Solution Key:

King Saud University
College of Sciences
Department of Mathematics
Math-244 (Linear Algebra); Final Exam; Semester 441

Max. Marks: 40 **Time: 3 hours**

Note: ***Attempt all the five questions. Scientific calculators are not allowed***!

Question 1 [Marks: 5× 1+5× 1]**:**

I. Choose the correct answer:

(i) If W is the subspace $\{(a,b,c,d) \in \mathbb{R}^4 : b = a - c\}$ of Euclidean space $\mathbb{R}^4$, then $\dim(W)$ is:

a) 1 b) 2 **c)** ✓ 3 d) 4.

(ii) If $rank(A) = 3$ where A is a matrix of size 5×9, then $nullity(A^T)$ is:

a) 1 b) ✓ 2 c) 3 d) 6.

(iii) If θ is the angle between A and B with respect to the standard inner product on M_{22} where $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$, then $\cos\theta$ is:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) ✓ 0.

(iv) The values of k for which the vectors $\boldsymbol{u} = (u_1 = 2, u_2 = -4)$ and $\boldsymbol{v} = (v_1 = 1, v_2 = 3)$ in $\mathbb{R}^2$ are orthogonal with respect to the inner product $\langle \boldsymbol{u},\boldsymbol{v} \rangle = 2u_1v_1 + ku_2v_2$:

a) $\frac{1}{\sqrt{2}}$ b) $\frac{1}{2}$ c) $\frac{15}{2\sqrt{30}}$ d) ✓ $\frac{1}{3}$.

(v) If $B = \{(2,1), (-3,4)\}$ and $C = \{(1,1), (0,3)\}$ are bases of $\mathbb{R}^2$, then the transition matrix ${}_{\boldsymbol{B}}\boldsymbol{P}_{\boldsymbol{C}}$ is:

a) $\begin{bmatrix} 7/11 & 1/11 \\ 9/11 & 6/11 \end{bmatrix}$ b) ✓ $\begin{bmatrix} 7/11 & 9/11 \\ 1/11 & 6/11 \end{bmatrix}$ c) $\begin{bmatrix} 7/11 & 9/11 \\ 6/11 & 1/11 \end{bmatrix}$ d) $\begin{bmatrix} 9/11 & 7/11 \\ 1/11 & 6/11 \end{bmatrix}$.

II. Determine whether the following statements are true or false; justify your answer.

(i) If $A, B \in M_n(\mathbb{R})$, then $det\left(A^{\mathrm{T}}B\right) = det\left(B^{\mathrm{T}}A\right)$.

True: $det(A^{\mathrm{T}}B) = det(A)det(B) = det(B^{\mathrm{T}}A)$.

(ii) A basis for solution space of the following linear system is $\{(4,1,0,0), (-3,0,1,0)\}$:

$$x_1 - 4x_2 + 3x_3 - x_4 = 0$$
$$2x_1 - 8x_2 + 6x_3 - 3x_4 = 0.$$

True: the solution space $= \{(4s - 3t, s, t, 0) : s, t \in \mathbb{R}\}$; $(4s - 3t, s, t, 0) = s(4,1,0,0) + t(-3,0,1,0)$ and $\{(4,1,0,0), (-3,0,1,0)\}$ is linearly independent.

(iii) If $W = \{A \in M_2(\mathbb{R}) : A \text{ is singular}\}$, then W is vector subspace of $M_2(\mathbb{R})$.

False: $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ are singular matrices but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is not singular.

(iv) If u, v and w are vectors in an inner product space such that $\langle u,v \rangle = 3$, $\langle v,w \rangle = -5$, $\langle u,w \rangle = -1$ and $\|u\| = 2$, then $\langle u - 2w, 3u + v \rangle = 25$.

False: $25 \neq 31$ ($\because \langle u - 2w, 3u + v \rangle = 3(2^2) + 3 + (-2)(3)(-1) + (-2)(-5) = 31$)

(v) If the characteristic polynomial of 2×2 matrix A is $q_A(\lambda) = \lambda^2 - 1$, then A is diagonalizable.

True: ∓ 1 are two different eigen-values of the 2×2 matrix $\boldsymbol{A}$.

