1. To design a lowpass digital filter with $\omega_p = 0.24\pi, \omega_s = 0.68\pi, \alpha_p = 1\text{dB}$, and $\alpha_s = 24\text{dB}$ using bilinear transformation $s \to k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$, we have to first design a prototype lowpass analog filter.

   (a) If the lowpass analog filter has a passband edge $F_p = 10\text{Hz}$, determine the value of *k*, and the stopband edge F_s of the analog prototype filter.

   (b) Using *k* =10 in the bilinear transformation, determine F_p and F_s of the analog prototype filter.

A. (a) $\Omega_p = k\tan\left(\frac{\omega_p}{2}\right)$

$$\Rightarrow 10 * 2\pi = k\tan(0.24\pi/2) \Rightarrow k = 158.6951$$

Thus, $\Omega_s = 158.6951\tan(\omega_s/2) = 288.6653\text{ rad/}s$

$$\Rightarrow F_s = \frac{288.6653}{2\pi} = 45.9425\text{Hz}$$

(b) $\Omega_p = 10\tan\left(\frac{\omega_p}{2}\right) = 3.9593$

$$\Rightarrow F_p = \frac{3.9593}{2\pi} = 0.6301\text{Hz}$$

$$\Omega_s = 10\tan\left(\frac{\omega_s}{2}\right) = 18.1899$$

$$\Rightarrow F_s = \frac{18.1899}{2\pi} = 2.8950\text{Hz}$$

2. A chebyshev lowpass analog filter meeting the analog specification of question 1(b) is given by

$$H_a(s) = \frac{15.4035s^{-2}}{1 + 4.3463s^{-1} + 17.2830s^{-2}}$$

Use bilinear transformation to transform the analog filter into the lowpass digital filter.

A:

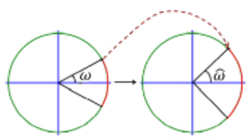
$$H(z) = \frac{15.4035\left(10\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^{-2}}{1 + 4.3463\left(10\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^{-1} + 17.2830\left(10\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)^{-2}}$$

$$= \frac{0.0958 + 0.1916z^{-1} + 0.0958z^{-2}}{1 - 1.0291z^{-1} + 0.4592z^{-2}}$$

$$= 0.0958 \cdot \frac{1 + 2z^{-1} + z^{-2}}{1 - 1.0291z^{-1} + 0.4592z^{-2}}$$

3. Let $H_{LP}(z)$ be an IIR lowpass transfer function with a zero (pole) at $z = z_k$. Let $H_D(\hat{z})$ denote the lowpass transfer function obtained by lowpass-to-lowpass transformation given by $z^{-1} = \frac{\hat{z}^{-1}-\alpha}{1-\alpha\hat{z}^{-1}}$, which moves the zero (pole) at $z = z_k$ of $H_{LP}(z)$ to a new location $\hat{z} = \hat{z}_k$. Express $\hat{z}_k$ in terms of z_k. If $H_{LP}(z)$ has a zero at $z = -1$, show that $H_D(\hat{z})$ also has a zero at $z = -1$.

$$z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}} \Leftrightarrow \hat{z}^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$



lowpass transfer function

A: A zero (pole) of $H_{LP}(z)$ is given by a factor $(z - z_k)$. After applying the lowpass to lowpass transformation, this factor becomes

$$\frac{1 - \alpha\hat{z}^{-1}}{\hat{z}^{-1} - \alpha} - z_k = \frac{\hat{z} - \alpha}{1 - \alpha\hat{z}} - z_k$$

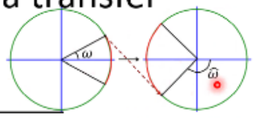
Hence the new location of the zero (pole) is given by the root of the equation

$$\hat{z} - \alpha - z_k + \alpha\hat{z}z_k = 0$$

$$\Longrightarrow \hat{z} = \hat{z}_k = \frac{a + z_k}{1 + az_k} \qquad \boxed{\hat{z}^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}}$$

When $z = -1$ is a zero, $\hat{z} = \frac{a-1}{1-a} = -1$ is also a zero.

