King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(4.2)

Equivalence Relations

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Equivalence Relations

DEFINITION 1 A relation on a set A is called an *equivalence relation* if it is *reflexive*, *symmetric*, and *transitive.*

DEFINITION 2 Two elements a and b that are related by an equivalence relation are called *equivalent*. the notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation

Equivalence Classes

DEFINITION 3 Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the ***equivalence class of a.*** The equivalence class of a with respect to R is denoted by $[a]$ for this equivalence class . $[a] = \{b \in A : a\ R\ b\}$

Equivalence Classes and Partitions

THEOREM 1 Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

$$(i)\ aRb \qquad (ii)\ [a] = [b] \qquad (iii)\ [a] \cap [b] \neq \emptyset$$

THEOREM 2 Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\Im = \{\{A_i : \emptyset \neq A_i \subseteq S , i \in I\}\}$ of the set S, there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes .

$$\forall\left(A_i, A_j \in \Im\right), (i \neq j \rightarrow A_i \cap A_j = \emptyset)$$

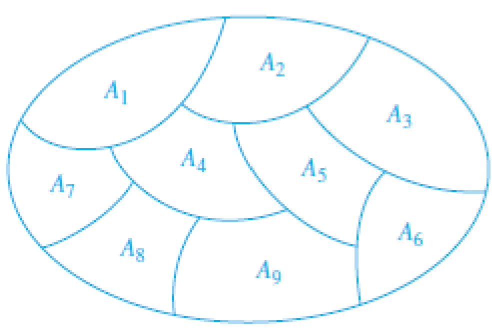

FIGURE 1 **A Partition of a Set.**

and

$$\bigcup_{i \in I} A_i = S$$

(Here the notation $\bigcup_{i \in I} A_i$ represents the union of the sets A_i for all $i \in I$.) Figure 1 illustrates the concept of a partition of a set.

EXAMPLE 1 Suppose that $S = \{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ forms a partition of S, because these sets are disjoint and their union is S.

$\mathfrak{I} = \{\{1,2,3\},\{4,5\},\{6\}\}$ where $\forall\left(A_i, A_j \in \mathfrak{I}\right), (i \neq j \rightarrow A_i \cap A_j = \emptyset)$ and $\bigcup_{i=1,2,3} A_i = S$

EXAMPLE 2 List the ordered pairs in the equivalence relation R produced by the partition $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5\}$, and $A_3 = \{6\}$ of $S = \{1, 2, 3, 4, 5, 6\}$, given in Example 1.

Solution: The subsets in the partition are the equivalence classes of R. The pair (a, b) ∈ R if and only if a and b are in the same subset of the partition. The pairs (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), and (3, 3) belong to R because $A_1 = \{1, 2, 3\}$ is an equivalence class; the pairs (4, 4), (4, 5), (5, 4), and (5, 5) belong to R because $A_2 = \{4, 5\}$ is an equivalence class; and finally the pair (6, 6) belongs to R because {6} is an equivalence class. No pair other than those listed belongs to R .

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5), (6,6)\}$

EXAMPLE 3 Let $\sim$ be a relation defined on $\mathbb{N} \times \mathbb{N}$ **,** such that:

$$(m,n),(p,q) \in \mathbb{N} \times \mathbb{N} \quad (m,n) \sim (p,q) \Leftrightarrow m + q = p + n$$

(i) Show that $\sim$ is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

(ii) Find the *equivalence classes* [(3,4)] , [(1,1)]

Solution:

(*i*)

(a) $\forall (m,n) \in \mathbb{N} \times \mathbb{N} \Rightarrow \because m + n = m + n \Rightarrow \therefore (m,n) \sim (m,n) \Rightarrow \therefore \sim$ *is reflexive*

(b) $(m,n),(p,q) \in \mathbb{N} \times \mathbb{N} : (m,n) \sim (p,q) \Rightarrow m + q = p + n$

$$\Rightarrow p + n = m + q$$

$$\Rightarrow (p,q) \sim (m,n) \Rightarrow \therefore \sim \text{ is symmetric}$$

(c) $(m,n),(p,q),(r,s) \in \mathbb{N} \times \mathbb{N} : (m,n) \sim (p,q) \Rightarrow m + q = p + n$

$$(p,q) \sim (r,s) \Rightarrow p + s = r + q$$

$$\Rightarrow m + q + p + s = p + n + r + q$$

$$\Rightarrow m + s = r + n \Rightarrow (m,n) \sim (r,s)$$

$\Rightarrow \therefore \sim$ is *transitive*

$\because \sim$ is *reflexive* , *symmetric* and *transitive* $\Rightarrow \therefore \sim$ is an *equivalence relation* on $\mathbb{N} \times \mathbb{N}$.

