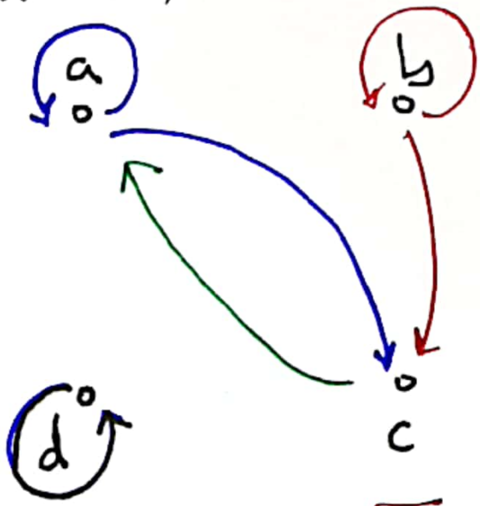
1. Let $T = \{(a,a),(a,c),(b,b),(b,c),(c,a),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$. Decide whether T is reflexive, symmetric, antisymmetric, transitive. (justify your answer)?

<u>Sol:</u>



① $\because (c,c) \notin T \Rightarrow \therefore T$ is not a refl.

② $\because (b,c) \in T$, but $(c,b) \notin T \Rightarrow \therefore T$ is not symm.

③ $\because (a,c) \wedge (c,a) \in T$, but $a \neq c$.

$\Rightarrow \therefore T$ is not antisymm.

④ $\because (b,c) \in T$ and $(c,a) \in T$

but $(b,a) \notin T \Rightarrow \therefore T$ is not transitive.

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2. Let R be a relation defined on the set $\mathbb{Z}^+ = \{1,2,3,\dots\}$ $= \mathbb{N}$

$$m, n \in \mathbb{Z}^+ \ , \ m\,R\,n \Leftrightarrow 6 | m\,n$$

Decide whether R is reflexive, symmetric, antisymmetric, transitive. (justify your answer)?

Sol. $mRn \Rightarrow 6 | m.n \Rightarrow mn = 6K : K \in \mathbb{N} = \mathbb{Z}^+$

① R is not refl. : $5 \not R 5 : 5(5) = 25 \neq 6(K)$

$: 6 \nmid 25$

② $m, n \in \mathbb{N} = \mathbb{Z}^+ : mRn \Rightarrow 6 | mn \Rightarrow 6 | nm$

$m.n = 6K \overset{\text{Symm.}}{\Rightarrow} n.m = 6K \Rightarrow nRm$

$\Rightarrow$ R is Symm.

③ R is not antisymm.

$3R2 ; 6 | (3)(2) \ , 6 | 6$ and $2R3 : 6 | (2)(3) , 6 | 6$

But $2 \neq 3$.

④ R is not transitive :

$3R2 : 6 | (3)(2)$

$\wedge$

$2R3 \quad 6 | (2)(3)$

but $3 \not R 3 : 6 \nmid 9 = (3)(3) \Rightarrow$ Not trans.

③

H.W.

3. Suppose T is a relation defined on the integers set $\mathbb{Z}$

$$m,n \in \mathbb{Z}, \quad m\,T\,n \Leftrightarrow m+n \leq 7$$

Determine whether the relation T is *reflexive*, *symmetric*, *antisymmetric*, and/or *transitive*.

Solution:

4. Let R be the relation defined on the set $\mathbb{N} = \{1,2,3,\ldots\}$, such that:

$a, b \in \mathbb{Z}$, $a\,R\,b \Leftrightarrow 5a \equiv b\ (mod\ 4) \Leftrightarrow 4|(5a - b)$, 4 devides $(5a - b)$

(1) *Show that R is an equivalence relation.*

(2) *Find the equivalence class* [1] .

Sol. $aRb \Leftrightarrow 4|(5a-b) \Rightarrow 5a-b = 4K : K \in \mathbb{Z}$

(i) $\forall a \in \mathbb{N}$, $5a - a = 4a \Rightarrow 4|(5a-a) \Rightarrow aRa$

$\therefore$ R is refl.