4. A second-order lowpass IIR digital filter with a 3-dB cutoff frequency at $\omega_c = 0.55\pi$ has a transfer function



$$G_{LP}(z) = \frac{0.34(1+z^{-1})^2}{1+0.1842z^{-1}+0.1776z^{-2}}$$

Design a second-order highpass filter $H_{HP}(z)$ with a 3-dB cutoff frequency at $\widehat{\omega}_c = 0.45\pi$ by the lowpass-to-highpass spectral transformation.

A: $\alpha = -\dfrac{\cos\left(\frac{\omega_c-(-\widehat{\omega}_c)}{2}\right)}{\cos\left(\frac{\omega_c+(-\widehat{\omega}_c)}{2}\right)} = -\dfrac{\cos\left(\frac{0.55\pi-(-0.45\pi)}{2}\right)}{\cos\left(\frac{0.55\pi+(-0.45\pi)}{2}\right)} = 0.$

Thus, $z^{-1} = -\dfrac{\hat{z}^{-1}+\alpha}{1+\alpha\hat{z}^{-1}} = -\hat{z}^{-1}$, therefore,

$$H_{HP}(z) = G_{LP}(z)|_{z^{-1}=-\hat{z}^{-1}} = \frac{0.34(1-z^{-1})^2}{1-0.1842z^{-1}+0.1776z^{-2}}$$

5. A third-order elliptic highpass filter with a passband edge at $\omega_p = 0.52\pi$ has a transfer function

$$G_{HP}(z) = \frac{0.2397(1-1.5858z^{-1}+1.5858z^{-2}-z^{-3})}{1+0.3272z^{-1}+0.7459z^{-2}+0.179z^{-3}}.$$

Design a highpass filter $H_{HP}(z)$ with a passband edge at $\widehat{\omega}_p = 0.48\pi$ by transforming the above highpass transfer function using the lowpass-to-lowpass spectral transformation.

A: $\alpha = \dfrac{\sin\left(\frac{\omega_p-\widehat{\omega}_p}{2}\right)}{\sin\left(\frac{\omega_p+\widehat{\omega}_p}{2}\right)} = \dfrac{\sin\left(\frac{0.52\pi-0.48\pi}{2}\right)}{\sin\left(\frac{0.52\pi+0.48\pi}{2}\right)} = 0.0628$

$$H_{HP}(z) = G_{HP}(z)\Big|_{z^{-1}=\frac{\hat{z}^{-1}-0.0628}{1-0.0628\hat{z}^{-1}}}$$

$$= \frac{0.3766-0.6803\hat{z}^{-1}+0.6803\hat{z}^{-2}-0.3766\hat{z}^{-3}}{1.3954+0.0705\hat{z}^{-1}+0.9783\hat{z}^{-2}+0.1892\hat{z}^{-3}}$$

6. Let $h_d[n], -\infty < n < \infty$, denote the impulse response samples of an ideal zero-phase lowpass filter with a frequency response $H_d(e^{j\omega})$. It has been shown that the frequency response $H(e^{j\omega})$ of the zero-phase FIR filter $h[n], -M < n < M$, obtained by multiplying $h_d[n]$ with a rectangular window $w_R[n], -M < n < M$, has the least integral-squared error $E_R = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$. Let E_{Hann} denote the integral-squared error if a length-$(2M+1)$ Hanning window is used to develop the FIR filter. Determine an expression for the excess error $E_{excess} = E_R - E_{Hann}$.

A: $E_R = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega$ (Parseval theorem)

$$= \sum_{n=-M}^{M} |h_d[n] - h_d[n]|^2 + \sum_{|n|>N} |h_d[n]|^2 = \sum_{|n|>N} |h_d[n]|^2$$

$$E_{Hann} = \sum_{n=-\infty}^{\infty} |h_d[n] - h_d[n] w_{Hann}[n]|^2$$

$$= \sum_{n=-M}^{M} \left|h_d[n] - h_d[n]\left(0.5 + 0.5\cos\frac{2\pi n}{2M+1}\right)\right|^2 + \sum_{|n|>N} |h_d[n]|^2$$

$$= \sum_{n=-M}^{M} \left|h_d[n] - h_d[n]\left(0.5 + 0.5\cos\frac{2\pi n}{2M+1}\right)\right|^2 + E_R$$

