II) Show that $4+9+14+\cdots+(5n-1)=\frac{n}{2}(3+5n)$, for all positive integers n.

by principle of mathematical induction

let $p(n)$: $4+9+14+\cdots+(5n-1)=\frac{n}{2}(3+5n)$, $n\geq 1$

① basis step: show that $p(1)$ is true.

$L.H.S=4$ $\quad R.H.S=\frac{1}{2}(3+5)=\frac{8}{2}=4$

$\because L.H.S=R.H.S$

$\therefore p(1)$ is true.

② inductive step: $\forall k\geq 1\,[P(k)\longrightarrow p(k+1)]$

suppose $p(k)$ is true and show that $p(k+1)$ is true.

suppose

$p(k)$: $4+9+14+\cdots+(5k-1)=\frac{k}{2}(3+5k)$

Show

$p(k+1)$: $4+9+14+\cdots+(5k-1)+(5(k+1)-1)\overset{?}{=}\frac{k+1}{2}(3+5(k+1))$

$L.H.S=\underbrace{4+9+14+\cdots+(5k-1)}+\underbrace{(5(k+1)-1)}$

$=\frac{k}{2}(3+5k)+(5(k+1)-1)$

$=\frac{k}{2}(3+5k)+(5k+5-1)$

$=\frac{3}{2}k+\frac{5}{2}k^2+5k+4$

$=\boxed{\frac{5}{2}k^2+\frac{13}{2}k+4}$

$R.H.S=\frac{k+1}{2}(3+5(k+1))$

$=\frac{k+1}{2}(3+5k+5)=\frac{k+1}{2}(5k+8)=\frac{5}{2}k^2+\frac{5}{2}k+\frac{8}{2}k+\frac{8}{2}$

$=\frac{k+1}{2}(5k+8).$ $\quad =\boxed{\frac{5}{2}k^2+\frac{13}{2}k+4}$

$\because L.H.S=R.H.S\longrightarrow p(k+1)$ is true.

③ Conclusion

$p(n)$ is true $\forall n\geq 1$

Question III:

Use mathematical induction to prove that $n! \geq 2^{n-1}$, for all $n = 1, 2, 3, \ldots\ldots$

by using principal of mathematical induction

let $p(n)$: " $n! \geqslant 2^{n-1}$ $n \geqslant 1$ "

1) Basis step: show that $p(1)$ is true.

$L.H.S = 1! = 1$
$R.H.S = 2^{1-1} = 2^0 = 1$
$1 \geqslant 1$
$\therefore p(1)$ is true.

2) Inductive step: $\forall k \geqslant 1 \ [p(k) \xrightarrow{?} p(k+1)]$

suppose $p(k)$ is true.

$p(k)$: $k! \geqslant 2^{k-1}$

show that $p(k+1)$ is true.

$p(k+1)$: $(k+1)! \geqslant 2^k$??

$(k+1)\,k!$

sine $p(k)$ is true.

$k! \geqslant 2^{k-1}$

$(k+1)\,k! \geqslant (k+1) \cdot 2^{k-1} \geqslant 2 \cdot 2^{k-1}$ sinc $k+1 \geqslant 2$

$1+1 \geqslant 2$
$2+1 \geqslant 2$
$\vdots$

Good Luck☺

$\rightarrow (k+1)\,k! \geqslant 2 \cdot 2^{k-1}$

$\rightarrow (k+1)! \geqslant 2^k$

$\therefore p(k+1)$ is true.

3) Conclusion: $p(n)$ is true $\forall n \geqslant 1$

Question II:

Let $\{a_n\}$ be a sequence defined inductively as

$$a_0 = 0, \qquad a_1 = 2, \qquad a_{n+1} = 4a_n - 3a_{n-1}, \ \forall n \geq 1.$$

Using **Strong Induction** prove that

$$a_n = 3^n - 1, \ \forall n \geq 0.$$

$a_0 = 0$

$a_1 = 2$

$a_{n+1} = 4a_n - 3a_{n-1} \quad n \geq 1$

$\boxed{a_{k+1} = 4a_k - 3a_{k-1}}$

$P(n): \quad a_n = 3^n - 1 \quad n \geq 0$

By strong induction.

① Basic step   show that $p(0)$, $p(1)$ true.

$p(0)$: $a_0 \overset{?}{=} 3^0 - 1$, $0 \overset{?}{=} 1 - 1$, $0 \overset{\checkmark}{=} 0$ $\therefore p(0)$ is true

$p(1)$: $a_1 \overset{?}{=} 3^1 - 1$, $2 \overset{?}{=} 3 - 1$, $2 \overset{\checkmark}{=} 2$ $\therefore p(1)$ is true.

② inductive step $\forall k \geq 0 \ [p(0) \wedge \cdots \wedge p(k) \overset{?}{\longrightarrow} p(k+1)]$

suppose that $p(k), p(k-1)$ are true

$p(k)$: $a_k = 3^k - 1$ ✓

$p(k-1)$: $a_{k-1} = 3^{k-1} - 1$ ✓

show that $p(k+1)$ is true.

