Dynamics of three link chains

1 Unbranched chains

1.1 planar 3 link parallel

See https://github.com/LinkageDynamics/open-chain/threelink dneplanar3a.tex for text expressions
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```
mfilename=mar21parpar3, outfilename=dneplanar3a.org, datetime=21-Mar-2025 17:22:59
         cog: 0
     inertia: 0
    numlinks: 3
     gravity: 'all'
```

$$M = \begin{bmatrix} a_1 + a_3 + 2\,L_1\,a_4\,c_{23} + 2\,L_2\,a_4\,c_3 + 2\,L_1\,L_2\,m_3\,c_2 + 2\,L_1\,L_{cog2}\,m_2\,c_2 & a_3 + L_1\,a_4\,c_{23} + 2\,L_2\,a_4\,c_3 + L_1\,L_2\,m_3\,c_2 + L_1\,L_{cog2}\,m_2\,c_2 & a_4\,\left(L_{cog3} + L_1\,c_{23} + L_2\,c_3\right) \\ a_3 + L_1\,a_4\,c_{23} + 2\,L_2\,a_4\,c_3 + L_1\,L_2\,m_3\,c_2 + L_1\,L_{cog2}\,m_2\,c_2 & a_3 + 2\,L_2\,a_4\,c_3 & a_4\,\left(L_{cog3} + L_2\,c_3\right) \\ a_4\,\left(L_{cog3} + L_1\,c_{23} + L_2\,c_3\right) & a_4\,\left(L_{cog3} + L_2\,c_3\right) & L_{cog3}\,a_4 \end{bmatrix}$$

where

$$a_1 = {L_1}^2\,m_2 + {L_1}^2\,m_3 + {L_{cog1}}^2\,m_1$$

$$a_3 = m_3\,{L_2}^2 + m_2\,{L_{cog2}}^2 + m_3\,{L_{cog3}}^2$$

$$a_4 = L_{cog3}\,m_3$$

$$G_x = \begin{bmatrix} -L_1\,m_2\,s_1 - L_1\,m_3\,s_1 - L_{cog1}\,m_1\,s_1 - L_{cog3}\,m_3\,s_{123} - L_2\,m_3\,s_{12} - L_{cog2}\,m_2\,s_{12} \\ -L_{cog3}\,m_3\,s_{123} - L_2\,m_3\,s_{12} - L_{cog2}\,m_2\,s_{12} \\ -L_{cog3}\,m_3\,s_{123} \end{bmatrix}$$

$$G_y = \begin{bmatrix} L_{cog3}\,m_3\,c_{123} + L_2\,m_3\,c_{12} + L_{cog2}\,m_2\,c_{12} + L_1\,m_2\,c_1 + L_1\,m_3\,c_1 + L_{cog1}\,m_1\,c_1 \\ L_{cog3}\,m_3\,c_{123} + L_2\,m_3\,c_{12} + L_{cog2}\,m_2\,c_{12} \\ L_{cog3}\,m_3\,c_{123} \end{bmatrix}$$

$$G_z = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -L_1\,\left(L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23}\right) & -L_{cog3}\,m_3\,\left(L_1\,s_{23} + L_2\,s_3\right) \\ L_1\,\left(L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23}\right) & 0 & -L_2\,L_{cog3}\,m_3\,s_3 \\ L_{cog3}\,m_3\,\left(L_1\,s_{23} + L_2\,s_3\right) & L_2\,L_{cog3}\,m_3\,s_3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2\,L_1\,\left(L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23}\right) & -2\,L_{cog3}\,m_3\,\left(L_1\,s_{23} + L_2\,s_3\right) & -2\,L_{cog3}\,m_3\,\left(L_1\,s_{23} + L_2\,s_3\right) \\ 0 & -2\,L_2\,L_{cog3}\,m_3\,s_3 & -2\,L_2\,L_{cog3}\,m_3\,s_3 \\ 2\,L_2\,L_{cog3}\,m_3\,s_3 & 0 & 0 \end{bmatrix}$$

1.2 perp par

See https://github.com/LinkageDynamics/open-chain/threelink dneperppar3.tex for text expressions
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```
mfilename=mar21perppar3, outfilename=dneperppar3.org, datetime=21-Mar-2025 17:33:00
         cog: 0
     inertia: 0
    numlinks: 3
     gravity: 'all'
```

