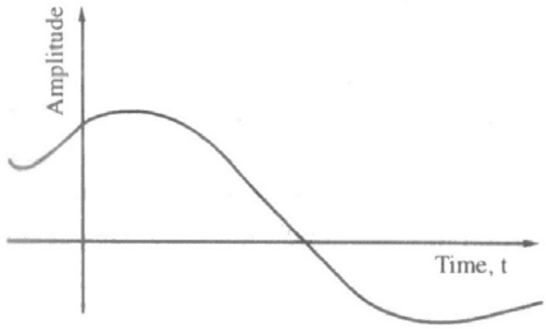
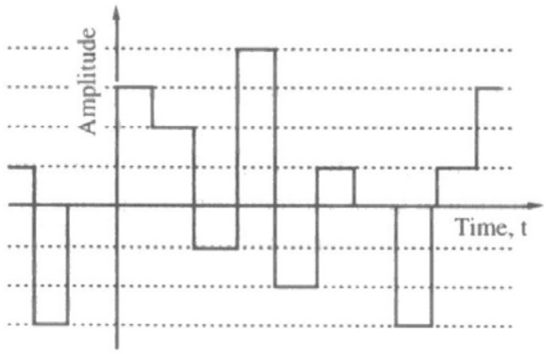
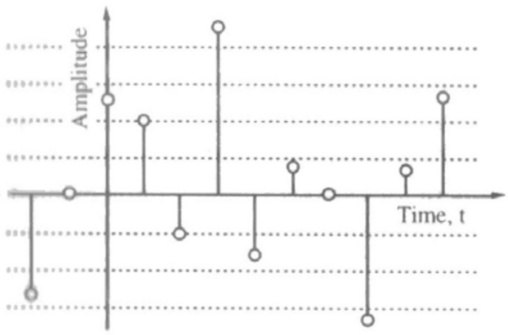
- A continuous-time signal with a continuous amplitude is usually called an **analog signal**.
  - A speech signal is an example of an analog signal.



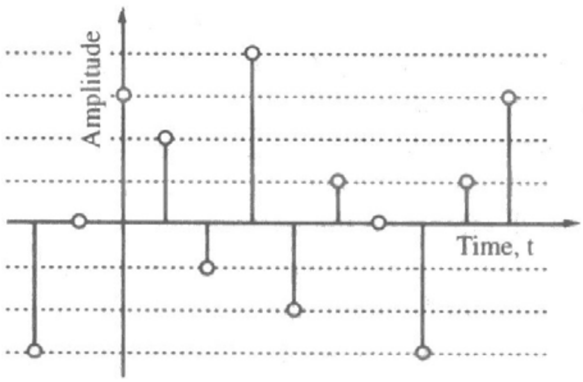
- A continuous-time signal with discrete valued amplitudes has been referred to as a **quantized boxcar signal**.
  - This type of signal occurs in digital electronic circuits where the signal is kept at fixed level (usually one of two values) between two instants of clocking.



- A discrete time signal with continuous valued amplitudes is called a **sampled-data signal**.
  - The amplitude of the signal may be any value.

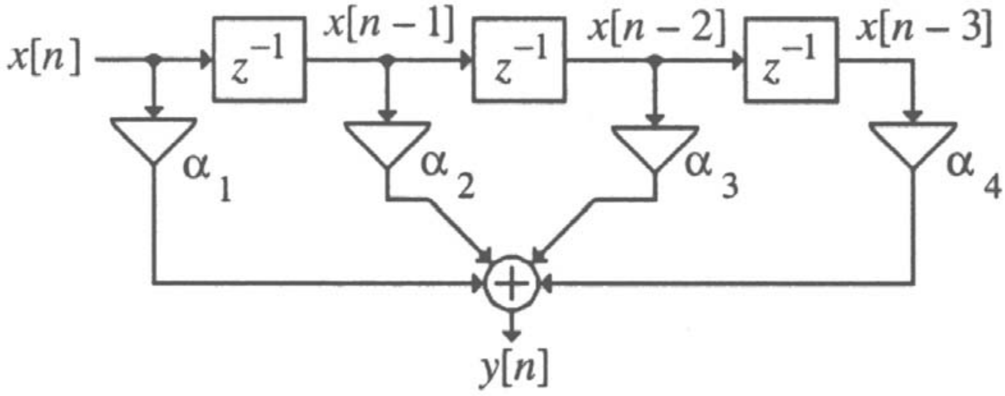


- A discrete time signal with discrete valued amplitudes represented by a finite number of digits is referred to as a **digital signal**.
  - A digital signal is thus a quantized sampled-data signal.



Combinations of Basic Operations

- Example:



$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

Classification of Sequences

- Based on Symmetry
  - Conjugate-symmetric: $x[n] = x^*[-n]$
  - Conjugate-antisymmetric: $x[n] = -x^*[-n]$
- Base on Periodicity
  - A sequence $\tilde{x}[n]$ satisfying

$$\tilde{x}[n] = \tilde{x}[n + kN] \quad \text{for all } n$$

is called a periodic sequence with a period N.

Basic Sequences

- Unit impulse $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
- Unit Step $\mu[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
- Exponential $x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$
- An arbitrary sequency $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$
- Discrete Sinusoids $x[n] = A\cos(\omega_0 n + \varphi)$
  - Periodicity with respect to time n
  - Periodicity with respect to frequency ω_0

Discrete Time System

- **Accumulator**
  - $y[n] = \sum_{l=-\infty}^{n} x[l] = y[n-1] + x[n] = y[-1] + \sum_{l=0}^{n} x[l]$
- ***M*-point Moving-Average Filter**
  - $y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$
- **Linear Interpolator (factor of 2)**
  - $y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$
- **Median Filter**
  - $y[n]=\text{med}\{x[n-k], \ldots, x[n-1], x[n], x[n+1], \ldots, x[n+k]\}$
- **Compressor**
  - $y[n] = x[Mn]$ for $M > 1$

Time domain representation of Discrete time System

- Input – output relation

Properties of Discrete time System

- Linearity
- Causality
- Memoryless
- Time-invariance
- BIBO-stability
- Passive and Lossless properties

Linear and Time-invariant (LTI) System

- What is impulse response
- How to find the impulse response of a given system
- Why impulse response can fully characterize an LTI system
- How to compute convolution
- How to compute the output of an LTI system, given the input sequence and impulse response
- Why only LTI system can use convolution to compute the output sequence
- BIBO and causal properties of LTI system
- General difference equation

Properties of LTI System

- An LTI system is BIBO stable iff $h[n]$ is absolutely summable

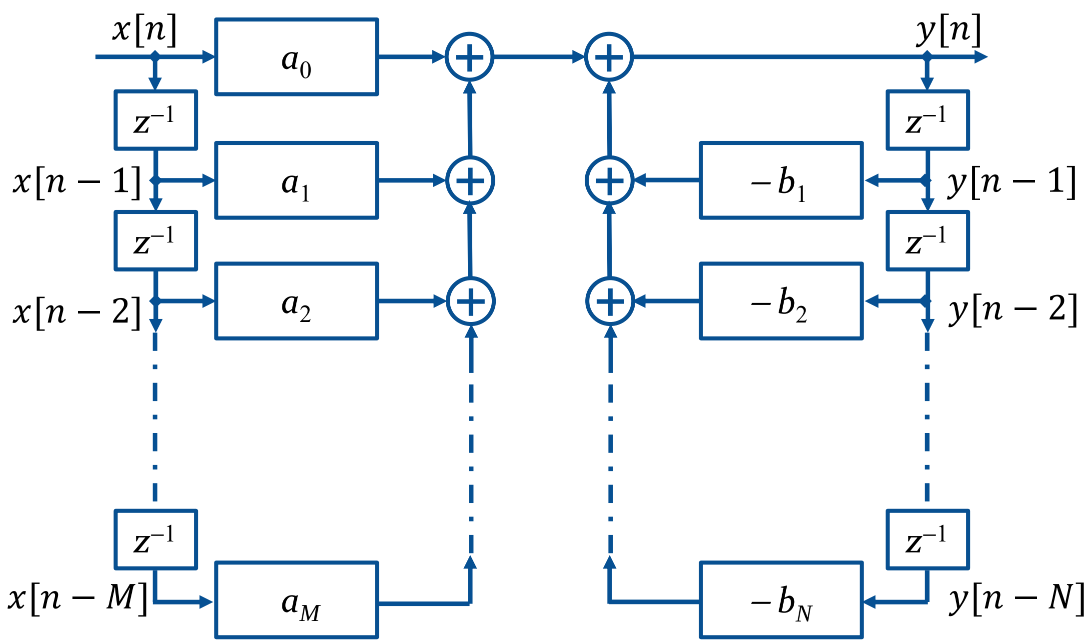
$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

- An LTI system is causal iff

$$h[k] = 0 \qquad \text{for } k < 0$$

Difference Equation

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$



Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{DTFT}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{Inverse DTFT}$$

- $X(e^{j\omega})$ is a complex function of the real variable ω

  $X(e^{j\omega}) = X_{\mathrm{re}}(e^{j\omega}) + jX_{\mathrm{im}}(e^{j\omega})$; $\quad X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$
- Symmetrical and periodic

