King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(4.1)

Relations and Their Operations

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2021

EXERCISES

1. Let R be a relation defined on the set $A = \{-2,-1,0,1,2,3,4\}$

$$m,n \in A \;,\quad m\,R\,n \Leftrightarrow n = m^2$$

(i) List all ordered pairs of R ?
(ii) Find the domain and the image of R ?
(iii) Represent R by the directed graph (diagraph) ?

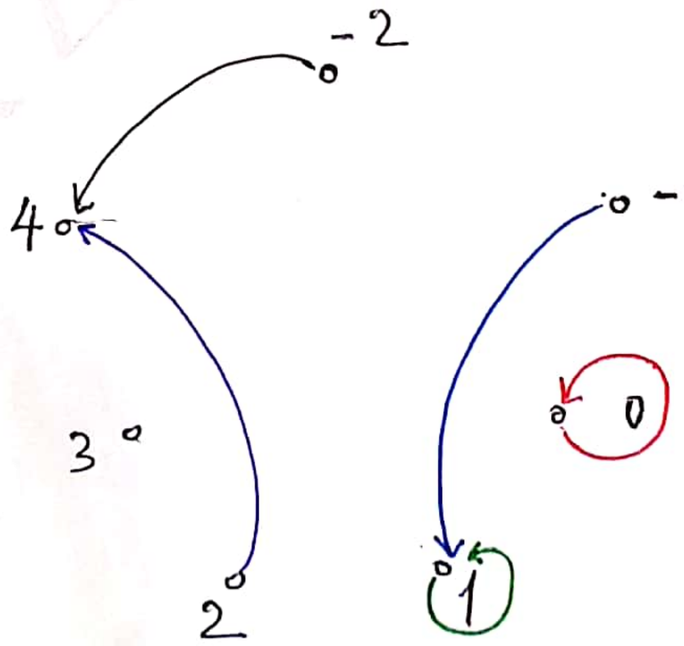
Solution:

(i) $R = \{(-2,4),(-1,1),(0,0),(1,1),(2,4)\}$ ✓ ✓

(ii) $\text{Domain } R = \{-2,-1,0,1,2\} \subseteq A$

$\text{Range } R = \{0,1,4\} \subseteq A$

(iii)



2. Let R be a relation defined on the set $A = \{1,2,3,4,5\}$

$$x, y \in A, \quad x\,R\,y \Leftrightarrow xy \leq 9$$

(i) List all ordered pairs of R ?
(ii) Find the domain and the image of R ?
(iii) Draw the directed graph (diagraph) that represents R ?
(iv) Represent R with a matrix ?

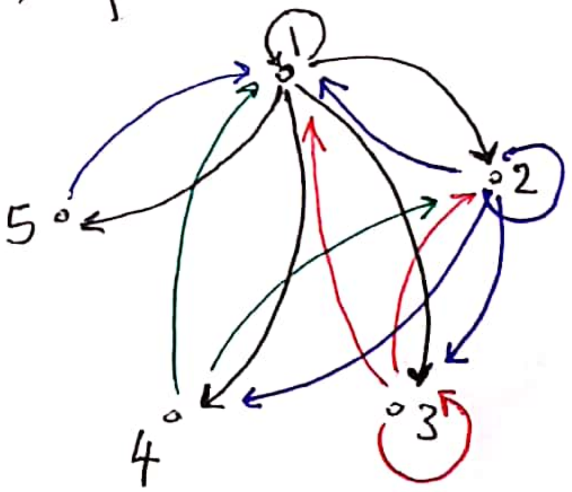
Solution:

(i) $R = \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(4,1),(4,2),(5,1)\}$

(ii) Domain $R = \{1,2,3,4,5\} = A$

Range $R = \{1,2,3,4,5\} = A$

(iii)



(iv)

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}_{5\times5} \end{array} .$$

واجب **3.** Let R be a relation defined on the set $A = \{-4,-3,-2,-1,0,1,2,3,4\}$

$$x,y \in A, \qquad x\,R\,y \Leftrightarrow y = 2x$$

(i) List all ordered pairs of R ?

(ii) Find the domain and the image of R ?

