An n^{th} order diff. Eqn.

$$F(x, y, y', \ldots, y^{(n)}) = C$$

Higher-Order Differential Equations

I differential equation of order n is called linear if it has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

Ex: $\sin x \, y'' + x y' + 4y = e^x$

A linear nth-order differential equation of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be **homogeneous,** whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous.**

Examples

- $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0 \quad \Longrightarrow$ Second Order & Homogeneous
- $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(x) \quad \Longrightarrow$ second Order & Non-Homogeneous
- $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \quad \Longrightarrow$ Third Order & Homogeneous
- $\frac{d^3y}{dx^3} + y = x \quad \Longrightarrow$ Third Order & Non-Homogeneous

We will discuss first the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0. \qquad \text{.........................}(1)$$

THEOREM 1 Principle of Superposition for Homogeneous Equations

Let y_1 and y_2 be two solutions of the homogeneous linear equation in (1) on the interval I. If c_1 and c_2 are constants, then the linear combination

$$y = c_1y_1 + c_2y_2$$

is also a solution of Eq. (.) on I.

DEFINITION Linear Independence of Two Functions

Two functions defined on an open interval I are said to be **linearly independent** on I provided that neither is a constant multiple of the other.

Theorem

y_1 **and** y_2 are linearly **dependent solutions if** $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$

$y_1 y_2' - y_2 y_1' = 0$

W is called the Wronskian of y_1 and y_2

- **Example:** **Prove that** $y_1 = sinx$ **and** $y_2 = cosx$ ***are independent functions.***
- **Solution:**
- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} sinx & cosx \\ cosx & -sinx \end{vmatrix} = -sin^2x - cos^2x = -1 \neq 0$

$\because W \neq 0$

$\therefore y_1 = sinx$ **and** $y_2 = cosx$ ***are independent functions.***

- **Example:** **Prove that** $y_1 = x$ **and** $y_2 = 3x$ ***are dependent functions.***
- **Solution:**
- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & 3x \\ 1 & 3 \end{vmatrix} = 3x - 3x = 0$

$\because W = 0 \qquad \therefore y_1 = x$ **and** $y_2 = 3x$ ***are dependent functions.***

- **Example:** **Prove that** $y_1 = x$ **and** $y_2 = x^2$ ***are independent functions.***
- **Solution:**
- $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$ for $x \neq 0$

$\because W \neq 0$

$\therefore y_1 = x$ **and** $y_2 = x^2$ ***are independent functions.***

THEOREM 4 General Solutions of Homogeneous Equations

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

with p and q continuous on the open interval I. If Y is any solution whatsoever of Eq on I, then there exist numbers c_1 and c_2 such that

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

for all x in I.

Example: Consider the differential equation

$$y'' - 4y = 0.$$

The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$, are linearly independent solutions of this equation (why?)

So the general solution of this equations is

$$y(x) = C_1e^{2x} + C_2e^{-2x}$$

Homogenous linear equation with constant coefficients

We begin by considering the special case of the second-order equation

$$ay'' + by' + cy = 0, \qquad (2)$$

where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$, then after substitution of $y' = me^{mx}$ and $y'' = m^2e^{mx}$, equation (2) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

As in the introduction we argue that because $e^{mx} \neq 0$ for all x, it is apparent that the only way $y = e^{mx}$ can satisfy the differential equation (2) is when m is chosen as a root of the quadratic equation

$$am^2 + bm + c = 0. \qquad (3)$$

Equation (3) is called the characeristic equation of the differential equation (2).

Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (2) corresponding to the three cases:

- m_1 and m_2 real and distinct $(b^2 - 4ac > 0)$,
- m_1 and m_2 real and equal $(b^2 - 4ac = 0)$, and
- m_1 and m_2 conjugate complex numbers $(b^2 - 4ac < 0)$.

