$\boxed{I}$ Vector spaces (Linear spaces)

Definition

we say that a set V is a vector space over the field F if we have:

(I) $(V, +)$ satisfies the following conditions:

(1) closed : $a, b \in V \Rightarrow a+b \in V$

(2) associetive : $(a+b)+c = a+(b+c)$ $\forall a, b, c \in V$.

(3) identity $0 \in V$: $a+0 = 0+a = a$ $\forall a \in V$

(4) inverse : $\forall a \in V$ there is $(-a) \in V$ where
$a+(-a) = (-a)+a = 0$

(5) Commutative : $a+b = b+a$

(II) the product $\boxed{\lambda a \in V}$ where $\lambda \in F$ and $a \in V$
which satisfies the following

(1) $1 \cdot a = a$ $\forall a \in V$

(2) $\lambda(a+b) = \lambda a + \lambda b$ $\forall \lambda \in F$ and $a, b \in V$

(3) $\lambda_1(\lambda_2 a) = (\lambda_1 \lambda_2) a$ $\forall \lambda_1, \lambda_2 \in F$ and $a \in V$

(4) $(\lambda_1 + \lambda_2) a = \lambda_1 a + \lambda_2 a$ $\forall \lambda_1, \lambda_2 \in F$ and $a \in V$.

and then, we write $\boxed{V(F)}$

Remark

For any $V(F)$, we have to know the definition of

① $v_1 + v_2 \in V$ $\forall v_1, v_2 \in V$

② $\lambda v \in V$ $\forall \lambda \in F$ and $v \in V$

Remark

In this course, F is either $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers)

Example $\mathbb{Z}(\mathbb{R})$ where $(\mathbb{Z},+)$ is the regular addition and $\lambda \cdot a$ is regular product where $\lambda \in \mathbb{R}$ and $a \in \mathbb{Z}$, is not Linear space

Let $\lambda = 1/2$ and $v = 3$ then

$\lambda v = \frac{3}{2} \notin \mathbb{Z}$

Example $\mathbb{R}(\mathbb{R})$ is Linear space.

Example $\mathbb{R}^2(\mathbb{R})$ is Linear space where

(i) $(a,b) + (c,d) = (a+c, b+d) \in \mathbb{R}^2$

(ii) $\lambda(a,b) = (\lambda a, \lambda b) \in \mathbb{R}^2$

(iii) $0 = (0,0)$

(iv) If $v = (a,b)$ then $-v = (-a,-b)$ because

$v + (-v) = (a,b) + (-a,-b) = (0,0) = 0$

Example $\mathbb{R}^n(\mathbb{R})$ is Linear space which is called Eucleaden space, Every $v \in \mathbb{R}^n$ is n-tuple

(i) $v = (a_1, a_2, -, a_n)$

(ii) $v_1 + v_2 = (a_1, -, a_n) + (b_1, -, b_n) = (a_1+b_1, -, a_n+b_n)$

(iii) $\lambda v = \lambda(a_1, -, a_n) = (\lambda a_1, -, \lambda a_n)$

(v) $0 = (0, -, 0)$

(iv) $-v = (-a_1, -, -a_n)$ where $v = (a_1, -, a_n)$

3

Example Let $F_5(x)$ be the set of all Polynomials of degree five. $F_5(x)(\mathbb{R})$ is not Linear space because

$$f(x) = x^5 + 2x - 3 \in F_5(x)$$
$$g(x) = -x^5 + 3x^2 \in F_5(x)$$

But $f(x) + g(x) = 3x^2 + 2x - 3 \notin F_5(x)$

Example $P_n(x)$ is the set of all Polynomials of degree $\leq n$. Then $P_n(x)(\mathbb{R})$ is Linear space such that

(1) $(a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) + (b_n x^n + b_{n-1} x^{n-1} + \dots + b_0)$
$= (a_n + b_n) x^n + (a_{n-1} + b_{n-1}) x^{n-1} + \dots + (a_0 + b_0)$

(2) $\lambda (a_n x^n + a_{n-1} x^{n-1} + \dots + a_0) = \lambda a_n x^n + \lambda a_{n-1} x^{n-1} + \dots + \lambda a_0$

(3) $0 = 0$ (as polynomial)

(4) $-(a_n x^n + \dots + a_0) = -a_n x^n - a_{n-1} x^{n-1} - \dots - a_0$

Remark

we write Linear spaces such as $\mathbb{R}^n$, $P_n(x)$ rather than $\mathbb{R}^n(\mathbb{R})$, $P_n(x)(\mathbb{R})$.

Definition

Let $V(F)$ be linear space where v_1, v_2 and $v_n \in V$. we say that w is Linear Combination of $v_1, v_2, \dots, v_n$ if there are $\lambda_1, \dots, \lambda_n \in F$ such that

$$w = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$$

Example : Let $\left.\begin{array}{l} v_1 = (1,3,2,1) \\ v_2 = (2,-2,-5,4) \\ v_3 = (2,-1,3,6) \end{array}\right\}$ be vectors of $\mathbb{R}^4$.

Is $u = (2,5,-4,0)$ a Linear combination of v_1, v_2 and v_3 ?

solution suppose that

$(2,5,-4,0) = \lambda_1 (1,3,2,1) + \lambda_2 (2,-2,-5,4) + \lambda_3 (2,-1,3,6)$

we can conclude the following system

$$\begin{aligned} \lambda_1 + 2\lambda_2 + 2\lambda_3 &= 2 \\ 3\lambda_1 - 2\lambda_2 - \lambda_3 &= 5 \\ 2\lambda_1 - 5\lambda_2 + 3\lambda_3 &= -4 \\ \lambda_1 + 4\lambda_2 + 6\lambda_3 &= 0 \end{aligned}$$

If the system is consistent then there are values of λ_1, λ_2 and λ_3, and hence u is Linear combination of v_1, v_2 and v_3.

