ZIEGLER'S EXAMPLE WITH A DIFFERENT SIGNATURE

DEXUAN HU

1. Introduction

The example about many-sorted circles is a classical one in the study of Lascar groups and Lascar strong types. It is the first example of a G-compact theory. Moreover, its variations are the only known examples of such theories. On the descriptive set theoretic side, the Borel cardinality of certain Lascar strong types of this theory is l^∞, which is considered quite complicated. However, a slight variation of the example gives rise to a Borel cardinality of E_0.

The original example starts with the following structure: circles, circular order on the circles, rotation functions and certain bonding maps between circles. We find that the topology of a circle never enters the argument, and one important type of relations, namely certain local orders, is not included in the language. Using the local orders, one can formulate relevant circular orders, rotation functions, even the relation of being in the same Lascar strong type. We will present essentially the same example from Zeigler, but with different underlying set and language.

2. Structure

We will be working in a multi-sorted setting. For each $n = 1, 2, \ldots$, let R_n be a binary relational symbol. Consider the cantor space $2^{\mathbb{N}}$ with the usual lexicographic order. Identify the end points $00\cdots$ and $11\cdots$ so that the order is circular. The interpretation of R_n on this set is the following: $R_n(\alpha, \beta)$ if and only if $a = b$, or there is a convex set S containing α and β such that

(1) the Lebesgue measure $\mu(S) < \frac{1}{2^n}$;
(2) $\alpha <_{lex} \beta$ if $00\cdots(11\cdots)$ is contained in S;
(3) $\beta <_{lex} \alpha$ if $00\cdots(11\cdots)$ is not contained in S.

Intuitively $R_n(\alpha, \beta)$ is a local segment of the circular order. Finally, we remove the set $C := \{\alpha : \exists N \forall n \geq N, \alpha(n) = \alpha(N)\}$ from $2^{\mathbb{N}}$. $2^{\mathbb{N}} \setminus C$ is the underlying set of our structure. R_n is interpreted as the restriction of R_n described above. We now use M_n to denote the structure $(2^{\mathbb{N}} \setminus C, R_n)$. Let T_n to be the theory of M_n, that is, all sentences satisfied by M_n.

With R_n, one can define the circular order and a special function:

$$
\begin{aligned}
&S_n(x,y,z) := x \neq y \neq z \bigwedge \\
&\qquad \forall z_1, z_2, \ldots, z_{2^n-1}(R_n(x,z_1) \wedge R_n(z_1,z_2) \wedge \cdots \wedge R_n(z_{2^n-1}, z) \\
&\qquad \longrightarrow (R_n(x,y) \wedge R_n(y,z_1)) \vee \cdots \vee (R_n(z_{2^n-1,y}) \wedge R_n(y,z)))
\end{aligned} \tag{1}
$$

$$g_n(x,y) := \neg R_n(x,y) \bigwedge \forall z(R_n(x,z) \wedge x \neq z) \longrightarrow R_n(z,y) \tag{2}$$

Note that S_n is the strict circular order formed by considering $<_{lex}$ and identifying endpoints. g_n is the function $\alpha \mapsto \delta(\alpha \upharpoonright n)^\frown \alpha \setminus n$, where δ is the map that sends a finite sequence of length n to the next finite sequence of length n in the lexicographic order.

The definitions of S_n and g_n use only the universal quantifiers. Now we show that S_n and g_n can also be existentially defined. To see this, note that M_n witnesses for any x, y and z, $S_n(x,y,z)$ if and only if $\neg S_n(y,x,z)$. Hence T_n contains this sentence, and one can define $S_n(x,y,z)$ as $\neg S_n(y,x,z)$, making the definition existential. For g_n, one can define the same function as

$$\begin{aligned}\exists z_1,\dots,z_{2^n-2}(&\neg R_n(x,y) \wedge \exists w_1 R_n(x,w_1) \wedge (w_1,y)\\ &\wedge \neg R_n(y,z_1) \wedge \exists w_2 R_n(y,w_2) \wedge (w_2,z_1)\\ &\dots\\ &\wedge \neg R_n(z_{2^n-2},x) \wedge \exists w_{2^n} R_n(z_{2^n-2},w_{2^n}) \wedge (w_{2^n},x))\end{aligned} \tag{3}$$

This gives an existential definition of g_n. Finally, we have

$$R_n(x,y) \text{ if and only if } x = y \vee S_n(x,y,g_n(x)) \vee S_n(y,x,g_n(y)). \tag{4}$$

Let M be the many-sorted structure $(M_n)_n$ where each M_n is a distinct sort. Let M^* be the monster model of $\mathrm{Th}(M)$. It is not hard to see $M^* = (M_n^*)_n$ with each M_n^* being the monster model of T_n.

