Solution of the Final Examination Math 244 Semester I, (1441, H)

Question 1 [2+2+3]

a) $|A| = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 3x-4 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 6x - 6$. Then $x = 2$.

b) Let $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$. $AX_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $AX_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is equivalent to: $\begin{cases} 2x+y &= 3 \\ x+y &= 2 \end{cases}$ and $\begin{cases} 2z+t &= 1 \\ z+t &= 2 \end{cases}$. Then $x = y = 1$, $z = -1, t = 3$ and $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

c) The augmented matrix is $\left[\begin{array}{ccc|c} m & 1 & 2 & 3 \\ m & m & 3 & 5 \\ 3m & m+2 & m+6 & 2m+9 \end{array}\right]$.

This matrix is row equivalent to $\left[\begin{array}{ccc|c} m & 1 & 2 & 3 \\ 0 & m-1 & 1 & 2 \\ 0 & m-1 & m & 2m \end{array}\right] \Longleftrightarrow \left[\begin{array}{ccc|c} m & 1 & 2 & 3 \\ 0 & m-1 & 1 & 2 \\ 0 & 0 & m-1 & 2(m-1) \end{array}\right]$.

If $m = 1$ there are infinitely many solutions.

Question 2 [2+3+(2+2)]

a) $|A| = 0$ then A cannot be a transition matrix between two bases of any $3-$dimensional vector space.

b) This system is equivalent to: $x - 3y + z = 0$. The set of solutions is $S = \{(3y - z, y, z) : \ y, z \in \mathbb{R}\} = \{y(3,1,0) + z(-1,0,1) : \ y, z \in \mathbb{R}\}$. Then $\{(3,1,0), (-1,0,1)\}$ is a basis for the solution space of linear system.

c) Consider the bases $B = \{u_1 = (1,1,0), u_2 = (1,1,1), u_3 = (1,0,1)\}$ and $C = \{v_1 = (1,-1,0), v_2 = (1,-1,1), v_3 = (-1,0,1)\}$ for $\mathbb{R}^3$.
Find the matrices ${}_C P_B$ and ${}_B P_C$. [${}_C P_B$ is the transition matrix from B to C.]
${}_C P_B = \begin{pmatrix} -3 & -4 & -2 \\ 2 & 3 & 2 \\ -2 & -2 & -1 \end{pmatrix}$ and ${}_B P_C = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

Question 3 [3+(2+2)]

a) $T(-5,1,3) = (0,0,0) \Longleftrightarrow \begin{cases} -5a+2b &= -3 \\ -5a-b &= -6 \\ -10a+b &= -9 \end{cases} \Longleftrightarrow a = b = 1.$

b) (i) $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$.

(ii) The rank of A is 3 and the nullity is 2.

Question 4 [(1+1)+2+2]

(i) $T(v_1) = -w_1 - 4w_2 + w_3 - 2w_4$, $T(v_3) = -2w_1 + w_2 + 4w_3 - 4w_4$.

(ii) $[T]_B^C = \begin{pmatrix} -1 & 2 & -2 \\ -4 & 1 & 1 \\ 1 & -1 & 4 \\ -2 & 3 & -4 \end{pmatrix}$.

(iii) $[T(v)]_C = \begin{pmatrix} 9 \\ 17 \\ -3 \\ 14 \end{pmatrix}$.

Question 5 [(2+2)+2+2+3]

a) (i) $v_4 = v_3 - v_1$ and $xv_1 + yv_2 + zv_3 = (0,0,0,0) \iff \begin{cases} x + z &= 0 \\ x + y &= 0 \\ -x + y + z &= 0 \\ y + z &= 0 \end{cases} \iff$
$x = y = z = 0$. Then $\{v_1, v_2.v_3\}$ is a basis for F.

(ii) $\langle v_1, v_2 \rangle = 0$ and $\langle v_1, v_3 \rangle = 0$.
$u_1 = \frac{1}{\sqrt{3}} v_1$, $u_2 = \frac{1}{\sqrt{3}} v_2$.
$\langle v_3, v_2 \rangle = 2$, $v_3 - \frac{2}{3} v_2 = \frac{1}{3}(3, -2, 1, 1)$. Then $u_3 = \frac{1}{\sqrt{15}}(3, -2, 1, 1)$.

b) If $X = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then $AX = \lambda X \iff a = 4$.

c) 3 is an eigenvalue of the matrix $A = \begin{pmatrix} 2 & -1 \\ 1 & b \end{pmatrix}$ if and only if $b = 4$.

d) The matrix is diagonalizable then there exists an invertible matrix P such that
$P^{-1}AP = D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}$. $A^{17} = PD^{17}P^{-1} = 3^{16}PDP^{-1} = 3^{16}A$.