Therefore, $E_{excess} = E_R - E_{Hann}$

$$= -\sum_{n=-M}^{M} \left|h_d[n](0.5 - 0.5\cos\frac{2\pi n}{2M+1})\right|^2$$

$$= -\sum_{n=1}^{M} \left|\frac{\sin n\omega_c}{n\pi}\left(1 - \cos\frac{2\pi n}{2M+1}\right)\right|^2$$

7. Design FIR filters with the smallest length meeting the following specification using the approach based on fixed window function.

- Design Step:
  - Determine ideal impulse response $h[n]$
  - Select window type
  - Determine window length $N = 2M + 1,\ \Delta\omega = c/M$
  - Determine window function $w[n]$
  - Time domain multiplication $h[n] \cdot w[n]$
  - Recover causality of the filter $\boxed{h[n-M] \cdot w[n-M]}$

(a) Lowpass filter,

$$\omega_p = 0.65\pi, \omega_s = 0.76\pi, \delta_p = 0.002, \delta_s = 0.004.$$

A: (a) $\Delta\omega = \omega_s - \omega_p = 0.11\pi,$

- $\delta_p = \delta_s = \delta$

$$\alpha_s = -20\log_{10} 0.002 = -53.98\text{dB},$$

$$\omega_c = \frac{\omega_s + \omega_p}{2} = 0.705\pi$$

- $H(e^{j\omega_c}) \cong 0.5$

Both Hamming window and Blackman-Harris window can meet the α_s specification, and Hamming window has a shorter filter length.

$\boxed{\text{length } N = 2M + 1}$

Type of Window	Window function		Resultant Filter	
	Main Lobe Width Δ_{ML}	Relative Side-lobe Level A_{sl}	Minimum Stop-band Attenuation δ	Transition Bandwidth $\Delta\omega$
Rectangular	$\frac{4\pi}{2M+1}$	13.3dB	20.9dB	$\frac{0.92\pi}{M}$
Hanning	$\frac{8\pi}{2M+1}$	31.5dB	43.9dB	$\frac{3.11\pi}{M}$
Hamming	$\frac{8\pi}{2M+1}$	42.7dB	54.5dB	$\frac{3.32\pi}{M}$
Blackman-Harris	$\frac{12\pi}{2M+1}$	58.1dB	75.3dB	$\frac{5.56\pi}{M}$

The filter length $N = 2M + 1$ is given by

$$\Delta\omega = \frac{3.32\pi}{M} \Rightarrow M = \frac{3.32\pi}{\Delta\omega} = \left\lceil \frac{3.32\pi}{0.11\pi} \right\rceil = 31$$

$$\Rightarrow N = 2M + 1 = 63$$

$$w_{Hamm}[n] = 0.54 + 0.46 \cos \frac{2\pi n}{63}$$

$$\Longrightarrow h[n] = h_d[n] w_{Hamm}[n]$$

$$= \frac{\sin(0.705\pi n)}{n\pi} \left(0.54 + 0.46 \cos \frac{2\pi n}{63} \right)$$

Thus, the causal lowpass FIR filter is given by:

$$h_c[n] = \frac{\sin(0.705\pi(n-31))}{(n-31)\pi} \left(0.54 + 0.46 \cos \frac{2\pi(n-31)}{63} \right)$$

(b) Highpass filter,

$$\omega_p = 0.58\pi, \omega_s = 0.42\pi, \delta_p = 0.008, \delta_s = 0.01.$$

A: The highpass filter $h[n]$ with the given specification may be obtained by $h[n] = h_d[n] * w[n]$, where $h_d[n]$ is an ideal highpass filter with the required cutoff frequency, and $w[n]$ is a proper window function.

Since $\Delta\omega = \omega_p - \omega_s = 0.16\pi$,

$$\alpha_s = -20 \log_{10} 0.008 = -41.94\text{dB},$$

$$\omega_c = \frac{\omega_s + \omega_p}{2} = 0.5\pi$$

Hann Window, Hamming window and Blackman-Harris window all can meet the α_s specification, and Hann window has the shortest filter length.

