Tutorial 12

1. Solution: ① Type I: symmetric impulse response, length is odd.

According to $H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \sum_{n=0}^{(N-1)/2} a[n]\cos(\omega n)$ and symmetric impulse response length is odd.

∴ the lowest-order of Type I is 2nd order, that is $N=3$.

∵ Either an even number or no zeros at $z=1$ and $z=-1$

∴ $N_1=2$ $N_2=0$ $N_3=0$ $N_4=0$ or $N_1=0$ $N_2=2$ $N_3=0$ $N_4=0$ or $N_1=0$ $N_2=0$ $N_3=1$ $N_4=0$

② Type II: $H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \sum_{n=1}^{\frac{N}{2}} b[n]\cos[\omega(n-\frac{1}{2})]$, symmetric impulse response and length is even.

∴ The lowest-order of Type II is 1st order, that is $N=2$.

∵ Either an even number or no zeros at $z=1$ and odd number of zeros at $z=-1$

∴ $N_1=1$. $N_2=0$ $N_3=0$ $N_4=0$.

③ Type III: $H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=1}^{\frac{N-1}{2}} c[n]\sin(\omega n)$, antisymmetric impulse response and length is odd.

∴ The lowest-order of Type III is 2nd order, that is $N=3$

∵ An odd number of zeros at $z=1$ and $z=-1$ ∴ $N_1=1$ $N_2=1$ $N_3=0$ $N_4=0$

④ Type IV: $H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=1}^{\frac{N}{2}} d[n]\sin[\omega(n-\frac{1}{2})]$ anti-symmetric impulse response and length is even.

∴ The lowest-order of Type IV is 1st-order, that is $N=2$

∵ ~~Either~~ An odd number of zeros at $z=1$ and either an even number or no zeros at $z=-1$

∴ $N_1=0$ $N_2=1$ $N_3=0$ $N_4=0$.

2. Solution: first-order lowpass IIR: $H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}$, $0<|\alpha|<1$.

~~He~~ $|H(e^{j\omega})|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}} = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$, 令 $|H(e^{j\omega})|^2 = \frac{1}{2}$, $\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$ (another is not satisfied)

$\omega_c = 0.42$ rad/samples, $\rightarrow \alpha = \frac{1-\sin(0.42\pi)}{\cos(0.42\pi)} \approx 0.6486$.

∴ $H(z) = 0.1757 \cdot \frac{1+z^{-1}}{1-0.6486z^{-1}}$.

3. Solution: (a) $|H_{BS}(e^{j\omega})|^2 = H_{BS}(z)H_{BS}(z^{-1})|_{z=e^{j\omega}} = \frac{1+\alpha}{2}\frac{1-2\beta e^{-j\omega}+e^{-j2\omega}}{1-\beta(1+\alpha)e^{-j\omega}+\alpha e^{-j2\omega}} \cdot \frac{1+\alpha}{2}\frac{1-2\beta e^{j\omega}+e^{j2\omega}}{1-\beta(1+\alpha)e^{j\omega}+\alpha e^{j2\omega}}$.

$= \left(\frac{1+\alpha}{2}\right)^2 \frac{(1-2\beta e^{-j\omega}+e^{-j2\omega})(1-2\beta e^{j\omega}+e^{j2\omega})}{[1-\beta(1+\alpha)e^{-j\omega}+\alpha e^{-j2\omega}][1-\beta(1+\alpha)e^{j\omega}+\alpha e^{j2\omega}]} = \left(\frac{1+\alpha}{2}\right)^2 \frac{2\cos(2\omega)-8\beta\cos\omega+2+4\beta^2}{2\alpha\cos(2\omega)-2\beta(1+\alpha)^2\cos\omega+1+\alpha^2+\beta^2(1+\alpha)^2}$

(b) let $|H_{BS}(e^{j\omega})|^2=0 \rightarrow 2\cos(2\omega_0)-8\beta\cos(\omega_0)+2+4\beta^2=0 \rightarrow \cos(2\omega_0)-4\beta\cos(\omega_0)+1+2\beta^2=0$.