(ii) $a, b \in \mathbb{N}$: $aRb \Rightarrow 5a-b = 4K : K \in \mathbb{Z}$

الهدف: $bRa \Rightarrow 5b - a = 4h$?

$b = 5a - 4K \overset{*5}{\Rightarrow} 5b = 25a - 20K$

$\overset{-a}{\Rightarrow} 5b - a = 24a - 20K = 4\underbrace{(6a-5K)}_{h \in \mathbb{Z}}$

$= 4K \Rightarrow \therefore bRa. \Rightarrow R$ is symm.

(iii) $a, b, c \in \mathbb{N}$:

الهدف: $aRc \Rightarrow 5a - c = 4L$?

$aRb \Rightarrow 5a - b = 4K_1$

$\wedge$ $\qquad : K_1, K_2 \in \mathbb{Z}$

$bRc \Rightarrow 5b - c = 4K_2$

$(+)$ $\overline{5a + 4b - c = 4K_1 + 4K_2}$

$5a - c = 4\underbrace{(K_1 + K_2 - b)}_{M \in \mathbb{Z}} = 4M \Rightarrow aRc$

$\therefore$ R is transitive.

$$R \begin{cases} \to \text{refl.} \\ \to \text{symm} \\ \to \text{transitive} \end{cases} \Rightarrow \therefore R \text{ is Equiv. Rel. on } \mathbb{N}.$$

(2)

$$[a] = \{x \in \mathbb{N} : aRx \Rightarrow 5a - x = 4K : K \in \mathbb{Z}\}$$

$$= \{x \in \mathbb{N} : x = 5a - 4k : k \in \mathbb{Z}\}.$$

$$[1] = \{x \in \mathbb{N} : x = 5(1) - 4K = 5 - 4K : k \in \mathbb{Z}\}$$

$$= \{1, 5, 9, 13, 17, \dots\}.$$

5. Let T be the relation defined on the Rational set $\mathbb{Q}$, such that:

$$x, y \in \mathbb{Q}\ ,\ x\,T\,y \Leftrightarrow (x-y) \text{ is even integer}$$

(i) *Show that* T *is an equivalence relation.*

(ii) *Find* $[0]$ *and* $[\frac{1}{2}]$.

Sol. $xTy \Leftrightarrow x-y=2m : m\in\mathbb{Z}$

① $\forall x\in\mathbb{Q}$, $x-x=0=2(0) \Rightarrow \therefore xTx$

$\Rightarrow \therefore T$ is refl.

② $x,y\in\mathbb{Q} : xTy \Rightarrow x-y=2m : m\in\mathbb{Z}$ | الهدف $yTx = y-x=2k$?

$*(-1) \Rightarrow y-x=2(-m)=2k : -m=k\in\mathbb{Z}$

$\therefore\ yTx \Rightarrow T$ is symm.

③ $x,y,z \in\mathbb{Q} : xTy \Rightarrow x-\cancel{y}=2m_1$ | الهدف $xTz \Rightarrow x-z=2h$?

$: m_1, m_2\in\mathbb{Z}$

$\wedge\ yTz \Rightarrow \cancel{y}-z=2m_2$

$(+)$

$\Rightarrow x-z=2\underbrace{(m_1+m_2)}_{h\in\mathbb{Z}}=2h \Rightarrow \therefore xTz$

$\therefore T$ is transitive.

T → refl, symm, transitive $\Rightarrow \therefore T$ is Equiv. Rel. on $\mathbb{Q}$

(ii) $[a] = \{x \in \mathbb{Q} : x \top a \Rightarrow x - a = 2m : m \in \mathbb{Z}\}$

$[a] = \{x \in \mathbb{Q} : x = a + 2m : m \in \mathbb{Z}\}$

$[0] = \{x \in \mathbb{Q} : x = 0 + 2m = 2m : m \in \mathbb{Z}\}$

$= \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$

$[\frac{1}{2}] = \{x \in \mathbb{Q} : x = \frac{1}{2} + 2m : m \in \mathbb{Z}\}$

$= \{\ldots, \frac{-5}{2}, \frac{-3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots\}$

H.W.

