Solution of the Final Examination Math 244 Semester I 1439-1440

Question 1 :

1. $AB + AC - D = 0 \iff A(B+C) = D \Rightarrow |A|\,|B+C| = |D|$.
   $B + C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$, then $|B+C| = -3$ and $|A| = -2$.

2. $RS + R - 2I = 0 \iff R(S+I) = 2I$. Then
   $R^{-1} = \dfrac{1}{2}(S+I) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix}$.

3. $a - b - 2c - 3d = 0 \iff a = b + 2c + 3d$. The matrices in W are in the form
   $\begin{pmatrix} b+2c+3d & b \\ c & d \end{pmatrix} = b\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.
   Then $\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\right\}$ is a basis of the vector subspace W.

Question 2 :

The extended matrix of the system is: $\left[\begin{array}{ccc|c} 1 & m & 2 & 3 \\ 4 & 6+m & -m & 13-m \\ 1 & 2(m-1) & m+4 & m+2 \end{array}\right]$.

The matrix $\left[\begin{array}{ccc|c} 1 & m & 2 & 3 \\ 0 & m-2 & m+2 & m-1 \\ 0 & 0 & 2(m-1) & m-1 \end{array}\right]$ is row equivalent to the extended matrix of the system.

a) If $m \neq 1$ and $m \neq 2$ the system has a unique solution.
b) If $m = 1$ the system has infinite solutions.
c) If $m = 2$ the system has no solution.

Question 3 :

1. If $u_1 - 2u_2 + 3u_6 = 5u_3 + 7u_4 - 6u_5$, then
   $u_1 - 2u_2 - 5u_3 - 7u_4 + 6u_5 + 3u_6 = 0$. This is a linear combination of the vectors $u_1, u_2, u_3, u_4, u_5, u_6$ which are linearly independent. This is impossible. Then $u_1 - 2u_2 + 3u_6 \neq 5u_3 + 7u_4 - 6u_5$.

2. (a) The standard matrix of T is $\begin{pmatrix} 1 & -2 & 1 & 3 \\ 2 & -3 & 0 & 2 \\ -1 & 0 & 3 & 5 \end{pmatrix}$.

   (b) The reduced echelon form of to the matrix of T is $\begin{pmatrix} 1 & 0 & -3 & -5 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then
   $\{(3,2,1,0), (5,4,0,1)\}$ is a basis for kernel T.

   (c) Using the reduced echelon form of to the matrix of T we deduce that
   $\{(1,2,-1), (2,3,0)\}$ is a basis for Image T.

Question 4 :

1. ${}_BP_C = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$. $[v]_B = {}_BP_C[v]_C = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2. $[T(w)]_B = [T]_B[w]_B = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$.

3. $T(u_1) = u_1 + u_2 + 2u_3 = au_1 - \frac{b}{5}u_2 + cu_3$. Then $a = 1$, $b = -5$, $c = 2$.

Question 5 :

1. $u_1 = \dfrac{1}{\sqrt{2}}(1,1,0)$, $\langle v_2, u_1 \rangle = \sqrt{2}$. Then $u_2 = (0,0,1)$.

2. (a) $\|u+v\|^2 = 11$.

   (b) $\langle u, v \rangle = 0$, then $\cos\theta = 0$ and $\theta = \frac{\pi}{2}$.

Question 6 :

1. The eigenvalues of B are $1, 2$. B is diagonalizable. $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$,

   $P = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = PDP^{-1}$. Then $B^{10} = PD^{10}P^{-1} = \begin{pmatrix} 1 & 2^{11}-2 \\ 0 & 2^{10} \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & -2 \end{pmatrix}$.

   (a) The characteristic polynomial of A is $q_A(\lambda) = (1-\lambda)^2(2+\lambda)$.

   (b) The eigenvalues of A are 1 and -2.
   The eigenspace E_1 is generated by the vector $(1,0,0)$
   and the eigenspace E_{-2} is generated by the vector $(1,0,-1)$.

   (c) A is not diagonalizable since $\dim(E_1) = 1$ and the algebraic multiplicity of 1 is 2.