Digital Signal Processing

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Textbook

- Sanjit K. Mitra, **Digital Signal Processing: A Computer-Based Approach**, 4th edition

Classical DSP Books

- Alan V. Oppenheim and Ronald W. Schafer, **Discrete-time Signal Processing**, Pearson
- Lawrence R. Rabiner and Bernard Gold, **Theory and Application of Digital Signal Processing**, Prince Hall
- J. Proakis, D. Manolakis, **Digital Signal Processing** 4th ed., Prentice-Hall, 2006

Contents

- Discrete-time signals and systems in the time domain
- Discrete-time signals and systems in the transform domain
  - Discrete-time Fourier transform
  - Discrete Fourier transform
  - z-Transform
  - Frequency response
- DSP algorithms implementation – Fast Fourier Transform (FFT)
- Digital filter structures
- IIR and FIR filter design

Assessment

- Assignments: 5%
- Two Quiz: 15%
- Laboratories: 30%
- Final Exam: 50%

Journals & Conferences in DSP

- Journals:
  - IEEE Transactions on Signal Processing (TSP)
  - IEEE Transactions on Circuits and Systems I (TCASI)
  - IEEE Transactions on Circuits and System II (TCASII)
  - IEEE Signal Processing Letter (SPL)
  - Signal Processing (Elsevier)
  - EURASIP Journal on Applied Signal Processing
  - Digital Signal Processing
  - Circuits Systems and Signal Processing (CSSP)

Journals & Conferences in DSP

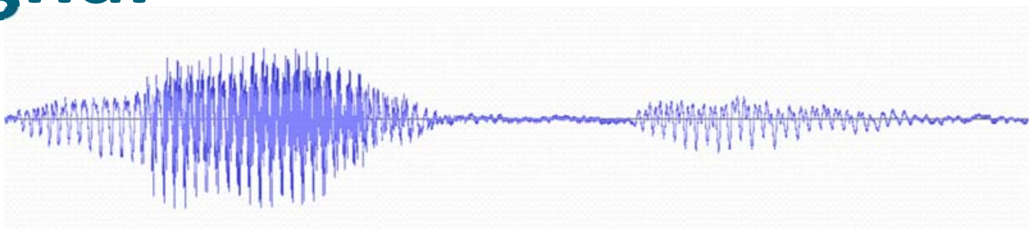
- Conference:
  - IEEE International Conference on Acoustic, Speech & Signal Processing (ICASSP)
  - IEEE International Symposium on Circuits & Systems (ISCAS)
  - European Signal Processing Conference (EUSIPCO)
  - International Conference on Digital Signal Processing (DSP)

Course Learning Outcomes

- CLO 1: I have an ability to **represent** discrete time signals and systems in time and frequency domain;
- CLO 2: I have an ability to **understand, represent, and analyse** linear time invariant discrete time systems in transformed domain by applying mathematics principles, such as **differential calculus, complex variables**.
- CLO 3: I have an ability to **analyse** digital filters and **design** digital filters to meet given specifications.
- CLO 4: I have an ability to **use a programming language** to conduct **analysis and design** of discrete-time signal processing systems to process discrete-time signals.

Lecture 1
Introduction

Signal



$f(t)$

$f(x, y)$

$f(t, x, y)$

Definition

- A signal can be defined simply as a mathematical function

$$y = f(x)$$

where x is the **independent variable** which specifies the domain of the signal.

- $y=\sin(\omega t)$ is a function of a variable in the **time domain** and is thus a time signal;
- An image $I(x,y)$ is in the spatial domain.

Signal Processing

- A signal carries information.
- The objective of signal processing:
  - Interpretation and information extraction. (e.g. speech recognition, machine learning, etc.)
  - Convert one signal to another. (e.g. filter, generate control command, etc.)

- Signal Processing concerns with:
  - The mathematical representation of the signal
  - The algorithmic operation carried out on the signal

Representation of Signal

- In terms of basis functions in the domain of original independent variable,
  - Time
  - Spatial, etc., or

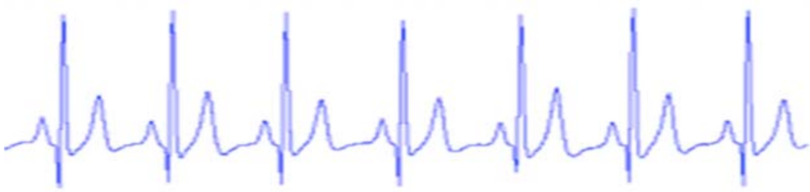
- In terms of basis functions in a transformed domain,
  - Discrete Fourier Transform
  - z transform, etc.

Classification of Signals

- Continuous vs. Discrete
- Real-valued vs. Complex-valued
- 1-D signal vs. 2-D signal vs. *M*-D signal
- Stationary vs. Non-stationary
- etc.

Characterization of Signals

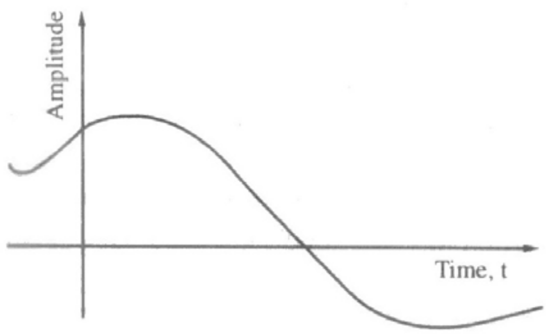
- The value of a signal at a specific value of the independent variable is called its **amplitude**.
- The variation of the amplitude as a function of the independent variable is called its **waveform**.
- Let's consider 1-D signal
  - The independent variable is usually labeled as **time**.



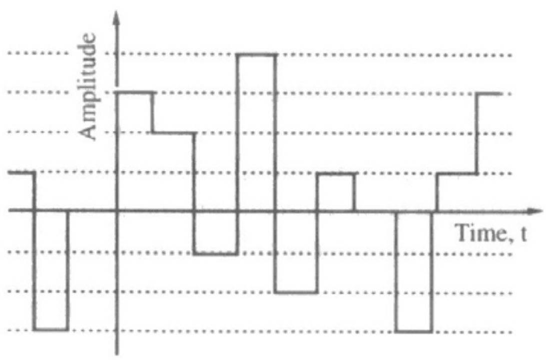
Continuous and Discrete Signals

- If the independent variable is continuous, the signal is called a **continuous-time (CT) signal**.
  - A continuous time signal is defined at every instant of time.
- If the independent variable is discrete, the signal is called a **discrete-time signal**.
  - A discrete time signal takes certain numerical values at specified discrete instants of time, and between these specified instants of time, the signal is **not defined**.
  - A discrete time signal is basically a sequence of numbers.

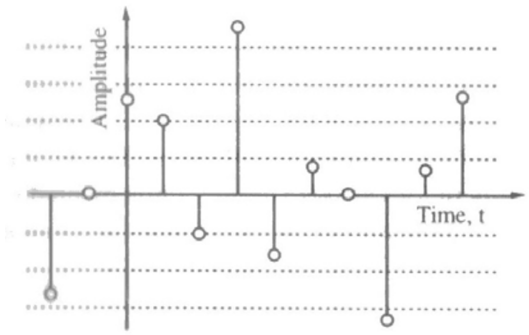
- A continuous-time signal with a continuous amplitude is usually called an **analog signal**.
  - A speech signal is an example of an analog signal.



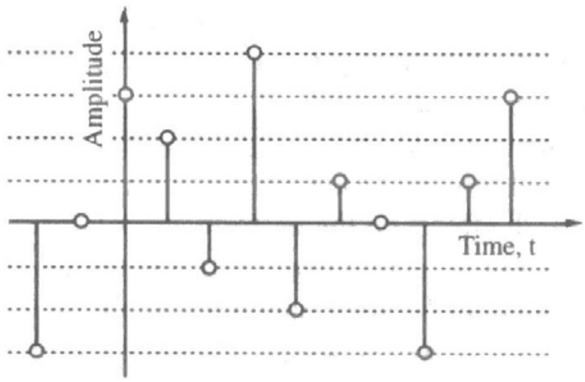
- A continuous-time signal with discrete valued amplitudes has been referred to as a **quantized boxcar signal**.
  - This type of signal occurs in digital electronic circuits where the signal is kept at fixed level (usually one of two values) between two instants of clocking.



- A discrete time signal with continuous valued amplitudes is called a **sampled-data signal**.
  - The amplitude of the signal may be any value.



- A discrete time signal with discrete valued amplitudes represented by a finite number of digits is referred to as a **digital signal**.
  - A digital signal is thus a quantized sampled-data signal.



Typical Signal Processing Operations

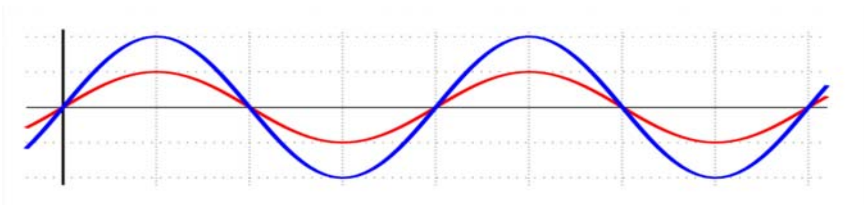
- In the case of analog signals, most signal processing operations are usually carried out in the **time domain**.
- In the case of discrete time signals, both **time domain and frequency domain** applications are employed.
- In either case, the desired operations are implemented by a combination of some **elementary operations** such as:
  - Simple time domain operations
  - Filtering
  - Amplitude modulation

Elementary Time-Domain Operations

- Three most basic time-domain signal operations
  - Scaling
  - Delay
  - Addition

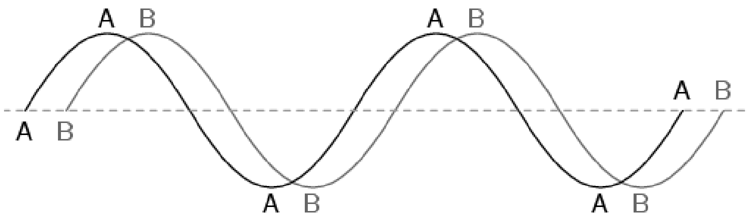
Scaling

- **Scaling** is simply the multiplication of a signal by a positive or a negative constant.
  - In the case of analog signal $x(t)$, the scaling operation generates a new signal $y(t) = \alpha x(t)$, where α is the multiplying constant.
  - The operation is called **amplification**, if $|\alpha| > 1$;
  - The operation is called **attenuation**, if $|\alpha| < 1$.



Delay

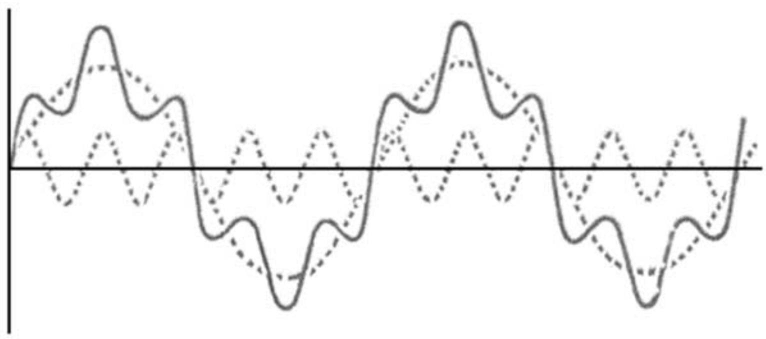
- **Delay** operation generates a signal that is delayed replica of the original signal.
  - In the case of analog signal $x(t)$, $y(t) = x(t - t_0)$ is the signal obtained by delaying $x(t)$ by the amount t_0, assuming $t_0 > 0$.



  - If $t_0 < 0$, it is an **advance** operation.

Addition

- **Addition** operation generates a new signal by the addition of signals. For instance, $y(t)$=$x_1(t)$+$x_2(t)$ is the signal generated by the addition of the three analog signals $x_1(t)$ and $x_2(t)$.



Why Learn DSP?

- Swiss-Army-Knife of modern EE
- Impacts all aspects of modern life
  - Communications (wireless, internet, GPS...)
  - Control and monitoring (cars, machines...)
  - Multimedia (mp3, cameras, videos, restoration ...)
  - Health (medical devices, imaging....)
  - Economy (stock market, prediction)
  - More....

DPS在信息技术中的地位和作用

信息获取 →	信息处理 →	信息传输 →	信息显示 →	信息应用
光电测试技术 检测技术 传感器技术	DSP技术与应用 误差理论与数据处理 测试信号分析与处理	网络与通信技术 仪器总线技术 接口电路	数据可视化 虚拟仪器技术	导航与控制技术 机器人技术 传感网与物联网 智能仪器 控制技术与系统

DPS在信息技术中的地位和作用

中国制造2025（高质量发展）离不开DSP技术的支撑

《中国制造2025》（高质量发展）是中国政府2015年3月提出的实施制造强国战略第一个十年的行动纲领。
围绕实现制造强国的战略目标，《中国制造2025》明确了9项战略任务和重点，提出了8个方面的战略支撑和保障。

五大工程：制造业创新中心建设工程、强化基础工程、智能制造工程、绿色制造工程、高端装备创新工程。
十个重点领域：新一代信息技术产业、高档数控机床和机器人、航空航天装备、海洋工程装备及高技术船舶、先进轨道交通装备、节能与新能源汽车、电力装备、农机装备、新材料、生物医药及高性能医疗器械。

Advantages of DSP

- Flexibility
- System/implementation does not age
- “Easy” implementation
- Reusable hardware
- Sophisticated processing
- Process on a computer
- (Today) Computation is cheaper and better

Example I: Audio Compression

- Compress audio by 10x without perceptual loss of quality.
- Sophisticated processing based on models of human perception
- 3MB files instead of 30MB -
  - Entire industry changed in less than 10 years!

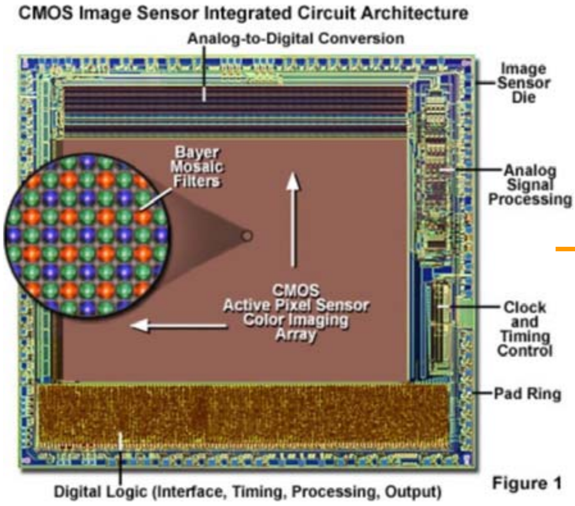
Historical Forms of Compression

- Morse code: dots (1 unit) Dashes (3 units)
  - Code Length inversely proportional to frequency
  - E (12.7%) = . (1 unit) Q (0.1%) = --.- (10 units)

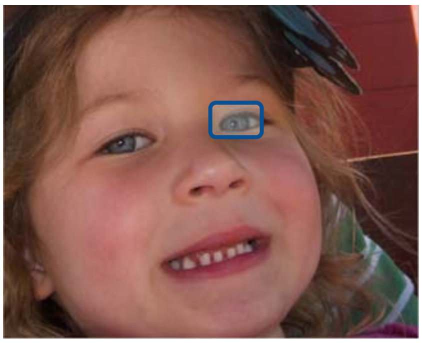
- "92 Code" - Used by Western-Union in 1859 to reduce BW on telegraph lines by numerical codes for frequently used phrases
  - 1 = wait a minute
  - 73 = Best Regards
  - 88 = Love and Kisses

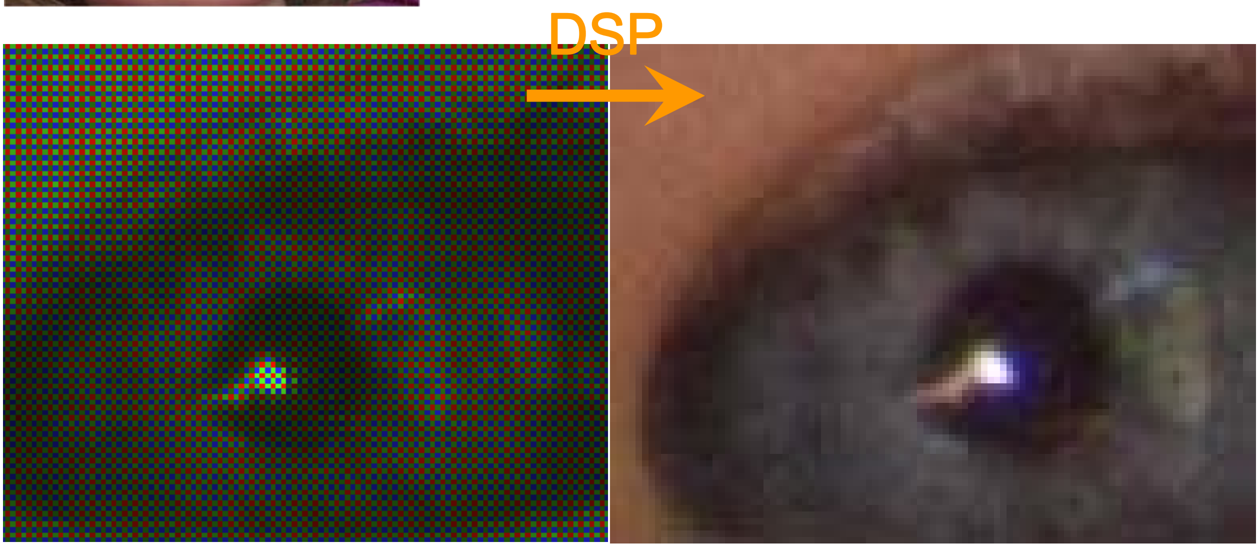
73	Best	Regards
--... ...--	-... . ... -	/ .-. . --. .- .-. -.. ...
19 units	59 units	

Example II: Digital Imaging Camera



Focus/exposure Control → Pre-processing → White Balancing → Demosaic → Color Transform → Post-processing → Compression

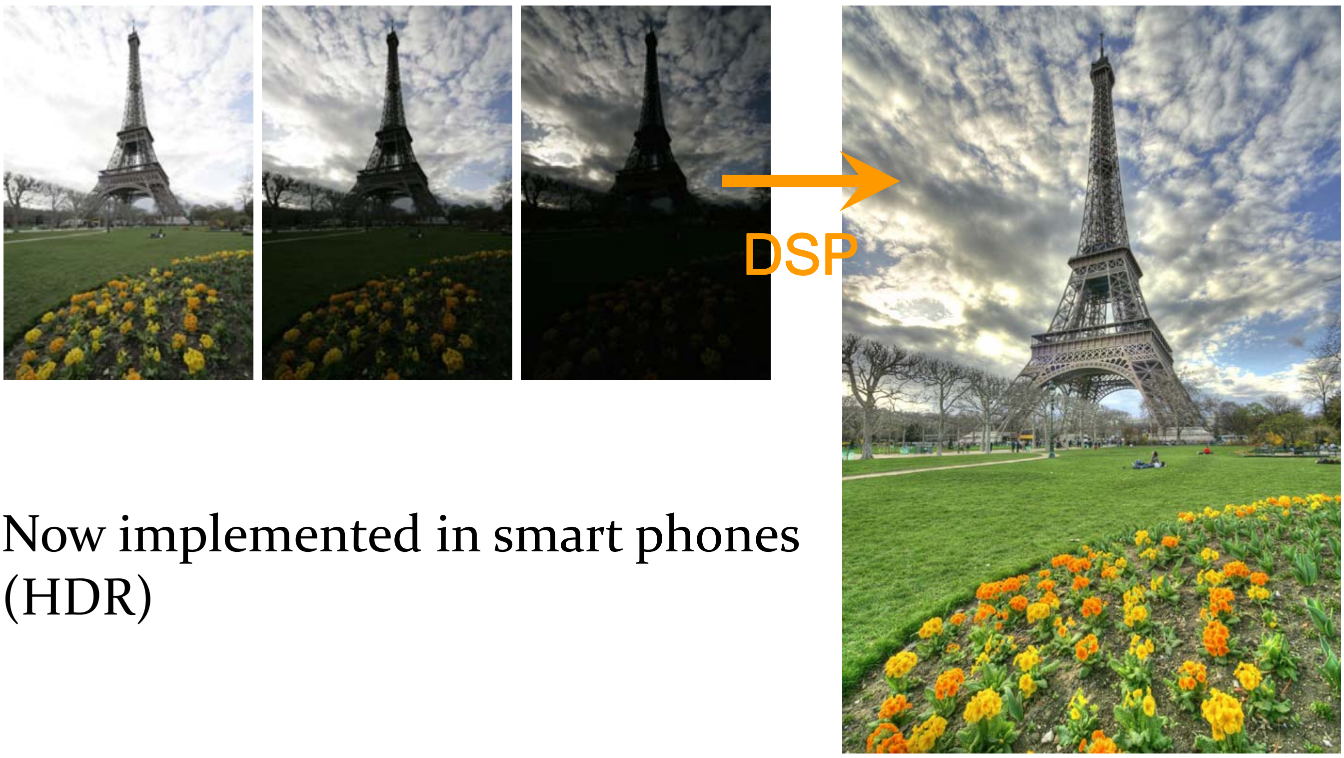




- Compression of 40x without perceptual loss of quality.

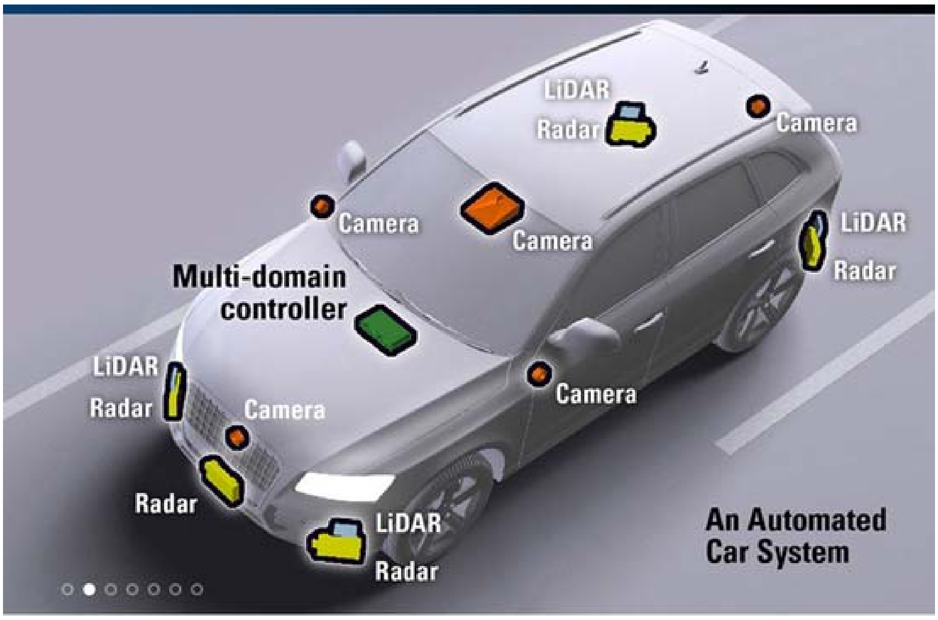
- Example of slight over compression:
  - difference enables x60 compression!

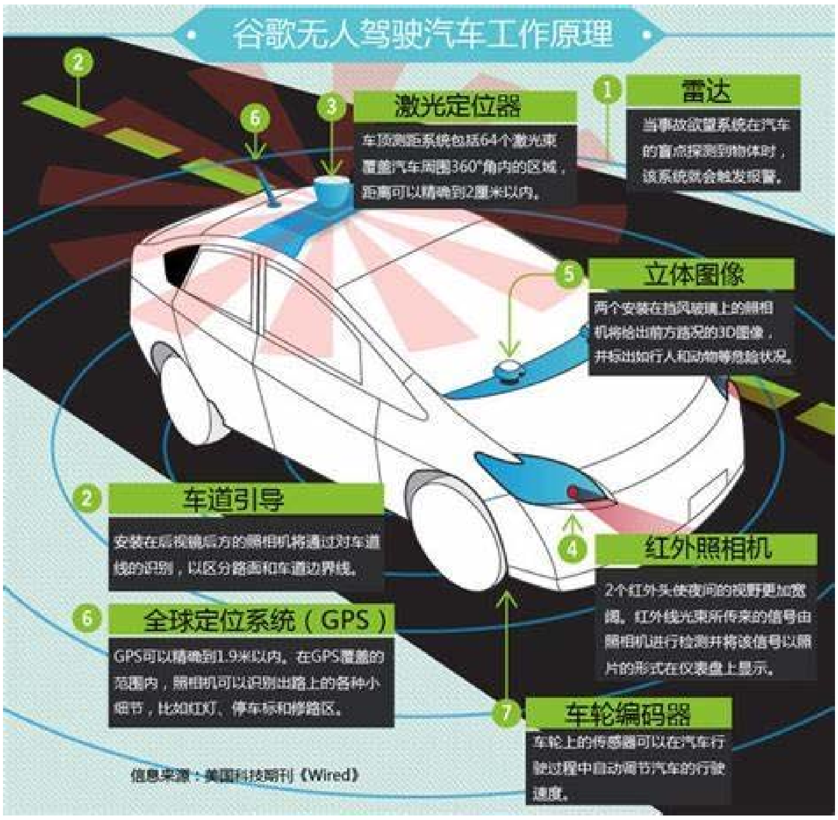
Computational Photography



Now implemented in smart phones (HDR)

Example III Auto Drive





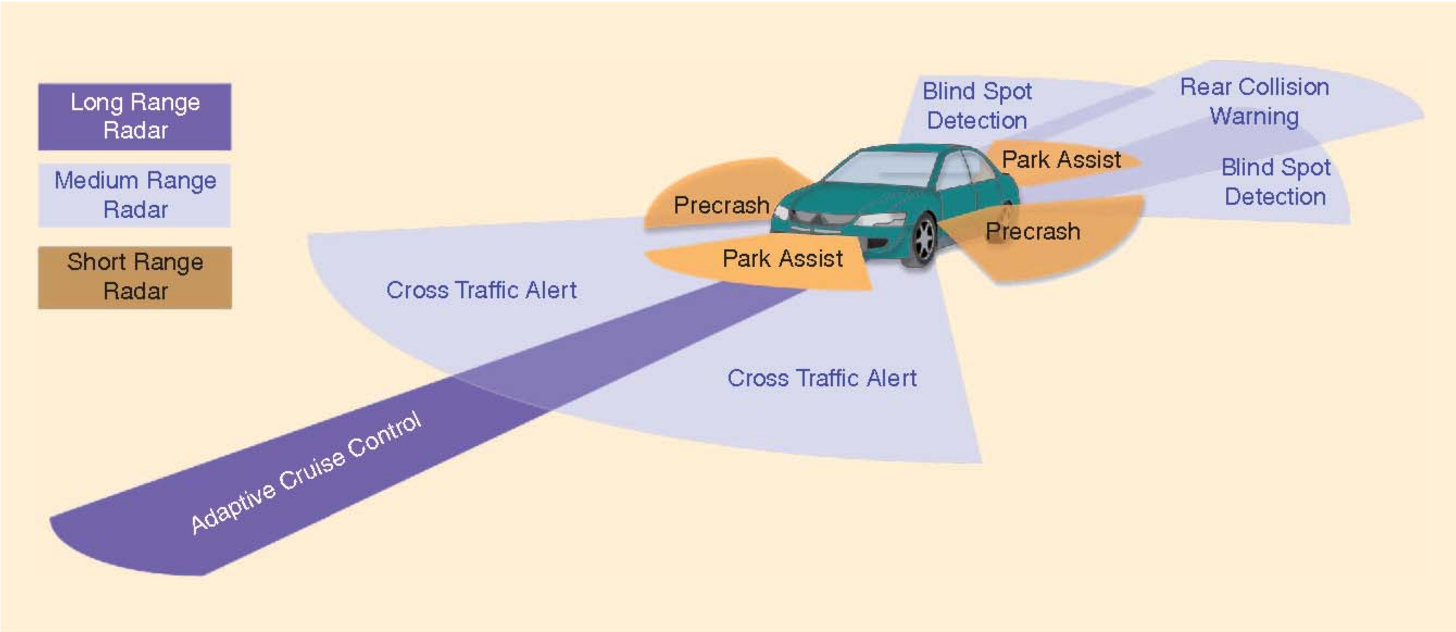


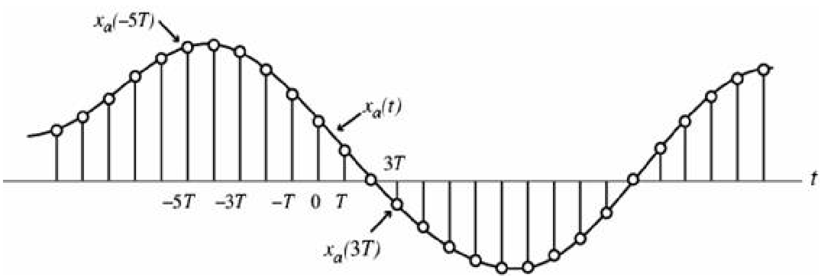
FIGURE 1. An ADAS consists of different range radars.