Solution:

4. Let R be a relation defined from the set $A = \{1,2,3,4\}$ to the set $B = \{2,3,4,5\}$

$$a\,R\,b \Leftrightarrow a + b = 5$$

(i) List all ordered pairs of R ?

(ii) Find the domain and the image of R ?

(iii) Represent R with a matrix ?

Solution:

(i) $R = \{(1,4),(2,3),(3,2)\}$

(ii) $\text{Domain } R = \{1,2,3\} \subseteq A$

$\text{Range } R = \{2,3,4\} \subseteq B$

(iii)

$$M_R = \begin{array}{c|cccc} {}^{B\rightarrow}_{A\downarrow} & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}_{4\times4}$$

5. Let R be a relation defined on the set $A = \{0,1,2,3\}$

$$a, b \in A\,, \qquad a\,R\,b \Leftrightarrow a + b = 4$$

(i) List all ordered pairs of R ?
(ii) Find the domain and the image of R ?
(iii) Draw the directed graph (diagraph) that represents R ?
(iv) Represent R with a matrix ?

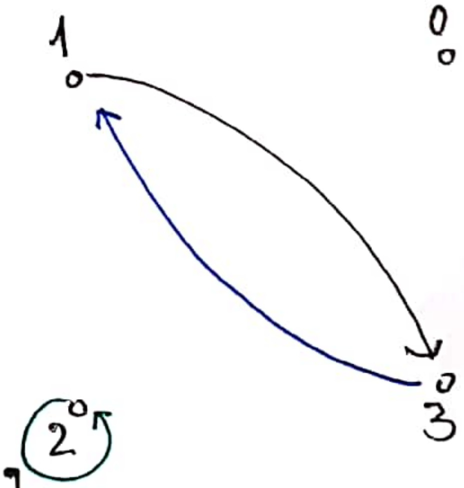
Solution:

(i) $R = \{(1,3),(2,2),(3,1)\}$

(ii) $\text{Domain } R = \{1,2,3\} \subseteq A$

$\text{Range } R = \{1,2,3\} \subseteq A$

(iii)



(iv)

$$M_R = \begin{array}{c|cccc|} {}_{A\rightarrow}\overset{A}{\downarrow} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 \end{array}_{4\times 4}$$

6. Let R be a relation defined on the set $A = \{2,3,4,5,6\}$

$$a, b \in A , \qquad a \, R \, b \Leftrightarrow a.b < 10$$

(i) List all ordered pairs of R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

(iv) Find $\mathbf{M}_R$.

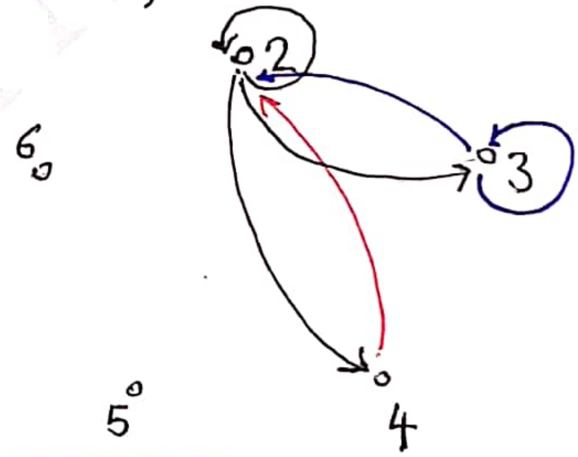
Solution:

(i) $R = \{(2,2),(2,3),(2,4),(3,2)(3,3),(4,2)\}$

(ii) Domain $R = \{2,3,4\} \subseteq A$

Range $R = \{2,3,4\} \subseteq A$

(iii)



(iv)

$$M_R = \begin{array}{c|ccccc} A & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \end{array}$$

5×5

واجب

7. Let R be a relation defined on the set $A = \{-2\,,-1\,,0,1,2,3,4\}$

$$a, b \in A\,, \qquad a\,R\,b \Leftrightarrow a^2 = b^2$$

(i) List all ordered pairs of R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

Solution:

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8. Let R be a relation defined on the set $A = \{-2,-1,0,1,2\}$

$$a,b \in A\ , \qquad a\ R\ b \Leftrightarrow a.b < 0$$

(i) List all ordered pairs of R ?
(ii) Find the domain and the image of R ?
(iii) Draw the directed graph (diagraph) that represents R ?
(iv) Find R^2 .