$\rightarrow 2\cos^2(\omega_0) - 4\beta\cos\omega_0+2\beta^2=0 \rightarrow \cos^2(\omega_0)-2\beta\cos\omega_0+\beta^2=0 \rightarrow (\cos\omega_0-\beta)^2=0 \rightarrow \cos\omega_0=\beta$

$\rightarrow \omega_0 = \cos^{-1}\beta$

(c) $|H_{BS}(e^{j\omega})|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{2\cos 2\omega - 8\beta\cos\omega + 4\beta^2 + 2}{2\alpha\cos 2\omega - 2\beta(1+\alpha)^2\cos\omega + 1 + \alpha^2 + \beta^2(1+\alpha)^2}$

when $\omega = 0$. $|H_{BS}(e^{j0})|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{2 - 8\beta + 4\beta^2 + 2}{2\alpha - 2\beta(1+\alpha)^2 + 1 + \alpha^2 + \beta^2(1+\alpha)^2} = \frac{(1+\alpha)^2}{4} \cdot \frac{4(\beta-1)^2}{(\beta-1)^2(1+\alpha)^2} = 1$.

when $\omega = \pi$ $|H_{BS}(e^{j\pi})|^2 = \frac{(1+\alpha)^2}{4} \frac{2 + 8\beta + 4\beta^2 + 2}{2\alpha + 2\beta(1+\alpha)^2 + 1 + \alpha^2 + \beta^2(1+\alpha)^2} = \frac{(1+\alpha)^2}{4} \frac{4(\beta+1)^2}{(\beta+1)^2(1+\alpha)^2} = 1$

(d) $|H_{BS}(e^{j\omega})|^2 = \frac{1}{2} \rightarrow \left(\frac{1+\alpha}{2}\right)^2 \frac{2\cos(2\omega) - 8\beta\cos\omega + 4\beta^2 + 2}{2\alpha\cos 2\omega - 2\beta(1+\alpha)^2\cos\omega + 1 + \alpha^2 + \beta^2(1+\alpha)^2} = \frac{1}{2}$

$\rightarrow 2(1+\alpha^2)\cos^2\omega - 2\beta(1+\alpha)^2\cos\omega + \beta^2(1+\alpha)^2 - (1-\alpha)^2 = 0$.

$\cos\omega_1 + \cos\omega_2 = \frac{-b}{a} = \frac{+\beta(1+\alpha)^2}{1+\alpha^2}$ $\quad \cos\omega_1 \cdot \cos\omega_2 = \frac{c}{a} = \frac{1}{2} \frac{(1+\alpha)^2\beta^2 - (1-\alpha)^2}{1+\alpha^2}$.

$\cos(\omega_2 - \omega_1) = \cos\omega_2\cos\omega_1 + \sin\omega_2\sin\omega_1 = \cos\omega_2\cos\omega_1 + \sqrt{(1-\cos^2\omega_2)(1-\cos^2\omega_1)} = \frac{2\alpha}{1+\alpha^2}$

$\rightarrow B_\omega = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$.

4. Solution: $\omega_0 = \cos^{-1}\beta \rightarrow \beta = \cos\omega_0 = \cos(0.35\pi) = 0.4540$.

$B_\omega = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right) \rightarrow \frac{2\alpha}{1+\alpha^2} = \cos B_\omega = \cos(0.15\pi) = 0.8910. \rightarrow \begin{cases} \alpha_1 = 1.6319 \text{ (not stable)} \\ \alpha_2 = 0.6128. \end{cases}$

~~Solution~~: $H_{BS}(z) = \frac{0.8064(1 - 0.908z^{-1} + z^{-2})}{1 - 0.7322z^{-1} + 0.6128z^{-2}}$

5. Solution: $|A_m(z)|^2 = A_m(z)A_m(z^{-1})\big|_{z=e^{j\omega}}$. $\because$ complex coefficient $\therefore A_m(z^{-1}) = A_m^*(z)$.

$\therefore |A_m(z)|^2 = A_m(z)A_m^*(z) = \frac{d_m^* + d_{m-1}^* z^{-1} + \cdots + d_1^* z^{-m+1} + z^{-m}}{1 + d_1 z^{-1} + \cdots + d_{m-1}z^{-m+1} + d_m z^{-m}} \cdot \frac{d_m + d_{m-1}z^1 + \cdots d_1 z^{m-1} + z^m}{1 + d_1^* z + \cdots + d_{m-1}^* z^{m-1} + d_m^* z^m}$

$= \frac{d_m^* + d_{m-1}^* z^{-1} + \cdots d_1^* z^{-m+1} + z^{-m}}{1 + d_1 z^{-1} + \cdots + d_{m-1}z^{-m+1} + d_m z^{-m}} \frac{z^m}{z^m} \cdot \frac{d_m z^{-m} + d_{m-1}z^{-m+1} + \cdots + 1}{1 + d_1^* z^{-m+1} + \cdots d_{m-1}^* z^{-1} + d_m^*} = 1$.

$\therefore A_m(z)$ is an all-pass filter.

~~$\because$~~ $\because A_m(z) = \frac{d_m^* + d_{m-1}^* z^{-1} + \cdots d_1^* z^{-m+1} + z^{-m}}{1 + d_1 z^{-1} + \cdots + d_{m-1}z^{-m+1} + d_m z^{-m}}$.