$p(k+1)$: $a_{k+1} = 3^{k+1} - 1$??

$$L.H.S = a_{k+1} = 4\underset{\text{فرض}}{a_k} - 3\underset{\text{فرض}}{a_{k-1}} = 4(3^k - 1) - 3(3^{k-1} - 1)$$

$$= 4 \cdot 3^k - 4 - \underbrace{3 \cdot 3^{k-1}}_{\text{جمع أسس}} + 3 = 4 \cdot 3^k - 3^k - 1$$

$$= 3 \cdot 3^k - 1 = 3^{k+1} - 1 = R.HS$$

$\therefore p(k+1)$ is true.

③ Conclusion: $p(n)$ is true $\forall n \geq 0$

(c) Let $\{a_n\}$ be a sequence of integers defined inductively as:

$$a_0 = 2, a_1 = 4 \qquad a_n = 4a_{n-1} - 3a_{n-2} \quad for\ n \geq 2$$

Prove, using <u>strong induction</u>, that $a_n = 1 + 3^n$ for all integers $n \geq 0$.

$a_0 = 2$

$a_1 = 4$

$a_n = 4a_{n-1} - 3a_{n-2} \quad n \geq 2$

$n = k+1$

$$\boxed{a_{k+1} = 4\underset{p(k)}{a_k} - 3\underset{p(k-1)}{a_{k-1}}}$$

$P(n);\ a_n = 1 + 3^n \quad n \geq 0$

by strong induction.

① basis step Show that $p(0)$ is true.

$P(0);\ a_0 \overset{?}{=} 1 + 3^0$

$L.H.S = a_0 = 2$

$R.H.S = 1 + 3^0 = 1 + 1 = 2$

$L.H.S = R.H.S$

$\therefore p(0)$ is true.

② Inductive step. $\forall k \geq 0\ [p(0) \wedge p(1) \wedge \cdots \wedge p(k)] \to p(k+1)]$

suppose that $p(k), p(k-1)$ are true

$P(k);\ a_k = 1 + 3^k$

$p(k-1);\ a_{k-1} = 1 + 3^{k-1}$

show that $p(k+1)$ is true.

$p(k+1);\ a_{k+1} = 1 + 3^{k+1}$??

$$\begin{aligned} L.H.S = a_{k+1} &= 4a_k - 3a_{k-1} \\ &= 4(1+3^k) - 3(1+3^{k-1}) \\ &= \underline{4} + 4.3^k \underline{-3} - 3.3^{k-1} \\ &= 1 + 4.3^k - 3^k \\ &= 1 + 3^k(4-1) \\ &= 1 + 3^k.3 = 1 + 3^{k+1} = R.H.S \end{aligned}$$

$\therefore p(k+1)$ is true.

③ Conclusion: $p(n)$ is true $\forall n \geq 0$.

II) A) Prove (by cases) that, for any integer n, the product $n(n+1)$ is even.

we have two cases

case 1 if n is even then $n(n+1)$ is even

suppose n is even and show that $n(n+1)$ is even

n is even

$\rightarrow n = 2k \quad k \in \mathbb{Z}$

$\rightarrow n+1 = 2k+1$

$\rightarrow n(n+1) = 2k(2k+1) = 4k^2 + 2k = 2(2k^2+k) = 2t$ even $\quad t = 2k^2+k$

Case 2 if n is odd then $n(n+1)$ is even

suppose n is odd

$\rightarrow n = 2k+1$

$n+1 = 2k+1+1 = 2k+2$

$n(n+1) = (2k+1)(2k+2) = 4k^2+4k+2k+2 = 4k^2+6k+2 = 2(2k^2+3k+1)$ even

B) A sequence $(a_n)_{n\geq 1}$ is defined by $a_1 = 3$ and $a_n = 7a_{n-1}$ for $n \geq 2$. Prove that $a_n = 3 \cdot 7^{n-1}$, for all $n \geq 1$.

$a_1 = 3$

$a_n = 7a_{n-1} \quad n \geq 2$

Prove that

$p(n): a_n = 3 . 7^{n-1} \quad n \geq 1$

$n = k+1$

$a_{k+1} = 7a_k$

by principal of mathematical induction.

let $p(n): a_n = 3 . 7^{n-1} \quad n \geq 1$

1) basis step show that $p(1)$ is true.

$a_1 \stackrel{?}{=} 3 . 7^{1-1}$

$3 = 3 . 7^0$

$3 = 3$ ✓ $\therefore p(1)$ is true.

2) inductive.step $\forall k \geq 1 \, [p(k) \xrightarrow{?} p(k+1)]$

suppose that $p(k)$ is true.

$p(k): a_k = 3 . 7^{k-1}$

show that $p(k+1)$ is true.

$p(k+1): a_{k+1} = 3 . 7^k$??

$L.H.S = a_{k+1} = 7a_k = 7(3 . 7^{k-1}) = 3 . 7^k = R.H.S$

$\therefore p(k+1)$ is true.