Solution:

9. Let R be a relation defined on the set $A = \{-2,-1,0,1,2\}$

$$a,b \in A, \qquad a\,R\,b \Leftrightarrow a.b \geq 2$$

(i) List all ordered pairs of R ?
(ii) Find the domain and the image of R ?
(iii) Draw the directed graph (diagraph) that represents R ?
(iv) Find R^2 .

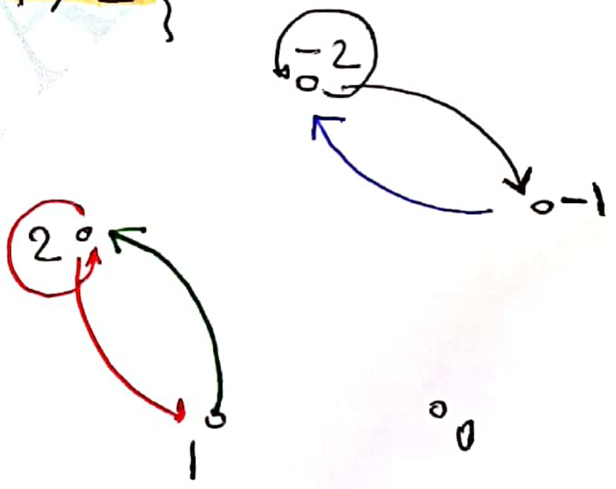
Solution:

(i) $R=\{(-2,-2),(-2,-1),(-1,-2),(1,2),(2,1),(2,2)\}$

(ii) Domain $R=\{-2,-1,1,2\}$

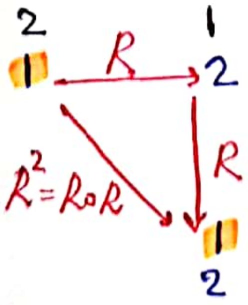
Range $R=\{-2,-1,1,2\}$

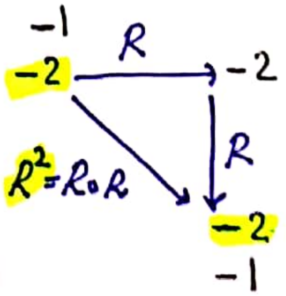
(iii)



(iv)

$R^2=R\circ R=\{(-2,-2),(-2,-1),(-1,-2),(-1,-1),$
$(1,1),(1,2),(2,2),(2,1)\}$.





10. Let $S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,3)\}$ be a relation on the set $B = \{1,2,3\}$

(i) Draw the directed graph (diagraph) that represents S ?

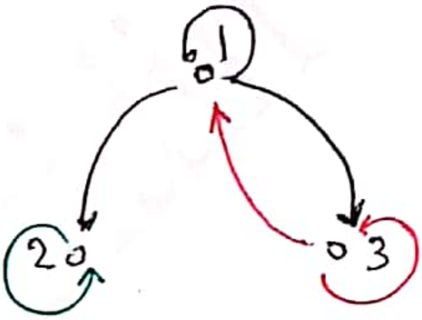
(ii) Find S^2 , S^{-1} , $\bar{S}$, $S o \bar{S}$, $\bar{S} o S$, $\bar{S} - S^{-1}$, $S o S^{-1}$, $S^{-1} o S$, S^3 , $S \cap S^{-1}$.

(iii) Find $\mathbf{M}_S$

Solution:

(iii) $$M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{ccc} 1 & 2 & 3 \\ \hline \end{array}$$

$$M_S = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}$$



$\bar{S} = B \times B - S = \{(2,1),(2,3),(3,2)\}$

$S^{-1} = \{(1,1),(2,1),(3,1),(2,2),(1,3),(3,3)\}$

$S^2 = S \circ S = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,2),(3,3)\}$

$S^3 = \underset{2\swarrow}{S^2} \circ \underset{\searrow 1}{S} = \{(1,1),(1,2),(1,3),(2,2),(3,1),(3,2),(3,3)\} = S^2$

$\underset{2\swarrow}{S} \circ \underset{\searrow 1}{\bar{S}} = \{(2,1),(2,2),(2,3),(3,2)\}$

