MATH 260

# Introduction to Differential Equations and Linear Algebra

## Chapter 1
## First Order Differential Equations

## 1.1 Differential Equations and Mathematical Models

The laws of the universe are written in the language of mathematics. Algebra is sufficient to solve many static problems, but the most interesting natural phenomena involve change and are described by equations that relate changing quantities.

Because the derivative $dx/dt = f'(t)$ of the function $f$ is the rate at which the quantity $x = f(t)$ is changing with respect to the independent variable $t$, it is natural that equations involving derivatives are frequently used to describe the changing universe. An equation relating an unknown function and one or more of its derivatives is called a **differential equation.**

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**Example 1** The differential equation

$$\frac{dx}{dt} = x^2 + t^2$$

involves both the unknown function $x(t)$ and its first derivative $x'(t) = dx/dt$. The differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$$

involves the unknown function $y$ of the independent variable $x$ and the first two derivatives $y'$ and $y''$ of $y$. ■

In algebra, we typically seek the unknown *numbers* that satisfy an equation such as $x^3+7x^2-11x+41=0$. By contrast, in solving a differential equation, we are challenged to find the unknown *functions* $y = y(x)$ for which an identity such as $y'(x) = 2xy(x)$—that is, the differential equation

$$\frac{dy}{dx} = 2xy$$

—holds on some interval of real numbers. Ordinarily, we will want to find *all* solutions of the differential equation, if possible.

**Example 2** If $C$ is a constant and

$$y(x) = Ce^{x^2}, \tag{1}$$

then

$$\frac{dy}{dx} = C\left(2xe^{x^2}\right) = (2x)\left(Ce^{x^2}\right) = 2xy.$$

Thus every function $y(x)$ of the form in Eq. (1) *satisfies*—and thus is a solution of—the differential equation

$$\frac{dy}{dx} = 2xy \tag{2}$$

for all $x$. In particular, Eq. (1) defines an *infinite* family of different solutions of this differential equation, one for each choice of the arbitrary constant $C$. By the method of separation of variables (Section 1.4) it can be shown that every solution of the differential equation in (2) is of the form in Eq. (1). ■

## Examples and Terminology

**Example 7** If $C$ is a constant and $y(x) = 1/(C-x)$, then

$$\frac{dy}{dx} = \frac{1}{(C-x)^2} = y^2$$

if $x \neq C$. Thus

$$y(x) = \frac{1}{C-x} \tag{8}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2 \tag{9}$$

on any interval of real numbers not containing the point $x = C$. Actually, Eq. (8) defines a *one-parameter family* of solutions of $dy/dx = y^2$, one for each value of the arbitrary constant or "parameter" $C$. With $C = 1$ we get the particular solution

$$y(x) = \frac{1}{1-x}$$

that satisfies the initial condition $y(0) = 1$. As indicated in Fig. 1.1.5, this solution is continuous on the interval $(-\infty, 1)$ but has a vertical asymptote at $x = 1$. ■

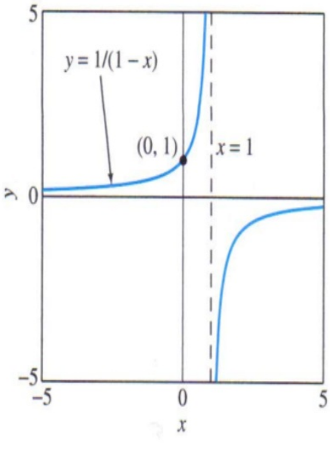



**FIGURE 1.1.5.** The solution of $y' = y^2$ defined by $y(x) = 1/(1-x)$.

**Example 8** Verify that the function $y(x) = 2x^{1/2} - x^{1/2}\ln x$ satisfies the differential equation

$$4x^2y'' + y = 0 \tag{10}$$

for all $x > 0$.

**Solution** First we compute the derivatives

$$y'(x) = -\tfrac{1}{2}x^{-1/2}\ln x \quad \text{and} \quad y''(x) = \tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}.$$

Then substitution into Eq. (10) yields

$$4x^2y'' + y = 4x^2\left(\tfrac{1}{4}x^{-3/2}\ln x - \tfrac{1}{2}x^{-3/2}\right) + 2x^{1/2} - x^{1/2}\ln x = 0$$

if $x$ is positive, so the differential equation is satisfied for all $x > 0$. ■

The fact that we can write a differential equation is not enough to guarantee that it has a solution. For example, it is clear that the differential equation

$$(y')^2 + y^2 = -1 \tag{11}$$

has *no* (real-valued) solution, because the sum of nonnegative numbers cannot be negative. For a variation on this theme, note that the equation

$$(y')^2 + y^2 = 0 \tag{12}$$

obviously has only the (real-valued) solution $y(x) \equiv 0$. In our previous examples any differential equation having at least one solution indeed had infinitely many.

The **order** of a differential equation is the order of the highest derivative that appears in it. The differential equation of Example 8 is of second order, those in Examples 2 through 7 are first-order equations, and

$$y^{(4)} + x^2 y^{(3)} + x^5 y = \sin x$$

is a fourth-order equation. The most general form of an ***n*th-order** differential equation with independent variable $x$ and unknown function or dependent variable $y = y(x)$ is

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, \tag{13}$$

where $F$ is a specific real-valued function of $n + 2$ variables.

Our use of the word *solution* has been until now somewhat informal. To be precise, we say that the continuous function $u = u(x)$ is a **solution** of the differential equation in (13) **on the interval** $I$ provided that the derivatives $u', u'', \ldots, u^{(n)}$ exist on $I$ and

$$F\left(x, u, u', u'', \ldots, u^{(n)}\right) = 0$$

for all $x$ in $I$. For the sake of brevity, we may say that $u = u(x)$ **satisfies** the differential equation in (13) on $I$.

**Example 9** If $A$ and $B$ are constants and

$$y(x) = A \cos 3x + B \sin 3x, \tag{14}$$

then two successive differentiations yield

$$\begin{aligned} y'(x) &= -3A \sin 3x + 3B \cos 3x, \\ y''(x) &= -9A \cos 3x - 9B \sin 3x = -9y(x) \end{aligned}$$

for all $x$. Consequently, Eq. (14) defines what it is natural to call a *two-parameter family* of solutions of the second-order differential equation

$$y'' + 9y = 0 \tag{15}$$

on the whole real number line. Figure 1.1.6 shows the graphs of several such solutions. ■

In this chapter we concentrate on *first-order* differential equations of the form

$$\frac{dy}{dx} = f(x, y). \tag{17}$$

We also will sample the wide range of applications of such equations. A typical mathematical model of an applied situation will be an **initial value problem**, consisting of a differential equation of the form in (17) together with an **initial condition** $y(x_0) = y_0$. Note that we call $y(x_0) = y_0$ an initial condition whether or not $x_0 = 0$. To **solve** the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{18}$$

means to find a differentiable function $y = y(x)$ that satisfies both conditions in Eq. (18) on some interval containing $x_0$.