  $|X(e^{j\omega})| = |X(e^{-j\omega})|$ and $\theta(\omega) = -\theta(-\omega)$ for real $x[n]$

  $X(e^{j\omega}) = X(e^{j(\omega+2k\pi)})$, i.e.,

  $$|X(e^{j\omega})|e^{j[\theta(\omega)+2k\pi]} = |X(e^{j\omega})|e^{j\theta(\omega)}$$

Properties of the DTFT

Assume: $x[n] \leftrightarrow X(e^{j\omega}),\ h[n] \leftrightarrow H(e^{j\omega}),\ y[n] \leftrightarrow Y(e^{j\omega})$

- **Time Reversal:** $\quad x[-n] \leftrightarrow X(e^{-j\omega})$
- **Time and Frequency Shifting**

  $x[n-n_d] \leftrightarrow e^{-j\omega n_d}X(e^{j\omega})$; $\quad e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$
- **Differentiation in frequency** $\quad nx[n] \leftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$
- **Convolution**

  $$y[n] = x[n] \circledast h[n] \Rightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

  $$y[n] = x[n]h[n] \Rightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$$
- Parseval's theory:

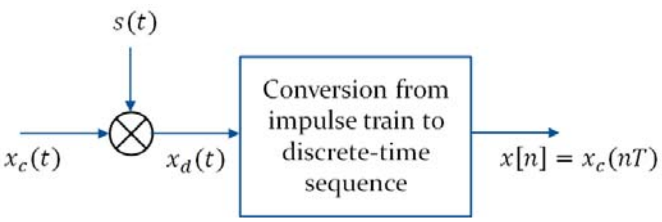
  $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

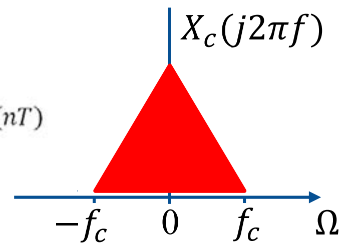
DTFT Convergence

Sequence		DTFT
$\alpha^n \mu[n], (\lvert\alpha\rvert < 1)$ Absolutely Summable	Sufficient $\longrightarrow$	$\dfrac{1}{1-\alpha e^{-j\omega}}$ Exist for all ω
$\mu[n]$ Neither absolutely summable, nor finite energy	Not necessary $\longrightarrow$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$ Not exist for $\omega = 0$
1 (for all n) Neither absolutely summable, nor finite energy		$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ Exist for all ω
$h_{LP}[n] = \dfrac{\sin\omega_c n}{\pi n}$ Finite energy	Sufficient $\longrightarrow$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, 0 \le \lvert\omega\rvert \le \omega_c\pi \\ 0, \omega_c\pi \le \lvert\omega\rvert \le \pi \end{cases}$ Exist for all ω

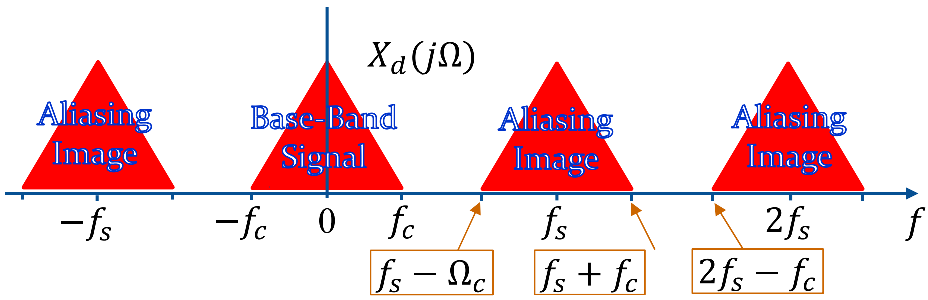
Effect of Time-Domain Sampling in Frequency Domain

$$X_d(j2\pi f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(2\pi j(f - nf_s))$$





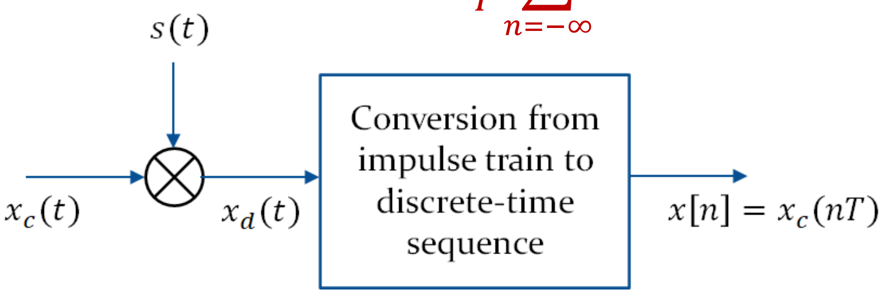
$X_c(j2\pi f)$, frequency spectrum of the original continuous time signal, the base-band signal



The condition to fully recover the continuous signal:

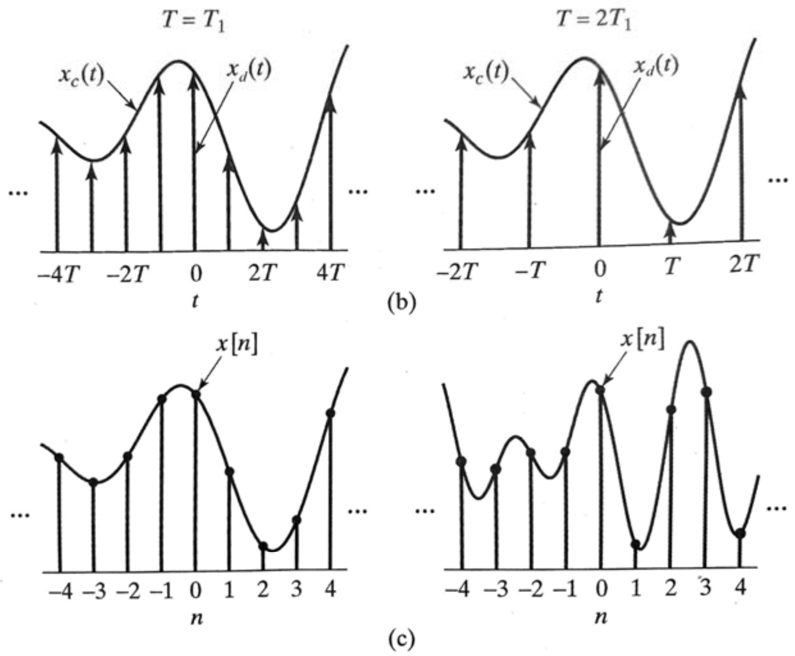
$$f_s \ge 2f_c$$

$$X_d(j2\pi f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(2\pi j(f - nf_s))$$



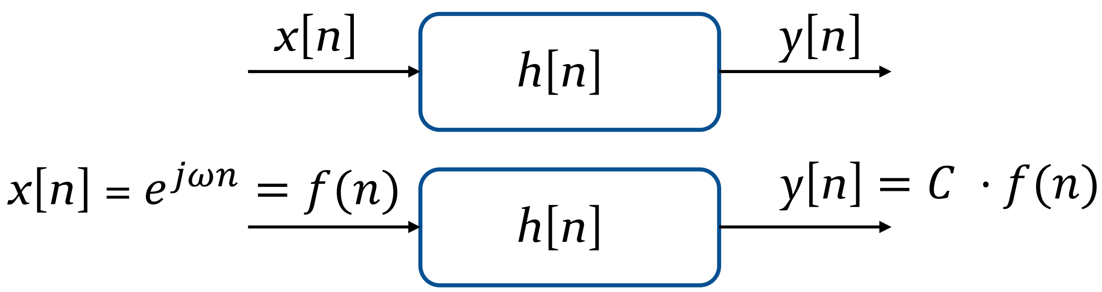
$X(e^{j\omega})$ is a frequency scaled version of $X_d(j\Omega)$, $\omega = \Omega T$

A normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_d(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$.



a result of the time normalization in the transformation from $x_d(t)$ to $x[n]$.