- **<u>Second order Homogeneous Equation</u>**

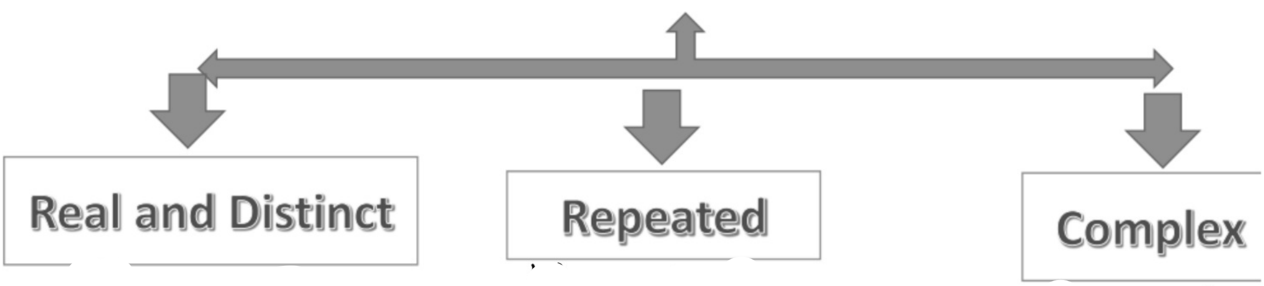
$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad (*)$$

Where a, b, c are real numbers.

Characteristic Equation for Eq. $(*)$ is

$$am^2 + bm + c = 0$$

There are three cases depend on the roots of Char. Eq.



- <u>First case</u> if the roots of characteristic equation m_1 **and** m_2 are real and distinct
- In this case the solution of Eq. (*) takes the form

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

- **<u>Example:</u> Solve**

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0 \qquad (*)$$

<u>Solution:</u>

Characteristic Equation for Eq. $(*)$ is

$$m^2 + 5m + 6 = 0$$

$$\Rightarrow \quad (m+2)(m+3) = 0$$

$$\Rightarrow \quad (m+2) = 0 \quad or \quad (m+3) = 0$$

$$\Rightarrow \quad m = -2 \quad or \quad m = -3$$

- The solution of Eq. (*) is

$$y = C_1 e^{-2x} + C_2 e^{-3x}$$

- **<u>Example:</u> Solve**

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0 \qquad (*)$$

<u>Solution:</u>

Characteristic Equation for Eq. $(*)$ is

$$m^2 + 5m - 6 = 0$$

$$\Rightarrow \quad (m-1)(m+6) = 0$$

$$\Rightarrow \quad (m-1) = 0 \quad or \quad (m+6) = 0$$

$$\Rightarrow \quad m = 1 \quad or \quad m = -6$$

- The solution of Eq. (*) is

$$y = C_1 e^{x} + C_2 e^{-6x}$$

- <u>Second case</u> If the roots of characteristic Eq. m_1 **and** m_2 are repeated (i.e $m_1 = m_2 = m$)
- In this case the solution of Eq. (*) takes the form

$$y = C_1 e^{mx} + C_2 x e^{mx}$$

$$ay'' + by' + cy = 0$$

$$am^2 + bm + c = 0$$

$$b^2 - 4ac = 0$$

$\Rightarrow$ one root m.

$\Rightarrow$ one solution e^{mx}

we can show that $x e^{mx}$ is also a solution

$$\therefore \boxed{y = C_1 e^{mx} + C_2 x e^{mx}}$$

- <u>Second case</u> If the roots of characteristic Eq. m_1 **and** m_2 are repeated (i.e $m_1 = m_2 = m$)
- In this case the solution of Eq. (*) takes the form

$$y = C_1 e^{mx} + C_2 x e^{mx}$$

- **<u>Example:</u> Solve**

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 \qquad (*)$$

<u>Solution:</u>

Characteristic Equation for Eq. (*) is

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)(m-2) = 0$$

$$\Rightarrow (m-2) = 0 \quad or \quad (m-2) = 0$$

$$\Rightarrow m = 2 \quad or \quad m = 2$$

- The solution of Eq. (*) is

$$y = C_1 e^{2x} + C_2 x e^{2x}$$

- <u>**Third case**</u> If the roots of characteristic Eq. m_1 **and** m_2 are Complex (i.e $m = A \pm Bi$)
- In this case the solution of Eq. (*) takes the form