Now, the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 3 & -2 & -1 & 5 \\ 2 & -5 & 3 & -4 \\ 1 & 4 & 6 & 0 \end{array}\right] \xrightarrow[\text{elimination}]{\text{after}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice here (number of Equation = number of veriables)

So, we have unique solution $\lambda_3 = -1$, $\lambda_2 = 1$, $\lambda_1 = 2$

Therefore $\boxed{u = 2v_1 + v_2 - v_3}$

** Linear subspaces

Definition Let $V(F)$ be a Linear space and $\phi \neq W \subseteq V$. we say that W is Linear subspace of V if $W(F)$ is Linear space.

Example Every Linear space V has two trivial subspace which are : 0 and V

Criterion Let V be a Linear space. Then

$$\left.\begin{array}{l} W \text{ is Linear} \\ \text{subspace of} \\ V \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \text{(i) } \phi \neq W \subseteq V \\ \text{(ii) } \forall\, v_1, v_2 \in W \text{ then } v_1 + v_2 \in W \\ \text{(iii) } \forall\, v \in W \text{ and } \lambda \in F, \text{ then } \lambda v \in W. \end{array}\right.$$

Example we know that $\mathbb{R}(\mathbb{R})$ is Linear space Let $w = \{3a \,;\, a \in \mathbb{R}\}$. Then w is Linear subspace for

(i) $0 = 3(0) \in w \Rightarrow w \neq \phi$

(ii) Let $v_1, v_2 \in w \Rightarrow v_1 = 3a \wedge v_2 = 3b$

Now, $v_1 + v_2 = 3a + 3b = 3(a+b) \in w$

(iii) Let $v \in w$ and $\lambda \in \mathbb{R}$. Then

$\lambda v = \lambda(3a) = 3(\lambda a) \in w$ ∎

Example : Let $W = \{(x,0) \,;\, x \in \mathbb{R}\}$. Then W is Linear subspace of $\mathbb{R}^2$ For :

(i) $0 = (0,0) \in W \Rightarrow W \neq \phi$

(ii) Let $v_1 = (a,0)$ and $v_2 = (b,0) \in W$. Then $v_1 + v_2 = (a+b, 0) \in W$

(iii) Let $v = (a,0)$ and $\lambda \in \mathbb{R}$. Then $\lambda v = \lambda(a,0) = (\lambda a, 0) \in W$

Example Let W be the set of all points lie on the line $y = 3x$. Then W is linear subspace of $\mathbb{R}^2$?
For:

we can describe W as follows:

$$W = \{(x, 3x) \; ; \; x \in \mathbb{R}\}$$

(i) $0 = (0,0) \in W \Rightarrow W \neq \emptyset$

(ii) Let $v_1, v_2 \in W \Rightarrow v_1 = (a, 3a)$, $v_2 = (b, 3b)$

So, $v_1 + v_2 = (a+b, 3a+3b) = (a+b, 3(a+b)) \in W$

(iii) Let $v \in W$ and $\lambda \in \mathbb{R} \Rightarrow \lambda v = \lambda (x, 3x)$

$= (\lambda x, \lambda 3x) = (\lambda x, 3(\lambda x)) \in W$

Example Let W_1 and W_2 be two linear subspace of V. Then

① $W_1 \cap W_2$ is linear subspace. For

(i) as $0 \in W_1$ and $0 \in W_2 \Rightarrow 0 \in W_1 \cap W_2 \Rightarrow$
$W_1 \cap W_2 \neq \emptyset$

(ii) Let $v_1, v_2 \in W_1 \cap W_2 \xrightarrow{\text{and}} \begin{cases} v_1, v_2 \in W_1 \rightarrow v_1 + v_2 \in W_1 \\ v_1, v_2 \in W_2 \rightarrow v_1 + v_2 \in W_2 \end{cases}$ and

So, $v_1 + v_2 \in W_1 \cap W_2$

(iii) Let $\lambda \in F$ and $v \in W_1 \cap W_2 \xrightarrow{\text{and}} \begin{cases} \lambda \in F \text{ and } v \in W_1 \\ \lambda \in F \text{ and } v \in W_2 \end{cases}$

$\Rightarrow$ and $\begin{cases} \lambda v \in W_1 \\ \lambda v \in W_2 \end{cases}$, so, $\lambda v \in W_1 \cap W_2$

② $W_1 \cup W_2$ is not necessarily linear subspace
<u>for example</u> $W_1 = \{3x \; ; \; x \in \mathbb{R}\}$ and $W_2 = \{2x \; ; \; x \in \mathbb{R}\}$ are two linear subspace of $\mathbb{R}$.
Let $v_1 = 2$ and $v_2 = 3$. clearly $v_1, v_2 \in W_1 \cup W_2$
but $v_1 + v_2 = 2 + 3 = 5 \notin W_1 \cup W_2$

Remark

$M_n(\mathbb{R})$ is the set of all squar matrices of size $n\times n$.
we know the sum of matrices and Product
scalar of matrix will make $M_n(\mathbb{R})$ is a Linear
space.