$$M = \begin{bmatrix} a_1 + a_2 + L_2\,a_4\,\cos\left(2\,\theta_2 + \theta_3\right) + \frac{L_{cog3}\,a_4\,\cos\left(2\,\theta_2 + 2\,\theta_3\right)}{2} + \frac{{L_2}^2\,m_3\,\cos\left(2\,\theta_2\right)}{2} + \frac{{L_{cog2}}^2\,m_2\,\cos\left(2\,\theta_2\right)}{2} + 2\,L_1\,a_4\,c_{23} + L_2\,a_4\,c_3 + 2\,L_1\,L_2\,m_3\,c_2 + 2\,L_1\,L_{cog2}\,m_2\,c_2 & 0 & 0 \\ 0 & a_3 + 2\,L_2\,a_4\,c_3 & a_4\,\left(L_{cog3} + L_2\,c_3\right) \\ 0 & a_4\,\left(L_{cog3} + L_2\,c_3\right) & L_{cog3}\,a_4 \end{bmatrix}$$

where

$$a_1 = {L_1}^2\,m_2 + {L_1}^2\,m_3 + \frac{{L_2}^2\,m_3}{2}$$

$$a_2 = m_1\,{L_{cog1}}^2 + \frac{m_2\,{L_{cog2}}^2}{2} + \frac{m_3\,{L_{cog3}}^2}{2}$$

$$a_3 = m_3\,{L_2}^2 + m_2\,{L_{cog2}}^2 + m_3\,{L_{cog3}}^2$$

$$a_4 = L_{cog3}\,m_3$$

$$G_x = \begin{bmatrix} -s_1\ (L_1\,m_2 + L_1\,m_3 + L_{cog1}\,m_1 + L_{cog3}\,m_3\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2) \\ -c_1\ (L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23}) \\ -L_{cog3}\,m_3\,s_{23}\,c_1 \end{bmatrix}$$

$$G_y = \begin{bmatrix} c_1\ (L_1\,m_2 + L_1\,m_3 + L_{cog1}\,m_1 + L_{cog3}\,m_3\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2) \\ -s_1\ (L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23}) \\ -L_{cog3}\,m_3\,s_{23}\,s_1 \end{bmatrix}$$

