[Solution Key] MATH-244 (Linear Algebra); Mid-term Exam; Semester 1 (1442)

Question 1: [Marks: 2+3]

a) **Let A** = $\begin{bmatrix} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{bmatrix}$ and **B** = $\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{bmatrix}$. Then show that the matrices **A** and **B** are row equivalent to each other.

Solution: $\because$ RREF(**A**) = $\begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ = RREF(**B**)

$\therefore$ **A** and **B** are row equivalent.

b) Give any two matrices **A** and **B** that satisfy:

trace (**A**+ **B**) = *trace* (**A**) + *trace* (**B**) and *trace* (**AB**) ≠ *trace* (**A**) *trace* (**B**).

Solution: A simple choice is: $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$:

Clearly, $\mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; so, *trace*(**A+B**) = 1+1 = *trace*(**A**) + *trace*(**B**).

And $\mathbf{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; so, *trace* (**AB**) = 0 ≠ 1 = 1×1 = *trace* (**A**) *trace* (**B**).

Question 2: [Marks: 2+3]

a) Let **A**, **B** ∈ $\mathbf{M_2}(\mathbb{R})$ with $|\mathbf{A}| = 3$ and $|\mathbf{B}| = 6$. Then evaluate $\left||\mathbf{A}|\boldsymbol{A^tB^2adj}(\boldsymbol{A}^2)\right|$.

Solution: $||\mathbf{A}|\boldsymbol{A^tB^2adj}(\boldsymbol{A}^2)| = |\mathbf{A}|^2\,|\mathbf{A}||\mathbf{B}|^2|\mathbf{adj}(|\mathbf{A}|^2)|| = |\mathbf{A}|^3|\mathbf{B}|^2|\mathbf{A}|^{4-2} = |\mathbf{A}|^5|\mathbf{B}|^2 = \mathbf{4\times 3^7}$.

b) Let **A** = $\begin{bmatrix} 1 & 0 & \delta \\ 2 & 1 & 2+\delta \\ 2 & 3 & \delta^2 \end{bmatrix}$. Find the values of δ if the matrix **A** is not invertible.

Solution: The given non-invertibility of the matrix A means $|\mathbf{A}| = 0$.

Hence, $\delta^2 + \delta - 6 = |\mathbf{A}| = 0$.

Thus, $\delta = -3, 2$.

Question 3: [Marks: 2+4]

a) Find the values of x and y if **A** = $\begin{bmatrix} - & 2 & - \\ - & x & - \\ - & y & - \end{bmatrix}$ and $\mathbf{A}^{-1}$ = $\begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ - & - & - \end{bmatrix}$.

Solution: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ - & - & - \end{bmatrix}\begin{bmatrix} - & 2 & - \\ - & x & - \\ - & y & - \end{bmatrix} = \begin{bmatrix} - & 6-2x+2y & - \\ - & 4+x+y & - \\ - & - & - \end{bmatrix}$.

So that $\begin{cases} 6 - 2x + 2y = 0 \\ 4 + x + y = 1. \end{cases}$ Hence, $x = 0$ and $y = -3$.

b) Find the value/s of α such that the following linear system:

$$\begin{aligned} x + 2y - z &= 2 \\ x - 2y + 3z &= 1 \\ x + 2y - (\alpha^2 - 3)z &= \alpha \end{aligned}$$

has: (i) no solution (ii) unique solution (iii) infinitely many solutions.

Solution: $\left[\begin{array}{ccc:c} 1 & 2 & -1 & 2 \\ 1 & -2 & 3 & 1 \\ 1 & 2 & -(\alpha^2-3) & \alpha \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 2 \\ 0 & -4 & 4 & -1 \\ 0 & 0 & 4-\alpha^2 & \alpha-2 \end{array}\right]$. Hence, answer to the three parts are: (i) $\alpha = -2$ (ii) $\alpha \in \mathbb{R}\backslash\{-2,2\}$ (iii) $\alpha = 2$.

