4.3

1. R Partial order on A.

   1. R is refl: $\forall x \in A \ni xRx$
   2. R is antisymm: $a, b \in A$: $\left.\begin{array}{l} aRb \\ \wedge \\ bRa \end{array}\right\} \Rightarrow a = b$
   3. R is transitive: $a, b, c \in A$: $\left.\begin{array}{l} aRb \\ \wedge \\ bRc \end{array}\right\} \Rightarrow aRc$

2. R is a total order

   (i) R is Partial order

   (ii) A is comparable $\Downarrow$ $a, b \in A$; $a \neq b$: $aRb \vee bRa$

1. Let R be a relation defined on the set $\mathbb{Z}^+ = \mathbb{N} = \{1,2,3 \dots\} : a,b \in \mathbb{Z}^+ = \mathbb{N}$, $a\,R\,b \Leftrightarrow a \mid b$

✓ (i) Show that R is a partial order relation on $\mathbb{N}$.

✓ (ii) In case R is defined on $\mathbb{Z}^*$, is R still a partial order relation on $\mathbb{Z}^*$, why? : $\mathbb{Z}^* = \mathbb{Z} - \{0\}$

✓ (iii) Draw the Hasse diagram representing the partial order relation R on the set $A = \{1,2,3,4,5,6,7,8,9,10,11,12\}$

✓ (iv) Decide whether R is total order relation on $\mathbb{Z}^+ = \mathbb{N}$, why?

✓ (v) In case R is defined on $B = \{1,3,9,27,81\}$, is R a total order relation on B, why?

✓ (vi) Draw the Hasse diagram representing the total order relation R on the set $B = \{1,3,9,27,81\}$

Sol. $a,b \in \mathbb{Z}^+$: $aRb \Leftrightarrow a \mid b \Rightarrow b = ma : m \in \mathbb{Z}^+$

(i) ① $\forall a \in \mathbb{Z}^+$, $a \mid a \Rightarrow \therefore aRa \Rightarrow \therefore$ R is refl.

② $a,b \in \mathbb{Z}^+$: $aRb \Rightarrow a \mid b \Rightarrow b = m_1 a$ $: m_1, m_2 \in \mathbb{Z}^+$

$\wedge$

$bRa \Rightarrow b \mid a \Rightarrow a = m_2 b$

$\Rightarrow a = m_2 (m_1 a) = m_1 . m_2\, a \Rightarrow m_1 . m_2 = 1$

$\Rightarrow \therefore m_1 = m_2 = 1 \Rightarrow \therefore \boxed{a = b}$

$\therefore$ R is antisymm.

③ $a,b,c \in \mathbb{Z}^+$: $aRb \Rightarrow a \mid b \Rightarrow b = k_1 a$ $: k_1, k_2 \in \mathbb{Z}$

$\wedge$

$bRc \Rightarrow b \mid c \Rightarrow c = k_2 b$

$\Rightarrow c = k_2 (k_1 a) = k_1 . k_2\, a = ka \Rightarrow \therefore a \mid c$

$\Rightarrow aRc$: $k_1 . k_2 = k \in \mathbb{Z}^+ \Rightarrow$ R is transitive.

1 ∧ 2 & 3 $\Rightarrow R$ is Partial order on $\mathbb{Z}^+ = \mathbb{N}$.

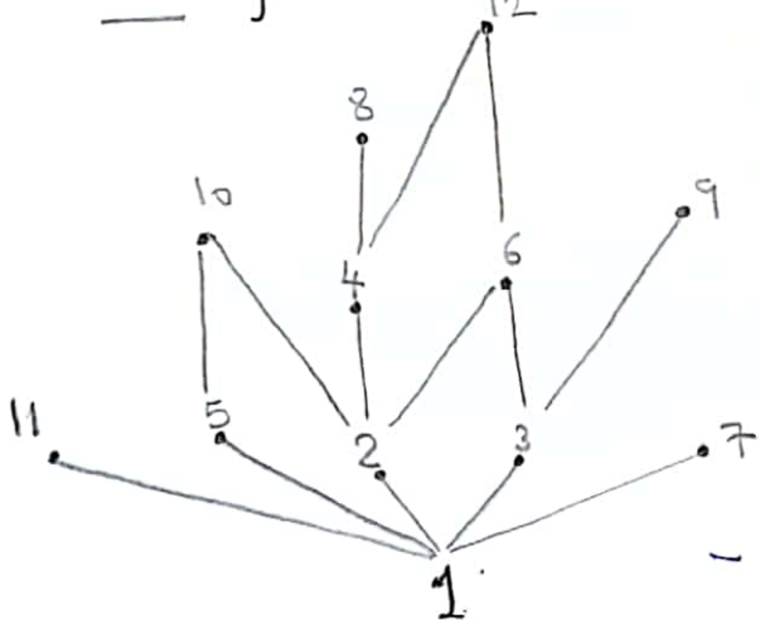
(ii) If R defined on $\mathbb{Z}^* \Rightarrow$.