(*ii*)

$$[(1,1)] = \{(a,b) \in \mathbb{N} \times \mathbb{N} : (a,b) \sim (1,1) \Rightarrow a+1 = b+1 \Rightarrow a = b\}$$

$$= \{(a,a) : a \in \mathbb{N}\} = \{(1,1),(2,2),(3,3),\dots\}$$

$$[(3,4)] = \{(a,b) \in \mathbb{N} \times \mathbb{N} : (a,b) \sim (3,4) \Rightarrow a+4 = b+3 \Rightarrow b = a+1\}$$

$$= \{(a,a+1) : a \in \mathbb{N}\} = \{(1,2),(2,3),(3,4),\dots\} \quad .$$

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EXAMPLE 4 Let $\boldsymbol{S}$ be a relation defined on $\mathbb{R}$ such that:

$$x,y \in \mathbb{R}, x\, \boldsymbol{S}\, y \Leftrightarrow x^2 - y^2 = 2(y-x) \Leftrightarrow x^2 + 2x = y^2 + 2y$$

(i) Show that $\boldsymbol{S}$ is an equivalence relation on $\mathbb{R}$.

(ii) Find the *equivalence classes* $[1]$, $[0]$

Solution:

(*i*)

(a) $\forall\, x \in \mathbb{R}$, $x^2 + 2x = x^2 + 2x \Rightarrow \therefore x\, S\, x \Rightarrow \therefore$ S is *reflexive*

(b) $x,y \in \mathbb{R}$, $x\, S\, y \Rightarrow x^2 + 2x = y^2 + 2y \Rightarrow y^2 + 2y = x^2 + 2x \Rightarrow y\, S\, x$
$\Rightarrow \therefore$ S is *symmetric*

(c) $x,y,z \in \mathbb{R} : x\, S\, y \Rightarrow x^2 + 2x = y^2 + 2y$

$y\, S\, z \Rightarrow y^2 + 2y = z^2 + 2z$

$\Rightarrow x^2 + 2x = y^2 + 2y = z^2 + 2z$

$\Rightarrow x^2 + 2x = z^2 + 2z$

$\Rightarrow x\, S\, z \Rightarrow \therefore$ S is *transitive*

$\because$ $\boldsymbol{S}$ is *reflexive* , *symmetric* and *transitive* $\Rightarrow \therefore$ $\boldsymbol{S}$ is an *equivalence relation* on $\mathbb{R}$.

(*ii*)

$$[0] = \{x \in \mathbb{R} : x\, S\, 0 \Rightarrow x^2 + 2x = 0^2 + 2(0)\}$$

$$= \{x \in \mathbb{R} : x(x+2) = 0\} = \{-2,0\}$$

$$[1] = \{x \in \mathbb{R} : x\, S\, 1 \Rightarrow x^2 + 2x = 1^2 + 2(1) = 3\}$$

$$= \{x \in \mathbb{R} : x^2 + 2x - 3 = 0 \Rightarrow (x+3)(x-1) = 0\} = \{-3,1\} \; .$$

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EXAMPLE 5 Let $\boldsymbol{T}$ be a relation defined on $\mathbb{Z}$ such that:

$$a, b \in \mathbb{Z}\,, \quad a\,T\,b \iff |a| = |b|$$

(i) Show that T is an equivalence relation .

(ii) Find $\mathfrak{I}(\mathrm{T})$.

