(Draft) KING SAUD UNIVERSITY
COLLEGE OF SCIENCES
DEPARTMENT OF MATHEMATICS

MATH-244 (Linear Algebra); Final Exam; Semester 1 (1443 H)

Max. Marks: 40 **Max. Time: 3 hours**

Note: ***Attempt all the five questions!***

Question 1 [4+2+2 marks]:

a) Find adjoint matrix and matrix of cofactors of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 2 \\ -2 & 3 & 6 \end{bmatrix}$ and also find A^{-1}.

b) Evaluate $det(det(A)B^2A^{-1})$, where **A** and **B** are square matrices of order 3 with $det(\mathbf{A})=\mathbf{3}$ and $det(\mathbf{B}) = \mathbf{2}$.

c) Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 6 & 3 \\ 0 & 2 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 4 \\ -1 & 0 & 8 \end{bmatrix}$. Explain why the matrices A and B are not row equivalent to each other?

Question 2 [5+3 marks]:

a) Find the values of α and β such that the following linear system:

$$\begin{aligned} x - 2y + 3z &= 4 \\ 2x - 3y + \alpha z &= 5 \\ 3x - 4y + 5z &= \beta \end{aligned}$$

has:

i) No solution;
ii) Infinitely many solutions.

b) Let $s_1 = 3 - 2x$, $s_2 = 2 + x$, $s_3 = 1 + x - x^2$, $s_4 = x + x^2 - x^3$. Find the values of a, b, c and d such that $1 - 6x - 3x^2 - 4x^3 = as_1 + bs_2 + cs_3 + ds_4$.

Question 3 [4+4 marks]:

a) Let $\{u_1 = (1,1,1,1), u_2 = (0,1,2,1), u_3 = (1,0,-2,3), u_4 = (1,1,2,-2)\}$ generate the subspace F of the Euclidean space $\mathbb{R}^4$. Then:

i) Find $dim(F)$

ii) Show that $(1,1,0,1) \notin F$.

b) Let $B = \{v_1 = (1,1,2), v_2 = (3,2,1), v_3 = (2,1,5)\}$ and $C = \{u_1, u_2, u_3,\}$ be two bases for $\mathbb{R}^3$ such that

$${}_BP_C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

is the transition matrix from C to B. Find the vectors u_1, u_2 *and* u_3.

Question 4 [(1+2)+(1+1)+3 marks]:

a) Let w_1= (0,0,1), w_2= (0,1,1), w_3= (1,1,1) be vectors in the Euclidean space $\mathbb{R}^3$. Then:
   i) Find the angle between w_1 and w_3.
   ii) By applying the Gram-Schmidt process on $\{w_1, w_2, w_3\}$ to find an orthonormal basis of the Euclidean space $\mathbb{R}^3$.

b) Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(x, y) = (x + 4y, 2x + 3y)$. Find:
   i) $Ker(T)$ ii) $dim\, \mathrm{Im}(T)$

c) Let the linear transformation $\mathrm{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by:

$$\mathrm{T}(x, y) = (x + 2y, x - y, 3\,x + y).$$

Find matrix of the transformation $[\mathrm{T}]_{\mathrm{B}}^{\mathrm{C}}$, where B and C are the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Question 5 [4 + 4 marks]:

a) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find eigenvalue/s of the matrix A and determine one basis of the corresponding eigenspace/s. Then, give reason for the non-diagonalizability of A.

b) Show that the matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$ diagonalizes the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ and then use this fact to compute A^{-1}.

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SOLUTION KEY: 1443/Semester-1/Math-244/Final Exam

Solution of Question 1:

a) $adj(A) = C^T = \begin{bmatrix} 30 & -12 & 4 \\ -16 & 6 & -2 \\ 18 & -7 & 2 \end{bmatrix}$ (2 marks)

and $|A| = -2$. (1 mark)

Hence, $A^{-1} = |A|^{-1} adj(A) = \begin{bmatrix} -15 & 6 & -2 \\ 8 & -3 & 1 \\ -9 & 7/2 & -1 \end{bmatrix}$ (1 mark)

b) $det(det(A)B^3A^{-1}) = (det(A))^3(det(B))^3(det(A))^{-1} = (det(A))^2(det(B))^3 = 36$ (2 marks)

c) $|A| = -2$ and $|B| = 0 \Longrightarrow A$ is invertible but B is non-invertible $\Longrightarrow$ A and B are not row equivalent. (2 marks)

Solution of Question 2:

a) $[A \vdots B] \sim \begin{bmatrix} 1 & -2 & 3 & \vdots & 4 \\ 0 & 1 & \alpha-6 & \vdots & -3 \\ 0 & 0 & 8-2\alpha & \vdots & \beta-6 \end{bmatrix}$. (2 marks)

Hence, the linear system has:

i) no solution if $\alpha = 4$ and $\beta \neq 6$; (1.5 marks)

ii) infinitely many solutions if $\alpha = 4$ and $\beta = 6$. (1.5 marks)

b) $a = 2, b = -3, c = 1, d = 4$. (3 marks)

Solution of Question 3.

a) i) $[u_1\ u_2\ u_3\ u_4] \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (R.E.F.) (1 marks)

$\Longrightarrow F = span\{u_1, u_2, u_3\} \Longrightarrow dim(F) = 3$. (1 marks)

ii) $\{u_1, u_2, u_3, (1,1,0,-1)\}$ is linearly independent and $F = span\{u_1, u_2, u_3\}$. So, $(1,1,0,1) \notin F$ (2 marks)

b) ${}_{B}P_{C} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} = [\,[u_1]_B\ \ [u_2]_B\ \ [u_3]_B\,]$. (1 mark)

Hence, $u_1 = 1v_1 + 1v_2 - 1v_3 = (2,2,-2)$. Similarly, $u_2 = (1,1,8)$ and $u_3 = (3,2,7)$. (3 marks)

Solution of Question 4:

a) i) $\theta = cos^{-1}\frac{1}{\sqrt{3}} \approx 0.955\ rad$ (1 mark)

ii) $u_1 = w_1 = (0,0,1)$, $u_2 = w_2 - \frac{\langle w_2, u_1\rangle}{\|u_1\|^2}u_1 = (0,1,0)$

and $u_3 = w_3 - \frac{\langle w_3, u_1\rangle}{\|u_1\|^2}u_1 - \frac{\langle w_3, u_2\rangle}{\|u_2\|^2}u_2 = (1,0,0)$. (3 marks)

So, $\{e_1 = u_1 = (0,0,1), e_2 = u_2 = (0,1,0), e_3 = u_3 = (1,0,0)\}$ is an orthonormal basis of $\mathbb{R}^3$.

b) i) $Ker(T) = \{(0,0)\}$ (1 mark)

ii) From Part i), T is one-one and so $Im(T) = \mathbb{R}^2$. Hence, $dim\ Im(T) = 2$. (1 mark)

c) $[T]_B^C = [\,[T(1,0)]_C\ \ [T(0,1)]_C\,] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}$. (2 marks)

Solution of Question 5.

a) Eigenvalues = 1, 1, 1 (1 mark)

$E_1 = span\{(0,0,1)\}$ (1 mark)

So, the algebraic multiplicity of the eigenvalue 1 is 3 which is different from its geometric multiplicity 1.

Hence, the given matrix A is not diagonalizable. (2 marks)

b) $P^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix}$, so that $D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. Thus, $A^{-1} = PD^{-1}P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ -1/2 & 0 & 1/2 \end{bmatrix}$. (1+1+2 marks)

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