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$\underset{2\swarrow}{\bar{S}} \circ \underset{\searrow 1}{S} = \{(1,1),(1,3),(1,2),(2,1),(2,3),(3,2)\}$

$\bar{S} - \bar{S}^{-1} = \{(2,3),(3,2)\}$

$S \cap S^{-1} = \{(1,1),(2,2),(1,3),(3,1),(3,3)\}$

$\underset{2\swarrow}{\bar{S}^{-1}} \circ \underset{\searrow 1}{S} = \{$

$\underset{2\swarrow}{S} \circ \underset{\searrow 1}{\bar{S}^{-1}} = \{$

* **11.** Let $S = \{(a,c),(b,a),(c,b)\}$ be a relation on the set $B = \{a,b,c\}$.

(i) Find $\mathbf{M}_S$?

(ii) Find $\bar{S} - S^{-1}$

(iii) Find S^2 , S^3

Solution:

12. Let S = $\{(a,b),(b,c),(c,d),(d,a)\}$ be a relation on the set $B = \{a,b,c,d\}$.

(i) Find $\mathbf{M}_S$?

(ii) Find S^2

(iii) Find $S^{-1} \circ S$

Solution:

13. Let $S = \{(1,v),(1,w),(2,u),(2,v),(3,w)\}$ and $T = \{(1,u),(1,w),(2,v),(2,w),(3,u),(3,v)\}$ are relations from the set $A = \{1,2,3\}$ to the set $B = \{u,v,w\}$.

(i) Find $\bar{S}$, $\bar{S} \cap T$, $T - \bar{S}$

(ii) Find $T^{-1} \circ S$

(iii) Find $S^{-1} \circ T$

Solution: (i) $\bar{S} = A \times B - S = \{(1,u),(2,w),(3,u),(3,v)\}$.

$$M_S = \begin{array}{c|ccc} & u & v & w \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

14. Let $R = \{(a,c),(a,b),(b,b)\}$ and
$S = \{(a,a),(a,c),(b,c),(c,a)\}$
are relations on the set $A = \{a\,,b\,,c\,\}$

✓(i) Find $(R\,o\,S)\cap R^{-1}$

✓(ii) Find $S^{-1}\circ R$

(iii) Find $\mathbf{M}_R$, $\mathbf{M}_S$, $\mathbf{M}_{R\cup S}$, $\mathbf{M}_{R\cap S}$, $\mathbf{M}_{R\circ S}$

Solution:

$$M_R = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\end{array} \qquad M_S = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\end{array}$$

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R\cap S} = M_R \wedge M_S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{\underset{\swarrow\ \ \searrow}{R\circ S}} = M_S \odot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

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$$M_{\underset{\underset{R\circ R}{\parallel}}{R^2}} = [M_R]^2 = M_R \odot M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R^{-1}} = M_R^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

واجب

15. Let $T = \{(1,2),(1,3),(2,2),(2,3)\}$ and $S = \{(1,1),(1,3),(2,1),(3,2)\}$ are relations on the set $E = \{1 , 2 , 3 \}$

(i) Find $T \, o \, S$, $\bar{T} \cap S$, $\bar{T} \, o \, \bar{S}$, $T^2 \, o \, S^{-1}$

(ii) Find $\mathbf{M}_T$, $\mathbf{M}_S$, $\mathbf{M}_{T \cup S}$, $\mathbf{M}_{T \cap S}$, $\mathbf{M}_{T \circ S}$

Solution:

16. Let $R = \{(a,c),(b,a),(b,b)\}$ and $S = \{(a,b),(b,b),(c,a)\}$ are relations on the set $A = \{a\ ,b\ ,c\}$

واجب

(i) Find $R^{-1}\ o\ S^{-1}$, $\overline{R} \cap S$, $R^2\ o\ S$

(ii) Find $\mathbf{M}_R$, $\mathbf{M}_S$, $\mathbf{M}_{R\cup S}$, $\mathbf{M}_{R\cap S}$, $\mathbf{M}_{R\circ S}$

Solution:

17. Let R be the relation represented by the matrix

$$\mathbf{M}_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find the matrix representing: a) R^{-1} b) $\bar{R}$ c) R^2 d) R^3