$$y = e^{Ax}[C_1 cos(Bx) + C_2 sin(Bx)]$$

ay''+by'+cy=0

$am^2+bm+c=0$

$b^2-4ac<0$

$m_1=A+Bi \quad m_2=A-Bi$

$y=C_1e^{(A+Bi)x}+C_2e^{(A-Bi)x}$

$e^{A+Bi}=e^Ae^{Bi}$

$e^{ti}=cost+i\,sint$

$y_1=e^{Ax}(cosBx+isinBx)$

$y_2=e^{Ax}(cosBx-isinBx)$

$(y_1+y_2)/2 =e^{Ax}cosBx$

$(y_1-y_2)/(2i) =e^{Ax}sinBx$

- <u>**Third case**</u> If the roots of characteristic Eq. m_1 **and** m_2 are Complex (i.e $m = A \pm Bi$)
- In this case the solution of Eq. (*) takes the form

$$y = e^{Ax}[C_1 cos(Bx) + C_2 sin(Bx)]$$

- <u>**Example:**</u> **Solve**

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0 \qquad (*)$$

<u>**Solution:**</u>

Characteristic Equation for Eq. $(*)$ is

$$m^2 + 2m + 2 = 0$$

$$a = 1, \qquad b = 2, \qquad c = 2$$

$$\Rightarrow \quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\Rightarrow \quad A = -1 \quad and \quad B = 1$$

- The solution of Eq. (*) is

$$y = e^{-x}[C_1 cos(x) + C_2 sin(x)]$$

- **Example:** **Solve**

$$\frac{d^2y}{dx^2} - 9y = 0 \qquad (*)$$

Solution:

Characteristic Equation for Eq. $(*)$ is

$$m^2 - 9 = 0$$

$$\Longrightarrow \qquad m^2 = 9$$

$$\Longrightarrow \quad \boldsymbol{m} = 3 \quad or \quad m = -3$$

- The solution of Eq. (*) is

$$\boldsymbol{y} = \boldsymbol{C_1}e^{3x} + \boldsymbol{C_2}e^{-3x}$$

- **Example:** **Solve**

$$\frac{d^2y}{dx^2} - \boldsymbol{6}\frac{\boldsymbol{dy}}{\boldsymbol{dx}} + 9y = 0 \qquad (*)$$

Solution:

Characteristic Equation for Eq. $(*)$ is

$$m^2 - \boldsymbol{6m} + 9 = \boldsymbol{0}$$

$$\Longrightarrow \quad (\boldsymbol{m} - 3)(m - 3) = 0$$

$$\Longrightarrow \quad (\boldsymbol{m} - \boldsymbol{3}) = 0 \quad or \quad (m - 3) = 0$$

$$\Longrightarrow \quad \boldsymbol{m} = 3 \quad or \quad m = 3$$

- The solution of Eq. (*) is

$$\boldsymbol{y} = \boldsymbol{C_1}e^{3x} + \boldsymbol{C_2}\boldsymbol{x}e^{3x}$$

- **Example:** **Solve**

$$\frac{d^2y}{dx^2} + 9y = 0 \qquad (*)$$

Solution:

Characteristic Equation for Eq. $(*)$ is

$$m^2 + 9 = 0$$

$$m^2 = -9$$

$$\Longrightarrow \quad \boldsymbol{m} = \pm\boldsymbol{3i}$$

$$\Longrightarrow \quad \boldsymbol{A} = \boldsymbol{0} \quad \boldsymbol{and} \quad \boldsymbol{B} = \boldsymbol{3}$$

- The solution of Eq. (*) is

$$\boldsymbol{y} = e^{\boldsymbol{0x}}[\boldsymbol{C_1 cos(3x)} + \boldsymbol{C_2 sin(3x)}]$$

$$\boldsymbol{y} = \boldsymbol{C_1 cos(3x)} + \boldsymbol{C_2 sin(3x)}$$

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$. **(b)** $y'' - 10y' + 25y = 0$ **(c)** $y'' + 4y' + 7y = 0$