Eigenfunction



$$y[n] = h[n] \circledast e^{j\omega n} = \left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

- Let

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

the **frequency response** of the LTI system with impulse response $h[n]$

Frequency-Domain Characterization of LTI System

$$y[n] = x[n] \circledast h[n] \qquad x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \qquad X(e^{j\omega}) \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow Y(e^{j\omega})$$

FIR system $$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}, \qquad y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k],$$

IIR system $$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{m=0}^{M} a_m e^{-j\omega m}}{\sum_{m=0}^{N} b_m e^{-j\omega m}}$$

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

DFT & IDFT

DFT:

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{j2\pi kn}{N}} = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]W_N^{kn},$$

where, $W_N = e^{-j2\pi/N}$, for $k = 0, 1, \dots, N-1$

IDFT:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}, \qquad n = 0, 1, \dots, N-1.$$

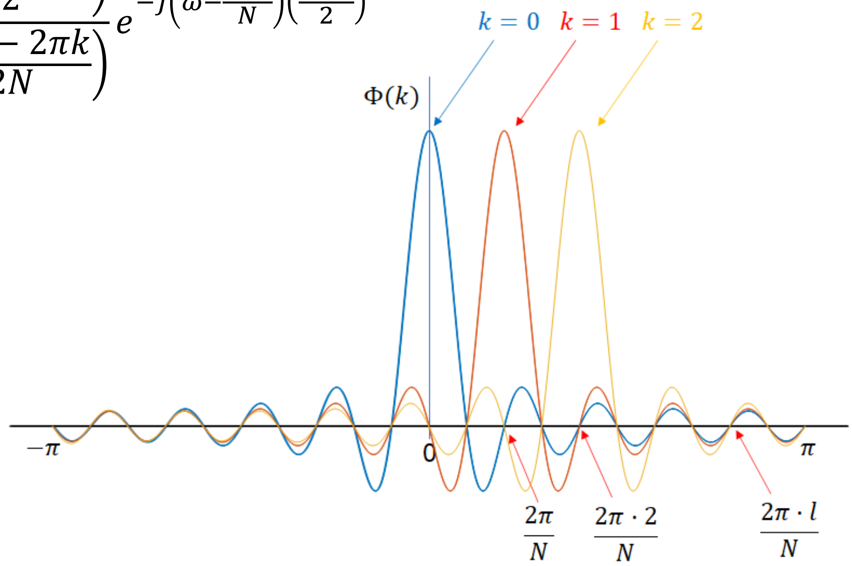
DTFT vs. DFT

- Q: Can we reconstruct the DTFT spectrum from the DFT?

$$x[n] \xrightarrow{DFT} X[k] \xrightarrow{?} X(e^{j\omega})$$

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j\left(\omega - \frac{2\pi k}{N}\right)\left(\frac{N-1}{2}\right)}$$

$$\Phi(k) = \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)}$$



Sampling the DTFT

- Consider a length M sequence $x[n]$ $(0 \leq n \leq M-1)$ going through the following transforms and operations:

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}) \xrightarrow[\text{}]{\substack{\text{Sample} \\ N \text{ points}}} Y[k] \xrightarrow{\substack{N\text{−point} \\ \text{IDFT}}} y[n]$$

Find the relation between $x[n]$ and $y[n]$.

-

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \qquad 0 \leq n \leq N-1$$

Finite-Length Sequence

- **circular time-reversal**

$$y[n] = x[\langle -n \rangle_N], \text{ for } 0 \le n \le N-1$$

- **circular conjugate (anti-)symmetric sequence**

$$x[n] = x^*[\langle -n \rangle_N] = x^*[\langle N-n \rangle_N]$$
$$x[n] = -x^*[\langle -n \rangle_N] = -x^*[\langle N-n \rangle_N]$$

- **circular Shift**

$$x_c[n] = x[\langle n-m \rangle_N]$$

- **circular convolution**

$$y_c[n] = x[n] \circledN h[n] \triangleq \sum_{m=0}^{N-1} x[m]h[\langle n-m \rangle_N]$$

Examples

- If $x[n] \leftrightarrow X[k]$, then $x^*[n] \leftrightarrow X^*[\langle -k \rangle_N]$
- If $x[n] \leftrightarrow X[k]$, then $X[\langle -k \rangle_N] = X[\langle N-k \rangle_N] = X^*[k]$
- If $x[n] \leftrightarrow X[k]$ for $0 \le n \le N-1$, and

  $g[n] = x[2n]$, $h[n] = x[2n+1]$, $g[n] \leftrightarrow G[k]$, $h[n] \leftrightarrow H[k]$ for $0 \le n \le N/2-1$

  then $X[k] = G\left[\langle k \rangle_{\frac{N}{2}}\right] + W_N^k H\left[\langle k \rangle_{\frac{N}{2}}\right],\ 0 \le k \le N-1$

Linear Convolution using DFT

- Zero-pad $x[n]$ by P–1 zeros

$$x_{zp}[n] = \begin{cases} x[n] & 0 \le n \le L-1 \\ 0 & L \le n \le L+P-2 \end{cases}$$

- Zero-pad $h[n]$ by L–1 zeros

$$h_{zp}[n] = \begin{cases} h[n] & 0 \le n \le P-1 \\ 0 & P \le n \le L+P-2 \end{cases}$$

- compute the linear convolution using a circular one with length $M = L+P-1$

$$y[n] = h[n] * x[n] = x_{zp}[n] \, \text{Ⓝ} \, h_{zp}[n]$$

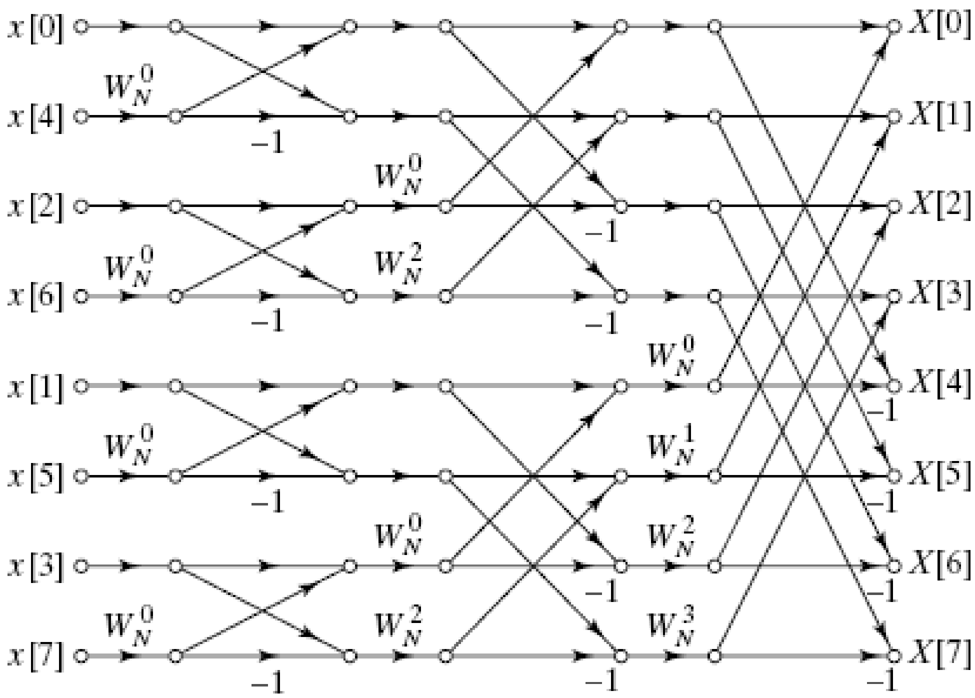
- implement a circular convolution using the DFT property:

$$h[n] * x[n] = x_{zp}[n] \, \text{Ⓝ} \, h_{zp}[n]$$
$$= \mathcal{IDFT}\{\mathcal{DFT}\{x_{zp}[n]\} \cdot \mathcal{DFT}\{h_{zp}[n]\}\}$$

Fourier-Domain Filtering

- Find the DTFT of the signal to get $X(e^{j\omega})$, multiply with $H(e^{j\omega})$ to obtain $Y(e^{j\omega})$, and find the IDTFT of $Y(e^{j\omega})$
- We can use DFT to compute $X(e^{j\omega})$ and $Y(e^{j\omega})$ at frequency values of $\omega = 2\pi k/N$, for , $k = 0, 1, \dots, N-1$
- This approach is equivalent to the **circular convolution** of the finite-length signal $x[n]$ and the finite-length ideal filter $h[n]$.
- However, the ideal filter has an infinite length impulse response. Sampling the Fourier transform to create DFT samples leads to the **time domain aliasing**.

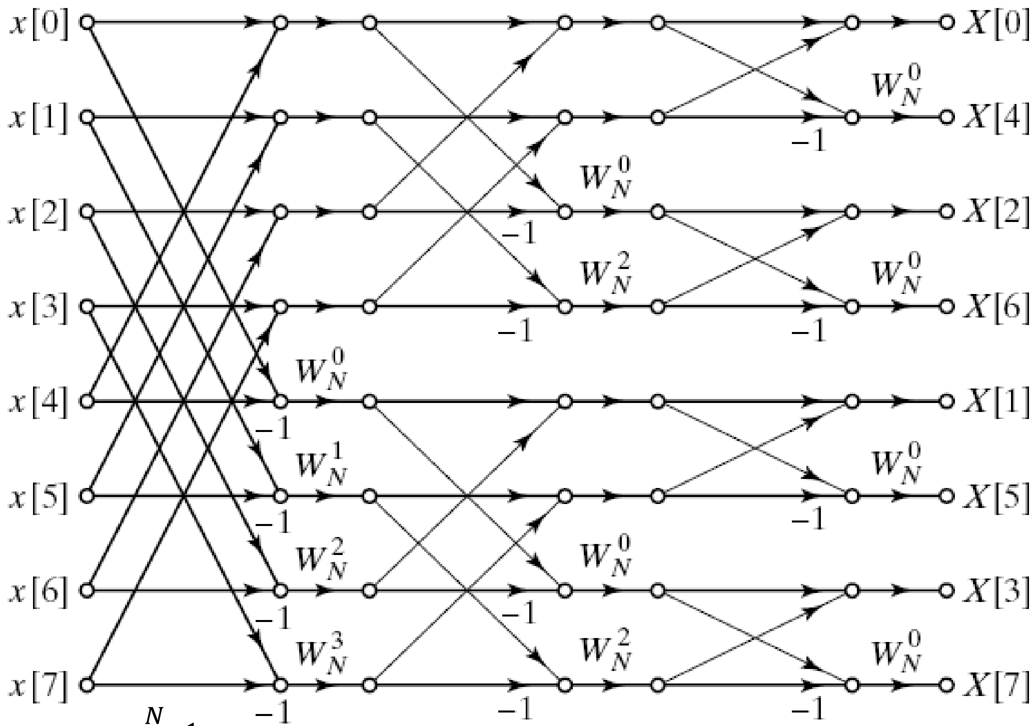
Decimation-in-time FFT of an 8-point DFT



$$X_0[k] = X_{00}[\langle k\rangle_{N/4}] + W_{N/2}^k\, X_{01}[\langle k\rangle_{N/4}], 0 \le k \le N/2 - 1$$

$$X_1[k] = X_{10}[\langle k\rangle_{N/4}] + W_{N/2}^k X_{11}[\langle k\rangle_{N/4}], 0 \le k \le N/2 - 1$$