The following facts about T_n will be useful.

Proposition 2.1. *(i) Under T_n, every formula is equivalent to a universal formula.*
(ii) Every submodel of M_n^* is an elementary submodel.

Proof. (i) We use the following strategy. First, we expand the language with two additional symbols B_n and g_n. The interpretation of these symbols is stated above. Let T_n' be the theory of M_n in the expanded language.

Assume that T_n' admits quantifier elimination. Given any $\{R_n\}$-formula ϕ, it is also an $\{R_n, B_n, g_n\}$-formula, therefore it is equivalent to a quantifier free $\{R_n, B_n, g_n\}$-formula. Now one just replace each instance of B_n and g_n by its definition. More precisely, if B_n or g_n appears in positive position, use the definition that only involves $\forall$ quantifier, and similarly if B_n or g_n appears in negative position, use the definition that only involves $\exists$ quantifier. In this way, we are done by obtaining a universal $\{R_n\}$-formula that is equivalent to ϕ.

Now, it suffices to show quantifier elimination for T_n'. Let M and N be models of T_n'. It is enough to show that finite partial isomorphisms from M to N form a back-and-forth system. Let γ be any finite partial isomorphism and let $a \in M$ such that $a \notin \mathrm{domain}(\gamma)$. If γ is empty, we can choose any $b \in N$ and define γ' to be $\gamma'(a) = (b)$ and $\gamma'(g_n^i(a)) = g_n^i(b)$ for $1 \leq i < n$. γ' is clearly a finite partial isomorphism that extends γ.

If γ is non-empty, we may assume domain(γ) is a nonempty finitely generated substructure of M, that is, a finite g_n-closed set in M. Enumerate domain(γ) with respect to the circular order as $\{a_1, a_2, \ldots, a_m\}$, then a must satisfy $B_n(a_j, a, a_{j+1})$ for some $1 \leq j \leq m-1$ or $B_n(a_m, a, a_1)$. By density of the circular order, we can choose $b \in N$ that is between $\gamma(a_j)$ and $\gamma(a_{j+1})$ (or $\gamma(a_m)$ and $\gamma(a_1)$). Now define γ' extending γ such that $\gamma'(a) = b$ and $\gamma'(g_n^i(a)) = g_n^i(b)$. It is easy to check γ' is a partial isomorphism between M and N. The "back" part is similar.

(ii) follows from (i). Too see this, let N be a submodel, $\phi(\bar{x})$ be any formula, $\bar{a}$ be a tuple from N. By (i), $\phi(\bar{x})$ is equivalent to a universal formula $\psi_1(\bar{x})$. Hence, if $M_n^* \models \psi_1(\bar{a})$, then $N \models \psi_1(\bar{a})$. On the other hand, since $\neg\phi(\bar{x})$ is equivalent to a universal formula, $\phi(\bar{x})$ is equivalent to an existential formula, say $\psi_2(\bar{x})$. Then, we have $N \models \psi_2(\bar{a})$ implies $M_n^* \models \psi_2(\bar{a})$. □

The next proposition is important for our analysis of Lascar strong types. It also justifies the choice of new language.

Proposition 2.2. *Let $a, b \in M_n^*$. Then a and b have the same type over some elementary submodel of M_n^* if and only if $R_n(a,b)$ or $R_n(b,a)$.*

Proof. Without loss of generality, assume $R_n(a,b)$. Consider the collection

$$I := \{c \in M_n^* : R_n(a,c) \wedge R_n(b,c)\}.$$

This is all the points in M_n^* that lie strictly between b and $g_n(a)$. Let X be the substructure with underlying set

$$\{d \in M_n^* : d = g_n^{(i)}(c), c \in I, 0 \leq i < 2^n\}.$$

Intuitively, X is the substructure generated by I using g_n. Although g_n is not formally in our language, M_n^* sees there is a unique element $g_n(c)$ that satisfies (2) for every $c \in M_n^*$. One checks that X is a submodel, and by 2.1(ii), an elementary submodel.

The backward direction is done if we show a, b satisfy the same set of formulas with parameters from X. Suppose $\phi(a)$ holds for some $\{R_n, X\}$-formula ϕ. Using (4) to replace all appearance of R_n, we get a $\{R_n, B_n, g_n, X\}$-formula $\phi'(a)$. By quantifier elimination in the proof of Proposition2.1, $\phi'(a)$ is equivalent to a quantifier free $\{R_n, B_n, g_n, X\}$-formula $\psi(a)$. However, X cannot distinguish a from b by a quantifier free $\{R_n, B_n, g_n\}$-formula. We have $\psi(b)$, therefore, $\phi(b)$ also holds.