$2m^2 - 5m - 3 = 0$

$m = \dfrac{5 \pm \sqrt{25 - (-24)}}{2(2)}$

$= \dfrac{5 \pm \sqrt{49}}{4}$

$3, -\frac{1}{2}$

$y = c_1 e^{3x} + c_2 e^{-\frac{1}{2}x}$

$m^2 - 10m + 25 = 0$

$(m-5)(m-5) = 0$

$m = 5, \quad m = 5$

$y = C_1 e^{5x} + C_2 x e^{5x}$

$m^2 + 4m + 7 = 0$

$m = \dfrac{-4 \pm \sqrt{4^2 - 4(1)(7)}}{2(1)} = \dfrac{-4 \pm \sqrt{-12}}{2}$

$= \dfrac{-4 \pm \sqrt{12}\, i}{2} = -2 \pm \sqrt{3}\, i$

$A = -2, \quad B = \sqrt{3}$

$y = e^{-2x}\left(C_1 \cos(\sqrt{3}x) + C_2 \sin(\sqrt{3}x)\right)$

Higher-Order Equations

In general, to solve an nth-order differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0,$$

where the a_i, $i = 0, 1, \ldots, n$ are real constants, we must solve an nth-degree polynomial equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0.$$

If all the roots of (12) are real and distinct, then the general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

It is somewhat harder to summarize the analogues of the other two cases because the roots of an characeristic equation of degree greater than two can occur in many combinations. When m_1 is a root of multiplicity k of an n^{th}-degree characeristic equation (that is, k roots are equal to m_1), it can be shown that the linearly independent solutions are

$$e^{m_1 x}, \quad x e^{m_1 x}, \quad x^2 e^{m_1 x}, \ldots, \quad x^{k-1} e^{m_1 x}$$

and the general solution must contain the linear combination

$$c_1 e^{m_1 x} + c_2 x e^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$

- **Example:** **Solve**

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 0 \qquad (*)$$

Solution:

Characteristic Equation for Eq. $(*)$ is

$$m^3 - 6m^2 + 5m = 0$$
$$m(m^2 - 6m + 5) = 0$$
$$\Longrightarrow \quad m(m-1)(m-5) = 0$$
$$\Longrightarrow m = 0 \;\; or \;\; (m-1) = 0 \;\; or \;\; (m-5) = 0$$
$$\Longrightarrow m = 0 \;\; or \;\; m = 1 \;\; or \;\; m = 5$$

- The solution of Eq. (*) is

$$y = C_1e^{0x} + C_2e^{1x} + C_3e^{5x}$$
$$= C_1 + C_2e^{x} + C_3e^{5x}$$

- **Example:** **Solve**

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 9\frac{dy}{dx} = 0 \qquad (*)$$

Solution:

Characteristic Equation for Eq. $(*)$ is

$$m^3 - 6m^2 + 9m = 0$$
$$m(m^2 - 6m + 9) = 0$$
$$\Longrightarrow \quad m(m-3)(m-3) = 0$$
$$\Longrightarrow m = 0 \;\; or \;\; (m-3) = 0 \;\; or \;\; (m-3) = 0$$
$$\Longrightarrow m = 0 \;\; or \;\; m = 3 \;\; or \;\; m = 3$$

- The solution of Eq. (*) is

$$y = C_1e^{0x} + C_2e^{3x} + C_3xe^{3x}$$
$$= C_1 + C_2e^{3x} + C_3xe^{3x}$$

Solve $y''' + 3y'' - 4y = 0$.

$m^3+3m^2-4=0$

$(m-1)(m^2+4m+4)=0$

$(m-1)(m+2)(m+2)=0$

$m=1,\ m=-2,\ m=-2$

$y=C_1e^x+C_2e^{-2x}+C_3xe^{-2x}$

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$.

$m^4+2m^2+1=0$

$(m^2+1)(m^2+1)=0$

$m^2= -1 \quad , \quad m^2= -1$

$m=\pm i \quad , \quad m=\pm i$

$y=e^{0x}(C_1\cos x+C_2\sin x)+xe^{0x}(C_3\cos x+C_4\sin x)$

$y=C_1\cos x+C_2\sin x+C_3x\cos x+C_4x\sin x$

EXERCISES

Find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$

2. $y'' - 3y' + 2y = 0$

3. $y'' + 8y' + 16y = 0$

4. $y'' - 10y' + 25y = 0$

4. $12y'' - 5y' - 2y = 0$

6. $3y'' + y = 0$

7. $y'' + 9y = 0$

8. $2y'' - 3y' + 4y = 0$

9. $y''' - y = 0$

10. $y''' - 5y'' + 3y' + 9y = 0$

11. $y''' + 3y'' - 4y' - 12y = 0$

12. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$

13. $y''' - 6y'' + 12y' - 8y = 0$

14. $y^{(4)} + y''' + y'' = 0$

15. $y^{(4)} - 2y'' + y = 0$

16. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$

17. $2\dfrac{d^5x}{ds^5} - 7\dfrac{d^4x}{ds^4} + 12\dfrac{d^3x}{ds^3} + 8\dfrac{d^2x}{ds^2} = 0$

Solve the given initial-value problem.