Decimation-in-frequency FFT of an 8-point DFT



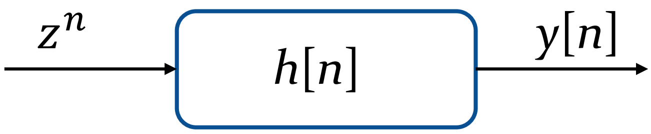
$$X[2r] = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + x\left[n + \frac{N}{2}\right]\right) X_{\frac{N}{2}}^{nr}, \qquad r = 0,1,2,\ldots,\frac{N}{2} - 1$$

$$X[2r+1] = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] - x\left[n + \frac{N}{2}\right]\right) W_N^n X_{\frac{N}{2}}^{nr}, \qquad r = 0,1,2,\ldots,\frac{N}{2} - 1$$

Cost to compute an N-point DFT

- Using the simplified butterfly computation, the number of complex multiplications performed at each stage is reduced to $N/2$. Thus the total numbers become $Nv/2 = \frac{N}{2}\log_2 N$
- By excluding trivial complex multiplications with $W_N^0 = 1$ and $W_N^{N/2} = -1$, the exact count of non-trivial complex multiplications are even less, given by $\frac{N}{2}(\log_2 N - 2) + 1$

z-Transform



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right) z^n = H(z)z^n$$

$$H(z) = Z\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m], \qquad H(z) \triangleq \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{\sum_{m=0}^{N} b_m z^{-m}}$$

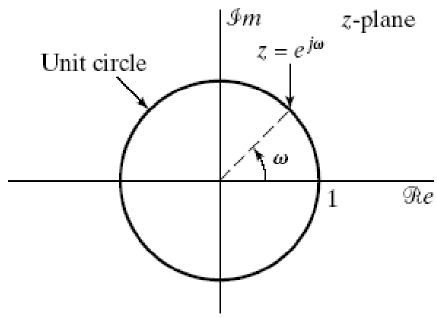
z-Transform

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Let $z = re^{j\omega}$, then the expression reduces to

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-j\omega n},$$

- If $r = 1$(i.e., $|z| = 1$), the z-transform reduces to DTFT.

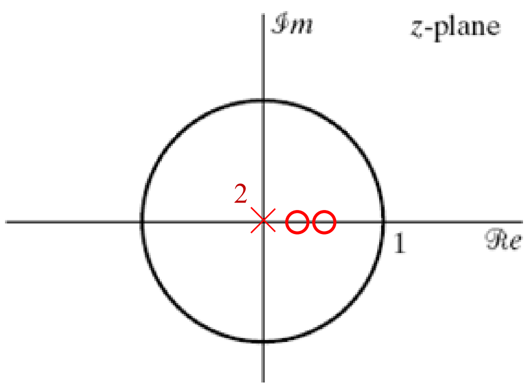


Rational z-transform

- LTI system with z-transforms represented as a rational function of z^{-1}

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1}z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1}z^{-(N-1)} + d_N z^{-N}}$$

$$H(z) = 6\frac{\left(z - \frac{1}{3}\right)\left(z - \frac{1}{2}\right)}{z^2}$$



Region of Convergence

- Example 1: Right-sided sequence $x[n] = a^n \mu[n]$

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{for } |az^{-1}| < 1$$

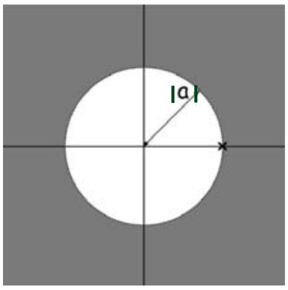
  - ROC = $\{z\!:\ |z| > |a|\}$

- Example 3: Left sided sequence $x[n] = -a^n \mu[-n-1]$

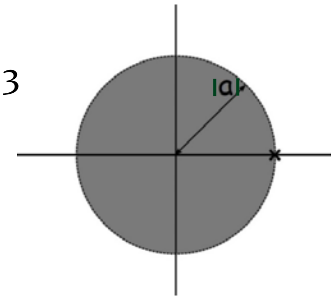
$$X(z) = \frac{1}{1 - az^{-1}}, \qquad \text{for } |a^{-1}z| < 1$$

  - ROC = $\{|z| < |a|\}$

Example 1



Example 3



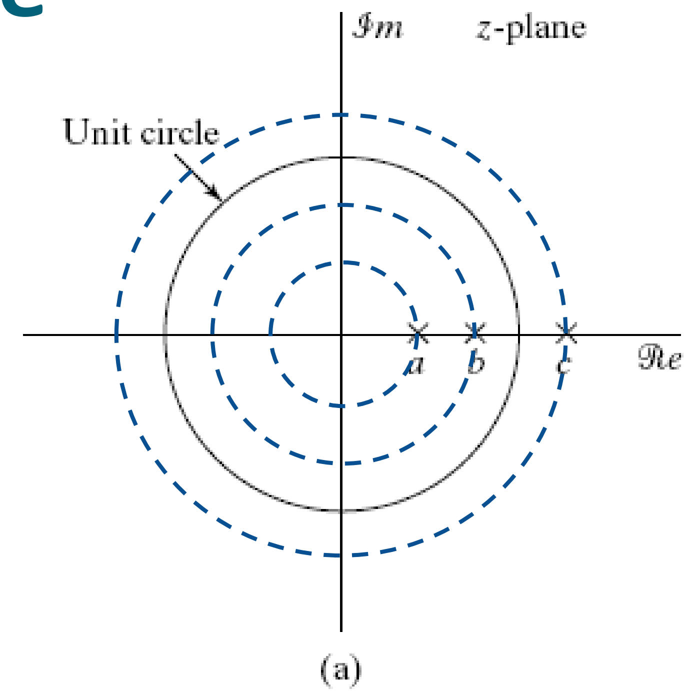
Properties of ROC

- In general, ROC of a z-transform is in a form:

$$R_{x^-} < |z| < R_{x^+}, \qquad \text{an annular region}$$

  - For right-sided sequences: $|z| > R_{\text{maxpole}}$
  - For left-sided: $|z| < R_{\text{minpole}}$.
  - For two-sided: $R_{x^-} < |z| < R_{x^+}$ or does not exist.
  - For finite duration sequences: the entire z-plane, except possibly z=0, z=∞

Example



(a) A system with three poles.

ROC for LTI System

- Consider the transfer function $H(z)$ of a linear system:
  - If the system is stable, ROC must include the unit circle.
  - If the system is causal, $|z| > R_{\text{maxpole}}$.
  - Therefore, **a stable causal LTI system has all poles inside unit circle**.

Inverse z-transform

- By Inspection

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, |z| < |\alpha| \to x[n] = \alpha^n \mu[n]$$

$$|z| > |\alpha| \to x[n] = -\alpha^n \mu[-n-1]$$

- Partial fraction expansion

$$X(z) = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}} = \sum_{l=0}^{M-N} \eta_l z^{-l} + \frac{P_1(z)}{D(z)} = \sum_{l=1}^{N} \left(\frac{\rho_l}{1 - \lambda_l z^{-1}} \right)$$

$$\rho_l = (1 - \lambda_l z^{-1})\, X(z)\Big|_{z=\lambda_l}$$

- Power series expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + \cdots$$

Frequency Response from Transfer Function

- If the ROC of $H(z)$ includes the unit circle

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})$$

$$= H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

- A stable causal LTI system has all poles inside unit circle.

