King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(3,3)

Methods of Proof

"Mathematical Induction"

(STRONG INDUCTION)

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Strong Induction

STRONG INDUCTION To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we complete two steps:

BASIS STEP: We verify that the proposition $P(1)$ is true.

INDUCTIVE STEP: We show that the conditional statement

$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Exercises

1. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\,,\quad a_2 = 6\quad,\qquad a_n = a_{n-1} + a_{n-2}\ (*)\qquad :\forall n \geq 3$$

Prove that $3|\,a_n$ for all positive integers n , $\forall n \geq 1$

Solution: Let $P(n)$ be the proposition , $P(n)$: $3|\,a_n$, $\Rightarrow a_n = 3c : c \in \mathbb{N}$

BASIS STEP: $When\ n = 1 \Rightarrow 3|\,a_1 \ :\ a_1 = 3 = 3(1) \Rightarrow\ \therefore\ P(1)\ \ is\ true$.

$When\ n = (2)$: $3|\,a_2\ :\ a_2 = 6 = 3(2)\ \Rightarrow\ \therefore\ P(2)\ is\ true$.

INDUCTIVE STEP: Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1)\,, P(k)$ All are true . (**)

$Our\ goal\ is\ to\ show\ that\ P(\,k+1\,)\ is\ also\ true$?

$a_{k+1} = 3c\ \Rightarrow\ 3|\,a_{k+1}$ (our goal) ??

$from\ (*)\ \Rightarrow\qquad a_{k+1} = a_k + a_{k-1}$ (**)

$\because\ P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

from $P(k) \Rightarrow\ 3|\,a_k\ \ \Rightarrow a_k = 3c_1 : c_1 \in \mathbb{N}$

from $P(k-1) \Rightarrow\ 3|\,a_{k-1}\ \ \Rightarrow a_{k-1} = 3c_2\ :\ c_2 \in \mathbb{N}$

$\xRightarrow{by\ subist.into(**)}$

$$a_{k+1} = a_k + a_{k-1} = 3c_1 + 3c_2 = 3(c_1 + c_2) = 3c$$

$$: c = (c_1 + c_2) \in \mathbb{N}$$

$\therefore\ a_{k+1} = 3c\ \Rightarrow\ 3|\,a_{k+1} \Rightarrow \therefore\ \ P(\,k+1\,)$ is true . #

2. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9 \,,\ a_1 = 15 \ ,\ a_n = \frac{a_{n-1}\, a_{n-2}}{3} + 6 \quad : \forall n \geq 2 \qquad (*)$$

Prove that $3|a_n$ for all nonnegative integers n , $\forall n \geq 0$

Solution: **S(1).** Let $P(n)$ be the proposition , $P(n)\!:\ 3|\, a_n \ ,\Rightarrow a_n = 3c : c \in \mathbb{N}$

S(2). *BASIS STEP: When* $= 0 \Rightarrow 3|\ a_0 \ :\ a_0 = 9 = 3(3) \Rightarrow \therefore P(0)$ *is true .*

$When\ n = (1) \Rightarrow \ 3|\ a_1 \ :\ a_1 = 15 = 3(5) \Rightarrow \therefore\ P(1)\ \ is\ true$.

S(3). *INDUCTIVE STEP:* Let $k \geq 1$ and assume that

$P(0), P(1), \ldots, P(k-2), P(k-1)\ , P(k)$ All are true . (**)

Our goal is to show that $P(k+1)$ is also true .

$3|\, a_{k+1} \ \Rightarrow\ a_{k+1} = 3c$ (our goal)??

$$from\ (*) \ \Rightarrow \quad a_{k+1} = \frac{a_k\, a_{k-1}}{3} + 6 \qquad (***)$$

$\because\ P(k)\ \&\ P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

$from\ \ P(k) \Rightarrow\ 3|\, a_k \ \Rightarrow a_k = 3c_1 : c_1 \in \mathbb{N}$

$from\ \ P(k-1) \Rightarrow\ 3|\, a_{k-1} \ \Rightarrow a_{k-1} = 3c_2 \ :\ c_2 \in \mathbb{N}$

$\xRightarrow{by\ subist.into(***)}$

$$a_{k+1} = \frac{a_k\, a_{k-1}}{3} + 6 = \frac{3c_1.\ 3c_2}{3} + 6 = 3(c_1.c_2) + 6$$

$$= 3(c_1.c_2 + 2) = 3M$$

$$: c_1.c_2 + 2 = M \in \mathbb{N}$$

$\because\ a_{k+1} = 3M \ \Rightarrow \therefore\ 3|\, a_{k+1} \Rightarrow \therefore\ \ P(\,k+1\,)$ is true .

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3. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 1\ ,\ u_2 = 2\ \ ,\ \ u_n = 2u_{n-1} - u_{n-2}\quad : \forall n \geq 3 \qquad (*)$$

Prove that $u_n = n$ <u>for all positive integers</u> n , $\forall n \geq 1$

Solution: Let $P(n)$ be the proposition , $P(n)$: $u_n = n$

BASIS STEP: When $n = 1 \Rightarrow$ $u_1 = 1 \Rightarrow \therefore P(1)$ *is true* .

