KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

MATH-244 (Linear Algebra); Final Exam; Semester 1 (1442 H)

Max. Marks: 40 Max. Time: 3 hours

Note: *Attempt all the five questions!*

Question 1 [3+2+3 marks]:

a) If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 3 \end{bmatrix}$, then find $\mathbf{A}^{-1}$.

b) Evaluate $\det(\det(\det(\det(\mathbf{A})\ \mathbf{A}^2)\ \mathbf{A})\ \mathbf{A}^{-1})$, where $\mathbf{A}$ is a square matrix of order 3 with $\det(\mathbf{A}) = 3$.

c) Let $\begin{bmatrix} 1 & 0 & 2 & 0 & -3 \\ 0 & 1 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$ be reduced row echelon form of the augmented matrix of linear system $\boldsymbol{AX} = \boldsymbol{B}$. Explain! Why the system $\boldsymbol{AX} = \mathbf{C}$ has a solution for any $\mathbf{C} \in \mathbb{R}^3$?

Question 2 [5+3 marks]:

a) Find the values of $\boldsymbol{\alpha}$ such that the following linear system:

$$\begin{aligned} x + y + z &= 0 \\ x + \alpha y + z &= 1 \\ x + y + (\alpha-2)^2 z &= 0 \end{aligned}$$

has:

i) No solution;
ii) Unique solution;
iii) Infinitely many solutions.

b) Let $\boldsymbol{v}_1 = (1, 2, 0, 3, -1)$, $\boldsymbol{v}_2 = (2, 4, 3, 0, 7)$, $\boldsymbol{v}_3 = (1, 2, 2, 0, 9)$, $\boldsymbol{v}_4 = (-2, -4, -2, -2, -3)$. **Find** a basis of the Euclidean space $\mathbb{R}^5$ which includes the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$.

Question 3 [2+3+3 marks]:

a) Let $\{\boldsymbol{x}, \boldsymbol{y}\}$ be linearly independent set of vectors in vector space $\mathbf{V}$. Determine whether the set $\{\mathbf{2}\boldsymbol{x},\ \boldsymbol{x} + \boldsymbol{y}\}$ is linearly independent or not?

b) Suppose $\boldsymbol{G}$ is a subspace of the Euclidean space $\mathbb{R}^{15}$ of dimension **3**, $\mathbf{S} = \{u, v, w\}$ and $\mathbf{Q}$ are two bases of the space $\boldsymbol{G}$ and ${}_{Q}P_{S} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ be the transition matrix from the basis $\mathbf{S}$ to the basis $\mathbf{Q}$. Find $[\mathbf{g}]_{\mathbf{Q}}$ where $\mathbf{g} = \mathbf{3}\boldsymbol{v} - \mathbf{5}\boldsymbol{u} + \mathbf{7}\boldsymbol{w}$.

c) Let $\boldsymbol{P}_2$ be the vector space of polynomials of degree $\leq \mathbf{2}$ with the inner product: $<\boldsymbol{p}, \boldsymbol{q}> = aa_1 + 2bb_1 + cc_1$ for all $\boldsymbol{p} = a + bx + cx^2$, $\boldsymbol{q} = a_1 + b_1x + c_1x^2 \in \boldsymbol{P}_2$. Find $\cos\theta$, where $\boldsymbol{\theta}$ is the angle between the polynomials $1 + x + x^2$ and $1 - x + 2x^2$.

Question 4 [3+1+4 marks]**:**

a) Find an orthonormal basis for the subspace **F**= **span(A)** of Euclidean space $\mathbb{R}^4$, where **A** = $\{x_1 = (1, 2, 3, 0),\ x_2 = (1, 2, 0, 0),\ x_3 = (1, 0, 0, 1)\}$.

b) Let $\mathbf{S}, \mathbf{T}$: $\mathbb{R}^n \to \mathbb{R}^n$ be the linear transformations such that:

$$\mathbf{S}(\boldsymbol{u}) = \mathbf{T}(\boldsymbol{u}), \mathbf{S}(\boldsymbol{v}) = \mathbf{T}(v) \text{ and } \mathbf{S}(\boldsymbol{w}) = \mathbf{T}(\boldsymbol{w}).$$

Show that $\mathbf{S}(\boldsymbol{x}) = \mathbf{T}(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbf{span}(\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\})$.

c) Let the linear transformation $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$\mathbf{T}(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{x} + 2\boldsymbol{y}, \boldsymbol{x} - \boldsymbol{y}, 3\,\boldsymbol{x} + \boldsymbol{y})$$

for all $\boldsymbol{v} = (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^2$. Find $[\mathbf{T}]_{\mathbf{B}}^{\mathbf{C}}$, $[\boldsymbol{v}]_{\mathrm{B}}$ and $[\mathbf{T}(\boldsymbol{v})]_{\mathrm{C}}$, where **B** = **{(1,-2), (2,3)}** and **C** = **{(1,1,1), (2,1,-1), (3,1,2)}** are bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Question 5 [2× 4 marks]**:**

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -3 & 0 & -2 \end{bmatrix}$. Then:

i) Show that 1 and -1 are the eigenvalues of A and find their algebraic and geometric multiplicities.

ii) Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

iii) Show that $\boldsymbol{A}^{-1}$exists and it is also diagonalizable.

iv) Compute the matrix A^{2020}.