6. Let R be the relation defined on the integers set $\mathbb{Z}$, such that:

$$x, y \in \mathbb{Z}, \quad x\,R\,y \Leftrightarrow 4|(3x+y) \quad , \quad 4 \text{ devides } (3x+y)$$

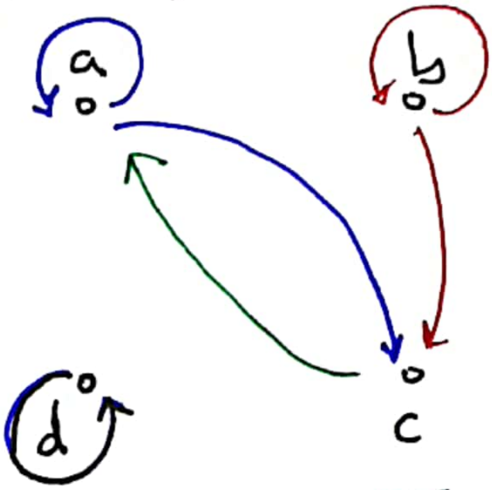
(*i*) *Show that R is an equivalence relation.*

(*ii*) *Find* [0] , [1] .

(*iii*) Determine whether $-2 \in [6]$

1. Let $T = \{(a,a),(a,c),(b,b),(b,c),(c,a),(d,d)\}$ be a relation defined on the set $A = \{a,b,c,d\}$. Decide whether T is reflexive, symmetric, antisymmetric, transitive. (justify your answer)?

Sol.



① $\because (c,c) \notin T \Rightarrow$
$\therefore T$ is not a refl.

② $\because (b,c) \in T$, but $(c,b) \notin T \Rightarrow \therefore T$ is not symm.

③ $\because (a,c) \wedge (c,a) \in T$, but $a \neq c$.
$\Rightarrow \therefore T$ is not antisymm.

④ $\because (b,c) \in T$ and $(c,a) \in T$
but $(b,a) \notin T \Rightarrow \therefore T$ is not transitive.

7. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$,

$$x, y \in \mathbb{N}\ ,\ x\,T\,y \Leftrightarrow \frac{x}{y}\ is\ odd$$

✓(i) Show that T is a partial order relation on $\mathbb{N}$.

✓(ii) Decide whether T is total ordering relation on $\mathbb{N}$, why?

(iii) Draw the Hasse diagram representing the partial order relation T on the set $A = \{1,2,3,4,5,6,9,10,12\}$.

Sol. (i) ① $\forall x \in \mathbb{N},\ \frac{x}{x} = 1$ is odd $\Rightarrow \therefore xTx$

$\therefore T$ is refl.

② $x, y \in \mathbb{N}: xTy \Rightarrow \frac{x}{y} = m_1$ (odd) $\in \mathbb{N}$.

$\wedge$

$yTx \Rightarrow \frac{y}{x} = m_2$ (odd) $\in \mathbb{N}$

(*)

$$\frac{x}{y} \cdot \frac{y}{x} = 1 = m_1 \cdot m_2 \Rightarrow \therefore m_1 = m_2 = 1$$

So $\Rightarrow \frac{x}{y} = 1 \Rightarrow \boxed{x = y} \Rightarrow T$ is antisymm.

③ $x, y, z \in \mathbb{N}: xTy \Rightarrow \frac{x}{y} = K_1$ (odd) $\in \mathbb{N}$

$\wedge$

$yTz \Rightarrow \frac{y}{z} = K_2$ (odd) $\in \mathbb{N}$

(*)

$$\frac{x}{\cancel{y}} \cdot \frac{\cancel{y}}{z} = \frac{x}{z} = K_1 K_2 = K \text{ (odd)} \in \mathbb{N} \Rightarrow \therefore xTz$$

$\Rightarrow \therefore T$ is transitive.

