King Saud University

College of Sciences

Department of Mathematics

Final Examination	Math-244	Semester I, (1441 H)	Max. Time: 3h

Note: Calculators are not allowed.

Question 1 [2+2+3]

a) Find the value of x so that $\det A = 6$, where $A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 3x-6 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

b) Let $X_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find the matrix A of order 2 such that $AX_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $AX_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

c) Let $m \in \mathbb{R}$ and consider the linear system:

$$\begin{cases} mx + y + 2z & = & 3 \\ mx + my + 3z & = & 5 \\ 3mx + (m+2)y + (m+6)z & = & 2m + 9. \end{cases}$$

Find the value(s) of m so that the system has infinitely many solutions.

Question 2 [2+3+(2+2)]

a) Show that the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 4 & 3 & -3 \end{pmatrix}$ cannot be a transition matrix between two bases of any 3−dimensional vector space.

b) Find a basis for the solution space of the following homogeneous linear system:

$$\begin{cases} x - 3y + z & = & 0 \\ 2x - 6y + 2z & = & 0 \\ 3x - 9y + 3z & = & 0. \end{cases}$$

c) Consider the bases $B = \{u_1 = (1,1,0), u_2 = (1,1,1), u_3 = (1,0,1)\}$ and $C = \{v_1 = (1,-1,0), v_2 = (1,-1,1), v_3 = (-1,0,1)\}$ for $\mathbb{R}^3$.
Find the matrices ${}_C P_B$ and ${}_B P_C$. [${}_C P_B$ is the transition matrix from B to C.]

Question 3 [3+(2+2)]

a) Let $T\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation such that:
$T(x, y, z) = (ax + 2by + z, ax - by + 2z, 2ax + by + 3z)$.
Find the values of a and b so that $(-5, 1, 3)$ is in the kernel of T.

b) Let $T\colon \mathbb{R}^5 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by the formula:

$$T(x_1, x_2, x_2, x_4, x_5) = (x_1 + x_2, x_2 + x_3 + x_4, x_4 + x_5)$$

(i) Find the matrix A of T relative to the standard bases of $\mathbb{R}^5$ and $\mathbb{R}^3$.

(ii) Find the rank and the nullity of the matrix A.

Question 4 [(1+1)+2+2]
Suppose $T\colon V \longrightarrow W$ is a linear transformation, $B = \{v_1, v_2, v_3\}$ is a basis for V and $C = \{w_1, w_2, w_3, w_4\}$ is a basis for W.
If $T(v_1) + T(v_2) = w_1 - 3w_2 + w_4$, $T(v_2) = 2w_1 + w_2 - w_3 + 3w_4$ and $T(v_2) + T(v_3) = 2w_2 + 3w_3 - w_4$.
Find:
(i) $T(v_1)$ and $T(v_3)$.
(ii) The matrix $[T]_B^C$.
(iii) $[T(v)]_C$, where $[v]_B = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$.

Question 5 [(2+2)+2+2+3]

a) Let F be the subspace of the Euclidean space $\mathbb{R}^4$ spanned by the following vectors: $v_1 = (1, 1, -1, 0)$, $v_2 = (0, 1, 1, 1)$, $v_3 = (1, 0, 1, 1)$, $v_4 = (0, -1, 2, 1)$.

(i) Show that $\{v_1, v_2.v_3\}$ is a basis for F.

(ii) Use Gram-Schmidt algorithm to convert the basis $\{v_1, v_2, v_3\}$ into an orthonormal basis for F.

b) Find a such that the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix $A = \begin{pmatrix} 1 & -1 \\ 2 & a \end{pmatrix}$.

c) If 3 is an eigenvalue of the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & b \end{pmatrix}$, find b.

d) Find A^{17}, where $A = \begin{pmatrix} 3 & 5 & -6 \\ 0 & 0 & -5 \\ 0 & 0 & -3 \end{pmatrix}$.