$$\left.\begin{array}{l} -1R1 \; : \; -1 \mid 1 \\ \wedge \\ 1R-1 \; : \; 1 \mid -1 \end{array}\right\} \text{ but } -1 \neq 1$$

$\Rightarrow \therefore R$ is not antisymm.

$\therefore$ R is not a Partial order on $\mathbb{Z}^*$

(iii) $R = \{(1,1),(2,2),(3,3),\ldots,(12,12),(1,2),(1,3),(1,4)$
$(1,5),(1,6),(1,7),(1,8),(1,9),(1,10),(1,11),(1,12)$
$(2,4),(2,6),(2,8),(2,10),(2,12)$
$(3,6),(3,9),(3,12),(4,8),(4,12),(5,10)$
$(6,12)\}$



– Hasse Diagram –

(iv) $3, 5 \in \mathbb{N}$, $3 \nmid 5$, $3 \not{R} 5$

$3 \neq 5$

and $5 \nmid 3$, $5 \not{R} 3$

$\therefore$ 3,5 incomparable $\Rightarrow \therefore$ R is not a total order on $\mathbb{N}$.

(v) $B = \{ \underset{3^0}{1}, \underset{3^1}{3}, \underset{3^2}{9}, \underset{3^3}{27}, \underset{3^4}{81} \}$ (Divisiors of 81)

$\because$ all elments of B are the powers of 3.

$\Rightarrow \therefore \forall\, a, b \in B,\ a \neq b \Rightarrow a \mid b$ or $b \mid a$.

$\therefore$ R is Partial order and B is comparable

$\Rightarrow \therefore$ R is a total order on B

$R = \{(1,1), (3,3), (9,9), (27,27), (81,81), (1,3), (1,9), (1,27), (1,81), (3,9), (3,27), (3,81), (9,27), (9,81), (27,81)\}$

(chain)
Hasse Diagram

2. Let R be a relation defined on the set $\mathbb{Q}^+$:

$$a, b \in \mathbb{Q}^+ \quad , a\,R\,b \Leftrightarrow \frac{a}{b} \in \mathbb{Z}^+$$

✓(i) Show that R is a partial order relation on $\mathbb{Q}^+$.

✓(ii) Decide whether R is total order relation on $\mathbb{Q}^+$, why?

(iii) Draw the Hasse diagram representing the partial order relation R on the set

$$A = \{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1,2,3,6\}$$

Sol. (i) ① $\forall a \in \mathbb{Q}^+$, $\frac{a}{a} = 1 \in \mathbb{Z}^+ \Rightarrow \therefore aRa \Rightarrow$ R is refl.

② $a, b \in \mathbb{Q}^+$: $aRb \Rightarrow \frac{a}{b} = m_1 \in \mathbb{Z}^+$

$\wedge$

$bRa \Rightarrow \frac{b}{a} = m_2 \in \mathbb{Z}^+$.

(*)

$\frac{a}{b} \cdot \frac{b}{a} = 1 = m_1 \cdot m_2 \Rightarrow \therefore m_1 = m_2 = 1$

$\Rightarrow \therefore \frac{a}{b} = 1 \Rightarrow \therefore \boxed{a = b} \Rightarrow$ R is antisymm.