Question 4: [Marks: 2+3+3]

a) Let $\mathbf{S} = \{(1,1,1,0), (1,2,3,1), (2,0,1,1)\}$ generates the subspace $\mathbf{F}$ of Euclidean space $\mathbb{R}^4$. Show that $(1,1,1,1) \notin \mathbf{F}$.

Solution: The set $\mathbf{S} \cup \{(1,1,1,1)\}$ is linearly independent in $\mathbb{R}^4$.

But $\mathbf{S}$ generates $\mathbf{F}$. Hence. $(1,1,1,1) \notin \mathbf{F}$.

b) Let $\mathbf{B} = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $\mathbf{C} = \{(1,1,1), (1,2,2), (1,1,2)\}$ be bases of the Euclidean space $\mathbb{R}^3$ and $[v]_B = [1 \quad 2 \quad 3]^T$. Then find the transition matrix ${}_{\mathbf{C}}P_{\mathbf{B}}$ and $[v]_C$.

Solution: ${}_{C}P_{B} = [[(1,0,0)]_C \quad [(0,1,0)]_C \quad [(0,0,1)]_C] = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

Hence, $[v]_C = {}_{C}P_{B}[v]_B = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} . \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

c) Let $\mathbf{A}^T = \begin{bmatrix} 1 & 2 & 2 & 0 & 0 \\ 2 & 4 & 5 & 2 & 2 \\ 1 & 2 & 3 & 2 & 2 \\ 3 & 6 & 4 & -3 & -4 \end{bmatrix}$. Then find: (i) a basis of $col(\mathbf{A})$ (ii) $rank(\mathbf{A})$ (iii) $nullity(\mathbf{A})$.

Solution: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 4 & 2 & 6 \\ 2 & 5 & 3 & 4 \\ 0 & 2 & 2 & -3 \\ 0 & 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is a REF of $\mathbf{A}$.

(i) $\{(1,2,2,0,0), (2,4,5,2,2), (3,6,4,-3,-4)\}$ is a basis of $col(\mathbf{A})$.

(ii) $rank(\mathbf{A})$ = dimension of $col(\mathbf{A}) = 3$.

(iii) $nullity(\mathbf{A})$ = no. of columns in $\mathbf{A} - rank(\mathbf{A}) = 4 - 3$.

Question 5: [Marks: 2+1+3]

Let $\mathbf{S} = \{\boldsymbol{v_1} = (1,-1,0,1), \boldsymbol{v_2} = (1,1,1,0), \boldsymbol{v_3} = (0,1,1,1)\}$ generates the subspace $\mathbf{W}$ of the Euclidean space $\mathbb{R}^4$. Then:

a) Show that $\mathbf{S}$ is a basis of $\mathbf{W}$.

Solution: $\mathbf{S}$ is linearly independent because $\alpha(1,-1,0,1) + \beta(1,1,1,0) + \gamma(0,1,1,1) = (0,0,0,0)$ implies $\alpha = \beta = \gamma = 0$.
Now, since $\mathbf{S}$ generates $\mathbf{W}$, $\mathbf{S}$ is a basis of $\mathbf{W}$.

b) Find the angle θ between the vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

Solution: Since $< \boldsymbol{v_1}, \boldsymbol{v_2} > = 0$, $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are orthogonal to each other. So, the asked $\theta = \frac{\pi}{2}$.

c) Apply the Gram-Schmidt process on $\mathbf{S}$ to obtain an orthonormal basis of $\mathbf{W}$.

Solution: The asked orthonormal basis of $\mathbf{W}$ is:

$$\left\{ \frac{1}{\sqrt{3}}(1,-1,0,1), \frac{1}{\sqrt{3}}(1,1,1,0), \frac{\sqrt{3}}{3\sqrt{5}}(-2,1,1,3) \right\}.$$

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