King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(3,3)

Methods of Proof

"Mathematical Induction"

(STRONG INDUCTION)

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Exercises

1. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\,, \quad a_2 = 6 \quad , \qquad a_n = a_{n-1} + a_{n-2} \;\; (*) \qquad :\forall n \geq 3$$

Prove that $3|\, a_n$ for all positive integers n , $\forall n \geq 1$

Solution: Let $P(n)$ be the proposition , $P(n)\colon\ 3|\, a_n \;,\; \Rightarrow a_n = 3c : c \in \mathbb{N}$

BASIS STEP: When $n = 1 \Rightarrow 3|\ a_1 \;\; : \;\; a_1 = 3 = 3(1) \;\Rightarrow\; \therefore\; P(1)$ *is true* .

When $n = (2) \Rightarrow \;\; 3|\ a_2 \;\; : \;\; a_2 = 6 = 3(2) \;\; \Rightarrow \;\therefore\; P(2)$ *is true* .

INDUCTIVE STEP: Let $k \geq 2$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\,,\, P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

$$a_{k+1} = 3c \;\;\Rightarrow\;\; 3|\, a_{k+1} \qquad (\text{ our goal })\;??$$

from $(*) \;\;\Rightarrow \qquad a_{k+1} = a_k + a_{k-1} \qquad (***)$

$\because\;\; P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow \;\; 3|\, a_k \;\; \Rightarrow a_k = 3c_1 : \; c_1 \in \mathbb{N}$

from $P(k-1) \Rightarrow \;\; 3|\, a_{k-1} \;\; \Rightarrow a_{k-1} = 3c_2 \;\; : \;\; c_2 \in \mathbb{N}$

$\xRightarrow{\text{by subist.into}(***)}$

$$a_{k+1} = a_k + a_{k-1} = 3c_1 + 3c_2 = 3(c_1 + c_2) = 3c$$

$$: c = (c_1 + c_2) \in \mathbb{N}$$

$\therefore\; a_{k+1} = 3c \;\;\Rightarrow\;\; 3|\, a_{k+1} \;\Rightarrow \therefore \;\; P(\,k+1\,)$ is true . #

2. Assume $\{a_n\}_{\text{n}=1}^{\infty}$ is a sequence defined as:

$$a_1 = 8\,,\ a_2 = 4\ ,\ a_n = a_{n-1} + a_{n-2}\ :\forall n \geq 3$$

Prove that a_n is even for $\forall n \geq 1$.

Solution:

11. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \quad a_1 = 2 \ , \quad a_n = 4a_{n-1} - 3a_{n-2} \quad (*) \qquad : \forall n \geq 2$$

Prove that $a_n = 3^n - 1$ for all integers n. $\forall n \geq 0$

Solution:

Let $P(n)$:

<u>*BASIS STEP*</u>*: When* $n = 0 \Rightarrow$

$When\ n = (1) \Rightarrow$

<u>*INDUCTIVE STEP*</u>*:* Let $k \geq 1$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\ , P(k)$ All are true . (**)

$Our\ goal\ is\ to\ show\ that\ P(\ k+1\ )\ is\ also\ true$?

$from\ \ (*) \ \Rightarrow$

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$

12. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\,,\ a_1 = 15\ ,a_2 = 3 \qquad a_n = \frac{a_{n-1}\,a_{n-2}\,a_{n-3}}{9} + 6 \quad : \forall n \geq 3 \ \ (*)$$

Prove that $3|a_n$ for all integers *n*. $\forall n \geq 0$

Solution:

Let $P(n)$:

<u>BASIS STEP</u>: *When* $n = 0 \ \Rightarrow$

$When\, n = 1 \ \Rightarrow$

$When\, n = (2) \Rightarrow$

<u>INDUCTIVE STEP</u>: Let $k \geq 2$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\ , P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

from (*) $\Rightarrow$

$\because$ $P(k)\ , P(k-1)$ & $P(k-2)$ all are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

from $P(k-2) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$

16. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 2\,,\ u_2 = 4 \quad ,\quad u_n = \frac{2u_{n-1} + u_{n-2} + 8}{3} \quad : \forall n \geq 3 \quad (*)$$

Prove that $u_n = 2n$ for all positive integers n. $\forall n \geq 1$

Solution:

Let $P(n)$:

<u>*BASIS STEP*</u>*: When* $n = 1 \Rightarrow$

$When\ n = (2) \Rightarrow$

<u>*INDUCTIVE STEP*</u>*:* Let $k \geq 2$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

from (*) $\Rightarrow$

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$

17. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \,, \quad a_1 = 2 \,, \; a_2 = 3 \,, \quad a_n = a_{n-1} + a_{n-2} + 2a_{n-3} \; : \forall n \geq 3 \; (*)$$

Prove that $a_n \leq 3^n$ for all integers $n \geq 0$.

Solution:

Let $P(n)$:

BASIS STEP: When $n = 0 \;\Rightarrow$

$When\; n = 1 \;\Rightarrow$

$When\; n = (2) \Rightarrow$

INDUCTIVE STEP: Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

from (*) $\Rightarrow$

$\because$ $P(k)\,, P(k-1)$ & $P(k-2)$ all are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

from $P(k-2) \Rightarrow$

$\xRightarrow{\text{by subist.into}(***)}$

18. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \,, \quad a_1 = 2 \quad , \quad a_n = 4a_{n-1} - 4a_{n-2} \quad : \forall n \geq 2 \quad (*)$$

Prove that $a_n = 2^n$ for all integers $n \geq 0$.

Solution:

Let $P(n)$:

<u>*BASIS STEP*</u>*: When* $n = 0 \;\Rightarrow$

$When\; n = (1) \Rightarrow$

<u>*INDUCTIVE STEP*</u>*:* Let $k \geq 1$ and assume that

$P(1), P(2), \dots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

from (*) $\Rightarrow$

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$

6. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 5\ ,\quad a_{n+1} = 2a_n + 3\,a_{n-1} \quad : \forall n \geq 2 \qquad (*)$$

Prove that $3^n \leq a_{n+1} \leq 2\,.\,3^n$ for all positive integers n.

Solution:

Let $P(n)$:

<u>*BASIS STEP*</u>*: When* $n = 1 \Rightarrow$

$When\ n = (2) \Rightarrow$

<u>*INDUCTIVE STEP*</u>*:* Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,,\, P(k)$ All are true . (**)

Our goal is to show that $P(\,k+1\,)$ *is also true* ?

from (*) $\Rightarrow$

$\because$ $P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$

9. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\ ,\ a_1 = 1\ \ ,\ \ a_n = 2a_{n-1} + a_{n-2}\ \ : \forall n \geq 2 \quad (*)$$

Prove that a_n is odd for all integers $n \geq 0$

Solution:

Let $P(n)$:

<u>BASIS STEP</u>: When $n = 0 \ \Rightarrow$

$When\ n = (1) \Rightarrow$

<u>INDUCTIVE STEP</u>: Let $k \geq 1$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\ , P(k)$ All are true . (**)

Our goal is to show that $P(\ k+1\ )$ *is also true* ?

from $(*)\ \Rightarrow$

$\because\ \ P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow$

from $P(k-1) \Rightarrow$

$\xRightarrow{by\ subist.into(***)}$