③ $a, b, c \in \mathbb{Q}^+$: $aRb \Rightarrow \frac{a}{b} = k_1 \in \mathbb{Z}^+$

$\wedge$

$bRc \Rightarrow \frac{b}{c} = k_2 \in \mathbb{Z}^+$

(*)

$\frac{a}{\cancel{b}} \cdot \frac{\cancel{b}}{c} = \frac{a}{c} = k_1 k_2 = k \in \mathbb{Z}^+$

$\Rightarrow \therefore aRc \Rightarrow$ R is transitive .

① $\wedge$ ② & ③ $\Rightarrow \therefore$ R is a Partial order on $\mathbb{Q}^+$

(ii) $5, 7 \in \mathbb{Q}^+$, $\frac{5}{7} \notin \mathbb{Z}^+ \Rightarrow \therefore 5 \not R 7$

$5 \neq 7$

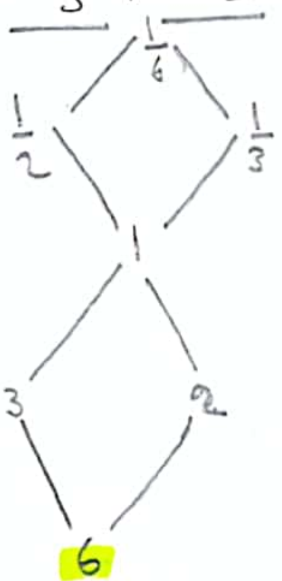
and

$\frac{7}{5} \notin \mathbb{Z}^+ \Rightarrow \therefore 7 \not R 5$.

$\therefore$ 5, 7 incomparable $\Rightarrow \therefore R$ is not a total order on $\mathbb{Q}^+$

(iii) $A = \left\{\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 6\right\}$.

$R = \{(\frac{1}{6}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (1,1), (2,2), (3,3), (6,6),$
$(6, \frac{1}{6}), (6, \frac{1}{3}), (6, \frac{1}{2}), (6,1), (6,2), (6,3),$
$(3, \frac{1}{6}), (3, \frac{1}{3}), (3, \frac{1}{2}), (3,1), (2, \frac{1}{6}), (2, \frac{1}{3}), (2, \frac{1}{2})$
$(2,1), (1, \frac{1}{6}), (1, \frac{1}{3}), (1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{6}), (\frac{1}{3}, \frac{1}{6})\}$



Hasse diagram.

3. Let R be a relation defined on the set $\mathbb{N} = \{1,2,3,\ldots\}$:

$$a,b \in \mathbb{N}\,,\quad a\,R\,b \Leftrightarrow \frac{b}{a} = 2^k \;:\; k \in \{0,1,2,\ldots\}$$

✓(i) Show that R is a partial ordering relation on $\mathbb{N}$.

✓(ii) Decide whether R is totally ordering relation on $\mathbb{N}$, why?

✓(iii) Draw the Hasse diagram representing the partial ordering relation R on the set $A = \{1,2,3,\ldots,12\}$

Sol. (i) ① $\forall\, a \in \mathbb{N}\,,\ \frac{a}{a} = 1 = 2^0 \Rightarrow \therefore aRa \Rightarrow$ R is refl.

② $a,b \in \mathbb{N} : aRb \Rightarrow \frac{b}{a} = 2^{k_1}$

$\wedge \qquad\qquad\qquad\qquad : k_1, k_2 \in \{0,1,2,\ldots\}$

$bRa \Rightarrow \frac{a}{b} = 2^{k_2}$

(*) $\overline{\frac{b}{a}\cdot\frac{a}{b} = \underset{\underset{2^0}{\shortparallel}}{1} = 2^{k_1}\cdot 2^{k_2} = 2^{k_1+k_2}}$

$\Rightarrow k_1 + k_2 = 0 \Rightarrow \boxed{k_1 = k_2 = 0} \Rightarrow \frac{a}{b} = 2^0 = 1$

$\therefore \boxed{a = b} \Rightarrow$ R is antisymm.