Lecture 2 Time Domain Representation of Discrete Time Signals

Discrete Time Signal

- Samples of a Continuous Time (CT) Signal

$$x[n] = x_a(nT),\ n = \ldots, -1, 0, 1, 2, \ldots$$

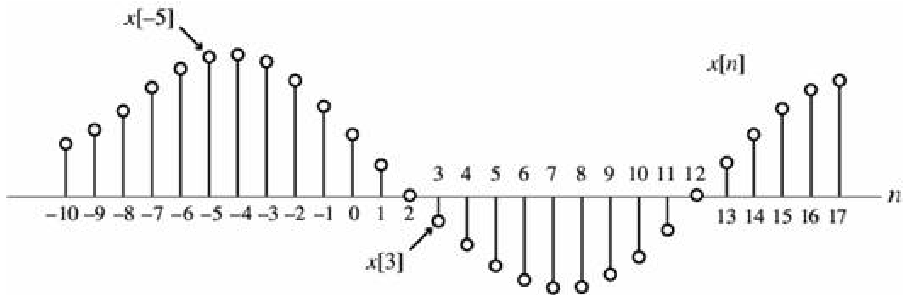


- The spacing T between two consecutive samples is called the sampling interval or sampling period
- Reciprocal of sampling interval T, denoted as F_T, is called the sampling frequency:

$$F_T = 1/T$$

Discrete Time Signal

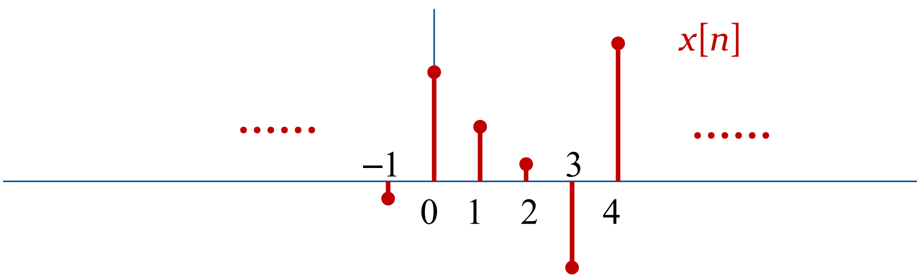
- Or, inherently discrete



- Examples?

Discrete Time Signal

- Signals represented as sequences of numbers, called **samples.**
- Sample value of a typical signal or sequence denoted as $x[n]$ with n being an integer in the range $-\infty \le n \le \infty$.
- $x[n]$ is called the n^{th} sample of the sequence.

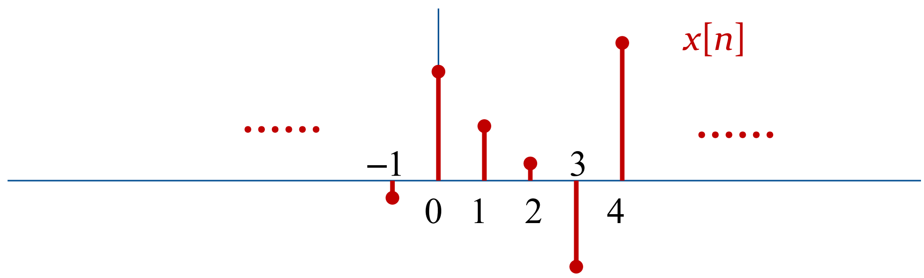


Discrete Time Signal

- $\{x[n]\}$ defined only for integer values of n and undefined for non-integer values of n.
- Discrete-time signal may also be written as a sequence of numbers inside braces:

$$\{x[n]\} = \{\ldots, -0.2, \underset{\uparrow}{2.2}, 1.1, 0.2, -1.9, 2.9, \ldots\}$$

placed under the sample at time index $n = 0$



Real and Complex Sequences

- $\{x[n]\}$ is a real sequence, if $x[n]$ is real for all values of n, otherwise, $\{x[n]\}$ is a complex sequence

- A complex sequence $\{x[n]\}$ can be written as

$$\{x[n]\} = \{x_{re}[n]\} + j\,\{x_{im}[n]\}$$

- Its complex conjugate is

$$\{x^*[n]\} = \{x_{re}[n]\} - j\,\{x_{im}[n]\}$$

Example:

- $\{x[n]\} = \{\cos 0.25n\}$ is a real sequence, while $\{y[n]\}=\{e^{j0.3n}\}$ is a complex sequence

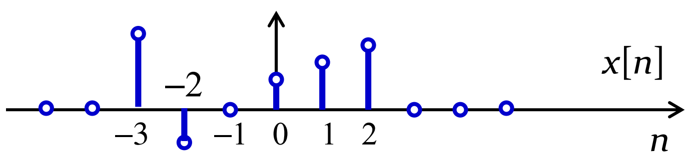
- We can write

$$\{y[n]\}=\{\cos 0.3n + j\sin 0.3n\} = \{\cos 0.3n\} + j\{\sin 0.3n\}$$

where $\{y_{re}[n]\}=\{\cos 0.3n\}$ and $y_{im}[n]=\{\sin 0.3n\}$

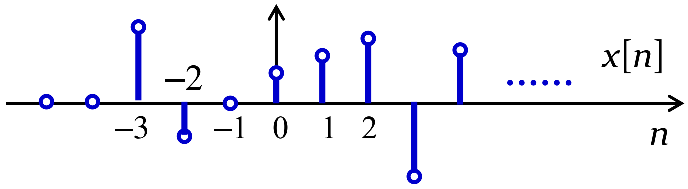
Length of a Discrete-Time Signal

- **Finite Length** (also called finite duration or finite extent)
  - Defined only for a finite time interval: $N_1 \le n \le N_2$, where $-\infty < N_1$ and $N_2 < \infty$ with $N_1 \le N_2$



  - Length of the above finite-length sequence is $N= N_2 - N_1 + 1$
  - Example: $x[n]=n^2$, $-3 \le n \le 4$ is a finite-length sequence of length $4 - (-3) + 1 = 8$

- **Infinite Length**
  - A right-sided sequence $\{x[n]\}$ has zero-valued samples for $n < N_1$



  - If $N_1 \geq 0$, a right-sided sequence is usually called a causal sequence.
  - A left-sided sequence $\{x[n]\}$ has zero-valued samples for $n > N_2$.
    - If $N_2 \leq 0$, a left-sided sequence is usually called an anti-causal sequence.
  - A general two-sided sequence is defined for all values of n in the range $-\infty < n < \infty$
    - Example: $\{y[n]\} = \{\cos 0.4n\}$ is a general infinite-length sequence

Operations on Sequences

- A **discrete-time system** operates on one (or more) sequence, called the **input sequence**, according some prescribed rules and develops another one (or more) sequence, called the **output sequence**, with more desirable properties

Operations on Sequences

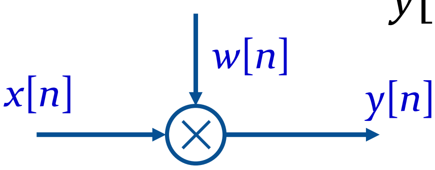
- For example, the input may be a signal corrupted with additive noise
- Discrete-time system is designed to generate an output by removing the noise component from the input
- In most cases, the operation defining a particular discrete-time system is composed of some **elementary operations**

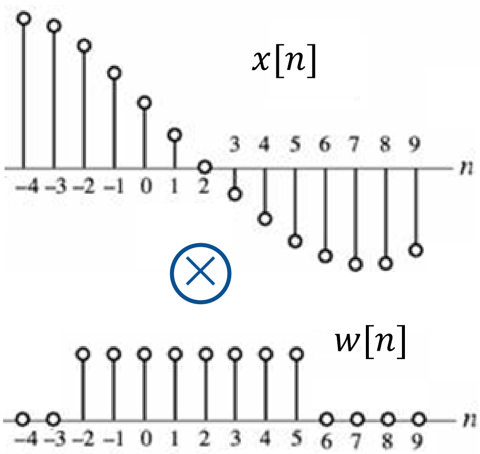
Elementary Operations

- **Product (modulation)** operation:

Modulator

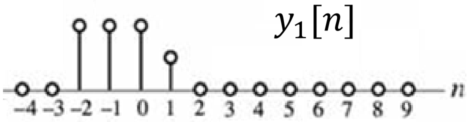
$$y[n] = x[n] \cdot w[n]$$

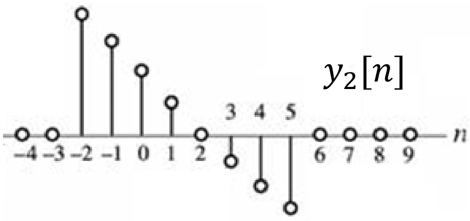




- An application is in forming a finite-length sequence from an infinite-length sequence by multiplying the latter with a finite-length sequence called a **window sequence**
- Process called **windowing**

Which one ??

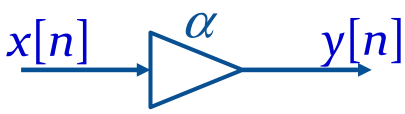




Elementary Operations

- **Multiplication operation:**

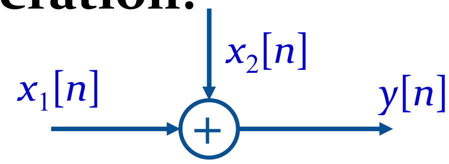
  Multiplier



$$y[n] = \alpha x[n]$$

- **Addition operation:**

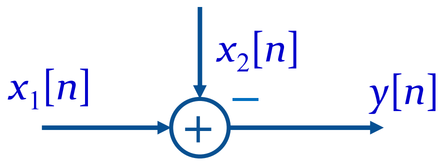
  Adder



$$y[n] = x_1[n] + x_2[n]$$

- **Subtraction operation**:

  Subtractor

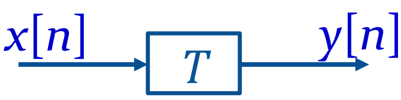


$$y[n] = x_1[n] - x_2[n]$$

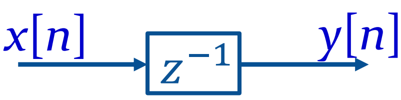
Elementary Operations

- **Time-shifting Operation:** $y[n] = x[n - n_0]$, where n_0 is an integer
- If $n_0 > 0$, it is delaying operation

  Unit Delay

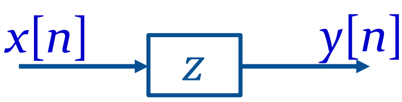


$$y[n] = x[n - 1]$$



- If $n_0 < 0$, it is an advance operation

  Unit Advance

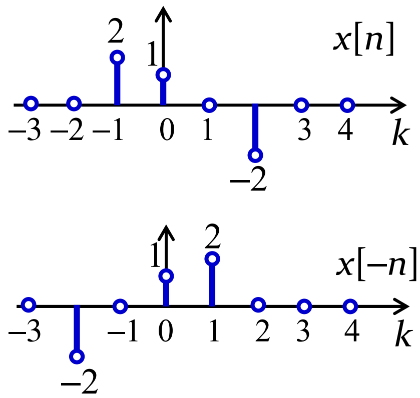


$$y[n] = x[n + 1]$$

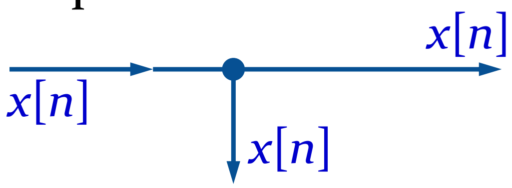
Elementary Operations

- **Time-reversal (folding) operation:**

$$y[n] = x[-n]$$

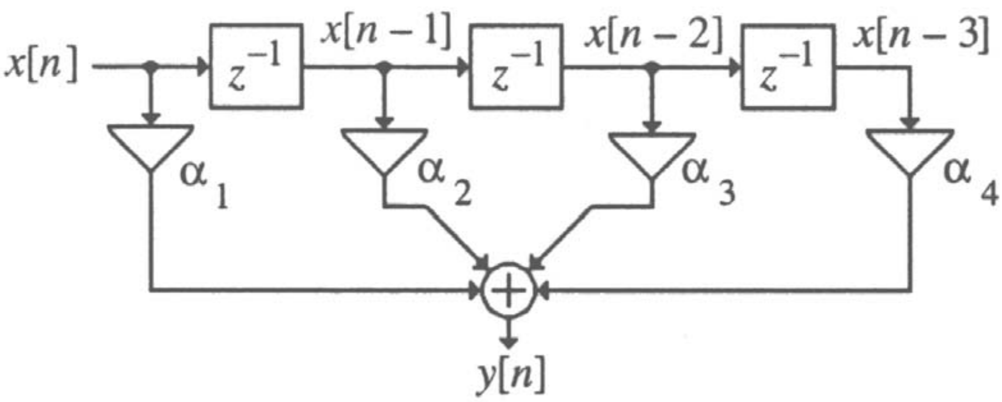


- **Branching operation:** Used to provide multiple copies of a sequence



Combinations of Basic Operations

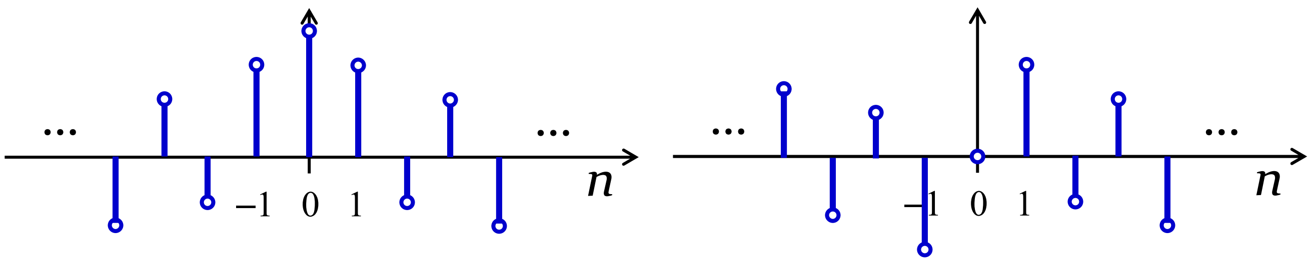
- Example:



$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

Classification of Sequences

- Based on Symmetry
  - Conjugate-symmetric: $x[n] = x^*[-n]$
    - Even sequence: a real conjugate-symmetric sequence
  - Conjugate-antisymmetric: $x[n] = -x^*[-n]$
    - Odd sequence: a real conjugate-antisymmetric sequence



Even sequence

Odd sequence

- Any complex sequence $x[n]$ can be expressed as

$$x[n] = x_{cs}[n] + x_{ca}[n],$$

where

$$x_{cs}[n] = \frac{1}{2}(x[n] + x^*[-n]),$$ Conjugate Symmetric part

$$x_{ca}[n] = \frac{1}{2}(x[n] - x^*[-n]),$$ Conjugate anti-symmetric part

- Any real sequence $x[n]$ can be expressed as

$$x[n] = x_{ev}[n] + x_{od}[n],$$

where

$$x_{ev}[n] = \frac{1}{2}(x[n] + x\,[-n]),$$ Even part

$$x_{od}[n] = \frac{1}{2}(x[n] - x[-n]),$$ Odd part

Example 1: Generation of Symmetric Parts of a Complex Sequence

- Sequence: $\{g[n]\} = \{0, 1+j4, -2+j3, 4-j2, -5-j6, -j2, 3\}$
- A: We form

$$\{g^*[n]\} = \{0, 1-j4, -2-j3, 4+j2, -5+j6, j2, 3\}, \text{ and}$$

$$\{g^*[-n]\} = \{3, j2, -5+j6, 4+j2, -2-j3, 1-j4, 0\},$$

- Thus:

$$g_{cs}[n] = \frac{1}{2}(g[n] + g^*[-n])$$

$$= \{1.5, 0.5 + j3, -3.5 + j4.5, 4, -3.5 - j4.5, 0.5 - j3, 1.5\}$$

$$g_{ca}[n] = \frac{1}{2}(g[n] - g^*[-n])$$

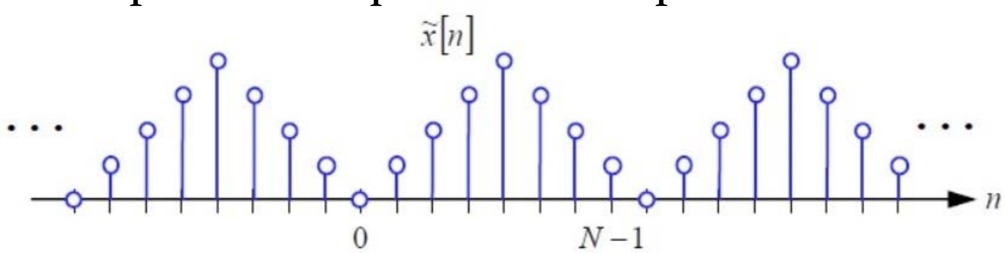
$$= \{-1.5, 0.5 + j, 1.5 - j1.5, -2j, -1.5 - j1.5, -0.5 + j, 1.5\}$$

Classification of Sequences

- Base on Periodicity
  - A sequence $\tilde{x}[n]$ satisfying

$$\tilde{x}[n] = \tilde{x}[n + kN] \quad \text{for all } n$$

is called a periodic sequence with a period N.

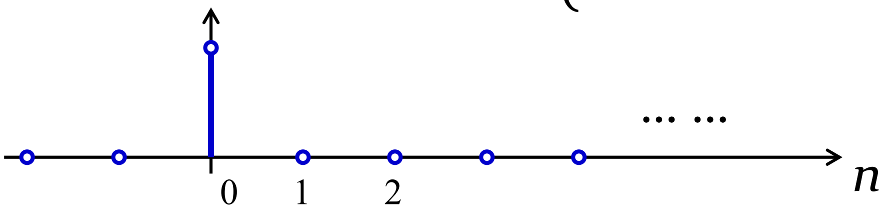


  - The fundamental period N_f of a periodic signal is the smallest value of N for which the above equation holds.

Basic Sequences

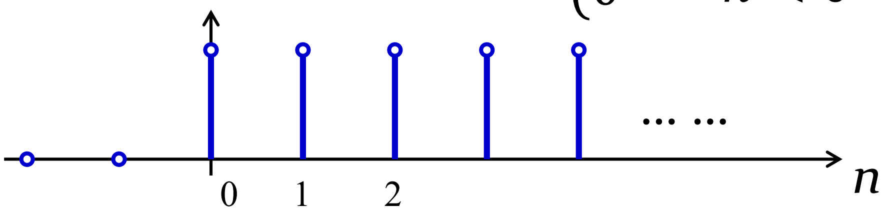
- Unit impulse

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



- Unit Step

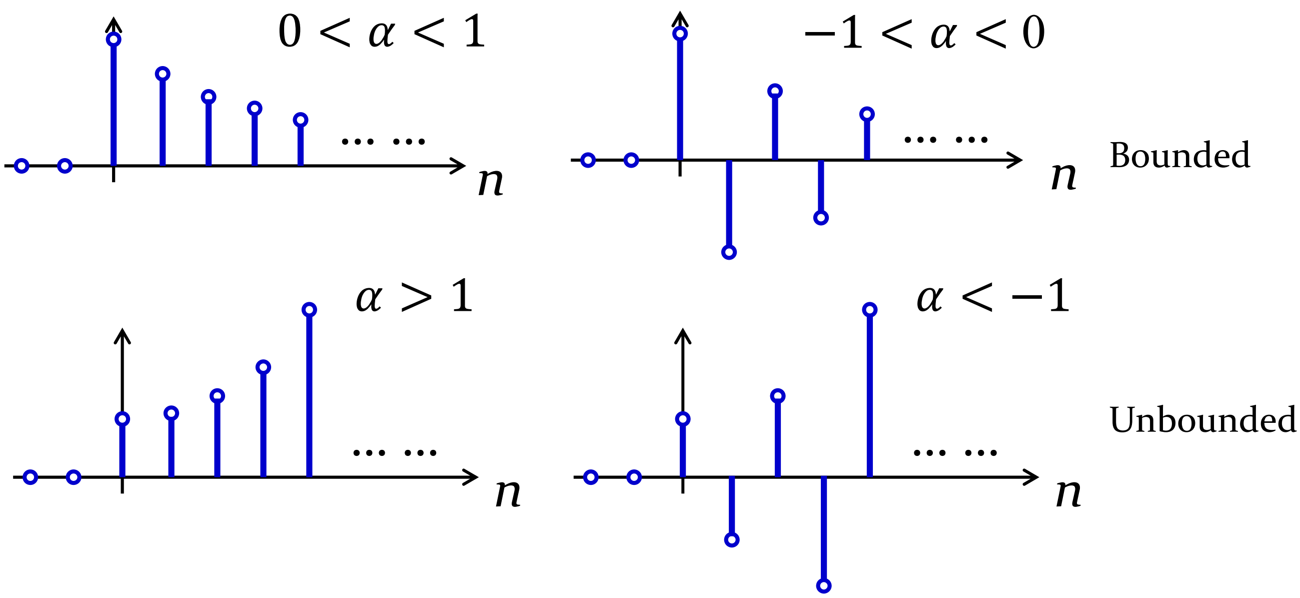
$$\mu[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Basic Sequences

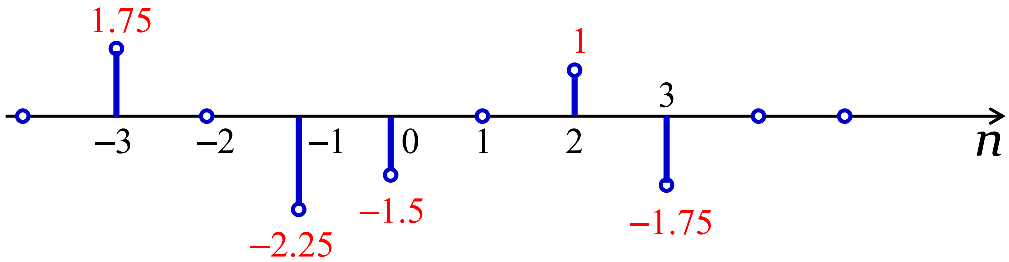
- Exponential

$$x[n] = \begin{cases} A\alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Representing an arbitrary sequence

- as a weighted sum of unit impulse and its delayed versions.



$$x[n] = 1.75\delta[n+3] - 2.25\delta[n+1] - 1.5\delta[n] + \delta[n-2] - 1.75\delta[n-3]$$

- A general form

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Discrete Sinusoids

$$x[n] = A\cos(\omega_0 n + \varphi)$$

or $x[n] = Ae^{j\omega_0 n + j\varphi}$

- Q: Period or not? $x[n] = x[n+N]$ for N integer.

Discrete Sinusoids

$$x[n] = A\cos(\omega_0 n + \varphi)$$

$$\text{or}\;\; x[n] = Ae^{j\omega_0 n + j\varphi}$$

- Q: Period or not? $x[n] = x[n+N]$ for N integer.
- A: Yes only if ω_0/π is rational (Different from CT!)

- To find fundamental period N
  - Find smallest integers K and N, satisfying:

$$\omega_0 N = 2\pi K$$

Discrete Sinusoids

- Example:

$$\cos\left(\frac{5}{7}\pi n\right) \qquad N = 14 \quad (K = 5)$$

$$\cos\left(\frac{1}{5}\pi n\right) \qquad N = 10 \quad (K = 1)$$

$$\cos\left(\frac{5}{7}\pi n\right) + \cos\left(\frac{1}{5}\pi n\right) \Rightarrow N = \mathrm{SCM}(14, 10) = 70$$

Discrete Sinusoids

- Another difference:
- Q: Which one is a higher frequency signal for $\sin(\omega_0 n)$?

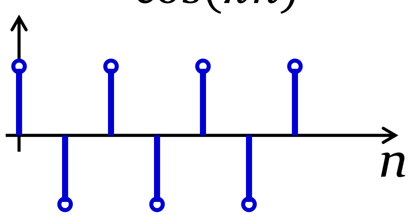
$$\omega_0 = \pi \qquad \text{or} \qquad \omega_0 = \frac{3}{2}\pi$$

Discrete Sinusoids

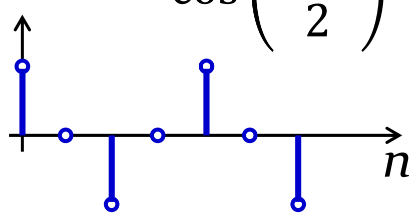
- Another difference:
- Q: Which one is a higher frequency signal?

$$\omega_0 = \pi \qquad \text{or} \qquad \omega_0 = \frac{3}{2}\pi$$

$\cos(\pi n)$



$\cos\left(\frac{3\pi n}{2}\right) = \cos\left(\frac{\pi n}{2}\right)$

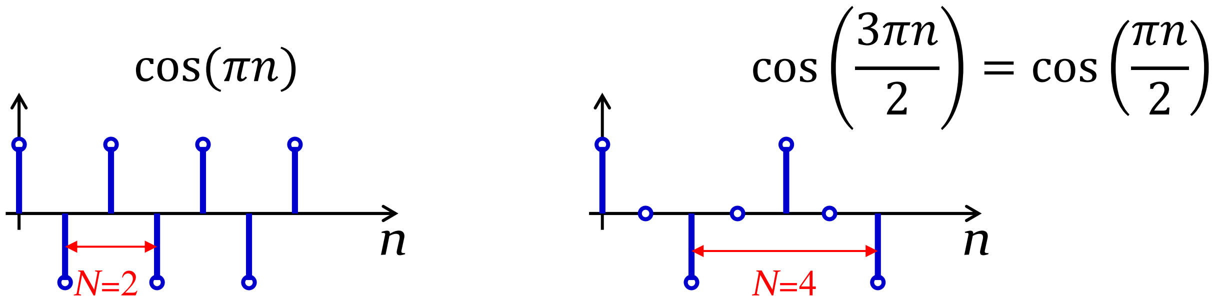


Discrete Sinusoids

- Another difference:
- Q: Which one is a higher frequency signal?

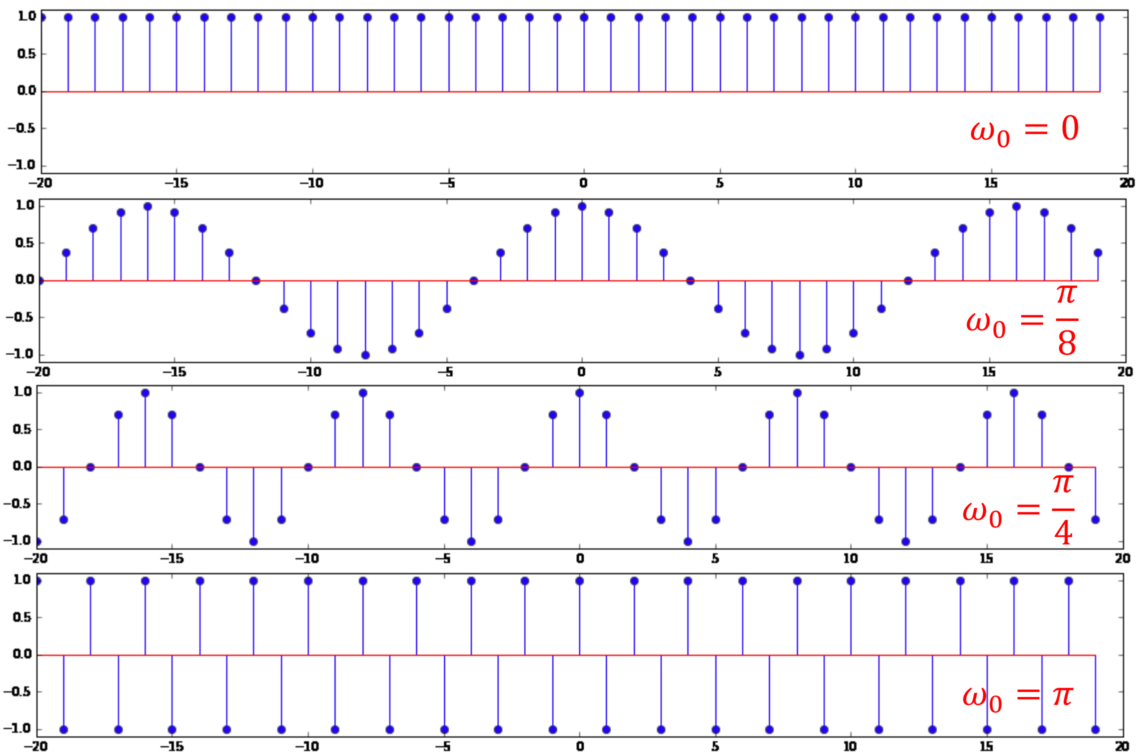
$$\omega_0 = \pi \qquad \text{or} \qquad \omega_0 = \frac{3}{2}\pi$$

- A: $\omega_0 = \pi$



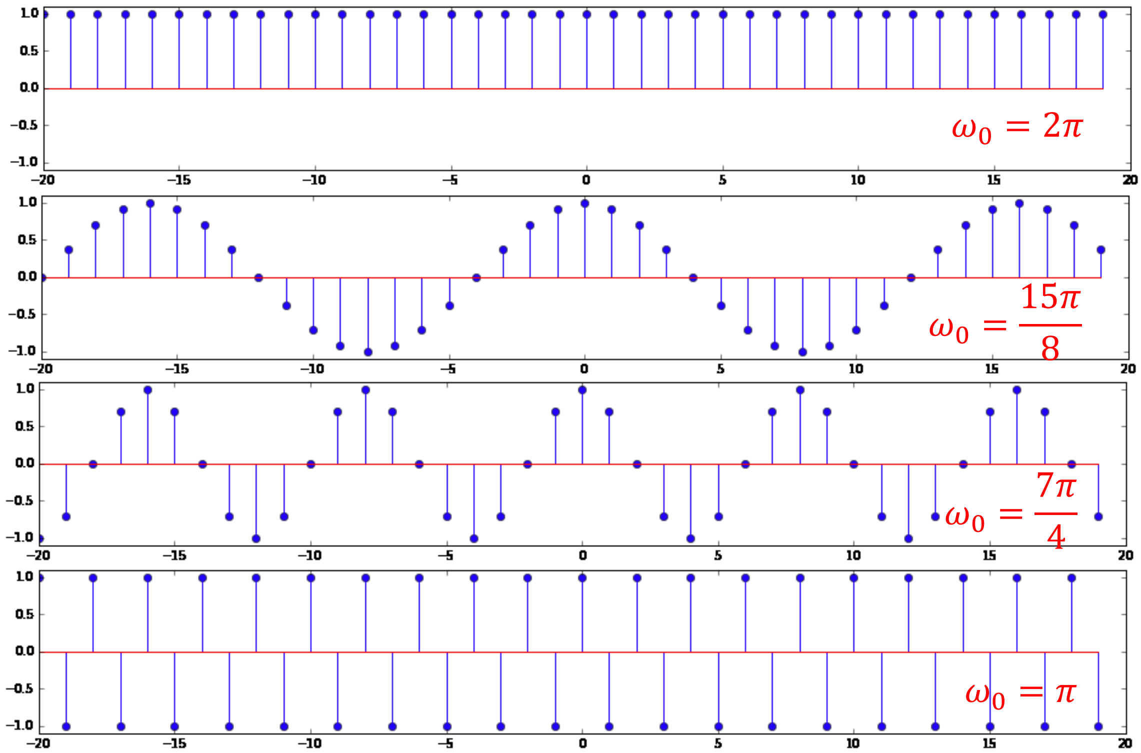
Discrete Sinusoids

$\cos(\omega_0 n)$



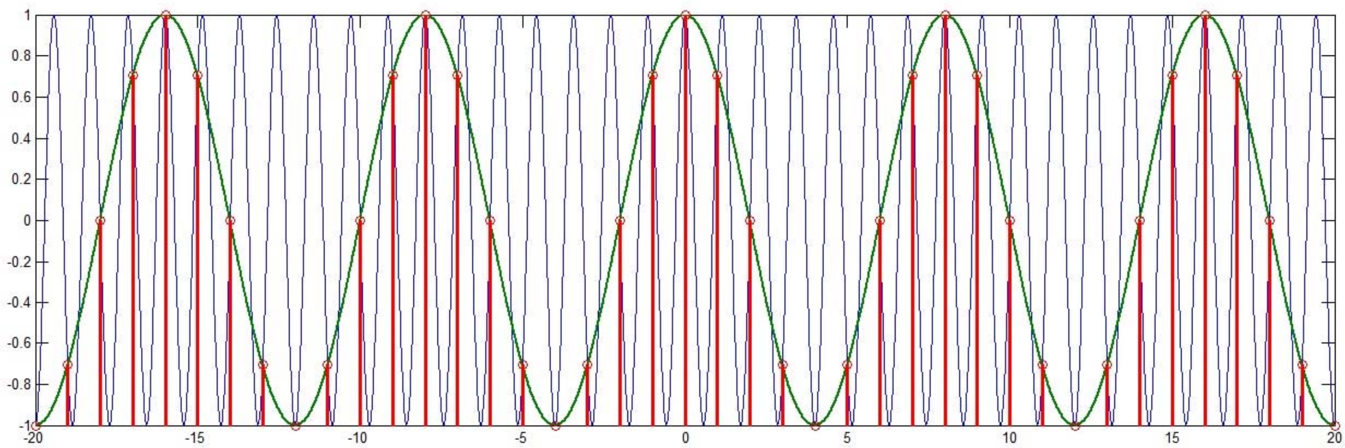
Discrete Sinusoids

$\cos(\omega_0 n)$



$\cos\left(\frac{7}{4}\pi t\right)$ $\quad$ $\cos\left(\frac{7\pi}{4}n\right)$ and $\cos\left(\frac{\pi}{4}n\right)$

$\cos\left(\frac{1}{4}\pi t\right)$



单选题 1分

设置

此题未设置答案，请点击右侧设置按钮

Is $x[n] = A\cos(\omega_0 n + \varphi)$ periodic or not?