Question 2 [Marks: 2+2+2]**:** Consider the matrices $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & -2 \end{bmatrix}$. Then:

a) Find A^{-1} by the elementary matrix method.

Solution: Since $\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 2 & -3 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 2 \end{array}\right]$, $A^{-1} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 1 & 2 & 2 & -3 \\ -1 & -1 & -1 & 2 \end{bmatrix}$.

b) Show that $nullity(A) \neq nullity(B)$.

Solution: Since $RREF(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 8 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$, $nullity(B) = 1$; but $nullity(A) = 0$ because A is invertible.

c) Find a basis for the null space spaces $N(B)$.

Solution: Since $N(B) = \{(-5t, 3t, -8t, 6t, t) : t \in \mathbb{R}\}$, $\{(-5, 3, -8, 6, 1)\}$ is a basis for $N(B)$.

Question 3 [Marks: 3+3]**:**

a) Find the values of x so that the set $\{ (1, -2, x), (1, -x, 2), (1, -4, 2x)\}$ is linearly independent in the Euclidean space $\mathbb{R}^3$.

Solution: $\because$ $\alpha(1, -2, x), +\beta(1, -x, 2) + \gamma(1, -4, 2x) = (0,0,0) \Rightarrow \begin{cases} \alpha + \beta + \gamma = 0 \\ -2\alpha - x\beta - 4\gamma = 0 \\ x\alpha + 2\beta + 2x\gamma = 0. \end{cases}$

$\therefore$ The given set would be linearly independent *iff* $\begin{vmatrix} 1 & 1 & 1 \\ -2 & -x & -4 \\ x & 2 & 2x \end{vmatrix} \neq 0$ *iff* $x \in \mathbb{R} \setminus \{\mp 2\}$.

b) Let $F = span(\{(1, -1, 0, 1), (0, 1, 0, -1), (-1, 2, 0, -1)\})$ in $\mathbb{R}^4$. Find a basis for F and show that $(0, 1, 0, 0) \in F$.

Solution: Since $\{(1, -1, 0, 1), (0, 1, 0, -1), (-1, 2, 0, -1)\}$ is linearly independent in $\mathbb{R}^4$, the same set is a basis of F. Next, we observe that $(0, 1, 0, 0) = (-1, 2, 0, -1) + (1, -1, 0, 1) \in F$.

Question 4: [Marks: 2+4]

a) Let u and v be any two vectors in an inner product space. Show that:

$$2(\|u\|^2 + \|v\|^2) = \|u + v\|^2 + \|u - v\|^2.$$

Solution: $\|u + v\|^2 + \|u - v\|^2 = \langle u + v, u + v \rangle + \langle u - v, u - v \rangle = 2 \langle u, u \rangle + 2 \langle v, v \rangle = 2(\|u\|^2 + \|v\|^2)$.

b) Let the set $B = \{u_1 = (1,0,0), u_2 = (3,1,-1), u_3 = (0,3,1)\}$ be linearly independent in the Euclidean inner product space $\mathbb{R}^3$. Construct an orthonormal basis for $\mathbb{R}^3$ by applying the Gram-Schmidt algorithm on B.

Solution: Put $\boldsymbol{e}_1 = \boldsymbol{v}_1 = \boldsymbol{u}_1 = (1,0,0)$. Then $\boldsymbol{v}_2 = \boldsymbol{u}_2 - < \boldsymbol{u}_2, \boldsymbol{e}_1 > \boldsymbol{e}_1 = \boldsymbol{u}_2 - 3\boldsymbol{e}_1 = (0,1,-1)$ and so $\boldsymbol{e}_2 = \frac{1}{||\boldsymbol{v}_2||}\boldsymbol{v}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Finally, $\boldsymbol{v}_3 = \boldsymbol{u}_3 - < \boldsymbol{u}_3, \boldsymbol{e}_2 > \boldsymbol{e}_2 - < \boldsymbol{u}_3, \boldsymbol{e}_1 > \boldsymbol{e}_1 = \boldsymbol{u}_3 - \sqrt{2}\boldsymbol{e}_2 = (\mathbf{0}, \mathbf{2}, \mathbf{2})$ **and so** $\boldsymbol{e}_3 = \frac{1}{||\boldsymbol{v}_3||}\boldsymbol{v}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Thus, $\{\boldsymbol{e}_1 = (1,0,0), \boldsymbol{e}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \boldsymbol{e}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\}$ is the required orthonoral basis of the inner product space $\mathbb{R}^3$.