Example: $W = \{A \in M_2(\mathbb{R}) : A\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} A\}$
is a Linear subspace. For:

(i) Since $0\cdot\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\cdot 0 = 0$, $0 \in W$. So, $W \neq \phi$

(ii) Let $A, B \in W$. The goal is to prove $A+B \in W$,
i.e. $(A+B)\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}(A+B)$. For that

$$\begin{aligned}
L.H.S &= (A+B)\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\\
&= A\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} + B\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\\
&= \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}A + \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}B \quad (\text{since } A, B \in W)\\
&= \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}(A+B) = R.H.S
\end{aligned}$$

(iii) Let $A \in W$ and $\lambda \in \mathbb{R}$. The goal is to prove
that $\lambda A \in W$; i.e., $(\lambda A)\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}(\lambda A)$
for that

$$\begin{aligned}
L.H.S &= \lambda A\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}\\
&= \lambda\begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}A \quad (\text{since } A \in W)\\
&= \begin{bmatrix}-1 & 1\\ 0 & 1\end{bmatrix}(\lambda A) \quad (\text{since } \lambda \text{ is scalar})\\
&= R.H.S
\end{aligned}$$

⑧ Let $A = \{ \begin{bmatrix} \alpha & \beta \\ \gamma & \sigma \end{bmatrix} ; \alpha+\beta = \gamma+\sigma \} \subseteq M_2(\mathbb{R})$. Then A is Linear subspace of $M_2(\mathbb{R})$.

Proof: (i) $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in A \Rightarrow A \neq \phi$

(ii) Let $v_1 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \sigma_1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \sigma_2 \end{bmatrix} \in W$.

Then, $\alpha_1+\beta_1 = \gamma_1+\sigma_1$ and $\alpha_2+\beta_2 = \gamma_2+\sigma_2$.

Now, $v_1+v_2 = \begin{bmatrix} \alpha_1+\alpha_2 & \beta_1+\beta_2 \\ \gamma_1+\gamma_2 & \sigma_1+\sigma_2 \end{bmatrix} \in A$ because

$$\begin{aligned}(\alpha_1+\alpha_2)+(\beta_1+\beta_2) &= (\alpha_1+\beta_1)+(\alpha_2+\beta_2) \\ &= (\gamma_1+\sigma_1)+(\gamma_2+\sigma_2) \\ &= (\gamma_1+\gamma_2)+(\sigma_1+\sigma_2)\end{aligned}$$

(iii) Let $v = \begin{bmatrix} \alpha & \beta \\ \gamma & \sigma \end{bmatrix} \in W$ and $\lambda \in \mathbb{R}$.

Now, $\lambda v = \begin{bmatrix} \lambda\alpha & \lambda\beta \\ \lambda\gamma & \lambda\sigma \end{bmatrix}$ where

$$\begin{aligned}\lambda\alpha+\lambda\beta &= \lambda(\alpha+\beta) \\ &= \lambda(\gamma+\sigma) \quad (\text{since } v \in W) \\ &= \lambda\gamma+\lambda\sigma\end{aligned}$$

So, $\lambda v \in W$. ∎

(9) ** Spanning set

Def: Let V be a vector space. The set $S = \{v_1, \dots, v_n\}$ of vectors of V is called a spanning set of V if every vector $v \in V$ is a linear combination of vectors of S.

Remarks:

(1) To prove that $S = \{v_1, \dots, v_n\}$ is spanning set of V, Let $v \in V$ and suppose that

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \quad \text{for some } \lambda_1, \dots, \lambda_n \in \mathbb{R}$$

If there is a solution for $\lambda_1, \dots, \lambda_n$ then v is linear combination of $S \Rightarrow S$ is spanning set

(2) Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of V we define the set

$$\text{Span}(S) = \{\lambda_1 v_1 + \dots + \lambda_n v_n : \lambda_1, \dots, \lambda_n \in \mathbb{R}\}$$

Notice that

(1) Span(S) is Linear subspace of V

(2) If $\text{span}(S) = V$ then S is spanning set of V

Theorem

Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of a Linear space V. Then span(S) is the smallest Linear subspace of V contains S, i.e., if W is Linear subspace contains S, then $\text{span}(S) \subseteq W$.

Example Let $\{v_1 = (1,1),\ v_2 = (1,-2)\} = S$. Does S span $V = \mathbb{R}^2$?

Solution Let $v = (a,b)$ be any vector of $\mathbb{R}^2$, and suppose that there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $(a,b) = \lambda_1 (1,1) + \lambda_2 (1,-2)$

Then, we have

$$\left.\begin{aligned} \lambda_1 + \lambda_2 &= a \\ \lambda_1 - 2\lambda_2 &= b \end{aligned}\right\}$$ it is non-Homogenous system

Notice that

$$|A| = \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = -3 \neq 0$$

So, A^{-1} existed

So, the system is consistent

Hence, $S = \{v_1, v_2\}$ spans $\mathbb{R}^2$ $\boxtimes$

Example

Does $\{1, 1-x, 1-x^2\}$ spans $P_2(x)$?

Solution Let ax^2+bx+c be any vector of $P_2(x)$ and suppose that there exists $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$ax^2+bx+c = \lambda_1(1) + \lambda_2(1-x) + \lambda_3(1-x^2)$$

Then we have :

$$\left.\begin{aligned} \lambda_1 + \lambda_2 + \lambda_3 &= c \\ -\lambda_2 \quad &= b \\ -\lambda_3 &= a \end{aligned}\right\} \Rightarrow$$ it is clear that the system has unique solution

$\Rightarrow \{1, 1-x, 1-x^2\}$ spans $P_2(x)$.

Example

Does $\left\{ \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix} \right\}$ spans $M_2(\mathbb{R})$?