A Yes

B No

C It depends on ω_0

D It depends on φ

提交

单选题 1分

设置

此题未设置答案，请点击右侧设置按钮

Determine the fundamental frequency of

$\cos\left(\frac{5}{7}\pi n\right) + \cos\left(\frac{1}{5}\pi n\right)$

A 14

B 10

C 2

D 70

提交

单选题 0.5分

设置

此题未设置答案，请点击右侧设置按钮

For $\cos(\omega_0 n)$, which one is a higher frequency signal?

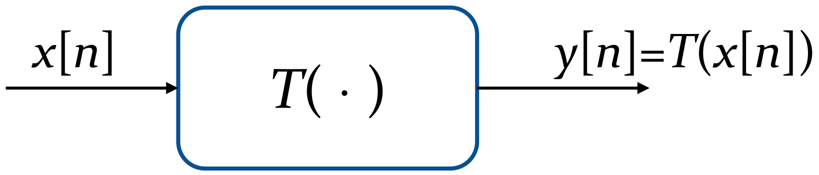
A $\omega_0 = \pi$

B $\omega_0 = \frac{3}{2}\pi$

提交

Lecture 3
Time Domain Representation of Discrete Time Systems

Discrete Time System



- The function of **discrete-time system** is to process a given sequence, called the **input sequence**, to generate another sequence, called the **output sequence**, with more desirable properties or to extract certain information about the input signal.

Examples

- **Accumulator**

$$y[n] = \sum_{l=-\infty}^{n} x[l]$$

$$= y[n-1] + x[n]$$

- The output $y[n]$ at time instant n is the sum of input sample values $x[n]$ and all past input samples.
- It accumulates all input sample values from $-\infty$ to n.

- Alternative input-output relation expression

$$y[n] = \sum_{l=-\infty}^{-1} x[l] + \sum_{l=0}^{n} x[l]$$

$$= y[-1] + \sum_{l=0}^{n} x[l]$$

Initial condition

A causal input sequence

Examples

- ***M*-point Moving-Average Filter**

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l]$$

- Used in smoothing random variation in data.
- If there is no bias in measurements, an improved estimate of noise data is obtained by simply increasing M.

- Alternative expression

$$y[n] = \frac{1}{M}\left(\sum_{l=1}^{M-1} x[n-l] + x[n] + x[n-M] - x[n-M]\right)$$

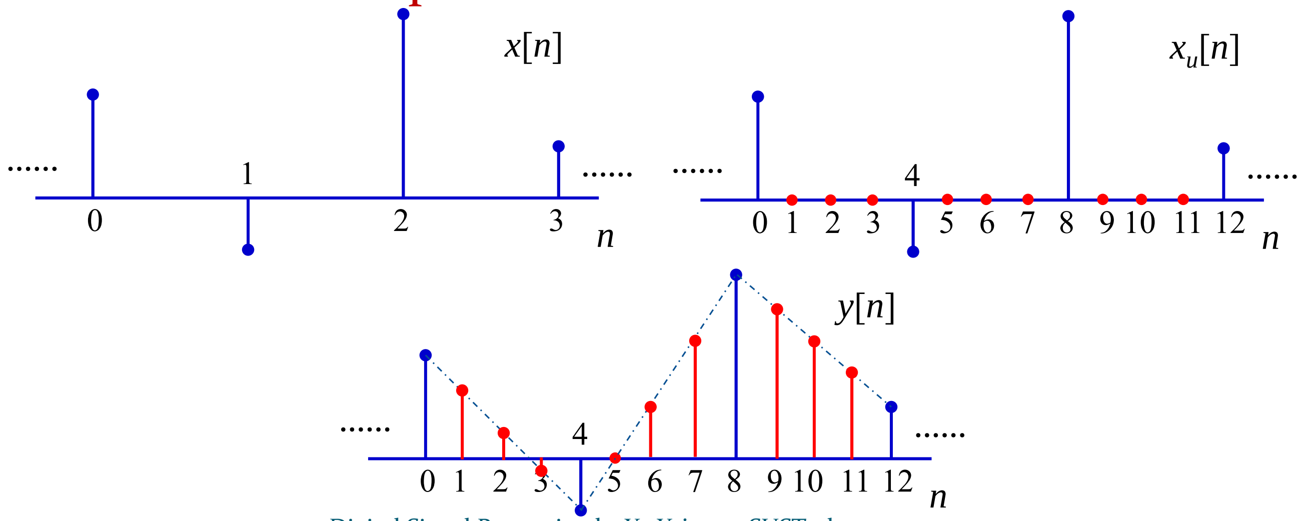
$$= \frac{1}{M}\left(\sum_{l=1}^{M} x[n-l] + x[n] - x[n-M]\right)$$

$$= \frac{1}{M}\left(\sum_{l=0}^{M-1} x[n-l-1] + x[n] - x[n-M]\right)$$

$$= y[n-1] + \frac{1}{M}(x[n] - x[n-M])$$

Examples

- **Linear Interpolator** – employed to estimate sample values between pairs of adjacent sample values of a discrete-time sequence
- Factor-of-4 interpolation



- Factor-of-2 interpolator

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

- Factor-of-3 interpolator

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-2] + x_u[n+2]) + \frac{2}{3}(x_u[n-1] + x_u[n+1])$$

- Factor-of-2 interpolation

Example: Median Filter

- **Median**
  - The **median** of a set of $(2k+1)$ numbers is the number such that k numbers from the set have values greater than this number, and the other K numbers have values smaller.
  - Median can be determined by rank-ordering the numbers in the set by their values and then choosing the number at the middle.
  - The median of a sequence is denoted as

$$\text{med}\{a_1, a_2, a_3, a_4, a_5\}$$

- Example: Consider the set of numbers

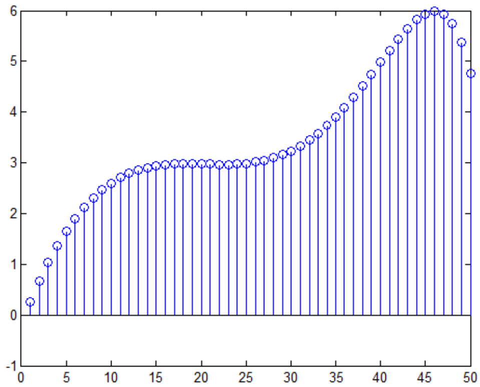
  $\{8, 2, 6, 12, -4\}$

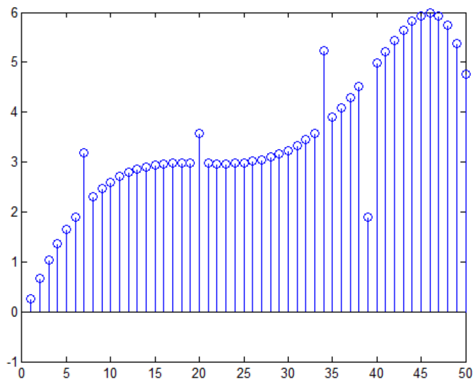
- Rank-ordered set: $\{-4, 2, 6, 8, 12\}$
- Hence: $\text{med}\{8, 2, 6, 12, -4\} = 6$

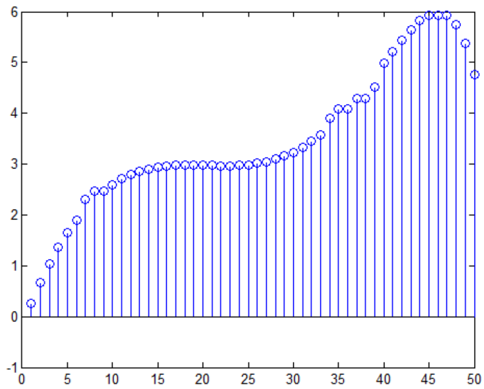
- **Median Filter**
  - Median filter is implemented by sliding a window of odd length over the input sequence $x[n]$ one sample at a time. The output $y[n]$ at the nth instant of the median filter with a window length-$(2k+1)$ is then given by

    $y[n]=\text{med}\{x[n-k], \ldots, x[n-1], x[n], x[n+1], \ldots, x[n+k]\}$.

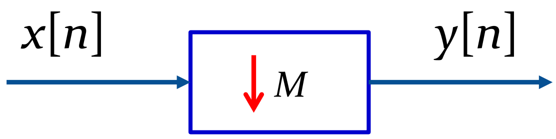
- Median Filter Example:
  - Find applications in removing additive random noise, which shows up as sudden large errors in the corrupted signal, for example signals corrupted by impulse noise.

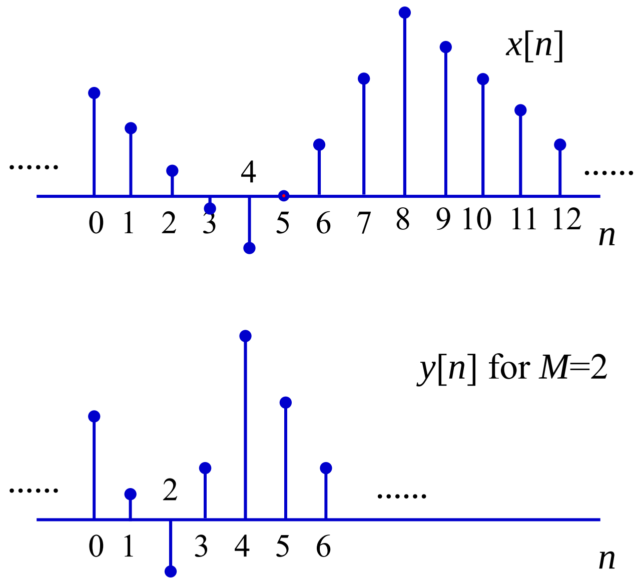






Compressor





- Compressor has an input-output relation given by

$$y[n] = x[Mn] \quad \text{for } M > 1$$

Properties of DT System

- Linearity
- Causality
- Memoryless
- Time-invariance
- BIBO-stability
- Passive and Lossless properties

多选题 0.5分 设置

此题未设置答案，请点击右侧设置按钮

Which of the following is a linear system

A $y[n] = x[n - n_d]$

B $y[n] = x[Mn]$ for $M > 1$

C $y[n] = x[n] + 3$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \dots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

多选题 1分 设置

此题未设置答案，请点击右侧设置按钮

Which of the following is a causal system

A $y[n] = x[n - n_d]$

B $y[n] = x[Mn]$ for $M > 1$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \dots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

多选题 1分

设置

此题未设置答案，请点击右侧设置按钮

Which of the following is memoryless?

A $y[n] = x[n - n_d]$

B $y[n] = x[Mn]$ for $M > 1$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

多选题 1分

设置

此题未设置答案，请点击右侧设置按钮

Which of the following is time-invariant?

A $y[n] = x[n - n_d]$

B $y[n] = \sum_{k=-\infty}^{n} x[k]$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

多选题 1分

设置

此题未设置答案，请点击右侧设置按钮

Which of the following is BIBO?

A $y[n] = x[n-n_d]$

B $y[n] = \sum_{k=-\infty}^{n} x[k]$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

多选题 1分

设置

此题未设置答案，请点击右侧设置按钮

Which of the following is passive?

A $y[n] = x[n-n_d]$

B $y[n] = \sum_{k=-\infty}^{n} x[k]$

C $y[n] = x[-n]$

D $y[n] = x^2[n]$

E $y[n] = \begin{cases} x\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots \\ 0, \quad \text{otherwise} \end{cases}$ for $L > 1$

提交

Properties of DT System

- **Linearity:**

  If $y_1[n] = T\{x_1[n]\}$, and $y_2[n] = T\{x_2[n]\}$

  - Superposition:

  $T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$

  - Homogeneity:

  $T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n]$

  Overall: $T\{a_1x_1[n] + a_2x_2[n]\} = a_1y_1[n] + a_2y_2[n]$

- The above property must hold for arbitrary constant a_1 and a_2, and for all possilbe input $x_1[n]$ and $x_2[n]$.

$$y[n] = \sum_{l=-\infty}^{n} x[l]$$

Linearity of Accumulator

- Let $y_1[n] = \sum_{l=-\infty}^{n} x_1[l]$, $y_2[n] = \sum_{l=-\infty}^{n} x_2[l]$
- For an input

$$x[n] = \alpha x_1[n] + \beta x_2[n]$$

- The output is

$$y[n] = \sum_{l=-\infty}^{n} (\alpha x_1[l] + \beta x_2[l])$$

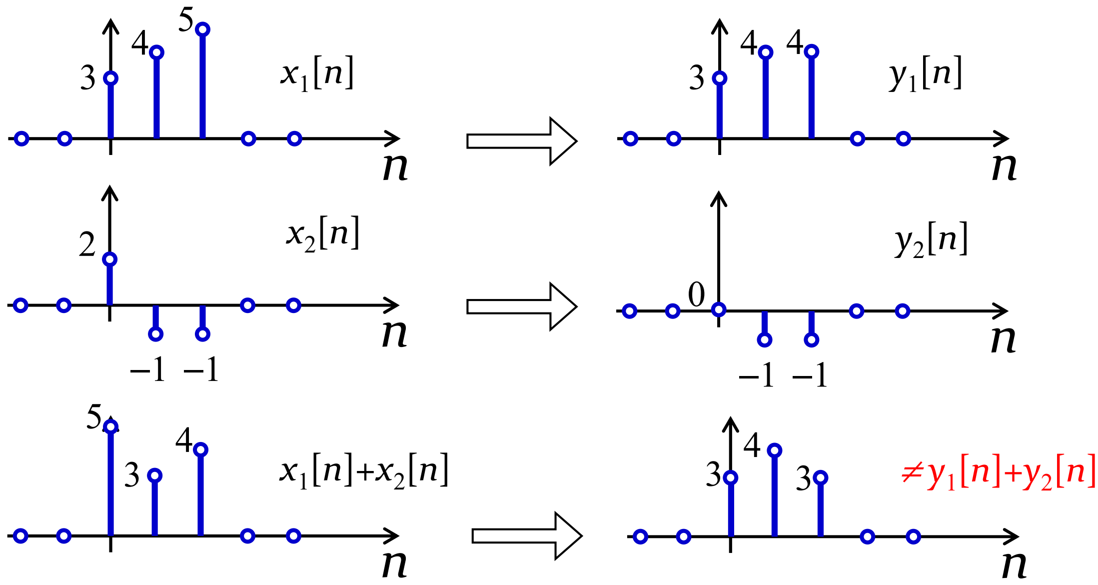
$$= \alpha \sum_{l=-\infty}^{n} x_1[l] + \beta \sum_{l=-\infty}^{n} x_2[l] = \alpha y_1[n] + \beta y_2[n]$$

- Hence, the above system is linear.

Linearity of Median Filter

- The median filter is a non-linear DT system
- $y[n]$ = MED{$x[n - k]$, ..., $x[n + k]$}.

Take $k = 1$



Properties of DT System (Cont.)

- **Causality:**
  - $y[n_0]$ depends only on $x[n]$ for $-\infty < n \le n_0$, and does not depends on input samples $n > n_0$.

- A non-causal system cannot be implemented because it uses future input signal to generate the current output signal.

- Example of causal systems:
$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$
$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$
$y[n] = y[n-1] + x[n]$
- Example of non-causal systems:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+1]) + \frac{2}{3}(x_u[n-2] + x_u[n+2])$$

Implementation of non-causal system

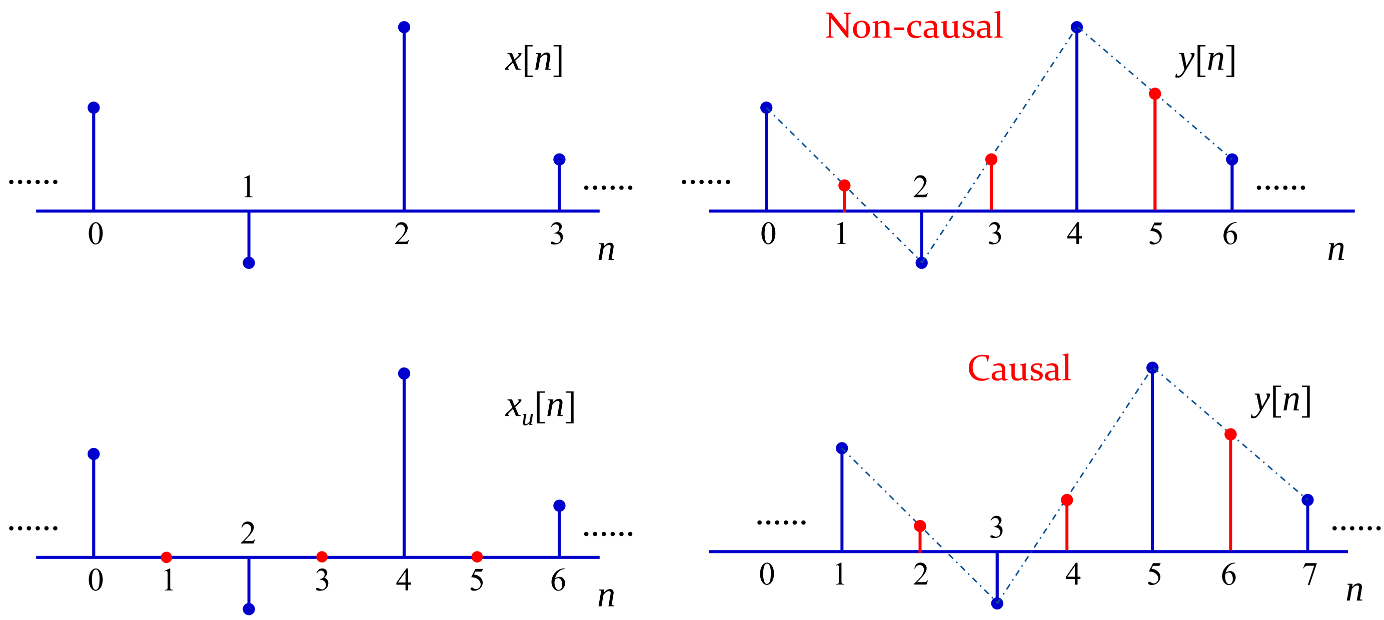
- A non-causal system may be implemented as a causal system by delaying the output by an appropriate number of samples
- For example a causal implementation of the factor-of-2 interpolator is given by

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

non-causal expressions of a factor-of-2 interpolator:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

Numerical Example



Properties of DT System (Cont.)

- **Memoryless:**
  - $y[n_0]$ depends only on $x[n_0]$, and does not depend on input samples for $n<n_0$.
  - Example of memoryless system: $y[n] = x[n]^2$
  - Example of memory system:

$$y[n] = a_1x[n] + a_2x[n-1] + a_3x[n-2] + a_4x[n-3]$$
$$y[n] = y[n-1] + x[n]$$

Properties of DT System (Cont.)

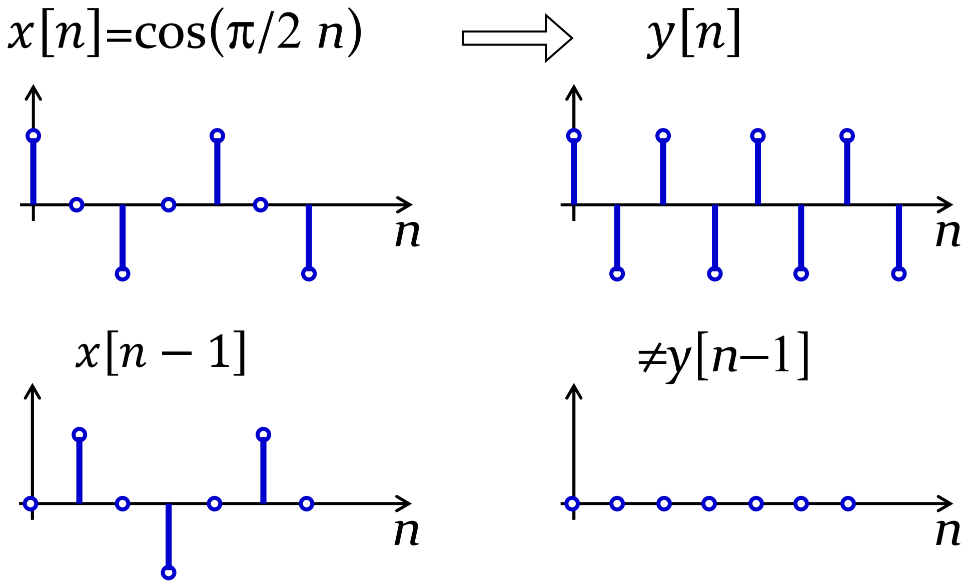
- **Time invariance:**

  If: $y[n] = T\{x[n]\}$

  Then: $y[n-n_0] = T\{x[n-n_0]\}$ for all integer n_0
- The above relation must hold for arbitrary input and its corresponding output.
- Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied.

Compressor is time-invariant or not? -- Numerical Example

- Suppose M=2, $y[n] = x[Mn]$



- Proof:
  - If an input $\{x[n]\}$ at time n produces an output $y[n]$, the system is time invariant if a time shifted input $x[n - n_0]$ produces a time shifted output $y[n - n_0]$, i.e. if

    $$x[n] \Rightarrow y[n]$$

    then $x[n - n_0] \Rightarrow y'[n] = y[n - n_0]$.
  - Consider the system $y[n] = x[Mn]$, we have

    $$x[n] \Rightarrow x[Mn] = y[n],$$

    $$x[n - n_0] \Rightarrow x[Mn - n_0] = y'[n].$$
  - $y[n]$ shifted by n_0 is $y[n - n_0] = x[M(n - n_0)] \neq x[Mn - n_0]$ $y'[n]$.
  - Therefor, $y[n] = x[Mn]$ is not time invariant if $M \neq 1$.

Properties of DT System (Cont.)

- **BIBO stability:**

  If: $|x[n]| \leq B_x < \infty \quad \forall n$

  Then: $|y[n]| \leq B_y < \infty \quad \forall n$
- Example – The M-point moving average filter is BIBO stable:
  - For a bounded input $|x[n]| \leq B_x$, we have

$$|y[n]| = \left| \frac{1}{M} \sum_{k=0}^{M-1} x[n-k] \right| \leq \frac{1}{M} \sum_{k=0}^{M-1} |x[n-k]| \leq \frac{1}{M} M B_x$$

$$= B_x$$

Properties of DT System (Cont.)

- **Passive and Lossless**

Passive if:
$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \le \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

Lossless if the inequality is satisfied with an equal sign for every input sequence,

- **A passive Discrete-time system**

$$y[n] = \alpha x[n-N]$$

- Since $\sum_{n=-\infty}^{\infty} |y[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n]|^2$

it is a passive system if $|\alpha| \le 1$ and is a lossless system if $|\alpha| = 1$

Summary

	Causal	Linear	Time-Invariant	Memory-less	BIBO Stable	Passive
Time Shift $y[n] = x[n-n_d]$						
Accumulator $y[n] = \sum_{k=-\infty}^{n} x[k]$						
Compressor $y[n] = x[Mn]$ for $M > 1$						
Up-sampler $y[n] = \begin{cases} x\left[\frac{n}{L}\right] & n = 0, \pm L, \pm 2l \\ 0 & \text{otherwise} \end{cases}$ for $L > 1$						

LTI Discrete Time Systems

- **Linear Time-Invariant (LTI) System** – a system satisfying both the linearity and the time-invariant properties.

- LTI systems are mathematically easy to analyze and characterize, and consequently easy to design.

- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades.

Impulse and Step Responses

- The response of a DT system to a unit impulse sequence $\{\delta[n]\}$ is called the **unit impulse response** or simply, the **impulse response**, denoted as $\{h[n]\}$.

- The response of a DT system to a unit step sequence $\{\mu[n]\}$ is called the **unit step response** or simply, the **step response**, denoted as $\{s[n]\}$.

Impulse Response

- Example – The impulse response of system

$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$

is obtained by setting $x[n] = \delta[n]$, resulting in

$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$

- The impulse response is thus a finite length sequence of length 4 given by

$$\{h[n]\} = \{a_1, a_2, a_3, a_4\}$$

Impulse Response

- Example – The impulse response of the discrete-time accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$

By setting $x[n]= \delta[n]$, we have

$$h(n) = \sum_{k=-\infty}^{n} \delta[k]$$

which is precisely the unit step sequence

Impulse Response

- Example – The impulse response $h[n]$ of factor-of-2 interpolator

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

is obtained by setting $x_u[n] = \delta[n]$, resulting in

$$h[n] = \delta[n-1] + \frac{1}{2}(\delta[n-2] + \delta[n])$$

- The impulse response is thus a finite length sequence of length 3 given by

$$h[n] = \{\underset{\uparrow}{0.5}, 1, 0.5\}$$

Time-Domain Characterization of LTI Discrete Time Systems

- Input-output relation – A consequence of the linear and time-invariant properties is that an LTI discrete time system is completely characterized by its impulse response

- In other words, knowing the impulse response one can compute the output of the system for an arbitrary input

Compute Impulse Response

- Let $h[n]$ denote the impulse response of a LTI discrete-time system.
- We compute its output $y[n]$ for the input:

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$$

- As the system is linear, we can compute its outputs for each term in the input separately and add the individual outputs to determine $y[n]$.

