$

Show that $a_n = 2^n + 3^n$ for all integers $n \geq 0$

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**34.** Assume $\{u_n\}_{n=0}^{\infty}$ is a sequence defined as:

$u_0 = 2$ , $u_1 = 6$ , $u_{n+1} = 3u_n + 10u_{n-1} - 12$ : $\forall n \geq 1$

Show that $u_n = 5^n + 1$ for all integers $n \geq 0$