Conversely, suppose $\neg R_n(a,b)$ and $\neg R_n(b,a)$. Note that any elementary submodel $N \prec M_n^*$ should intersect $\{c : R_n(a,c)\}$. Too see this, Let $c \in N$ be in the intersection. We have $R_n(a,c)$ but $\neg R_n(b,c)$. Therefore, a, b cannot have the same type over N. This finishes the proof since N is arbitrary. □

We obtain a quick corollary from the proposition above.

Corollary 2.3. *For each n, any two elements of M_n^* are Lascar strong related, but their Lascar distance can be $\geq 2^{n-1}$.*

Now we add function symbols f_n to the language. For each n, f_n is interpreted as the map from M_{n+1} to M_n that deletes the 0-th digit:

$$f_n(\alpha)(n) = \alpha(n+1).$$

We need the following facts about f_n.

(1) f_n is surjective and two-to-one.
(2) f_n preserves R_n, that is, $R_{n+1}(a, b)$ implies $R_n(f_n(a), f_n(b))$.
(3) Conversely, suppose $R_n(a, b)$. Let $\{a_1, a_2\}$ be the preimage of a, $\{b_1, b_2\}$ be the preimage of b. Then $R_{n+1}(a_1, b_i)$ for some i, and $R_{n+1}(a_2, b_{3-i})$.

Intuitively f_n describes how to wrap the circular order on M_{n+1} twice to form a doubling of M_n. We call f_n bonding maps.

Let M' be the new multi-sorted structure with these bonding maps. Let M'^* be the saturated model of the theory of M'. Finally, let X be the set of infinite tuples $(a_n)_n$ where $a_n \in M^*_n$ for each n and $f_n(a_{n+1}) = a_n$ for all n. We call sequences in X inverse limit sequences.

Quantifier elimination up to universal quantifier also holds for $Th(M')$. To show that we need a model theoretic fact.

Proposition 2.4 (Corollary 3.1.6 of [5])**.** *Let T be a theory, Suppose that for all quantifier-free formulas $\phi(\overline{v}, w)$, if $M, N \models T$, A is a common substructure of M and N, $\overline{a}$ is a tuple from A, and there is $b \in M$ such that $M \models \phi(\overline{a}, b)$, then there is $c \in N$ such that $N \models \phi(\overline{a}, c)$. Then T has quantifier elimination.*

The test will fail with only the relational symbols R_n and bonding maps f_n because a substructure is not necessarily closed under g_n, although g_n is definable from R_n. Hence, we use the same strategy as before, we add additional symbols B_n and g_n to the language, show that the new theory has quantifier elimination, and use the appropriate $\forall$ or $\exists$ definitions to obtain universal R_n, f_n-formulas.

From now, suppose we are working in the language $\{R_n, B_n, g_n, f_n\}$. Note that the monster model M'^* is also a monster model of the theory of M' in the expanded language. The following lemma is useful.

Lemma 2.5. *If M is an elementary submodels of M'^*, A is a substructure of M, $\phi(\overline{v}, w)$ is a quantifier free formula, $\overline{a}$ is a tuple from A. Then the solution to $\phi(\overline{a}, w)$ in M is of the form a union of elements in A and intervals with endpoints in A.*

Proof. The proof is a straightforward induction on the complexity of formula. Note that $\phi(\overline{v}, w)$ is quantifier free so we only deal with propositional symbols. The only non-trivial thing is that there are formulas with terms $f_n(g^k_{n+1}(w))$, but one can always write another formula with the same set of solutions to w but without f_n. This is done by taking preimage of f_n of appropriate elements in $\overline{a}$ and write the formula that w satisfies with repects to those points in the preimage. □

Proposition 2.6. *The theory of M' in the language $\{R_n, B_n, g_n, f_n\}$ has quantifier elimination.*

Proof. Let M and N be any elementary submodels of M'^* and A be their common substructure. Let $\phi(\overline{v}, w)$ be a quantifier free formula and $\overline{a}$ be a tuple from A. If $\phi(\overline{a}, w)$ has a solution in M, by Lemma 2.5, this means the solution set, which is a union of elements in A and intervals with endpoints in A, is non-empty. Since A is also a subset of N and the order on N is dense, N should contain a solution to $\phi(\overline{a}, w)$. Then the conclusion follows immediately from Proposition 2.4. □