Question 5: [Marks: (4+2) + (2+2+$\overset{2}{\cancel{1}}$)]

a) Let $\boldsymbol{B} = \{(1,1,0), (0,1,1), (1,0,1)\}$ be a basis for $\mathbb{R}^3$, $\boldsymbol{C} = \{x+1, x-1, x^2+1\}$ be a basis for P_2 (the vector space of all real polynomials (in variable x) of degree ≤ 2 . Let $\boldsymbol{T}: \mathbb{R}^3 \to P_2$ be the linear transformation: $\boldsymbol{T}(a,b,c) = (a+b) + (b+c)x + (a+c)x^2$, $\forall (a,b,c) \in \mathbb{R}^3$.

(i) Find the values of q, r, s in the transformation matrix $[\boldsymbol{T}]_{\boldsymbol{B}}^{\boldsymbol{C}} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$ with respect to the bases $\boldsymbol{B}$ and $\boldsymbol{C}$.

Solution: Since $[\boldsymbol{T}]_{\boldsymbol{B}}^{\boldsymbol{C}} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$, we get $[T(1,1,0)]_{\boldsymbol{C}} = \begin{bmatrix} 1 \\ r \\ 1 \end{bmatrix}$, $[T(0,1,1)]_{\boldsymbol{C}} = \begin{bmatrix} q \\ 1 \\ 1 \end{bmatrix}$ and $[T(1,0,1)]_{\boldsymbol{C}} = \begin{bmatrix} 0 \\ 1 \\ s \end{bmatrix}$.

Now, $T(1,1,0) = 2 + x + x^2 = 1(x+1) + 0(x-1) + 1(x^2+1)$ gives $\begin{bmatrix} 1 \\ r \\ 1 \end{bmatrix} = [T(1,1,0)]_{\boldsymbol{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; hence, $\boldsymbol{r} = \mathbf{0}$.

Similarly, $q = \mathbf{1}$ and $\boldsymbol{s} = \mathbf{2}$.

(ii) Find the coordinate vector $[T(1,1,1)]_{\boldsymbol{C}}$.

Solution: Since $[\boldsymbol{T}]_{\boldsymbol{B}}^{\boldsymbol{C}} = \begin{bmatrix} 1 & q & 0 \\ r & 1 & 1 \\ 1 & 1 & s \end{bmatrix}$ and $[(1,1,1)]_{\boldsymbol{B}} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we get $[T(1,1,1)]_{\boldsymbol{C}} = [\boldsymbol{T}]_{\boldsymbol{B}}^{\boldsymbol{C}}[(1,1,1)]_{\boldsymbol{B}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

b) Consider the matrix $A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ is diagonalizable.

(i) Show that the matrix A is diagonalizable.

Solution: The given matrix $A = \begin{bmatrix} 1 & 7 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ being upper triangular has eigen-values -1, 1 and 2; so, it is diagonalizable.

(ii) Find an invertible matrix P and a diagonal matrix D satisfying $P^{-1}AP = D$.

Solution: $\boldsymbol{P} = \begin{bmatrix} \mathbf{1} & \mathbf{7} & \frac{+7}{3\cancel{2}} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix}$ **with** $\boldsymbol{P}^{-1} = \begin{bmatrix} \mathbf{1} & -\mathbf{7} & \frac{-7}{\cancel{6}} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$ **and** $\boldsymbol{D} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix}$.

(iii) Find A^7.

Solution: $\boldsymbol{A}^7 = \boldsymbol{P}\,\boldsymbol{D}^7\,\boldsymbol{P}^{-1} = \begin{bmatrix} \mathbf{1} & \mathbf{7} & \frac{-7}{2} \\ 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2^7 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\mathbf{7} & \frac{-7}{6} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$.

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