Linear Phase FIR Filter

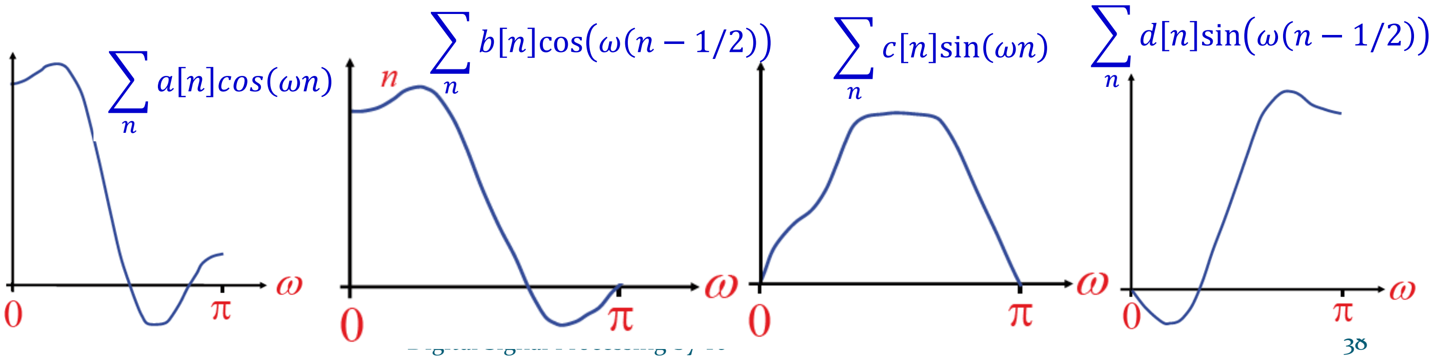
- Its frequency response is given by $e^{-j\left(\frac{N-1}{2}\omega-\beta\right)}R(\omega)$, where $R(\omega)$ is a real function.
- The group delay is $-d\{\theta(\omega)\} = \frac{N-1}{2} = \alpha$.
- Simple example

$$H(z) = \frac{1+z^{-1}}{2} \leftrightarrow \{h[n]\} = \left\{\underset{\uparrow}{\frac{1}{2}}, \frac{1}{2}\right\} \leftrightarrow e^{-j\omega/2}\cos(\omega/2)$$

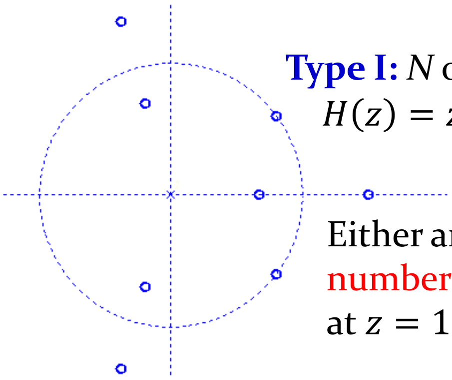
$$H(z) = \frac{1-z^{-1}}{2} \leftrightarrow \{h[n]\} = \left\{\underset{\uparrow}{\frac{1}{2}}, -\frac{1}{2}\right\} \leftrightarrow e^{j(\pi/2-\omega/2)}\sin(\omega/2)$$

Four type linear phase FIR filters

- Classified based on the symmetry of impulse response and parity of filter length
  - Type I: Odd length, symmetric impulse response
  - Type II: Even length, symmetric impulse response
  - Type III: Odd length, anti-symmetric impulse response
  - Type IV: Even length, anti-symmetric impulse response
- Frequency response of the 4 type FIR filters

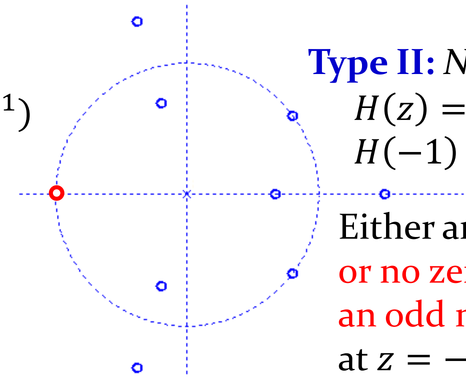


Zero-Locations of FIR Filters



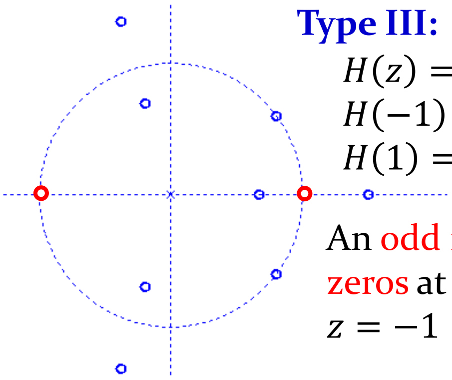
Type I: N odd
$H(z) = z^{-(N-1)}H(z^{-1})$

Either an even number or no zeros at $z = 1$ and $z = -1$



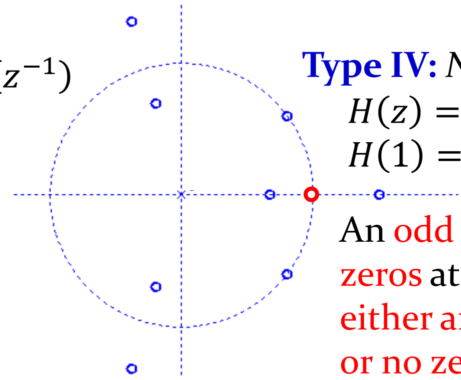
Type II: N even
$H(z) = z^{-(N-1)}H(z^{-1})$
$H(-1) = -H(-1)$

Either an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$



Type III: N odd
$H(z) = -z^{-(N-1)}H(z^{-1})$
$H(-1) = -H(-1)$
$H(1) = -H(1)$

An odd number of zeros at $z = 1$ and $z = -1$



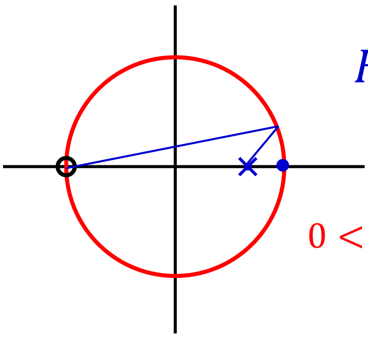
Type IV: N even
$H(z) = -z^{-(N-1)}H(z^{-1})$
$H(1) = -H(1)$

An odd number of zeros at $z = 1$ and either an even number or no zeros at $z = -1$

Simple and improved IIR filters

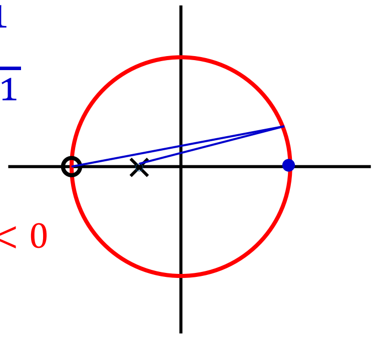
$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \qquad 0 < |\alpha| < 1,$$

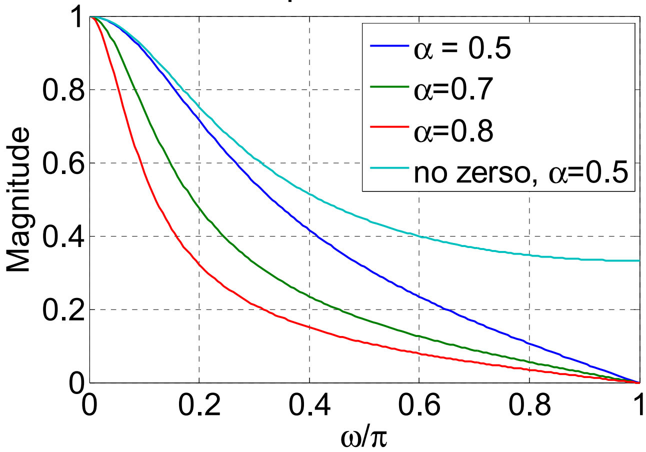
$$H(z) = \frac{1-\alpha}{2}\,\frac{1 + z^{-1}}{1 - \alpha z^{-1}}$$

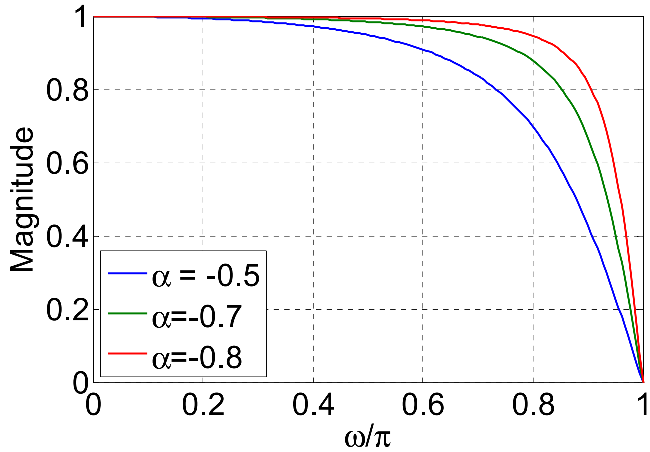


$0 < \alpha < 1$

$-1 < \alpha < 0$







Bandpass IIR Digital Filter

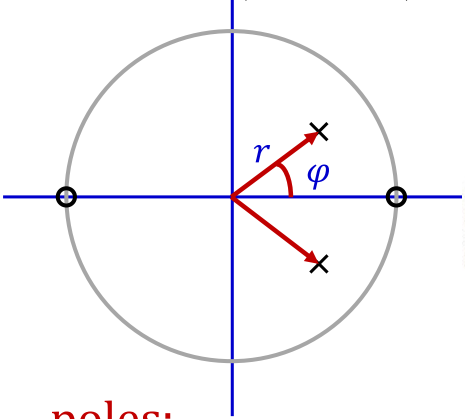
- A 2nd-order general form

$$H_{BP}(z) = \frac{K(1 - z^{-2})}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$