Solution Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any vector of $M_2(\mathbb{R})$, and suppose that there exists $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \lambda_1 \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix} + \lambda_4 \begin{bmatrix} 0 & 8 \\ 2 & 1 \end{bmatrix}$$

Then, we have

$$\left.\begin{aligned} -\lambda_2 - 2\lambda_3 \qquad &= a \\ 2\lambda_1 + 3\lambda_2 \qquad + 8\lambda_4 &= b \\ \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 &= c \\ 2\lambda_2 + 3\lambda_3 + \lambda_4 &= d \end{aligned}\right\}$$ Non-Homogenous square-system

we will examine $|A| = \begin{vmatrix} 0 & -1 & -2 & 0 \\ 2 & 3 & 0 & 8 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 1 \end{vmatrix}$ $\underline{\underline{-2R_3+R_2}}$

$$= \begin{array}{c} + \\ - \\ + \\ \\ \end{array} \begin{vmatrix} 0 & -1 & -2 & 0 \\ 0 & 1 & -2 & 4 \\ 1 & 1 & 1 & 2 \\ 0 & 2 & 3 & 4 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -1 & -2 & 0 \\ 1 & -2 & 4 \\ 2 & 3 & 4 \end{vmatrix} \quad \frac{\underline{\underline{R_1+R_2}}}{2R_1+R_3}$$

$$= \begin{vmatrix} -1 & -2 & 0 \\ 0 & -4 & 4 \\ 0 & -1 & 4 \end{vmatrix} = (-1) \begin{vmatrix} -4 & 4 \\ -1 & 4 \end{vmatrix}$$

$$= 12 \neq 0$$

So, the system has unique solution

Hence S spans $M_2(\mathbb{R})$ ■

** Linear Space of Solutions of $AX=0$.

Theorem: Let $AX=0$ be a Homogenous system of Linear equations. Then The solution set of such system is Linear subspace of $\mathbb{R}^n$

Proof Let $S = \{ v \in \mathbb{R}^n : v \text{ is a solution} \}$
$= \{ v \in \mathbb{R}^n : Av = 0 \}$

(i) $0 \in \mathbb{R}^n$ is a solution because $A \cdot 0 = 0 \Rightarrow 0 \in S$
$\Rightarrow S \neq \emptyset$

(ii) Let $v_1, v_2 \in S$. Our goal is to Prove that $v_1 + v_2 \in S$; i.e., $A(v_1 + v_2) = 0$. For that

$$\begin{aligned} L.H.S = A(v_1+v_2) &= Av_1 + Av_2 \\ &= 0 + 0 \quad (\text{as } v_1, v_2 \in S) \\ &= 0 = R.H.S \end{aligned}$$

(iii) Let $v \in S$ and $\lambda \in \mathbb{R}$. Our goal is to Prove that $\lambda v \in S$; i.e. $A(\lambda v) = 0$. For that

$$L.H.S = A(\lambda v) = \lambda(Av) = \lambda(0) = 0 = R.H.S$$

Hence, S is a Linear subspace of $\mathbb{R}^n$.

Example: Find the span set of the space of the solutions of the following system:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right] ?$$

Solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that number of Parameters $= \boxed{3} - \boxed{1} = 2$

Let $x = s$, $y = t$. Then $z = \frac{-s+2t}{3}$ (since $x - 2y + 3z = 0$)

So, $S = \left\{ \begin{bmatrix} s \\ t \\ \frac{-s+2t}{3} \end{bmatrix} : s,t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} s \\ 0 \\ \frac{-s}{3} \end{bmatrix} + \begin{bmatrix} 0 \\ t \\ \frac{2t}{3} \end{bmatrix} : s,t \in \mathbb{R} \right\}$

$= \left\{ s\begin{bmatrix} 1 \\ 0 \\ \frac{-1}{3} \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} : s,t \in \mathbb{R} \right\} \Rightarrow \text{span}(V) = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{-1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{2}{3} \end{bmatrix} \right\}$

★ stander span set of some famous Linear space:

$span(\mathbb{R}^2) = \{(1,0), (0,1)\}$ becaus for every

$(a,b) \in \mathbb{R}^2 \Rightarrow (a,b) = (a,0) + (0,b)$

$= a(1,0) + b(0,1)$.

$span(\mathbb{R}^3) = \{(1,0,0), (0,0,1), (0,0,1)\}$

$span(M_2(\mathbb{R})) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$span(P_2(x)) = \{1, x, x^2\}$

$span(P_1(x)) = \{1, x\}$

Scanned with CamScanner

1

** Linear Depenedency

Definition

Let $\{v_1, v_2, \dots, v_n\}$ be a set of vectors of a Linear space.

(i) If $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ iff $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ then $\{v_1, v_2, \dots, v_n\}$ is Linear independente where $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$

(ii) If $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$ has a non-zero values of $\lambda_1, \dots, \lambda_n$ then $\{v_1, \dots, v_n\}$ is Linear dependente

Remark

To examine $\{v_1, \dots, v_n\}$ is Linear independente or not,

STEP1 : suppose $\lambda_1 v_1 + \dots + \lambda_n v_n = 0$.

STEP2 : Deduce Homogenous system

STEP3 : If it has unique (zero) solution, i.e., $\lambda_1 = \dots = \lambda_n = 0$ then $\{v_1, \dots, v_n\}$ is Linear independe. otherwise, The system has non-zero solution, and then it is Linear Dependonte.

(Ex) Let $\{(6,2,1), (-1,3,2)\}$ be a set of vectors of $\mathbb{R}^3$. Does the set be Linear independe ?

Solution suppose that $\lambda_1 (6,2,1) + \lambda_2 (-1,3,2) = (0,0,0)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then

$$\left.\begin{aligned} 6\lambda_1 - \lambda_2 &= 0 \\ 2\lambda_1 + 3\lambda_2 &= 0 \\ \lambda_1 + 2\lambda_2 &= 0 \end{aligned}\right\}$$ rectaigular - Homogenous system

$$\left[\begin{array}{cc|c} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow{\text{REEF}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$ has unique solution $\lambda_1 = \lambda_2 = 0$

$\therefore \{(6,2,1), (-1,3,2)\}$ is Linear independent.