$T \to$ refl, antisymm., transitive $\Big\} \Rightarrow T$ is Partial order Rel. on $\mathbb{N}$

(ii) $2, 3 \in \mathbb{N}$, $2 \nmid 3$: $\frac{2}{3} \notin \mathbb{N}$ (and not odd)

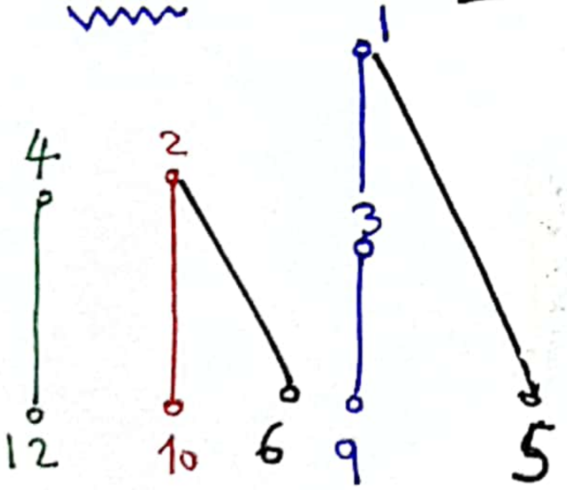
: $2 \neq 3$ $\quad \wedge$

$3 \nmid 2$: $\frac{3}{2} \notin \mathbb{N}$ (and " ").

$\therefore$ 2, and 3 incomparable.

$\Rightarrow \therefore T$ is not a total order.

(iii) $T = \{(1,1), (2,2), \ldots, (12,12), (12,4), (10,2), (9,3)$
$(9,1), (6,2), (5,1), (3,1)\}$



Hasse Diagram.

8. Let $T = \{(x,x), (x,z), (y,x), (y,y), (y,z), (z,z)\}$ be a relation defined on the set $B = \{x, y, z\}$

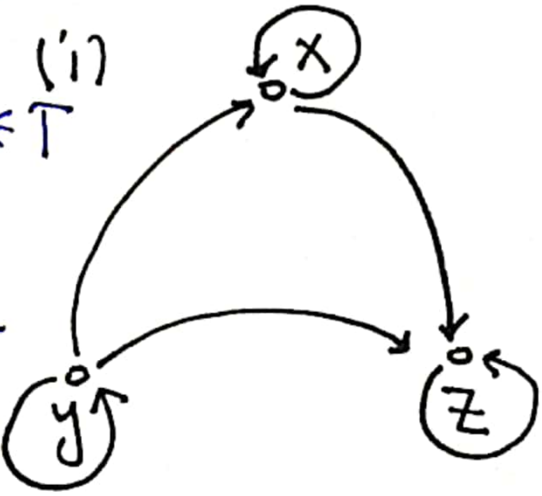
(i) Represent the relation T by diagram .

(ii) Show that T is a partial order relation on B .

(iii) Decide whether T is total order relation on A. why?

(iv) Draw the Hasse diagram representing the partial order relation T on the set B .

(i)



(ii) ① $(x,x), (y,y)$ and $(z,z) \in T$

$\therefore$ T is refl.

② $(x,z) \in T$, but $(z,x) \notin T$

$(y,z) \in T$, but $(z,y) \notin T$

and $(y,x) \in T$, but $(x,y) \notin T$

$\therefore$ T is antisymm.

③ $(y,x) \in T$ and $(x,z) \in T$ $\overset{\text{also}}{\Rightarrow}$ $(y,z) \in T \Rightarrow T$ is transitive.

①, ② and ③ $\Rightarrow$ T is Partial order Rel. on B .

(iii) $\because (y,x) \in T,\ (x,z) \in T$

and $(y,z) \in T$: $x \neq y \neq z$

$\Rightarrow (B,T)$ Comparable.

$\Rightarrow \therefore T$ is a **total order**.

(iv)

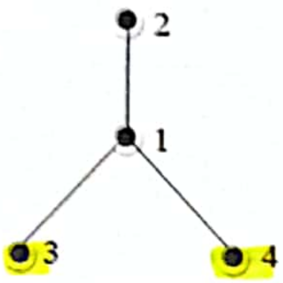
Hasse Diagram
(chain).