③ $a,b,c \in \mathbb{N} ; aRb \Rightarrow \frac{b}{a} = 2^{h_1}$

$\wedge \qquad\qquad\qquad\qquad : h_1, h_2 \in \{0,1,2,\ldots\}$

$bRc \Rightarrow \frac{c}{b} = 2^{h_2}$

(*) $\overline{\frac{\cancel{b}}{a}\cdot\frac{c}{\cancel{b}} = \frac{c}{a} = 2^{h_1+h_2} = 2^{h} \Rightarrow aRc}$

$; h_1 + h_2 = h \in \{0, 1, 2, \ldots\}$

$\Rightarrow \therefore$ R is transitive.

1 & 2 & 3 $\Rightarrow \therefore$ R is a Partial order on N.

(ii) $7, 9 \in \mathbb{N}$ $\quad \frac{7}{9} \neq 2^K \Rightarrow 7 \not R 9$

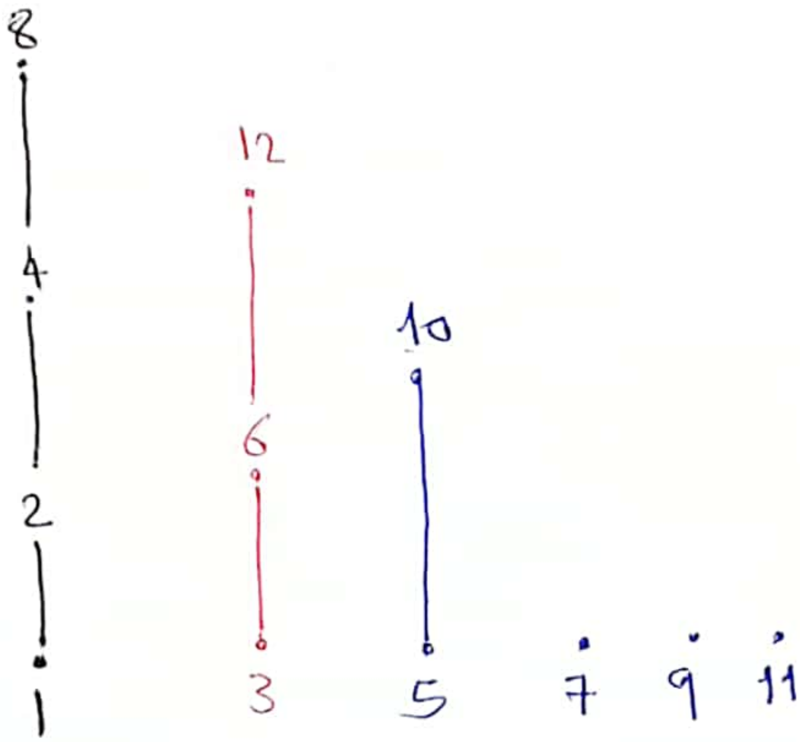
$7 \neq 9$ $\quad ; K \in \{0, 1, 2, \ldots\}$

and $\frac{9}{7} \neq 2^K \Rightarrow 9 \not R 7$

$\therefore$ 7, 9 are incomparable $\Rightarrow$ R is not a total order on $\mathbb{N}$.

(iii) $A = \{1, 2, 3, \ldots, 12\}$.

$R = \{(1,1), (2,2) \ldots, (12,12), (1,2), (1,4), (1,8)$
$(2,4), (2,8), (3,6), (3,12), (4,8), (5,10)$
$(6,12)\}$



Hasse diagram.

4. Let T be a relation defined on the set $\mathbb{Z}$:

$$x,y \in \mathbb{Z}\ ,\quad x\,T\,y \Leftrightarrow x-y=2k \qquad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial ordering relation on $\mathbb{Z}$.

(ii) Decide whether T is totally ordering relation on $\mathbb{Z}$. why?

(iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A=\{0,1,2,3\}$.

Solution: (i) ① $\forall\, x \in \mathbb{Z}$, $x-x=0=2(0) \Rightarrow \therefore xTx \Rightarrow$ T is refl.

② $x,y \in \mathbb{Z}$: $xTy \Rightarrow x-y=2k_1$

$\wedge$: $k_1, k_2 \in \{0,1,2,\dots\} = \mathbb{W}$

$yTx \Rightarrow y-x=2k_2$

$(+)$ ———————————————

$0 = 2(k_1+k_2) \Rightarrow k_1+k_2=0$

$\Rightarrow \therefore k_1=k_2=0 \Rightarrow \therefore x-y=0 \Rightarrow \boxed{x=y}$

$\therefore$ T is antisymm.