(ex) Let $S = \{ (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}), (3, 4, 7/2), (-\frac{3}{2}, 6, 2) \}$ be a set of vectors of $\mathbb{R}^3$. Determine, if S is Linear independent or dependent ?

Solution

suppose that $\lambda_1 (\frac{3}{4}, \frac{5}{2}, \frac{3}{2}) + \lambda_2 (3, 4, 7/2) + \lambda_3 (-\frac{3}{2}, 6, 2) = (0, 0, 0)$

then,

$$\left.\begin{array}{lcl} \frac{3}{4}\lambda_1 + 3\lambda_2 - \frac{3}{2}\lambda_3 = 0 & \Longleftrightarrow & 3\lambda_1 + 12\lambda_2 - 6\lambda_3 = 0 \\ \frac{5}{2}\lambda_1 + 4\lambda_2 + 6\lambda_3 = 0 & \Longleftrightarrow & 5\lambda_1 + 8\lambda_2 + 12\lambda_3 = 0 \\ 3/2\,\lambda_1 + 7/2\,\lambda_2 + 2\lambda_3 = 0 & \Longleftrightarrow & 3\lambda_1 + 7\lambda_2 + 4\lambda_3 = 0 \end{array}\right\}$$

It is square - Homogenous system.

So, $|A| = \begin{vmatrix} 3 & 12 & -6 \\ 5 & 8 & 12 \\ 3 & 7 & 4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 4 & -2 \\ 5 & 8 & 12 \\ 3 & 7 & 4 \end{vmatrix} \quad \underline{\underline{\begin{matrix} -5R_1 + R_2 \\ -3R_1 + R_3 \end{matrix}}}$

$$= 3 \begin{vmatrix} 1 & 4 & -2 \\ 0 & -12 & 22 \\ 0 & -5 & 10 \end{vmatrix}$$

$$= 3(1) \begin{vmatrix} -12 & 22 \\ -5 & 10 \end{vmatrix} \neq 0$$

So, A^{-1} is existed $\Rightarrow$ the zero - solution is the unique solution

$\Rightarrow$ S is Linear independent.

(ex) Let $S = \{ 1 + x^2 , 2 + x + x^2 \}$ be a set of vectors of $P_2(x)$. Determine, if S is Linear independent or not ?

Solution suppose that $\lambda_1 (1 + x^2) + \lambda_2 (2 + x + x^2) = 0 + 0x + 0x^2$

$$\Rightarrow \left.\begin{array}{r} \lambda_1 + 2\lambda_2 = 0 \\ \lambda_2 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{array}\right\}$$

It is clear that $\lambda_1 = \lambda_2 = 0$ is the unique solution.

Hence, S is Linear independent ∎

3

Basis and Dimension

Definition Let $S = \{v_1, \dots, v_n\}$ be a set of vectors of Linear space V. Then

S is a basis of V iff $\begin{cases} ① \ S \text{ spans } V. \\ ② \ S \text{ is Linear independent.} \end{cases}$

Remarks

① If S is a basis of a Linear space V, then $|S|$ is called the dimension of V, $\text{Dim}(V)$, .

② If $\text{Dim}(V) < \infty$, then V is finite dimensional.
If $\text{Dim}(V) = \infty$, then V is infinite dimensional.

③ Every vector space has atleast a basis which is not necessarily unique

standered basis of some famous Linear spaces :

Linear space	Basis	Dim
$\mathbb{R}^2$	$B = \{(1,0), (0,1)\}$	2
$\mathbb{R}^3$	$B = \{(1,0,0), (0,1,0), (0,0,1)\}$	3
$M_2(\mathbb{R})$	$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$	4
$P_1(x)$	$B = \{1, x\}$	2
$P_2(x)$	$B = \{1, x, x^2\}$	3

** Some Properties

Property (1) Let B_1 and B_2 be two basis of ~~the~~ a Linear space V, then

$$|B_1| = |B_2| = Dim(V)$$

Property (2) Let S be a set of vectors of a linear space V such that $0 \in S$, then S is Linear dependent which implies that S is not basis

Property (3) Let S be a set of vectors of a Linear space V where there exists a vector $v \in S$ can be written as Linear Combination of the remain vectors in S. Then S is Linear dependent which implies that S is not basis

for example $\{ \overset{v_1}{(2,-4)}, \overset{v_2}{(-1,2)} \} \subseteq \mathbb{R}^2$.
Notice that $(2,-4) = -2(-1,2)$. So,
$\{(2,-4), (-1,2)\}$ is Linear dependent.

Property (4) Let S be a set of vector of a Linear space V. If S is Linear dependent then there exists $v \in S$ such that v is a Linear Combination of the remain vectors in S

for example $S = \{2x, x^2, 5\} \subseteq P_2(x)$.
Notice that

$$5 \neq \lambda_1(2x) + \lambda_2(x^2)$$
$$2x \neq \lambda_1(5) + \lambda_2(x^2)$$
$$x^2 \neq \lambda_1(5) + \lambda_2(2x)$$

So, S is Linear indepent.

Property 5 Let W be a linear subspace of V then $Dim(W) \leq Dim(V)$.

Property 6 Let B be a spanning set of a vector space V. If S is Linear independent then $|S| \leq |B|$

for example

Let $Dim(V) = 3$ and S be a set of vectors where $|S| = 4$. Then S is Linear dependent (because B_v is spanning set of V and $|B_v| = 3$)

$$\boxed{\begin{array}{l}\textbf{Remark} \ \text{If } |S| > Dim(V) \text{ then } S \text{ is Linear dependent} \\ \text{If } S \text{ is Linear indepent then } |S| \leq Dim(V)\end{array}}$$

Property 7 Let S be a set of vectors of a Linear space V. Then

How to prove S is a basis if you know $Dim(V)$

(1) If $|S| = Dim(V)$ and S is Linear indepent then S is a basis.