**Example 10** Given the solution $y(x) = 1/(C - x)$ of the differential equation $dy/dx = y^2$ discussed in Example 7, solve the initial value problem

$$\frac{dy}{dx} = y^2, \quad y(1) = 2.$$

**Solution** We need only find a value of $C$ so that the solution $y(x) = 1/(C - x)$ satisfies the initial condition $y(1) = 2$. Substitution of the values $x = 1$ and $y = 2$ in the given solution yields

$$2 = y(1) = \frac{1}{C - 1},$$

so $2C - 2 = 1$, and hence $C = \frac{3}{2}$. With this value of $C$ we obtain the desired solution

$$y(x) = \frac{1}{\frac{3}{2} - x} = \frac{2}{3 - 2x}.$$

Figure 1.1.7 shows the two branches of the graph $y = 2/(3 - 2x)$. The left-hand branch is the graph on $(-\infty, \frac{3}{2})$ of the solution of the given initial value problem $y' = y^2$, $y(1) = 2$. The right-hand branch passes through the point $(2, -2)$ and is therefore the graph on $(\frac{3}{2}, \infty)$ of the solution of the different initial value problem $y' = y^2$, $y(2) = -2$. ■

# Problems

*In Problems 1 through 12, verify by substitution that each given function is a solution of the given differential equation. Throughout these problems, primes denote derivatives with respect to x.*

**1.** $y' = 3x^2$; $y = x^3 + 7$

**2.** $y' + 2y = 0$; $y = 3e^{-2x}$

$y' = -6e^{-2x}$

$y' + 2y = -6e^{-2x} + 2(3e^{-2x}) = 0$

**3.** $y'' + 4y = 0$; $y_1 = \cos 2x$, $y_2 = \sin 2x$

**4.** $y'' = 9y$; $y_1 = e^{3x}$, $y_2 = e^{-3x}$

**7.** $y'' - 2y' + 2y = 0$; $y_1 = e^x \cos x$, $y_2 = e^x \sin x$

**8.** $y'' + y = 3\cos 2x$, $y_1 = \cos x - \cos 2x$, $y_2 = \sin x - \cos 2x$

**9.** $y' + 2xy^2 = 0$; $y = \dfrac{1}{1+x^2}$

**10.** $x^2y'' + xy' - y = \ln x$; $y_1 = x - \ln x$, $y_2 = \dfrac{1}{x} - \ln x$

**11.** $x^2y'' + 5xy' + 4y = 0$; $y_1 = \dfrac{1}{x^2}$, $y_2 = \dfrac{\ln x}{x^2}$

*In Problems 13 through 16, substitute $y = e^{rx}$ into the given differential equation to determine all values of the constant r for which $y = e^{rx}$ is a solution of the equation.*

$y = e^{rx}$

**13.** $3y' = 2y$

**14.** $4y'' = y$

**15.** $y'' + y' - 2y = 0$

**16.** $3y'' + 3y' - 4y = 0$

14

$4y'' = y$

$y' = re^{rx}$

$y'' = r^2e^{rx}$

$4r^2e^{rx} = e^{rx}$

$4r^2 = 1$

$r^2 = \frac{1}{4}$

$r = \pm\frac{1}{2}$

$y = e^{\frac{1}{2}x}$, $y = e^{-\frac{1}{2}x}$

*In Problems 17 through 26, first verify that $y(x)$ satisfies the given differential equation. Then determine a value of the constant C so that $y(x)$ satisfies the given initial condition. Use a computer or graphing calculator (if desired) to sketch several typical solutions of the given differential equation, and highlight the one that satisfies the given initial condition.*

**17.** $y' + y = 0$; $y(x) = Ce^{-x}$, $y(0) = 2$

**18.** $y' = 2y$; $y(x) = Ce^{2x}$, $y(0) = 3$

**19.** $y' = y + 1$; $y(x) = Ce^x - 1$, $y(0) = 5$

**20.** $y' = x - y$; $y(x) = Ce^{-x} + x - 1$, $y(0) = 10$

**22.** $e^y y' = 1$; $y(x) = \ln(x + C)$, $y(0) = 0$

**23.** $x\dfrac{dy}{dx} + 3y = 2x^5$; $y(x) = \frac{1}{4}x^5 + Cx^{-3}$, $y(2) = 1$

**25.** $y' = 3x^2(y^2 + 1)$; $y(x) = \tan(x^3 + C)$, $y(0) = 1$

**43. (a)** If $k$ is a constant, show that a general (one-parameter) solution of the differential equation

$$\frac{dx}{dt} = kx^2$$

is given by $x(t) = 1/(C - kt)$, where $C$ is an arbitrary constant.

## 1.2 Integrals as General and Particular Solutions

The first-order equation $dy/dx = f(x, y)$ takes an especially simple form if the right-hand-side function $f$ does not actually involve the dependent variable $y$, so

$$\frac{dy}{dx} = f(x). \tag{1}$$

In this special case we need only integrate both sides of Eq. (1) to obtain

$$y(x) = \int f(x)\,dx + C. \tag{2}$$

This is a **general solution** of Eq. (1), meaning that it involves an arbitrary constant $C$, and for every choice of $C$ it is a solution of the differential equation in (1). If $G(x)$ is a particular antiderivative of $f$—that is, if $G'(x) \equiv f(x)$—then

$$y(x) = G(x) + C. \tag{3}$$

To satisfy an initial condition $y(x_0) = y_0$, we need only substitute $x = x_0$ and $y = y_0$ into Eq. (3) to obtain $y_0 = G(x_0) + C$, so that $C = y_0 - G(x_0)$. With this choice of $C$, we obtain the **particular solution** of Eq. (1) satisfying the initial value problem

$$\frac{dy}{dx} = f(x), \quad y(x_0) = y_0.$$

**Example 1** Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

Integration of both sides of the differential equation as in Eq. (2) immediately yields the general solution

$$y(x) = \int (2x + 3)\, dx = x^2 + 3x + C.$$

Figure 1.2.3 shows the graph $y = x^2 + 3x + C$ for various values of $C$. The particular solution we seek corresponds to the curve that passes through the point $(1, 2)$, thereby satisfying the initial condition

$$y(1) = (1)^2 + 3 \cdot (1) + C = 2.$$

It follows that $C = -2$, so the desired particular solution is

$$y(x) = x^2 + 3x - 2. \quad \blacksquare$$

## Problems

*In Problems 1 through 10, find a function $y = f(x)$ satisfying the given differential equation and the prescribed initial condition.*

1. $\frac{dy}{dx} = 2x + 1;\ y(0) = 3$ $\rightarrow$ ① $y = \int (2x+1)\,dx = x^2 + x + C$