Compute Impulse Response

- Since the system is time-invariant, we have

Input		Output
$\delta[n+2]$	$\rightarrow$	$h[n+2]$
$\delta[n-1]$	$\rightarrow$	$h[n-1]$
$\delta[n-2]$	$\rightarrow$	$h[n-2]$
$\delta[n-5]$	$\rightarrow$	$h[n-5]$

Compute Impulse Response

- Likewise, as the system is linear, we have

Input		Output
$0.5\delta[n+2]$	$\rightarrow$	$0.5h[n+2]$
$1.5\delta[n-1]$	$\rightarrow$	$1.5h[n-1]$
$\delta[n-2]$	$\rightarrow$	$h[n-2]$
$0.75\delta[n-5]$	$\rightarrow$	$0.75h[n-5]$

- Hence, because of the linear property, we get

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$

Compute Impulse Response

- Recall, an arbitrary input sequence $x[n]$ can be expressed as a linear combination of delayed and advanced unit impulse sequence in the form

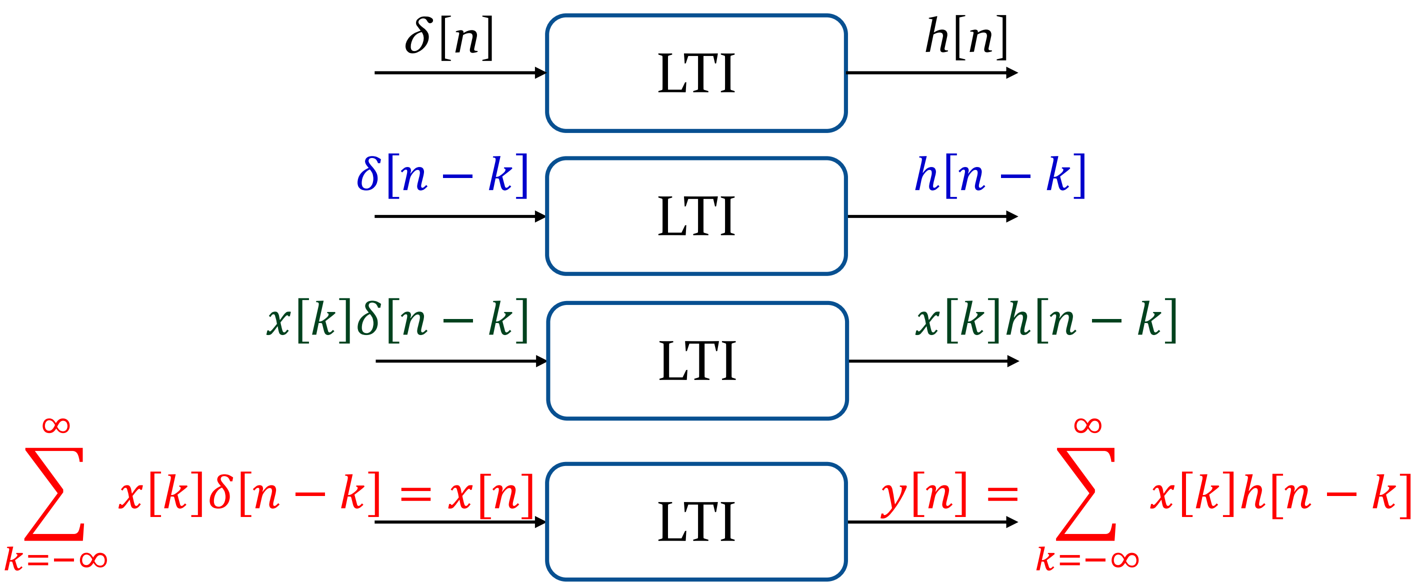
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- The response of the LTI system to an input $x[k]\delta[n-k]$ will be $x[k]h[n-k]$
- Hence, the overall output is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Compute Impulse Response

- The impulse response $h[n]$ completely characterizes an LTI system. "DNA of LTI"



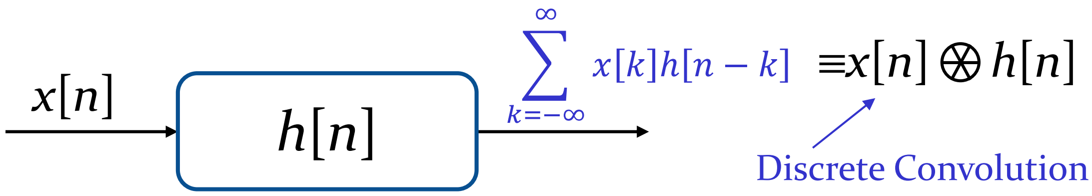
Compute Impulse Response

- Mathematically,

$$y[n] = \text{LTI}\{x[n]\} = \text{LTI}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\text{LTI}\{\delta[n-k]\} = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Discrete (Linear) Convolution



$$x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Sum of weighted and delayed impulse response

$$= \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] \circledast x[n]$$

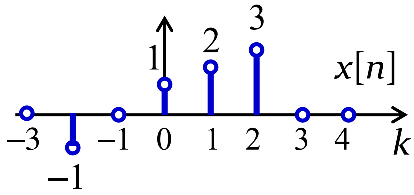
The above summation is defined to be the convolution of the sequences $x[n]$ and $h[n]$ and represented compactly as

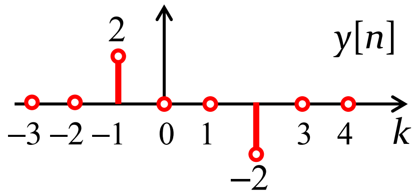
$$y[n] = x[n] \circledast h[n]$$

主观题 8分

设置

Find the convolution of $x[n]$ and $y[n]$ given below





$$z[n] = x[n] \circledast y[n] = ??$$

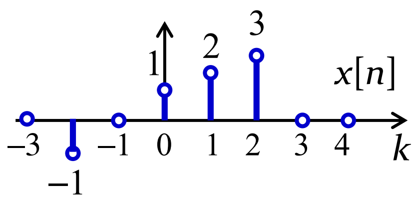
正常使用主观题需2.0以上版本雨课堂

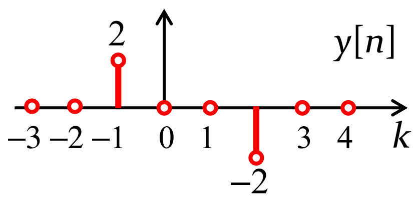
作答

填空题 2分

设置

Find the convolution of $x[n]$ and $y[n]$ given below





$z[n] = x[n] \circledast y[n] =$[填空1]

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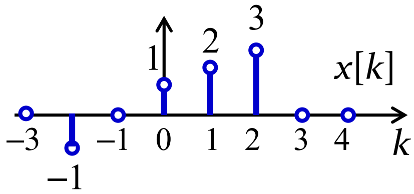
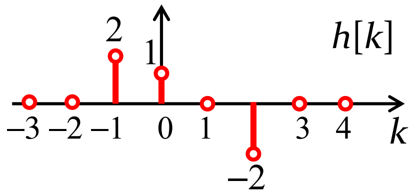
作答

An Illustrative Example

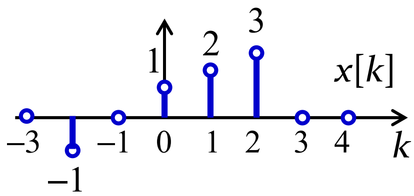
Compute the convolution of $x[n]$ and $h[n]$

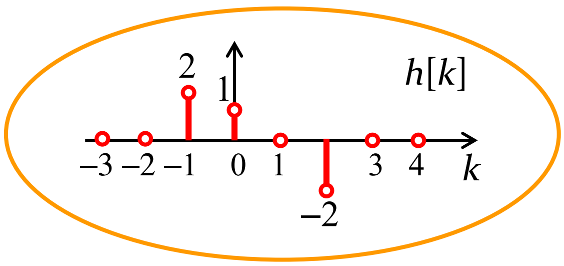


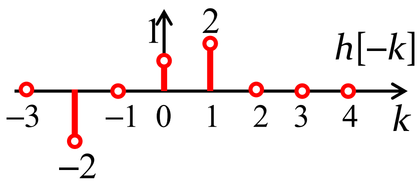


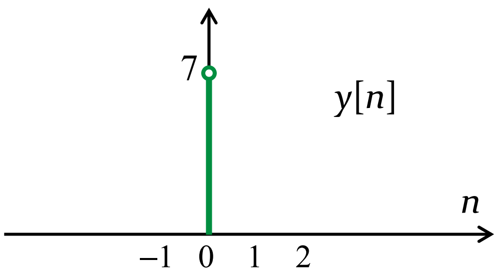
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$









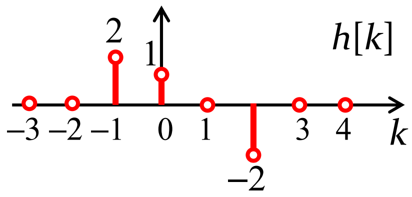
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

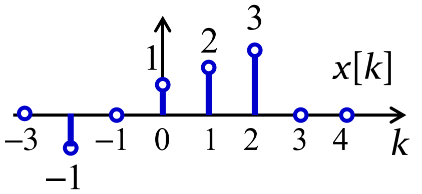
$$y[0] = \sum_{k=-\infty}^{\infty} x[k]h[-k]$$

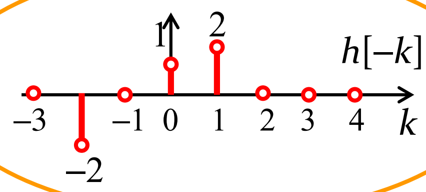
$$= x[-2]h[2] + x[-1]h[1] + x[0]h[0]$$

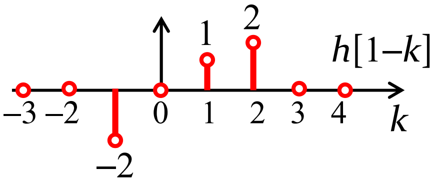
$$+x[1]h[-1] + x[2]h[-2]$$

$$= -1 \times (-2) + 1 \times 1 + 2 \times 2 = 7$$

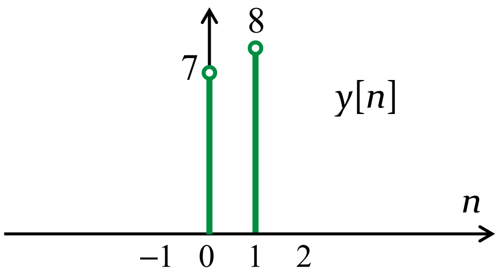






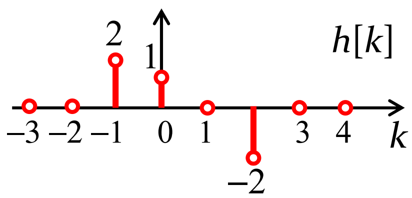


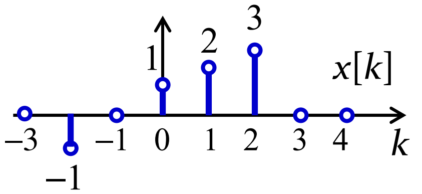
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

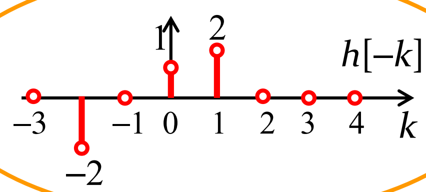


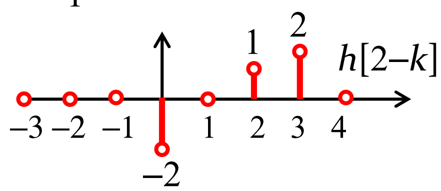
$$y[1] = \sum_{k=-\infty}^{\infty} x[k]h[1-k]$$

$$= x[-2]h[3] + x[-1]h[2] + x[0]h[1]$$
$$+x[1]h[0] + x[2]h[-1]$$
$$= 2\times1+3\times2 = 8$$

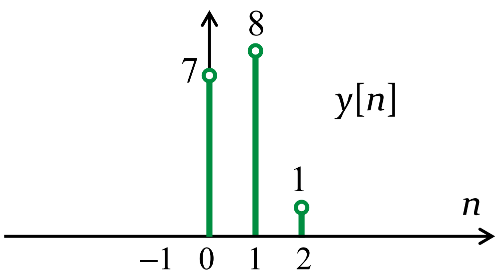






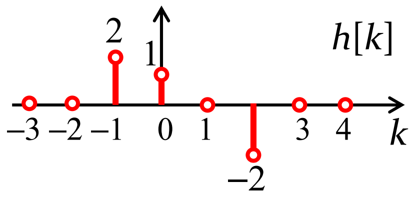


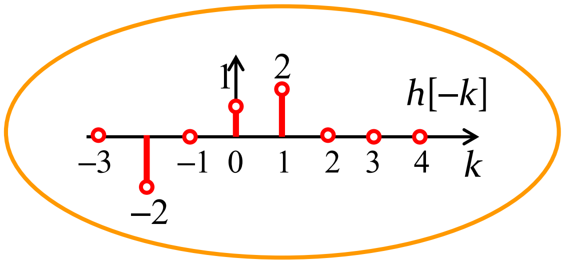
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

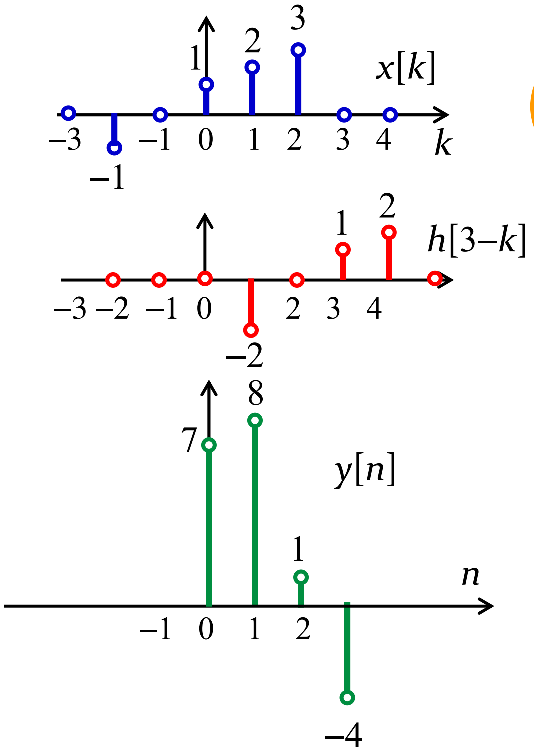


$$y[2] = \sum_{k=-\infty}^{\infty} x[k]h[2-k]$$

$$= x[-2]h[4] + x[-1]h[3] + x[0]h[2]$$
$$+x[1]h[1] + x[2]h[0]$$
$$= 1\times(-2)+3\times1=1$$



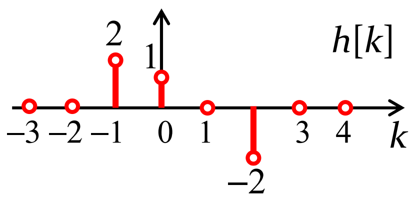


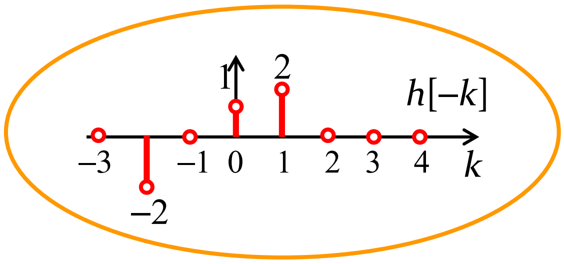


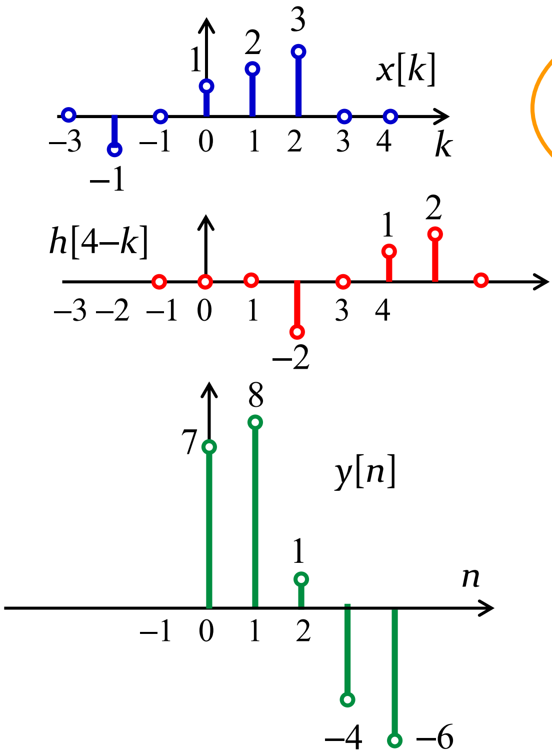
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[3] = \sum_{k=-\infty}^{\infty} x[k]h[3-k]$$

$$= x[-2]h[5] + x[-1]h[4] + x[0]h[3]$$
$$+x[1]h[2] + x[2]h[1]$$
$$= 2\times(-2) = -4$$



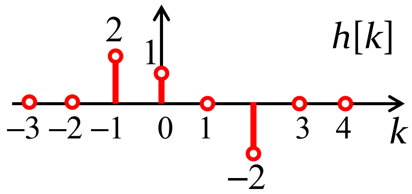


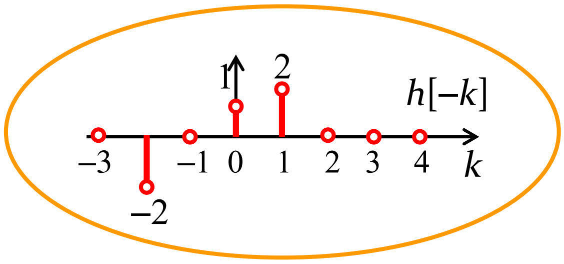


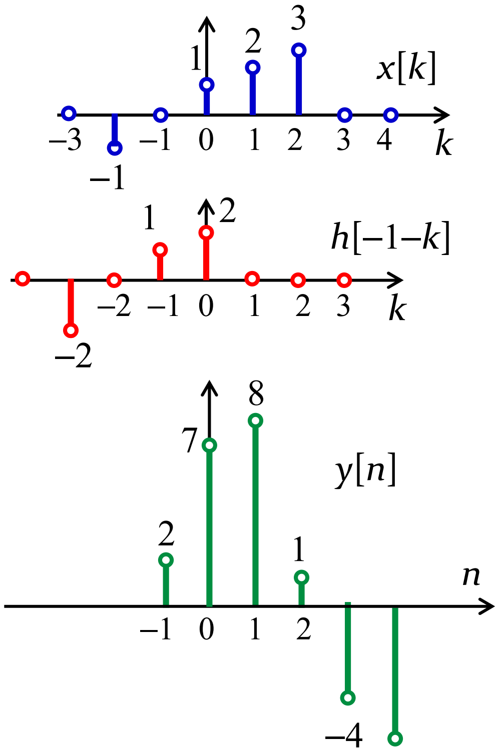
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[4] = \sum_{k=-\infty}^{\infty} x[k]h[4-k]$$

$$= x[-2]h[6] + x[-1]h[5] + x[0]h[4]$$
$$+x[1]h[3] + x[2]h[2]$$
$$= 3\times(-2) = -6$$



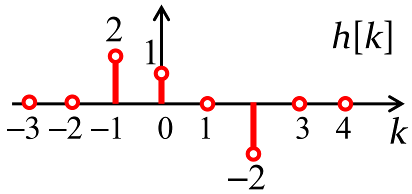


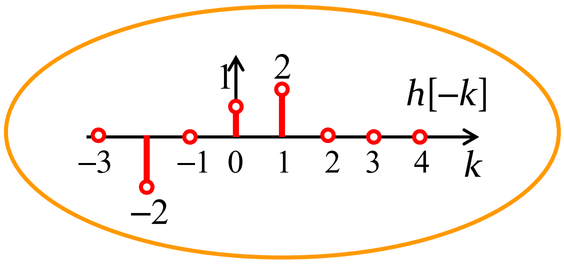


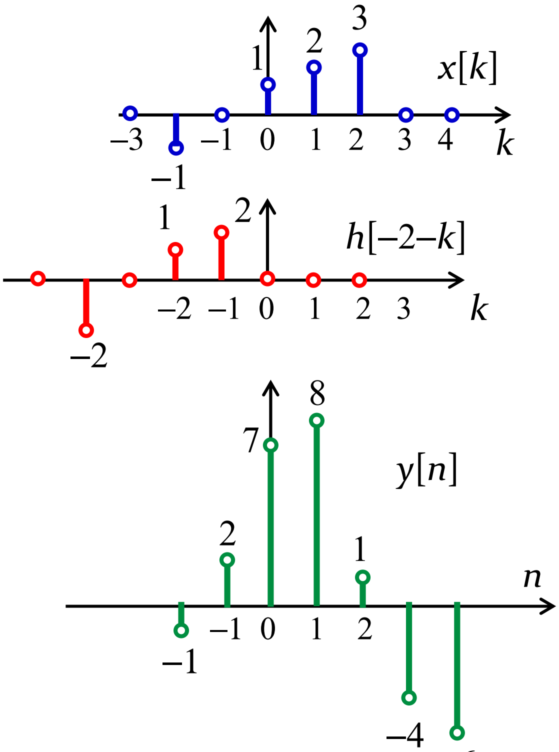
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$y[-1] = \sum_{k=-\infty}^{\infty} x[k]h[-1-k]$$

$$= x[-2]h[1] + x[-1]h[0] + x[0]h[-1]$$
$$+x[1]h[-2] + x[2]h[-3]$$
$$= 1 \times 2 = 2$$



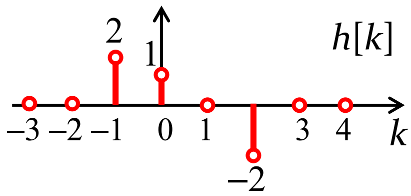


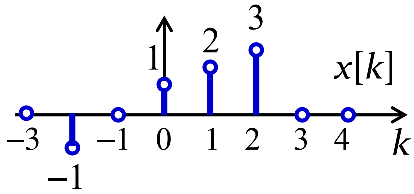


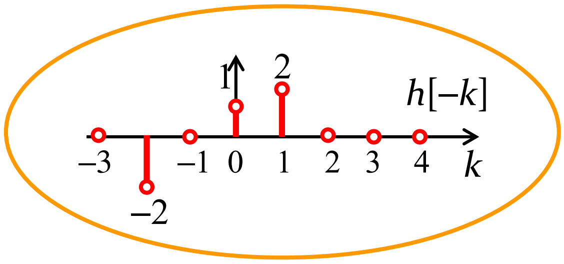
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

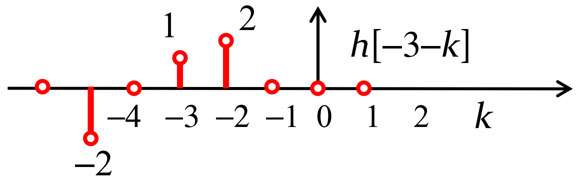
$$y[-2] = \sum_{k=-\infty}^{\infty} x[k]h[-2-k]$$

$$= x[-2]h[0] + x[-1]h[-1] + x[0]h[-2]$$
$$+x[1]h[-3] + x[2]h[-4]$$
$$= -1 \times 1 = -1$$

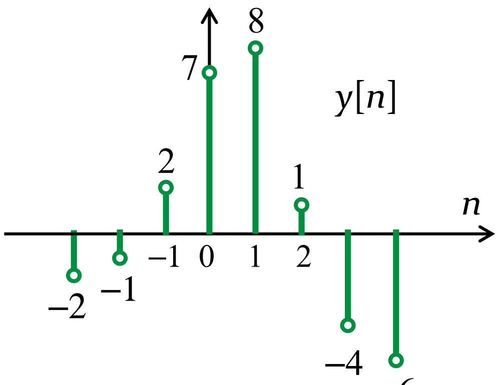








$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$



$$y[-3] = \sum_{k=-\infty}^{\infty} x[k]h[-3-k]$$

$$= x[-2]h[-1] + x[-1]h[-2] + x[0]h[-3]$$
$$+x[1]h[-4] + x[2]h[-5]$$
$$= -1 \times 2 = -2$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Computation of Convolution

- Fold $h[k]$ with respect to the origin to obtain $h[-k]$.
- Form the product of the corresponding samples of sequences $h[-k]$ and $x[k]$, and sum all product terms to obtain $y[0]$.
- Shift $h[-k]$ to the right by n samples (more specifically, shift to right by $|n|$ samples if $n>0$ and shift to left by $|n|$ samples if $n<0$).
- Form the product of the corresponding samples of sequences $h[n-k]$ and $x[k]$, and sum all product terms to obtain $y[n]$.
- Overall, fold, shift, product, and sum

- <u>**Note**</u>: The sum of indices of each sample product inside the convolution sum is equal to the index of the sample being generated by the convolution operation
  - For example, the computation of $y[-3]$ in the previous example involves the products, $x[-2]h[-1]$, $x[-1]h[-2]$, $x[0]h[-3]$, $x[1]h[-4]$, and $x[2]h[-5]$
  - The sum of indices in each of these products is equal to -3
- In general, if the lengths of the two sequences being convolved are M and N, then the sequence generated by the convolution is of length $M+N-1$

Example:

- If $x[n] \xrightarrow{} \boxed{h[n]} \xrightarrow{} y[n]$

  i.e., $y[n] = x[n] \circledast h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

- How about $x[n] \xrightarrow{} \boxed{h[n-m]} \xrightarrow{} y_1[n]$

$$y_1[n] = x[n] \circledast h[n-m] = \sum_{k=-\infty}^{\infty} x[k]h[n-m-k]$$

$$= y[n-m]$$

Properties of Convolution

- Communitive: $x[n] \circledast h[n] = h[n] \circledast x[n]$
- Associative: $x[n] \circledast (h[n] \circledast g[n]) = (x[n] \circledast h[n]) \circledast g[n]$
- Linear:

  $$x[n] \circledast (\alpha h[n] + \beta g[n]) = \alpha x[n] \circledast h[n] + \beta x[n] \circledast g[n]$$
- Sequence shifting is equivalent to convolve with a shifted impulse

  $$x[n-d] = x[n] \circledast \delta[n-d]$$

Computation of LTI System Output

- In practice, if either the input or the impulse response, or both of them, are finite length, the convolution can be used to compute the output sample, as it involves a finite sum of products.
- If both the input sequence and the impulse response sequence are of infinite length, convolution can **NOT** be used to compute the output.
- For system characterized by an infinite impulse response sequence, an alternate time-domain description involving a finite sum of products have to be considered.

BIBO Stability of LTI Systems

- Recall, BIBO stability condition:
  - iff $\{y[n]\}$ remains bounded for any bounded input sequence $\{x[n]\}$
- An LTI system is BIBO stable iff $h[n]$ is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

BIBO Stability of LTI Systems

- Proof: "if" **(Sufficient Condition)**

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]|$$

$$= \sum_{k=-\infty}^{\infty} |h[k]| \underbrace{|x[n-k]|}_{\leq B_x}$$

$$\leq B_x \underbrace{\sum_{k=-\infty}^{\infty} |h[k]|}_{< \infty} \leq B_y$$

BIBO Stability of LTI Systems

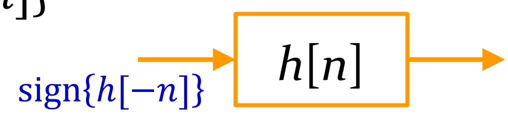
- Proof: "only if" **(Necessary Condition)**

suppose $\sum_{k=-\infty}^{\infty} |h[k]| = \infty$, show that for that $h[k]$, there always exists a bounded $x[n]$ that gives unbounded $y[n]$.

- Let:

Assume $h[n]$ is real sequence

$$x[n] = \frac{h[-n]}{|h[-n]|} = \mathrm{sign}\{h[-n]\}$$



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$y[0] = \sum_{k=-\infty}^{\infty} h[k]x[-k] = \sum_{k=-\infty}^{\infty} \frac{h[k]h[k]}{|h[k]|} = \sum_{k=-\infty}^{\infty} |h[k]| = \infty$$

BIBO Stability of LTI Systems

- Example – Consider an LTI discrete-time system with an impulse response $h[n] = \alpha^n \mu[n]$
- For this system:

$$S = \sum_{n=-\infty}^{\infty} |\alpha^n|\mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|}, \text{if } |\alpha| < 1$$

- Therefor $S < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable.
- If $|\alpha| = 1$, the system is not BIBO stable.

Causality of LTI Systems

- An LTI system is causal iff

$$h[k] = 0 \qquad \text{for } k < 0$$

- Proof: $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

$$= \sum_{k=-\infty}^{-1} h[k]x[n-k] + \sum_{k=0}^{\infty} h[k]x[n-k]$$

Sufficient: Since $h[k] = 0$ for $k < 0$, the first term is 0. So, $y[n_0]$ is independent of $x[n_0+1]$, $x[n_0+2]$, … .
Necessary: The system is causal, i.e., $y[n_0]$ independent of $x[n_0+1]$, $x[n_0+2]$, … , implying that the first term is equal to 0. Since $x[n]$ may not be equal to 0, $h(k)$ must be equal to 0 for $k < 0$

Examples

- **Causal LTI system**
  - Accumulator: $y[n] = \sum_{l=-\infty}^{n} x[l]$

    $h[n] = \mu[n]$, a causal impulse response sequence

- **Non causal LTI system**
  - Factor-of-2 interpolator

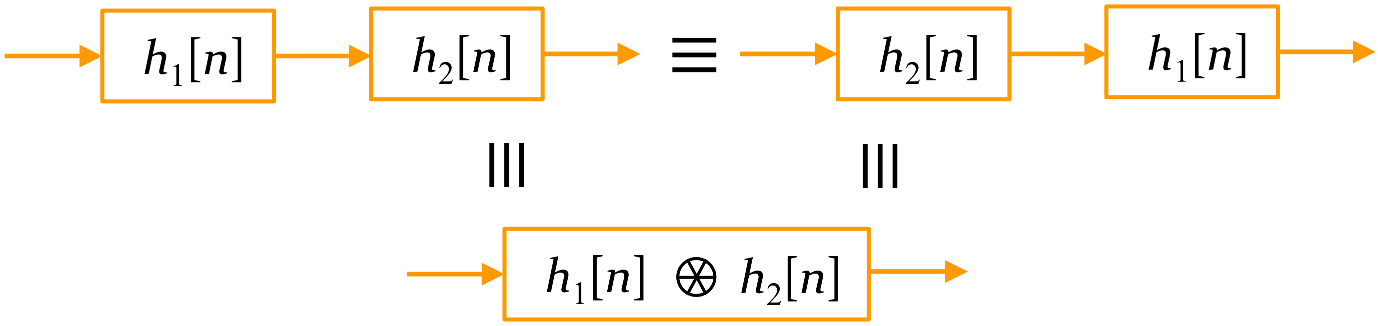
$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$h[n] = \{0.5, \underset{\uparrow}{1}, 0.5\}$$

a non causal impulse response sequence

Simple Interconnection Schemes

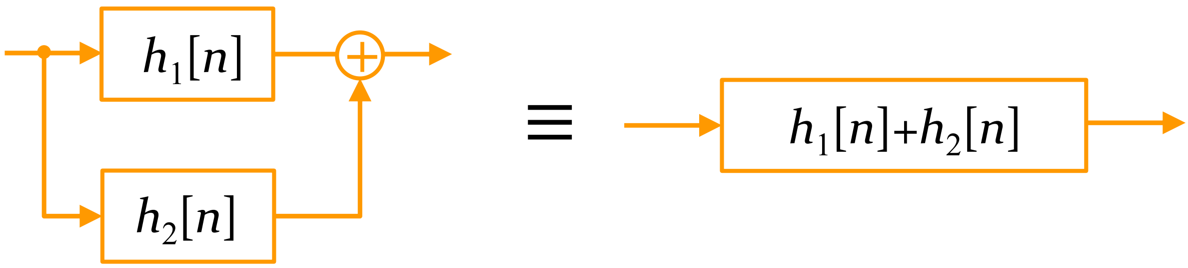
- Cascade Connection



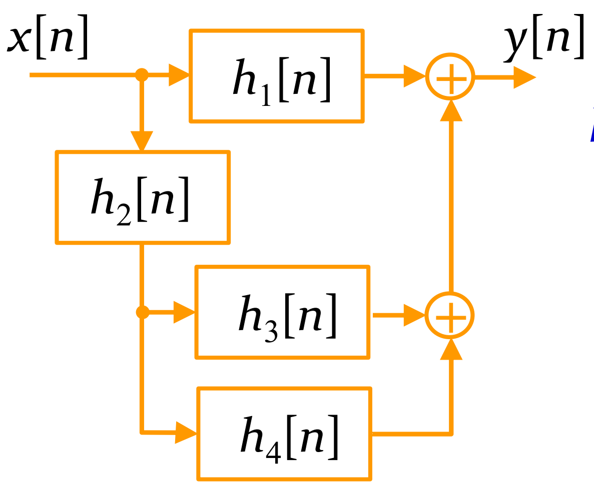
- If $h_1[n] \circledast h_2[n] = \delta[n]$

  system $h_1[n]$ is said to be the **inverse** of system $h_2[n]$, and vice versa.

