KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

MATH-244 (Linear Algebra); Final Exam; Semester 432

Max. Marks: 40 **Max. Time: 3 hours**

Note: ***Attempt all the five questions!***

Question 1 [3+2+2 marks]:

a) If $A^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, then find $adj(adj(A))$.

b) Find the values of k that makes the matrix $\begin{bmatrix} 2 & 3k-2 \\ k^2 & -1 \end{bmatrix}$ symmetric.

c) Let $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 4 \\ 0 & 0 & 6 \end{bmatrix}$. Explain! Why the matrix B can be expressed as a product of elementary matrices?

Question 2 [3+3 marks]:

a) Solve the linear system of equations with augmented matrix:

$$[A:B] = \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 1 \\ -1 & 2 & 0 & -2 & 2 \\ 3 & 1 & 0 & 3 & -1 \end{array}\right].$$

b) Solve the following linear system of equations by Cramer's Rule:

$$\begin{aligned} x - y \quad\quad &= 1 \\ -2x + 3y - 4z &= 0 \\ -2x + 3y - 3z &= 1 \end{aligned}$$

Question 3 [2+2+2+3 marks]:

a) Show that $E = \{ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 : a, b \in \mathbb{R}\}$ is a subspace of the real vector space P_7 of polynomials with *degree* ≤ 7.

b) Find a basis and dimension of the vector space E.

c) Show that $< (x_1, x_2, x_3), (y_1, y_2, y_3) > = x_1y_1 + 2x_2y_2 + 3x_3y_3$ defines an inner product on the vector space $\mathbb{R}^3$.

d) Find an orthogonal basis of $\mathbb{R}^3$, with respect to the inner product defined above in Part c), by using the Gram-Schmidt algorithm on $\{u_1 = (1,1,1),\ u_2 = (1,1,0),\ u_3 = (0,1,0)\}$.

Question 4 [3+3+3 marks]:

Let $\boldsymbol{B} = \{u_1 = (1,-1), u_2 = (1,1)\}$ and $\boldsymbol{C} = \{v_1 = (1,1,0), v_2 = (1,0,1), v_3 = (1,1,1)\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Let $\mathbf{T}: \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $\mathbf{T}(1,-1) = (3,5,2)$, $\mathbf{T}(1,1) = (2,-1,-3)$. Then find:

a) $\mathbf{T}(\mathbf{1},\mathbf{0})$ and $\mathbf{T}(\mathbf{0},\mathbf{1})$.
b) The matrix $[\mathbf{T}]_{\boldsymbol{B}}^{\boldsymbol{C}}$ of the linear transformation $\mathbf{T}$ with respect to the bases $\boldsymbol{B}$ and $\boldsymbol{C}$.
c) The coordinate vector $[\mathbf{T}(0,1)]_C$ by using $[\mathbf{T}]_{\boldsymbol{B}}^{\boldsymbol{C}}$.

Question 5 [2+2+2+3 marks]:

Let $\boldsymbol{\lambda_1 = 0}$, $\boldsymbol{\lambda_2 = 1}$ and $\boldsymbol{\lambda_3 = -1}$ be the eigenvalues of 8×8 matrix $\boldsymbol{A}$ with algebraic multiplicities **3**, **2** and **3**, respectively. Let $\mathbf{dim}(\boldsymbol{E}_{\lambda_1}) = \mathbf{3}$, $\mathbf{dim}(\boldsymbol{E}_{\lambda_2}) = \mathbf{2}$ and $\mathbf{dim}(\boldsymbol{E}_{\lambda_3}) = \mathbf{3}$, where $\boldsymbol{E}_{\lambda_j}$ denotes the eigenspace with respect to the eigenvalue $\boldsymbol{\lambda_j}$.

a) Find the characteristic polynomial $\boldsymbol{q_A(\lambda)}$ of the matrix $\boldsymbol{A}$.
b) Explain, why the matrix $\boldsymbol{A}$ is diagonalizable?
c) Find the diagonal matrix $\boldsymbol{D}$ such that $\boldsymbol{A} = \boldsymbol{PDP^{-1}}$, where $\boldsymbol{P}$ is an invertible matrix.
d) Find $\boldsymbol{A^{11}}$.