Corollary 2.7. *Th(M') in the language $\{R_n, f_n\}$ has quantifier elimination up to $\forall$, that is, every formula is equivalent to a universal formula.*

Proof. By Proposition2.6, every $\{R_n, f_n\}$-formula is equivalent to a quantifier free $\{R_n, B_n, g_n, f_n\}$-formula. Then we can substitute appropriate definitions of B_n and g_n to get a universal formula. □

By the same argument as Proposition2.1(ii), an immediate consequence of this corollary is that every submodel of M'^* is an elementary submodel. Note that $N \subset M'^*$ is a submodel if and only if N_n is a submodel of M'^*_n for each sort n, and N is closed under taking f_n and preimage f_n^{-1}. This also gives a characterization of elementary submodels of M'^*.

Lemma 2.8. *Let (a_n), (b_n) be two inverse limit sequences from M'^* such that $R_n(a_n, b_n)$ for all n, then there is an automorphism of M'^* fixing an elementary submodel that maps (a_n) to (b_n).*

Proof. For each sort n, consider the interval $I_n := \{c \in M'^*_n : R_n(a_n, c) \wedge R_n(b_n, c)\}$. Take the substructure N_n "generated" by I_n using g_n as before. Then N_n is a submodel of M'^*_n. Moreover, by property (2) and (3) of f_n, $f_n(N_{n+1})$ is contained in N_n and $f_n^{-1}(N_n)$ is contained in N_{n+1}. By the characterization of elementary submodel above, $(N_n)_n$ is an elementary submodel. Furthermore, (a_n) and (b_n) have the same type over $(N_n)_n$. By saturation of the monster model M'^*, there is an automorphism of M'^* fixing $(N_n)_n$ that maps (a_n) to (b_n). □

Now we can compute the Borel cardinality of the lascar strong type of the inverse limit sequences. First, fix a countable elementary submodel M^0 of M'^* with the underlying set contained in the original structure M', that is, every element from M^0 is a binary sequence. Then, we consider the type space, with parameters from M^0, of the inverse limit sequences, denoted by $S_X(M^0)$. Note that this space is a closed subspace of $S_{x_0,x_1,\ldots}(M^0)$ where each x_i is from sort i. Hence, $S_X(M^0)$ is a compact Polish space.

Let $\equiv_L$ be the equivalence relation on $S_X(M^0)$, defined as follow:

$$p \equiv_L q \iff \exists a, b \in M'^* \text{ such that } a \models p, b \models q, \text{ and } a, b \text{ are Lascar related .}$$

This is the same equivalence relation "induced" by the Lascar strong types as in Definition 2.1 of [2].

We introduce another equivalence relation on $2^{\mathbb{N}}$. Fix an $n \geq 1$, consider all the finite sequences of length n as vertices of some graph. We then define an edge between two sequences if and only if they are adjacent in the reversed lexicographic order. By the reversed lexicographic order, we mean the order defined as

$s_0s_1\cdots s_n \leq t_0t_1\cdots t_n$ if and only if $s_ns_{n-1}\cdots s_0 \leq_{\text{lex}} t_nt_{n-1}\cdots t_0$. We add one more edge between $00\cdots 0$ and $11\cdots 1$ and obtain a cyclic graph on 2^n vertices. The graph distance between two vertices s, t is denoted by $d_n^g(s,t)$. Finally, we can define a relation E_g on $2^{\mathbb{N}}$:

$$\alpha\, E_g\, \beta \Longleftrightarrow \exists K\, \forall n\, d_n^g(\alpha \restriction n, \beta \restriction n) \leq K.$$

Two sequence are E_g-related if and only if the graph distances of their finite initial segments are bounded. E_g is clearly a Borel equivalence relation on $2^{\mathbb{N}}$.

The next proposition connects $\equiv_L$ with E_g.

Proposition 2.9. *$\equiv_L$ is Borel reducible to E_g via a continuous surjective map.*

Proof. We define a concrete $\phi : S_X(M^0) \longrightarrow 2^{\mathbb{N}}$ that is continuous and surjective, then check that ϕ is a reduction.