$\therefore$ $y[n] = \sum_{m=0}^{M} d_{m-m} x[n-m] - \sum_{m=0}^{M} d_m y[n-m]$.

According to the expression, $y[n]$ ~~is~~ depends on the current input and the past input and output

$\therefore$ $A_m(z)$ is a causal filter

$\therefore A_m(z)$ is an causal ~~filter~~ all pass filter.

6. Solution: (a). $1-1.6z^{-1}+2z^{-2}=0 \Rightarrow z_1=0.8+\sqrt{1.36}j$ $z_2=0.8-\sqrt{1.36}j \Rightarrow$ outside unit circle

$1+1.6z^{-1}+z^{-2}=0 \Rightarrow z_3=-0.8+0.6j$ $z_4=-0.8-0.6j$.

$1+z^{-1}=0 \Rightarrow z_5=1$ $\quad 1-0.8z^{-1}+0.5z^{-2}=0 \Rightarrow z_6=0.4+\sqrt{0.34}j$. $z_7=0.4-\sqrt{0.34}j \Rightarrow$ inside the unit circle.

$\therefore 1-1.6z^{-1}+2z^{-2}=0 \Rightarrow (z-0.8-\sqrt{1.36}j)(z-0.8+\sqrt{1.36}j) \Rightarrow \frac{1}{a_1^*}=0.8+\sqrt{1.36}j$ $\frac{1}{a_2^*}=0.8-\sqrt{1.36}j$

$\therefore H_1(z)=2.5(1+1.6z^{-1}+z^{-2})(1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})(1-1.6z^{-1}+2z^{-2})\frac{1-a_1z^{-1}}{1-a_1z^{-1}}\cdot\frac{1-a_2z^{-1}}{1-a_2z}$

$=2.5(1+1.6z^{-1}+z^{-2})(1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})(1-1.6z^{-1}+2z^{-2})\frac{\frac{1}{2}z^{-2}-0.8z^{-1}+1}{\frac{1}{2}z^{-2}-0.8z^{-1}+1}$

$=2.5(1+1.6z^{-1}+z^{-2})(1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})(z^{-2}-1.6z^{-1}+2)\frac{1-1.6z^{-1}+2z^{-2}}{z^{-2}-1.6z^{-1}+2}=H_2(z)\frac{1-1.6z^{-1}+2z^{-2}}{z^{-2}-1.6z^{-1}+2}$.

$\therefore H_2(z)=2.5(1+1.6z^{-1}+z^{-2})(1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})(z^{-2}-1.6z^{-1}+2)$.

(b) According to (a), $1-0.8z^{-1}+0.5z^{-2}=0 \Rightarrow (z-0.4-j\sqrt{0.34})(z-0.4+j\sqrt{0.34}) \Rightarrow \frac{1}{a_3^*}=0.4+j\sqrt{0.34}$ $\frac{1}{a_4^*}=0.4-j\sqrt{0.34}$

$H_1(z)=2.5(1-1.6z^{-1}+2z^{-2})(1+1.6z^{-1}+z^{-2})(1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})\cdot\frac{1-a_3z^{-1}}{1-a_3z^{-1}}\cdot\frac{1-a_4z^{-1}}{1-a_4z^{-1}}$

$=2.5(1-1.6z^{-1}+2z^{-2})(1+1.6z^{-1}+z^{-2})(1+z^{-1})(z^{-2}-0.8z^{-1}+0.5)\frac{1-0.8z^{-1}+0.5z^{-2}}{z^{-2}-0.8z^{-1}+0.5}$.

$H_3(z)=2.5(1-1.6z^{-1}+2z^{-2})(1+1.6z^{-1}+z^{-2})(1+z^{-1})(z^{-2}-0.8z^{-1}+0.5)$.

(c) Because there are three combinations having the same magnitude response, the other zeros are all on unit circle $\therefore$ there are no other length 8 FIR filter having same magnitude response.

7. Solution: $2.2+5z^{-1}=0 \Rightarrow z_1=\frac{-5}{2.2}=\frac{1}{a_1^*} \Rightarrow a_1=\frac{-2.2}{5}$, $1-3.1z^{-1}=0 \Rightarrow z=3.1=\frac{1}{a_2^*} \Rightarrow a_2=\frac{1}{3.1}$

$1+0.81z^{-1}=0 \Rightarrow z_3=-0.81$ inside the unit circle., $1-0.62z^{-1}=0 \Rightarrow z_4=0.62$ inside the unit circle.

