King Saud University

College of Science

Department of Mathematics

151 Math Exercises

(4.4)

Properties of Relations

Malek Zein AL-Abidin

$\frac{1441\text{هـ}}{2020}$

DEFINITION 1 Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set T of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,T\,b$ to denote that $(a, b) \in T$ and $a \not T b$ to denote that $(a, b) \notin T$. Moreover, when (a, b) belongs to T, a is said to be **related to** b by T.
Binary relations represent relationships between the elements of two sets.

EXAMPLE 1 Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0T\,a$, but that $1 \not T b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs.

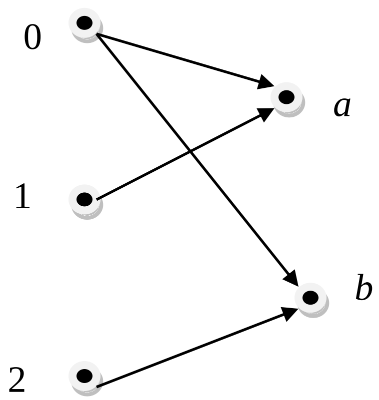


FIGURE 1 **Displaying the Ordered Pairs in the Relation *T***

Relations on a Set

Relations from a set A to itself are of special interest.

DEFINITION 2 A *relation on a set A* is a relation from A to A. In other words, a relation on a set A is a subset of $A \times A$.

EXAMPLE 2 Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation

$$R = \{(a, b) \mid a \text{ divides } b\}?$$

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$.

The pairs in this relation are displayed graphically form in Figure 2.

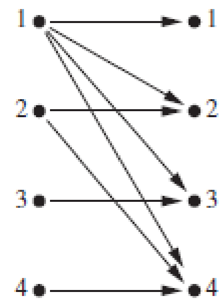
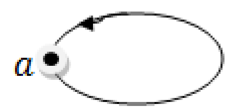


FIGURE 2 **Displaying the Ordered Pairs in the Relation *R* from Example 2.**

Properties of Relations

DEFINITION 3 A relation R on a set A is called *reflexive* if

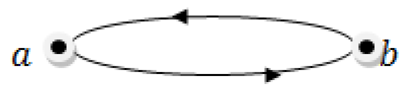
$(a, a) \in R$ for every element $a \in A$. $\forall\, a \in A\ , aRa$



EXAMPLE 3 $R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is *reflexive*

DEFINITION 4 A relation R on a set A is called *symmetric*

if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$. $\forall \mathrm{a}, \mathrm{b} \in \mathrm{A}\ ,\ \mathrm{aRb} \Rightarrow \mathrm{bRa}$



EXAMPLE 4 $R = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$ is *symmetric*

DEFINITION 5 A relation R on a set A such that for all $a, b \in A$,

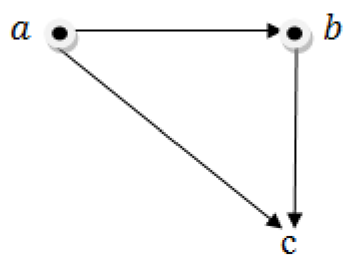
if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

EXAMPLE 5 $a \leq b \ \wedge\ b \leq a \ \Rightarrow\ a = b \ : a, b \in A \quad \therefore \leq$ is *antisymmetric*.

DEFINITION 6 A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

$$aRb \ \& \ bRc \ \Rightarrow \ aRc$$

$$(a,b) \in R \ \& \ (b,c) \in R \ \Rightarrow \ (a,c) \in R \ , \forall \ a,b,c \in A$$



EXAMPLE 6 $a|b \ \wedge \ b|c \ \Rightarrow \ a|c$, $\therefore \ |$ is *transitive* #

Representing Relations Using Digraphs

DEFINITION 10 A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a, b), and the vertex b is called the terminal vertex of this edge.

EXAMPLE 1 Determine whether the relation for the directed graphs shown in Figure 3 is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution: Because there are loops at every vertex of the directed graph of R, it is *reflexive*. R is *neither symmetric* nor antisymmetric because there is an edge from a to b but not one from b to a, but there are edges in both directions connecting b and c. Finally, R is *not transitive* because there is an edge from a to b and an edge from b to c, but no edge from a to c.

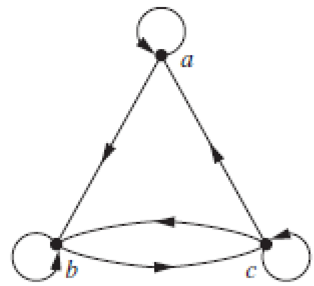


FIGURE 3 **A Directed Graph of the relation R**

#

EXAMPLE 2 Determine whether the relation for the directed graphs shown in Figure 4 is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive.*

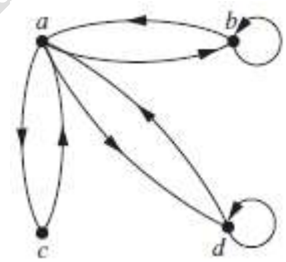


FIGURE 4 **A Directed Graph of the relation R**

Solution: Because loops are not present at all the vertices of the directed graph of *S*, this relation is ***not reflexive.***

It is ***symmetric*** and ***not antisymmetric***, because every edge between distinct vertices is accompanied by an edge in the opposite direction. It is also not hard to see from the directed graph that *S* is ***not transitive***, because (c, a) and (a, b) belong to *S*, but (c, b) does not belong to *S*.