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Note: *In answering the questions other than Question 2/b), if a student has used some method/s different from the one/s being preferred and used in this solution key then the respective instructor should award the student appropriately.*

Solution of Question 1: a) $|A^{-1}| = 1$ so that $|A| = 1$. [1 mark]

Hence, $adj(adj(A)) = |A|^{3-2}A = A$ [1 mark]

$$= (A^{-1})^{-1} = |A|\begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix} = \begin{bmatrix}-1 & 1 & 1\\ 1 & -1 & 0\\ 1 & 0 & -1\end{bmatrix}.$$ [1 mark]

b) By the symmetricity, $\begin{bmatrix}2 & 3k-2\\ k^2 & -1\end{bmatrix} = \begin{bmatrix}2 & k^2\\ 3k-2 & -1\end{bmatrix}$. [1 mark]

So, $k^2 - 3k + 2 = 0$. Thus, $k = 1, 2$. [1 mark]

c) $|B| = -30 \neq 0 \Leftrightarrow B$ is invertible $\Leftrightarrow$ $\boldsymbol{B}$ is a product of elementary matrices. [0.5+0.5+1 mark]

Solution of Question 2: a) $[A{:}B] = \left[\begin{array}{cccc|c}1 & -1 & 0 & 1 & 1\\ -1 & 2 & 0 & -2 & 2\\ 3 & 1 & 0 & 3 & -1\end{array}\right] \sim \left[\begin{array}{cccc|c}1 & -1 & 0 & 1 & 1\\ 0 & 1 & 0 & -1 & 3\\ 0 & 0 & 0 & 1 & -4\end{array}\right]$. [1.5 marks]

Hence, solution set of the system $= \{(4, -1, t, -4) \colon\ t \in \mathbb{R}\}$. [1.5 marks]

b) Let $\boldsymbol{A}$ denote the matrix of coefficients. Then:

$|A| = \begin{vmatrix}1 & -1 & 0\\ -2 & 3 & -4\\ -2 & 3 & -3\end{vmatrix} = 1,\ |A_x| = \begin{vmatrix}1 & -1 & 0\\ 0 & 3 & -4\\ 1 & 3 & -3\end{vmatrix} = 7, |A_y| = \begin{vmatrix}1 & 1 & 0\\ -2 & 0 & -4\\ -2 & 1 & -3\end{vmatrix} = 6,\ |A_z| = \begin{vmatrix}1 & -1 & 1\\ -2 & 3 & 0\\ -2 & 3 & 1\end{vmatrix} = 1.$ [2 marks]

Hence, $x = \frac{|A_x|}{|A|} = 7,\ y = \frac{|A_y|}{|A|} = 6,\ z = \frac{|A_z|}{|A|} = 1.$ [1 mark]

Solution of Question 3:

a) The zero polynomial $= ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 \in \boldsymbol{E}$ with $a = b = 0 \in \mathbb{R}$. [0.5 mark]

Next, for any $a_1, a_2, b_1, b_2, \lambda, \delta \in \mathbb{R}$, we have: [2.5 marks]

$$\lambda(a_1x - 2a_1x^4 + (a_1 - b_1)x^6 + (3a_1 + 2b_1)x^7) + \delta(a_2x - 2a_2x^4 + (a_2 - b_2)x^6 + (3a_2 + 2b_2)x^7)$$

$$= (\lambda a_1 + \delta a_2)\,x - 2(\lambda a_1 + \delta a_2)x^4 + ((\lambda a_1 + \delta a_2) - (\lambda b_1 + \delta b_2))x^6 + (3(\lambda a_1 + \delta a_2) + 2(\lambda b_1 + \delta b_2))x^7 \in \boldsymbol{E}.$$

Hence, $\boldsymbol{E}$ is a vector subspace of $\boldsymbol{P_7}$.

b) $ax - 2ax^4 + (a-b)x^6 + (3a+2b)x^7 = a(x - 2x^4 + x^6 + 3x^7) + b(-x^6 + 2x^7)$ for all $a, b \in \mathbb{R}$. [0.5mark]

Moreover, $x - 2x^4 + x^6 + 3x^7$ and $-x^6 + 2x^7$are linearly independent polynomials in $\boldsymbol{E}$. [0.5 mark]