- Parallel Connection



Analysis of Cascade and Parallel connections



$$y[n] = x[n] \circledast h[n]$$

$$h[n]=h_1[n] + h_2[n] \circledast (h_3[n]+h_4[n])$$

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

主观题 1分 设置

$$h[n]=h_1[n] + h_2[n] \circledast (h_3[n]+h_4[n])$$

$$h_1[n] = \delta[n] + \frac{1}{2}\delta[n-1]$$

$$h_2[n] = \frac{1}{2}\delta[n] - \frac{1}{4}\delta[n-1]$$

$$h_3[n] = 2\delta[n]$$

$$h_4[n] = -2\left(\frac{1}{2}\right)^n \mu[n]$$

What is the equivalent $h[n]$? Use $d[n]$ to represent $\delta[n]$ if the later is not able to be keyed in.

作答

General Difference Equation

- An important subclass of LTI system is characterized by a linear constant-coefficient difference equation of the form:

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

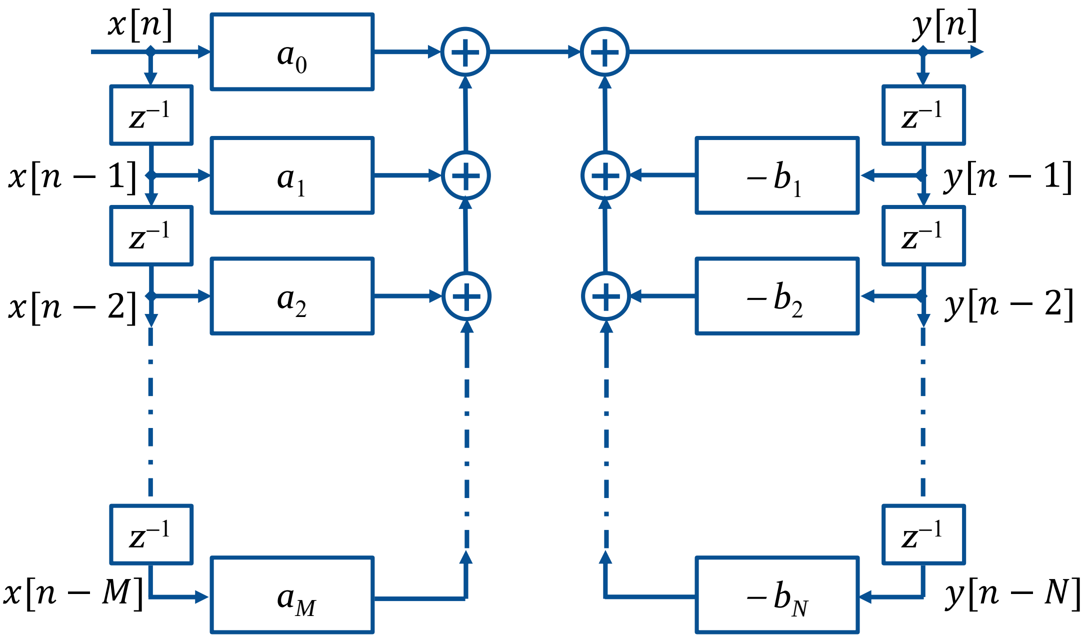
- $y[n]$ can be computed recursively

$$b_0 y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

Signal-flow graph

- When b_0 is normalized to $b_0 = 1$

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$



Classification of LTI System

- Based on Impulse Response Length
  - If $h[n]$ is of **finite** length, then it is known as a **finite impulse response (FIR)** discrete time system.

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

  - Examples: Moving averaged filter, factor-2-interpolator

$$y[n] = \sum_{k=0}^{4} \frac{1}{5} x[n-k]$$

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

- If $h[n]$ is of **infinite** length, then it is know as a **infinite impulse response (IIR)** discrete-time system.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- If causal (system causal, input sequence causal),

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k]$$

- The class of IIR system we are concerned with is the causal system characterized by the linear constant coefficient difference equation.
- Example: accumulator

$$y[n] = \sum_{l=-\infty}^{n} x[l] = y[n-1] + x[n]$$

- Based on the Output Calculation Process
  - **Non-recursive** discrete time system: Computation of output samples involves only the present and past input samples. Example: FIR system.
  - **Recursive** discrete time system: Computation of output samples involves NOT ONLY the present and past input samples, but also the **past output** samples. Example: IIR system implemented using difference equation.
  - It's possible to implement an FIR system using a recursive scheme, e.g., $y[n] = y[n-1] + \frac{1}{M}(x[n] - x[n-M])$

- Based on the Coefficient Values
  - **Real coefficient** filters.
  - **Complex coefficient** filters.

Lecture 4
Frequency Domain Representation of Discrete Time Signal

- The frequency domain representation of discrete time sequence is the **discrete-time Fourier transform (DTFT).** DTFT is a frequency analysis tool for aperiodic discrete-time signals

- This transform maps a time-domain sequence into a continuous function of the frequency variable ω.

Review of Continuous-Time Fourier Transform (CTFT)

- CTFT:

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt .$$

Fourier Spectrum
Or Spectrum

- Inverse CTFT:

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} d\Omega .$$

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

Definition of Discrete-Time Fourier Transform (DTFT)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

DTFT

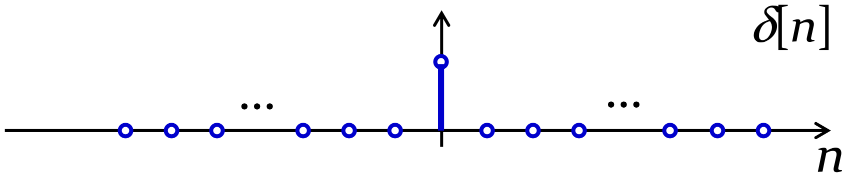
where, ω is a continuous variable in the range $-\infty < \omega < \infty$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Inverse DTFT

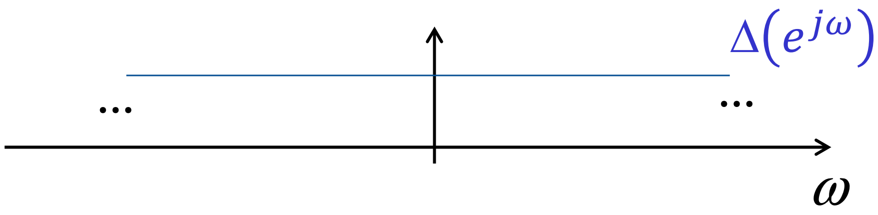
Why one is sum and the other integral?

Example 1



DTFT:

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \sum_{n=0} e^{-j\omega n} = 1$$



Example 2

- Causal sequence $x[n] = \alpha^n \mu[n]$, $|\alpha|<1$

  Its DTFT is given by

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

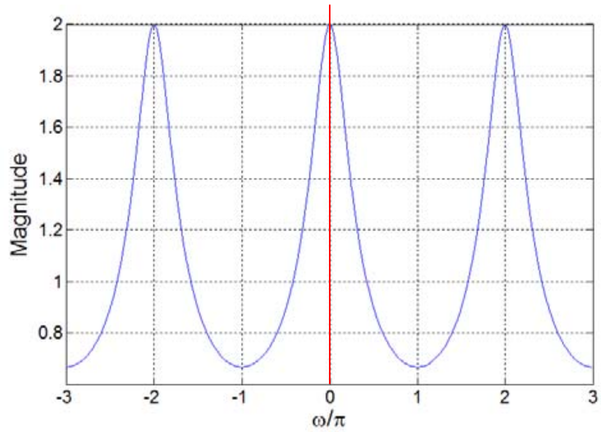
as $\left|\alpha e^{-j\omega}\right| = |\alpha|<1$

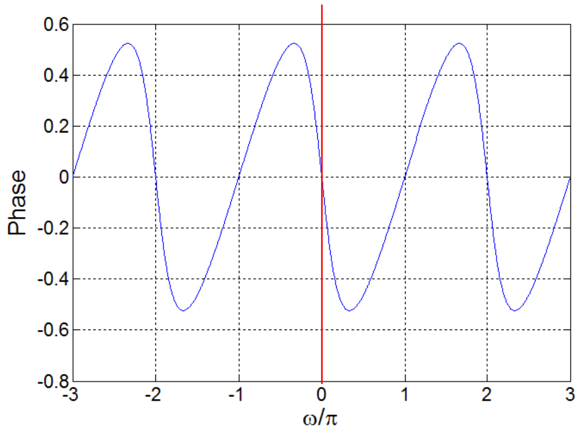
$$\boxed{\text{Recall: } 1 + q + q^2 + \cdots + q^{\infty} = \frac{1}{1-q} \qquad \text{for } |q| < 1}$$

Example 2 (Cont'd)

- The magnitude and phase of DTFT of

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$





$|X(e^{j\omega})| = |X(e^{-j\omega})|$ $\theta(\omega) = -\theta(-\omega)$

DTFT

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω, and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$$

- $X(e^{j\omega})$ can alternatively be expressed as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

where, $\theta(\omega) = \arg\{X(e^{j\omega})\}$

  - $|X(e^{j\omega})|$ is called the magnitude function
  - $\theta(\omega)$ is called the phase function
  - In applications where DTFT is called Fourier spectrum, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called magnitude and phase spectra

Symmetry of DTFT

- For **a real sequence $x[n]$**, $\left|X(e^{j\omega})\right|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω, whereas $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω.
- Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$$
$$= \left\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right\}^{*}, \text{for real } x[n]$$
$$= X^{*}(e^{j\omega})$$

- Therefore, $\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$ and $\theta(\omega) = -\theta(-\omega)$

Periodicity of DTFT

- $X(e^{j\omega}) = X(e^{j(\omega+2k\pi)})$, i.e. $\left|X(e^{j\omega})\right|e^{j[\theta(\omega)+2k\pi]} = \left|X(e^{j\omega})\right|e^{j\theta(\omega)}$ for any integer k.
- Proof:

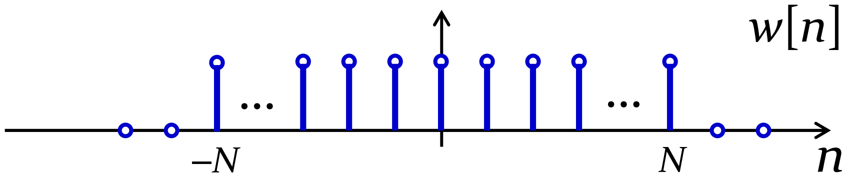
$$X(e^{j(\omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$= X(e^{j\omega})$$

- In other words, the phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT.
- Unless otherwise stated, we assume that the phase function $\theta(\omega)$ is restricted to the range of

$$-\pi < \theta(\omega) < \pi$$

called the **principle value**.

Example 3



DTFT:

$$W\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty} w[n]e^{-j\omega k}=\sum_{k=-N}^{N} e^{-j\omega k}$$

$$=e^{-j\omega N}\left(1+e^{j\omega}+e^{j2\omega}+\cdots+e^{j2N\omega}\right)$$

Recall: $1+q+q^2+\cdots+q^M=\dfrac{1-q^{M+1}}{1-q}$ $\quad q=e^{j\omega}$ $\quad M=2N$

Example 3 Cont.

DTFT: $W\left(e^{j\omega}\right)=e^{-j\omega N}\left(1+e^{j\omega}+e^{j2\omega}+\cdots+e^{j2N\omega}\right)$

$$=e^{-j\omega N}\frac{1-e^{j\omega(2N+1)}}{1-e^{j\omega}}$$

$$=\frac{e^{-j\omega N}-e^{j\omega N}e^{j\omega}}{1-e^{j\omega}} \qquad \times\frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}$$

$$=\frac{e^{-j\omega\left(N+\frac{1}{2}\right)}-e^{j\omega\left(N+\frac{1}{2}\right)}}{e^{-j\frac{\omega}{2}}-e^{j\frac{\omega}{2}}}$$

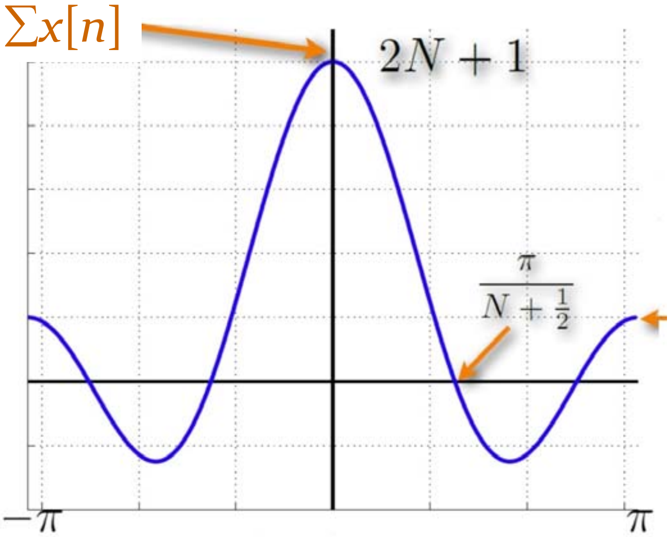
$$=\frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

Periodic Sinc

Example 3 Cont.

$$W(e^{j\omega}) = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)} \longrightarrow 2N+1 \quad \text{as } \omega \longrightarrow 0$$

Also = $\sum x[n]$



=1 if N is even. Why?

Example of Inverse DTFT

- **Find the inverse DTFT of**

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- A:

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \boxed{\frac{\sin \omega_c n}{\pi n}}, \text{for } n \neq 0$$

$$h_{LP}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

Properties of the DTFT

- **Linearity:**

Let $\quad g[n] \leftrightarrow G(e^{j\omega})$ and $h[n] \leftrightarrow H(e^{j\omega})$

Then $\quad \alpha g[n] + \beta h[n] \leftrightarrow \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$

- **Periodicity:** $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Properties of the DTFT Cont.

- **Time Reversal:**

Let $\quad x[n] \leftrightarrow X(e^{j\omega})$

Then $\quad x[-n] \leftrightarrow X(e^{-j\omega})$

$= X^*(e^{j\omega})$ if $x[n]$ is real

If $x[n]$=$x[-n]$ and $x[n]$ is real, then

$X(e^{j\omega}) = X^*(e^{j\omega}) \;\; \rightarrow \;\; X(e^{j\omega})$ is real

- Q: Suppose $x[n] \leftrightarrow X(e^{j\omega})$, $x[n] \in \mathcal{R}eal$

$$? \leftrightarrow \mathcal{R}e\{X(e^{j\omega})\}$$

- A: Decompose $x[n]$ to even and odd functions

$$x[n]=x_e[n]+x_o[n],$$

where
$$x_e[n] = \frac{1}{2}(x[n] + x[-n])$$
$$x_o[n] = \frac{1}{2}(x[n] - x[-n])$$

$$\boxed{x_e[n]} \leftrightarrow X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{-j\omega})\right)$$
$$= \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{j\omega})\right) = \mathcal{R}e\left\{X(e^{j\omega})\right\}$$

$$x_o[n] \leftrightarrow X_o(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) - X(e^{-j\omega})\right) = j\mathcal{I}m\left\{X(e^{j\omega})\right\}$$

Properties of the DTFT Cont.

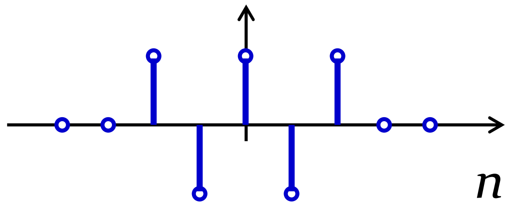
- **Time and Frequency Shifting**

Let
$$x[n] \leftrightarrow X(e^{j\omega})$$
Then
$$x[n-n_d] \leftrightarrow e^{-j\omega n_d}X(e^{j\omega})$$
$$e^{j\omega_0 n}x[n] \leftrightarrow X(e^{j(\omega-\omega_0)})$$

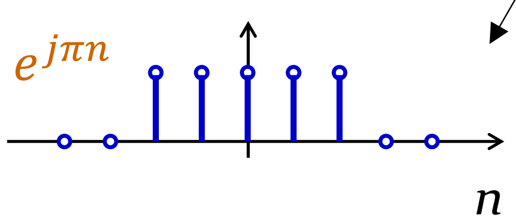
Example 4

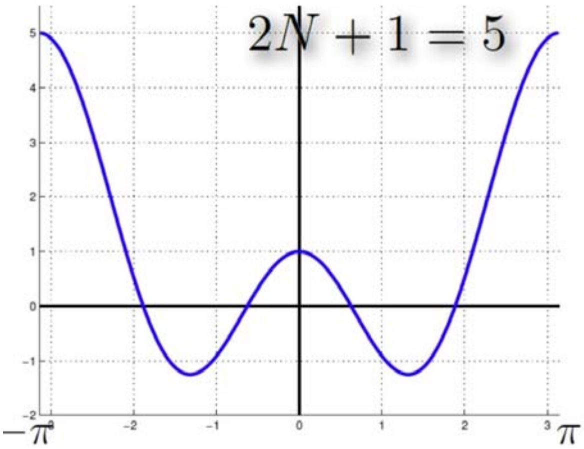
$$W(e^{j\omega}) = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

What is the DTFT of:



$\times\, e^{j\pi n}$





$$W(e^{j\omega}) = \frac{\sin\left((\omega-\pi)\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega-\pi}{2}\right)}$$

Properties of the DTFT Cont.

- **Differentiation in frequency**

Let $x[n] \leftrightarrow X(e^{j\omega})$

Then $nx[n] \leftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$

Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Differentiate both side to get $\dfrac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$

Multiply both side by j, we get $j\dfrac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$

Example 5

- Determine DTFT $Y(e^{j\omega})$ of
$$y[n] = (n+1)\alpha^n \mu[n], \qquad |\alpha| < 1$$
- Let $x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$
- We can therefore write
$$y[n] = nx[n] + x[n]$$
- From example 2, we have known that the DTFT of $x[n]$ is given by
$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

- Using the differentiation in frequency, we observe that DTFT of $nx[n]$ is given by,
$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$
- Next, using linear property of DTFT, we arrive at
$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

Properties of the DTFT Cont.

- **Convolution**

  Let $x[n] \leftrightarrow X(e^{j\omega})$ and $h[n] \leftrightarrow H(e^{j\omega})$

  - DTFT convolution theorem

  If $y[n] = x[n] \circledast h[n]$

  Then $y[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

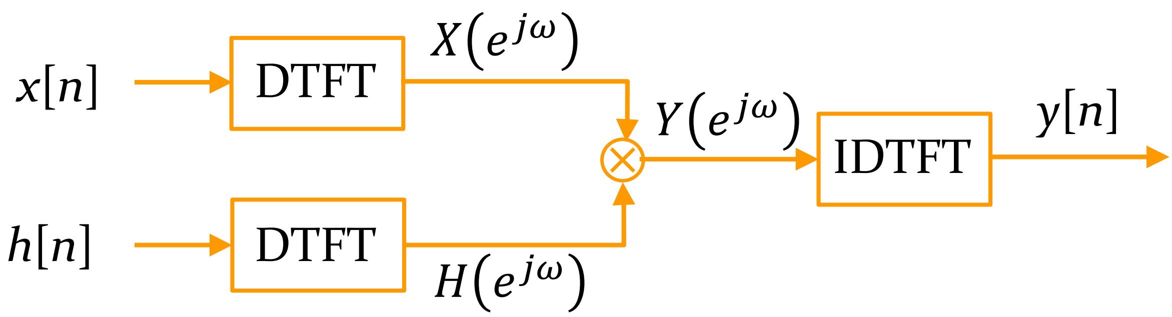
  - **DTFT convolution modulation**

  If $y[n] = x[n]h[n]$

  Then $y[n] \leftrightarrow Y(e^{j\omega}) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$

Linear Convolution Using DTFT

- Linear convolution $y[n]$ of the sequence $x[n]$ and $h[n]$ can be performed as follows:
  - Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively.
  - Form DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
  - Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$



Properties of the DTFT Cont.

- **Parseval's theorem**

Let $x[n] \leftrightarrow X(e^{j\omega}) \qquad h[n] \leftrightarrow H(e^{j\omega})$

Then $$\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H^*(e^{j\omega})d\omega$$

Proof: $$\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})e^{-j\omega n}d\omega\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\,X(e^{j\omega})d\omega$$

Energy & Energy Density Spectrum

- Energy: $E_g = \sum_{-\infty}^{\infty}|x[n]|^2$
- According to Parseval's theorem, when $h[n] = x[n]$,

$$E_g = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\boxed{\left|X(e^{j\omega})\right|^2}d\omega$$

- Energy spectral density:

$$S_{xx}(\omega) = \left|X(e^{j\omega})\right|^2$$

Energy Spectral Density

- Example – Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \qquad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega t})\right|^2 d\omega$$

where

$$H_{LP}(e^{j\omega t}) = \begin{cases} 1, & 0 \leq |\omega| \leq \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

- Therefore:

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

Symmetry Relations If $x[n]$ is real

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Conjugate Symmetric	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$
	$X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$
	$\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Symmetry Relations

If $x[n]$ is complex

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x_{\text{re}}[n]$	$X_{\text{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$jx_{\text{im}}[n]$	$X_{\text{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\text{cs}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{ca}}[n]$	$jX_{\text{im}}(e^{j\omega})$

Band-Limited DT Signals

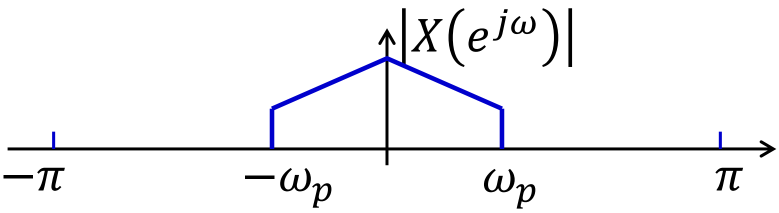
- The spectrum of a DT signal is a periodic function of ω with a period of 2π. Thus a **full-band** signal has a spectrum occupying the frequency range $-\pi < \omega \le \pi$.
- A **band-limited** DT signal has a spectrum that is limited to a portion of the frequency range $-\pi < \omega \le \pi$.
- An ideal real band-limited signal:

$$|X(e^{j\omega})| = \begin{cases} 0, & 0 \le |\omega| < \omega_a \\ \text{non zero}, & \omega_a \le |\omega| \le \omega_b \\ 0, & \omega_b \le |\omega| \le \pi \end{cases}$$

Classification of Band-Limited Signal

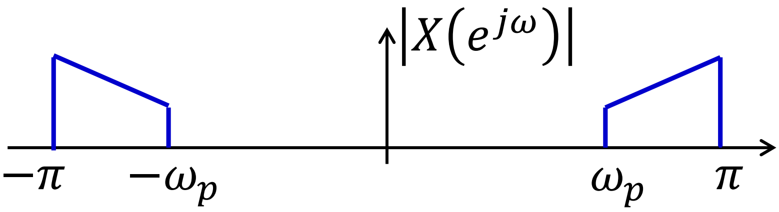
- Lowpass real signal

  Bandwidth: ω_p



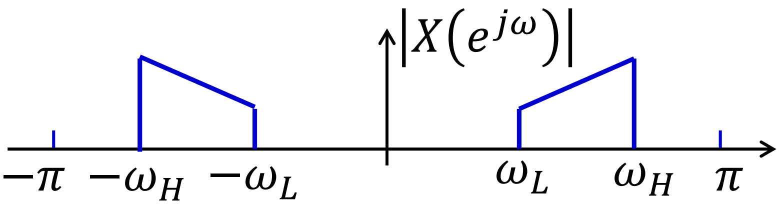
- Highpass real signal

  Bandwidth: $\pi - \omega_p$



- Bandpass real signal

  Bandwidth: $\omega_H - \omega_L$



DTFT Convergence Condition

- The infinite series

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  may or may not converge.

- If it converges for all value of ω, then the DTFT $X(e^{j\omega})$ exists.
- Consider the finite sum:

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

- Strong convergence: $X(e^{j\omega})$ **converge uniformly, i.e.,**

$$\lim_{K\to\infty} X_K(e^{j\omega}) = X(e^{j\omega})$$

  - $x[n]$ absolutely summable $\Longrightarrow$ $X(e^{j\omega})$ **exist and converge uniformly**

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Longrightarrow |X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\le \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  - This is a **sufficient condition**, not necessary.

Example

$x[n] = \alpha^n \mu[n]$, $|\alpha|<1$ is absolutely summable, as

$$\sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and therefore is DTFT $X(e^{j\omega})$ converge to $\frac{1}{1-\alpha e^{-j\omega}}$ uniformly.

Since $$\sum_{n=-\infty}^{\infty}|x[n]|^2 \leq \left(\sum_{-\infty}^{\infty}|x[n]|\right)^2$$

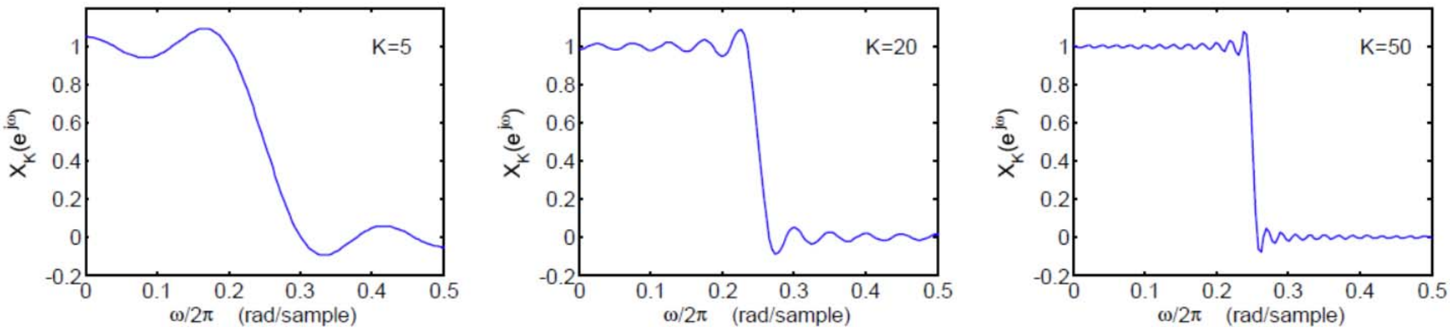
- an absolutely summable sequence has always a finite energy,
- However, a finite energy sequence is not necessary absolutely summable.

Example: a sequence $x[n] = \frac{1}{n}\mu[n-1]$ has a finite energy, as $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$, however, it is not absolutely summable as $\sum_{n=1}^{\infty}\left|\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ does not converge.

- Weak convergence: $X(e^{j\omega})$ converge mean square
$$\lim_{K\to\infty}\int_{-\pi}^{\pi}\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega = 0$$
  - $x[n]$ finite energy $\Longrightarrow X(e^{j\omega})$ converge mean square.
$$\sum_{n=-\infty}^{\infty}|x[n]|^2 < \infty \Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 < \infty$$
  - The absolute value of the error $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ may not go to zero when K goes to infinite.

Example

- $h_{LP}[n] = \frac{\sin 0.5\pi n}{\pi n} \leftrightarrow H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le 0.5\pi \\ 0, & 0.5\pi \le |\omega| \le \pi \end{cases}$
- $\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{\omega_c}{\pi} < \infty$, but not absolutely summable.

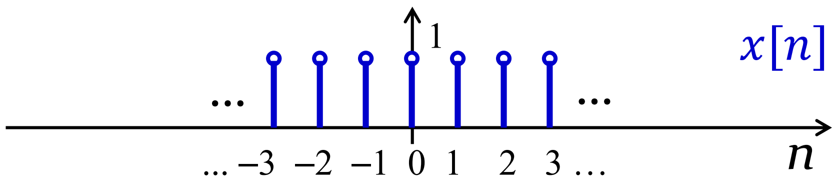


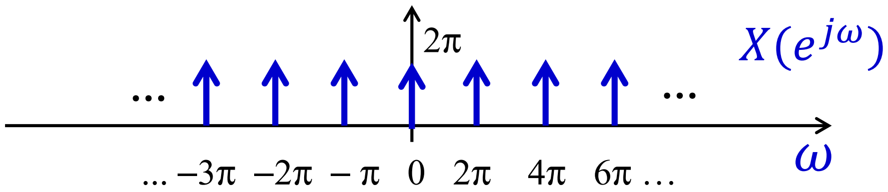
- **Gibbs phenomenon**: mean square converges ~~at each ω~~ as $K \to \infty$, but peak error does not get smaller.

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$ and the sinusoidal sequence $\cos(\omega_0 n + \varphi)$.
- For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$.