$r = \sqrt{\alpha}$

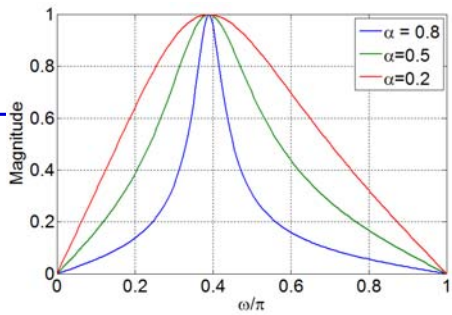
$\varphi = \cos^{-1}\left(\frac{\beta(1+\alpha)}{2\sqrt{\alpha}}\right)$

For stability, we have $r < 1 \;\rightarrow\; |\alpha| < 1$



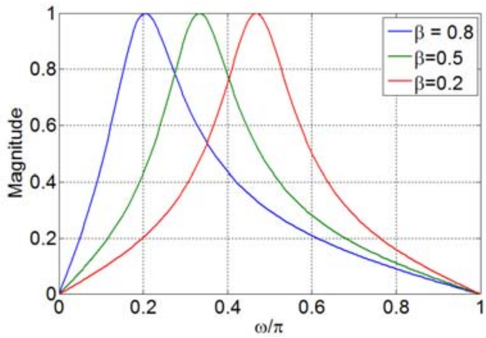
poles:
$z = re^{\pm j\varphi}$

$\beta = 0.34$



$\omega_0 = \cos^{-1}\beta$

$\alpha = 0.6$

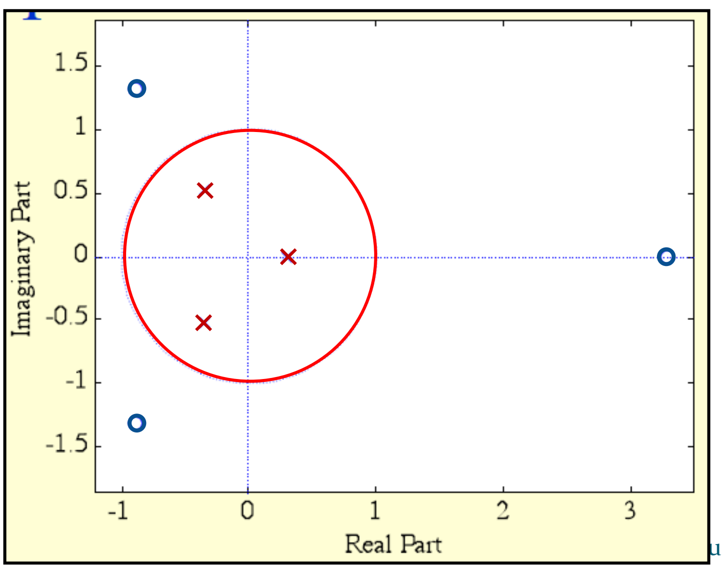


$B_w = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$

Allpass Filter

- An M-th order causal **real-coefficient** allpass transfer function is of form

$$A_M(z) = \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}} = \frac{z^{-M}D_M(z^{-1})}{D_M(z)}$$

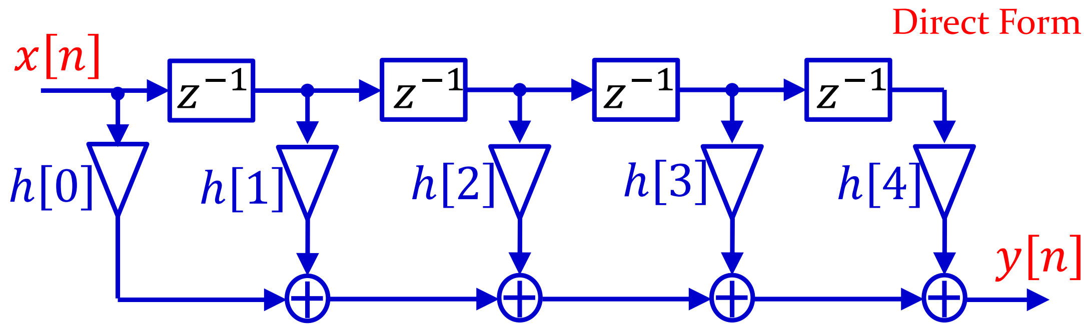


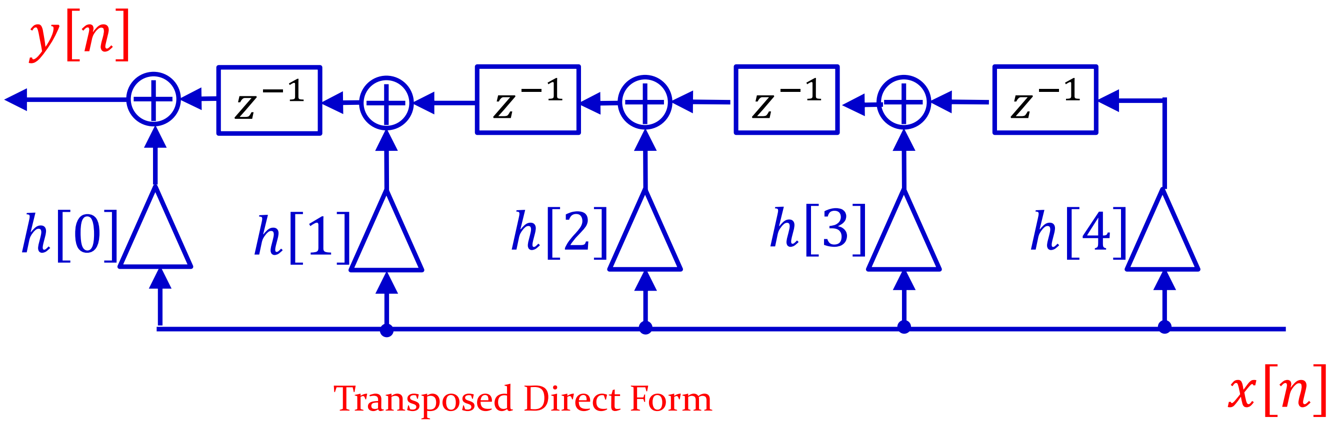
$$\left|A\left(e^{j\omega}\right)\right|^2 = 1, \text{ for all } \omega$$

The poles of a causal stable transfer function must lie inside the unit circle in z-plane

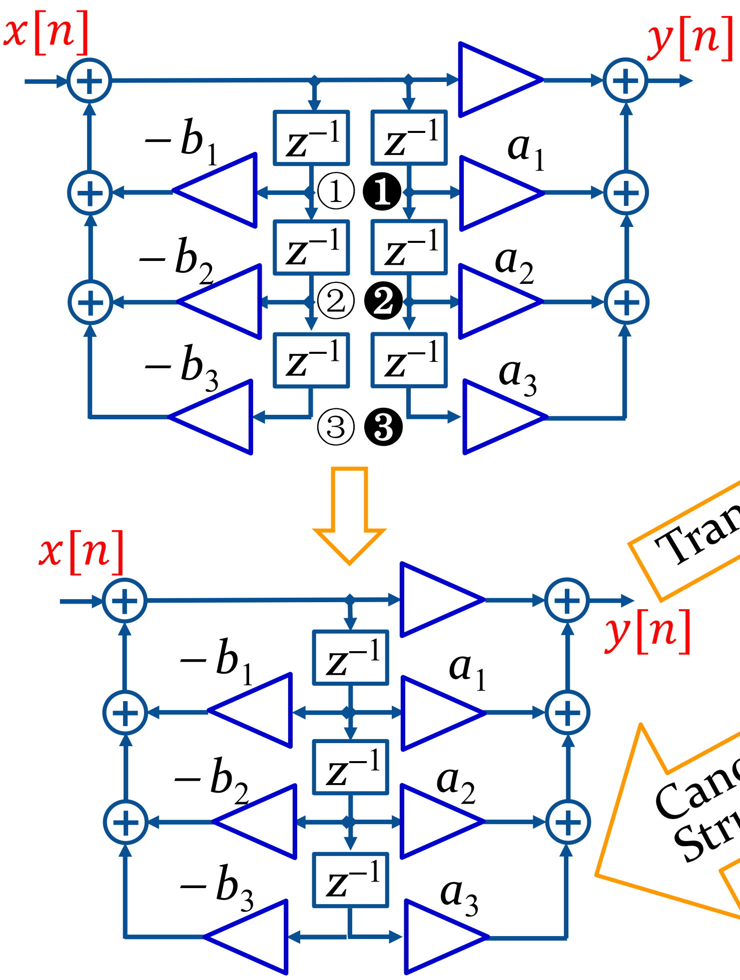
- FIR Filter Structures
  - Direct form and
  - Transposed direct form