18. $y'' + 16y = 0, \quad y(0) = 2, y'(0) = -2$

19. $\dfrac{d^2y}{d\theta^2} + y = 0, \quad y(\pi/3) = 0, y'(\pi/3) = 2$

20. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0, \quad y(1) = 0, y'(1) = 2$

21. $4y'' - 4y' - 3y = 0, \quad y(0) = 1, y'(0) = 5$

22. $y'' + y' + 2y = 0, \quad y(0) = y'(0) = 0$

23. $y'' - 2y' + y = 0, \quad y(0) = 5, y'(0) = 10$

20

$m^2 - 4m - 5 = 0$

$(m-5)(m+1) = 0$

$m = -1, \quad m = 5$

$y = c_1 e^{-x} + c_2 e^{5x}$

$0 = c_1 e^{-1} + c_2 e^{+5}$

$y' = -c_1 e^{-x} + 5c_2 e^{5x}$

$2 = -c_1 e^{-1} + 5c_2 e^{5}$

$c_1 e^{-1} + c_2 e^{5} = 0$

$-c_1 e^{-1} + 5c_2 e^{5} = 2$

$6c_2 e^{5} = 2$

$\boxed{c_2 = \frac{1}{3} e^{-5}}$

$\Rightarrow c_1 e^{-1} + \frac{1}{3} e^{-5} e^{5} = 0$

$\Rightarrow c_1 =$

$c_1 = \frac{e}{3}$

$\boxed{y = \frac{e}{3} e^{-x} + \frac{e^{-5}}{3} e^{5x}}$

Nonhomogeneous Higher-Order Differential Equations

A linear nth-order nonhomogeneous differential equation has the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

with $g(x)$ not identically zero

For example

- $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \sin(x) \quad \Rightarrow$ second Order & Non-Homogeneous
- $\dfrac{d^3y}{dx^3} + y = x \quad \Rightarrow$ Third Order & Non-Homogeneous

Theorem **General Solution—Nonhomogeneous Equations**

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on I. Then the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

We see that the general solution of a nonhomogeneous linear equation consists of the sum of two functions: y_c

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x) = y_c(x) + y_p(x).$$

The linear combination $y_c(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$, , which is the general solution of associated homogeneous differential equation, is called the complementary function.

In other words, to solve a nonhomogeneous linear differential equation, we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation. The general solution of the nonhomogeneous equation is then

$$\begin{aligned} y &= \textit{complementary function} + \textit{any particular solution} \\ &= y_c + y_p. \end{aligned}$$

The first of two ways we shall consider for obtaining a particular solution y_p for a nonhomogeneous linear diffetrential equations is called **the method of undetermined coefficients**. The underlying idea behind this method is a conjecture about the form of y_p, an educated guess really, that is motivated by the kinds of functions that make up the input function g(x). The general method is limited to linear nonhomogeneous differential equations where

• the coefficients a_i, i = 0, 1, . . . , n are constants and

• g(x) is a constant k, a polynomial function, an exponential function e^{ax}, a sine or cosine function $\sin bx$ or $\cos bx$, or finite sums and product of these functions.

The set of functions that consists of constants, polynomials, exponentials e^{ax}, sines, and cosines has the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials e^{ax}, sines, and cosines. Because the linear combination of derivatives

$a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p$ must be identical to g(x), it

seems reasonable to assume that y_p has the same form as g(x).

First we deal with the cases when no function in the assumed particular solution is a solution of the associated homogeneous differential equation.

Case 1: If g(x) is a polynomial then we assume y_p is a polynomial with the same degree

Example: Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6$.