$\therefore H(z)=\frac{(2.2+5z^{-1})(1-3.1z^{-1})}{(1+0.81z^{-1})(1-0.62z^{-1})}\cdot\frac{1-a_1z^{-1}}{1-a_1z^{-1}}\cdot\frac{1-a_2z^{-1}}{1-a_2z^{-1}}=\frac{(1+\frac{2.2}{5}z^{-1})(1-\frac{1}{3.1}z^{-1})}{(1+0.81z^{-1})(1-0.62z^{-1})}\times\frac{(2.2+5z^{-1})(1-3.1z^{-1})}{(1+\frac{2.2}{5}z^{-1})(1-\frac{1}{3.1}z^{-1})}$.

$\because$ According to the problem, $H(z)G(z)$ is an allpass filter.

$\because \frac{(2.2+5z^{-1})(1-3.1z^{-1})}{(1+\frac{2.2}{5}z^{-1})(1-\frac{1}{3.1}z^{-1})}$ is an allpass filter $\therefore G(z)=\frac{(1+0.81z^{-1})(1-0.62z^{-1})}{(1+\frac{2.2}{5}z^{-1})(1-\frac{1}{3.1}z^{-1})}$

All poles of $G(z)$ are in the unit circle and $G(z)$ is casual.

$\therefore G(z)$ is what we want.

8. Solution: $Y(z) = G(z)[X(z) - KY(z)] \rightarrow Y(z)(1+KG(z)) = G(z)X(z) \rightarrow H(z) = \dfrac{G(z)}{1+KG(z)}$.

$$\rightarrow H(z) = \frac{\dfrac{z^{-2}}{1+1.5z^{-1}+0.5z^{-2}}}{1+\dfrac{Kz^{-2}}{1+1.5z^{-1}+0.5z^{-2}}} = \frac{z^{-2}}{(K+0.5)z^{-2}+1.5z^{-1}+1} = \frac{1}{(K+0.5)+1.5z+z^{2}}$$

poles: $z = \dfrac{-1.5 \pm \sqrt{1.5^2-4(K+0.5)}}{2} = -0.75 \pm \dfrac{\sqrt{0.25-4K}}{2}$, $z_1 = -0.75 + \dfrac{\sqrt{0.25-4K}}{2}$, $z_2 = -0.75 - \dfrac{\sqrt{0.25-4K}}{2}$

if z are real, $0.25-4K>0 \rightarrow K<\frac{1}{16}$. ~~$|z_1|$ and $|z_2|$ are less than~~ $\left|z_1 = -0.75+\dfrac{\sqrt{0.25-4K}}{2}\right| < 1$.

$z_2 = -0.75 - \dfrac{\sqrt{0.25-4K}}{2} > -1 \rightarrow K>0 \rightarrow 0<K<\frac{1}{16}$.

if z are complex. $|z|^2<1 \rightarrow 0.75^2 + \left(\dfrac{\sqrt{0.25-4K}}{2}\right)^2 < 1 \rightarrow K<0.5$

$\rightarrow K \in (0, 0.5)$.

9. Solution: According to 8Q8. $H(z) = \dfrac{G(z)}{1+G(z)C(z)} \Rightarrow C(z) = \dfrac{G(z)-H(z)}{H(z)G(z)}$.

$$\rightarrow C(z) = \frac{\dfrac{1.2+1.8z^{-1}}{1+0.7z^{-1}+0.8z^{-2}} - \dfrac{z^{-1}+1.35z^{-2}+0.9z^{-3}+0.3375z^{-4}}{0.3+0.5z^{-1}+0.505z^{-2}+0.375z^{-3}+0.21z^{-4}}}{\dfrac{z^{-1}+1.35z^{-2}+0.9z^{-3}+0.3375z^{-4}}{0.3+0.5z^{-1}+0.505z^{-2}+0.375z^{-3}+0.21z^{-4}} \cdot \dfrac{1.2+1.8z^{-1}}{1+0.7z^{-1}+0.8z^{-2}}} = \frac{1.2+0.4667z^{-1}-1.8333z^{-2}-4.2867z^{-3}-3.75z^{-4}-1.9275z^{-5}-0.9z^{6}}{1+2.3667z^{-1}+3.65z^{-2}+3.7617z^{-3}+2.9217z^{-4}+1.49z^{-5}+0.56z^{6}}.$$

or $$C(z) = \frac{0.3+0.1167z^{-1}-0.4533z^{-2}-1.0717z^{-3}-0.9338z^{-4}-0.4819z^{-5}-0.225z^{-6}}{z^{-1}+2.85z^{-2}+2.925z^{-3}+1.6875z^{-4}+0.5063z^{-5}}.$$