③ $x,y,z \in \mathbb{Z}$: $xTy \Rightarrow x-\cancel{y}=2h_1$: $h_1, h_2 \in \{0,1,2,\dots\}$

$\wedge$

$yTz \Rightarrow \cancel{y}-z=2h_2$

$(+)$ ———————————————

$x-z=2(h_1+h_2)=2h$

$\therefore xTz$: $h_1+h_2=h \in \{0,1,2,\dots\}$

$\therefore$ T is transitive.

1 $\wedge$ 2 & 3 $\Rightarrow$ $\therefore$ T is a Partial order on $\mathbb{Z}$.

(ii) $-2, 3 \in \mathbb{Z}$ $\quad -2 \not{T}\, 3 : -2-3 = \underbrace{-5}_{odd} \neq 2K$ $\; ; K \in \{0,1,2,\dots\}$

$\wedge$

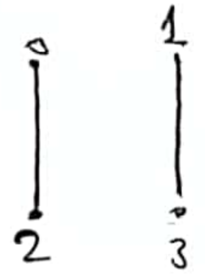
$3 \not{T} -2 : \overbrace{3+2}^{3-(-2)=} = \underbrace{5}_{odd} \notin 2K$

$\therefore$ $-2, 3$ incomparable $\Rightarrow$

$\therefore$ T is not a total order on $\mathbb{Z}$.

(iii) $A = \{0, 1, 2, 3\}$.

$T = \{(0,0), (1,1), (2,2), (3,3), (2,0), (3,1)\}$



Hasse diagram.

H.W
Same #3

5. Let T be a relation defined on the set $\mathbb{Z}^* = \{\dots,-2,-1,1,2,\dots\}$:

$$a,b \in \mathbb{Z}^*, a\,T\,b \Leftrightarrow \frac{a}{b} = 3^k \quad : k \in \{0,1,2,\dots\}$$

(i) Show that T is a partial order relation on $\mathbb{Z}^*$.

(ii) Decide whether T is total order relation on $\mathbb{Z}^*$, why?

(iii) Draw the Hasse diagram representing the partial order relation T on the set $A = \{-27,-18,-9,-6,-3,1,2,3,6,9\}$.

Solution :

6. Let T be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$,

$$x, y \in \mathbb{N}, \quad x\,T\,y \Leftrightarrow x = y^k \quad : k \in \{0,1,2,\dots\}$$

✓ (i) Show that T is a partial ordering relation on $\mathbb{N}$.

✓ (ii) Decide whether T is totally ordering relation on $\mathbb{N}$. why?

✓ (iii) Draw the Hasse diagram representing the partial ordering relation T on the set $A = \{1,2,3,4\}$.

Solution : (i) ① $\forall x \in \mathbb{N},\ x = x^1 \Rightarrow \therefore x\,T\,x \Rightarrow$ T is refl.

② $x, y \in \mathbb{N}$: $x\,T\,y \Rightarrow x = y^{k_1}$ $\quad : k_1, k_2 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,x \Rightarrow y = x^{k_2}$

$y^1 = (y^{k_1})^{k} = y^{k_1 k_2} \Rightarrow k_1 \cdot k_2 = 1$.

$\Rightarrow \therefore k_1 = k_2 = \boxed{1} \Rightarrow \boxed{x = y}$

$\therefore$ T is antisymm.

③ $x, y, z \in \mathbb{N}$: $x\,T\,y \Rightarrow x = y^{h_1}$ $\quad : h_1, h_2 \in \{0,1,2,\dots\}$

$\wedge$

$y\,T\,z \Rightarrow y = z^{h_2}$

$\Rightarrow x = (z^{h_2})^{h_1} = z^{h_1 \cdot h_2} = z^{h}$ $\quad : h_1 \cdot h_2 = h \in \{0,1,2,\dots\}$

$\Rightarrow x\,T\,z \Rightarrow \therefore$ T is transitive.

① & ② & ③ $\Rightarrow \therefore$ T is a Partial order on $\mathbb{N}$.