   $3 = 0 + 0 + C \Rightarrow C = 3$

   $\boxed{y = x^2 + x + 3}$

2. $\frac{dy}{dx} = (x - 2)^2;\ y(2) = 1$ $\rightarrow$

3. $\frac{dy}{dx} = \sqrt{x};\ y(4) = 0$

4. $\frac{dy}{dx} = \frac{1}{x^2};\ y(1) = 5$

5. $\frac{dy}{dx} = \frac{1}{\sqrt{x+2}};\ y(2) = -1$

6. $\frac{dy}{dx} = x\sqrt{x^2 + 9};\ y(-4) = 0$

7. $\frac{dy}{dx} = \frac{10}{x^2 + 1};\ y(0) = 0$

8. $\frac{dy}{dx} = \cos 2x;\ y(0) = 1$ $\rightarrow$ $y = \int \cos 2x\,dx = \frac{\sin 2x}{2} + C$

9. $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}};\ y(0) = 0$

10. $\frac{dy}{dx} = xe^{-x};\ y(0) = 1$

② $y = \int (x-2)^2\,dx = \frac{(x-2)^3}{3} + C$

$1 = \frac{(2-2)^3}{3} + C \Rightarrow C = 1$

$\boxed{y = \frac{(x-2)^3}{3} + 1}$

**10.** $\dfrac{dy}{dx} = xe^{-x};\ y(0) = 1$

$$y = \int x e^{-x}\,dx$$

$$u = x, \quad dv = e^{-x}\,dx$$

$$du = dx, \quad v = -e^{-x}$$

$$y = -xe^{-x} - \int -e^{-x}\cdot 1\cdot dx$$

$$\boxed{y = -xe^{-x} - e^{-x} + C}$$

$$\int u\,dv = uv - \int v\,du$$

$$\begin{array}{ccc} x & \oplus & e^{-x} \\ 1 & \ominus & -e^{-x} \\ 0 & & e^{-x} \end{array}$$

$$1 = 0 - e^{-0} + C$$

$$C = 2$$

$$y = -xe^{-x} - e^{-x} + 2$$

In general A first order differential equation has one of the following forms

$$F(x, y, y') = 0$$

$$\frac{dy}{dx} = f(x, y)$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

## 1.4 Separable Equations

***Definition***
A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be separable or to have separable variables.

For example, the equation $\dfrac{dy}{dx} = y^2xe^{3x+4y}$ is separable because we can write it as

$$\frac{dy}{dx} = y^2xe^{3x+4y} = \underset{g(x)}{(xe^{3x})}\underset{h(y)}{(y^2e^{4y})}$$

But the equation $\dfrac{dy}{dx} = y + \sin x$ is not separable.

Observe that by dividing by the function $h(y)$, we can write a separable equation $dy/dx = g(x)h(y)$ as

$$p(y)\frac{dy}{dx} = g(x), \tag{2}$$

where, for convenience, we have denoted $1/h(y)$ by $p(y)$. From this last form we can see immediately that (2) reduces to (1) when $h(y) = 1$.

Now if $y = \phi(x)$ represents a solution of (2), we must have $p(\phi(x))\phi'(x) = g(x)$, and therefore

$$\int p(\phi(x))\phi'(x)\,dx = \int g(x)\,dx. \tag{3}$$

But $dy = \phi'(x)\,dx$, and so (3) is the same as

$$\int p(y)\,dy = \int g(x)\,dx \qquad \text{or} \qquad H(y) = G(x) + c, \tag{4}$$

where $H(y)$ and $G(x)$ are antiderivatives of $p(y) = 1/h(y)$ and $g(x)$, respectively.

- **To Solve the Separable Equation**

1. **Separate the variable**
2. **Integrate the both sides**

**Example:** Solve

$$xdy + ydx = 0$$

**Solution**

**Dividing both sides by $xy$**

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating

$$\int \frac{dy}{y} + \int \frac{dx}{x} = c \quad \Rightarrow \quad ln\,|y| + ln\,|x| = c$$

$$\Rightarrow \quad ln\,|yx| = c$$

**The solution of the given equation is**

$$y = \frac{C}{x}$$

- **To Solve the Separable Equation**

1. **Separate the variable**
2. **Integrate the both sides**

**Example:** Solve

$$xdy + ydx = 0$$

$$xdy = -ydx$$

$$\int \frac{dy}{y} = \int -\frac{dx}{x} \rightarrow \text{separable.}$$

$$\ln|y| = -\ln|x| + C$$

$$e^{\ln|y|} = e^{-\ln|x| + C}$$

$$|y| = \frac{1}{|x|} \cdot e^{C}$$

Let $\pm e^{C} = A$

$$\boxed{y = \frac{A}{x}}$$

**Example :** solve

$$\frac{dy}{dx} = xy$$

**Solution**

$$dy = xydx$$

**Dividing both sides by $y$**

$$\frac{dy}{y} = xdx$$

Integrating

$$\int \frac{dy}{y} = \int xdx + c \quad \Longrightarrow \quad ln\,|y| = \frac{1}{2}x^2 + c$$

**The solution of the given equation is**

$$ln\,|y| = \frac{1}{2}x^2 + c$$

$$y = Ae^{x^2/2}$$

**Example** Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\begin{aligned}
\int \frac{dy}{y} &= \int \frac{dx}{1 + x} \\
\ln|y| &= \ln|1 + x| + c_1 \\
y &= e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1} && \leftarrow \text{laws of exponents} \\
&= |1 + x|\,e^{c_1} && \leftarrow \begin{cases} |1 + x| = 1 + x, & x \geq -1 \\ |1 + x| = -(1 + x), & x < -1 \end{cases} \\
&= \pm e^{c_1}(1 + x).
\end{aligned}$$

Relabeling $\pm e^{c_1}$ as $c$ then gives $y = c(1 + x)$.

**Example** Solve $\dfrac{dy}{dx} = \dfrac{4 - 2x}{3y^2 - 5}$.

**Solution**

When we separate the variables and integrate both sides, we get

$$\int (3y^2 - 5)\, dy = \int (4 - 2x)\, dx;$$
$$y^3 - 5y = 4x - x^2 + C.$$

This equation is not readily solved for $y$ as an explicit function of $x$.

**Example :** solve

$$xydy + (y^2 + 1)dx = 0$$

$$\frac{\cancel{x} y\,dy}{(y^2+1)} = -\frac{\cancel{(y^2+1)}\,dx}{x}$$

$$\frac{1}{2}\int \frac{2y\,dy}{y^2+1} = \int -\frac{dx}{x}$$

$$\frac{1}{2}\ln|y^2+1| = -\ln|x| + C$$

**Example :** solve

$$(x + xy)dy + dx = 0$$

**Solution**

$$x(1 + y)dy + dx = 0$$

$(x+xy)dy = -dx$

$x(1+y)dy = -dx$

$\int (1+y)\,dy = \int \frac{-dx}{x}$

$y + \frac{y^2}{2} = -\ln|x| + C$

**Dividing both sides by** $x$

$$(1 + y)dy + \frac{dx}{x} = 0$$

**Integrating**

$$\int (1 + y)dy + \int \frac{dx}{x} = c$$

$$\Rightarrow \quad \frac{1}{2}y^2 + y + ln\,|x| = c$$

**The solution of the given equation is**

$$\frac{1}{2}y^2 + y + ln\,|x| = c$$

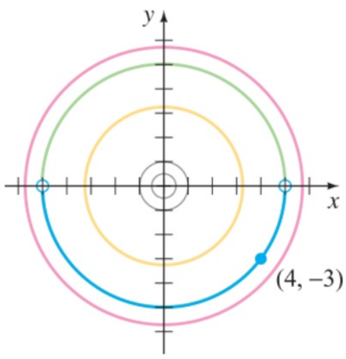
**Example** Solve the initial-value problem $\frac{dy}{dx} = -\frac{x}{y}, \quad y(4) = -3.$