Solution:

(i)

1- $\forall a \in \mathbb{Z}\,, \because \ |a| = |a| \Rightarrow \therefore a\,T\,a \Rightarrow \therefore \ T$ *is reflexive*

2- $a, b \in \mathbb{Z}\,, \ a\,T\,b \iff |a| = |b| \Rightarrow |b| = |a| \Rightarrow b\,T\,a \Rightarrow \therefore T$ *is symmetric*

3- $a, b, c \in \mathbb{Z}\,, \ a\,T\,b \iff |a| = |b|$

&

$b\,T\,c \iff |b| = |c|$

$|a| = |b| = |c| \Rightarrow |a| = |c| \Rightarrow \therefore a\,Tc \Rightarrow \therefore \ T$ *is transitive*

$\because$ $\boldsymbol{T}$ *is reflexive , symmetric and transitive* $\Rightarrow \therefore$ $\boldsymbol{T}$ *is an equivalence relation on* $\mathbb{Z}$.

(ii)

$$[a] = \{b \in \mathbb{Z} : \ |a| = |b|\} = \{b \in \mathbb{Z} : a = \pm b\}$$
$$= \{a, -a\}$$

$$\therefore \ \mathfrak{I}(\mathbb{Z}) = \{[a] : a \in \mathbb{Z}\} = \{\{0\}, \{-1,1\}, \{-2,2\}, \dots\} \quad .$$

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EXERCISES

1. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$$a, b \in \mathbb{Z}, a\ R\ b \iff 6a \equiv b\ (\,mod\ 5\,) \iff 5|(6a-b)\,, \quad 5 \text{ devides } (\,6a-b\,)$$

(i) *Show that R is an equivalence relation.*

(ii) *Find the equivalence class* $[0]$.

(iii) *Decide whether* $9 \in [4]$.

Solution: (i)

2. Let S be the relation defined on the set $A = \{-2, -1, 0, 1, 2\}$, such that:

$a, b \in A$, $a\ S\ b \Leftrightarrow 3|(a + 2b)$, 3 devides $(a + 2b)$

(i) *Show that S is an equivalence relation.*

(ii) *Find all equivalence classes .*

Solution :

3. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$m, n \in \mathbb{Z}$, $\quad m\, R\, n \iff 4 | (m - n + 8)$, $\quad$ 4 devides $(m - n + 8)$

(i) *Show that R is an equivalence relation.*

(ii) *Show that* [10] = [-6] .

Solution :

4. Assume T is an equivalence relation defined on the set $A = \{a, b, c, d, e\}$,

and the matrix of T given such that

$$M_T = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the number of equivalence classes .

Solution :

5. Let $T = \{(a,a),(b,b),(b,d),(c,c),(d,b),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$

(i) *Represent T by the directed graph (diagraph)*

(ii) *Show that R is an equivalence relation.*

(iii) *Find all equivalence classes .*

Solution :

6. Let R be the relation defined on the set $\mathbb{N} = \{1,2,3,\ldots\}$, such that:

$$a, b \in \mathbb{N} \quad , \quad a\,R\,b \Leftrightarrow \left(\sqrt{a} - \sqrt{b}\right) \in \mathbb{Z} \quad , \quad \left(\sqrt{a} - \sqrt{b}\right) \text{ is integer}$$

(i) *Show that R is an equivalence relation.*

(ii) *Decide whether* $4 \in [9]$.

Solution :

7. Let S be the relation defined on the set $\mathbb{N} = \{1,2,3,\ldots\}$, such that:

$a, b \in \mathbb{Z}$, $a\ R\ b \Leftrightarrow 5a \equiv b\ (\,mod\ 4\,) \Leftrightarrow 4|(5a - b)$, 4 devides $(\,5a - b\,)$

(i) *Show that* S *is an equivalence relation.*

(ii) *Find the equivalence class* $[1]$.

Solution:

8. Let S be the relation defined on the Rational set $\mathbb{Q}$, such that:

$$x, y \in \mathbb{Q}\ ,\ x\,S\,y \Leftrightarrow x - y \in \mathbb{Z} \quad , \quad (x - y)\ is\ integer$$

(i) *Show that S is an equivalence relation.*

(ii) *Decide whether* $\frac{9}{4} \in [\frac{1}{2}]$ *or not ?*

Solution: (i)

9. Let S be the relation defined on the set E = $\{2a \mid a \in \mathbb{Z}\}$ (even Integers set), such that:

$m, n \in E$, $m\, S\, n \Leftrightarrow 4 \mid (m+n)$, 4 devides $(m+n)$

(i) *Show that* S *is an equivalence relation.*

(ii) *Find the equivalence class* $[0]$.