$$G_z = \begin{bmatrix} 0 \\ L_{cog3}\,m_3\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2 \\ L_{cog3}\,m_3\,c_{23} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 \\ \frac{m_3\ \sin(2\,\theta_2)\,{L_2}^2}{2} + m_3\ \sin\left(2\,\theta_2 + \theta_3\right)\ L_2\,L_{cog3} + L_1\,m_3\,s_2\,L_2 + \frac{m_2\ \sin(2\,\theta_2)\,{L_{cog2}}^2}{2} + L_1\,m_2\,s_2\,L_{cog2} + \frac{m_3\ \sin(2\,\theta_2+2\,\theta_3)\,{L_{cog3}}^2}{2} + L_1\,m_3\,s_{23}\,L_{cog3} & 0 & -L_2\,L_{cog3}\,m_3\,s_3 \\ \frac{L_{cog3}\,m_3\,(L_{cog3}\ \sin(2\,\theta_2+2\,\theta_3)+2\,L_1\,s_{23}+L_2\,s_3+L_2\ \sin(2\,\theta_2+\theta_3))}{2} & L_2\,L_{cog3}\,m_3\,s_3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{{L_{cog3}}^2\,m_3\ \sin(\theta_2+2\,\theta_3)}{2} - \frac{{L_{cog3}}^2\,m_3\,s_2}{2} - \frac{{L_2}^2\,m_3\ \sin(2\,\theta_2)}{2} - \frac{{L_{cog2}}^2\,m_2\ \sin(2\,\theta_2)}{2} - \frac{{L_{cog3}}^2\,m_3\ \sin(2\,\theta_2+2\,\theta_3)}{2} - L_1\,L_{cog3}\,m_3\,s_{23} + \frac{L_2\,L_{cog3}\,m_3\,s_{23}}{2} - L_1\,L_2\,m_3\,s_2 - L_1\,L_{cog2}\,m_2\,s_2 + L_1\,L_{cog3}\,m_3\,s_3 - \frac{L_2\,L_{cog3}\,m_3\ \sin(\theta_2-\theta_3)}{2} - L_2\,L_{cog3}\,m_3\ \sin\left(2\,\theta_2 + \theta_3\right) & -2\,L_{cog3}\,m_3\,c_3\,s_2\ (L_1 + L_{cog3}\,c_{23} + \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$t_1 = \frac{{L_{cog3}}^2\,m_3\ \sin\left(\theta_2 + 2\,\theta_3\right)}{2} - \frac{{L_{cog3}}^2\,m_3\,s_2}{2} - \frac{{L_2}^2\,m_3\ \sin\left(2\,\theta_2\right)}{2} - \frac{{L_{cog2}}^2\,m_2\ \sin\left(2\,\theta_2\right)}{2} - \frac{{L_{cog3}}^2\,m_3\ \sin\left(2\,\theta_2 + 2\,\theta_3\right)}{2} - L_1\,L_{cog3}\,m_3\,s_{23} + \frac{L_2\,L_{cog3}\,m_3\,s_{23}}{2} - L_1\,L_2\,m_3\,s_2 - L_1\,L_{cog2}\,m_2\,s_2 + L_1\,L_{cog3}\,m_3\,s_3 - \frac{L_2\,L_{cog3}\,m_3\ \sin\left(\theta_2 - \theta_3\right)}{2} - L_2\,L_{cog3}\,m_3\ \sin\left(2\,\theta_2 + \theta_3\right)$$

$$t_2 = L_1\,L_{cog3}\,m_3\,s_3 - L_1\,L_2\,m_3\,s_2 - L_1\,L_{cog2}\,m_2\,s_2 - L_1\,L_{cog3}\,m_3\,s_{23} - L_2\,L_{cog3}\,m_3\ \sin\left(2\,\theta_2 + \theta_3\right)$$

$$t_3 = \frac{{L_{cog3}}^2\,m_3\ \sin\left(\theta_2 + 2\,\theta_3\right)}{2} - \frac{{L_{cog3}}^2\,m_3\,s_2}{2} - \frac{{L_2}^2\,m_3\ \sin\left(2\,\theta_2\right)}{2} - \frac{{L_{cog2}}^2\,m_2\ \sin\left(2\,\theta_2\right)}{2} - \frac{{L_{cog3}}^2\,m_3\ \sin\left(2\,\theta_2 + 2\,\theta_3\right)}{2}$$

$$B1 = \begin{bmatrix} \frac{{L_{cog3}}^2\,m_3\ \sin(\theta_2+2\,\theta_3)}{2} - \frac{{L_{cog3}}^2\,m_3\,s_2}{2} - \frac{{L_2}^2\,m_3\ \sin(2\,\theta_2)}{2} - \frac{{L_{cog2}}^2\,m_2\ \sin(2\,\theta_2)}{2} - \frac{{L_{cog3}}^2\,m_3\ \sin(2\,\theta_2+2\,\theta_3)}{2} - L_1\,L_{cog3}\,m_3\,s_{23} + \frac{L_2\,L_{cog3}\,m_3\,s_{23}}{2} - L_1\,L_2\,m_3\,s_2 - L_1\,L_{cog2}\,m_2\,s_2 + L_1\,L_{cog3}\,m_3\,s_3 - \frac{L_2\,L_{cog3}\,m_3\ \sin(\theta_2-\theta_3)}{2} - L_2\,L_{cog3}\,m_3\ \sin\left(2\,\theta_2 + \theta_3\right) \\ 0 \\ 0 \end{bmatrix}$$

$$B2 = \begin{bmatrix} -2\,L_{cog3}\,m_3\,c_3\,s_2\ (L_1 + L_{cog3}\,c_{23} + L_2\,c_2) \\ 0 \\ 0 \end{bmatrix}$$

$$B3 = \begin{bmatrix} 0 \\ -2\,L_2\,L_{cog3}\,m_3\,s_3 \\ 0 \end{bmatrix}$$

1.3 3 link perpz parallel

See https://github.com/LinkageDynamics/open-chain/threelink dneperzpar3.tex for text expressions
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mfilename=mar21perzpar3, outfilename=dneperzpar3.org, datetime=21-Mar-2025 16:03:37
         cog: 0
     inertia: 0
    numlinks: 3
     gravity: 'all'
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$$M = \begin{bmatrix} a_1 + a_2 - L_2\,a_4\ \cos\left(2\,\theta_2 + \theta_3\right) - \frac{L_{cog3}\,a_4\ \cos(2\,\theta_2+2\,\theta_3)}{2} - \frac{{L_2}^2\,m_3\ \cos(2\,\theta_2)}{2} - \frac{{L_{cog2}}^2\,m_2\ \cos(2\,\theta_2)}{2} + L_2\,a_4\,c_3 & L_1\ (a_4\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2) & L_1\,a_4\,c_{23} \\ L_1\ (a_4\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2) & a_3 + 2\,L_2\,a_4\,c_3 & a_4\ (L_{cog3} + L_2\,c_3) \\ L_1\,a_4\,c_{23} & a_4\ (L_{cog3} + L_2\,c_3) & L_{cog3}\,a_4 \end{bmatrix}$$

where

$$a_1 = {L_1}^2\,m_2 + {L_1}^2\,m_3 + \frac{{L_2}^2\,m_3}{2}$$

$$a_2 = m_1\,{L_{cog1}}^2 + \frac{m_2\,{L_{cog2}}^2}{2} + \frac{m_3\,{L_{cog3}}^2}{2}$$

$$a_3 = m_3\,{L_2}^2 + m_2\,{L_{cog2}}^2 + m_3\,{L_{cog3}}^2$$

$$a_4 = L_{cog3}\,m_3$$

$$G_x = \begin{bmatrix} -L_1\,m_2\,s_1 - L_1\,m_3\,s_1 - L_{cog1}\,m_1\,s_1 - L_2\,m_3\,c_1\,s_2 - L_{cog2}\,m_2\,c_1\,s_2 - L_{cog3}\,m_3\,c_1\,c_2\,s_3 - L_{cog3}\,m_3\,c_1\,c_3\,s_2 \\ -s_1\,\left(L_{cog3}\,m_3\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2\right) \\ -L_{cog3}\,m_3\,c_{23}\,s_1 \end{bmatrix}$$

$$G_y = \begin{bmatrix} L_1\,m_2\,c_1 + L_1\,m_3\,c_1 + L_{cog1}\,m_1\,c_1 - L_2\,m_3\,s_1\,s_2 - L_{cog2}\,m_2\,s_1\,s_2 - L_{cog3}\,m_3\,c_2\,s_1\,s_3 - L_{cog3}\,m_3\,c_3\,s_1\,s_2 \\ c_1\,\left(L_{cog3}\,m_3\,c_{23} + L_2\,m_3\,c_2 + L_{cog2}\,m_2\,c_2\right) \\ L_{cog3}\,m_3\,c_{23}\,c_1 \end{bmatrix}$$