Hence, $\{x - 2x^4 + x^6 + 3x^7,\ -x^6 + 2x^7\}$ is a basis of $\boldsymbol{E}$ and $dim(\boldsymbol{E}) = 2$. [0.5+0.5 mark]

c) For any $(x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) \in \mathbb{R}^3;\ \alpha \in \mathbb{R}$, we have

$$< (x_1, x_2, x_3), (y_1, y_2, y_3) > = x_1y_1 + 2x_2y_2 + 3x_3y_3 = < (y_1, y_2, y_3), (x_1, x_2, x_3) >;$$

$$< (x_1,x_2,x_3) + (y_1,y_2,y_3),(z_1,z_2,z_3) > = < (x_1+y_1,x_2+y_2,x_3+y_3),(z_1,z_2,z_3) >$$

$$= (x_1+y_1)z_1 + 2(x_2+y_2)z_2 + 3(x_3+y_3)z_3 = x_1z_1 + 2x_2z_2 + 3x_3z_3 + y_1z_1 + 2y_2z_2 + 3y_3z_3$$

$$= < (x_1,x_2,x_3),(z_1,z_2,z_3) > + < (y_1,y_2,y_3),(z_1,z_2,z_3) > ;$$

$$< \alpha(x_1,x_2,x_3),(y_1,y_2,y_3) > = < (\alpha x_1,\alpha x_2,\alpha x_3),(y_1,y_2,y_3) > = (\alpha x_1)y_1 + 2(\alpha x_2)y_2 + 3(\alpha x_3)y_3$$

$$= \alpha(x_1y_1 + 2x_2y_2 + 3x_3y_3 = \alpha < (x_1,x_2,x_3),(y_1,y_2,y_3) >;$$

$$< (x_1,x_2,x_3),(x_1,x_2,x_3) > = x_1^2 + 2x_2^2 + 3x_3^2 \geq 0.$$

Finally, $< (x_1,x_2,x_3),(x_1,x_2,x_3) > = 0 \Leftrightarrow x_1^2 + 2x_2^2 + 3x_3^2 = 0 \Leftrightarrow (x_1,x_2,x_3) = (0,0,0)$. Q.E.D. [2 marks]

d) $v_1 = u_1 = (1,1,1)$; $v_2 = u_2 - \frac{<u_2,\ v_1>}{\|v_1\|^2} v_1 = (\frac{1}{2},\frac{1}{2},\frac{-1}{2})$; [0.5+1 mark]

$v_3 = u_3 - \frac{<u_3,\ v_1>}{\|v_1\|^2} v_1 - \frac{<u_3,\ v_2>}{\|v_2\|^2} v_2 = (\frac{-2}{3},\frac{1}{3},0)$. [1.5 marks]

Solution of Question 4: a) $(1,0) = \alpha(1,-1) + \beta(1,1) \Rightarrow \alpha + \beta = 1,\ -\alpha + \beta = 0 \Rightarrow \alpha = \beta = \frac{1}{2}$.
So that $(1,0) = \frac{1}{2}(1,-1) + \frac{1}{2}(1,1)$. Similarly, $(0,1) = -\frac{1}{2}(1,-1) + \frac{1}{2}(1,1)$. [1 mark]

Hence, $T(1,0) = T\left(\frac{1}{2}(1,-1) + \frac{1}{2}(1,1)\right) = \frac{1}{2}T(1,-1) + \frac{1}{2}T(1,1) = \frac{1}{2}((3,5,2) + (2,-1,-3)) = \frac{1}{2}(5,4,-1)$ and $T(0,1) = \frac{1}{2}(-1,-6,-5)$. [2 marks]

b) $(3,5,2) = T(1,-1) = \alpha(1,1,0)) + \beta(1,0,1) + \gamma(1,1,1) \Rightarrow \alpha + \beta + \gamma = 3,\ \alpha + \gamma = 5,\beta + \gamma = 2$

$\Rightarrow \alpha = 1, \beta = -2, \gamma = 4$. So, $[T(u_1)]_C = [T(1,-1)]_C = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Similarly, $[T(u_1)]_C = \begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$. [2 marks]

Hence, $[T]_B^C = [\,[T(u_1)]_C\ \ [T(u_1)]_C] = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 4 & -6 \end{bmatrix}$. [1 mark]

c) $[T(0,1)]_C = [T]_B^C\,[(0,1)]_B = \begin{bmatrix} 1 & 5 \\ -2 & 3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{5}{2} \\ -5 \end{bmatrix}$. [1.5+1+0.5 marks]

Solution of Question 5: a) $q_A(\lambda) = (\lambda - 0)^3(\lambda - 1)^2(\lambda + 1)^3 = \lambda^3(\lambda - 1)^2(\lambda + 1)^3$. [2 marks]

b) Because algebraic multiplicity of eigenvalue λ_j = the geometric multiplicity of $\lambda_j, \forall j = 1,2,3$. [2 marks]

c) $D = \begin{bmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&-1 \end{bmatrix}$ [2 marks]

d) $A^{11} = (PDP^{-1})^{11} = PD^{11}P^{-1} = PDP^{-1} = A$. [3 marks]

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