**SOLUTION** Rewriting the equation as $y\,dy = -x\,dx$, we get

$$\int y\,dy = -\int x\,dx \qquad \text{and} \qquad \frac{y^2}{2} = -\frac{x^2}{2} + c_1.$$

We can write the result of the integration as $x^2 + y^2 = c^2$ by replacing the constant $2c_1$ by $c^2$. This solution of the differential equation represents a family of concentric circles centered at the origin.

Now when $x = 4$, $y = -3$, so $16 + 9 = 25 = c^2$. Thus the initial-value problem determines the circle $x^2 + y^2 = 25$ with radius 5. Because of its simplicity we can solve this implicit solution for an explicit solution that satisfies the initial condition. We saw this solution as $y = \phi_2(x)$ or $y = -\sqrt{25 - x^2}$, $-5 \le x \le 5$ in Example 3 of Section 1.1. A solution curve is the graph of a differentiable function. In this case the solution curve is the lower semicircle, shown in dark blue in Figure containing the point $(4, -3)$.



*Find general solutions (implicit if necessary, explicit if convenient) of the differential equations in Problems 1 through 18. Primes denote derivatives with respect to x.*

**1.** $\frac{dy}{dx} + 2xy = 0$

**3.** $\frac{dy}{dx} = y \sin x$

**4.** $(1+x)\frac{dy}{dx} = 4y$

**5.** $2\sqrt{x}\,\frac{dy}{dx} = \sqrt{1-y^2}$

**6.** $\frac{dy}{dx} = 3\sqrt{xy}$

$$\int y^{-\frac{1}{2}} dy = \int 3x^{\frac{1}{2}} dx$$

**8.** $\frac{dy}{dx} = 2x \sec y$

$$\int \frac{dy}{\sec y} = \int 2x\,dx$$

$$\int \cos y\,dy = \int 2x\,dx$$

$$+\sin y = x^2 + C$$

**10.** $(1+x)^2\frac{dy}{dx} = (1+y)^2$

**12.** $yy' = x(y^2+1)$

$$y\frac{dy}{dx} = x(y^2+1)$$

$$\int \frac{1}{2}\,\frac{2y\,dy}{y^2+1} = \int x\,dx$$

$$\frac{1}{2}\ln(y^2+1) = \frac{x^2}{2} + C.$$

**16.** $(x^2+1)(\tan y)y' = x$

$$\int \tan y\,dy = \int \frac{x\,dx}{x^2+1}$$

$$\ln|\sec y| = \frac{1}{2}\ln(x^2+1) + C$$

*Find explicit particular solutions of the initial value problems in Problems 19 through 28.*

$\ln(AB) = \ln A + \ln B.$

**19.** $\dfrac{dy}{dx} = ye^x, \quad y(0) = 2e$

$$\frac{dy}{y} = e^x dx$$

$$\ln|y| = e^x + C$$

$$\ln|2e| = e^0 + C$$

$$\ln 2 + \ln e = 1 + C$$

$$\ln 2 \cancel{+1} = \cancel{1} + C$$

$$\boxed{\ln|y| = e^x + \ln 2}$$

**20.** $\dfrac{dy}{dx} = 3x^2(y^2 + 1), \quad y(0) = 1$

$$\int \frac{dy}{y^2+1} = \int 3x^2 dx$$

$$\tan^{-1} y = x^3 + C$$

$$\tan^{-1} 1 = 0 + C$$

$$C = \pi/4$$

$$\tan^{-1} y = x^3 + \pi/4$$

**22.** $\dfrac{dy}{dx} = 4x^3y - y, \quad y(1) = -3$

## 1.5 Linear First-Order Equations

**Definition**

The first order deferential equation

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (*)$$

Is called linear in $y$.

**Example**

- The following equations are Linear

(1) $\frac{dy}{dx} + xy = 2x$ **(2)** $\frac{dy}{dx} + \frac{y}{x} = \boldsymbol{x^2}$ (3) $xdy + (x + \boldsymbol{y})dx = 0$

- While the following equations are not Linear

(1) $xdy + (x - \boldsymbol{y^2})dx = 0$ **(2)** $\mathbf{sin}(\boldsymbol{y})dy + (x + y)dx = 0$

With the aid of the appropriate integrating factor, there is a standard technique for solving the **linear first-order equation**

$$\frac{dy}{dx} + \underline{P(x)}y = Q(x) \qquad (3)$$

on an interval on which the coefficient functions $P(x)$ and $Q(x)$ are continuous. We multiply each side in Eq. (3) by the integrating factor

$I(x) = e^{\int P(x)dx}$

$$\rho(x) = e^{\int P(x)\,dx}. \qquad (4)$$

The result is

$$e^{\int P(x)\,dx}\frac{dy}{dx} + P(x)e^{\int P(x)\,dx}y = Q(x)e^{\int P(x)\,dx}. \qquad (5)$$

the left-hand side is the derivative of the *product* $y(x) \cdot e^{\int P(x)\,dx}$, so Eq. (5) is equivalent to

$$\frac{d}{dx}\left[y(x) \cdot e^{\int P(x)\,dx}\right] = Q(x)e^{\int P(x)\,dx}.$$

Integration of both sides of this equation gives

$$y(x)e^{\int P(x)\,dx} = \int \left(Q(x)e^{\int P(x)\,dx}\right) dx + C.$$

$\int p(x)\,dx$

$$\frac{dy}{dx} + p(x)y = Q(x)$$

$$I = e^{\int p(x)\,dx}$$ integral factor

$$I' = \underline{p(x)\,e^{\int p(x)\,dx}} = p(x)\,I$$

$$I(x)\frac{dy}{dx} + I\,p(x)\,y = I\,Q(x)$$

$$I\,y' + I'y = I(x)\,Q(x)$$

$$(I\,y)' = I\,Q$$

$$I\,y = \int I\,Q\,dx \Rightarrow y = \frac{1}{I}\int I\,Q\,dx$$

Finally, solving for $y$, we obtain the general solution of the linear first-order equation in (3):

$$y(x) = e^{-\int P(x)\,dx}\left[\int \left(Q(x)e^{\int P(x)\,dx}\right)dx + C\right]. \qquad (6)$$

This formula should **not** be memorized. In a specific problem it generally is simpler to use the *method* by which we developed the formula. That is, in order to solve an equation that can be written in the form in Eq. (3) with the coefficient functions $P(x)$ and $Q(x)$ displayed explicitly, you should attempt to carry out the following steps.