Given a type $p \in S_X(M^0)$, for each n, let $A_n := \{a \in M^0 : R_n(x_n, a) \in p\}$ and $B_n := \{b \in M^0 : R_n(b, x_n) \in p\}$. If (r_n) is a realization of p, then A_n collects all elements on the right of r_n that is close to r_n, and similarly B_n collects all elements on the left of r_n that is closed to r_n, so A_n and B_n are a local "cut". There are three possible cases:

- i A_n and B_n have non-empty intersection. Then there is some $a \in M^0$ such that $R_n(x,a)$ and $R_n(a,x)$ are both in p. This implies a is the n-th sort of any realization of p.
- ii A_n and B_n do not intersect, but there exists $\alpha \in A_n$ and $\beta \in B_n$ with $\alpha \restriction n = \beta \restriction n$.
- iii A_n and B_n do not intersect, and for all $\alpha \in A_n$ and $\beta \in B_n$ we always have $\alpha \restriction n \neq \beta \restriction n$.

If case i holds, let $\phi(p)(n) = a(0)$. If case ii is true, there are three possible initial segments of elements in A_n and B_n, say $s, r, t \in 2^n$. Let r be the initial segment that appears in both A_n and B_n. We now define $\phi(p)(n) = r(0)$. If iii is true, then there only two possible initial segments of elements in A_n and B_n, say $s, t \in 2^n$ respectively. In this case, we let $\phi(p)(n) = s(0)$.

We have defined the map ϕ from $S_X(M^0)$ to $2^{\mathbb{N}}$. To see why it is onto, we start with any $\alpha \in M_1^0$. Its preimage under f_1 will be $0^\frown\alpha$ and $1^\frown\alpha$, so we can choose either one, say $i^\frown\alpha$. Similarly, for the preimage of $i^\frown\alpha$ under f_2, one can freely choose $0^\frown i^\frown\alpha$ or $1^\frown i^\frown\alpha$. Therefore by varying inverse sequences from M^0 with arbitrary first digits of n-th sort, one get all of $2^{\mathbb{N}}$. Continuity follows from the fact that the first n digits of $\phi(p)$ depends only on the first n sorts, so given $s \in 2^n$, one can write finitely many formulas to guarantee that $\phi(p) \restriction n = s$.

It remains to show that ϕ is a reduction.

$(\Rightarrow)$: Suppose $\phi(p)$ and $\phi(q)$ are not E_g related. Then for any realization (a_n) of p and (b_n) of q, the Lascar distance between (a_n) and (b_n) is unbounded. This is because on each sort n, the Lascar distance between a_n and b_n is greater than or equal to $d_n^g(\phi(p) \restriction n, \phi(q) \restriction n) - 1$, so the Lascar distance is unbounded as n increases.

($\Leftarrow$) Suppose $\phi(p)$ and $\phi(q)$ are E_g related, we need to show that for some (any) realizations (a_n) of p and (b_n) of q, (a_n) can be mapped to (b_n) by a finite sequence of automorphisms of M'^*, each fixing a small elementary submodel.

Let $k = \max\{d_n^g(\phi(p) \restriction n, \phi(q) \restriction n)\}$. Suppose this maximum distance is first obtained by $\phi(p) \restriction m, \phi(q) \restriction m$ for some m. Then the d_{m+1}^g distance between $\phi(p) \restriction m+1$ and $\phi(q) \restriction m+1$ is k or $2^m - k$. $2^m - k$ is greater or equal to k by induction. Hence, we must have $d_{m+1}^g(\phi(p) \restriction m+1, \phi(q) \restriction m+1) = k$. By the same argument on longer initial segments, we see that d_n^g stabilizes at k after m.

Without loss of generality, assume for all $n \geq m$, the shorter arc on the cyclic graph from $\phi(p) \restriction n$ to $\phi(q) \restriction n$ is clockwise. Fix realizations $(a_n) \models p$ and $(b_n) \models q$. There exists a finite sequence, of length k, of inverse limit sequences $(c_n^1), (c_n^2), \ldots, (c_n^k)$ from M'^* such that

$$R_n(a_n, c_n^1) \wedge R_n(c_n^1, c_n^2) \wedge \cdots \wedge R_n(c_n^k, b_n)$$

for all n. By Lemma2.8, there is an automorphism fixing small elementary submodels that maps (a_n) to (c_n^1). Moreover, the same holds for pairs $(c_n^i), (c_n^{i+1})$ and $(c_n^k), (b_n)$. Composi ng all the $k+1$ automorphisms, we see that (a_n) and (b_n) have the same Lascar strong type. $\square$

One thing to point out is that $\equiv_L$ is Borel bi-reducible to E_g, because the reduction ϕ is a continuous surjective map between compact Polish spaces, therefore it has a Borel section, and this Borel function witnesses the reduction from E_g to $\equiv_L$. This particular case of bi-reducibility appears in other proofs concerning the Borel cardinality of Lascar strong types.

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Department of Mathematics, Cornell University, Ithaca, NY 14853
Email address: dh623@cornell.edu