9. Let T be a partial ordering relation defined on the set $A = \{1,2,3,4\}$

shown in the given Hasse diagram

√(i) List all ordered pairs of T.

(ii) Decide whether T is totally ordering relation on A. why?



(i)

$T = \{(1,1),(2,2),(3,3),(4,4),(3,1),(1,2),(3,2),(4,1),(4,2)\}$

(ii) $3 \neq 4$, $(3,4) \notin T$ and $(4,3) \notin T$ } 3, 4 are incomparable

$\Downarrow$

$\therefore$ T is not a total order.

10. Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots, -2, -1, 1, 2, \dots\} = \mathbb{Z} - \{0\}$

$$x, y \in \mathbb{Z}^*, \quad x T y \Leftrightarrow x = y^{2k+1} \quad : k \in \{0,1,2,\dots\} = \mathbb{W}$$

√ (i) Show that T is a partial order relation on $\mathbb{Z}^*$.

(ii) Decide whether T is total order relation on $\mathbb{Z}^*$. why?

(iii) Draw the Hasse diagram representing the partial order relation T on the set $A = \{1,2,3,4,8,27\}$.

Sol. (i) ① $\forall x \in \mathbb{Z}^*$, $x = x^{2(0)+1} \Rightarrow \therefore xTx \Rightarrow T$ is refl.

② $x, y \in \mathbb{Z}^*$: $xTy \Rightarrow x = y^{2K_1+1}$ ⊛ $: K_1, K_2 \in \{0,1,2,\dots\}$

$\wedge$

$yTx \Rightarrow y = x^{2K_2+1}$ (⊛⊛)

هدفي:

$x = y$?

$\Rightarrow K_1 = K_2 = 0$?

By subst. ⊛ into (⊛⊛) $\Rightarrow y = \left(y^{2K_1+1}\right)^{(2K_2+1)}$

$$y^1 = y^{(2K_1+1)(2K_2+1)} \Rightarrow$$

$$(2K_1+1)(2K_2+1) = 1 \Rightarrow$$

$\therefore 2K_1 + 1 = 1$ and $2K_2 + 1 = 1$

$2K_1 = 0 \Rightarrow \boxed{K_1 = 0}$ and $2K_2 = 0 \Rightarrow \boxed{K_2 = 0}$

from (⊛) $\Rightarrow x = y^{2(0)+1} = y^1 = y \Rightarrow \therefore \boxed{x = y}$

$\therefore T$ is antisymm.

3. $x, y, z \in \mathbb{Z}^*$:

$xTy \Rightarrow x = y^{2h_1+1}$: $h_1, h_2 \in \{0, 1, 2, \ldots\}$

$\wedge$

$yTz \Rightarrow y = z^{2h_2+1}$

الهدف :

$xTz \Rightarrow x = z^{2h+1}$?

$(a^b)^c = a^{bc}$

By subst. $\Rightarrow x = \left(z^{2h_2+1}\right)^{(2h_1+1)}$

$\Rightarrow x = z^{(2h_2+1)(2h_1+1)}$

$= z^{4h_1h_2+2h_1+2h_2+1} = z^{2\overbrace{(2h_1h_2+h_1+h_2)}^{= h \in \{0,1,2,\ldots\}}+1}$

$\Rightarrow \therefore x = z^{2h+1} \Rightarrow xTz$.

: $2h_1h_2+h_1+h_2 = h \in \{0, 1, 2, \ldots\}$

$\therefore T$ is transitive.

$T \Rightarrow$ refl, antisymm, Transitive $\Big\} \Rightarrow \therefore T$ is a Partial order rel. on $\mathbb{Z}^*$.

(ii) $3, 5 \in \mathbb{Z}^*$, $3 \neq 5^{2K+1}$, $3 \not T 5$

$3 \neq 5$ and $5 \neq 3^{2K+1}$, and $5 \not T 3$

$: K \in \{0, 1, 2, \ldots\}$.

3, 5 incomparable.

$\Downarrow$

$\therefore T$ is not a total order.

