Ejercicio 1. Considerar la fórmula numérica siguiente:

$$\int_{-1}^{1} f(x)(1-x^2)\ dx = \alpha_0 f(-1) + \alpha_1 f(0) + \alpha_2 f(1) + R(f)$$

1. Hallar los valores de α_0, α_1 y α_2.

   Calculamos cada uno de los polinomios básicos de Lagrange:

$$\ell_0(x) = \prod_{\substack{j=0\\ j\neq 0}}^{2} \frac{x-x_j}{x_0-x_j} = \frac{x-x_1}{x_0-x_1}\cdot\frac{x-x_2}{x_0-x_2} = \frac{x(x-1)}{(-1)(-2)} = \frac{x(x-1)}{2}$$
$$\ell_1(x) = \prod_{\substack{j=0\\ j\neq 1}}^{2} \frac{x-x_j}{x_1-x_j} = \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2} = \frac{(x+1)(x-1)}{1(-1)} = -(x+1)(x-1)$$
$$\ell_2(x) = \prod_{\substack{j=0\\ j\neq 2}}^{2} \frac{x-x_j}{x_2-x_j} = \frac{x-x_0}{x_2-x_0}\cdot\frac{x-x_1}{x_2-x_1} = \frac{x(x+1)}{(2)1} = \frac{x(x+1)}{2}$$

   Multiplicamos ahora cada uno de los polinomios por la función peso:

$$\ell_0(x)(1-x^2) = -\ell_0(x)(x+1)(x-1) = -\frac{x(x-1)^2(x+1)}{2} = -\frac{1}{2}(x^4-x^3-x^2+x)$$
$$\ell_1(x)(1-x^2) = -\ell_1(x)(x+1)(x-1) = (x+1)^2(x-1)^2 = x^4-2x^2+1$$
$$\ell_2(x)(1-x^2) = -\ell_2(x)(x+1)(x-1) = -\frac{x(x+1)^2(x-1)}{2} = -\frac{1}{2}(x^4+x^3-x^2-x)$$

   Calculamos ahora los valores de α_0, α_1 y α_2:

$$\alpha_0 = \int_{-1}^{1} \ell_0(x)(1-x^2)\ dx = -\frac{1}{2}\int_{-1}^{1}(x^4-x^3-x^2+x)\ dx = -\left(\frac{1}{5}-\frac{1}{3}\right) = \frac{2}{15}$$
$$\alpha_1 = \int_{-1}^{1} \ell_1(x)(1-x^2)\ dx = \int_{-1}^{1}(x^4-2x^2+1)\ dx = 2\left(\frac{1}{5}-\frac{2}{3}+1\right) = \frac{16}{15}$$
$$\alpha_2 = \int_{-1}^{1} \ell_2(x)(1-x^2)\ dx = -\frac{1}{2}\int_{-1}^{1}(x^4+x^3-x^2-x)\ dx = \frac{2}{15}$$

   Por tanto, tenemos que:

$$\begin{aligned}\int_{-1}^{1} f(x)(1-x^2)\ dx &\approx \int_{-1}^{1}\left[\ell_0(x)f(-1)+\ell_1(x)f(0)+\ell_2(x)f(1)\right](1-x^2)\ dx\\ &= \left(\int_{-1}^{1}\ell_0(x)(1-x^2)\ dx\right)f(-1) + \left(\int_{-1}^{1}\ell_1(x)(1-x^2)\ dx\right)f(0)+\\ &\qquad + \left(\int_{-1}^{1}\ell_2(x)(1-x^2)\ dx\right)f(1)\\ &= \frac{2}{15}f(-1)+\frac{16}{15}f(0)+\frac{2}{15}f(1)\end{aligned}$$

2. Hallar una expresión del error $R(f)$.

   Tenemos que la expresión del error cometido al aproximar $f(x)$ por el polinomio de interpolación de Lagrange de grado 2 es:

$$E(x) = f[-1,0,1,x]\Pi(x) \qquad \text{donde} \qquad \Pi(x) = \prod_{j=0}^{2}(x-x_j) = x(x-1)(x+1)$$

   Por tanto:

$$R(f) = L(E) = \int_{-1}^{1} E(x)(1-x^2)\, dx = \int_{-1}^{1} f[-1,0,1,x]\Pi(x)(1-x^2)\, dx$$

   Sabemos que $\Pi(x)$ cambia de signo en $x=0$. Para evitar esto, hacemos uso de que:

$$f[-1,0,0,1,x] = \frac{f[-1,0,1,x]-f[-1,0,0,1]}{x-0} \Longrightarrow$$
$$\Longrightarrow f[-1,0,1,x] = f[-1,0,0,1,x]x + f[-1,0,0,1]$$

   Por tanto, tenemos que:

$$\begin{aligned} R(f) &= \int_{-1}^{1} \left(f[-1,0,0,1,x]x + f[-1,0,0,1]\right) x(x-1)(x+1)(1-x^2)\, dx = \\ &= \int_{-1}^{1} \left(f[-1,0,0,1,x]x^2 + f[-1,0,0,1]x\right)(x-1)^2(x+1)^2\, dx = \\ &= \int_{-1}^{1} f[-1,0,0,1,x]x^2(x-1)^2(x+1)^2\, dx + f[-1,0,0,1]\int_{-1}^{1} x(x-1)^2(x+1)^2\, dx \end{aligned}$$

   Por el Teorema del Valor Medio Integral Generalizado, $\exists \mu \in [-1,1]$ tal que:

$$\begin{aligned} R(f) &= f[-1,0,0,1,\mu]\int_{-1}^{1} x^2(x-1)^2(x+1)^2\, dx + f[-1,0,0,1]\int_{-1}^{1} x(x-1)^2(x+1)^2\, dx = \\ &= f[-1,0,0,1,\mu]\int_{-1}^{1}(x^6-2x^4+x^2)\, dx + f[-1,0,0,1]\int_{-1}^{1}(x^5-2x^3+x)\, dx = \\ &= 2f[-1,0,0,1,\mu]\left(\frac{1}{7}-\frac{2}{5}+\frac{1}{3}\right) + f[-1,0,0,1]\cdot 0 \\ &= \frac{16}{105}\cdot f[-1,0,0,1,\mu] \end{aligned}$$

   Suponiendo $f \in C^4[-1,1]$, $\exists \xi \in [-1,1]$ tal que:

$$R(f) = \frac{16}{105}\cdot\frac{f^{(4)}(\xi)}{4!} = \frac{2}{315}\cdot f^{(4)}(\xi)$$