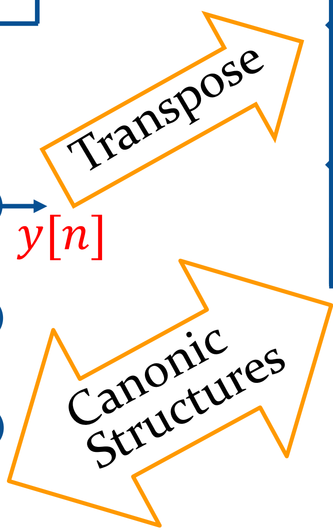
$$Y(z) = H(z)X(z) = \sum_{k=0}^{N-1} h[k]z^{-k}X(z)$$

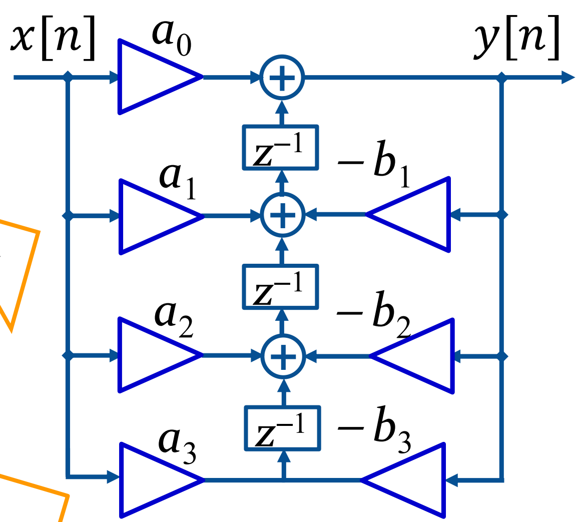




IIR Filter Structures

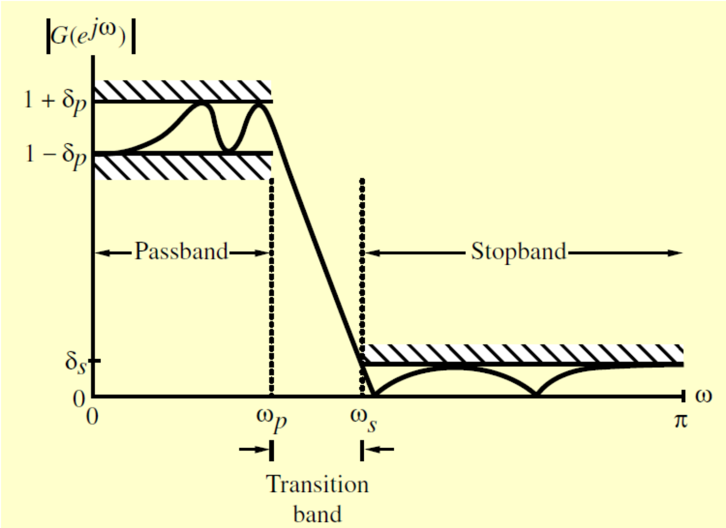






Cascade Structure
Parallel Structure

Typical magnitude Specifications



- **Passband:**

  $\omega \leq \omega_p$,

  $1-\delta_p \leq |G(e^{j\omega})| \leq 1+\delta_p$
- **Stopband:**

  $\omega_s \leq \omega \leq \pi$,

  $|G(e^{j\omega})| \leq \delta_s$
- **Transition band:**

  $\omega_p < \omega < \omega_s$,

  arbitrary response

IIR Filter Design

Transform $H(S)$ into the desired digital transfer function $G(z)$

- ✓ Imaginary ($j\Omega$) axis in the s-plane be mapped onto the **unit circle** of the z-plane
- ✓ Left-half of the s-plane be mapped **inside the unit circle**

Bilinear Transformation

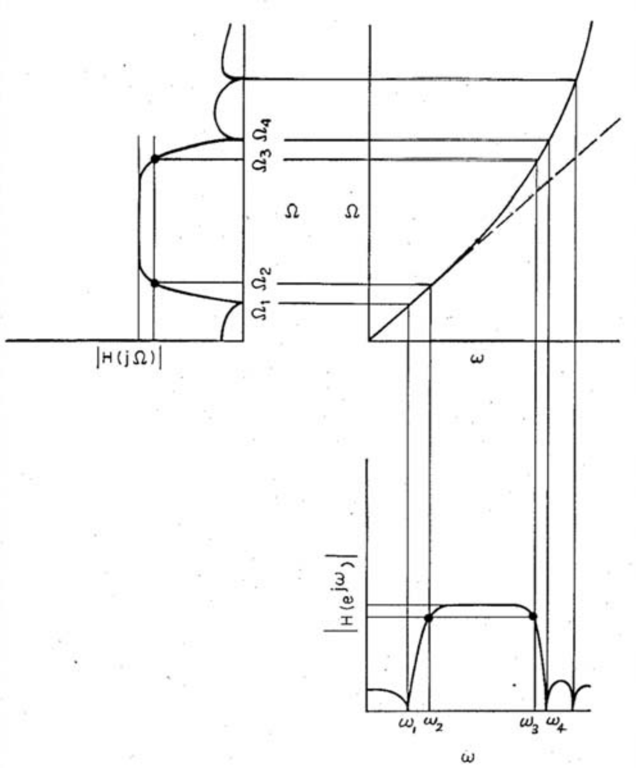
$$s = k\left(\frac{1-z^{-1}}{1+z^{-1}}\right), k > 0, \qquad \text{or } z = \frac{k+s}{k-s}$$

- Thus, relation between $G(z)$ and $H(s)$ is then given by

$$G(z) = H(s)\Big|_{s=k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Frequency Warping

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- • Effect of warping

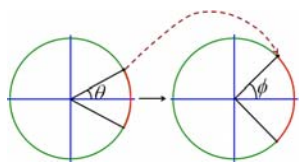


$$\omega = 2\tan^{-1}\frac{\Omega}{k}, \text{ or}$$

$$\Omega = k\tan\left(\frac{\omega}{2}\right)$$

Spectral Transformation

- **Lowpass to lowpass transformation:**

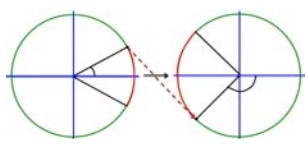


$$z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha\hat{z}^{-1}} \Leftrightarrow \hat{z}^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$

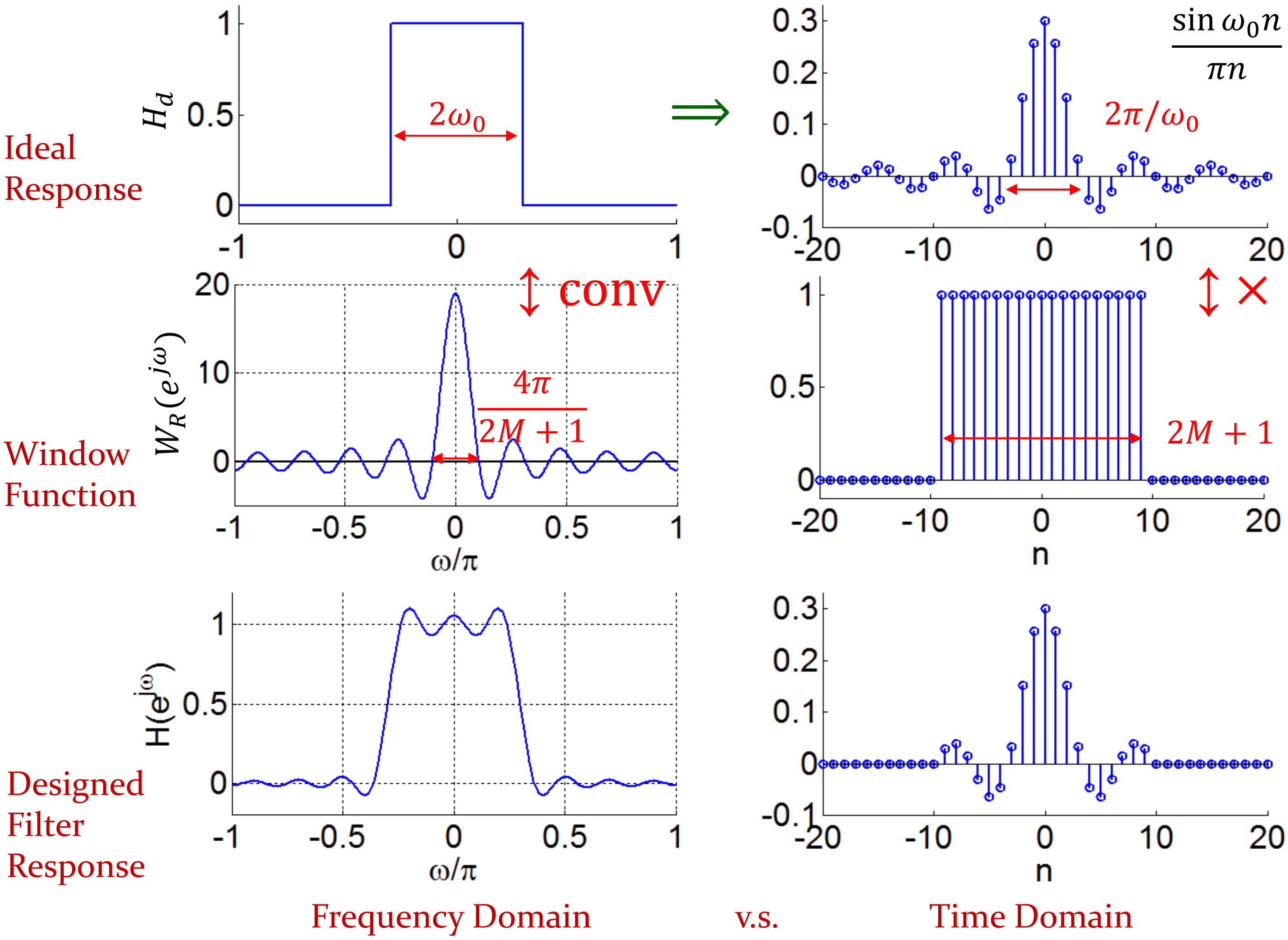
  - $z = e^{j\omega} \Longrightarrow |\hat{z}| = 1; \tan\left(\frac{\omega}{2}\right) = \left(\frac{1+\alpha}{1-\alpha}\right)\tan\left(\frac{\hat{\omega}}{2}\right);\ \alpha = \frac{\sin\left(\frac{\omega-\hat{\omega}}{2}\right)}{\sin\left(\frac{\omega+\hat{\omega}}{2}\right)}$