## To Solve the Linear Equation

**1. We put the linear equation in this form**

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (*)$$

**2. We determine the integral factor**

$$I = e^{\int p(x)dx}$$

**3. The solution of the linear equation (*) takes the form**

$$I.y = \int I.Q(x)dx + c$$

**Example :** solve

$$\frac{dy}{dx} + \frac{1}{x} y = \boldsymbol{x^2} \qquad (*)$$

**Solution**

Let $P(x) = \frac{1}{x}$ *and* $Q(x) = \boldsymbol{x^2}$

**The integral factor is**

$$I = \boldsymbol{e^{\int p(x)dx}} = \boldsymbol{e^{\int \frac{1}{x} dx}} = \boldsymbol{e^{lnx}} = \boldsymbol{x}$$

**The solution of the linear equation (*) takes the form**

$$\boldsymbol{I}.y = \int \boldsymbol{I}.Q(x)\boldsymbol{dx} + \boldsymbol{c} \quad \Rightarrow \quad x.y = \int \boldsymbol{x}.\boldsymbol{x^2 dx} + \boldsymbol{c}$$

$$\Rightarrow \quad x.y = \int \boldsymbol{x^3 dx} + \boldsymbol{c}$$

$$\Rightarrow \quad x.y = \frac{1}{4}\boldsymbol{x^4} + \boldsymbol{c}$$

$$y = \frac{x^3}{4} + \frac{c}{x}$$

**Example :** solve

$$x\frac{dy}{dx} + 2y = \boldsymbol{6x^4} \qquad (*)$$

**Solution**

**Dividing Eq.(*) by** $x$

$$\frac{dy}{dx} + \frac{2}{x} y = \boldsymbol{6x^3}$$

Let $P(x) = \frac{2}{x}$ *and* $Q(x) = \boldsymbol{6x^3}$

**The integral factor is**

$$I = \boldsymbol{e^{\int p(x)dx}} = \boldsymbol{e^{\int \frac{2}{x} dx}} = \boldsymbol{e^{2lnx}} = \boldsymbol{e^{lnx^2}} = \boldsymbol{x^2}$$

**The solution of the linear equation (*) takes the form**

$$\boldsymbol{I}.y = \int \boldsymbol{I}.Q(x)\boldsymbol{dx} + \boldsymbol{c} \quad \Rightarrow \quad \boldsymbol{x^2}.y = \int \boldsymbol{x^2}.\boldsymbol{6x^3 dx} + \boldsymbol{c}$$

$$\Rightarrow \quad \boldsymbol{x^2}.y = \int 6\boldsymbol{x^5 dx} + \boldsymbol{c}$$

$$\Rightarrow \quad \boldsymbol{x^2}.y = \boldsymbol{x^6} + \boldsymbol{c}$$

$$y = x^4 + c/x^2$$

**Example :** solve

$$xdy + (2y - \mathbf{6x^4})dx = 0$$

**Solution: Dividing both sides by** $\mathrm{d}x$

$$\Longrightarrow \qquad x\frac{dy}{dx} + 2y - \mathbf{6x^4} = \mathbf{0}$$

$$\Longrightarrow \qquad x\frac{dy}{dx} + 2y = \mathbf{6x^4} \qquad (*)$$

**Dividing Eq.(*) by** $x$

$$\frac{dy}{dx} + \frac{2}{x}y = \mathbf{6x^3}$$

Let $\quad P(x) = \frac{2}{x} \quad and \quad Q(x) = \mathbf{6x^3}$

**The integral factor is**

$$I = e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2lnx} = e^{lnx^2} = x^2$$

**The solution of the linear equation (*) takes the form**

$$I.y = \int I.Q(x)dx + c \quad \Rightarrow \quad x^2.y = \int x^2.6x^3dx + c$$

$$\Rightarrow \quad x^2.y = \int 6x^5dx + c$$

$$\Rightarrow \quad x^2.y = x^6 + c$$

$$y = x^4 + \frac{c}{x^2}$$

**Example** Solve the initial value problem

$$\frac{dy}{dx} - y = \frac{11}{8}e^{-x/3}, \quad y(0) = -1.$$

① $\dfrac{dy}{dx} - y = \dfrac{11}{8} e^{-x/3}$

$P(x) = -1$, $Q(x) = \dfrac{11}{8} e^{-x/3}$

② $I = e^{\int P(x)\,dx} = e^{\int -1\,dx} = e^{-x}$

③ $I \cdot y = \int I \cdot Q(x)\,dx + C$

$e^{-x} \cdot y = \int e^{-x} \cdot \dfrac{11}{8} e^{-x/3}\,dx + C$

$e^{-x} y = \dfrac{11}{8}\int e^{-x} \cdot e^{-x/3}\,dx + C$

$-x - \dfrac{x}{3}$

$-\dfrac{4x}{3}$

$e^{-x} \cdot y = \dfrac{11}{8}\int e^{-\frac{4x}{3}}\,dx + C$

$e^{-x} \cdot y = \dfrac{11}{8} \cdot \dfrac{e^{-\frac{4}{3}x}}{-4/3} + C$

$e^{-x} y = -\dfrac{33}{32} e^{-4/3\,x} + C$

$\dfrac{1}{e^{-x}} = e^{x}$

$$\boxed{y = -\frac{33}{32} e^{-\frac{1}{3}x} + Ce^{x}}$$

$y(0) = -1$

$$\Rightarrow \quad -1 = -\frac{33}{32} e^{0} + C e^{0}$$

$$-1 = -\frac{33}{32} + C$$

$$C = \frac{1}{32}.$$

Solution of IVP

$$\boxed{y = -\frac{33}{32} e^{-\frac{1}{3}x} + \frac{1}{32} e^{x}}$$

**Example** Find a general solution of

$$(x^2 + 1)\frac{dy}{dx} + 3xy = 6x.$$

① $$\frac{dy}{dx} + \frac{3x}{x^2+1} y = \frac{6x}{x^2+1}$$

$$P(x) = \frac{3x}{x^2+1}, \quad Q(x) = \frac{6x}{x^2+1}$$

② $$I = e^{\int P(x)\,dx} = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\int \frac{2x}{x^2+1}dx}$$

$$= e^{\frac{3}{2}\ln(x^2+1)}$$

$$= (x^2+1)^{3/2}$$

③ $I \cdot y = \int I \cdot Q(x)\,dx + C$

$(x^2+1)^{3/2} \cdot y = \int (x^2+1)^{3/2} \cdot \frac{6x}{(x^2+1)} + C$

$(x^2+1)^{3/2} y = \int (x^2+1)^{\frac{1}{2}} 6x\,du + C$

$u = x^2+1$
$du = 2x$

$(x^2+1)^{3/2} y = 3 \frac{(x^2+1)^{3/2}}{3/2} + C \leftarrow$

$y = 2 + C / (x^2+1)^{3/2}$

*Find general solutions of the differential equations in Problems 1 through 25. If an initial condition is given, find the corresponding particular solution. Throughout, primes denote derivatives with respect to x.*