Example 5: DTFT of *x*[*n*]=1 for all *n*

- $x[n] = 1 = \sum_{k=-\infty}^{\infty} \delta[n-k]$
- It is more convenient to prove that the inverse DTFT of $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ is 1





- *Proof:* The inverse DTFT of $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ is evaluated as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)\right)e^{j\omega n}d\omega$$

$$= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)\right)e^{j\omega n}d\omega$$

- From the sifting property, we have

$$\left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)\right)e^{j\omega n} = \left(\sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)\right)e^{j2\pi kn}$$

$$= \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

We have used $e^{j2\pi kn} = 1$ for all n here

- When we integrate the sequence of impulse from $-\pi$ to π, we have only the impulse at $\omega = 0$.
- Hence

$$\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)\right)e^{j\omega n}d\omega$$
$$=\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)\right)d\omega$$
$$=\int_{-\pi}^{\pi}\delta(\omega)\,d\omega=\int_{-\infty}^{-\pi}\delta(\omega)\,d\omega+\int_{-\pi}^{\pi}\delta(\omega)\,d\omega+\int_{\pi}^{\infty}\delta(\omega)\,d\omega$$
$$=\int_{-\infty}^{\infty}\delta(\omega)\,d\omega=1\quad\text{for all } n$$

Example 6: DTFT of $\mu[n]$

- Let $\mu[n]=u_1[n]+u_2[n]$, where

$$u_1[n]=\frac{1}{2},\qquad \text{for } -\infty<n<\infty$$

and

$$u_2[n]=\begin{cases}\dfrac{1}{2} & \text{for } n\geq 0\\ -\dfrac{1}{2} & \text{for } n<0\end{cases}$$

Therefore, we have

$$\delta[n]=u_2[n]-u_2[n-1]$$

- Using $\delta[n] \leftrightarrow 1$ and $u_2[n] - u_2[n-1] \leftrightarrow U_2(e^{j\omega}) - e^{-j\omega}U_2(e^{j\omega}) = U_2(e^{j\omega})(1-e^{-j\omega})$, we have

$$1 = U_2(e^{j\omega})(1-e^{-j\omega})$$

i.e.

$$U_2(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} \quad \text{for } \omega \neq 0$$

Since

$$u_1[n] \leftrightarrow \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k) = U_1(e^{j\omega})$$

we have

$$\mu[n] \leftrightarrow \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k) \quad \text{for } \omega \neq 0$$

DTFT Convergence

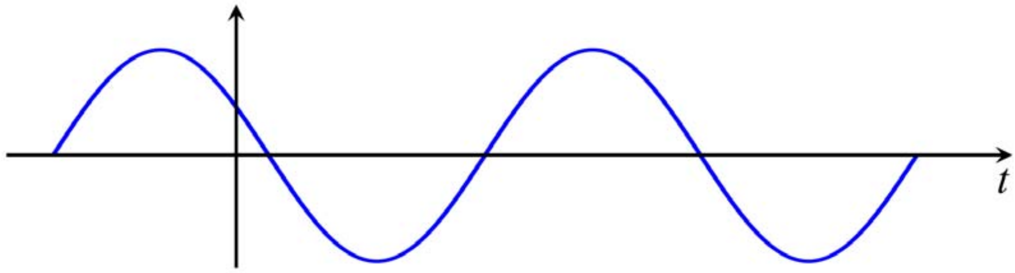
Sequence		DTFT
$\alpha^n\mu[n], (\lvert\alpha\rvert < 1)$ Absolutely Summable	Sufficient $\longrightarrow$	$\frac{1}{1-\alpha e^{-j\omega}}$ Exist for all ω
$\mu[n]$ Neither absolutely summable, nor finite energy	Not necessary $\longrightarrow$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$ Not exist for $\omega = 0$
1 (for all n) Neither absolutely summable, nor finite energy		$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ Exist for all ω
$h_{LP}[n] = \frac{\sin\omega_c n}{\pi n}$ Finite energy	Sufficient $\longrightarrow$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, 0 \le \lvert\omega\rvert \le \omega_c\pi \\ 0, \omega_c\pi \le \lvert\omega\rvert \le \pi \end{cases}$ Exist for all ω

Commonly used DTFT pairs

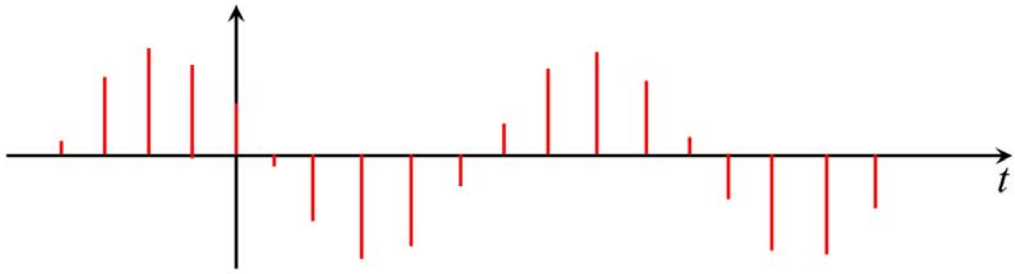
Sequence	DTFT
$\delta[n]$	1
$\mu[n]$	$\dfrac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega+2\pi k)$
1 (for all n)	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-2\pi k)$
$\alpha^n\mu[n], (\lvert\alpha\rvert<1)$	$\dfrac{1}{1-\alpha e^{-j\omega}}$

Effect of Time-Domain Sampling in Frequency Domain

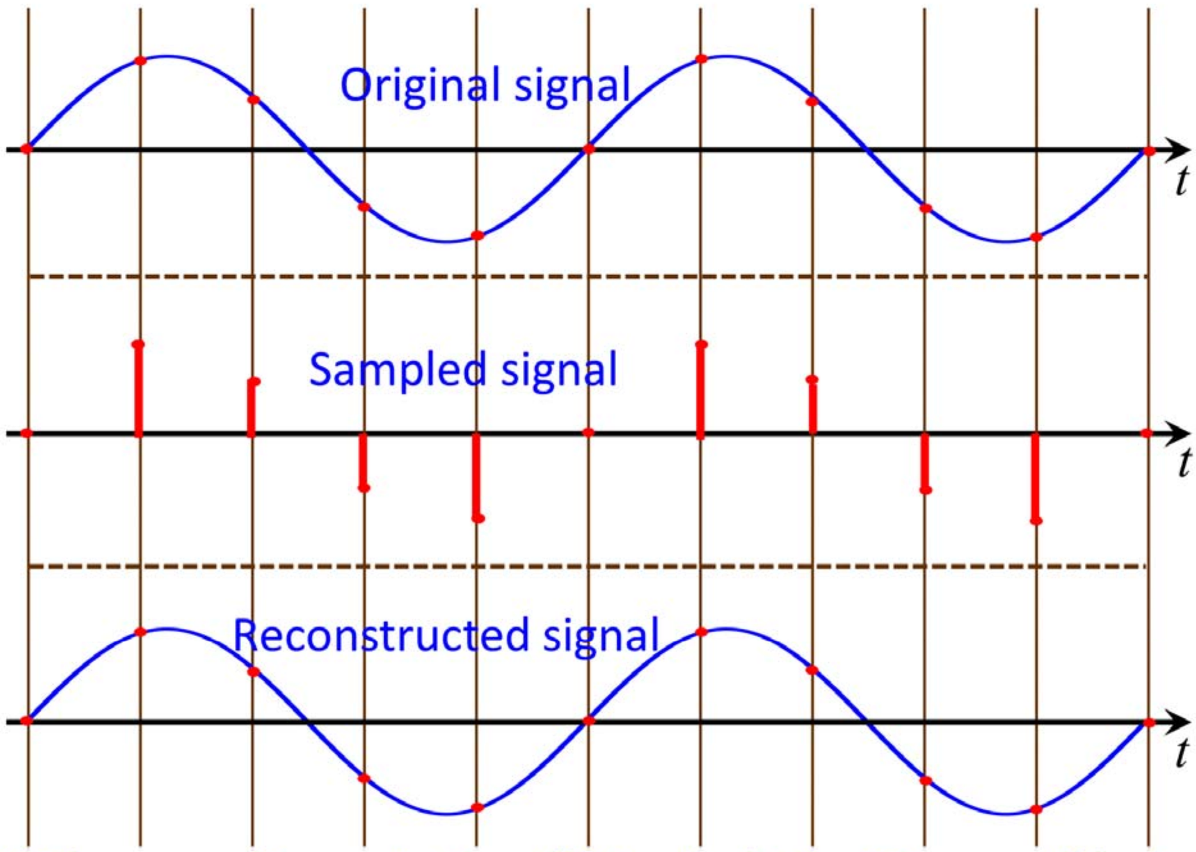
- Questions to be answered
  - When discrete signal is obtained by sampling, can we recover the original continuous signal from the discrete signal?
  - What is the condition that we can recover the continuous signal?
  - Relation Between Continuous and Discrete Signal Spectrum



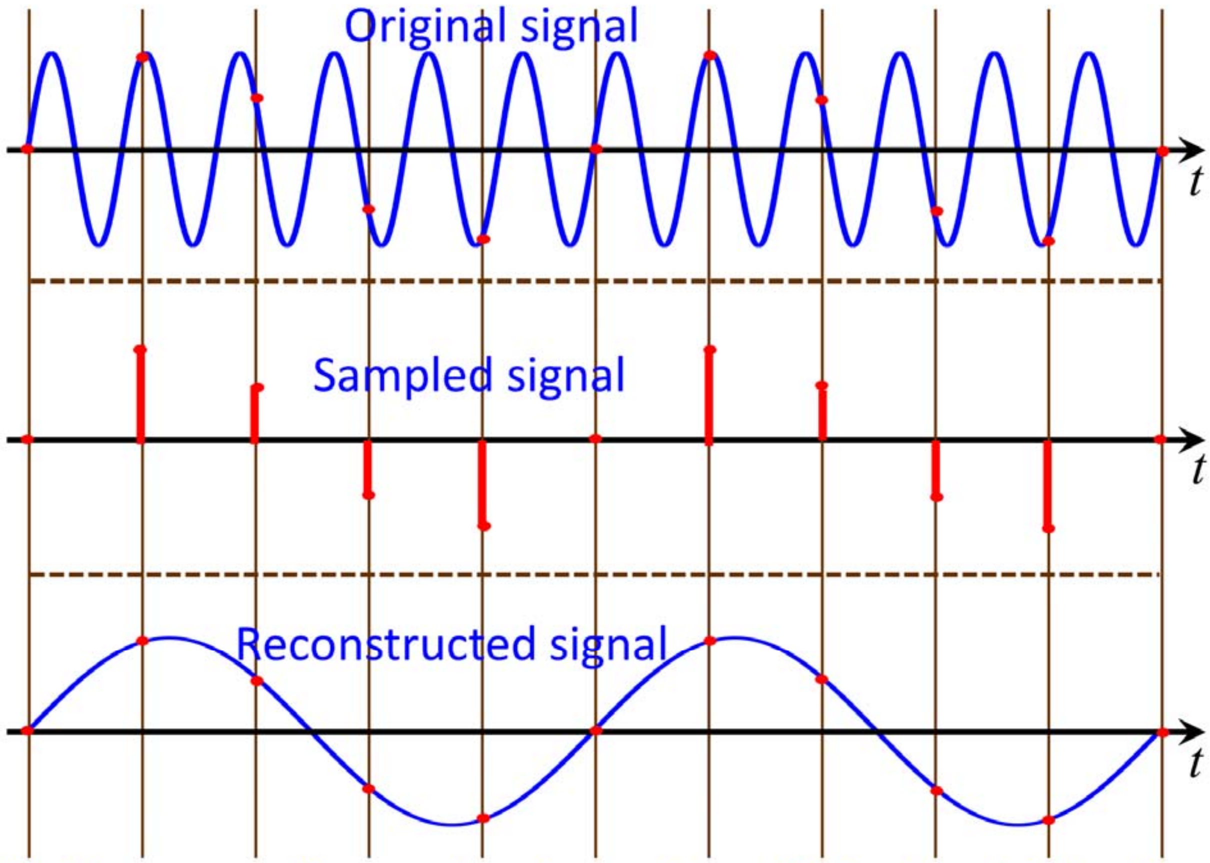
A continuous time signal may be sampled to produce a discrete time signal.



If the rate of sampling is very high, it is obvious by inspection that the sampled result will resemble that of the original continuous time signal.

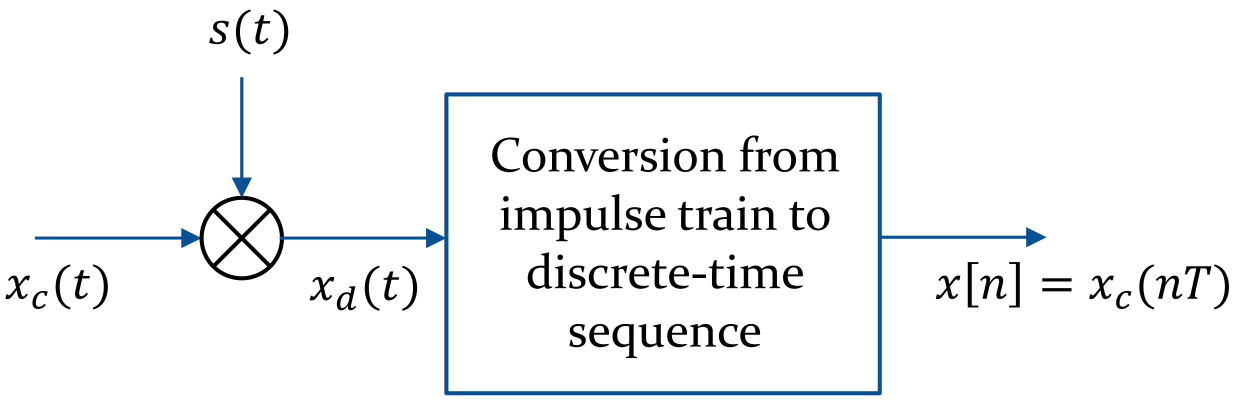


If the sampling rate is sufficiently high, it is possible to reconstruct the original signal from the sampled signal.



If the sampling rate is not sufficiently high, the reconstructed signal is different from the original signal.

Sampling with a periodic impulse train



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ where } \delta(t) = \infty \text{ for } t = 0$$
$$= 0 \text{ for } t \neq 0$$

and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

Frequency-Domain Representation of Sampling

Let $x_c(t)$ be continuous time signal

Let the sampled version of $x_c(t)$ be denoted by $x_d(t)$.

Let the sampling interval be T.

Let $\Omega_s = 2\pi/T$.

Let $X_c(j\Omega)$ be the Fourier transform of $x_c(t)$.

Let $X_d(j\Omega)$ be the Fourier transform of $x_d(t)$.

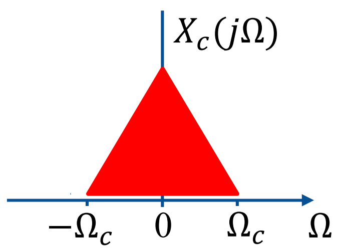
It can be shown that

$$X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

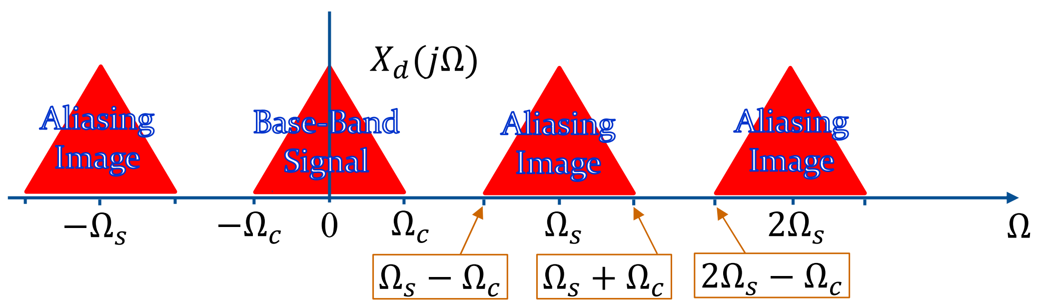
$1/T$ indicates that the magnitude of $X_d(j\omega)$ increases with sampling density $1/T$

$X_c(j(\Omega - n\Omega_s))$ is $X_c(j\Omega)$ shifted along the Ω-axis by $n\Omega_s$

$$X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

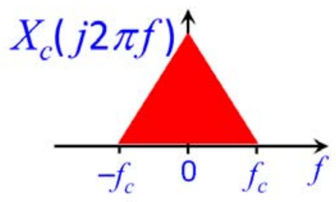


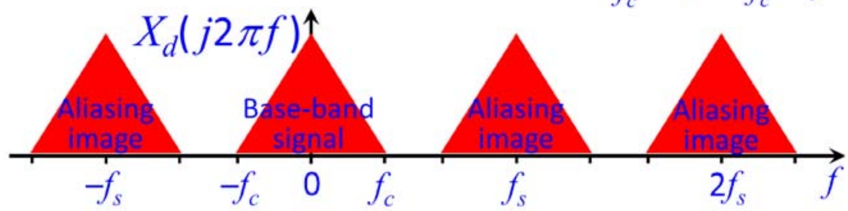
$X_c(j\Omega)$, frequency spectrum of the original continuous time signal, the base-band signal



"Hz" is often used as frequency unit in communication systems. Hence, replacing ω by $2\pi f$ we have

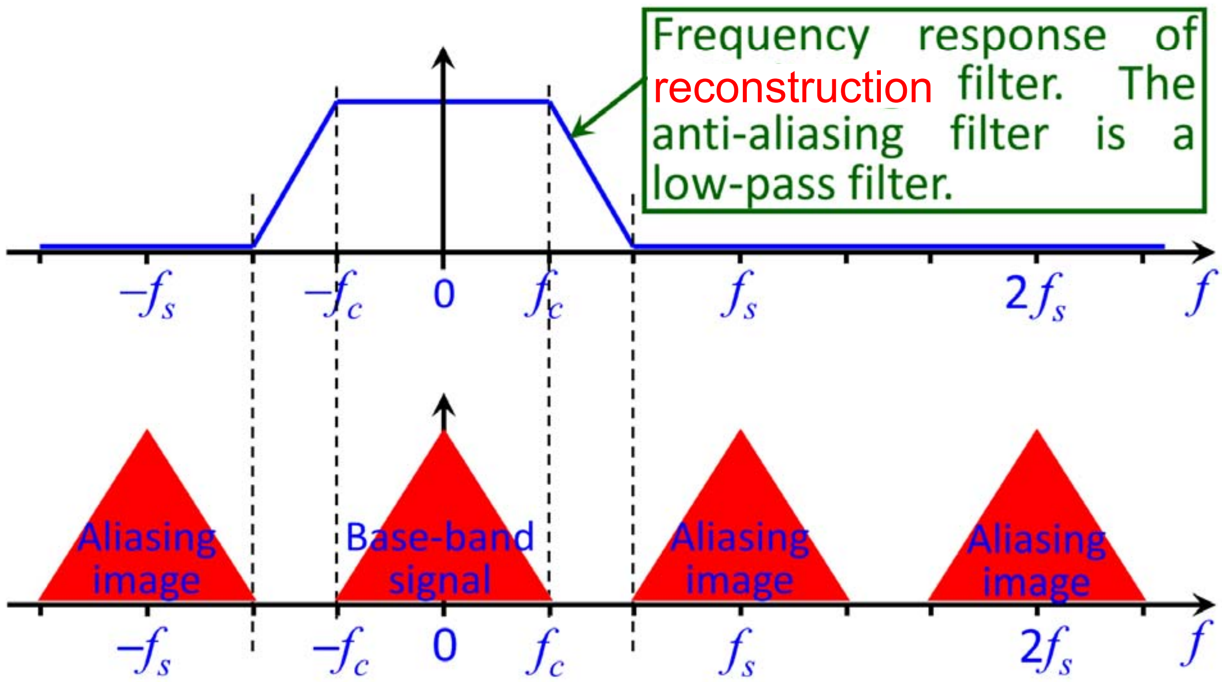
$$X_d(j2\pi f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(j2\pi(f - nf_s)\right)$$

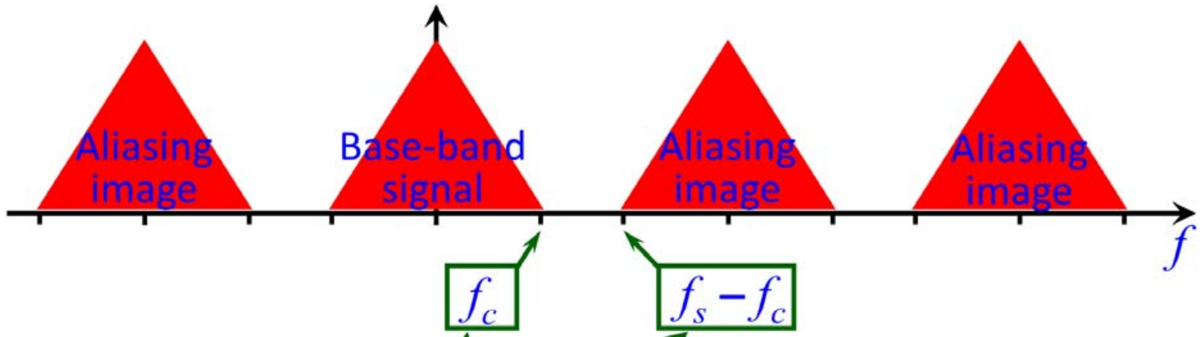




When a continuous time signal (the base-band signal) is sampled at a rate of f_s samples per second, the frequency spectrum of the sampled signal is that of the base-band signal plus duplicates (aliasing) of that of the base-band signal centred at kf_s where k = ..., −1, 0, 1, 2, 3, ...

The original continuous time signal can be recovered by removing the aliasing images by low-pass filtering. The low-pass filter is called a reconstruction filter.





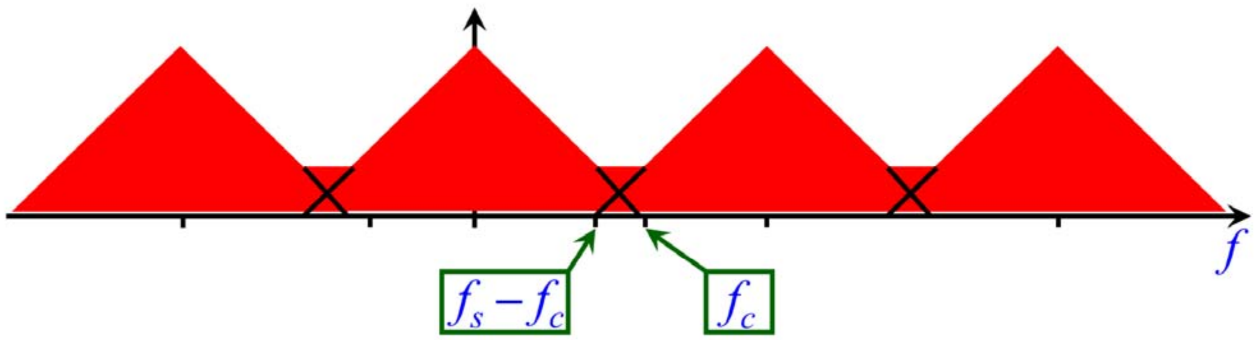
If the original continuous time signal can be completely recovered, the aliasing is not a serious problem. Recovering of the original continuous time signal is possible if and only if there is no overlap between the base-band signal and the aliasing images.

$f_s - f_c > f_c$, i.e. $f_s > 2f_c$.

Nyquist Frequency & Nyquist Rate

- The highest frequency (f_c) contained in a continuous signal $x_c(t)$ is usually called **Nyquist frequency**, which determines the minimum sampling frequency ($f_s = 2f_c$) that must be used to fully recover $x_c(t)$ from its sampled version.
- The minimum sampling frequency (or sampling rate), $f_s = 2f_c$, required to avoid aliasing from irrecoverable problem is called **Nyquist rate**.

If $f_s - f_c < f_c$, i.e. $f_s < 2f_c$,

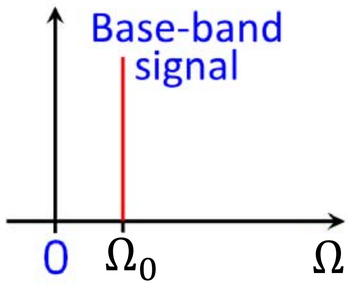


Example:

The output of an ideal low-pass filter with band-edges at $\pm\pi$ radians per second is sampled at a rate of f_s samples per second. What is the minimum f_s in order to avoid aliasing from causing irrecoverable problem?

Since the band-edges of the ideal low-pass filter is $\pm\pi$ radians per second, the highest frequency component of the low-pass filter output is π radians per second. We have $\Omega = 2\pi f$. Thus, the highest frequency component of the low-pass filter output is $\pi/(2\pi)$ = ½ Hz. The sampling rate must be greater than twice the maximum frequency. Hence, minimum sampling rate is 1 sample per second.

Spectral lines of a sampled complex sinusoid.

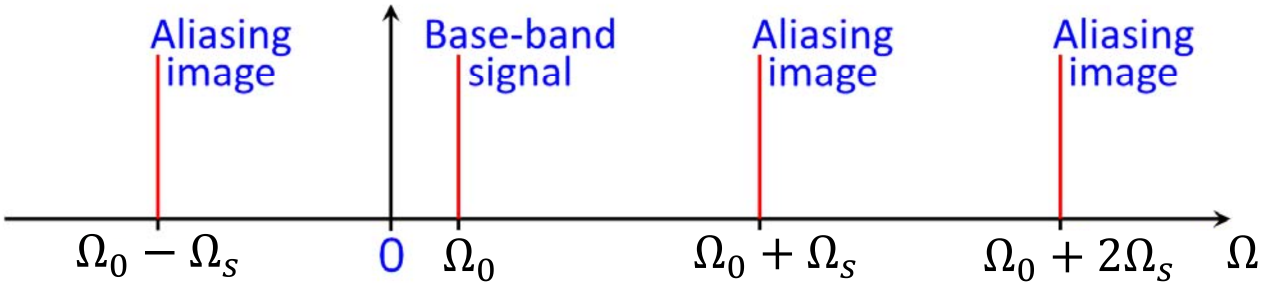


$$x_c(t) = e^{j\Omega_0 t} \Leftrightarrow X_c(j\Omega)$$
$$X_c(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$$

Sampling interval = T

$$\Omega_s = \frac{2\pi}{T}$$

$$X_d(j\Omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\Omega - \Omega_0 - n\Omega_s)$$



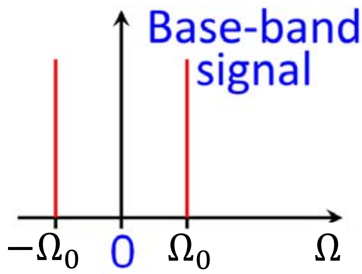
Spectral lines of a sampled sinusoid.

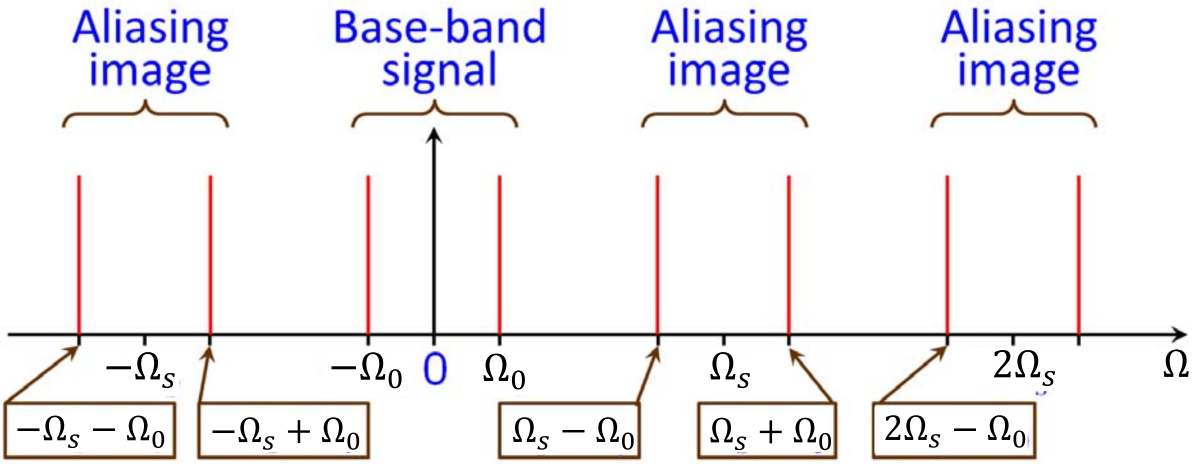
$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

Sampling interval = T

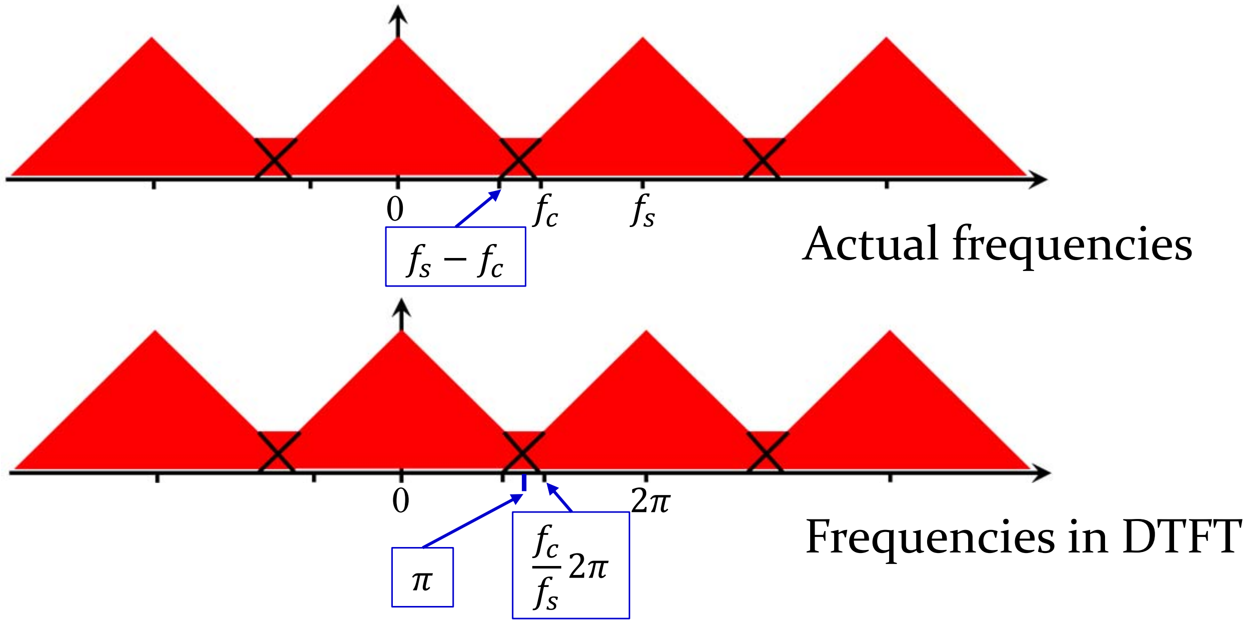
$$\Omega_s = \frac{2\pi}{T}$$

$$X_d(j\Omega) = \frac{\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\Omega \pm \Omega_0 - n\Omega_s)$$



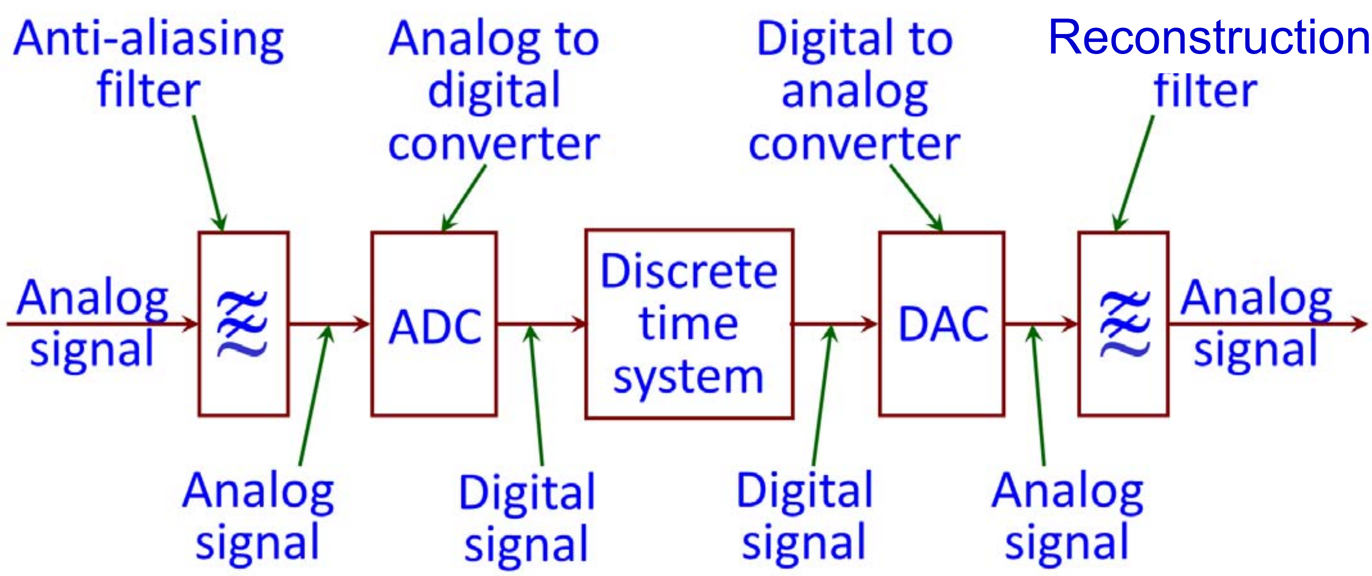


Normalization of frequency in DTFT



- Actual sampling frequency $f_s \leftrightarrow 2\pi$ in DTFT
- Normalization of frequency: $f \leftrightarrow \frac{f}{f_s}2\pi$

A typical discrete time system configuration



Examples of sampling rate:

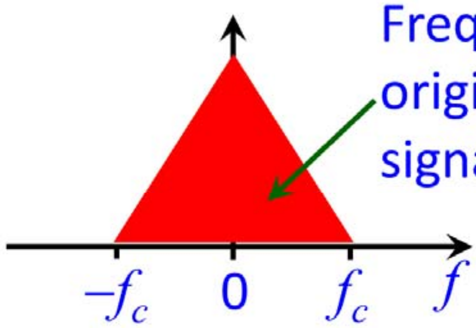
(a) CD : 44.1 kHz.