EXERCISES

1. Let $T = \{(a,a),(a,b),(b,b),(c,c)\}$ be a relation defined on the set $A = \{a,b,c\}$. Decide whether T is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering relation . Why?

Solution:

1- $\because\ (a,a),(b,b),(c,c) \in T \Rightarrow \quad T \ \ is\ reflexive$

2- $\because (a,b) \in T \ \wedge \ (b,a) \notin T \Rightarrow \quad T \ is\ not\ symmetric$

3- $\because (a,b) \in T \ \wedge \ (b,a) \notin T \Rightarrow \quad T \ is\ antisymmetric$

3- $\because (a,a) \in T \ \wedge \ (a,b) \in T \ \Rightarrow (a,b) \in T$

& $\because (a,b) \in T \ \wedge \ (b,b) \in T \ \Rightarrow (a,b) \in T \ \Rightarrow \quad T \ \ is\ transitive$

$\because T \ \ is\ reflexive \ ,antisymmetric \ and\ transitive$

$\Rightarrow \therefore T \ \ is\ partial\ ordering\ relation\ .$

#

2. Let $R = \{(a,a),(b,b),(c,c),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$. Decide whether R is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering , totally ordering relation . Why?

Solution:

1- $\because (a,a),(b,b),(c,c),(d,d) \in R \Rightarrow \therefore R$ *is reflexive*

2- $\because (a,a) \wedge (a,a) \in R$ & $(b,b) \wedge (b,b) \in R$

& $(c,c) \wedge (c,c) \in R$ & $(d,d) \wedge (d,d) \in R \Rightarrow \therefore R$ *is symmetric*

3- $\because (a,a) \wedge (a,a) \in R \Rightarrow \therefore a = a$, also same for $(b,b),(c,c),(d,d)$

$\therefore R$ *is antisymmetric*

4- $\because (a,a) \wedge (a,a) \in R \Rightarrow (a,a) \in R$, also same for $(b,b),(c,c),(d,d)$

$\therefore R$ *is transitive*

5- $\because R$ *is reflexive , symmetric and transitive*

$\Rightarrow \therefore R$ *is equivalence relation*

6- $\because R$ *is reflexive , antisymmetric and transitive*

$\Rightarrow \therefore R$ *is partial ordering relation*

7- $\because (a,b) \wedge (b,a) \notin R \Rightarrow a$ *and* b *incomparable*

$\Rightarrow \therefore R$ *is not totally ordering relation*

Finally R *is equivalence relation & partial ordering relation .*

#

3. Let $R = \{(x, x)\}$ be a relation defined on the set $A = \{x\}$.

Decide whether R is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering , totally ordering relation . Why?

4. Let R be a relation defined on the set $\mathbb{Z}^+ = \{1,2,3,\dots\}$

$$m,n \in \mathbb{Z}^+ \ , \qquad m\,R\,n \Leftrightarrow m+n=20$$

Decide whether R is reflexive , symmetric , antisymmetric , transitive , equivalence , partial ordering relation . Why?

Solution :

1- $\because\ 5+5 \neq 20 \Rightarrow (5,5) \notin R \Rightarrow \therefore\ R$ *is not reflexive*

2- $m,n \in \mathbb{Z}^+ \ , \ m\,R\,n \Leftrightarrow m+n=20$

$$\xRightarrow{(\text{commutative})} n+m=20 \Rightarrow \therefore\ n\,R\,m \Rightarrow \therefore\ R \text{ is symmetric}$$

3- $\because\ 7\,R\,13 : 7+13=20 \ \wedge\ 13\,R\,7 : 13+7=20$

but $7 \neq 13 \Rightarrow \therefore\ R$ *is not antisymmetric*.

4- $\because\ 8\,R\,12 : 8+12=20 \ \wedge\ 12\,R\,8 : 12+7=20$

But $(8,8) \notin R : 8+8=16 \neq 20 \Rightarrow \therefore\ R$ *is not transitive*.

Finally , R *is only symmetric* .

#

5. Let $T = \{(a,a),(a,c),(b,b),(b,c),(c,a),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$. Decide whether T is reflexive , symmetric , antisymmetric ,transitive . Why?

6. Let R be a relation defined on the set $A = \{0,1,2,3\}$

$$a, b \in A \quad , \quad a\,R\,b \Leftrightarrow a \leq 2b$$

(i) List all the ordered pairs of R .

(ii) Represent R in a diagram .

(iii) Decide whether R is reflexive , symmetric , antisymmetric , transitive . Why?

7. Let R be a relation defined on the set $\mathbb{Z}^+ = \{1,2,3,\dots\}$

$$m,n \in \mathbb{Z}^+ \quad , \; m\,R\,n \iff 6|m\,n$$

Decide whether R is reflexive , symmetric , antisymmetric , transitive . Why?