- Lowpass to highpass transformation

$$z^{-1} = -\frac{\hat{z}^{-1} + \alpha}{1 + \alpha\hat{z}^{-1}} \Leftrightarrow \hat{z}^{-1} = -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$



  - $z = e^{j\omega} \Longrightarrow |\hat{z}| = 1; \text{ctan}\left(\frac{\omega}{2}\right) = \left(\frac{-1+\alpha}{1+\alpha}\right)\tan\left(\frac{\hat{\omega}}{2}\right); \alpha = -\frac{\cos\left(\frac{\omega-\hat{\omega}}{2}\right)}{\cos\left(\frac{\omega+\hat{\omega}}{2}\right)}$



Properties of Fixed Windows

Type of Window	Window function		Resultant Filter	
	Main Lobe Width Δ_{ML}	Relative Side-lobe Level A_{sl}	Minimum Stop-band Attenuation δ	Transition Bandwidth $\Delta\omega$
Rectangular	$\frac{4\pi}{2M+1}$	13.3dB	20.9dB	$\frac{0.92\pi}{M}$
Hanning	$\frac{8\pi}{2M+1}$	31.5dB	43.9dB	$\frac{3.11\pi}{M}$
Hamming	$\frac{8\pi}{2M+1}$	42.7dB	54.5dB	$\frac{3.32\pi}{M}$
Blackman-Harris	$\frac{12\pi}{2M+1}$	58.1dB	75.3dB	$\frac{5.56\pi}{M}$

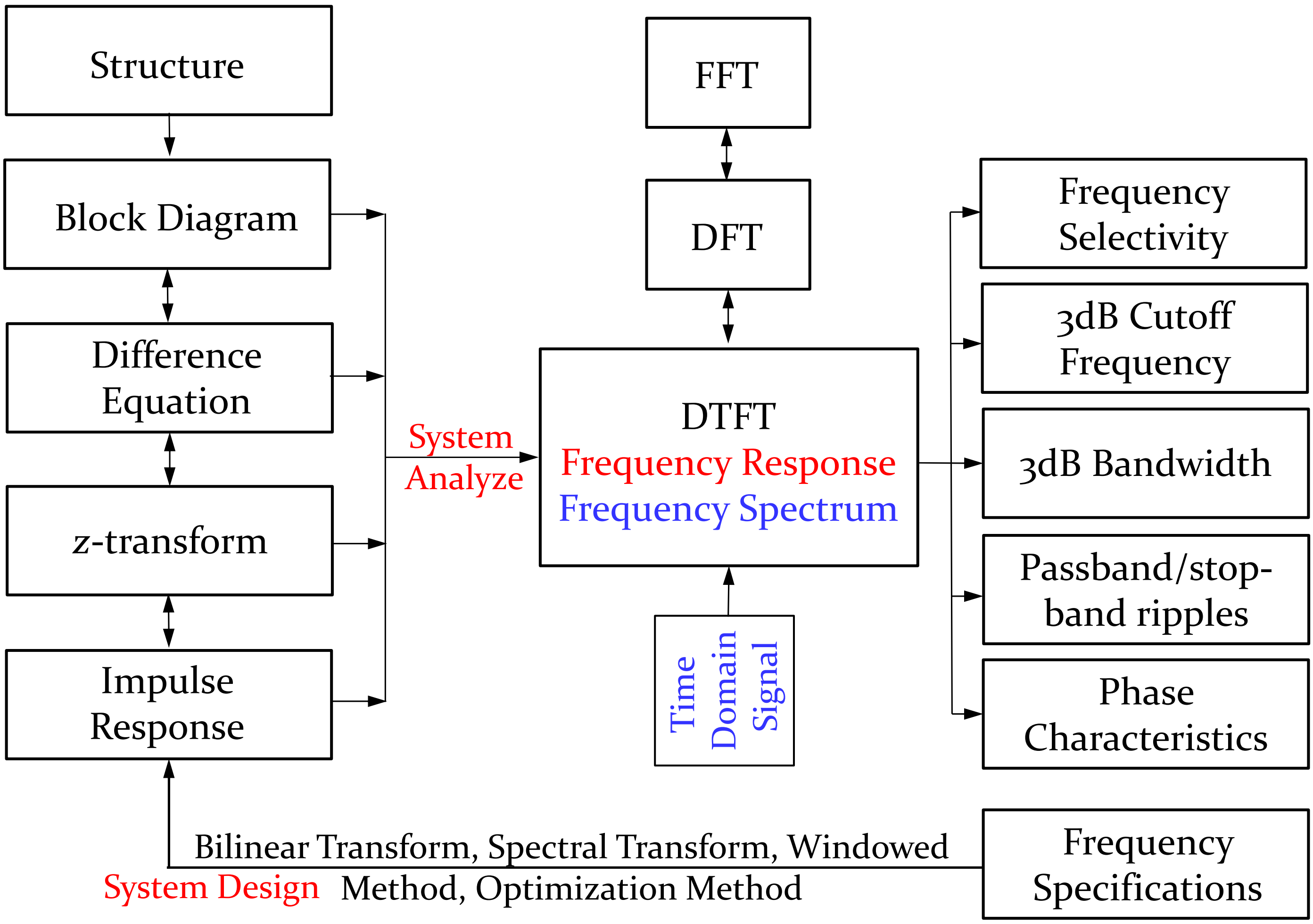
- δ is independent from M, or ω_c, and is essentially constant.
- $\Delta\omega = \frac{c}{M}$

Course Learning Outcomes

- CLO 1: I have an ability to **represent** discrete time signals and systems in time and frequency domain;
- CLO 2: I have an ability to **understand, represent, and analyse** linear time invariant discrete time systems in transformed domain by applying mathematics principles, such as **differential calculus, complex variables**.
- CLO 3: I have an ability to **analyse** digital filters and **design** digital filters to meet given specifications.
- CLO 4: I have an ability to **use a programming language** to conduct **analysis and design** of discrete-time signal processing systems to process discrete-time signals.

Exam coverage

- Convolution, circular convolution, DTFT, DFT, IDTFT, IDFT.
- Difference equation ↔ z-transform↔ impulse response ↔ frequency response ↔ structure block diagram
- z-transform and LTI system, BIBO stable, zeros and poles, ROC.
- Frequency response: magnitude response, bandwidth, 3-db cutoff frequency.
- FIR filter, simple IIR filter, allpass filter
- minimum phase, maximum phase
- Bilinear transformation for the design of IIR filter spectrum transform.
- Window method for the design of FIR filter



- $e^{x} = \cos x + j\sin x, \cos x = \frac{e^{x}+e^{-x}}{2}, \ \sin x = \frac{e^{x}-e^{-x}}{2j}$
- $\left|H\left(e^{j\omega}\right)\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = H(z)H^*(z)\big|_{z=e^{j\omega}}$

$$\xRightarrow{\boxed{\text{real impulse response}}} = H\left(e^{j\omega}\right)H\left(e^{-j\omega}\right)$$
$$= H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

- Revisit trigonometric formulas
- $\Omega = 2\pi f \xleftarrow{\text{analog}} f \text{ in Hz} \xrightarrow{\text{digital}} \omega = 2\pi \frac{f}{f_s}$
- When compute sin, cos values using calculator, note it is for degree or for radian.
- Filter length N or filter order N.
- **Skip difficult questions and answer as many questions as possible in the first round answering**.

Reminder:

You should manually practice the tutorial questions, instead of only reading or looking at the solutions.

2\. 微信端：通过微信进入“南方科技大学”微信企业号--教学质量管理平台。

3\. 在“我的任务-待评”中填写并提交本学期所选课程的所有听课评教表。操作指南请见附件（或扫描以下二维码获取）。**特别提醒：因本学期考试课程考查申请也使用该系统，请选择待评任务中首行显示教师姓名的任务进行评价，首行显示课程名称的任务无需填写评价（如下图所示）。**