(b) Digital audio tape : 48 kHz.

(c) Telephone system : 8 KHz.

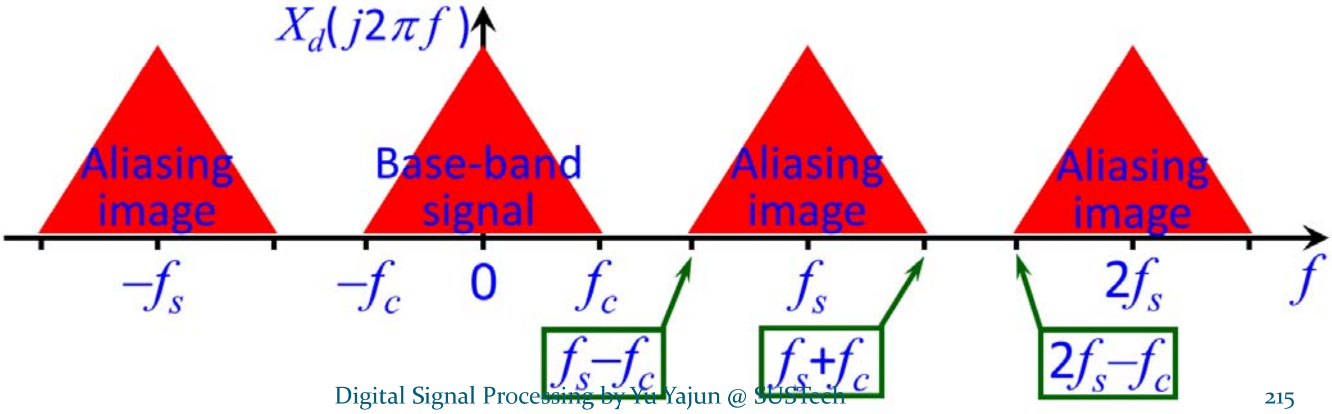
Summary:

1.

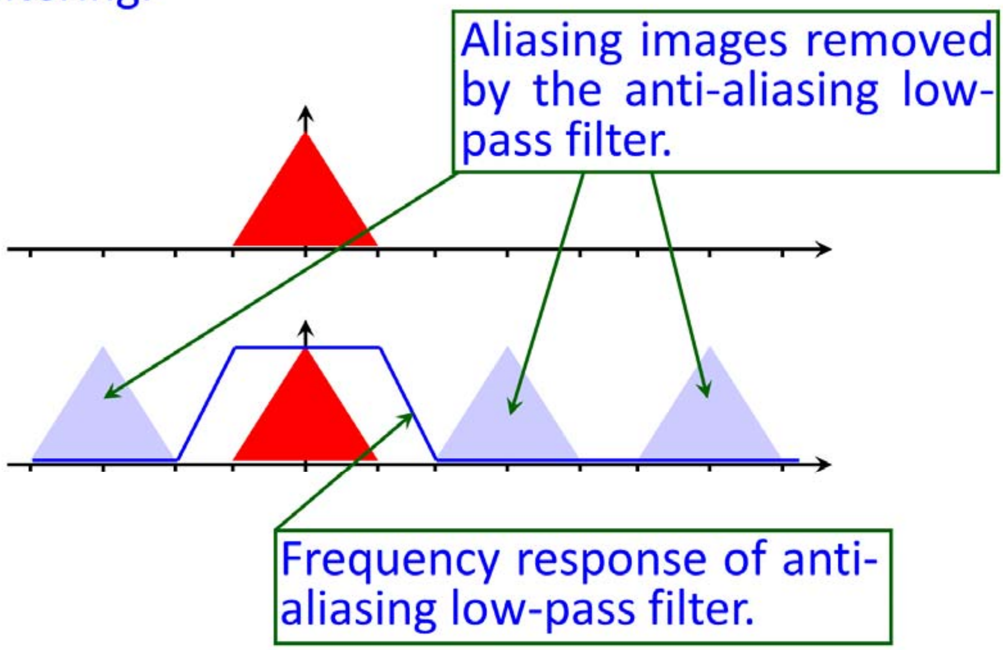


Frequency spectrum of the original continuous time signal.

Frequency spectrum of the sampled discrete time signal.

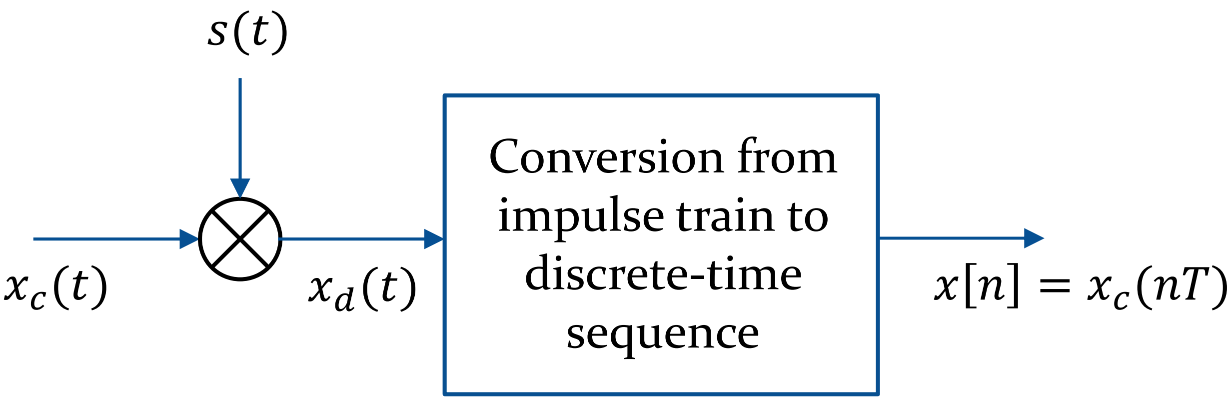


2. Information in the original continuous time signal can be recovered from the sampled signal by low-pass filtering.



3. Nyquist rate = 2 × maximum frequency.

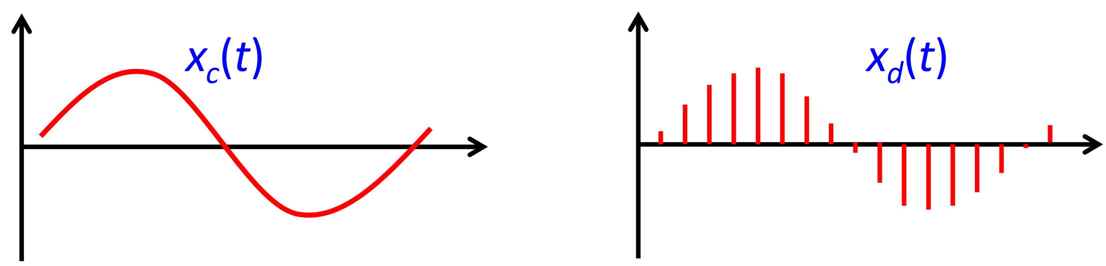
Mathematical Derivation



To prove $X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\big(j(\Omega - n\Omega_s)\big)$

$$x_d(t) = x_c(t)s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Approach 1:



Recall, the Fourier Series of

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}, \quad \text{where } \Omega_s = \frac{2\pi}{T}$$

Thus,

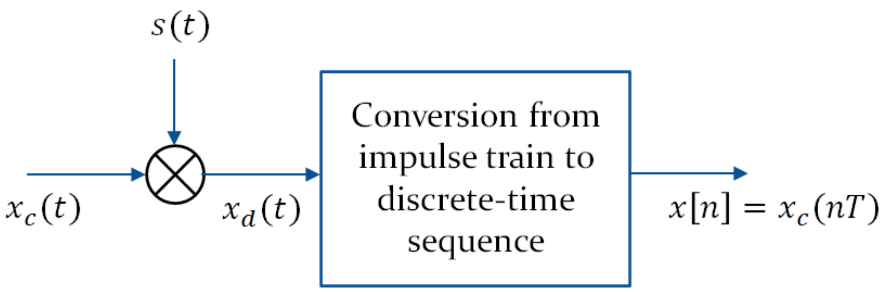
$$x_d(t) = x_c(t) \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t},$$

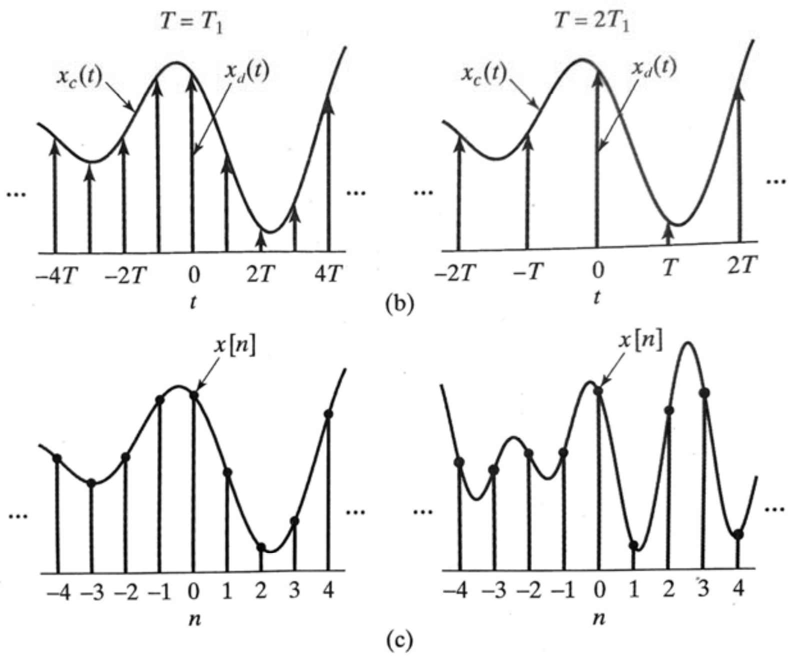
Now, look at the spectrum of the transformed signal. Using the convolution property, we have

$$X_d(j\Omega) = \frac{1}{2\pi}\frac{1}{T}X_c(j\Omega) \circledast \sum_{n=-\infty}^{\infty} 2\pi\delta(\Omega - n\Omega_s)$$

$$= \frac{1}{T}\int_{-\infty}^{\infty} X_c(j\varphi) \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_s - \varphi)\, d\varphi$$

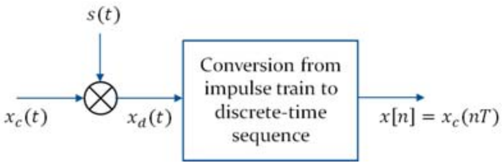
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\big(j(\Omega - n\Omega_s)\big)$$





Since $x_d(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$

we have, $$X_d(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn},$$



Furthermore, since $x[n] = x_c(nT)$, and

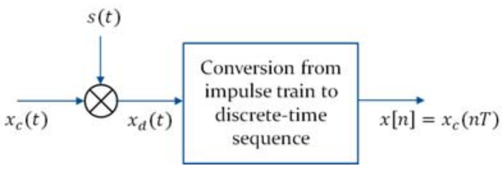
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

it follows that

$$X_d(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T}).$$

Consequently,

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$



$$X(e^{j\Omega T}) = X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

- $X(e^{j\omega})$ is a frequency scaled version of $X_d(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$.
- This scaling can alternatively be thought of as a normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_d(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$.
- The normalization in the transformation from $X_d(j\Omega)$ to $X(e^{j\omega})$ is directly a result of the time normalization in the transformation from $x_d(t)$ to $x[n]$.

Fourier transform of a discrete time signal.

Let $x_c(t)$ be a continuous time signal.

Let the sampled version of $x_c(t)$ be denoted by $x_d(t)$.

Let the sampling interval be T.

Let $\omega_s = 2\pi/T$.

Let $X_c(j\omega)$ be the Fourier transform of $x_c(t)$.

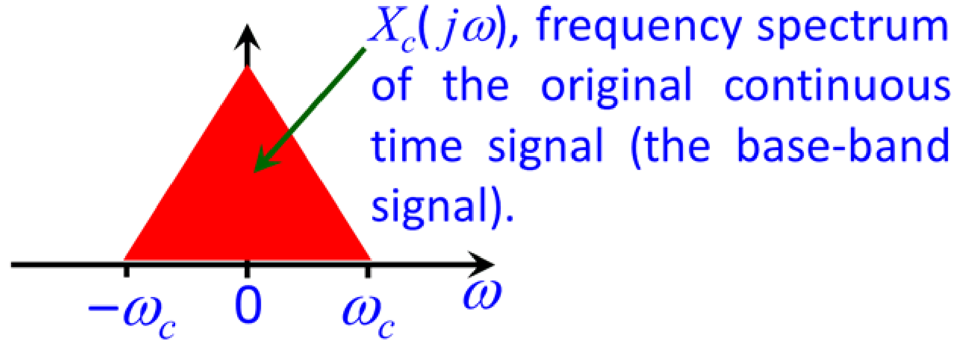
Let $X_d(j\omega)$ be the Fourier transform of $x_d(t)$.

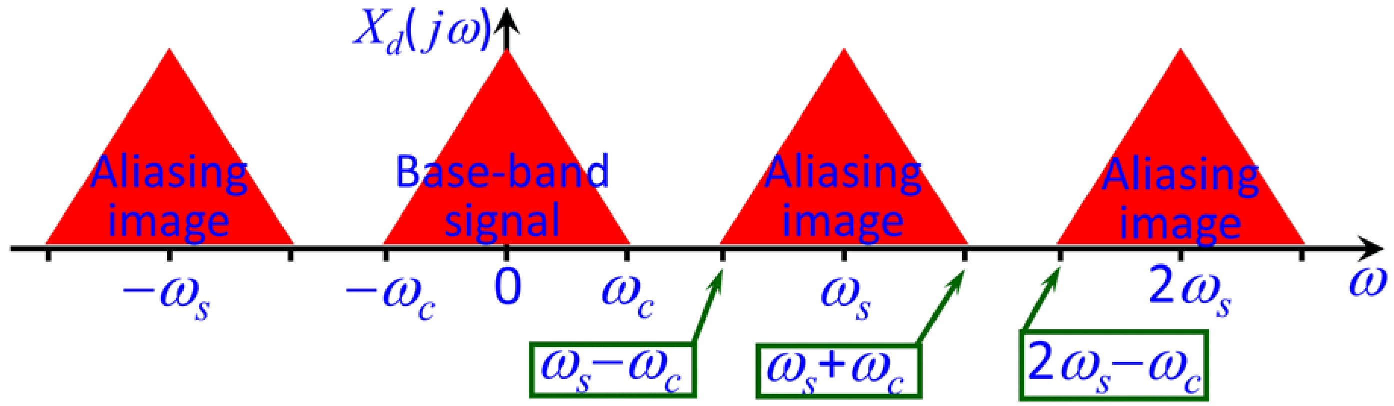
It can be shown that $X_d(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\omega - n\omega_s))$

$X_c(j(\omega - n\omega_s))$ is $X_c(j\omega)$ shifted along the ω-axis by $n\omega_s$.

$1/T$ indicates that the magnitude of $X_d(j\omega)$ increases with sampling density $1/T$.

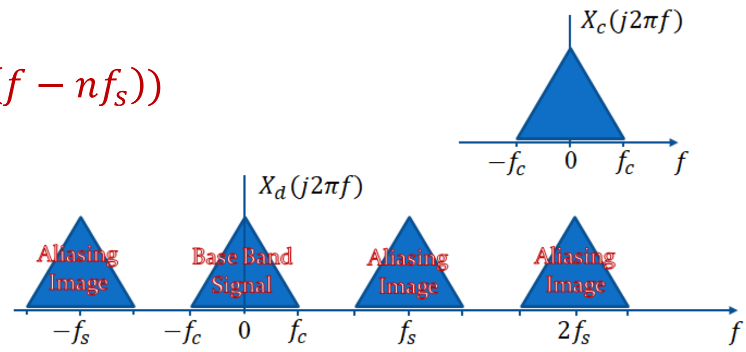
$$X_d(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\omega - n\omega_s))$$





"Hz" is often used as frequency unit in communication systems. Hence, replacing Ω by $2\pi f$ we have

$$X_d(j2\pi f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j2\pi(f - nf_s))$$



When a continuous time signal (the base-band signal) is sampled at a rate of f_s samples per second, the frequency spectrum of the sampled signal is that of the base-band signal plus duplicates (aliasing) of that of the base-band signal plus duplicates (aliasing) of that of the base-band signal centered at kf_s where $k = \cdots, -1, 0, 1, 2, 3, \ldots$.

*Approach 2:

- Fourier transform pair

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}dt$$

- The discrete time signal $x_d(t)$is given by

$$x_d(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT)$$

- Hence the Fourier transform of $x_d(t)$ is

$$X_d(j\Omega) = \int_{-\infty}^{\infty} x_d(t)e^{-j\Omega t}dt$$

$$= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x_c(t)\delta(t-nT)e^{-j\Omega t}dt = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega nT}$$

- Let $x_d[n] = x_c(nT)$. Replace the notation $X_d(j\Omega)$ by $X_d\left(e^{j\omega T}\right)$. Thus, we have

$$X_d\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\omega nT}$$

- Multiply both sides by $e^{j\omega mT}$ and integrate with respect to ω from $-\frac{\pi}{T}$ to $\frac{\pi}{T}$.

$$\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_d(e^{j\omega T}) e^{j\omega mT} d\omega = \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \sum_{n=-\infty}^{\infty} x_d[n] e^{-j\omega nT}\, e^{j\omega mT} d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_d[n] \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} e^{j\omega(m-n)T} d\omega = \frac{2\pi}{T} x_d[m]$$

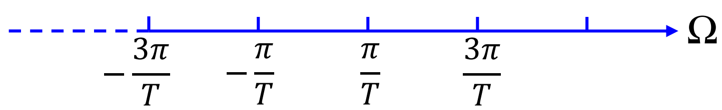
- Therefore,

$$x_d[m] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_d(e^{j\omega T}) e^{j\omega mT} d\omega$$

- Since $x_d[n] = x_c(t)|_{t=nT}$, it is possible to relate $X_d(e^{j\omega T})$ and $X_c(\Omega)$ by evaluating $x_c(t)$ at $t = nT$. Thus,

$$x_d[n] = x_c(nT) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c(\Omega)\, e^{j\Omega nT} d\Omega$$

$$= \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \int_{(2m-1)\frac{\pi}{T}}^{(2m+1)\frac{\pi}{T}} X_c(\Omega) e^{j\Omega nT} d\Omega$$



$$= \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega + \frac{2\pi}{T} m\right) \right] e^{j\Omega nT} d\Omega$$

- Comparing $x_d[m] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X_d\left(e^{j\omega T}\right) e^{j\omega m T} d\omega$ and $x_d[n] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \left[\frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega + \frac{2\pi}{T} m\right)\right] e^{j\Omega n T} d\Omega$, yields

$$X_d\left(e^{j\omega T}\right) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X_c\left(\Omega + \frac{2\pi}{T} m\right)$$

Lecture 5
Frequency Domain Representation of Discrete Time Systems

Signal & Frequency Components

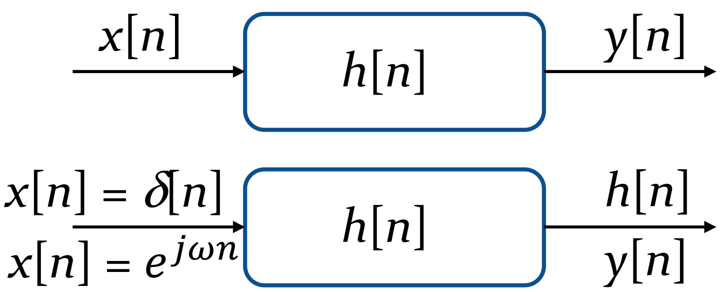
- Continuous periodic signal ⇨ Discrete frequency components (Fourier Series)
- Continuous non-periodic signal ⇨ Continuous frequency components (CTFT)
- Discrete-time signal ⇨ Continuous frequency components (DTFT)

Linear Combination

- When a signal can be represented as a linear combination of complex exponentials :

$$x[n] = \sum_k a_k e^{j\omega_k n}$$

knowing the response of the LTI system to a single sinusoidal signal, we can determine its response to more complicated signals by making use of superposition property.



$$y[n] = h[n] \circledast e^{j\omega n} = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= \left(\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

- Let

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Eigenfunction

- Then, we can write

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

- Thus, for a complex exponential input signal $e^{j\omega n}$, the output of an LTI discrete-time system is also a complex exponential signal of the same frequency multiplied by a complex constant $H(e^{j\omega})$.
- If applying a function as an input to a system, and the output of the system is the same function multiplied by a constant, such function is an **eigenfunction** of the system.
- So, $e^{j\omega n}$, is an eigenfunction of the system.

The Frequency Response

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

is defined to be the **frequency response** of the LTI system with impulse response $h[n]$

- $H(e^{j\omega}) = H_{\mathrm{re}}(e^{j\omega}) + jH_{\mathrm{im}}(e^{j\omega})$
- $H(e^{j\omega}) = |H(e^{j\omega})|e^{j\theta(\omega)}$, where, $\theta(\omega) = \arg\{H(e^{j\omega})\}$
- $|H(e^{j\omega})|$: **magnitude response**
- $\theta(\omega)$: **phase response**

Example

- Consider the ideal delay system defined by

$$y[n] = x[n - n_d], \text{for constant integer } n_d$$

- With input $x[n] = e^{j\omega n}$, we have

$$y[n] = e^{j\omega(n-n_d)} = e^{-j\omega n_d} e^{j\omega n}$$

The frequency response of the ideal delay is therefore

$$H(e^{j\omega}) = e^{-j\omega n_d}$$

- An alternative method: the impulse response of the ideal delay is $h[n] = \delta[n - n_d]$. So the frequency response of the ideal delay system is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}$$

Frequency Response in Decibels

- Gain Function:

$$\mathcal{G}(\omega) = 20\log_{10}|H(e^{j\omega})|$$

the unit is in dB

- Attenuation (or loss function):

$$\mathcal{A}(\omega) = -20\log_{10}|H(e^{j\omega})|$$

is the negative of the gain function.

Symmetry of frequency Response

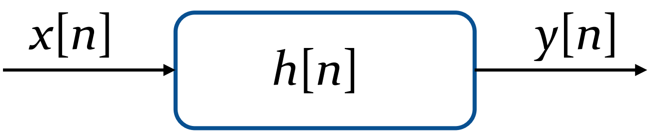
- Due to DTFT, for a real impulse response $h[n]$, $H(e^{j\omega})$ is conjugate symmetric, i.e.,
  - $H(e^{j\omega}) = H^*(e^{-j\omega})$, or
  - $|H(e^{j\omega})| = |H(e^{-j\omega})|$, and $\theta(\omega) = -\theta(-\omega)$, or
  - $H_{\text{re}}(e^{j\omega})$ is even and $H_{\text{im}}(e^{j\omega})$ is odd.

- For a real symmetric impulse response,
  - $H(e^{j\omega})$ is real and symmetric.