(ii) $3,5 \in \mathbb{N}$, $3 \neq 5^K$ and $\nexists K \in \{0,1,2\ldots\}$ $\Rightarrow 3 \not T 5$

$3 \neq 5$

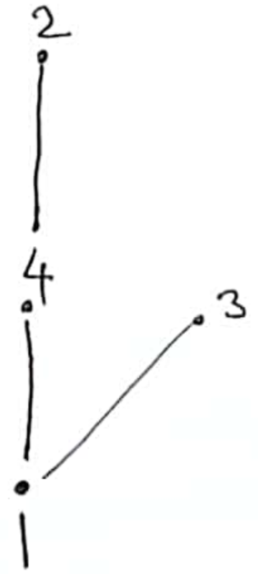
$5 \neq 3^K \Rightarrow 5 \not T 3$.

$\therefore$ 3,5 incomparable.

$\Rightarrow \therefore T$ is not a total order on $\mathbb{N}$.

(iii) $A = \{1,2,3,4\}$.

$T = \{(1,1),(2,2),(3,3),(4,4),(4,2),(1,2),(1,3),(1,4)\}$



Hasse diagram.

7. Let $T = \{(x,x),(x,z),(y,x),(y,y),(y,z),(z,z)\}$ be a relation defined on the set $B = \{x,y,z\}$

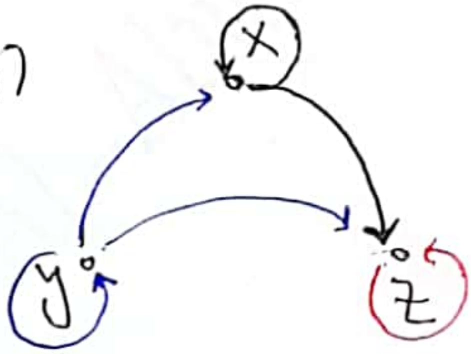
✓ (i) Represent the relation T by diagram .

✓ (ii) Show that T is a partial order relation on B .

✓ (iii) Decide whether T is total order relation on A. why?

✓ (iv) Draw the Hasse diagram representing the partial order relation T on the set B .

(i)



(ii) ① $\because (x,x), (y,y)$ and $(z,z) \in T$

$\therefore T$ is refl.

② $\because (y,x) \in T$ but $(x,y) \notin T \Rightarrow T$ is antisymm.

(also for the others)

③ $\because (y,x) \wedge (x,z) \rightarrow (y,z) \in T \Rightarrow T$ is transitive

$\therefore T$ is a partial order.

(iii) $\because (y,x), (x,z), (y,z) \in T \Rightarrow$ C is comparable

$\therefore T$ is a total order.

Hasse diagram

H.W. **8.** Let S be a relation defined on the set $\mathbb{N} = \{1,2,3,\dots\}$:

$$x, y \in \mathbb{N}\ ,\ x\,S\,y \Leftrightarrow \frac{x}{y} \ \textit{is odd}$$

(i) Show that S is a partial order relation on $\mathbb{N}$.

(ii) Decide whether S is total order relation on $\mathbb{N}$, why?

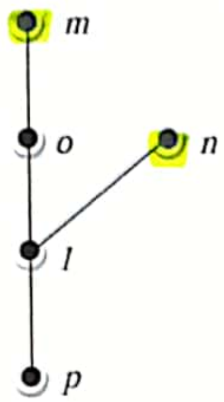
(iii) Draw the Hasse diagram representing the partial order relation S on the set $A = \{1,2,6\}$.

Solution: (i)

9. Let T be a partial order relation defined on the set $C = \{l, m, n, o, p\}$ shown in the given Hasse diagram

✓ (i) List all ordered pairs of T.

✓ (ii) Decide whether T is totally order relation on C. why?



(i)

$T = \{(l,l), (m,m), (n,n), (o,o), (p,p), (p,l), (p,n), (p,o), (p,m), (l,n), (l,o), (l,m), (o,m)\}$

(ii) $m, n \in C$, $m \neq n$, $(m,n) \wedge (n,m) \notin T. \Rightarrow$

$\therefore$ m, n incomparable $\Rightarrow \therefore T$ is not a ~~Partial~~ Total order on C.