Solution :

10. Let R be the relation defined on the Rational set $\mathbb{Z}^* = \mathbb{Z} - \{0\}$, such that:

$$x, y \in \mathbb{Z}^* \ , \quad x\, R\, y \iff xy > 0$$

(i) *Show that* S *is an equivalence relation.*

(ii) *Find the equivalence classes* $[-1]$, $[1]$

.

Solution: (i)

11. Let $S = \{(a,a),(a,c),(b,b),(b,e),(c,a),(c,c),(d,d),(e,b),(e,e)\}$ be a relation defined on the set $A = \{a,b,c,d,e\}$

(i) *Show that S is an equivalence relation.*

(ii) *Find all equivalence classes .*

Solution: (i)

1- $\because$ $(a,a),(b,b),(c,c),(d,d),(e,e) \in S \Rightarrow \therefore$ S is *reflexive*

2- $\because (a,c)\,,(c,a) \in S$ & $(b,e)\,,(e,b) \in S \Rightarrow \therefore$ S is *symmetric*

3- $\because (a,c)\,,(c,a),(a,a) \in S$

& $(b,e)\,,(e,b),(b,b) \in S$

& $(e,b),(b,e),(e,e) \in S$

& $(c,a),(a,c),(c,c) \in S$ $\Rightarrow \therefore$ S is *transitive*

$\because$ S is *reflexive* , *symmetric* and *transitive* $\Rightarrow \therefore$ S is an *equivalence relation* on A .

(ii) $[a] = \{a,c\}$

$[b] = \{b,e\}$

$[d] = \{d\}$

$\Rightarrow$ $\mathfrak{J}(\mathrm{S}) = \{\{a,c\},\{\mathrm{b},\mathrm{e}\},\{\mathrm{d}\}\}$

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12. Let S be the relation defined on the set $A = \{0,1,2,3,4\}$, such that:

$a, b \in A$, $a\ S\ b \iff 3|(2a + b)$, 3 devides $(\ 2a + b\ )$

(i) *Show that S is an equivalence relation.*

(ii) *Find the equivalence classes* $[0]$, $[1]$

(iii) *Find the number of equivalence classes of the relation S .*

Solution :

13. Let R be the relation defined on the set $\mathbb{N} = \{1,2,3, \dots\}$, such that:

$$x, y \in \mathbb{N}, \quad x\,R\,y \iff x + y \quad \textit{is even} \quad .$$

(i) *Show that S is an equivalence relation.*

(ii) *Find the equivalence class* $[2]$.

(iii) *Decide whether* $4 \in [11]$ *or not ?*

Solution: (i)

1- $\forall x \in \mathbb{N}, \; x + x = 2x \;\; (\textit{is even}) \Rightarrow \therefore \; x\,R\,x \Rightarrow \therefore \;\; \mathrm{R} \;\; \textit{is reflexive}$

2- $x, y \in \mathbb{N}, \quad x\,R\,y \iff x + y = 2m \quad (\textit{is even}) : \mathrm{m} \in \mathbb{N}$

$\Rightarrow y + x = 2m \quad (\textit{is even}) \Rightarrow y\,R\,x \quad \therefore \quad \mathrm{R} \;\; \textit{is symmetric}$

3- $x, y, z \in \mathbb{N}, \; x\,R\,y \iff x + y = 2m_1 \quad (\textit{is even}) : \mathrm{m}_1 \in \mathbb{N}$

&

$y\,R\,z \iff y + z = 2m_2 \quad (\textit{is even}) : \mathrm{m}_2 \in \mathbb{N}$

$(+) \Rightarrow$ ____________________

$x + 2y + z = 2m_1 + 2m_2$

$\Rightarrow \; x + z = 2(\mathrm{m}_1 + m_2 - y) = 2m \;\; (\textit{is even})$

$: m = (\mathrm{m}_1 + m_2 - y) \in \mathbb{N} \qquad \Rightarrow \therefore \; x\,R\,z \;\; \Rightarrow \therefore \quad \mathrm{R} \;\; \textit{is transitive}$

$\because \;\; R$ *is reflexive , symmetric and transitive* $\Rightarrow \therefore \;\; R$ *is an equivalence relation on* $\mathbb{N}$.