When $n = (2) \Rightarrow$ $u_2 = 2 \Rightarrow \therefore P(2)$ *is true* .

INDUCTIVE STEP: Let $k \geq 2$ and assume that

$P(1), P(2), \ldots, P(k-2), P(k-1), P(k)$ All are true (**)

Our goal is to show that $P(k+1)$ is also true .

$$u_{k+1} = k+1 \qquad \text{(our goal) ??}$$

from (*) $\Rightarrow$ $u_{k+1} = 2u_k - u_{k-1}$ (***)

$\because$ $P(k)$ & $P(k-1)$ both are true, (from inductive hypothesis **) $\Rightarrow$

$from\ \ P(k) \Rightarrow \quad u_k = k$

$from\ \ P(k-1) \Rightarrow \quad u_{k-1} = k-1$

$\xRightarrow{by\ subist.into(***)}$

$$u_{k+1} = 2u_k - u_{k-1} = 2k - (k-1) = 2k - k + 1 = k+1$$

$\because u_{k+1} = k+1 \Rightarrow \therefore \quad P(\,k+1\,)$ is true .

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Exercises

4. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3\ ,\ \ a_2 = 6\ \ ,\ a_n = a_{n-1} + a_{n-2}\ :\forall n \geq 3$$

Prove that $3 | a_n$ for all integers $n \geq 1$.

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5. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 8\ ,\ a_2 = 4\ \ ,\ a_n = a_{n-1} + a_{n-2}\ :\forall n \geq 3$$

Prove that a_n is even for all integers $n \geq 1$.

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6. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9\ ,\ a_1 = 15\ \ ,\ \ a_n = \frac{a_{n-1}\, a_{n-2}}{3} + 6\ \ :\forall n \geq 2$$

Prove that $3|a_n$ for all integers $n \geq 0$

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7. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 1\ ,\ u_2 = 2\ \ ,\ \ u_n = 2u_{n-1} - u_{n-2}\ \ :\forall n \geq 3$$

Prove that $u_n = n$ for all positive integers *n*.

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8. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = -1\ ,\ \ a_2 = -\frac{1}{2}\ ,\ a_3 = -\sqrt{10}\ \ ,\ \ a_{n+1} = a_n . a_{n-1} . a_{n-2}\ \ :\forall n \geq 3$$

Prove that $a_n \leq 0$ for all positive integers *n*.

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9. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\ ,\ a_2 = 5\ \ ,\ \ a_{n+1} = 2a_n + 3\,a_{n-1}\ \ :\forall n \geq 2$$

Prove that $3^n \leq a_{n+1} \leq 2 \,.\, 3^n$ for all positive integers *n*.

10. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \quad a_n = 2a_{n-1} + 1 \ : \forall n \geq 1$$

Prove that $a_n = 2^{n+1} - 1$ for all integers $n \geq 0$

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11. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 3 \ , \ a_2 = 9 \ , \ a_3 = 15 \ , \quad a_{n+1} = a_n + a_{n-1} + a_{n-2} \ : \forall n \geq 3$$

Show that a_n is an integer divisible by 3, for all integers $n \geq 1$

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12. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1 \ , \ a_1 = 1 \ , \quad a_{n+1} = 2a_n + a_{n-1} \ : \forall n \geq 1$$

Show that a_n is odd for all integers $n \geq 0$

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13. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 0 \ , \ a_1 = 4 \ , \quad a_{n+1} = -2a_n + 3\,a_{n-1} \ : \forall n \geq 1$$

Show that $a_n = 1 - (-3)^n$ for all integers $n \geq 0$

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14. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 0 \ , \quad a_1 = 2 \ , \ a_{n+1} = 4a_n - 3a_{n-1} \ : \forall n \geq 1$$

Show that $a_n = 3^n - 1$ for all integers $n \geq 0$

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15. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 9 \ , \ a_1 = 15 \ , \ a_2 = 3 \qquad a_n = \frac{a_{n-1}\,a_{n-2}\,a_{n-3}}{9} + 6 \ : \forall n \geq 3$$

Show that $3|a_n$ for all integers $n \geq 0$

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16. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 1 \ , u_2 = 2 \ , u_3 = 3 \ , \ u_n = 3u_{n-1} - u_{n-2} - u_{n-3} - 2 : \forall n \geq 4$$

Show that $u_n = n$ for all integers $n \geq 1$.

17. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\; a_2 = 2\,, a_3 = 3\;,\quad a_n = \frac{a_{n-1} + a_{n-2} + a_{n-3}}{3} \quad : \forall n \geq 4$$

Show that $1 \leq a_n \leq 3$ for all integers $n \geq 1$

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18. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = \frac{3}{4}\,,\; u_2 = \frac{8}{13}\;,\quad u_n = \frac{3\,u_{n-1} + 2\,u_{n-2} - 3}{3} \quad : \forall n \geq 3$$

Show that $u_n < 1$ for all integers $n \geq 1$

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19. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 2\,,\; u_2 = 4\;,\quad u_n = \frac{2u_{n-1} + u_{n-2} + 8}{3} \quad : \forall n \geq 3$$

Show that $u_n = 2n$ for all positive integers *n*.

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20. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\quad a_1 = 2\;,\; a_2 = 3\;,\quad a_n = a_{n-1} + a_{n-2} + 2a_{n-3} \;: \forall n \geq 3$$

Show that $a_n \leq 3^n$ for all integers $n \geq 0$

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21. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\quad a_1 = 2\;,\quad a_n = 4a_{n-1} - 4a_{n-2} \quad : \forall n \geq 2$$

Prove that $a_n = 2^n$ for all nonnegative integers *n*.

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22. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\quad a_1 = 1\;,\quad a_n = 4a_{n-1} - 4a_{n-2} \quad : \forall n \geq 2$$

Show that $a_n = 2^n - n2^{n-1}$ for all integers $n \geq 0$

23. Assume $\{a_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$a_1 = 1\,,\ a_2 = 3 \quad , \quad a_n = a_{n-1} + a_{n-2} \qquad : \forall n \geq 3$$

Prove that $a_n = (\frac{1+\sqrt{5}}{2})^n + (\frac{1-\sqrt{5}}{2})^n$ for all positive integers *n*.

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24. Assume $\{a_n\}_{n=1}^{\infty}$ is a "Fibonacci" sequence defined as:

$$a_1 = 1\,,\ a_2 = 1 \quad , \quad a_n = a_{n-1} + a_{n-2} \qquad : \forall n \geq 3$$

Prove that $a_n \leq (\frac{1+\sqrt{5}}{2})^n$ for all positive integers *n*.

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25. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 1\,,\ a_1 = 1\ , a_2 = 3 \ , \quad a_n = a_{n-1} + a_{n-2} + a_{n-3} \quad : \forall n \geq 3$$

Show that $a_n < 3^n$ for all integers $n \geq 0$

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26. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 2\,,\ a_1 = 4\ , a_2 = 6 \ , \quad a_n = 5a_{n-3} \quad : n = 3,4,5,\ldots$$

Show that $2|a_n$ for all integers $n \geq 0$

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27. Assume $\{a_n\}_{n=1}^{\infty}$ is a "Fibonacci" sequence defined as:

$$a_1 = 1\,,\ a_2 = 2 \quad , \quad a_n = 2a_{n-1} + a_{n-2} \qquad : \forall n \geq 2$$

Prove that $a_n \leq (\frac{5}{2})^n$ for all positive integers *n*.

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28. Assume $\{u_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$u_0 = 2\,,\ u_1 = 3 \quad , \quad u_{n+1} = 3u_n - 2u_{n-1} - 1 \ : \forall n \geq 1$$

Show that $u_n = n + 2$ for all integers $n \geq 0$

29. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 0\ ,\ u_2 = 1\quad ,\quad u_{n+1} = 3u_n - 2u_{n-1} - 1\ \ for\ \ n = 2,3,4,\ldots$$

Show that $u_n = n - 1$ for all integers $n \geq 1$

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30. Assume $\{u_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$u_0 = 12\ ,\ u_1 = 21\quad ,\quad u_{n+1} = \frac{(u_n)^2\ u_{n-1}}{9}\quad for\quad n = 1,2,3,\ldots$$

Show that u_n is an integer divisible by 3, for all integers $n \geq 0$

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31. Assume $\{u_n\}_{n=1}^{\infty}$ is a sequence defined as:

$$u_1 = 2\ ,\ u_2 = 5\quad ,\quad u_{n+1} = 2u_n - u_{n-1} + 2\quad for\ \ n = 2,3,4,\ldots$$

Show that $u_n = n^2 + 1$ for all integers $n \geq 1$

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32. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 2\ ,\quad a_1 = 4\ ,\ a_{n+1} = 4a_n - 3a_{n-1}\ \ : \forall n \geq 1$$

Show that $a_n = 1 + 3^n$ for all integers $n \geq 0$

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33. Assume $\{a_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$a_0 = 2\ ,\quad a_1 = 5\ ,\ a_{n+1} = 5a_n - 6a_{n-1}\ \ : \forall n \geq 1$$

Show that $a_n = 2^n + 3^n$ for all integers $n \geq 0$

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34. Assume $\{u_n\}_{n=0}^{\infty}$ is a sequence defined as:

$$u_0 = 2\ ,\quad u_1 = 6\ ,\ u_{n+1} = 3u_n + 10u_{n-1} - 12\ \ : \forall n \geq 1$$

Show that $u_n = 5^n + 1$ for all integers $n \geq 0$