First Find y_c:

$y'' + 4y' - 2y = 0$

$m^2 + 4m - 2 = 0$

$m = \dfrac{-4 \pm \sqrt{16+8}}{2} = \dfrac{-4 \pm \sqrt{24}}{2} = -2 \pm \sqrt{6}$

$m_1 = -2 + \sqrt{6}$ $\quad m_2 = -2 - \sqrt{6}$

$y_c = C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(2-\sqrt{6})x}$

$y(x) = 2x^2 - 3x + 6$

let $y_p = Ax^2 + Bx + C$

$y'_p = 2Ax + B$

$y''_p = 2A$ $\quad y'' + 4y' - 2y = 2x^2 - 3x + 6.$

$2A + 4(2Ax + B) - 2(Ax^2 + Bx + C) = 2x^2 - 3x + 6$

$\Rightarrow \quad -2Ax^2 + (8A - 2B)x + 2A + 4B - 2C = 2x^2 - 3x + 6$

$-2A = 2 \Rightarrow \boxed{A = -1}$

$8A - 2B = -3 \Rightarrow -2B = 5 \Rightarrow B = -5/2.$

$2A + 4B - 2C = 6 \Rightarrow -2C = 6 + 2 + 10$

$C = -9.$

General Solution is

$y = y_c + y_p$

$= C_1 e^{(-2+\sqrt{6})x} + C_2 e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9$

Case 2: *If* g(x)=ae^{bx} *weassume* $y_p = Ae^{bx}$.

Example: Solve $y''-5y'+6y=8e^{4x}$

To find y_c:

$$y''-5y'+6y=0$$

Char. Eq. is

$$m^2-5m+6=0$$

$$(m-2)(m-3)=0$$

$$\rightarrow m=2 \text{ or } m=3$$

the Complementary solution is

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

To find y_p:

let $y_p = Ae^{4x}$

$$y_p' = 4Ae^{4x}$$

$$y_p'' = 16Ae^{4x}$$

$\because$ y_p is a solution of Eq. ⊛

$$\therefore \; y_p''-5y_p'+6y_p = 8e^{4x}$$

$$16Ae^{4x}-5(4Ae^{4x})+6(Ae^{4x}) = 8e^{4x}$$

$$\rightarrow 2Ae^{4x} = 8e^{4x} \rightarrow 2A=8 \rightarrow A=4$$

$$\therefore \; y_p = 4e^{4x}$$

the general solution of Eq. ⊛ is

$$y = y_c + y_p = c_1e^{2x}+c_2e^{3x}+4e^{4x}$$

Type of $g(x)$	Form of y_p
polynomial of degree k	polynomial of degree k
ae^{bx}	Ae^{bx}
$\sin bx$	$A\sin bx+B\cos bx$
$\cos bx$	$A\sin bx+B\cos bx$

General case: If $g(x)$ is sum or product of any two functions we assume y_p the sum or product of the corresponding forms; for instance

if $g(x)=x+2e^{4x}$ we assume $y_p=Ax+B+Ce^{4x}$

if $g(x)=xe^{4x}$ we assume $y_p=(Ax+B)e^{4x}$

Example:

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}.$$

SOLUTION

Corresponding to $3x^2$ we assume $\quad y_{p_1} = Ax^2 + Bx + C.$

Corresponding to $-5\sin 2x$ we assume $\quad y_{p_2} = E\cos 2x + F\sin 2x.$

Corresponding to $7xe^{6x}$ we assume $\quad y_{p_3} = (Gx + H)e^{6x}.$

The assumption for the particular solution is then

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}.$$

No term in this assumption duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$.

Example: Solve $y''-4y=3xe^x$

Find y_c $\qquad y''-4y=0$

$$m^2-4=0$$

$$m=\pm 2$$

$$y_c = c_1e^{2x} + c_2e^{-2x}.$$

$$y_p = (Ax+B)e^x$$

$$y'_p = (Ax+B)e^x + Ae^x$$

$$y''_p = (Ax+B)e^x + 2Ae^x$$

$$(Ax+B)\cancel{e^x} + 2A\cancel{e^x} - 4(Ax+B)\cancel{e^x} = 3x\cancel{e^x}$$

-3Ax+2A-3B=3x

$\Rightarrow \quad -3A = 3 \quad \Rightarrow \quad \boxed{A = -1}$

$2A - 3B = 0$

$-2 - 3B = 0 \quad \Rightarrow \quad \boxed{B = -\frac{2}{3}}$

$\therefore \quad y = c_1 e^{2x} + c_2 e^{-2x} + \left(-x - \frac{2}{3}\right)e^{x}$