(2) If $|S| = Dim(V)$ and S is spanning set of V then S is a basis.

Property 8 Let S be Linear indepent ~~of ve~~ set of a Linear space V where $|S| < Dim(V)$. Then there exists a basis B of V such that $S \subseteq B$.

(Ex) find a basis of $\mathbb{R}^2$ contains $v = (1,1)$?

Notice that $Dim(V) = Dim(\mathbb{R}^2) = 2$.

So, choose $v' = (0,1)$. It is clear that $v' \neq \lambda v$ and $v \neq \lambda' v'$ for any $\lambda, \lambda' \in \mathbb{R}$

So, $\{v, v'\}$ is Linear indepent $\Rightarrow \{v, v'\}$ is basis.

(Ex) find a basis of $\mathbb{R}^3$ contains $v = (1,1,1)$.

Solution

we know that $\text{Dim}(\mathbb{R}^3) = 3$

so, choose $v_1 = (1,0,0)$

$v_2 = (0,1,0)$

then $S = \{(1,1,1), (1,0,0), (0,1,0)\}$ is Linear independent because

(After forming $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$)

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0$$

so, the system has unique solution (zero solution).

Theorem A squar matrix is invertible iff it is Columns are Linear independent. iff it is rows are Linear independent

Theorem If A is row-echelon form matrix then the non-zero rows is Linear independent

for example

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is row-echelon form

Then, the set $\{(1,2,3,4), (0,0,1,1)\}$ is Linear independent in $\mathbb{R}^4$.

** Rank of matrix

$\text{Rank}(A) = \text{Dim}(V)$ where V is the Linear space spanned by columns of A.

How to find rank A ?

To find Rank(A) : step 1 : A in row-echolon form

step 2 : Rank(A) = number of non-zero rows.

7

** Row space, column space and Null space:

Def: Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be $n \times m$ matrix.

The vectors

$r_1 = (a_{11}, a_{12}, \dots, a_{1n})$

$\vdots$

$r_m = (a_{m1}, a_{m2}, \dots, a_{mn})$

are vectors in R^n which called row-vectors

The vectors

$c_1 = \begin{bmatrix} a_{11} \\ | \\ a_{m1} \end{bmatrix}$ --- $c_n = \begin{bmatrix} a_{1n} \\ | \\ a_{mn} \end{bmatrix}$ are vectors in R^m

which called colum vectors

(1) The subspace which spanned by row-vectors ~~are~~ is called row space of A.

(2) The subspace which spanned by column-vectors is called column space of A

(3) The space of solutions of $AX=0$ is called the null space of A

** Finding a basis of null space of matrix

(ex) find a basis of null space of

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

solution we have to find the solution of $AX=0$ by gauss method

$$S = \left\{ \begin{bmatrix} -3s-4t-2k \\ s \\ -2t \\ t \\ k \\ 0 \end{bmatrix} ;\ s, t \text{ and } k \in \mathbb{R} \right\}$$

~~note~~

$$= \begin{bmatrix} -3s \\ s \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -4t \\ 0 \\ -2t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2k \\ 0 \\ 0 \\ 0 \\ k \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} k \Rightarrow S = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

the basis

Remark

(1) Let A be a matrix. Then the row factors which has Leaders 1 (as echolon form) is the basis of the row space of A.

(2) Let A be a matrix. Then the column with leaders 1 (as echolon form) is the basis of the column space.

(Ex) If $A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. It is clear that A is in Row echolon form. So,

$$B_{\text{row-space}} = \{(1,-2,5,0,3),\ (0,1,3,0,0),\ (0,0,0,1,0)\}$$

$$B_{\text{colum-space}} = \{(1,0,0,0),\ (-2,1,0,0),\ (0,0,1,0)\}$$

** Basis of the space spanned by a set of vectors

Example Let $S = \{ v_1 = (1,2,2,-1),\ v_2 = (-3,-6,-6,3)$
$v_3 = (4,9,9,-4),\ v_4 = (-2,-1,-1,2)$
$v_5 = (5,8,9,-5),\ v_6 = (4,2,7,-4) \}$

find the Basis of Span(S) ?

Answer: write the vectors as columns

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

write A as row echolon form $A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$$B = \{(1,0,0,0),\ (4,1,0,0),\ (5,-2,1,0)\}$$

Scanned with CamScanner

Rule Let A be a matrix. Then

$$\text{Dim}(\text{row space of } A) = \text{Dim}(\text{Column space of } A)$$

Definition

① $\text{Dim}(\text{null space of } A) = \text{nullity}(A)$

② $\text{Dim}(\text{row space}) = \text{Dim}(\text{column space}) = \text{rank}(A)$

Example find the rank and nullity of the following matrix:

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution

STEP1 : write A on the row-echolon form.

$$A = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

STEP2 : To find nullity (A) :

(i) find the solutions of $AX=0$.