8. Suppose T is a relation defined on the integers set $\mathbb{Z}$

$$m, n \in \mathbb{Z}, \quad m\,T\,n \Leftrightarrow m + n \geq 2$$

Decide whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

9. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\} : m\,T\,n \Leftrightarrow m < n$
Decide whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

10. Let R be a relation defined on the set $A = \{1,2,3,4,5\}$

$$x, y \in A, \qquad x\, R\, y \Leftrightarrow xy \leq 9$$

(i) List all the ordered pairs of the relation R ?

(ii) Draw the directed graph (diagraph) that represents R ?

Solution:

11. Suppose R is a relation defined on the set $A = \{-2,-1,0,1,2\}$, as

$$x, y \in A, \quad x\ R\ y \iff |x-y| < 2$$

(i) List all the ordered pairs of the relation R ?

(ii) Draw the directed graph (diagraph) that represents R

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

12. Let R be a relation defined on the set $A = \{0,1,2,3\}$

$$a, b \in A\ , \qquad a\ R\ b \Leftrightarrow a + b = 4$$

(i) List all the ordered pairs of the relation R ?

(ii) Draw the directed graph (diagraph) that represents R ?

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

13. Let R be a relation defined on the set $A = \{2,3,4,5,6\}$

$$a, b \in A , \qquad a\ R\ b \Leftrightarrow a.b < 10$$

(i) List all the ordered pairs of the relation R ?

(ii) Draw the directed graph (diagraph) that represents R ?

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

14. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2, 3, 4\}$

$$a, b \in A, \quad a\,R\,b \Leftrightarrow a^2 = b^2$$

(i) List all the ordered pairs of the relation R ?

(ii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

and/or *transitive*.

Solution:

15. Let R be a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$

$$a, b \in A , \qquad a\ R\ b \Leftrightarrow a.b < 0$$

(i) List all the ordered pairs of the relation R ?

(ii) Draw the directed graph (diagraph) that represents R ?

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

16. Let *R* be a relation defined on the set $A = \{-2, -1, 0, 1, 2\}$

$$a, b \in A, \qquad a\ R\ b \Leftrightarrow a.b \geq 2$$

(i) List all the ordered pairs of the relation *R* ?

(ii) Draw the directed graph (diagraph) that represents *R* ?

(iii) Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

17. Let $S = \{(1,1), (1,2), (1,3), (2,2), (3,1), (3,3)\}$ be a relation on the set $B = \{1 , 2 , 3 \}$

(i) Draw the directed graph (diagraph) that represents S ?

(ii) Determine whether the relation S is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*

Solution:

18. Suppose R is a relation defined on the integers set $\mathbb{Z}^+ = \{1,2,3,\dots\}$

$$m,n \in \mathbb{Z}^+ , \quad m\,R\,n \iff m+n=20$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

1- $\because \; 5+5 \neq 20 \Rightarrow (5,5) \notin R \Rightarrow \therefore \; R \text{ is } irreflexive$.

2- $m\,,n \in \mathbb{Z}^+ \;, \quad m\,R\,n \iff m+n=20$

$$\xRightarrow{\text{comutative}} \qquad n+m=20 \Rightarrow \therefore \; n\,R\,m \Rightarrow \therefore \; R \text{ is } symmetric$$

3- $\because \; 7\,R\,13 \;: 7+13=20 \quad \wedge \quad 13\,R\,7 \;: 13+7=20$

But $7 \neq 13 \Rightarrow \therefore \; R \text{ is not } antisymmetric$

4- $\because \; 8\,R\,12 \;: \; 8+12=20 \quad \wedge \quad 12\,R\,8 \;: \; 12+7=20$

$but\ (8,8) \notin R \;: 8+8=16 \neq 20 \Rightarrow \therefore \; R \text{ is not } transitive.$

$\therefore \; R \text{ is } symmetric$ only .

19. Suppose T is a relation defined on the integers set $\mathbb{Z}$

$m, n \in \mathbb{Z}, \quad m\,T\,n \Leftrightarrow m + n \leq 7$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

20. Let T be a relation defined on the set $\mathbb{N} = \{ 1 , 2 , 3, \ldots \}$,

$$m, n \in \mathbb{N} , \quad m\, T\, n \Leftrightarrow m + n > 3$$

Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

21. Let T be a relation defined on the integers set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$$m, n \in \mathbb{Z}, \quad m\,T\,n \Leftrightarrow m + n \quad is\ odd$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

22. Let R be a relation defined on the integers set $\mathbb{N} = \{\,1,2,3,\ldots\}$

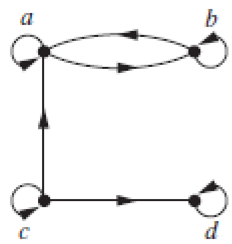
$$x, y \in \mathbb{N}\,, \quad x\,R\,y \;\Leftrightarrow\; x < y$$

Determine whether the relation R is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

23. Determine whether the relation for the directed graphs shown in the Figure is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:



24. Determine whether the relation for the directed graphs shown in the Figure is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution: