

Lecture 4

Frequency Domain Representation of Discrete Time Signal

- The frequency domain representation of discrete time sequence is the **discrete-time Fourier transform (DTFT)**. DTFT is a frequency analysis tool for **aperiodic** discrete-time signals

- This transform maps a time-domain sequence into a **continuous** function of the frequency variable ω .


Review of Continuous-Time Fourier Transform (CTFT)

- CTFT: for continuous ~~aperiodic~~ signal

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt.$$

- Inverse CTFT:

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega.$$

$$x_a(t) \xrightarrow{\text{CTFT}} X_a(j\Omega)$$

Fourier Spectrum
Or Spectrum

$$x_a(t) \xrightarrow{\text{CTFT}} X_a(j\Omega)$$

Definition of Discrete-Time Fourier Transform (DTFT) for discrete ~~aperiodic~~ signal

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \text{DTFT}$$

where, ω is a ~~continuous~~ variable in the range $-\infty < \omega < \infty$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \quad \text{Inverse DTFT}$$

Why one is sum and the other integral?

$x[n]$ is discrete.

$X(e^{j\omega})$ is continuous

Fourier Series (for continuous ~~periodic~~ signal):

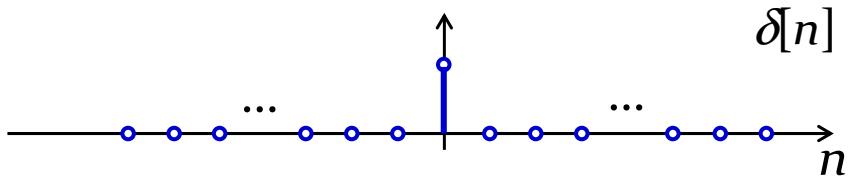
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\frac{2\pi}{T})t}$$

Fourier Series Coefficients:

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_T x(t) e^{-jk(\frac{2\pi}{T})t} dt$$

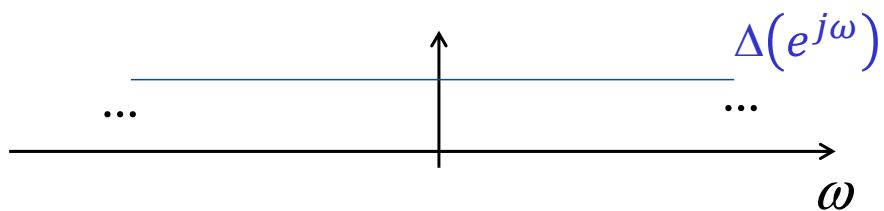
$$x(t) \xleftrightarrow{FS} a_k.$$

Example 1



DTFT:

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n} = 1$$



Example 2

- Causal sequence $x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$

Its DTFT is given by

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1 - \alpha e^{-j\omega}}$$

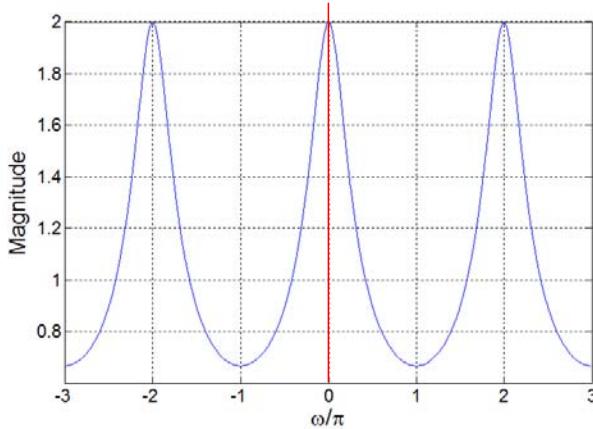
as $|\alpha e^{-j\omega}| = |\alpha| < 1$

$$\text{Recall: } 1 + q + q^2 + \cdots + q^{\infty} = \frac{1}{1 - q} \quad \text{for } |q| < 1$$

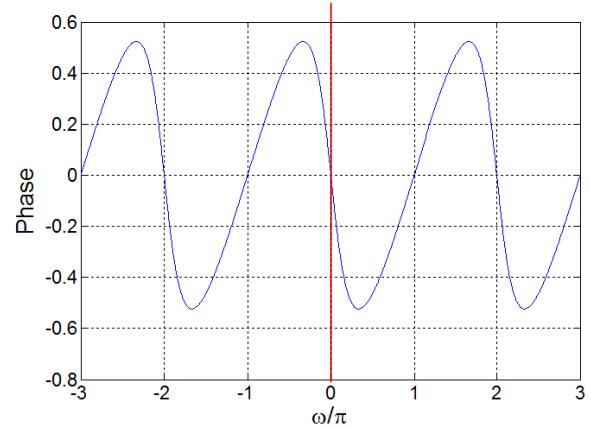
Example 2 (Cont'd)

- The magnitude and phase of DTFT of

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$



$$|X(e^{j\omega})| = |X(e^{-j\omega})|$$



$$\theta(\omega) = -\theta(-\omega)$$

DTFT

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω , and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$$

- $X(e^{j\omega})$ can alternatively be expressed as

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega)} \text{ 相位方程}$$

where, $\theta(\omega) = \arg\{X(e^{j\omega})\}$

- $|X(e^{j\omega})|$ is called the magnitude function
- $\theta(\omega)$ is called the phase function
- In applications where DTFT is called Fourier spectrum, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called magnitude and phase spectra

Symmetry of DTFT

For a **real sequence $x[n]$** , $|X(e^{j\omega})|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω , whereas $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω .

- Proof:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ X(e^{-j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} \\ &= \left\{ \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right\}^*, \text{ for real } x[n] \\ &= \underline{X^*(e^{j\omega})} \end{aligned}$$

- Therefore, $|X(e^{j\omega})| = |X(e^{-j\omega})|$ and $\theta(\omega) = -\theta(-\omega)$

Periodicity of DTFT

- $X(e^{j\omega}) = X(e^{j(\omega+2k\pi)})$, i.e. $|X(e^{j\omega})|e^{j[\theta(\omega)+2k\pi]} = |X(e^{j\omega})|e^{j\theta(\omega)}$ for any integer k .
- Proof:

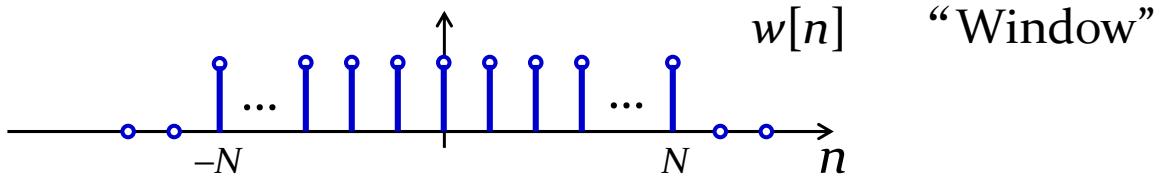
$$\begin{aligned} X(e^{j(\omega+2k\pi)}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= X(e^{j\omega}) \end{aligned}$$

- In other words, the phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT.
- Unless otherwise stated, we assume that the phase function $\theta(\omega)$ is restricted to the range of

$$-\pi < \theta(\omega) < \pi$$

called the **principle value**.

Example 3



DTFT:

$$\begin{aligned}
 W(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} w[n]e^{-j\omega k} = \sum_{k=-N}^{N} e^{-j\omega k} \\
 &= e^{-j\omega N}(1 + e^{j\omega} + e^{j2\omega} + \dots + e^{j2N\omega})
 \end{aligned}$$

Recall: $1 + q + q^2 + \dots + q^M = \frac{1 - q^{M+1}}{1 - q}$

$q = e^{j\omega}$

 $M = 2N$

Example 3 Cont.

DTFT: $W(e^{j\omega}) = e^{-j\omega N}(1 + e^{j\omega} + e^{j2\omega} + \dots + e^{j2N\omega})$

$$= e^{-j\omega N} \frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

★ $= \frac{e^{-j\omega N} - e^{j\omega N} e^{j\omega}}{1 - e^{j\omega}}$

$\times \frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}$

$$= \frac{e^{-j\omega(N+\frac{1}{2})} - e^{j\omega(N+\frac{1}{2})}}{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}}$$

$$= \frac{\sin\left(\omega\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

← → Periodic Sinc

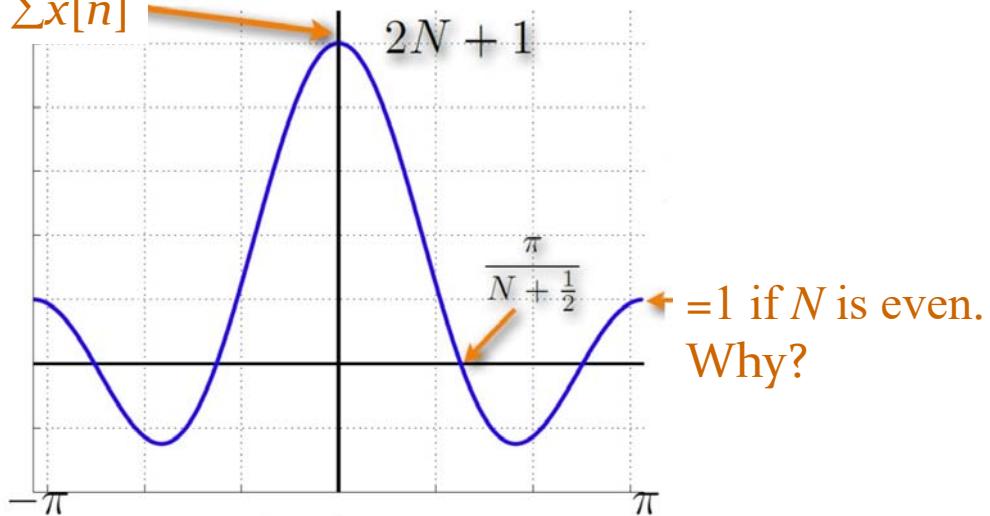
remember

$N \uparrow$, 时域宽, 频域窄

Example 3 Cont.

$$W(e^{j\omega}) = \frac{\sin\left(\omega\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)} \rightarrow 2N+1 \text{ as } \omega \rightarrow 0$$

Also = $\sum x[n]$



Example of Inverse DTFT

- Find the inverse DTFT of

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

A: 当\omega=0时需要单独计算

$$h_{LP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

当\omega=0时
该步不成立

$$\ominus \frac{1}{2\pi} \left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn} \right) = \frac{\sin \omega_c n}{\pi n}, \text{ for } n \neq 0$$

$$h_{LP}[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{LP}(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

Properties of the DTFT

- **Linearity:**

Let $g[n] \leftrightarrow G(e^{j\omega})$ and $h[n] \leftrightarrow H(e^{j\omega})$
Then $\alpha g[n] + \beta h[n] \leftrightarrow \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$

- **Periodicity:** $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Properties of the DTFT Cont.

- **Time Reversal:**

Let $x[n] \leftrightarrow X(e^{j\omega})$
Then $x[-n] \leftrightarrow X(e^{-j\omega})$

$\underbrace{x[-n]}_{\cong x[n]e^{-j\omega n}} \underbrace{\leftrightarrow X(e^{-j\omega})}_{= X^*(e^{j\omega})} \text{ if } x[n] \text{ is real}$

proof?

{ If $x[n] = x[-n]$ and $x[n]$ is real, then

~~•~~ $X(e^{j\omega}) = X^*(e^{j\omega}) \rightarrow X(e^{j\omega})$ is real
 $\cong X(e^{-j\omega}) \checkmark$

Q: Suppose $x[n] \leftrightarrow X(e^{j\omega})$, $x[n] \in \mathcal{R}eal$

$$? \leftrightarrow \mathcal{R}e\{X(e^{j\omega})\}$$

- A: Decompose $x[n]$ to even and odd functions

$$x[n] = x_e[n] + x_o[n],$$

where $x_e[n] = \frac{1}{2}(x[n] + x[-n])$

$$x_o[n] = \frac{1}{2}(x[n] - x[-n])$$

$$\begin{aligned} x_e[n] \leftrightarrow X_e(e^{j\omega}) &= \frac{1}{2}(X(e^{j\omega}) + X(e^{-j\omega})) \\ &= \frac{1}{2}(X(e^{j\omega}) + X^*(e^{j\omega})) = \mathcal{R}e\{X(e^{j\omega})\} \end{aligned}$$

$$x_o[n] \leftrightarrow X_o(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) - X(e^{-j\omega})) = j\mathcal{I}m\{X(e^{j\omega})\}$$

Properties of the DTFT Cont.

- Time and Frequency Shifting

Let $x[n] \leftrightarrow X(e^{j\omega})$

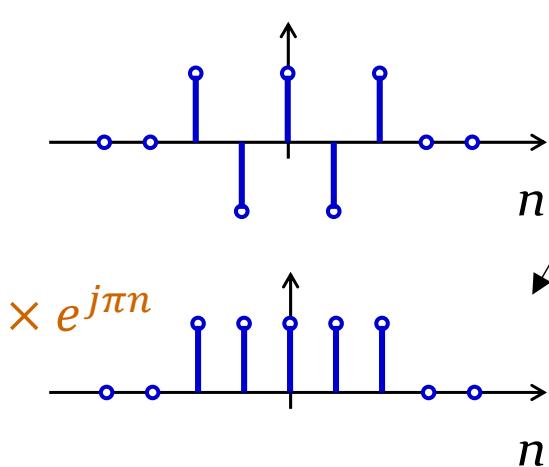
Then $x[n - n_d] \leftrightarrow e^{-j\omega n_d} X(e^{j\omega})$

$$e^{j\omega_0 n} x[n] \leftrightarrow X(e^{j(\omega - \omega_0)})$$

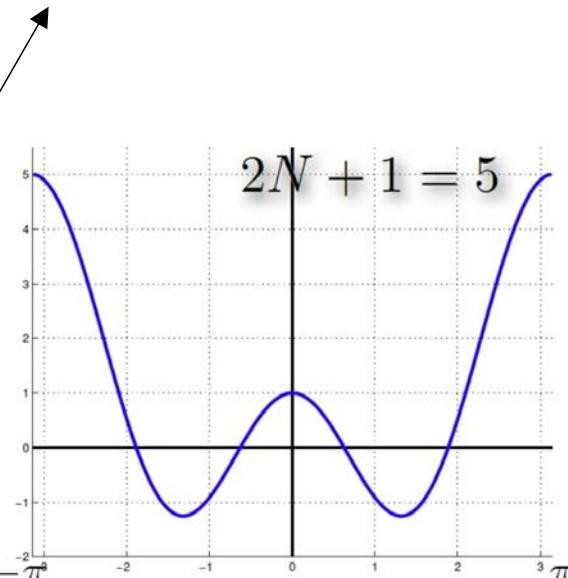
Example 4

?

What is the DTFT of:



$$W(e^{j\omega}) = \frac{\sin\left(\omega\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$



$$W(e^{j\omega}) = \frac{\sin\left((\omega - \pi)\left(N + \frac{1}{2}\right)\right)}{\sin\left(\frac{\omega - \pi}{2}\right)}$$

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Properties of the DTFT Cont.

- **Differentiation in frequency**

Let $x[n] \leftrightarrow X(e^{j\omega})$

Then $nx[n] \leftrightarrow j \frac{dX(e^{j\omega})}{d\omega}$

Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Differentiate both side to get $\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$

Multiply both side by j , we get $j \frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$

Example 5

- Determine DTFT $Y(e^{j\omega})$ of
 $y[n] = (n + 1)\alpha^n \mu[n], \quad |\alpha| < 1$
- Let $x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$
- We can therefore write
$$y[n] = nx[n] + x[n]$$
- From example 2, we have known that the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

- Using the differentiation in frequency, we observe that DTFT of $nx[n]$ is given by,

$$j \frac{dX(e^{j\omega})}{d\omega} = j \frac{d}{d\omega} \left(\frac{1}{1 - \alpha e^{-j\omega}} \right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$$

- Next, using linear property of DTFT, we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

Properties of the DTFT Cont.

- **Convolution**

Let $x[n] \leftrightarrow X(e^{j\omega})$ and $h[n] \leftrightarrow H(e^{j\omega})$

- DTFT convolution theorem

If $y[n] = x[n] \otimes h[n]$

Then $y[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

- **DTFT convolution modulation**

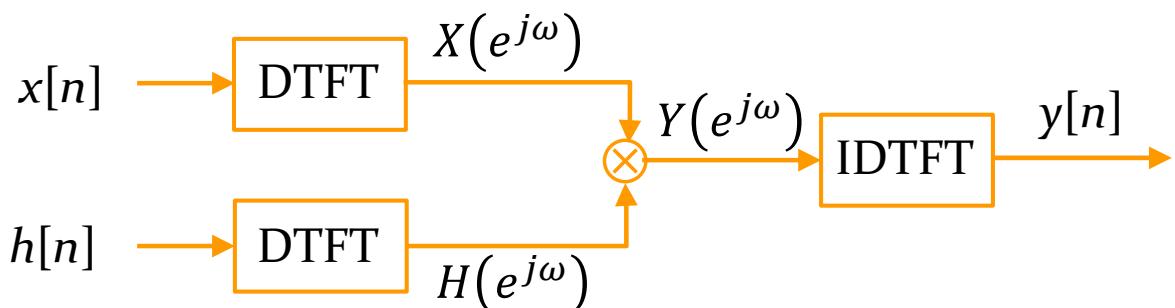
If $y[n] = x[n]h[n]$

Then $y[n] \leftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$

Linear Convolution Using DTFT

- Linear convolution $y[n]$ of the sequence $x[n]$ and $h[n]$ can be performed as follows:

- Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively.
- Form DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$



Properties of the DTFT Cont.

- Parseval's theorem

Let $x[n] \leftrightarrow X(e^{j\omega})$ $h[n] \leftrightarrow H(e^{j\omega})$

Then $\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})H^*(e^{j\omega})d\omega$

Proof:
$$\begin{aligned} \sum_{n=-\infty}^{\infty} x[n]h^*[n] &= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} H^*(e^{j\omega})e^{-j\omega n} d\omega \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H^*(e^{j\omega}) \left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right) d\omega \\ &\star = \frac{1}{2\pi} \int_{-\pi}^{\pi} H^*(e^{j\omega}) X(e^{j\omega}) d\omega \end{aligned}$$

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能量在時域和頻域上保持不變。

Energy & Energy Density Spectrum

- Energy: $E_g = \sum_{n=-\infty}^{\infty} |x[n]|^2$
- According to Parseval's theorem, when $h[n] = x[n]$,

$$E_g = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

- Energy spectral density:

$$\star S_{xx}(\omega) = |X(e^{j\omega})|^2$$

虽然序列不收敛，但能量可能有限。

Energy Spectral Density

- Example – Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \quad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{LP}(e^{j\omega t})|^2 d\omega$$

where

$$H_{LP}(e^{j\omega t}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

- Therefore:

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

Symmetry Relations

If $x[n]$ is real

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Conjugate Symmetric	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$ $X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$ $ X(e^{j\omega}) = X(e^{-j\omega}) $ $\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Symmetry Relations

If $x[n]$ is complex

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x_{\text{re}}[n]$	$X_{\text{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$jx_{\text{im}}[n]$	$X_{\text{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\text{cs}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{ca}}[n]$	$jX_{\text{im}}(e^{j\omega})$

Band-Limited DT Signals

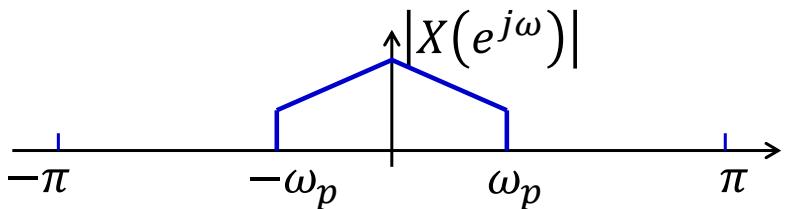
- The spectrum of a DT signal is a  periodic function of ω with a period of 2π . Thus a **full-band** signal has a spectrum occupying the frequency range $-\pi < \omega \leq \pi$.
- A **band-limited** DT signal has a spectrum that is limited to a portion of the frequency range $-\pi < \omega \leq \pi$.
- An ideal  band-limited signal:

$$|X(e^{j\omega})| = \begin{cases} 0, & 0 \leq |\omega| < \omega_a \\ \text{non zero}, & \omega_a \leq |\omega| \leq \omega_b \\ 0, & \omega_b \leq |\omega| \leq \pi \end{cases}$$

Classification of Band-Limited Signal

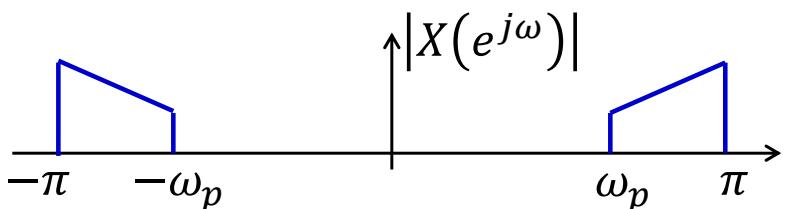
- Lowpass real signal

Bandwidth: ω_p



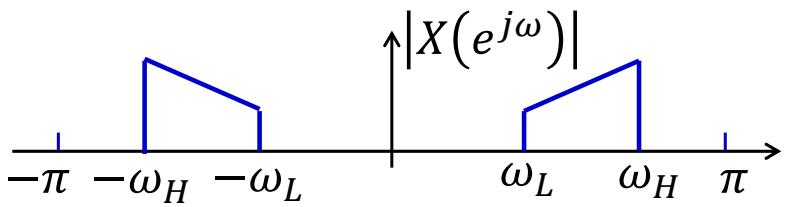
- Highpass real signal

Bandwidth: $\pi - \omega_p$



- Bandpass real signal

Bandwidth: $\omega_H - \omega_L$



DTFT Convergence Condition

- The infinite series

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

⊗ may or may not converge.

- If it converges for all value of ω , then the DTFT $X(e^{j\omega})$ exists.
- Consider the finite sum:

$$X_K(e^{j\omega}) = \sum_{n=-K}^K x[n]e^{-j\omega n}$$

- Strong convergence: $X(e^{j\omega})$ converge uniformly, i.e.,

$$\lim_{K \rightarrow \infty} X_K(e^{j\omega}) = X(e^{j\omega})$$

- $x[n]$ absolutely summable $\Rightarrow X(e^{j\omega})$ exist and converge uniformly

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |x[n]| < \infty \Rightarrow |X(e^{j\omega})| &= \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \end{aligned}$$

- This is a **sufficient condition**, not necessary.

Example

$x[n] = \alpha^n \mu[n]$, $|\alpha| < 1$ is absolutely summable, as

$$\sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1 - |\alpha|} < \infty$$

and therefore is DTFT $X(e^{j\omega})$ converge to $\frac{1}{1 - \alpha e^{-j\omega}}$ uniformly.

Since $\sum_{n=-\infty}^{\infty} |x[n]|^2 \leq \left(\sum_{-\infty}^{\infty} |x[n]| \right)^2$

an absolutely summable sequence has always a finite energy,

However, a finite energy sequence is not necessarily absolutely summable.

Example: a sequence $x[n] = \frac{1}{n} \mu[n - 1]$ has a finite energy, as $\sum_{n=1}^{\infty} \left(\frac{1}{n} \right)^2 = \frac{\pi^2}{6}$, however, it is not absolutely summable as $\sum_{n=1}^{\infty} \left| \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ does not converge.

Weak convergence: $X(e^{j\omega})$ converge mean square

$$\lim_{K \rightarrow \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_K(e^{j\omega})|^2 d\omega = 0$$

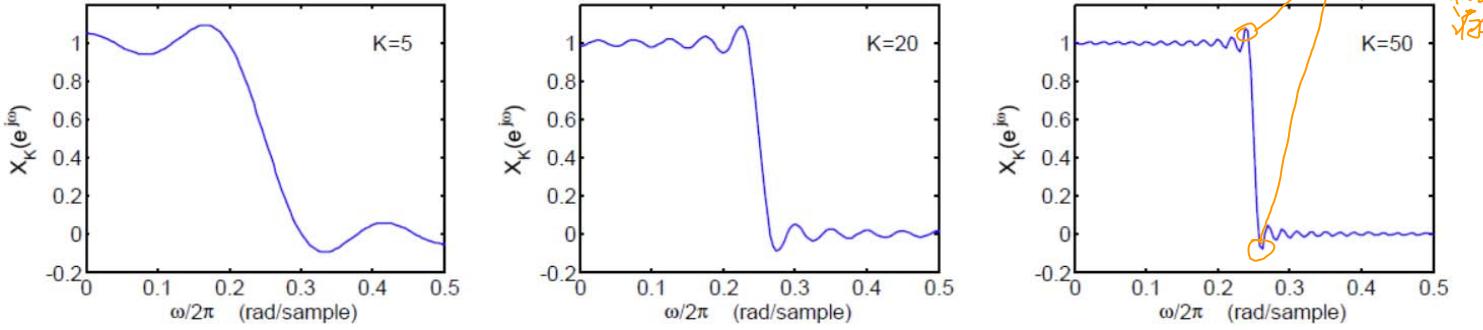
• $x[n]$ finite energy $\Rightarrow X(e^{j\omega})$ converge mean square.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega < \infty$$

• The absolute value of the error $|X(e^{j\omega}) - X_K(e^{j\omega})|$ may not go to zero when K goes to infinite.

Example

- $h_{LP}[n] = \frac{\sin 0.5\pi n}{\pi n} \leftrightarrow H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq |\omega| \leq 0.5\pi \\ 0, & 0.5\pi \leq |\omega| \leq \pi \end{cases}$ ②
分析
- $\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{\omega_c}{\pi} < \infty$, but not absolutely summable. ↑



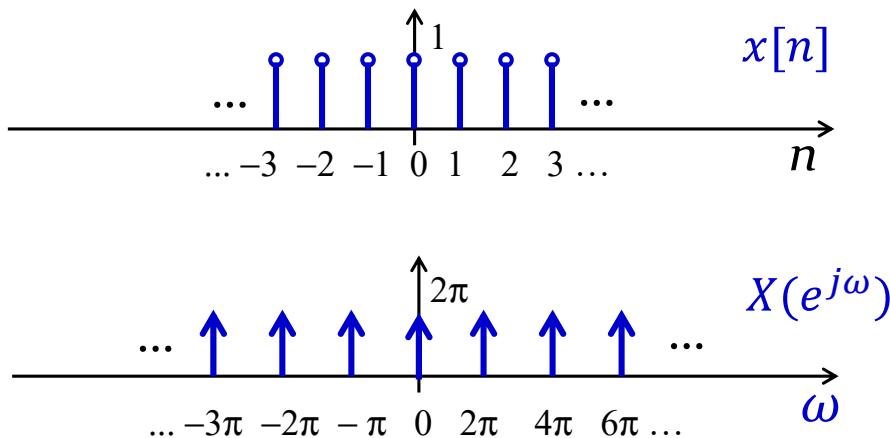
- **Gibbs phenomenon**: mean square converges ~~at each ω~~ as $K \rightarrow \infty$, but peak error does not get smaller.

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$ and the sinusoidal sequence $\cos(\omega_0 n + \varphi)$.
- For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$.

$$x[n] \longleftrightarrow X(e^{j\omega}) \quad \text{一一对应}$$

Example 5: DTFT of $x[n]=1$ for all n

- $x[n] = 1 = \sum_{k=-\infty}^{\infty} \delta[n - k]$
- It is more convenient to prove that the inverse DTFT of $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ is 1



- *Proof:* The inverse DTFT of $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ is evaluated as

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k) \right) e^{j\omega n} d\omega \\ &= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right) e^{j\omega n} d\omega \end{aligned}$$

- From the sifting property, we have

$$\begin{aligned} \left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right) e^{j\omega n} &= \left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right) e^{j2\pi kn} \\ &= \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \end{aligned}$$

We have used $e^{j2\pi kn} = 1$ for all n here

- When we integrate the sequence of impulse from $-\pi$ to π , we have only the impulse at $\omega = 0$.
- Hence

$$\begin{aligned}
 & \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right) e^{j\omega n} d\omega \\
 &= \int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \right) d\omega \\
 &= \int_{-\pi}^{\pi} \delta(\omega) d\omega = \int_{-\infty}^{-\pi} \delta(\omega) d\omega + \int_{-\pi}^{\pi} \delta(\omega) d\omega + \int_{\pi}^{\infty} \delta(\omega) d\omega \\
 &= \int_{-\infty}^{\infty} \delta(\omega) d\omega = 1 \text{ for all } n
 \end{aligned}$$

Example 6: DTFT of $\mu[n]$

- Let $\mu[n] = u_1[n] + u_2[n]$, where

$$u_1[n] = \frac{1}{2}, \quad \text{for } -\infty < n < \infty$$

and

$$u_2[n] = \begin{cases} \frac{1}{2} & \text{for } n \geq 0 \\ -\frac{1}{2} & \text{for } n < 0 \end{cases}$$

Therefore, we have

$$\delta[n] = u_2[n] - u_2[n - 1]$$

- Using $\delta[n] \leftrightarrow 1$ and $u_2[n] - u_2[n-1] \leftrightarrow U_2(e^{j\omega}) - e^{-j\omega}U_2(e^{j\omega}) = U_2(e^{j\omega})(1 - e^{-j\omega})$, we have
- $$1 = U_2(e^{j\omega})(1 - e^{-j\omega})$$

i.e.

$$U_2(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \text{ for } \omega \neq 0$$

Since

$$u_1[n] \leftrightarrow \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k) = U_1(e^{j\omega})$$

we have

 $\mu[n] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k) \text{ for } \omega \neq 0$

DTFT Convergence

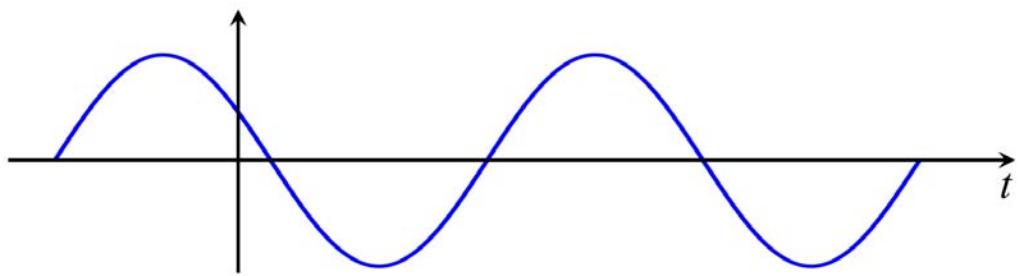
Sequence	DTFT
$\alpha^n \mu[n], (\alpha < 1)$ Absolutely Summable	 $\xrightarrow{\text{充分条件}} \frac{1}{1 - \alpha e^{-j\omega}}$ Exist for all ω
$\mu[n]$ Neither absolutely summable, nor finite energy	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$ Not exist for $\omega = 0$
1 (for all n) Neither absolutely summable, nor finite energy	 $\xrightarrow{\text{Not necessary}} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ Exist for all ω
$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n}$ Finite energy	 $\xrightarrow{\text{充分条件}} H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \leq \omega \leq \omega_c \pi \\ 0, & \omega_c \pi \leq \omega \leq \pi \end{cases}$ Exist for all ω

Commonly used DTFT pairs

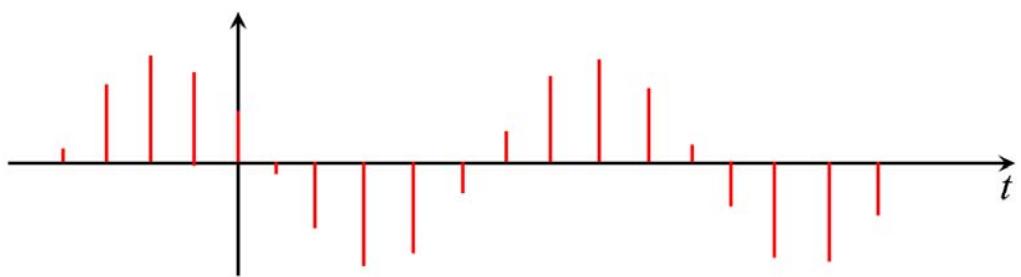
Sequence	DTFT
$\delta[n]$	1
$\mu[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
 1 (for all n)	 $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$
$\alpha^n \mu[n], (\alpha < 1)$	$\frac{1}{1 - \alpha e^{-j\omega}}$

Effect of Time-Domain Sampling in Frequency Domain

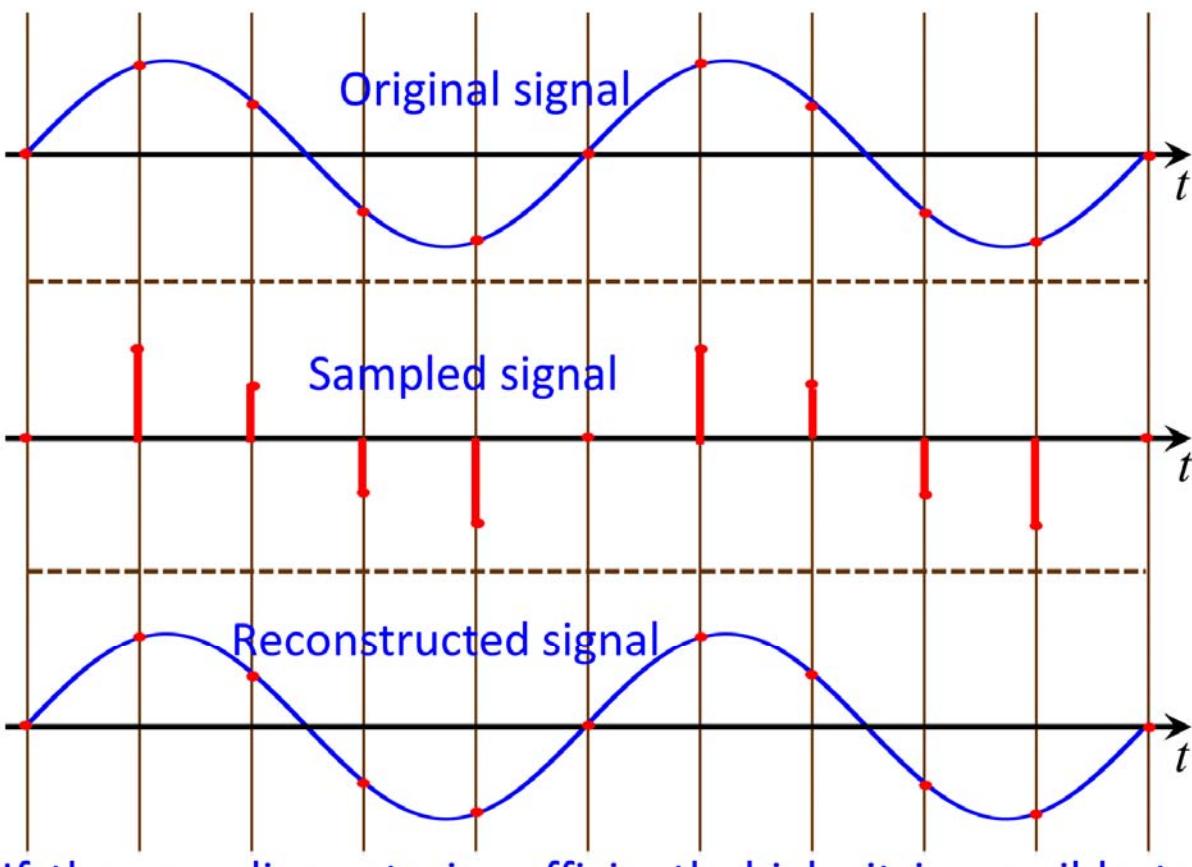
- Questions to be answered
 - When discrete signal is obtained by sampling, can we recover the original continuous signal from the discrete signal?
 - What is the condition that we can recover the continuous signal?
 - Relation Between Continuous and Discrete Signal Spectrum



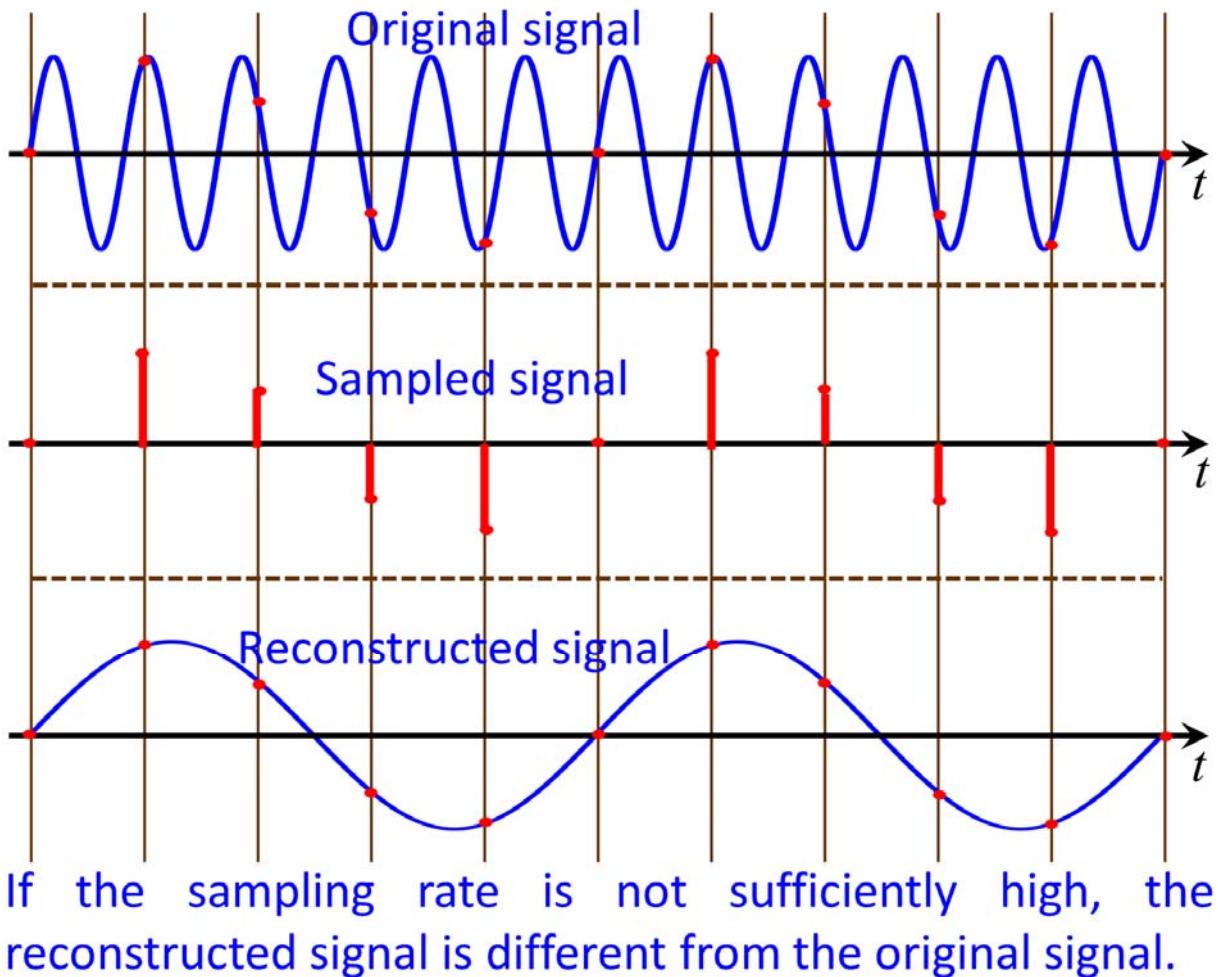
A continuous time signal may be sampled to produce a discrete time signal.



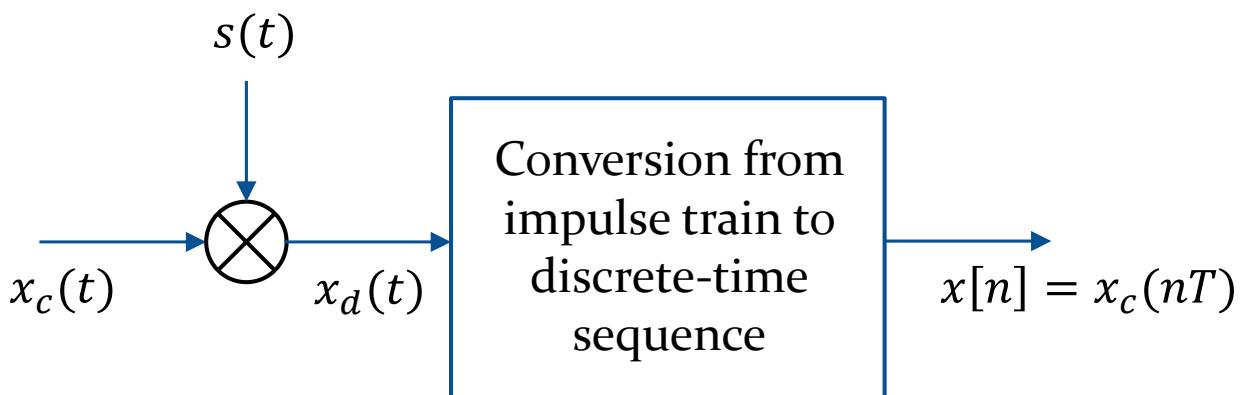
If the rate of sampling is very high, it is obvious by inspection that the sampled result will resemble that of the original continuous time signal.



If the sampling rate is sufficiently high, it is possible to reconstruct the original signal from the sampled signal.



Sampling with a periodic impulse train



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ where } \delta(t) = \infty \text{ for } t = 0 \\ = 0 \text{ for } t \neq 0$$

$$\text{and } \int_{-\infty}^{\infty} \delta(t) dt = 1$$

Frequency-Domain Representation of Sampling

Let $x_c(t)$ be continuous time signal

Let the sampled version of $x_c(t)$ be denoted by $x_d(t)$.

Let the sampling interval be T .

Let $\Omega_s = 2\pi/T$. \rightarrow 采样频率

Let $X_c(j\Omega)$ be the Fourier transform of $x_c(t)$.

Let $X_d(j\Omega)$ be the Fourier transform of $x_d(t)$. \rightarrow 采样信号

It can be shown that

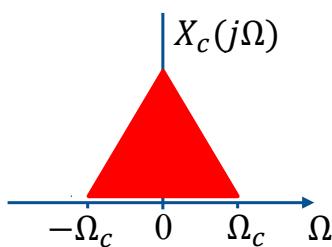
$$\text{Proof} \quad X_d(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

1/T indicates that the magnitude of $X_d(j\omega)$ increases with sampling density $1/T$

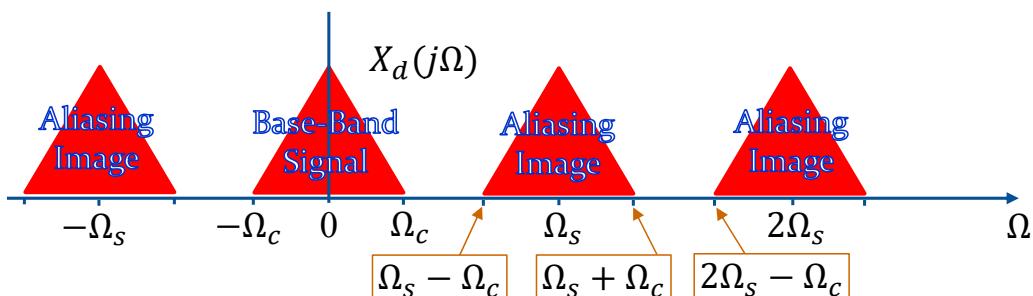
$X_c(j(\Omega - n\Omega_s))$ is $X_c(j\Omega)$ shifted along the Ω -axis by $n\Omega_s$

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$$X_d(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

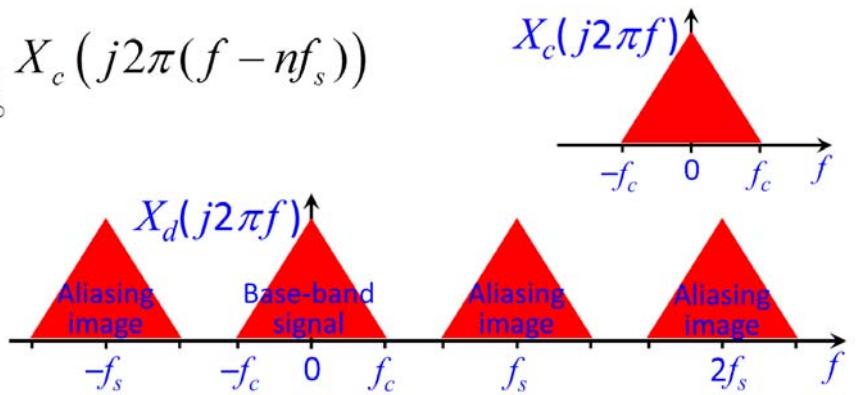


$X_c(j\Omega)$, frequency spectrum of the original continuous time signal, the base-band signal



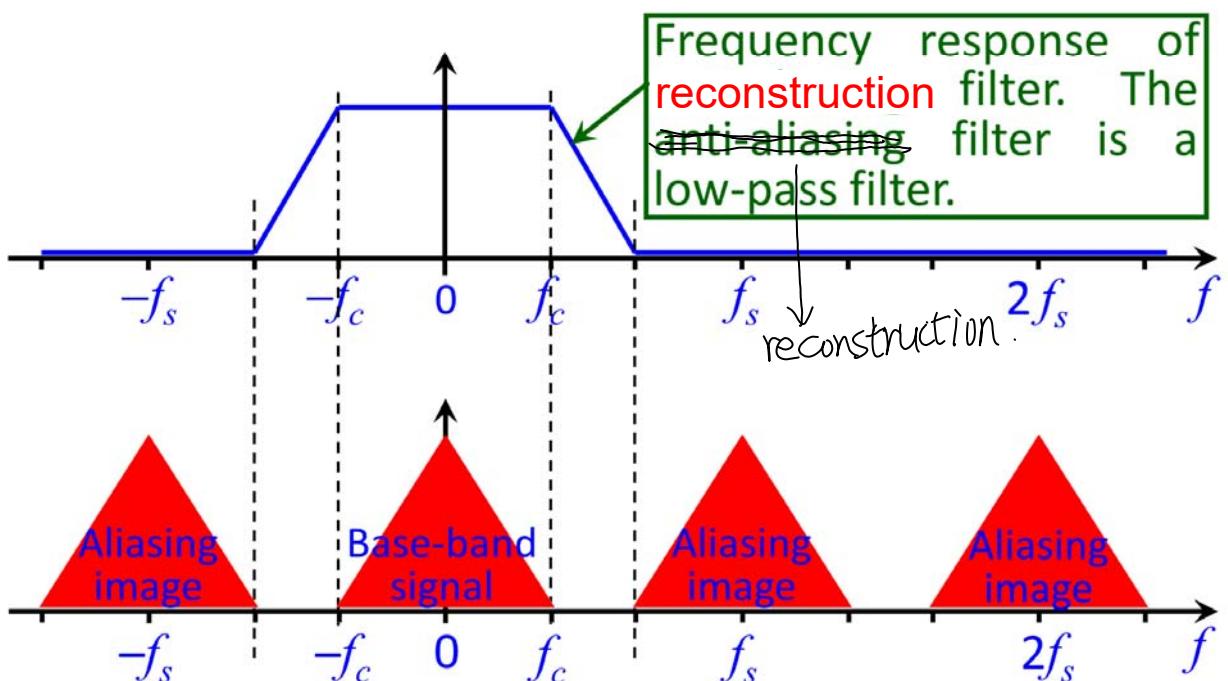
“Hz” is often used as frequency unit in communication systems. Hence, replacing ω by $2\pi f$ we have

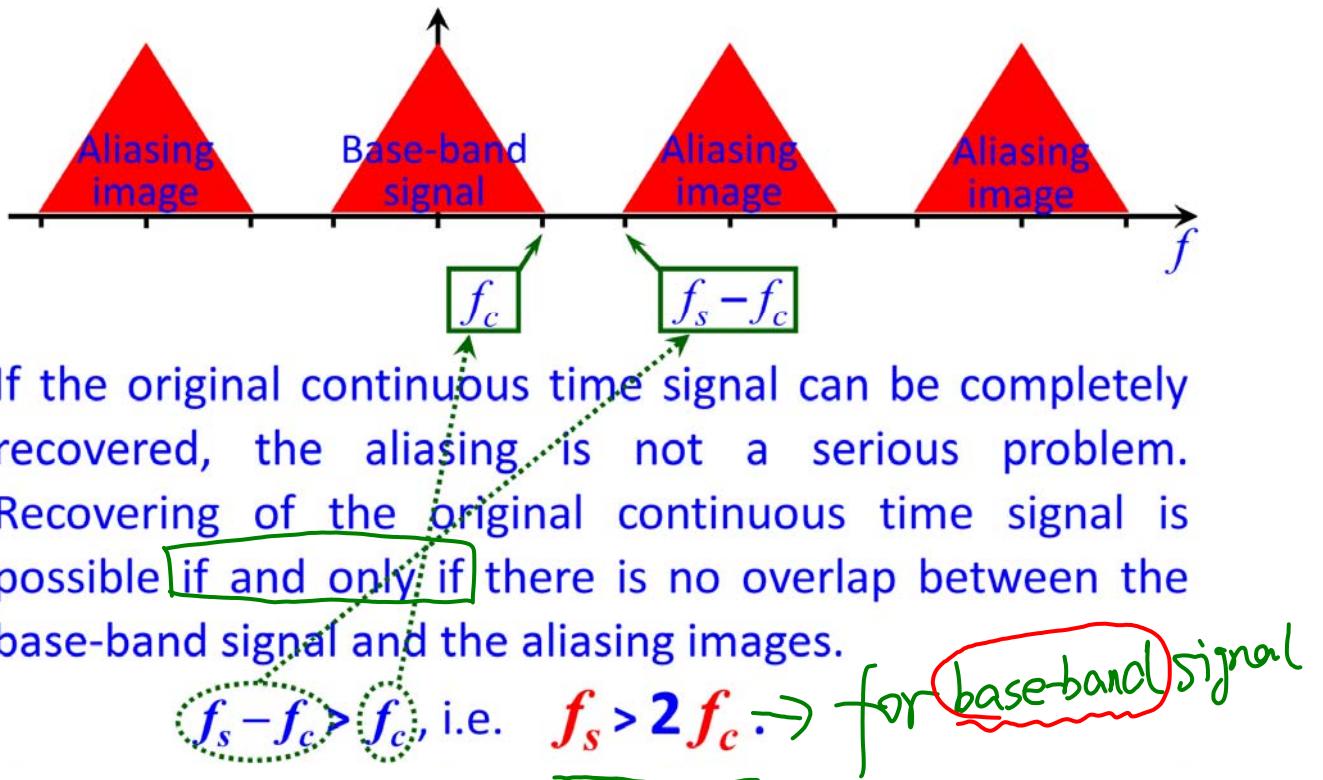
$$X_d(j2\pi f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j2\pi(f - nf_s))$$



When a continuous time signal (the base-band signal) is sampled at a rate of f_s samples per second, the frequency spectrum of the sampled signal is that of the base-band signal plus duplicates (aliasing) of that of the base-band signal centred at kf_s where $k = \dots, -1, 0, 1, 2, 3, \dots$

The original continuous time signal can be recovered by removing the aliasing images by low-pass filtering. The low-pass filter is called a **reconstruction filter**.

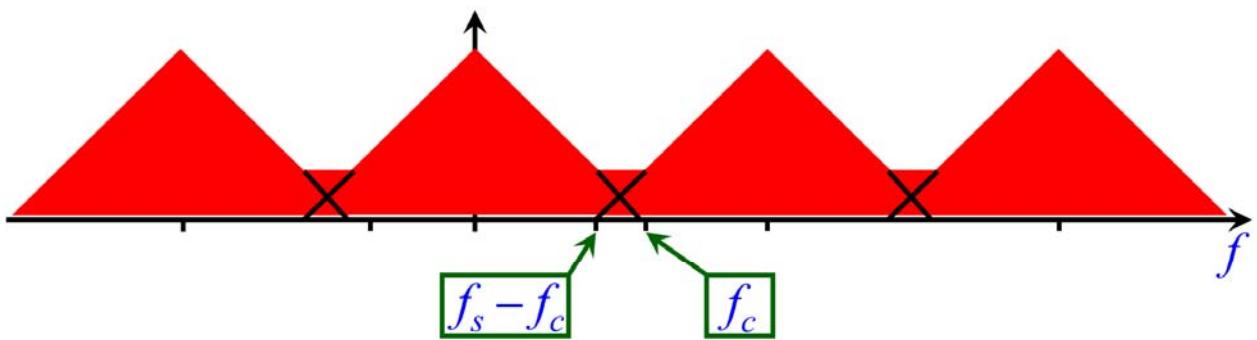




Nyquist Frequency & Nyquist Rate

- The highest frequency (f_c) contained in a continuous signal $x_c(t)$ is usually called **Nyquist frequency**, which determines the minimum sampling frequency ($f_s = 2f_c$) that must be used to fully recover $x_c(t)$ from its sampled version.
- The minimum sampling frequency (or sampling rate), $f_s = 2f_c$, required to avoid aliasing from irrecoverable problem is called **Nyquist rate**.

If $f_s - f_c < f_c$, i.e. $f_s < 2f_c$,

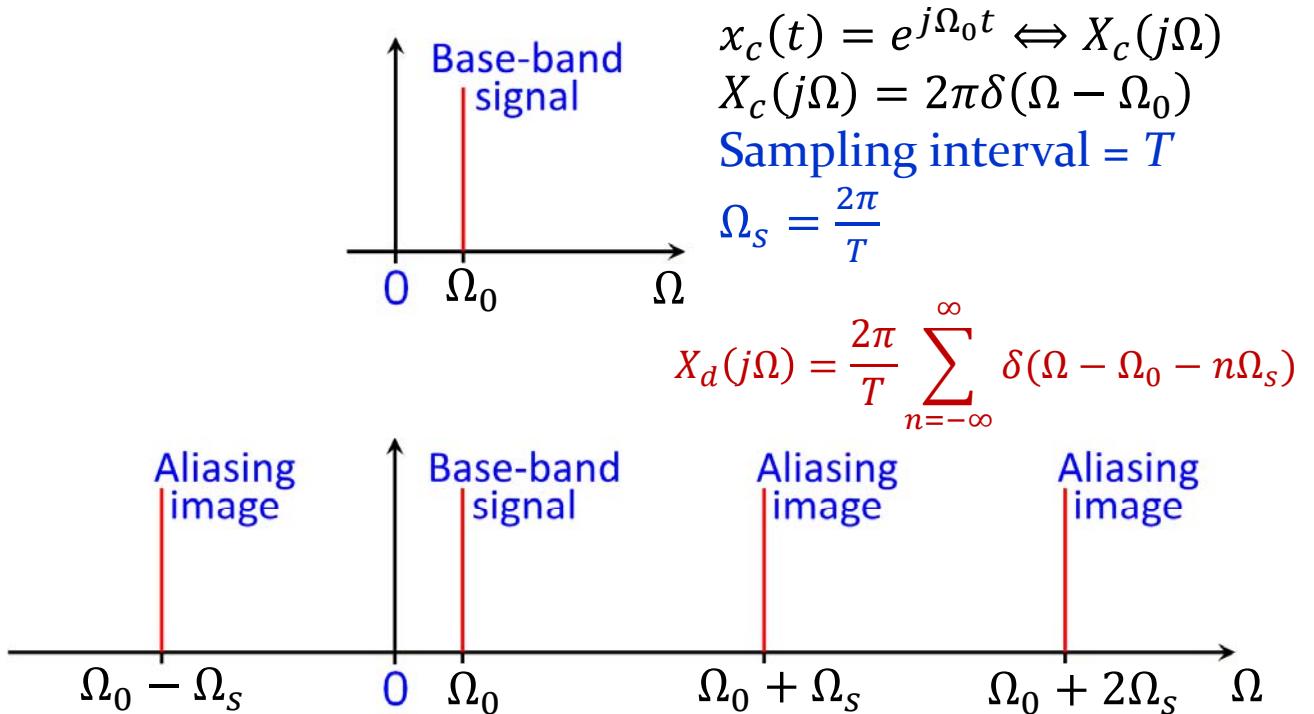


Example:

The output of an ideal low-pass filter with band-edges at $\pm\pi$ radians per second is sampled at a rate of f_s samples per second. What is the minimum f_s in order to avoid aliasing from causing irrecoverable problem?

Since the band-edges of the ideal low-pass filter is $\pm\pi$ radians per second, the highest frequency component of the low-pass filter output is π radians per second. We have $\Omega = 2\pi f$. Thus, the highest frequency component of the low-pass filter output is $\pi/(2\pi) = 1/2$ Hz. The sampling rate must be greater than twice the maximum frequency. Hence, minimum sampling rate is 1 sample per second.

Spectral lines of a sampled complex sinusoid.



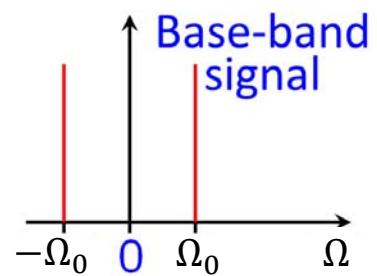
Spectral lines of a sampled sinusoid.

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

Sampling interval = T

$$\Omega_s = \frac{2\pi}{T}$$

$$X_d(j\Omega) = \frac{\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega \pm \Omega_0 - n\Omega_s)$$

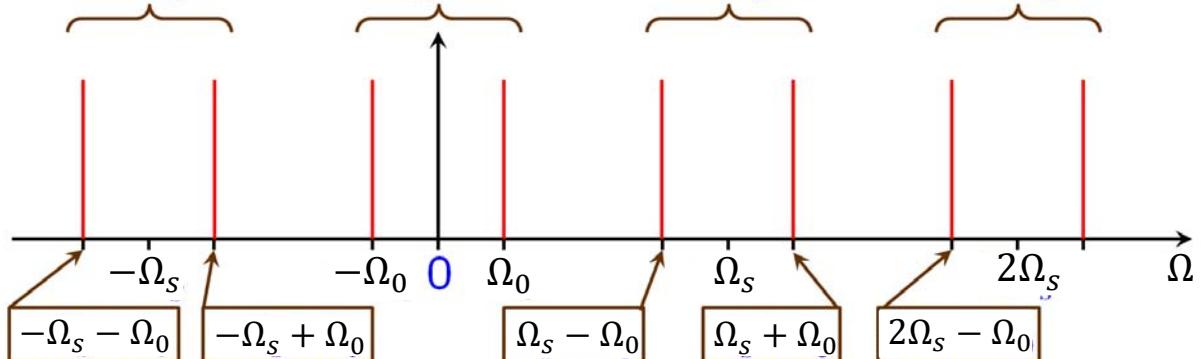


Aliasing image

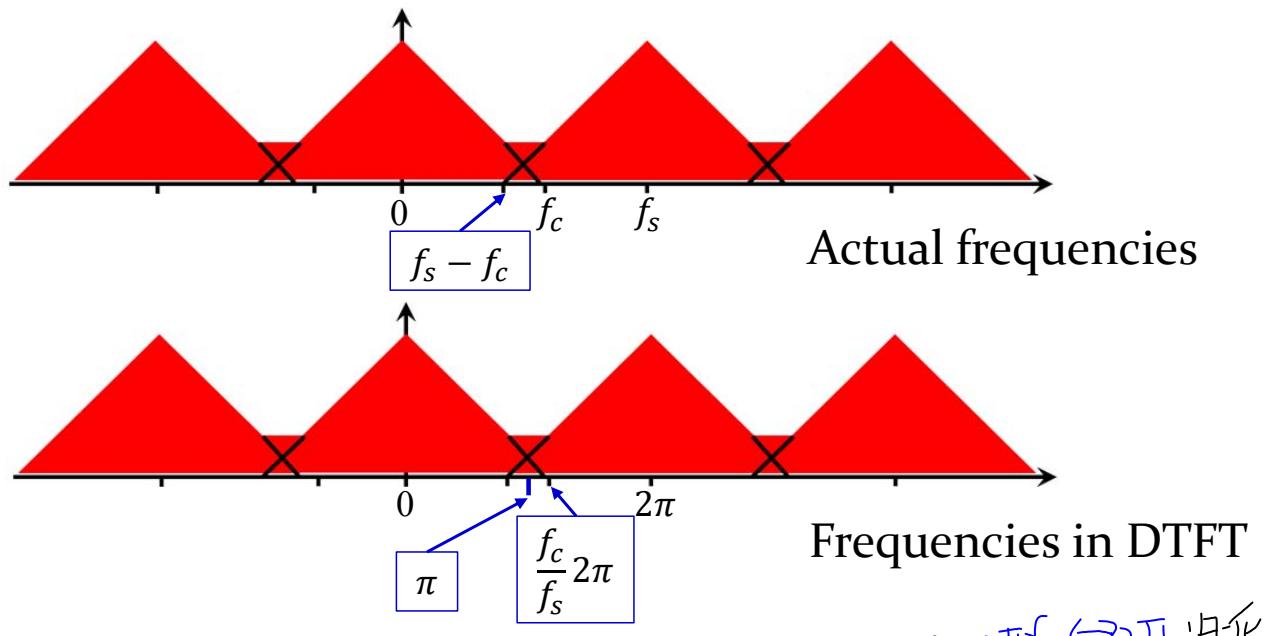
Base-band signal

Aliasing image

Aliasing image



Normalization of frequency in DTFT

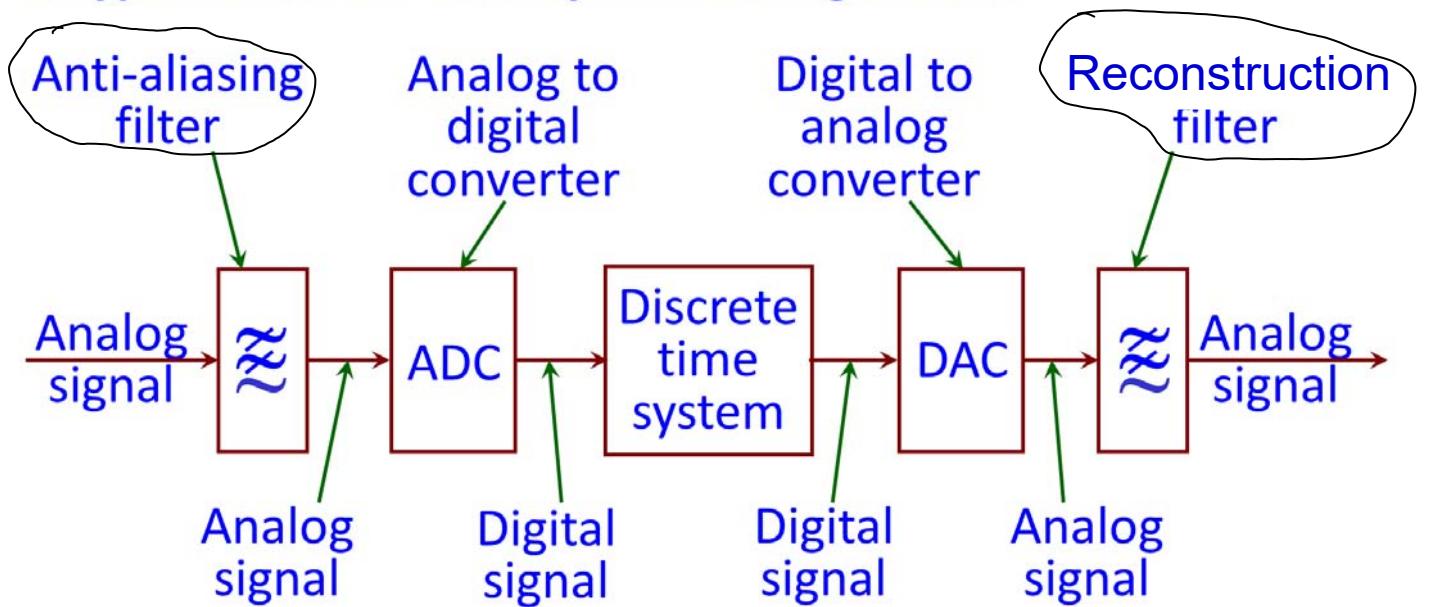


- Actual sampling frequency $f_s \leftrightarrow 2\pi$ in DTFT
- Normalization of frequency: $f \leftrightarrow \frac{f}{f_s} 2\pi$

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A typical discrete time system configuration

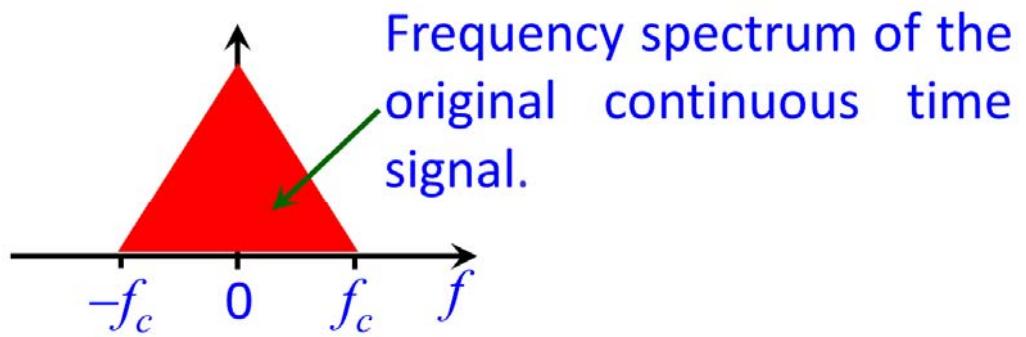


Examples of sampling rate:

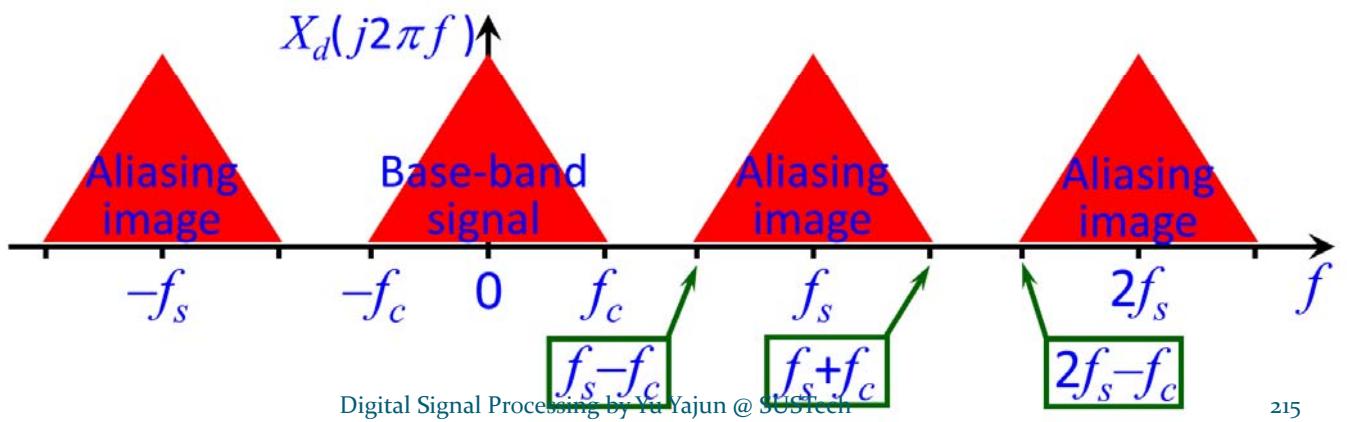
- CD : 44.1 kHz.
- Digital audio tape : 48 kHz.
- Telephone system : 8 KHz.

Summary:

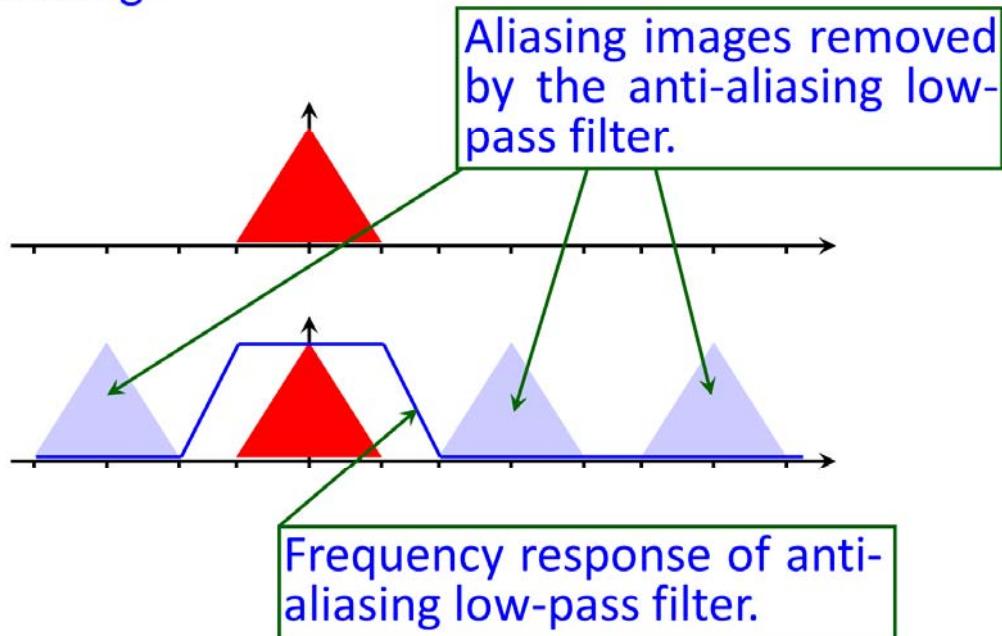
1.



Frequency spectrum of the sampled discrete time signal.

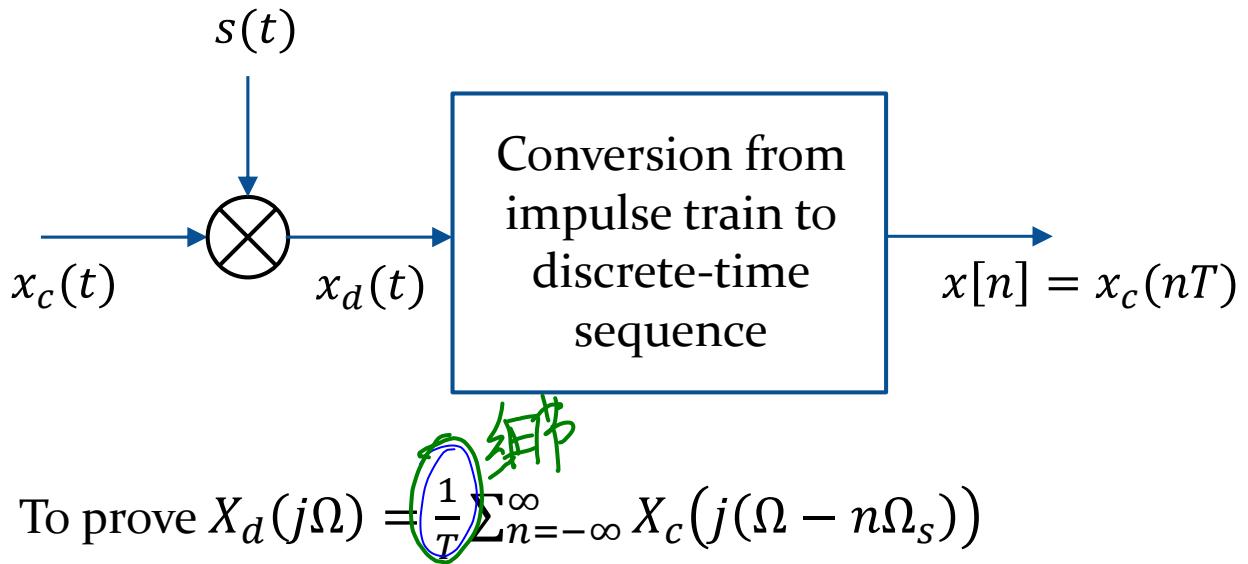


2. Information in the original continuous time signal can be recovered from the sampled signal by low-pass filtering.



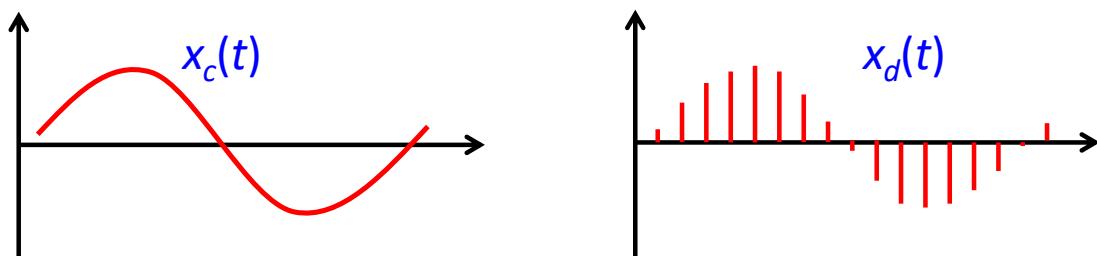
3. Nyquist rate = $2 \times$ maximum frequency.

Mathematical Derivation



$$x_d(t) = x_c(t)s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Approach 1:



Recall, the Fourier Series of

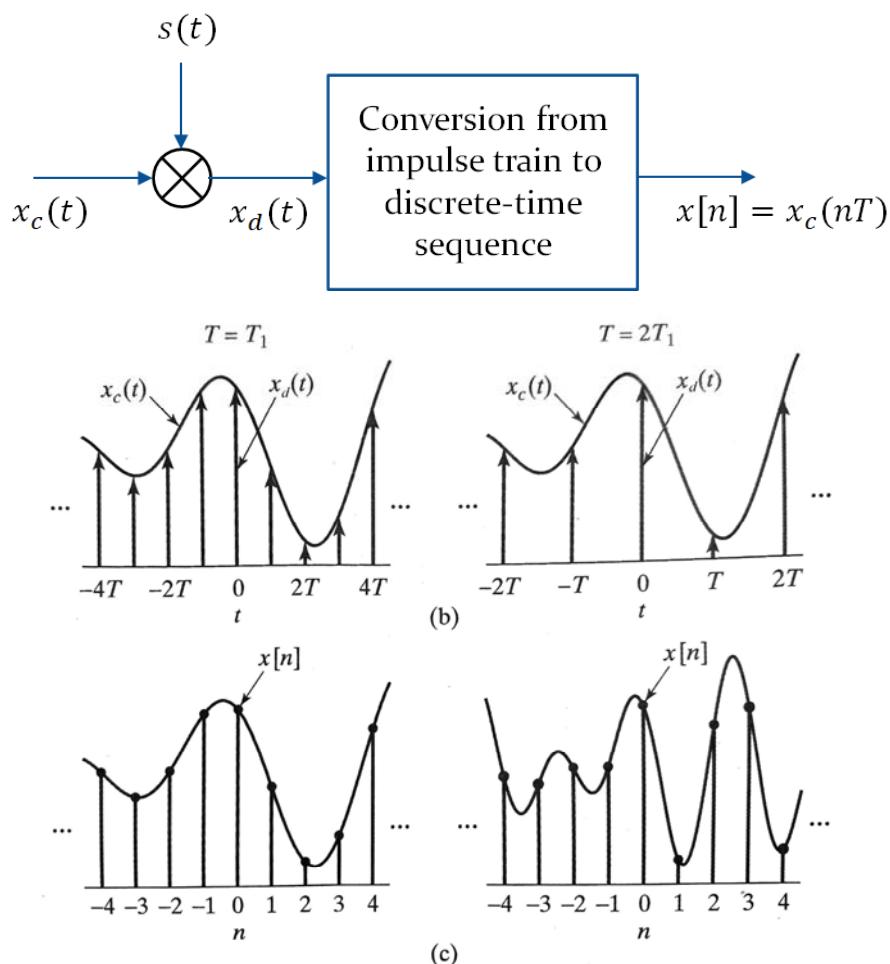
$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}, \quad \text{where } \Omega_s = \frac{2\pi}{T}$$