**1.** $y' + y = 2,\ y(0) = 0$

**3.** $y' + 3y = 2xe^{-3x}$

**4.** $y' - 2xy = e^{x^2}$

**6.** $xy' + 5y = 7x^2,\ y(2) = 5$

$I = e^{\int -2x\,dx} = e^{-x^2}$

$I \cdot y = \int e^{-x^2} \cdot e^{x^2}\,dx + C$

$e^{-x^2} y = x + C \Rightarrow \boxed{y = xe^{x^2} + Ce^{x^2}}$

**9.** $xy' - y = x$, $y(1) = 7$

**12.** $xy' + 3y = 2x^5$, $y(2) = 1$

**15.** $y' + 2xy = x$, $y(0) = -2$

**18.** $xy' = 2y + x^3 \cos x$

$$y' - \frac{2}{x}\cdot y = x^2 \cos x$$

$$I = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2} = \frac{1}{x^2}$$

$$I \cdot y = \int I Q \, dx + C$$

$$\frac{1}{x^2} y = \int \frac{1}{\cancel{x^2}} \cancel{x^2} \cos x \, dx + C$$

$$\frac{1}{x^2} y = \sin x + C \Rightarrow y = x^2 \sin x + Cx^2 .$$

**20.** $y' = 1 + x + y + xy$, $y(0) = 0$

① $y' - y - xy = 1 + x$

$y' - (1+x)y = 1 + x$

② $\vdots$

③ $\vdots$

**22.** $y' = 2xy + 3x^2 \exp(x^2)$, $y(0) = 5$

**23.** $xy' + (2x - 3)y = 4x^4$

*Solve the differential equations in Problems 26 through 28 by regarding y as the independent variable rather than x.*

**26.** $(1-4xy^2)\frac{dy}{dx}=y^3$

**27.** $(x+ye^y)\frac{dy}{dx}=1$

$$\frac{1}{1-4xy^2}\frac{dx}{dy}=\frac{1}{y^3}$$

$$\frac{dx}{dy}+p(y)x=Q(y)$$

$$\frac{dx}{dy}=\frac{1-4xy^2}{y^3}$$

$$\frac{dx}{d\cancel{x}y}+\frac{4}{y}x=\frac{1}{y^3}$$

$$I=e^{\int\frac{4}{y}dy}=e^{4\ln y}=y^4$$

$$I\cdot x=\int I\,Q(\cancel{x}y)\,dy+C \Rightarrow y^4\cdot x=\int y^4\cdot\frac{1}{y^3}dy+C \;\ldots$$

## 1.6 Substitution Methods and Exact Equations

### Homogeneous Functions

**Definition:** The functions $f(x, y)$ *is called homogeneous of degree* $n$ *if it satisfies*

$$f(\lambda x, \lambda y) = \lambda^n f(x, y).$$

**Example**

- The following functions are homogeneous

$(1) f(x, y) = xy$ **(2)** $f(x, y) = \boldsymbol{x^2 + y^2}$ (3) $f(x, y) = \boldsymbol{x^3 + y^3 - 5xy^2}$

(4) $f(x, y) = \frac{y}{x}$ (5) $f(x, y) = \frac{\boldsymbol{x^2+y^2}}{x}$ (6) $f(x, y) = \sin(\frac{y}{x})$

- While the following functions are not homogeneous

(1) $f(x, y) = x - \boldsymbol{y^2}$ **(2)**$f(x, y) = \boldsymbol{\sin(y)}$ **(3)** $f(x, y) = \boldsymbol{x^2 + y^2}$**+5**

- (4) $f(x, y) = \boldsymbol{e^{x+y}}$

### Homogeneoues Equation

**Definition:** The first order deferential equation

$$M(x, y)dx + N(x, y)dy = 0 \qquad (*)$$

Is called Homogeneous if the both functions $M(x, y)$ and $N(x, y)$ are homogeneous with the same degree.

**Example**

- The following equations are Homogeneous
- (1) $(\mathrm{x} - \mathrm{y})dx + xdy = 0$ **(2)** $\mathrm{x}ydy + (\boldsymbol{y^2 - x^2})dx = 0$

$(3)\sin(\frac{y}{x})dx + dy = 0$

- While the following equations are not Homogeneous
- (1) $xdy + (x - \boldsymbol{y^2})dx = 0$ **(2)** $\boldsymbol{\sin(y)}dy + (x + y)dx = 0$

To Solve the Homogeneous Equation

1. We put the Homogeneous equation $(*)$ in this form

$$\frac{dy}{dx}=\frac{-M(x,y)}{N(x,y)} \qquad (I)$$

2. We put $\boldsymbol{y=ux}$ and differentiate it to get $\frac{dy}{dx}=u+x\frac{du}{dx}$

3. Then we substitute into equation $(I)$ to get

$$u+x\frac{du}{dx}=\frac{-M(x,\boldsymbol{ux})}{N(x,\boldsymbol{ux})}$$

4. After abbreviation we get a separable equation

5. We solve the separable equation

6. We substitute $\mathrm{u}=\frac{y}{x}$ into the solution of the separable solution to get the solution of homogeneous equation.

**Example :** solve

$$\mathrm{x}y dy+(\boldsymbol{x^2-y^2})dx=0 \qquad (*)$$

**Solution :** This equation is homogeneous of degree two

- $\frac{dy}{dx}=\frac{-(x^2-y^2)}{\mathrm{xy}} \quad \Longrightarrow \quad \frac{dy}{dx}=\frac{y^2-x^2}{\mathrm{xy}} \qquad \text{(I)}$
- Let $\boldsymbol{y=ux} \quad \Longrightarrow \quad \frac{dy}{dx}=u+x\frac{du}{dx}$
- Substitute into Eq. (I)
- $u+x\frac{du}{dx}=\frac{u^2x^2-x^2}{ux^2} \quad \Longrightarrow \quad u+x\frac{du}{dx}=\frac{x^2(u^2-1)}{ux^2}=\frac{u^2-1}{u}=u-\frac{1}{u}$

$$\Longrightarrow \quad u+x\frac{du}{dx}=u-\frac{1}{u} \quad \Longrightarrow \quad x\frac{du}{dx}=-\frac{1}{u} \quad \text{(Separable Eq.)}$$

$$\Longrightarrow \quad udu=-\frac{1}{x}dx \quad \Longrightarrow \quad \int udu=-\int\frac{1}{x}dx+c$$

$$\Longrightarrow \quad \frac{1}{2}u^2=-lnx+c$$

Hence the solution of the homogeneous Eq. (*) is $\frac{1}{2}\left(\frac{y}{x}\right)^2=-lnx+c$

**Example :** solve

$$x\frac{dy}{dx} = y + \sqrt{x^2 - y^2} \qquad (*)$$

**Solution :** This equation is homogeneous of degree one

$\Rightarrow \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2-y^2}}{x} \qquad$ (I)

- Let $y = ux \quad \Rightarrow \quad \frac{dy}{dx} = u + x\frac{du}{dx}$
- Substitute into Eq. (I)
- $u + x\frac{du}{dx} = \frac{ux+\sqrt{x^2-x^2u^2}}{x} \qquad \Rightarrow u + x\frac{du}{dx} = \frac{ux+x\sqrt{1-u^2}}{x} = u + \sqrt{1-u^2}$

$\Rightarrow \; u + x\frac{du}{dx} = u + \sqrt{1-u^2} \quad \Rightarrow \; x\frac{du}{dx} = \sqrt{1-u^2} \qquad$ (Separable Eq.)