$$G_z = \begin{bmatrix} 0 \\ L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23} \\ L_{cog3}\,m_3\,s_{23} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -L_1\,\left(L_2\,m_3\,s_2 + L_{cog2}\,m_2\,s_2 + L_{cog3}\,m_3\,s_{23}\right) & -L_1\,L_{cog3}\,m_3\,s_{23} \\ -\frac{m_3\,\sin(2\,\theta_2)\,{L_2}^2}{2} - m_3\,\sin\left(2\,\theta_2 + \theta_3\right)\,L_2\,L_{cog3} - \frac{m_2\,\sin(2\,\theta_2)\,{L_{cog2}}^2}{2} - \frac{m_3\,\sin(2\,\theta_2+2\,\theta_3)\,{L_{cog3}}^2}{2} & 0 & -L_2\,L_{cog3}\,m_3\,s_3 \\ -\frac{L_{cog3}\,m_3\,\left(L_{cog3}\,\sin(2\,\theta_2+2\,\theta_3) - L_2\,s_3 + L_2\,\sin(2\,\theta_2+\theta_3)\right)}{2} & L_2\,L_{cog3}\,m_3\,s_3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{{L_2}^2\,m_3\,\sin(2\,\theta_2)}{2} - {L_{cog2}}^2\,m_2\,s_2 - \frac{{L_{cog3}}^2\,m_3\,s_2}{2} - \frac{{L_{cog3}}^2\,m_3\,\sin(\theta_2+2\,\theta_3)}{2} - {L_2}^2\,m_3\,s_2 + \frac{{L_{cog2}}^2\,m_2\,\sin(2\,\theta_2)}{2} + \frac{{L_{cog3}}^2\,m_3\,\sin(2\,\theta_2+2\,\theta_3)}{2} - \frac{3\,L_2\,L_{cog3}\,m_3\,s_{23}}{2} - \frac{L_2\,L_{cog3}\,m_3\,\sin(\theta_2-\theta_3)}{2} + L_2\,L_{cog3}\,m_3\,\sin\left(2\,\theta_2 + \theta_3\right) & 2\,L_{cog3}\,m_3\,c_2\,c_3\,\left(L_{cog3}\,s_{23} + L_2\,s_2\right) & -2\,L_1\,L_{cog3}\,m_3\,s_{23} \\ 0 & 0 & -2\,L_2\,L_{cog3}\,m_3\,s_3 \\ 0 & 0 & 0 \end{bmatrix}$$

2 Branched chains (trees)

2.1 3 link planar branched

See https://github.com/LinkageDynamics/open-chain/threelink dnetypeA3.tex for text expressions

$$M = \begin{bmatrix} a_1 + 2\,L_1\,L_{cog2}\,m_2\,c_2 + 2\,L_1\,L_{cog3}\,m_3\,c_3 & L_{cog2}\,m_2\,\left(L_{cog2} + L_1\,c_2\right) & L_{cog3}\,m_3\,\left(L_{cog3} + L_1\,c_3\right) \\ L_{cog2}\,m_2\,\left(L_{cog2} + L_1\,c_2\right) & {L_{cog2}}^2\,m_2 & 0 \\ L_{cog3}\,m_3\,\left(L_{cog3} + L_1\,c_3\right) & 0 & {L_{cog3}}^2\,m_3 \end{bmatrix}$$

where

$$a1 = {L_1}^2\,m_1 + {L_1}^2\,m_2 + {L_1}^2\,m_3 + {L_{cog2}}^2\,m_2 + {L_{cog3}}^2\,m_3$$

$$G_y = \begin{bmatrix} L_{cog2}\,m_2\,c_{12} + L_{cog3}\,m_3\,c_{13} + L_1\,m_1\,c_1 + L_1\,m_2\,c_1 + L_1\,m_3\,c_1 \\ L_{cog2}\,m_2\,c_{12} \\ L_{cog3}\,m_3\,c_{13} \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & -L_1\,L_{cog2}\,m_2\,s_2 & -L_1\,L_{cog3}\,m_3\,s_3 \\ L_1\,L_{cog2}\,m_2\,s_2 & 0 & 0 \\ L_1\,L_{cog3}\,m_3\,s_3 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -2\,L_1\,L_{cog2}\,m_2\,s_2 & -2\,L_1\,L_{cog3}\,m_3\,s_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$