The filter length $N = 2M + 1$ is given by

$$\Delta\omega = \frac{3.11\pi}{M} \Rightarrow M = \frac{3.11\pi}{\Delta\omega} = \left\lceil \frac{3.11\pi}{0.16\pi} \right\rceil = 20$$

$$\Rightarrow N = 2M + 1 = 41$$

$$w_{Hann}[n] = 0.5 + 0.5 \cos \frac{2\pi n}{41}$$

The ideal highpass filter has the impulse response

$h_d[n] = 1 - \frac{\omega_c}{\pi}$, for $n = 0$, and $h_d[n] = -\frac{\sin \omega_c n}{n\pi}$, for $n \neq 0$, and $h[n] = h_d[n] w_{Hann}[n]$

Thus, the causal highpass FIR filter is given by:

$$h_c[n] = h[n - 20] = h_d[n - 20] w_{Hann}[n - 20]$$

$$= \begin{cases} \left(1 - \frac{0.5\pi}{\pi}\right)\left(0.5 + 0.5 \cos \frac{2\pi(n-20)}{41}\right) = 0.5, & \text{for } n = 20 \\ -\frac{\sin 0.5\pi(n-20)}{(n-20)\pi}\left(0.5 + 0.5 \cos \frac{2\pi(n-20)}{41}\right), & \text{for } n \neq 20 \end{cases}$$

(c) bandpass filter, $\omega_{p1} = 0.4\pi, \omega_{p2} = 0.55\pi, \omega_{s1} = 0.25\pi, \omega_{s2} = 0.75\pi, \delta_p = 0.02, \delta_{s1} = 0.006, \quad \delta_{s2} = 0.008$, where δ_{s1} and δ_{s2} are, respectively, the ripple in the lower and upper stopbands.

A: A bandpass filter may be constructed by the difference of two lowpass filters with specifications of:

Filter 1: $\omega_p = 0.25\pi, \omega_s = 0.4\pi, \delta_p = 0.006, \delta_s = 0.008$,

$$\Delta\omega = 0.15\pi, \omega_c = 0.325\pi$$

$$\alpha_s = -20 \log_{10} \frac{0.006}{2} = -50.45\text{dB};$$

Filter 2: $\omega_p = 0.55\pi, \omega_s = 0.75\pi, \delta_p = 0.006, \delta_s = 0.008$,
$\Delta\omega = 0.2\pi, \omega_c = 0.65\pi$

$$\alpha_s = -20 \log_{10} \frac{0.006}{3} = -50.45\text{dB}.$$

- Thus, Hamming window can meet the α_s specification, and has shorter filter length
- Taking the narrower $\Delta\omega = 0.15\pi$ as the transition width, the filter length $N = 2M + 1$ is given by

$$\Delta\omega = \frac{3.32\pi}{M} \Rightarrow M = \left\lceil \frac{3.32\pi}{0.15\pi} \right\rceil = 23 \Rightarrow N = 2M + 1 = 47$$

$$w_{Hamm}[n] = 0.54 + 0.46 \cos \frac{2\pi n}{47}$$

Thus, the causal bandpass FIR filter is given by:

$$h_c[n] = (h_{d1}[n] - h_{d2}[n]) * w_{Hamm}[n]$$
$$= \left(\frac{\sin(0.65\pi(n - 23))}{(n - 23)\pi} - \frac{\sin(0.325\pi(n - 23))}{(n - 23)\pi} \right) \left(0.54 + 0.46 \cos \frac{2\pi(n - 23)}{47} \right)$$

(d) bandstop filter,

$\omega_{p1} = 0.33\pi, \omega_{p2} = 0.8\pi, \omega_{s1} = 0.5\pi, \omega_{s2} = 0.7\pi, \delta_{p1} = 0.04, \delta_{p2} = 0.04, \quad \delta_s = 0.03$, where δ_{p1} and δ_{p2} are, respectively, the ripple in the lower and upper passbands.