Solution:

$$a)\ M_{R^{-1}} = M_R^T = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$b)\ M_{\bar{R}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$c)\ M_{R^2} = M_R \odot M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$d)\ M_{R^3} = M_R \odot M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Notice: $R^{n+1} = R^n \circ R \Rightarrow R^3 = R^2 \circ R$

18. Let R_1 and R_2 are relations on a set A represented by the matrices

$$\mathbf{M}_{R_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{R_2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the matrices representing

a) $R_1 \cup R_2$ b) $R_1 \cap R_2$ c) $R_1 \circ R_2$ d) $R_2 \circ R_1$

Solution:

6. Let R be a relation defined on the set $A = \{2,3,4,5,6\}$

$$a, b \in A, \quad a\,R\,b \Leftrightarrow a.b < 10$$

(i) List all ordered pairs of R ?

(ii) Find the domain and the image of R ?

(iii) Draw the directed graph (diagraph) that represents R ?

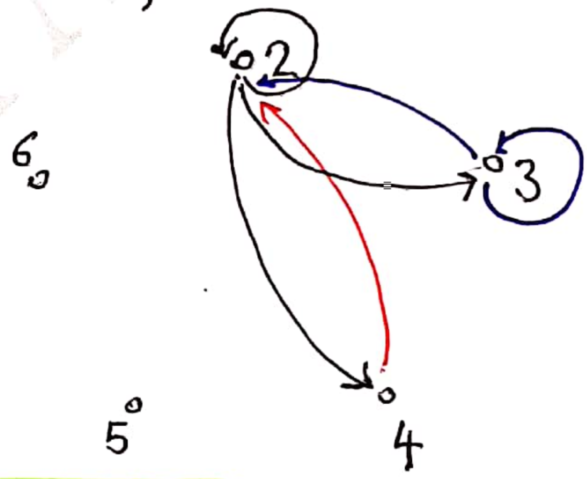
(iv) Find $\mathbf{M}_R$.

Solution: (i) $R = \{(2,2),(2,3),(2,4),(3,2)(3,3),(4,2)\}$

(ii) $\text{Domain } R = \{2,3,4\} \subseteq A$

$\text{Range } R = \{2,3,4\} \subseteq A$

(iii)



(iv)

$$M_R = \begin{array}{c|ccccc} A & 2 & 3 & 4 & 5 & 6 \\ \hline 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \end{array}$$

5X5

$R^2 = R \circ R = \{(2,2),(2,3),(2,4),(3,2),(3,3),(3,4),(4,2),(4,3),(4,4)\}$

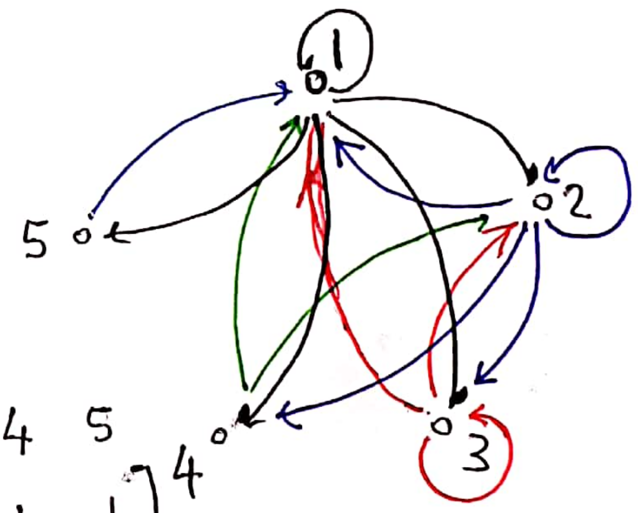
Q_2:

(i) $R=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),$
$(2,4),(3,1),(3,2),(3,3),(4,1),(4,2),(5,1)\}$

(ii) Domain $R=\{1,2,3,4,5\}=A$

Range $R=\{1,2,3,4,5\}=A$

(iii)



$$M_R=\begin{array}{c|ccccc} \overset{A\downarrow}{A\rightarrow} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 & 0 \\ 4 & 1 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Q. 4 :

(i) $R = \{(1,4), (2,3), (3,2)\}$.