$\Rightarrow \quad \frac{1}{\sqrt{1-u^2}}du = \frac{1}{x}dx \quad \Rightarrow \; \int \frac{1}{\sqrt{1-u^2}}du = \int \frac{1}{x}dx + c$

$\Rightarrow \quad \sin^{-1}u = \ln x + c$

Hence the solution of the homogeneous Eq. (*) is

$$\sin^{-1}\left(\frac{y}{x}\right) = \ln x + c$$

***Example:*** Solve the differential equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2.$$

***Solution***: This equation is homogeneous of degree 2.

$$\frac{dy}{dx} = \frac{4x^2 + 3y^2}{2xy}$$

Take the substitution $\quad y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}, \quad v = \frac{y}{x}, \quad$ and $\quad \frac{1}{v} = \frac{x}{y}.$

$$v + x\frac{dv}{dx} = \frac{4x^2 + 3x^2v^2}{2xvx}$$

$$v + x\frac{dv}{dx} = \frac{4 + 3v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{4 + 3v^2}{2v} - v$$

$$x\frac{dv}{dx} = \frac{4 + v^2}{2v}$$

$$\Rightarrow \int \frac{2v}{v^2+4}\,dv = \int \frac{1}{x}\,dx;$$

$$\Rightarrow \quad \ln(v^2+4) = \ln|x| + C.$$

$$\Rightarrow \ln\left(\frac{y^2}{x^2}+4\right) = \ln|x| + C$$

***Example:*** Solve the initial value problem

$$x\frac{dy}{dx} = y + \sqrt{x^2-y^2}, \quad y(x_0)=0, \text{ where } x_0 > 0.$$

***Solution***: We divide both sides by $x$ and find that

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1-\left(\frac{y}{x}\right)^2},$$

so we make the substitutions

$$y = vx, \quad \frac{dy}{dx} = v + x\frac{dv}{dx}, \quad v = \frac{y}{x}, \quad \text{and} \quad \frac{1}{v} = \frac{x}{y}.$$

we get

$$v + x\frac{dv}{dx} = v + \sqrt{1-v^2};$$

$$\int \frac{1}{\sqrt{1-v^2}}\,dv = \int \frac{1}{x}\,dx;$$

$$\sin^{-1} v = \ln x + C.$$

We need not write $\ln|x|$ because $x > 0$ near $x = x_0 > 0$.

Now note that $v(x_0) = y(x_0)/x_0 = 0$, so $C = \sin^{-1} 0 - \ln x_0 = -\ln x_0$. Hence

$$v = \frac{y}{x} = \sin(\ln x - \ln x_0) = \sin\left(\ln \frac{x}{x_0}\right),$$

and therefore

$$y(x) = x \sin\left(\ln \frac{x}{x_0}\right)$$

## Exact Equations

**Definition** The first order deferential equation

$$M(x,y)dx + N(x,y)dy = 0 \qquad (*)$$

Is called **exact if**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

**Example:** Prove that the following equation

$$xdy + (y + \sin x)dx = 0$$

Is exact

**Solution:** **Let** $M(x,y) = y + \sin x$ **and** $N(x,y) = x$

$$\Longrightarrow \quad \frac{\partial M}{\partial y} = 1 \quad and \quad \frac{\partial N}{\partial x} = 1$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \ \textit{the given equation is exact}$$

- To Solve the Exact Equation

1. *Since the given equation is exact*, then there is a function $\mathbf{f}(x, y)$ such that

$$\frac{\partial f}{\partial x} = \mathbf{M(x, y)} \qquad (*)$$

$$and \quad \frac{\partial f}{\partial y} = N(x, y) \qquad (**)$$

**2. Integrate (*) w. r . to x**

$$f(x, y) = \int \mathbf{M(x, y)}\, dx + g(y)$$

**3. D.w.r.to y**

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\int \mathbf{M(x, y)}\, dx + g(y)\right]$$

**4. From (**)**

$$N(x, y) = \frac{\partial}{\partial y}\left[\int \mathbf{M(x, y)}\, dx + g(y)\right]$$

**5. After abbreviation we get the solution**

$$f(x, y) = C$$

**Example :** Solve

$$(\boldsymbol{x}^2 + 2xy)dx + (\boldsymbol{x}^2 + \mathbf{y})\mathbf{dy} = \mathbf{0}$$

**Solution**

Let $\mathbf{M} = \boldsymbol{x}^2 + 2xy \qquad and \qquad N = (\boldsymbol{x}^2 + \mathbf{y})$

$$\Longrightarrow \qquad \frac{\partial M}{\partial y} = 2x \quad and \qquad \frac{\partial N}{\partial x} = \mathbf{2x}$$

$\because \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \therefore$ ***the given equation is exact***

*Since the given equation is exact*, then there is a function $\mathbf{f}(x, y)$ such that

$$\frac{\partial f}{\partial x} = \boldsymbol{x}^2 + 2xy \qquad (*)$$

$$and \qquad \frac{\partial f}{\partial y} = \boldsymbol{x}^2 + \mathbf{y} \qquad (**)$$

**Integrate (*) w. r . to x**

$$f(x, y) = \int (\boldsymbol{x}^2 + 2xy)\, dx + g(y)$$

$$\Rightarrow f(x, y) = \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{x}^3 + \boldsymbol{x}^2 y + g(y) \qquad (I)$$

**D.w.r.to y**

$$\frac{\partial f}{\partial y} = x^2 + g'(y)$$

**From (**)**

$$x^2 + y = x^2 + g'(y)$$

$$\Longrightarrow \quad y = g'(y)$$

$$\Longrightarrow \quad g(y) = \frac{1}{2}y^2$$

Substituting into (I) we get

$$f(x,y) = \frac{1}{3}x^3 + x^2 y + \frac{1}{2}y^2$$

The solution of the exact equation is

$$\frac{1}{3}x^3 + x^2 y + \frac{1}{2}y^2 = c$$

**Example :** Solve

$$(sinx + 2y)dx + (2x + y^2 + 3)dy = 0$$

**Solution**

Let $M = sinx + 2y$ and $N = 2x + y^2 + 3$

$$\Longrightarrow \quad \frac{\partial M}{\partial y} = 2 \quad and \quad \frac{\partial N}{\partial x} = 2$$

$$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \therefore \ \textit{the given equation is exact}$$