3) $\therefore p(n)$ is true $\forall n \geq 1$

C) **Use the first principle of mathematical induction to show that 2 is a divisor of $n^2 + n$, for every integer $n \geq 1$.**

$P(n)$: $\exists t \in \mathbb{Z}$ s.t $n^2+n=2t$ $n \geq 1$

by principle of mathematical induction

1) basis step show that $P(1)$ is true.

$L.H.S = 1^2+1 = 2 = 2 \cdot 1$ $\therefore t=1$

$\therefore P(1)$ is true.

2) inductive step $\forall k \geq 1$ $[P(k) \overset{?}{\rightarrow} P(k+1)]$

suppose $P(k)$ is true.

$P(k)$: $k^2+k = 2t'$ $t' \in \mathbb{Z}$

show that $P(k+1)$ is true.

$P(k+1)$: $(k+1)^2+(k+1) = 2t''$ $t'' \in \mathbb{Z}$??

$$L.H.S = (k+1)^2 + k+1$$
$$= k^2+2k+1+k+1$$
$$= k^2+k+2k+2$$
$$= 2t'+2k+2$$
$$= 2\underbrace{(t'+k+1)}_{t''}$$

$\therefore P(k+1)$ is true.

3) Conclusion:

$\forall n \geq 1$ $P(n)$ is true.

C. Use Mathematical Induction to prove that

$$\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+---+\frac{1}{(n-1)(n)}=\frac{n-1}{n}, \qquad n\geq 2.$$

Proof: by using principle of mathematical induction.

Let $P(n): \frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots+\frac{1}{(n-1)(n)}=\frac{n-1}{n} \quad n\geq 2$

① B.S show that $p(2)$ is true.

$L.H.S=\frac{1}{(1)(2)}=\frac{1}{2} \qquad R.H.S=\frac{2-1}{2}=\frac{1}{2}$

$\because L.H.S=R.H.S$

$\therefore p(2)$ is true.

② I.S $\forall k\geq 2\,[P(k)\overset{?}{\longrightarrow}P(k+1)]$

suppose that $p(k)$ is true.

$P(k): \frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots+\frac{1}{(k-1)(k)}=\frac{k-1}{k} \quad k\geq 2$

show that $p(k+1)$ is true.

$P(k+1): \frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots+\frac{1}{(k-1)(k)}+\frac{1}{k(k+1)}=\frac{k}{k+1}$??

$$L.H.S=\underbrace{\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots+\frac{1}{(k-1)(k)}}_{\text{فرض}}+\underbrace{\frac{1}{k(k+1)}}$$

$$=\frac{k-1}{k}+\frac{1}{k(k+1)}=\frac{(k-1)(k+1)+1}{k(k+1)}=\frac{k^2+\cancel{k}-\cancel{k}\cancel{-1}\cancel{+1}}{k(k+1)}$$

$$=\frac{k^{\cancel{2}}}{\cancel{k}(k+1)}=\frac{k}{k+1}=R.H.S$$

$\therefore p(k+1)$ is true.

③ Conclusion $p(n)$ is true $\forall n\geq 2$

Question IV

(1) Let $\{a_n\}$ be a sequence of integers defined inductively as:

$a_0 = 0, a_1 = 4$, and $a_{n+1} = -2a_n + 3a_{n-1}$. Use strong induction to prove that $a_n = 1 - (-3)^n$ for all $n \geq 2$.

$a_0 = 0$
$a_1 = 4$

$p(n) : a_n = 1 - (-3)^n \quad n \geq 2$

$a_{n+1} = -2a_n + 3a_{n-1} \quad n \geq 1$
$a_2 = -2a_1 + 3a_0$
$= -2(4) + 3 \cdot 0 = -8$

$a_{k+1} = -2a_k + 3a_{k-1}$

by strong induction.

1) Basis step show that $p(2)$ is true.

L.H.S $= a_2 = -8$
R.H.S $= 1 - (-3)^2 = 1 - 9 = -8$
$=$ $\therefore p(2)$ is true.

2) inductive step: $\forall k \geq 2 \, [p(2) \wedge \dots \wedge p(k) \longrightarrow p(k+1)]$

suppose that $p(k), p(k-1)$ are true show that $p(k+1)$ is true

$p(k): a_k = 1 - (-3)^k$
$p(k-1): a_{k-1} = 1 - (-3)^{k-1}$
$p(k+1) = a_{k+1} = 1 - (-3)^{k+1}$??

L.H.S $= a_{k+1} = -2a_k + 3a_{k-1}$
$= -2(1 - (-3)^k) + 3(1 - (-3)^{k-1})$
$= \underline{-2} + 2(-3)^k + \underline{3} - 3(-3)^{k-1}$
$= 1 - (-3)^{k-1}(-2(-3) + 3) = 1 - (-3)^{k-1}(9) = 1 - (-3)^{k-1}(-3)^2$
$= 1 - (-3)^{k+1} =$ R.H.S

$\therefore p(k+1)$ is true.

3) conclusion $p(n)$ is true $\forall n \geq 2$

(2) Show that $\forall m, n \in \mathbb{N}, \; mn > m + n$ is false