Thus,

$$x_d(t) = x_c(t) \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t},$$

Now, look at the spectrum of the transformed signal. Using the convolution property, we have

$$\begin{aligned}
 X_d(j\Omega) &= \frac{1}{2\pi} \frac{1}{T} X_c(j\Omega) \oplus \sum_{n=-\infty}^{\infty} 2\pi\delta(\Omega - n\Omega_s) \\
 &= \frac{1}{T} \int_{-\infty}^{\infty} X_c(j\varphi) \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_s - \varphi) d\varphi \\
 &= \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))
 \end{aligned}$$



Since $x_d(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$

we have, $X_d(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega T n},$

Furthermore, since $x[n] = x_c(nT)$, and

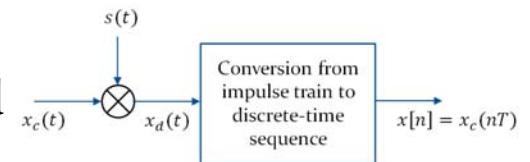
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

it follows that

$$X_d(j\Omega) = X(e^{j\omega}) \Big|_{\omega=\Omega T} = X(e^{j\Omega T}).$$

Consequently,

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$



$$X(e^{j\Omega T}) = X_d(j\Omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

- $X(e^{j\omega})$ is a **frequency scaled version** of $X_d(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$
- This scaling can alternatively be thought of as a **normalization** of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_d(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$.

 The normalization in the transformation from $X_d(j\Omega)$ to $X(e^{j\omega})$ is directly a result of the time normalization in the transformation from $x_d(t)$ to $x[n]$.

Fourier transform of a discrete time signal.

Let $x_c(t)$ be a continuous time signal.

Let the sampled version of $x_c(t)$ be denoted by $x_d(t)$.

Let the sampling interval be T .

Let $\omega_s = 2\pi/T$.

Let $X_c(j\omega)$ be the Fourier transform of $x_c(t)$.

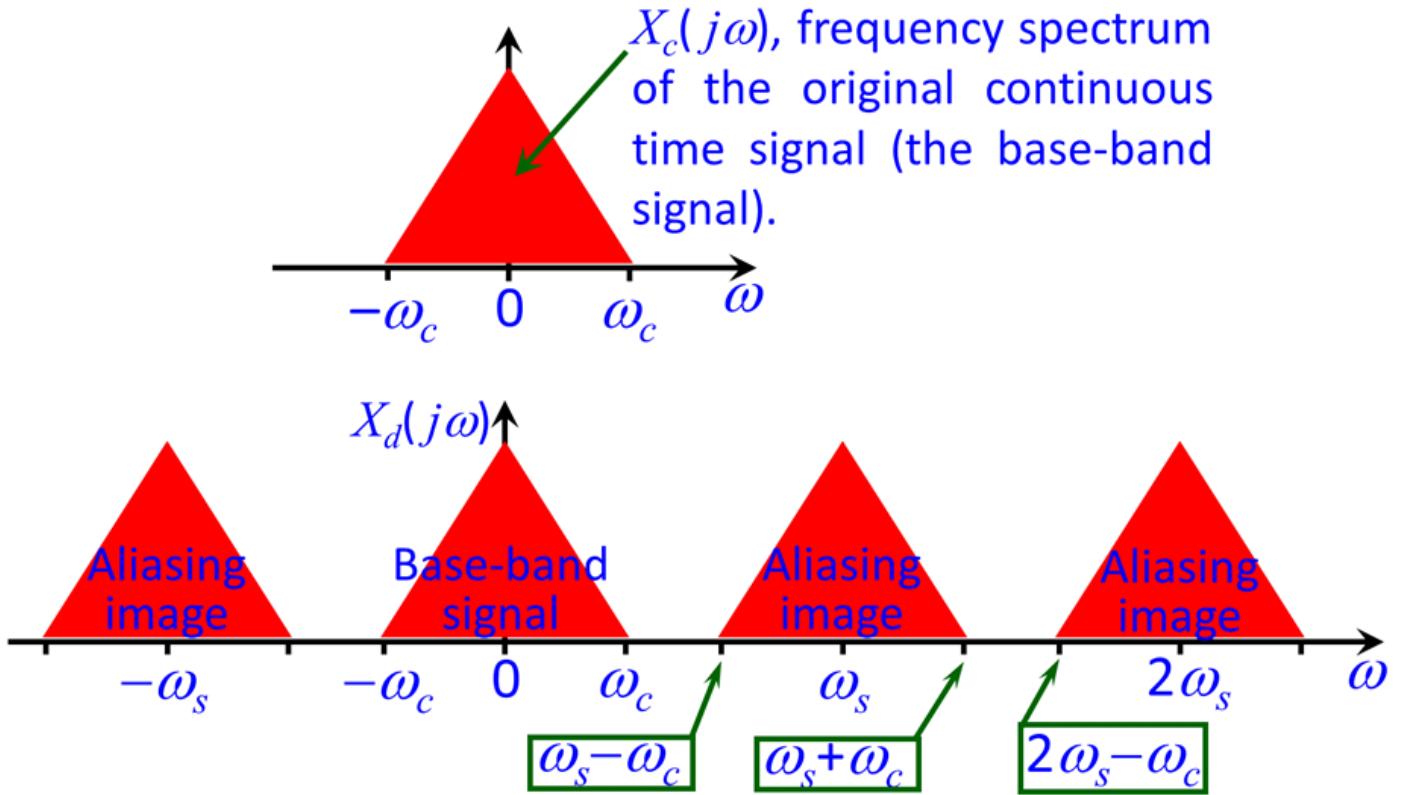
Let $X_d(j\omega)$ be the Fourier transform of $x_d(t)$.

It can be shown that $X_d(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\omega - n\omega_s))$

1/T indicates that the magnitude of $X_d(j\omega)$ increases with sampling density 1/T.

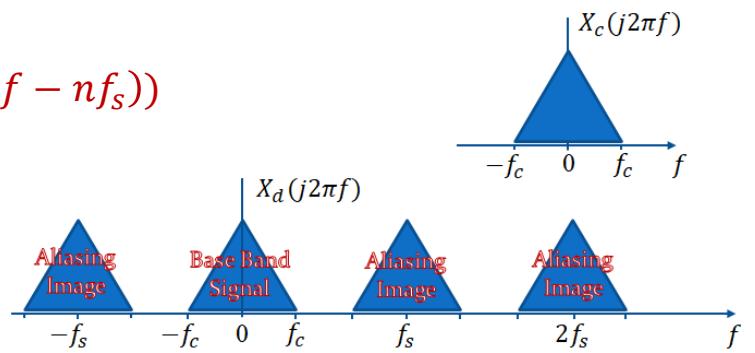
$X_c(j(\omega - n\omega_s))$ is $X_c(j\omega)$ shifted along the ω -axis by $n\omega_s$.

$$X_d(j\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j(\omega - n\omega_s))$$



“Hz” is often used as frequency unit in communication systems. Hence, replacing Ω by $2\pi f$ we have

$$X_d(j2\pi f) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X_c(j2\pi(f - nf_s))$$



When a continuous time signal (the base-band signal) is sampled at a rate of f_s samples per second, the frequency spectrum of the sampled signal is that of the base-band signal plus duplicates (aliasing) of that of the base-band signal plus duplicates (aliasing) of that of the base-band signal centered at kf_s where $k = \dots, -1, 0, 1, 2, 3, \dots$.

*Approach 2:

- Fourier transform pair

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t} dt$$

- The discrete time signal $x_d(t)$ is given by

$$x_d(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)$$

- Hence the Fourier transform of $x_d(t)$ is

$$X_d(j\Omega) = \int_{-\infty}^{\infty} x_d(t)e^{-j\Omega t} dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x_c(t)\delta(t - nT)e^{-j\Omega t} dt$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x_c(t)\delta(t - nT)e^{-j\Omega t} dt = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT}$$

- Let $x_d[n] = x_c(nT)$. Replace the notation $X_d(j\Omega)$ by $X_d(e^{j\omega T})$. Thus, we have

$$X_d(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega nT}$$

- Multiply both sides by $e^{j\omega mT}$ and integrate with respect to ω from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$.

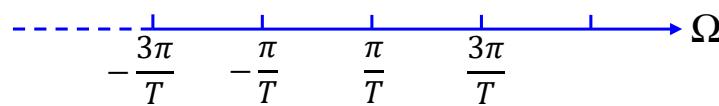
$$\begin{aligned} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_d(e^{j\omega T}) e^{j\omega mT} d\omega &= \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\omega nT} e^{j\omega mT} d\omega \\ &= \sum_{n=-\infty}^{\infty} x_d[n] \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{j\omega(m-n)T} d\omega = \frac{2\pi}{T} x_d[m] \end{aligned}$$

- Therefore,

$$x_d[m] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_d(e^{j\omega T}) e^{j\omega mT} d\omega$$

- Since $x_d[n] = x_c(t)|_{t=nT}$, it is possible to relate $X_d(e^{j\omega T})$ and $X_c(\Omega)$ by evaluating $x_c(t)$ at $t = nT$. Thus,

$$\begin{aligned} x_d[n] &= x_c(nT) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Omega) e^{j\Omega nT} d\Omega \\ &= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{(2m-1)\frac{\pi}{T}}^{(2m+1)\frac{\pi}{T}} X_c(\Omega) e^{j\Omega nT} d\Omega \end{aligned}$$



$$= \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega + \frac{2\pi}{T}m\right) \right] e^{j\Omega nT} d\Omega$$

- Comparing $x_d[m] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_d(e^{j\omega T}) e^{j\omega mT} d\omega$ and
 $x_d[n] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega + \frac{2\pi}{T}m\right) \right] e^{j\Omega nT} d\Omega$, yields

$$X_d(e^{j\omega T}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega + \frac{2\pi}{T}m\right)$$