(ii) $[2] = \{x \in \mathbb{N} : x\,R\,2\}$

$= \{x \in \mathbb{N} : x + 2 = 2m : m \in \mathbb{N}\}$

$= \{x \in \mathbb{N} : x = 2m - 2 = 2(m-1) = 2k : k = m - 1 \in \mathbb{N}\}$

$= \{x \in \mathbb{N} : x = 2k \;, k \in \mathbb{N}\} = \{2,4,6, \dots\}$

(iii) $\because \; 4 + 11 = 15 \quad (\textit{is an odd}) \Rightarrow \quad \therefore \; 4 \notin [\,11\,]$

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14. Let S be the relation defined on the Rational set $\mathbb{Q}$, such that:

$$a, b \in \mathbb{Q} \ , \ a\, S\, b \ \Leftrightarrow \ a - b = 2k \ : \ k \in \mathbb{Z} \ , \ (a - b) \text{ is even integer } .$$

(i) *Show that* S *is an equivalence relation.*

(ii) *Show that* [m] = [0] *for every even integer* m *,and* [n] = [1] *for every odd integer* n .

Solution: (i)

(ii) $[0] = \{x \in \mathbb{Q} : x\, S\, 0\}$

$= \{x \in \mathbb{Q} : x - 0 = x = 2k = m \text{ is an even integer}\}$

$= [m] : m \text{ is an even integer}$

$[1] = \{x \in \mathbb{Q} : x\, S\, 1\}$

$= \{x \in \mathbb{Q} : x - 1 = 2k \text{ is an even integer}\}$

$= \{x \in \mathbb{Q} : x = 2k + 1 = n \text{ is an odd integer}\}$

$= [n] : n \text{ is an odd integer}$

#

15. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$$a, b \in \mathbb{Z}, a\,R\,b \Leftrightarrow a \equiv b\ (\,mod\ 7\,) \Leftrightarrow 7|(a-b)\ , \quad 7 \text{ devides } (\,a-b\,)$$

(i) *Show that R is an equivalence relation.*

(ii) *Decide whethere* $9 \in [2]$.

(iii) *If R is defined on the set $A = \{1,5,12,22,35,41,55\}$, find all equivalence classes .*

Solution: (i) $a, b \in \mathbb{Z}, a\,R\,b \Leftrightarrow a \equiv b\ (\,mod\ 7\,) \Leftrightarrow 7|(a-b) \Rightarrow a-b = 7m : m \in \mathbb{Z}$

1- $\forall\, a \in \mathbb{Z},\ 7|(a-a) = 0 \Rightarrow \therefore a\,R\,a \Rightarrow \therefore\ R$ is $reflexive$

2- $a, b \in \mathbb{Z}, a\,R\,b \Leftrightarrow a \equiv b\ (\,mod\ 7\,) \Leftrightarrow 7|(a-b) \Rightarrow a-b = 7m : m \in \mathbb{Z}$

(multiply both sides by -1) $\Rightarrow b-a = 7(-m) \Rightarrow 7|(b-a) \Rightarrow b\,R\,a \Rightarrow \therefore\ R$ is $symmetric$

3- $a, b, c \in \mathbb{Z}, a\,R\,b \Leftrightarrow a \equiv b\ (\,mod\ 7\,) \Leftrightarrow 7|(a-b) \Rightarrow a-b = 7m_1 : m_1 \in \mathbb{Z}$

&

$b\,R\,c \Leftrightarrow b \equiv c\ (\,mod\ 7\,) \Leftrightarrow 7|(b-c) \Rightarrow b-c = 7m_2 : m_2 \in \mathbb{Z}$

$(+) \Rightarrow$ ______________________

$a-c = 7(m_1+m_2) = 7m$

$: m = (m_1+m_2) \in \mathbb{Z} \quad \Rightarrow 7|(a-c) \Rightarrow a\,R\,c \Rightarrow \therefore\ R$ is $transitive$

$\because\ R$ is $reflexive$, $symmetric$ and $transitive \Rightarrow \therefore\ R$ is an equivalence relation on $\mathbb{Z}$.