A: The bandstop filter may be constructed by the sum of a highpass filter and a lowpss filter with specifications of:

Lowpass Filter: $\omega_p = 0.33\pi, \omega_s = 0.5\pi, \delta_p = 0.04, \delta_s = 0.03$

$$\Delta\omega = 0.17\pi, \omega_c = 0.415\pi$$

$$\alpha_s = -20 \log_{10} \frac{0.03}{2} = -36.48\text{dB};$$

Highpass Filter: $\omega_p = 0.8\pi, \omega_s = 0.7\pi, \delta_p = 0.04, \delta_s = 0.03$

$$\Delta\omega = 0.1\pi, \omega_c = 0.75\pi$$

$$\alpha_s = -20 \log_{10} \frac{0.03}{2} = -36.48\text{dB}.$$

- Thus, Hann window can meet the α_s specification, and has the shortest filter length
- Taking the narrower $\Delta\omega = 0.1\pi$ as the transition width, the filter length $N = 2M + 1$ is given by

$$\Delta\omega = \frac{3.11\pi}{M} \Rightarrow M = \left\lceil \frac{3.11\pi}{0.1\pi} \right\rceil = 32 \Rightarrow N = 2M + 1 = 65$$

$$w_{Hamm}[n] = 0.5 + 0.5\cos\frac{2\pi n}{65}$$

Thus, the causal bandstop FIR filter is given by:

$$h_c[n] = h[n-32] = h_d[n-32]w_{Hann}[n-32]$$

$$= \begin{cases} \left(\frac{0.415\pi}{\pi} + 1 - \frac{0.75\pi}{\pi}\right)\left(0.5 + 0.5\cos\frac{2\pi(n-32)}{65}\right) = 0.58, \\ \text{for } n = 32 \\ \frac{\sin 0.415\pi(n-32) - \sin 0.75\pi(n-32)}{(n-32)\pi}\left(0.5 + 0.5\cos\frac{2\pi(n-20)}{65}\right) \\ \text{for } n \neq 32 \end{cases}$$

8. A lowpass FIR filter of order $N = 71$ is to be designed with a transition band given by $\omega_s - \omega_p = 0.04\pi$ with minimax criteria. Determine the approximate value of the stopband attenuation α_s in dB and the corresponding stopband ripple δ_s of the designed filter if the filter order is estimated using each of the following formulas: (a) Kaiser's formula, (b) Bellanger's formula. *Assume the passband and stopband ripples to be the same.*

A: (a) Kaiser's formula:

$$N \cong \frac{-20\log_{10}\left(\sqrt{\delta_p\delta_s}\right) - 13}{\frac{14.6(\omega_s - \omega_p)}{2\pi}}$$

$$\Longrightarrow \delta_s = 10^{\frac{14.6(\omega_s-\omega_p)N}{2\pi(-20)}+\frac{13}{-20}} = 10^{\frac{14.6\times 0.04\pi\times 71}{-40\pi}+\frac{13}{-20}}$$
$$= 10^{-1.6866} = 0.0206$$

$$\Longrightarrow \alpha_s = -20\log_{10} 0.02 = 33.73\text{dB}$$

(b) Bellanger's formula

$$N \cong -\frac{2\log_{10}(10\delta_p\delta_s)}{3(\omega_s-\omega_p)/2\pi} - 1$$

$$\Longrightarrow \delta_s = \sqrt{0.1\times 10^{\frac{3(\omega_s-\omega_p)(N+1)}{2\pi(-2)}}} = \sqrt{0.1\times 10^{\frac{3\times 0.04\pi\times 72}{-4\pi}}}$$
$$= \sqrt{0.1\times 10^{-2.16}} = 0.0263$$

$$\Longrightarrow \alpha_s = -20\log_{10} 0.0263 = 31.6\text{dB}$$

9. Repeat Problem 8 if the filter is designed using the Kaiser's window-based method. α_s, δ_s

A:

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)} + 1$$

$$\Longrightarrow \alpha_s = (N-1)\times 2.285(\Delta\omega) + 8$$

$$= 70\times 2.285\times 0.04\pi + 8 = 28.1\text{dB}$$

$$\Longrightarrow \delta_s = 10^{\frac{\alpha_s}{-20}} = 10^{\frac{28.1}{-20}} = 0.039$$