$N(E) = 2$
$N(V) = 6$ $\Rightarrow N(P) = 6-2 = 4$

we have two equations

$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$

$x_2 - 2x_3 - 12x_4 - 16x_5 + 5x_6 = 0$

Let $x_3 = r$ $\quad x_4 = s$ $\quad x_5 = t$ $\quad x_6 = u$

$\Rightarrow x_1 = 4r + 28s + 37t - 13u$

$x_2 = 2r + 12s + 16t - 5u$

(ii) Hence

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \underbrace{\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + s \underbrace{\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_2} + t \underbrace{\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3} + u \underbrace{\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_4}$$

$B_{\text{null-space}} = \{v_1, v_2, v_3, v_4\} \Rightarrow \text{nullity}(A) = 4$

STEP3 To find Rank (A) :

$$B_{\text{row-space}} = \{ (-1, 0, 4, -28, -37, 13), (0, 1, -2, -12, -16, 5) \}$$

Hence $\text{rank}(A) = \text{Dim}(\text{row-space}) = 2$

Rule

Let $A_{m \times n}$, then

$$\text{rank}(A) + \text{nullity}(A) = n \quad \text{(number of columns)}$$

Rule

Let A be matrix

$$\text{rank}(A) = \text{rank}(A^t)$$

example

Let $\underset{5 \times 10}{A}$ where $\text{rank}(A) = 4$. Find $\text{nullity}(A^t)$?

solution

Notice that $\text{size}(A^t) = 10 \times 5$

$\text{rank}(A^t) = \text{rank}(A) = 4$

Now

$\text{rank}(A^t) + \text{nullity}(A^t) = 5 \Rightarrow \text{nullity}(A^t) = 5 - 4 = 1$

III Some Excersices :

① Let $M = \{ (x,y,z) : 2x - y + z = 0 \}$.

(i) Prove M is Linear subspace of $\mathbb{R}^3$?
(ii) find $Dim(M)$

Solution

(i) Homework !

(ii) notice that every $(x,y,z) \in M$ is a solution of $2x - y + z = 0$

$$\left.\begin{matrix} N(E) = 1 \\ N(V) = 3 \end{matrix}\right\} \Rightarrow N(P) = 2$$

Let $x = t$, $y = s$

then $z = -2t + s$

So, $M = \left\{ \begin{bmatrix} t \\ s \\ -2t+s \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

$= \left\{ \begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix} + \begin{bmatrix} 0 \\ s \\ s \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

$= \left\{ t \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} ; s,t \in \mathbb{R} \right\}$

Hence $B = \{ (1,0,-2) , (0,1,1) \}$ spans M

since $(1,0,-2) \neq \lambda (0,1,1) \quad \forall \lambda \in \mathbb{R}$

then B is Linear independant

Therefore, B is basis to $M \Rightarrow Dim(M) = 2$

② Let $M = \{ a + bx + cx^2 + dx^3 : a + b = c - 2d = 0 \}$

(i) Prove M is Linear subspace ?
(ii) find $Dim(M)$?

Solution

(i) Home work!

(ii) We have $a + b = 0 \Rightarrow a = -b$ and
$c - 2d = 0 \Rightarrow c = 2d$

So, any Polynomial belongs to M will be written as :

$-b + bx + 2dx^2 + dx^3$
$= b \underbrace{(1-x)}_{\boxed{v_1}} + d \underbrace{(2x^2 - x^3)}_{\boxed{v_2}}$

Scanned with CamScanner

So, $B = \{ v_1 = -1 + x \ , \ v_2 = 2x^2 + x^3 \}$ spans M

Notice that $v_1 \neq \lambda v_2 \quad \forall \lambda \in \mathbb{R}$

Hence, B is Linear independent

Therefore, B is basis of $M \Rightarrow Dim(M) = 2$

③ Does $S = \{ (1,3,-1), (0,1,5), (2,2,3) \}$ be a basis of $\mathbb{R}^3$?

Solution

Notice that $|S| = 3 = Dim(\mathbb{R}^3)$. So, it is enough to study if S is Linear independent or not.

suppose that

$\lambda_1 (1,3,-1) + \lambda_2 (0,1,5) + \lambda_3 (2,2,3) = (0,0,0)$. Then, we have

$$\left.\begin{array}{l} \lambda_1 + \quad + 2\lambda_3 = 0 \\ 3\lambda_1 + \lambda_2 + 2\lambda_3 = 0 \\ -\lambda_1 + 5\lambda_2 + 3\lambda_3 = 0 \end{array}\right\} \text{ square + Homogenous system}$$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 3 & 1 & 2 \\ -1 & 5 & 3 \end{vmatrix} \xrightarrow[R_1 + R_3]{-3R_1 + R_2} \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & -4 \\ 0 & 5 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -4 \\ 5 & 5 \end{vmatrix} = 25 \neq 0$$

So, we have only the zero solution

Hence, S is Linear independent $\Rightarrow$ S is a basis of $\mathbb{R}^3$.

Definition

Let $B = \{v_1, \dots, v_n\}$ be a basis of V. If $v \in V$ where $v = a_1 v_1 + \dots + a_n v_n$ then $(a_1, \dots, a_n)$ is called the coordinate of v.

Notice that: the coordinate of v by using the basis B is unique.

Scanned with CamScanner

(4) Let $S = \{(1,3), (0,1)\}$ be a basis of $\mathbb{R}^2$. Find the coordinate of $(2,5)$?

solution

since $(2,5) \in \mathbb{R}^2$, $(2,5) = \lambda_1 (1,3) + \lambda_2 (0,1)$

So, $\left.\begin{array}{l} \boxed{\lambda_1 = 2} \\ 3\lambda_1 + \lambda_2 = 5 \end{array}\right\} \Rightarrow \boxed{\lambda_2 = -1}$

Hence $(2,-1)$ is the coordinat of $(2,5)$ respects to the basis S.

** Coordinates and change of basis

Let $B = \{v_1, \dots, v_n\}$ be a basis of V. If $v \in V$ then $v = a_1 v_1 + \cdots + a_n v_n$ where $a_1, \dots, a_n \in \mathbb{R}$.
The coordinate vector ~~of~~ of v respects to B is $[v]_B = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = (a_1, a_2, \dots, a_n)$

Example Let $\{(1,2), (-1,4)\} = S$ be a set of vectors of $\mathbb{R}^2$. (1) Prove that S is a basis ?
(2) find $[(5,6)]_B$?