(ii) $\because\ 9-2 = 7 \Rightarrow 7|(9-2) \Rightarrow \therefore\ 9\,R\,2 \ \therefore\ 9 \in [\,2\,]$

(iii)

$[1] = \{1,22\}$

$[5] = \{5,12\}$

$[35] = \{35\}$

$[41] = \{41,55\}$

$$\Im(A) = \{\{1,22\},\{5,12\},\{35\},\{41,55\}\}$$

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16. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$$a, b \in \mathbb{Z}\,,\ \ a\,R\,b \ \Leftrightarrow a^2 \equiv b^2\ (\,mod\ 7\,) \Leftrightarrow 7|(a^2 - b^2)\,, \quad , \quad 7 \text{ devides } (\,a^2 - b^2\,)$$

(i) *Show that R is an equivalence relation.*

(ii) *Find* $\Im(\mathbb{Z})$.

Solution:

17. Let S be a relation defined on $\mathbb{R}^* \times \mathbb{R}^*$, such that:

$$(x,y),(a,b) \in \mathbb{R}^* \times \mathbb{R}^*, (x,y)\ S\ (a,b) \Leftrightarrow xy(a^2+b^2) = ab(x^2+y^2)$$

$$\Leftrightarrow \frac{xy}{x^2+y^2} = \frac{ab}{a^2+b^2}$$

(i) *Show that S is an equivalence relation.*

(ii) *Find the equivalence classes* $[(3,4)]$, $[(2,1)]$

Solution:

18. Let T be the relation defined on the integers set $\mathbb{Z}$, such that:

$$x, y \in \mathbb{Z}, \quad x\, T\, y \iff |x-3| = |y-3|$$

(i) *Show that* T *is an equivalence relation.*

(ii) *Find* $[0]$, $[3]$, $[-2]$

(ii) *Find* $\Im(\mathbb{Z})$.

Solution:

19. Let $\sim$ be a relation defined on $\mathbb{Z} \times \mathbb{Z}^{+}$, such that:

$$(m,n),(p,q) \in \mathbb{Z} \times \mathbb{Z}^{+} \qquad (m,n) \sim (p,q) \Leftrightarrow mq = pn \Leftrightarrow \frac{m}{n} = \frac{p}{q}$$

(i) *Show that* $\sim$ *is an equivalence relation.*

(ii) *Find the equivalence classes* $[(3,4)]$, $[(1,2)]$

Solution:

20. Let T be the relation defined on the Rational set $\mathbb{Q}$, such that:

$$x, y \in \mathbb{Q} \ , \ x \, T \, y \Leftrightarrow x - y \in \mathbb{Z} \quad , \ (x - y) \ is \ integer$$

(i) *Show that* T *is an equivalence relation.*

(ii) *Find* $[0]$ *and* $[\frac{1}{2}]$.

Solution:

21. Let $\boldsymbol{S}$ be a relation defined on $\mathbb{R}$ such that:

$$x, y \in \mathbb{R}\ ,\ x\, S\, y \Leftrightarrow x - y \in \mathbb{Q} \quad , \quad (x - y)\ is\ rational\ .$$

(i) *Show that S is an equivalence relation.*

(ii) *Find* $[0]$

Solution:

22. Let T be a relation defined on $\mathcal{B} = \mathbb{R} \times \mathbb{R}$ **,** such that:

$$(a,b),(c,d) \in \mathcal{B} = \mathbb{R} \times \mathbb{R}, \ (a,b)\,T\,(c,d) \Leftrightarrow b - a^2 = d - c^2$$

(i) *Show that T is an equivalence relation.*

(ii) *Find the equivalence classes* $[(0,0)]$, $[(1,2)]$

Solution: (i)

1- $\forall (a,b) \in \mathbb{R} \times \mathbb{R}, \ b - a^2 = b - a^2 \Rightarrow (a,b)\,T\,(a,b) \Rightarrow \therefore \ T \text{ is } reflexive$

2- $(a,b),(c,d) \in \mathbb{R} \times \mathbb{R}, \ (a,b)\,T\,(c,d) \Leftrightarrow b - a^2 = d - c^2$

$$\Rightarrow d - c^2 = b - a^2$$

$$\Rightarrow (c,d)\,T\,(a,b) \Rightarrow \therefore \ T \text{ is } symmetric$$

3-

$$(a,b),(c,d),(e,f) \in \mathbb{R} \times \mathbb{R}, \ (a,b)\,T\,(c,d) \Leftrightarrow b - a^2 = d - c^2$$