(ii) Domain $R = \{1, 2, 3\} \subseteq A$

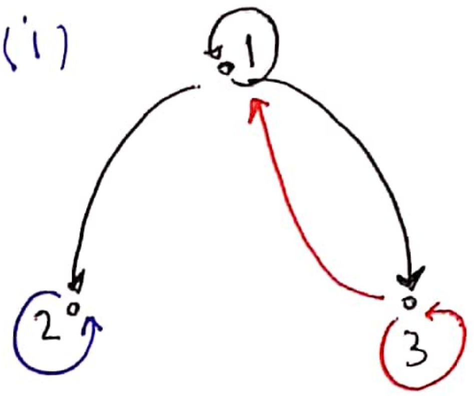
Range $R = \{2, 3, 4\} \subseteq B$.

(iii)

$$M_R = \begin{array}{c|cccc} \overset{A\downarrow}{B\rightarrow} & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}_{4\times 4}$$

#10)

(i)



(ii)

$$M_S = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{array}$$

$\overline{S} = B \times B - S = \{(2,1),(2,3),(3,2)\}$.

$S^{-1} = \{(1,1),(2,1),(3,1),(2,2),(1,3),(3,3)\}$.

$\overline{S} - S^{-1} = \{(2,3),(3,2)\}$., $\overline{S} \cap S^{-1} = \{(2,1)\}$.

$S \cap S^{-1} = \{(1,1),(2,2),(3,3)\}$

$S^2 = S \circ S = \{(1,1),(1,2),(1,3)(2,2),(3,1),(3,2),(3,3)\}$

$S^3 = \underset{\swarrow 2}{S^2} \circ \underset{\searrow 1}{S} = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3)\} = S^2$

$\underset{\swarrow 2}{S} \circ \underset{\searrow 1}{\overline{S}} = \{(2,1),(2,2),(2,3),(3,2)\}$.

#

$\underset{\swarrow 2}{\overline{S}} \circ \underset{\searrow 1}{S} = \{(1,1),(1,3),(1,2),(2,1),(2,3),(3,2)\}$

$\underset{\swarrow 2}{S} \circ \underset{\searrow 1}{S^{-1}} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

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$\underset{\swarrow 2}{S^{-1}} \circ \underset{\searrow 1}{S} = \{(1,1),(1,3),(1,2),(2,1),(2,2),(3,1),(3,3)\}$

$\overline{S} \cup S^{-1} = \{(1,1),(2,1),(3,1),(2,2),(2,3),(1,3),(3,2),(3,3)\}$

$\underset{\swarrow 2}{(\overline{S} \cup S^{-1})} \circ \underset{\searrow 1}{S} = \{(1,1),(1,3),(1,2),(2,1),(2,2),(2,3),(3,1),(3,3),(3,2)\}$

13. Let $S = \{(1,v),(1,w),(2,u),(2,v),(3,w)\}$ and $T = \{(1,u),(1,w),(2,v),(2,w),(3,u),(3,v)\}$ are relations from the set $A = \{1,2,3\}$ to the set $B = \{u,v,w\}$.

(i) Find $\bar{S}$, $\bar{S} \cap T$, $T - \bar{S}$

(ii) Find $T^{-1} \circ S$

(iii) Find $S^{-1} \circ T$

Solution: (i) $\bar{S} = A \times B - S = \{(1,u),(2,w),(3,u),(3,v)\}$.

$$M_S = \begin{array}{c|ccc} & u & v & w \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{array}$$

(ii) ?

(ii) $T^{-1} = \{(u,1),(w,1),(v,2),(w,2),(u,3),(v,3)\}$

Q ⑭ $R^{-1} = \{(c,a),(b,a),(b,b)\}$, $S^{-1} = \{(a,a),(c,a),(c,b),(a,c)\}$

$\underset{2}{\underset{\swarrow}{R}} \circ \underset{1}{\underset{\searrow}{S}} = \{(a,c),(a,b),(c,c),(c,b)\}$.

$(R \circ S) \cap R^{-1} = \{\ \} = \phi$

$\underset{2}{\underset{\swarrow}{S^{-1}}} \circ \underset{1}{\underset{\searrow}{R}} = \{(a,a),(a,b), \qquad \}$