*Since the given equation is exact*, then there is a function $f(x,y)$ such that

$$\frac{\partial f}{\partial x} = sinx + 2y \qquad (*)$$

$$and \qquad \frac{\partial f}{\partial y} = 2x + y^2 + 3 \qquad (**)$$

**Integrate (*) w. r . to x**

$$f(x,y) = \int (sinx + 2y)\, dx + g(y)$$

$$\Longrightarrow f(x,y) = -cosx + 2yx + g(y) \qquad (I)$$

**D.w.r.to y**

$$\frac{\partial f}{\partial y} = 2x + g'(y)$$

**From (**)**

$$\cancel{2x} + y^2 + 3 = \cancel{2x} + g'(y)$$

$$\Longrightarrow y^2 + 3 = g'(y)$$

$$\Longrightarrow \quad g(y) = \frac{1}{3}y^3 + 3y$$

Substituting into (I) we get

$$f(x,y) = -\cos x + 2yx + \frac{1}{3}y^3 + 3y$$

The solution of the exact solution is

$$-\cos x + 2yx + \frac{1}{3}y^3 + 3y = c$$

Example Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0.$$

Let $M(x, y) = 6xy - y^3$ and $N(x, y) = 4y + 3x^2 - 3xy^2$. The given equation is exact because

$$\frac{\partial M}{\partial y} = 6x - 3y^2 = \frac{\partial N}{\partial x}.$$

Integrating $\partial F/\partial x = M(x, y)$ with respect to $x$, we get

$$F(x, y) = \int (6xy - y^3)\,dx = 3x^2y - xy^3 + g(y).$$

Then we differentiate with respect to $y$ and set $\partial F/\partial y = N(x, y)$. This yields

$$\frac{\partial F}{\partial y} = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2,$$

and it follows that $g'(y) = 4y$. Hence $g(y) = 2y^2 + C_1$, and thus

$$F(x, y) = 3x^2y - xy^3 + 2y^2 + C_1.$$

Therefore, a general solution of the differential equation is defined implicitly by the equation

$$3x^2y - xy^3 + 2y^2 = C$$

## Exercises

*Find general solutions of the differential equations*

$$xy' = y + 2\sqrt{xy}$$

Hom. Let $y = ux$

$$\frac{dy}{dx} = \frac{y + 2\sqrt{xy}}{x}$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = \frac{ux + 2\sqrt{x\,ux}}{x}$$

$$u + x\frac{du}{dx} = u + 2\sqrt{u}$$

$$u^{-\frac{1}{2}}\,du = 2\frac{dx}{x}$$

$$2\sqrt{u} = 2\ln|x| + C$$

$$2\sqrt{y/x} = 2\ln|x| + C$$

$$xy^2y' = x^3 + y^3$$

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2}$$

$(x-y)y' = x+y$

$yy' + x = \sqrt{x^2+y^2}$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+y^2} - x}{y}$$

$x^2y' = xy + x^2e^{y/x}$

$$\frac{dy}{dx} = \frac{xy + x^2 e^{y/x}}{x^2} \rightarrow \text{- - - -}$$

Let $y = ux$

$\downarrow$

$\vdots$

$$\frac{dy}{dx} = \frac{x+3y}{3x+y}$$

determine whether the given differential equation is exact. If it is exact, solve it.

$(2x - 1)\, dx + (3y + 7)\, dy = 0$

$M = 2x - 1$

$N = 3y + 7$

$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$

$\frac{\partial f}{\partial x} = 2x - 1$

$f = x^2 - x + g(y)$

$f = x^2 - x + g(y)$

$\frac{\partial f}{\partial y} = g'(y)$

$3y + 7 = g'(y)$

$g(y) = \frac{3}{2} y^2 + 7y$

$x^2 - x + \frac{3}{2}y^2 + 7y = C$

separable.

$\int (3y + 7)\, dy = \int -(2x - 1)\, dx$

$\frac{3y^2}{2} + 7y = -x^2 + x + C$

$(5x + 4y)\, dx + (4x - 8y^3)\, dy = 0$

$(\sin y - y \sin x)\, dx + (\cos x + x \cos y - y)\, dy = 0$

$M(x, y) = siny - ysinx$

$N(x, y) = cosx + xcosy - y$

$\frac{\partial M}{\partial y} = cosy - sinx = \frac{\partial N}{\partial x} \quad \Rightarrow \quad Exact$

$\frac{\partial f}{\partial x} = siny - ysinx$

$f(x, y) = xsiny + ycosx + g(y)$

$f(x, y) = xsiny + ycosx + g(y)$

$N(x, y) = \frac{\partial f}{\partial y} = xcosy + cosx + g'(y)$

$cosx + xcosy - y = xcosy + cosx + g'(y)$

$g'(y) = -y$

$g(y) = \int -ydy = -\frac{y^2}{2}$

The general solution is

$xsiny + ycosx + -\frac{y^2}{2} = C$

$(x^2 - y^2)\, dx + (x^2 - 2xy)\, dy = 0$

$(x - y^3 + y^2 \sin x)\, dx = (3xy^2 + 2y \cos x)\, dy$

$(x^3 + y^3)\, dx + 3xy^2\, dy = 0$

$x \frac{dy}{dx} = 2xe^x - y + 6x^2$ LIATE

$x\, dy - (2xe^x - y + 6x^2)\, dx = 0$

$M = -(2xe^x - y + 6x^2)$, $N = x$

$\frac{\partial N}{\partial x} = 1$ $\frac{\partial M}{y} = 1$ Exact.

| | | |
|---|---|---|
| $-2x$ | $+$ | $e^x$ |
| $-2$ | $-$ | $e^x$ |
| $0$ | | $e^x$ |

$$F = \int -(2xe^x - y + 6x^2)\, dx$$
$$= \int (-2xe^x + y - 6x^2)\, dx$$
$$= -2xe^x + 2e^x + yx - 2x^3 + g(y)$$
$$\frac{\partial F}{\partial y} = +\cancel{x} + g'(y) = \cancel{x}$$
$$g'(y) = 0 \Rightarrow g(y) = 0 + C_1$$
$$\boxed{-2xe^x + 2e^x + yx - 2x^3 = C}$$

solve the given initial-value problem.

$$(x+y)^2\,dx + (2xy + x^2 - 1)\,dy = 0, \quad y(1) = 1$$

$$\underbrace{(4y + 2t - 5)}_{4}\,dt + (6\underbrace{y + 4t}_{4} - 1)\,dy = 0, \quad y(-1) = 2$$

$$F = \int (4y + 2t - 5)\,dt$$

$$= 4yt + t^2 - 5t + g(y)$$

$$\frac{\partial F}{\partial y} = \cancel{4t} + g'(y) = \cancel{4t} + 6y - 1$$

$$g(y) = 3y^2 - y$$

$$4yt + t^2 - 5t + 3y^2 - y = C$$