$= c_1 e^{2x} + c_2 e^{-2x} - xe^{x} - \frac{2}{3}e^{x}.$

Next we deal with the cases when a function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

Suppose again that g(x) consists of m terms of the kinds disscussed before, and suppose further that the usual assumption for a particular solution is

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m},$$

where the y_{p_i}, $i = 1, 2, \ldots, m$ *are the trial particular solution forms corresponding to these terms. If any* y_{p_i} *contains terms that duplicate terms in* y_c, *then that* y_{p_i} *must be multiplied by* x^n, *where n is the smallest positive integer that eliminates that duplication.*

Example: Solve $y''-2y'=5e^{2x}$

Find y_c :

$$m^2-2m=0$$

$$m(m-2)=0$$

$$m=0,\ 2$$

$$y_c=c_1e^{0x}+c_2e^{2x}$$

$$y_c=c_1+c_2e^{2x}$$

$y_p=Axe^{2x}$

$y_p'=2Axe^{2x}+Ae^{2x}$

$y_p''=4Axe^{2x}+4Ae^{2x}$

We multiply by x to eleminate the duplication e^{2x}

$$4A\cancel{e^{2x}}+4Ax\cancel{e^{2x}}-2(A\cancel{e^{2x}}+2Ax\cancel{e^{2x}})=5\cancel{e^{2x}}$$

$$\cancel{(4A-4A)x}+2A=5$$

$$2A=5$$

$$A=5/2$$

$$\Rightarrow y_p=5/2\,xe^{2x}$$

$$y=y_c+y_p=c_1+c_2e^{2x}+\frac{5}{2}xe^{2x}.$$

Example: Solve $y'' + y = 4x + 10\sin x$

Find y_c: $\quad y'' + y = 0$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i =$$

$$y_c = e^{0x}[C_1\cos x + C_2\sin x].$$

$$y_c = C_1\cos x + C_2\sin x$$

$$y_p = (Ax+B) + (C\sin x + D\cos x)\,x$$

$$= Ax + B + Cx\sin x + Dx\cos x$$

$$y'_p = A + C\sin x + Cx\cos x + D\cos x - Dx\sin x$$

$$y''_p = 2C\cos x - Cx\sin x - 2D\sin x - Dx\cos x$$

$$2C\cos x - \cancel{Cx\sin x} - 2D\sin x - \cancel{Dx\cos x}$$
$$+ \underline{Ax + B} + \cancel{Cx\sin x} + \cancel{Dx\cos x} = \underline{4x} + 10\sin x$$

A=4, B=0

2C=0 ⇨ C=0

-2D=10 ⇨ D=-5

$y=C_1\cos x+C_2\sin x+4x-5x\cos x$

Example: Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

Find y_c: $y'' - 6y' + 9y = 0$

$$m^2 - 6m + 9 = 0$$

$$(m-3)(m-3) = 0$$

$$m = 3$$

$$y_c = \underline{C_1 e^{3x} + C_2 x e^{3x}}$$

$$y_p = Ax^2 + Bx + C + D x^2 e^{3x}$$

$$y_p' = 2Ax + B + 2Dxe^{3x} + 3Dx^2e^{3x}$$

$$y_p'' = 2A + 2De^{3x} + 12Dxe^{3x} + 9Dx^2e^{3x}$$

Cont. : Find A, B, C, D

$$y = y_c + y_p$$

EXERCISES

$$y'' + 3y' + 2y = 6$$

$$y'' + y' - 6y = 2x$$

$$y'' - 8y' + 20y = 100x^2 - 26xe^x$$

$$4y'' - 4y' - 3y = \cos 2x$$

$$y'' + 2y' = 2x + 5 - e^{-2x}$$

$$y'' - 16y = 2e^{4x}$$

$$y'' + 4y = 3 \sin 2x$$

$$y'' - 4y = (x^2 - 3) \sin 2x$$

$$y'' + y = 2x \sin x$$

$$y'' - 2y' + 5y = e^x \cos 2x$$

$$y'' + 2y' + y = \sin x + 3 \cos 2x$$

$$y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$$

$$y''' - 2y'' - 4y' + 8y = 6xe^{2x}$$

$$y''' - 6y'' = 3 - \cos x$$

$$y^{(4)} - y'' = 4x + 2xe^{-x}$$