(iii) $A = \{1, 2, 3, 4, 8, 27\}$.

$T = \{(1,1), (2,2), (3,3), (4,4), (8,8), (27,27),$
$(27, 3), (8, 2)\}$

Hasse Diagram.

11. Let R be a relation defined on the set $\mathbb{Z}^+$: $a,b \in \mathbb{Z}^+$, $a\,R\,b \Leftrightarrow a^2 \mid b^2$

(i) Show that R is a partial order relation on $\mathbb{Z}^+$.

(ii) In case R is defined on $\mathbb{Z}$, decide whether R a partial order relation on $\mathbb{Z}$, why?

(iii) Draw the Hasse diagram representing the partial order relation R on the set
$A = \{2,3,4,6,8,9\}$

(iv) Decide whether R is total order relation on $\mathbb{Z}^+$, why?

Solution: (i)

1- $\forall a \in \mathbb{Z}^+,\ a^2 \mid a^2 \Rightarrow a\,R\,a \Rightarrow R$ is reflexive

2- $a,b \in \mathbb{Z}^+$, $a\,R\,b \Leftrightarrow a^2 \mid b^2 \Rightarrow b^2 = ma^2 \quad : m \in \mathbb{Z}^+$

$\wedge$

$b\,R\,a \Leftrightarrow b^2 \mid a^2 \Rightarrow a^2 = nb^2 \quad : n \in \mathbb{Z}^+$

$b^2 = mnb^2 \Rightarrow mn = 1 \Rightarrow m = n = 1$

$a^2 = b^2 \overset{\sqrt{\ }}{\Longrightarrow} a = b \Rightarrow R$ is antisymmetric

3- $a,b,c \in \mathbb{Z}^+$, $a\,R\,b \Leftrightarrow a^2 \mid b^2 \Rightarrow b^2 = ma^2 : m \in \mathbb{Z}^+$

$\wedge$

$b\,R\,c \Leftrightarrow b^2 \mid c^2 \Rightarrow c^2 = nb^2 : n \in \mathbb{Z}^+$

$c^2 = mna^2 \Rightarrow a^2 \mid c^2 \Rightarrow a\,R\,c \Rightarrow R$ is transitive

$\because$ R is reflexive, antisymmetric and transitive

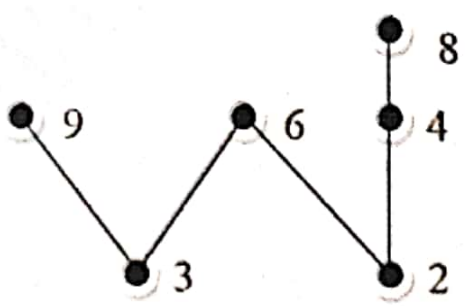
$\therefore$ R is partial ordering relation on $\mathbb{Z}^+$.

(ii) $-2, 2 \in \mathbb{Z}$

$-2\,R\,2 : (-2)^2 \mid 2^2 \wedge -2\,R\,2 : 2^2 \mid (-2)^2$ but $-2 \neq 2 \Rightarrow R$ is not antisymmetric

$\therefore$ R is not partial ordering relation on $\mathbb{Z}$

(iii) $R = \{(2,2), \dots, (9,9), (2,4), (2,6), (2,8), (3,6), (3,9), (4,8)\}$



$$2,3 \in \mathbb{Z}^+ , 2^2 \nmid 3^2 \wedge 3^2 \nmid 2^2 \Rightarrow \therefore 2,3 \text{ incomparable}$$

$$\Rightarrow \therefore T \text{ is not total ordering relation}$$

H.W:

12. Let $T = \{(1,1),(1,3),(2,1),(2,2),(2,3),(3,3),(4,1),(4,3),(4,4)\}$ be a relation defined on the set

$A = \{1,2,3,4\}$

(v) Represent the relation T by diagram .

(vi) Show that T is a partial order relation on A .

(vii) Decide whether T is total order relation on A. why?

(viii) Draw the Hasse diagram representing the partial order relation T on the set A .