Solution 1. Homework!

2. Suppose that $(5,6) = \lambda_1 (1,2) + \lambda_2 (-1,4)$

then, we have $\lambda_1 - \lambda_2 = 5$
$2\lambda_1 + 4\lambda_2 = 6$

Therefore $\lambda_1 = \frac{13}{3}$, $\lambda_2 = -\frac{2}{3}$

Hence $[(5,6)]_B = \begin{bmatrix} \frac{13}{3} \\ \frac{-2}{3} \end{bmatrix}$

Example Let $B = \{v_1 = 3, v_2 = -1 + x, v_3 = x^2\}$ be a set of vetors of $P_2(x)$. (1) Prove that B is basis.
(2) find $[1 - x^2]_B$?

Solution (1) Notice that $|B| = 3 = \operatorname{Dim}(P_2(x))$
So, It is enough to prove that B is Linear indep. to show that B is basis. For that

suppose that

$$\lambda_1 (3) + \lambda_2 (-1+x) + \lambda_3 (x^2) = 0 + 0x + 0x^2$$

Then, we have

$$\left.\begin{aligned} 3\lambda_1 - \lambda_2 &= 0 \\ \lambda_2 &= 0 \\ \lambda_3 &= 0 \end{aligned}\right\}$$

Hence $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Therefore, B is Linear indep - which implies that B is basis.

2. suppose that $1-x^2 = \lambda_1 (3) + \lambda_2 (-1+x) + \lambda_3 (x^2)$

Then

$$\left.\begin{aligned} 3\lambda_1 - \lambda_2 &= 1 \\ \boxed{\lambda_2 = 0} \\ \boxed{\lambda_3 = -1} \end{aligned}\right\} \Rightarrow \boxed{\lambda_1 = \frac{1}{3}}$$

Hence $[(1-x^2)]_B = \begin{bmatrix} 1/3 \\ 0 \\ -1 \end{bmatrix}$

** Transmission matrix

Let B_1 and B_2 be two basis of a Linear space V. The following example will show you how to find transmission matrix from B_1 to B_2 and the converse

Example: Let $B = \{(2,1), (0,3)\}$ and $B' = \{(-1,0), (3,3)\}$ be two basis of $\mathbb{R}^2$. Find the transmission matrix from B into B'?

solution step 1: find $[(2,1)]_{B'}$. For that suppose that $(2,1) = \lambda_1 (-1,0) + \lambda_2 (3,3)$.

Then $\left.\begin{aligned} -\lambda_1 + 3\lambda_2 &= 2 \\ 3\lambda_2 &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} \lambda_2 &= 3 \\ \lambda_1 &= -1 \end{aligned}$

Hence $[(2,1)]_{B'} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

step 2: find $[(0,3)]_{B'}$. By the same method, we will deduce that

$$[(0,3)]_{B'} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

There fore ;

$$\underset{B' B}{P} = \text{transmission matrix from } B \text{ into } B' = \begin{bmatrix} -1 & 3 \\ -\frac{1}{3} & 1 \end{bmatrix}$$

Rule

Let B_1 and B_2 be two basis of a Linear space V and $v \in V$. Then

$$[v]_{B_2} = {}_{B_2}P_{B_1} \, [v]_{B_1}$$

Example Let $S = \{ \frac{1}{2}, -x, 2x^2 \}$ be a basis of $P_2(x)$. If $[f(x)]_S = (1, 2, -1)$ and ${}_{S^*}P_S = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$, find $[f(x)]_{S^*}$ where S^* is another basis of $P_2(x)$. Find $f(x)$?

solution

(i) $$[f(x)]_{S^*} = {}_{S^*}P_S \, [f(x)]_S$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

(ii) $$f(x) = \boxed{1}\left(\tfrac{1}{2}\right) + \boxed{2}(-x) + \boxed{-1}(2x^2)$$

$$= \frac{1}{2} - 2x - 2x^2 .$$

Example Let $S = \{ (2,1), (0,3) \}$ be a basis of $\mathbb{R}^2$. If S^* is another basis of $\mathbb{R}^2$ such that ${}_{S^*}P_S = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. find S^*

solution From ${}_{S^*}P_S$, we can deduce that

$$[(2,1)]_{S^*} = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \text{ and}$$

$$[(0,3)]_{S^*} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

• So, if $S^* = \{(a,b), (c,d)\}$ we have that

$$\left.\begin{aligned} (2,1) &= \frac{1}{\sqrt{5}}(a,b) - \frac{2}{\sqrt{5}}(c,d) \\ (0,3) &= \frac{2}{\sqrt{5}}(a,b) + \frac{1}{\sqrt{5}}(c,d) \end{aligned}\right\}$$

which implies that

$$\frac{1}{\sqrt{5}}a - \frac{2}{\sqrt{5}}c = 2 \quad \cdots (1)$$

$$\frac{1}{\sqrt{5}}b - \frac{2}{\sqrt{5}}d = 1 \quad \cdots (2)$$

$$\frac{2}{\sqrt{5}}a + \frac{1}{\sqrt{5}}c = 0 \quad \cdots (3)$$

$$\frac{2}{\sqrt{5}}b + \frac{1}{\sqrt{5}}d = 3 \quad \text{---} (4)$$

• Solve the system (1) and (3) and the system (2), (4) to get

$$(a,b) = \left(\frac{2}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$$

$$(c,d) = \left(\frac{4}{\sqrt{5}}, -1/\sqrt{5}\right) \quad \blacksquare$$

Rules

(1) $P_{S'S^*} \cdot P_{S^*S} = P_{S'S}$

(2) $P_{S^*S} \cdot P_{SS^*} = I \iff P_{SS^*} = \left(P_{S^*S}\right)^{-1}$