Q1(a):

$$A=\begin{bmatrix}1&-1&4&5\\-2&1&-11&-8\\-1&2&2&2\end{bmatrix}\xrightarrow[1R_{13}]{2R_{12}}\begin{bmatrix}1&-1&4&5\\0&-1&-3&2\\0&1&6&7\end{bmatrix}$$

$$\xrightarrow{1R_{23}}\begin{bmatrix}1&-1&4&5\\0&-1&-3&2\\0&0&3&9\end{bmatrix}\xrightarrow[1/3R_3]{-1R_2}\begin{bmatrix}1&-1&4&5\\0&1&3&-2\\0&0&1&3\end{bmatrix}$$

$$\xrightarrow{1R_{21}}\begin{bmatrix}1&0&7&3\\0&1&3&-2\\0&0&1&3\end{bmatrix}\xrightarrow[-7R_{31}]{-3R_{32}}\begin{bmatrix}1&0&0&-18\\0&1&0&-11\\0&0&1&3\end{bmatrix}=RREF(A)$$

$$B=\begin{bmatrix}1&-1&3&2\\0&1&4&1\\1&0&8&6\end{bmatrix}\xrightarrow{-1R_{13}}\begin{bmatrix}1&-1&3&2\\0&1&4&1\\0&1&5&4\end{bmatrix}$$

$$\xrightarrow[1\mathrm{R}_{21}]{-1R_{23}}\begin{bmatrix}1&0&7&3\\0&1&4&1\\0&0&1&3\end{bmatrix}\xrightarrow[1\mathrm{R}_{21}]{-4R_{32}}\begin{bmatrix}1&0&0&-18\\0&1&0&-11\\0&0&1&3\end{bmatrix}$$

$$=RREF(B)=RREF(A)$$

So A and B are row equivalents.

(b): Choose $A=B=I_2$, $tr(A+B)=2=tr(A)+tr(B)$

but $tr(AB)=tr(I_2)=2\neq 4=tr(A)tr(B)$

Q2(a): $||A|A^TB^2adj(A^2)|=|A|^2|A^T||B^2||\ adj(A^2)|$

$=|A|^2|A||B|^2||A^2|A^{-2}|=|A|^3|B|^2||A|^2A^{-2}|$

$=|A|^3|B|^2|A|^4|A^{-2}|=|A|^3|B|^2|A|^4|A|^{-2}$

$=|A|^5|B|^2=3^5(6^2)=3^7(4)=8748$

(b):

$$0=\begin{vmatrix}1 & 0 & \delta\\ 2 & 1 & 2+\delta\\ 2 & 3 & \delta^2\end{vmatrix}\overset{-2R_{12}}{\underset{-2R_{13}}{=}}\begin{vmatrix}1 & 0 & \delta\\ 0 & 1 & 2-\delta\\ 0 & 3 & \delta^2-2\delta\end{vmatrix}\overset{-3R_{23}}{=}\begin{vmatrix}1 & 0 & \delta\\ 0 & 1 & 2-\delta\\ 0 & 0 & \delta^2+\delta-6\end{vmatrix}$$

$$=\delta^2+\delta-6=(\delta+3)(\delta-2)$$

$$\Rightarrow \delta=2,-3$$

Q3(a): $A^{-1}A=I \Longrightarrow 6-x+2y=0,\ 4+x+y=1$. So $10+3y=1$ and hence $y=-3$, so $x=0$.

(b)

$$\begin{bmatrix}1 & 2 & -1 & 2\\ 1 & -2 & 3 & 1\\ 1 & 2 & -(\alpha^2-3) & \alpha\end{bmatrix}\xrightarrow[-1R_{13}]{-1R_{12}}\begin{bmatrix}1 & 2 & -1 & 2\\ 0 & -4 & 4 & -1\\ 0 & 0 & -\alpha^2+4 & \alpha-2\end{bmatrix}$$

(i) If $\alpha=-2$

(ii) If $\alpha\in R-\{2,-2\}$

(iii) If $\alpha=2$

Q4(a):

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[(-1)R_{13}]{(-1)R_{12}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow[(-1)R_{24}]{(-2)R_{23}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\xrightarrow{(-1)R_{34}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(1/3)R_{3}} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

So SU{(1,1,1,1)} is linearly independent. But S generates F, hence, (1,1,1,1)∉F.

(b):

$$\left[C \mid B\right] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-1)R_{13}]{(-1)R_{12}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[(-1)R_{23}]{(-1)R_{21}} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{(-1)R_{31}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] = \left[I \mid {}_{C}P_{B}\right]$$

$${}_{C}P_{B}[v]_{B} = [v]_{C}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

(c):

$$A=\begin{bmatrix}1&2&1&3\\2&4&2&6\\2&5&3&4\\0&2&2&-3\\0&2&2&-4\end{bmatrix}\xrightarrow[(-2)R_{13}]{(-2)R_{12}}\begin{bmatrix}1&2&1&3\\0&0&0&0\\0&1&1&-2\\0&2&2&-3\\0&2&2&-4\end{bmatrix}$$

$$\xrightarrow[(-2)R_{35}]{(-2)R_{34}}\begin{bmatrix}1&2&1&3\\0&0&0&0\\0&1&1&-2\\0&0&0&1\\0&0&0&0\end{bmatrix}\xrightarrow{R_{24}}\begin{bmatrix}1&2&1&3\\0&0&0&1\\0&1&1&-2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$\xrightarrow{R_{23}}\begin{bmatrix}1&2&1&3\\0&1&1&-2\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

(i) $\{(1,2,2,0,0)^T,(2,4,5,2,2)^T,(3,6,4,-3,-4)^T\}$

(ii) Rank(A)=3

(iii) Nullity(A)=n-rank(A)=4-3=1

Q5(a):

$$\begin{bmatrix}1&1&0\\-1&1&1\\0&1&1\\1&0&1\end{bmatrix}\xrightarrow[(-1)R_{14}]{1R_{12}}\begin{bmatrix}1&1&0\\0&2&1\\0&1&1\\0&-1&1\end{bmatrix}\xrightarrow[(-1)R_{34}]{(-1)R_{32}}\begin{bmatrix}1&1&0\\0&1&0\\0&1&1\\0&-2&0\end{bmatrix}\xrightarrow[2R_{24}]{\substack{(-1)R_{21}\\(-1)R_{23}}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

So S is a basis of W.

(b):

$$\cos(\theta) = \frac{<v_1, v_2>}{\|v_1\|\|v_2\|} = \frac{1-1+0+0}{3(3)} = 0$$

$$\theta = \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ$$

(c): $u_1=v_1=(1,-1,0,1)$

$$u_2 = v_2 - \frac{<v_2, u_1>}{\|u_1\|^2} u_1 = (1,1,1,0)$$

$$u_3 = v_3 - \frac{<v_3, u_1>}{\|u_1\|^2} u_1 - \frac{<v_3, u_2>}{\|u_2\|^2} u_2$$

$$= (0,1,1,1) - 0 - \frac{2}{3}(1,1,1,0) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 1)$$

$$q_3 = 3u_3 = (-2,1,1,3)$$

$$w_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}}(1,-1,0,1)$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3}}(1,1,1,0)$$

$$w_3 = \frac{q_3}{\|q_3\|} = \frac{1}{\sqrt{15}}(-2,1,1,3)$$