&

$$(c,d)\,T\,(e,f) \Leftrightarrow d - c^2 = f - e^2$$

$$b - a^2 = d - c^2 = f - e^2 \Rightarrow b - a^2 = f - e^2 \Rightarrow (a,b)\,T\,(e,f) \Rightarrow \therefore \ T \text{ is } transitive$$

$\because$ T *is reflexive , symmetric and transitive* $\Rightarrow \therefore$ T *is an equivalence relation on* $\mathbb{R} \times \mathbb{R}$.

(ii)

$$[(0,0)] = \{(x,y) \in \mathbb{R} \times \mathbb{R} : (x,y)\,T\,(0,0)\}$$
$$= \{(x,y) \in \mathbb{R} \times \mathbb{R} : y - x^2 = 0 - 0^2 = 0\}$$
$$= \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$$

$$[(1,2)] = \{(x,y) \in \mathbb{R} \times \mathbb{R} : (x,y)\,T\,(1,2)\}$$
$$= \{(x,y) \in \mathbb{R} \times \mathbb{R} : y - x^2 = 2 - 1^2 = 1\}$$
$$= \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2 + 1\}$$

#

23. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$$x, y \in \mathbb{Z}, \quad x\,R\,y \iff 4|(3x+y) \quad , \quad 4 \text{ devides } (\,3x+y\,)$$

(i) *Show that* R *is an equivalence relation.*

(ii) *Find* $[0]$, $[1]$.

(iii) Determine whether $-2 \in [6]$

Solution :

24. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$$a, b \in \mathbb{Z}, a\ R\ b \Leftrightarrow a \equiv 4b\ (\,mod\ 3\,) \Leftrightarrow 3|(a-4b)\,, \quad 3 \text{ devides } (\,a-4b\,)$$

(i) *Show that R is an equivalence relation.*

(ii) *Find the equivalence class* $[0]$.

(iii) *Decide whether* $2 \in [5]$?

25. Let S be a relation defined on $A \times A$ **,** where $A = \{1,2,3,4,5\}$ such that:

$$(a,b),(c,d) \in A \times A \,,(a,b)\; S\;(c,d) \;\Leftrightarrow a+b=c+d$$

(i) *Show that* S *is an equivalence relation.*

(ii) *Find the equivalence classes* $[(3,3)]$, $[(5,5)]$, $[(2,3)]$

Solution: (i)

1- $\forall (a,b) \in A \times A\,,\; a+b=a+b \;\Rightarrow\; (a,b)\;S\;(a,b) \Rightarrow \therefore\; S$ is *reflexive*

2- $(a,b),(c,d) \in A \times A\,,(a,b)\;S\;(c,d) \;\Leftrightarrow a+b=c+d$

$$\Rightarrow\; c+d=a+b \;\Rightarrow\; (c,d)\;S\;(a,b) \;\Rightarrow \therefore\; S \text{ is } symmetric$$

3- $(a,b),(c,d),(e,f) \in A \times A\,,\quad (a,b)\;S\;(c,d) \;\Leftrightarrow a+b=c+d$

&

$(c,d)\;S\;(e,f) \;\Leftrightarrow c+d=e+f$

$$a+b=c+d=\;e+f \;\Rightarrow\; a+b=\;e+f \;\Rightarrow\; (a,b)\;S\;(e,f) \;\Rightarrow \therefore\; S \text{ is } transitive$$

$\because\; S$ is *reflexive* , *symmetric* and *transitive* $\Rightarrow \therefore\; S$ *is an equivalence relation on* $A \times A$.

(ii)

$$[(3,3)] = \{(3,3),(1,5),(2,4),(5,1),(4,2)\}$$

$$[(5,5)] = \{(5,5)\}$$

$$[(2,3)] = \{(1,4),(2,3),(4,1),(3,2)\}$$

#

26. Let R be the relation defined on the integers set $\mathbb{N} = \{1,2,3,\ldots\}$, such that:

$$a, b \in \mathbb{N} \ , \ a\ R\ b \Leftrightarrow ab = k^2 \quad : k \in \{1,2,3,\ldots\}$$

(i) *Show that* R *is an equivalence relation.*

(ii) *Find the equivalence class* $[1]$.

Solution: (i)

1- $\forall\ a \in \mathbb{N} \quad , \quad aa = a^2 \quad \Rightarrow \therefore a\ R\ a \quad \Rightarrow \quad \therefore \quad \text{R } is\ reflexive$

2- $a, b \in \mathbb{N} \ , \ a\ R\ b \Leftrightarrow \quad ab = k^2 \qquad : k \in \{1,2,3,\ldots\}$

$$\Rightarrow ba = k^2 \quad \Rightarrow b\ R\ a \ \Rightarrow \therefore \quad \text{R } is\ symmetric$$

3- $a, b, c \in \mathbb{N} \ , \ a\ R\ b \Leftrightarrow \quad ab = {k_1}^2 \qquad : k_1 \in \{1,2,3,\ldots\} \quad (1)$

&

$$b\ R\ c \Leftrightarrow \quad bc = {k_2}^2 \qquad : k_2 \in \{1,2,3,\ldots\} \quad (2)$$

$(1) \times (2) \Rightarrow$ ____________________

$$ab^2c = {k_1}^2{k_2}^2$$

$$\Rightarrow \qquad ac = \frac{{k_1}^2{k_2}^2}{b^2} = \left(\frac{k_1\, k_2}{b}\right)^2 = k^2$$

$: \dfrac{k_1\, k_2}{b} = k \quad : \quad (k\ is\ a\ positive\ integer, cause\ b\ devides\ both\ \ k_1\ and\ k_2\ )$

$$\Rightarrow \quad ac = k^2 \quad \Rightarrow \quad a\ R\ c \quad \Rightarrow \quad \therefore \quad R \text{ is } transitive\ .$$

$\because \ R$ is $reflexive$, $symmetric$ and $transitive \quad \Rightarrow \therefore \quad R \ \ is\ an\ equivalence\ relation\ on$

(ii)

$$\begin{aligned}[1] &= \{a \in \mathbb{N} : a\ R\ 1\} \\ &= \{a \in \mathbb{N} : a(1) = a = k^2 : k \in \mathbb{N}\} \\ &= \{1,4,9,16,25,\ldots\}\end{aligned}$$

#

27. Let T be the equivalence relation defined on the set $A = \{1,2,3,4\}$, where $\{1,3\}, \{2\}, \{4\}$ are equivalence classes. Represent T in ordered pairs .

28. Let T be the equivalence relation defined on the set $A = \{1,2,3,4,5,6,7,8\}$, where $\Im(A) = \{\{1\}, \{2,3\}, \{4,5,6\}, \{7,8\}\}$. Represent T in ordered pairs .

29. Let R be a relation on $\mathbb{Z}$ such that $a, b \in \mathbb{Z}\, , a\ R\ b\ if\ \ and\ only\ if\ 2|(a^2 + b^2)$

(i) Show that R is an equivalence relation

(ii) Show that $[x] = [-x]$ for all integers x .

(iii) Determine whether $2 \in [-4]$

(iv) Show that $[7] \cap [10] = \emptyset$

30. Let T be a relation on $\mathbb{Z}^* = \mathbb{Z} - \{0\}$ such that

$$a, b \in \mathbb{Z}^*, a\,T\,b \quad if\ and\ only\ if \quad ab > 0$$

(i) Show that T is an equivalence relation

(ii) Find $[1]$ and $[-1]$.

31. Let T be a relation on $\mathbb{Z}$ such that

$$a,b \in \mathbb{Z}\,,\, a\;T\;b \quad if\ and\ only\ if \quad a^2 - b^2 = a - b$$

(i) *Show* that T is an equivalence relation

(ii) Find $[0]$ and $[-1]$.

32. Let $\sim$ be the equivalence relation on $B = \{1,2,3,4,5\}$ such that

$1 \sim 5$, $3 \sim 4$ and $2 \nsim 4$

(i) List all ordered pairs of $\sim$

(ii) Find the (distinct) equivalence classes of $\sim$