Frequency-Domain Characterization of LTI DT System

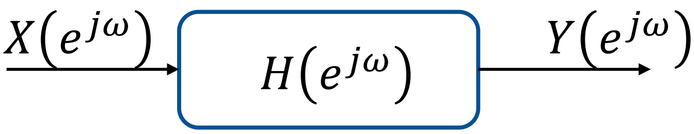
- For LTI system in time domain, we have

$$y[n] = x[n] \circledast h[n]$$



- Applying convolution property of DTFT, we have

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$



Frequency Response of FIR System

- The time-domain input-output relation of FIR system:

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k], \quad N_1 < N_2$$

- Applying DTFT on both sides, we arrive at

$$Y(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}X(e^{j\omega}),$$

- The frequency response of FIR system is given by

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k},$$

Frequency Response of IIR System

- The time-domain input-output relation of IIR system

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

- Applying DTFT on both sides, we arrive at

$$\sum_{m=0}^{N} b_m e^{-j\omega m} Y(e^{j\omega}) = \sum_{m=0}^{M} a_m e^{-j\omega m} X(e^{j\omega})$$

- The frequency response of IIR system is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{m=0}^{M} a_m e^{-j\omega m}}{\sum_{m=0}^{N} b_m e^{-j\omega m}}$$

Example

- Determine the frequency response of the M-point moving average filter.
- Since the input-output relation is given by:

$$y[n] = \frac{1}{M}\sum_{l=0}^{M-1} x[n-l]$$

- the impulse response is given by:

$$h[n] = \frac{1}{M}\sum_{l=0}^{M-1} \delta[n-l] = \begin{cases} \frac{1}{M}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

- Thus, the frequency response is given by

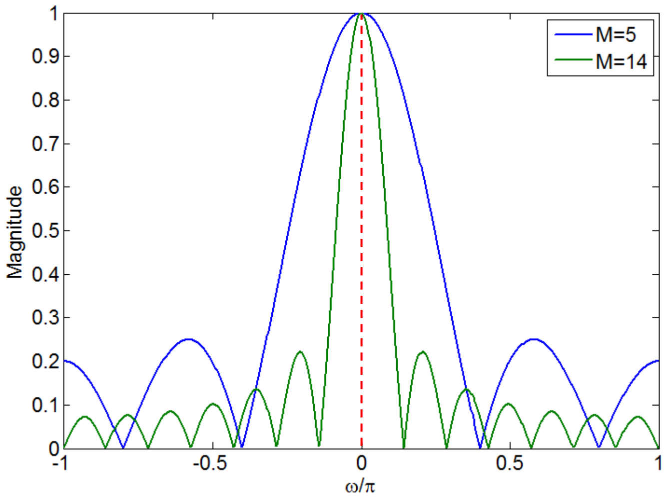
$$H(e^{j\omega}) = \frac{1}{M}\sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1}{M}\cdot\frac{1-e^{-jM\omega}}{1-e^{-j\omega}}$$

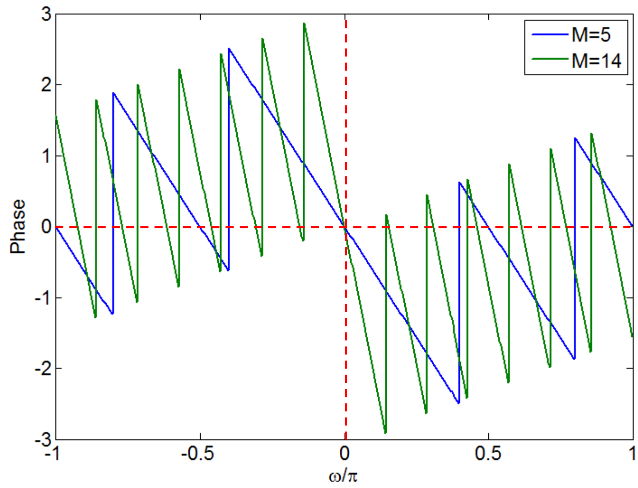
$$= \frac{1}{M}\cdot\boxed{\frac{\sin(M\omega/2)}{\sin(\omega/2)}}\boxed{e^{-j(M-1)\omega/2}}$$

$$|H(e^{j\omega})| = \frac{1}{M}\left|\frac{\sin\left(\frac{M\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right|$$

$$\theta(\omega) = \frac{-(M-1)\omega}{2} + \pi\sum_{k=1}^{\lfloor M/2\rfloor}\mu\left(\omega - \frac{2\pi k}{M}\right)$$

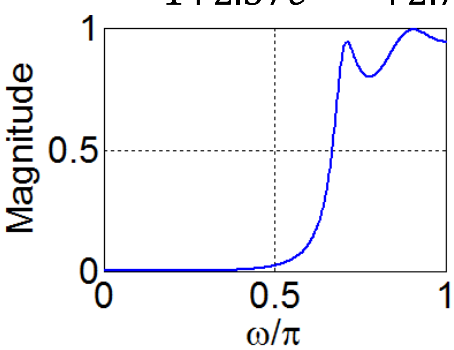
- The plots of the magnitude response and phase response of the *M*-point moving average filter, for *M*=5 and *M*=14

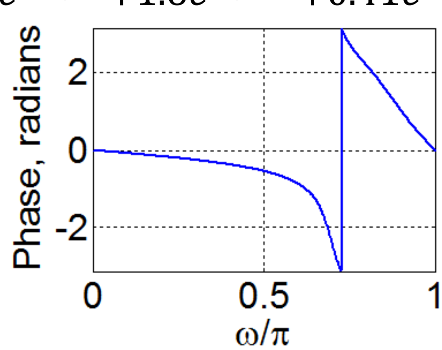




Unwrapped Phase Function

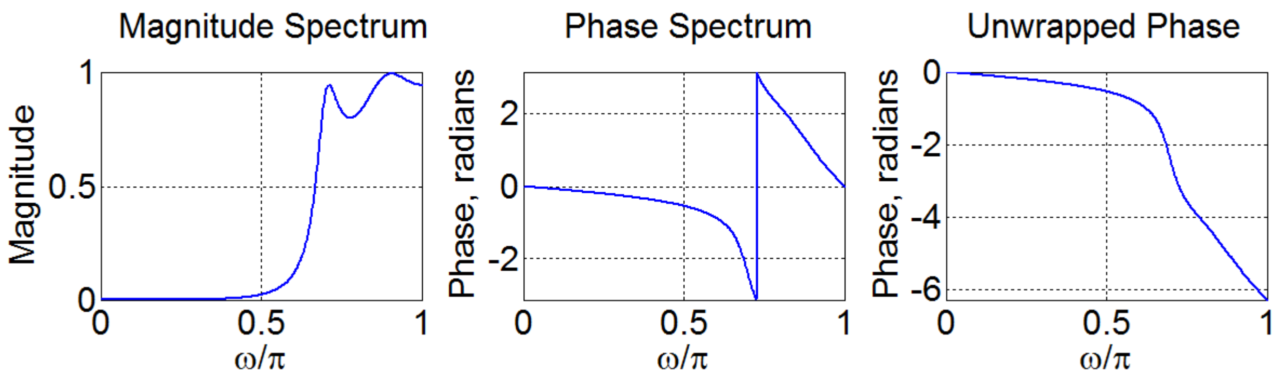
- The principle value of phase function is defined to within a range $[-\pi, \pi]$.
- The phase function of DTFT thus computed exhibits discontinuity of 2π radians in plots.
- Example: $X(e^{j\omega}) = \dfrac{0.008-0.033e^{-j\omega}+0.05e^{-j2\omega}-0.033e^{-j3\omega}+0.008e^{-j4\omega}}{1+2.37e^{-j\omega}+2.7e^{-2j\omega}+1.6e^{-j3\omega}+0.41e^{-j4\omega}}$





Unwrapped Phase Function

- The process to remove the 2π discontinuity is called **unwrapping the phase.**

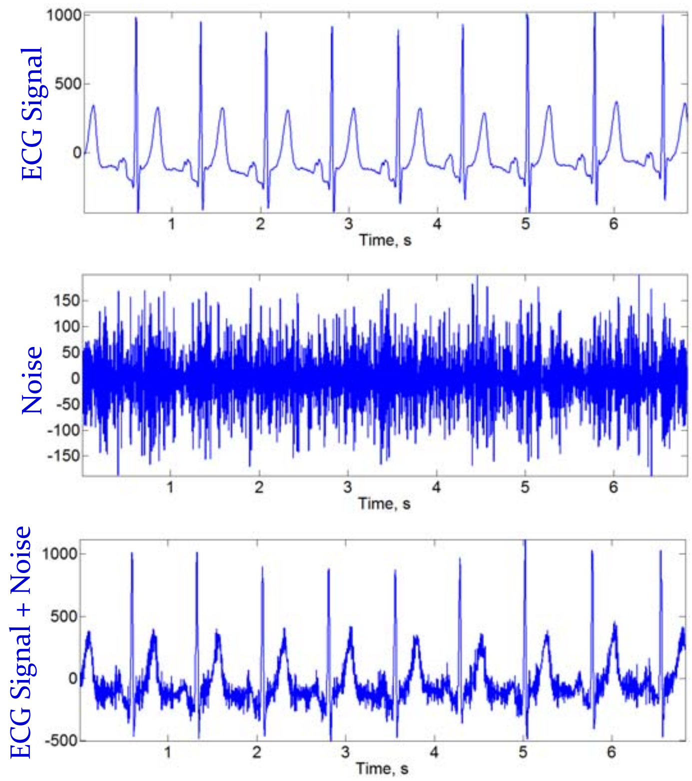


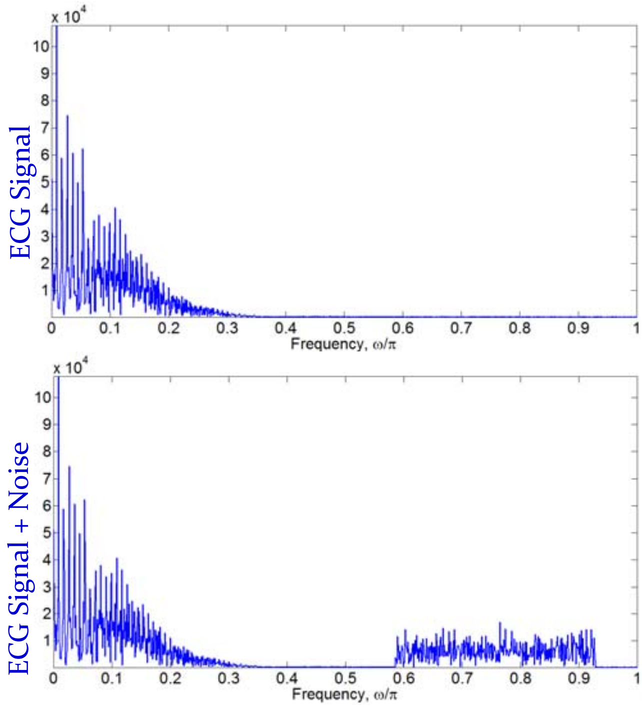
The Concept of Filtering

- One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components.

- Such systems are called digital filters and are one of main devices in digital signal processing

- A time domain signal with noise
- Their frequency spectrum





The Concept of Filtering

- Any discrete-time signal may be expressed as

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Any frequency component $e^{j\omega n}$ may be scaled by a frequency response $H(e^{j\omega})$ at frequency ω, such that the frequency component is passed without distortion, or attenuated.
- For example, if we have an ideal LTI system with magnitude response given by

$$|H(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

A Simple Example

- We apply an input $x[n]$ to the system, where
$$x[n] = A\cos\omega_1 n + B\cos\omega_2 n,$$
$$0 < \omega_1 < \omega_c < \omega_2 < \pi$$
- Because of linearity, the output of the system is
$$y[n] = A\left|H\left(e^{j\omega_1}\right)\right|\cos(\omega_1 n + \theta(\omega_1)) + B\left|H\left(e^{j\omega_2}\right)\right|\cos(\omega_2 n + \theta(\omega_2))$$
- As $\left|H\left(e^{j\omega_1}\right)\right| = 1$, and $\left|H\left(e^{j\omega_2}\right)\right| = 0$, the output reduces to $y[n] = A\cos(\omega_1 + \theta(\omega_1))$
- The LTI system acts like a lowpass filter.

Design Example

- Design a very simple digital filter.
- **Requirement:** An input, consisting of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample, is to be filtered to keep the high-frequency component, but block the low-frequency component.
- For simplicity, we assume a filter of length 3 with an impulse response: $h[0]=h[2]=\alpha$, and $h[1]=\beta$.

- The input-output relation in time-domain would be:

$$y[n] = h[0]x[n] + h[1]x[n-1] + h[2]x[n-2]$$
$$= \alpha x[n] + \beta x[n-1] + \alpha x[n-2]$$

- **Design objective:** Choose suitable values of α and β, such that the output contains only a sinusoidal sequence with an angular frequency 0.4 rad/sample.
- Now the frequency response of the filter is given by,

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$
$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$
$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega} = (2\alpha\cos\omega + \beta)e^{-j\omega}$$

Magnitude response

- To block the low-frequency component, let

$$H(e^{j0.1}) = (2\alpha\cos(0.1) + \beta) = 0$$

- To pass the high-frequency component, let

$$H(e^{j0.4}) = (2\alpha\cos(0.4) + \beta) = 1$$

- Result in:

$$\alpha = -6.76185, \qquad \beta = 13.456335$$

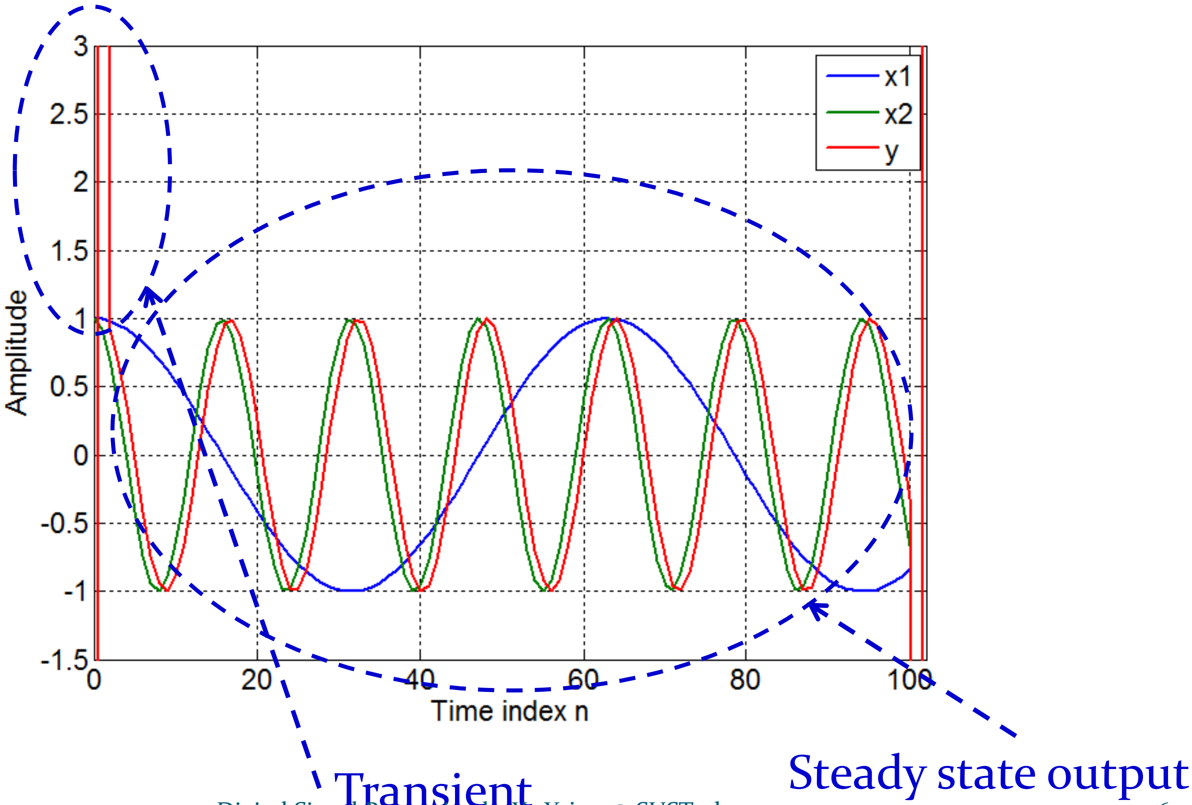
i.e., $h[n] = \{-6.76185, 13.456335, -6.76185\}$, for $n = 0, 1, 2$

- So the designed filter has the input-output relation in time-domain given by

$$y[n] = -6.76185(x[n] + x[n-2]) + 13.456335x[n-1]$$

and the input is $x[n] = (\cos(0.1n) + \cos(0.4n))\mu[n]$

- Input and output sequences in time-domain



Phase Delay and Group Delay

- Re-examine a system with an input of pure sinusoidal signal

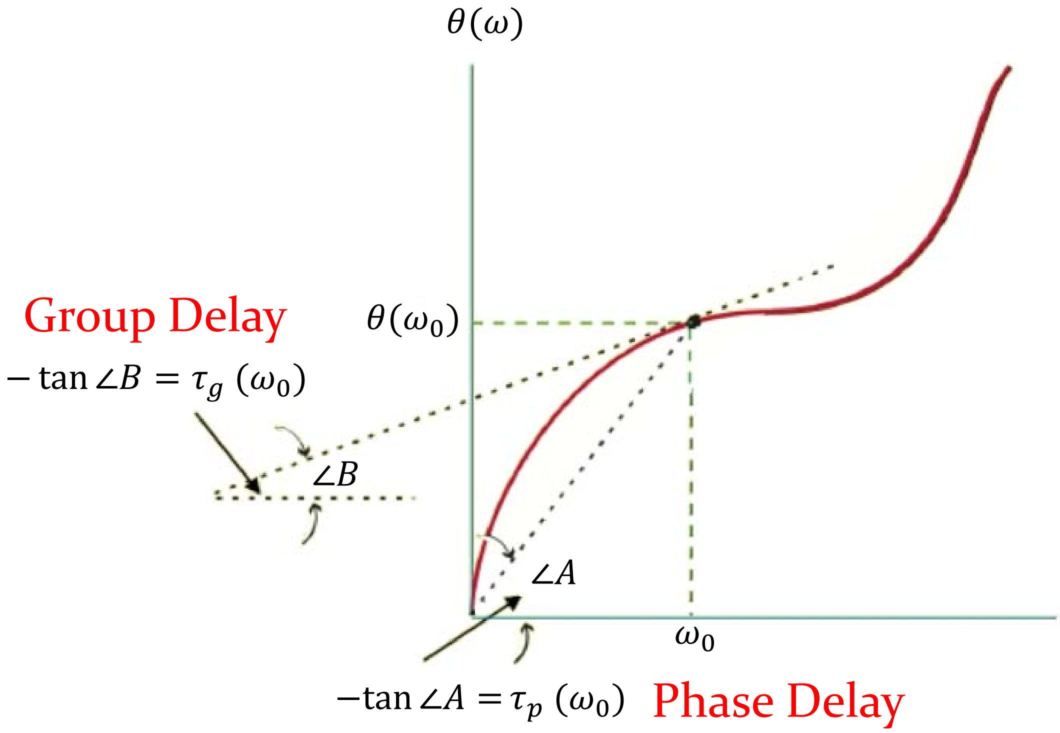
$$
\begin{aligned}
y[n] &= h[n] \circledast A\cos(\omega_0 n + \varphi) \\
&= A\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}\right)\cos(\omega_0 n + \varphi) \\
&= A\left|H\left(e^{j\omega_0}\right)\right|\cos(\omega_0 n + \theta(\omega_0) + \varphi) \\
&= A\left|H\left(e^{j\omega_0}\right)\right|\cos\left(\omega_0\left[n + \boxed{\frac{\theta(\omega_0)}{\omega_0}}\right] + \varphi\right)
\end{aligned}
$$

$\boxed{-\tau_p(\omega_0)}$

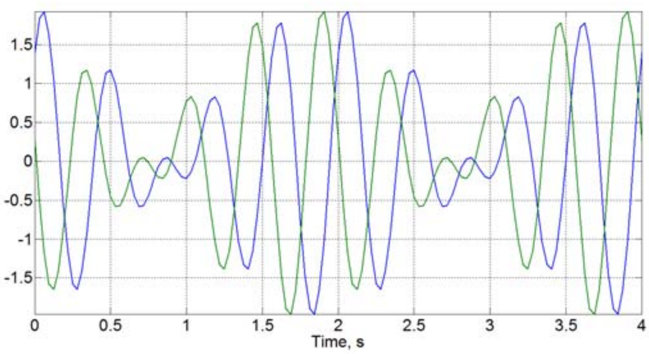
- Define **phase delay** and **group delay**, respectively, as

$$
\tau_p(\omega_0) = -\frac{\theta(\omega_0)}{\omega_0}, \qquad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}.
$$

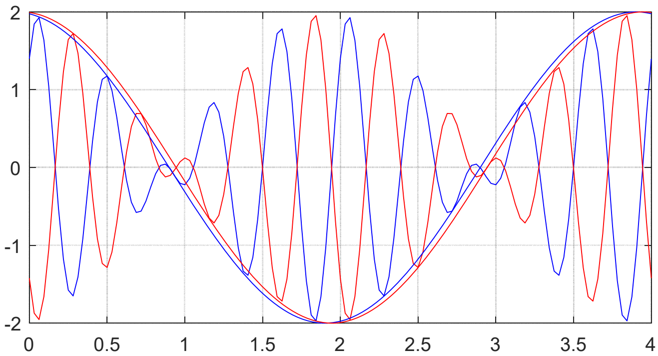
A Graphic Comparison



*Physical Meanings



Phase Delay: Delay of Samples



Group Delay: Delay of Envelopes

- $T = \frac{1}{32}, \omega_1 = 4\pi, \omega_2 = 5\pi$
- Blue Signal:

$\sin(\omega_1 n + 0.2\pi) + \sin(\omega_2 n + 0.3\pi)$

- Green Signal:

$$\sin(\omega_1 n + 0.2\pi + 5\omega_1 T) + \sin(\omega_2 n + 0.3\pi + 5\omega_2 T)$$

$$\theta(\omega_1) = 5\omega_1 T,\ \theta(\omega_2) = 5\omega_2 T,$$

$$\tau_p(\omega_1) = \tau_p(\omega_2) = -5T$$

$$\tau_g(\omega_1) = \tau_g(\omega_2) = -5T$$

- Red Signal:

$$\sin(4\pi n + 0.2\pi + 5\omega_1 T + 0.4\pi) + \sin(5\pi n + 0.3\pi + 5\omega_2 T + 0.2\pi)$$

$$\theta(\omega_1) = 5\omega_1 T + 0.4\pi,\ \theta(\omega_2) = 5\omega_2 T + 0.2\pi\ ,$$

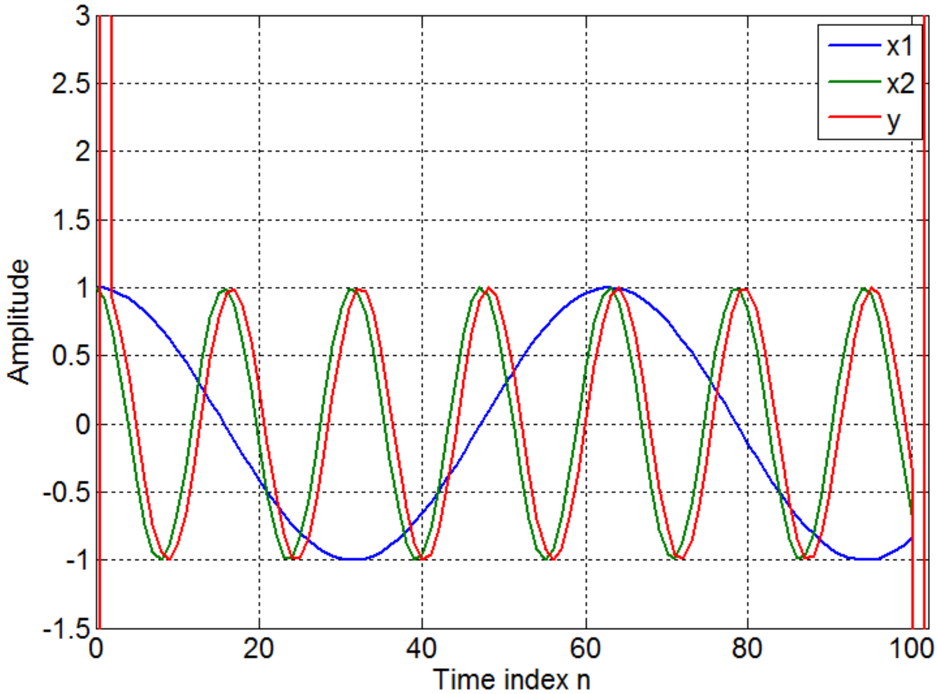
$$\tau_p(\omega_1) = -5T - 0.1$$

$$\tau_p(\omega_2) = -5T - 0.04$$

$$\tau_g(\omega_1) = \tau_g(\omega_2) = -\frac{\Delta\theta(\omega)}{\Delta\omega} - \frac{\theta(\omega_2) - \theta(\omega_1)}{\omega_2 - \omega_1}$$

$$= -(5T - 0.2) = 1.4T$$

- $H\left(e^{j\omega}\right) = (2\alpha\cos\omega + \beta)e^{-j\omega}$,
- $\tau_p(\omega) = -\frac{\theta(\omega)}{\omega} = 1, \qquad \tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega} = 1$



Lecture 6
Discrete Fourier Transform (DFT) and its Fast Implementation

DFT Definition

DFT: $$X[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}}, \qquad k = 0, 1, \dots, N-1$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \Big|_{\omega = \frac{2\pi k}{N}} = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$

Using: $W_N = e^{-j2\pi/N}$, we can rewrite the DFT as

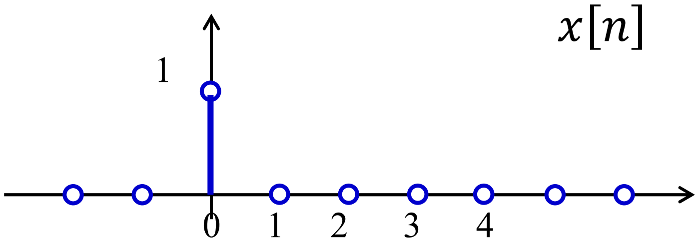
$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, 1, \dots, N-1.$$

IDFT: $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, \qquad n = 0, 1, \dots, N-1.$$

DTFT vs. DFT

- Both apply to discrete time signal
- DTFT is for infinite length of discrete time signal
- DFT is for finite length of discrete time signal
- For a length-N sequence, N values of $X\left(e^{j\omega}\right)$, at N distinct frequency points, $\omega = \omega_k$, $k = 0, 1, \ldots, N-1$, are sufficient to determine $x[n]$, uniquely.
- Q: Can we reconstruct the DTFT spectrum (continuous in ω) from the DFT?

Example 1



$x[n] \qquad 0 \le n \le N-1$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, 1, \ldots, N-1$$

$$= x[0] W_N^0 = 1$$

N point DFT

Example 2

- Consider a length N sequence defined for $0 \le n \le N\!-\!1$

$$g[n] = \cos\left(\frac{2\pi r}{N} n\right), \qquad 0 \le r \le N-1$$

- **Solution:** Since

$$g[n] = \frac{1}{2}\left(e^{\frac{j2\pi r}{N}n} + e^{-\frac{j2\pi r}{N}n}\right) = \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

- Thus,

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn} = \frac{1}{2}\sum_{n=0}^{N-1}\left(W_N^{-(r-k)n} + W_N^{(r+k)n}\right)$$

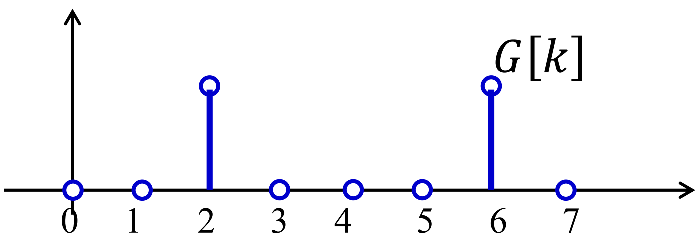
$$G[k] = \frac{1}{2}\sum_{n=0}^{N-1}\left(W_N^{-(r-k)n} + W_N^{(r+k)n}\right)$$

- Making use of the identity:

$$\sum_{n=0}^{N-1} W_N^{-(k-l)n} = \begin{cases} N, & \text{for } k-l = mN, m \text{ an integer} \\ 0, & \text{otherwise} \end{cases}$$

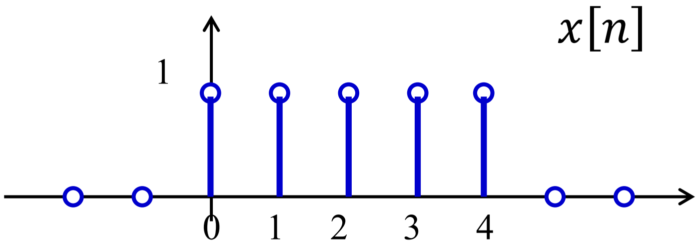
- We get

$$G[k] = \begin{cases} N/2, & k = r \\ N/2, & k = N-r \\ 0, & \text{otherwise} \end{cases} \qquad 0 \le k \le N-1$$



for $N = 8$, $r = 2$

Example 3



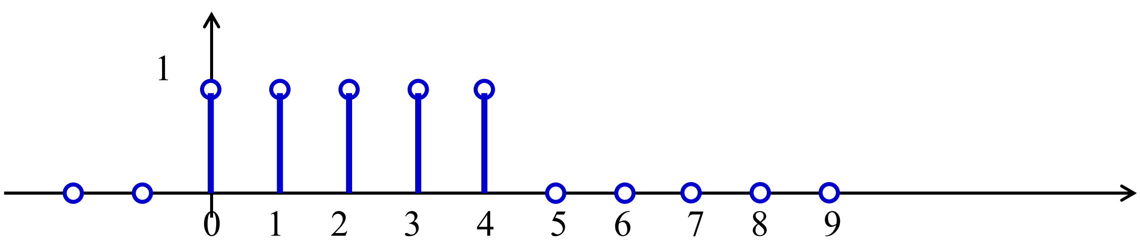
- Take N=5

$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n] W_5^{kn}, & k = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$= 5\delta[k]$$

5 point DFT

- Q: What if we take N = 10?

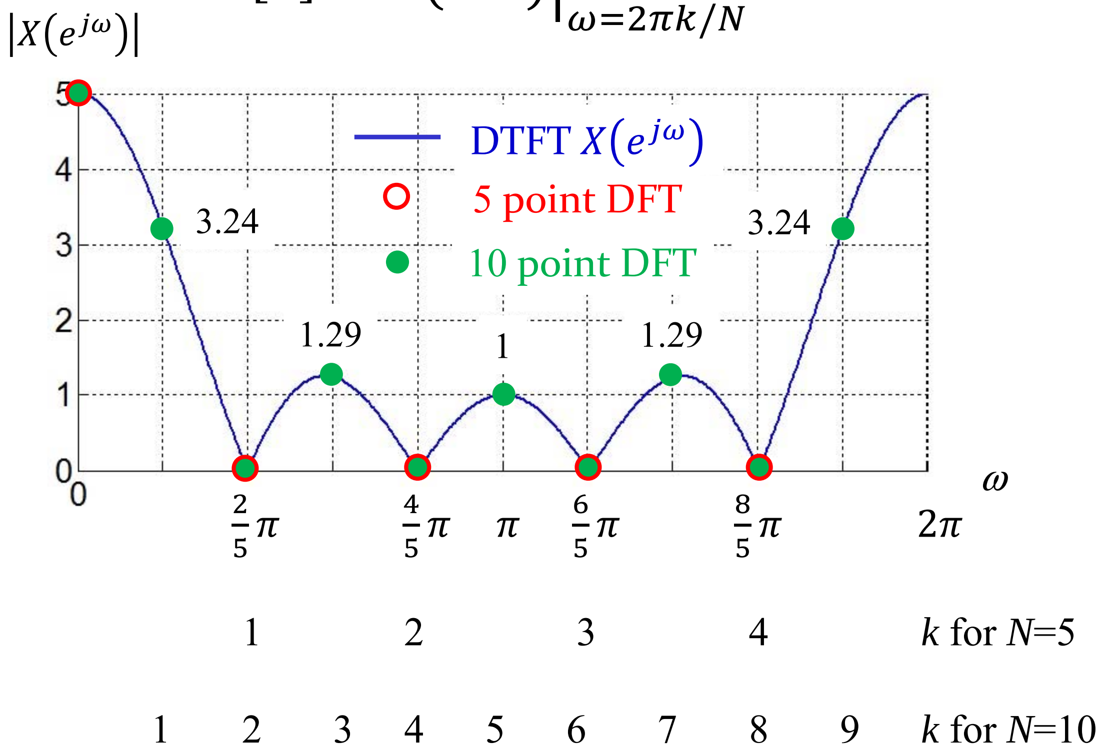


$$X[k] = \begin{cases} \sum_{n=0}^{4} x[n] W_{10}^{kn}, & k = 0, 1, 2, \dots, 9 \\ 0 & \text{otherwise} \end{cases}$$

10 point DFT

$$= \begin{cases} e^{-j\frac{4\pi}{10}k} \dfrac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{10}\right)} & k = 0, 1, 2, \dots, 9 \\ 0 & \text{otherwise} \end{cases}$$

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$



Four Types of Fourier Transform

Time Domain	Non-Periodic	Periodic	
Continuous	Continuous Time Fourier Transform (CTFT)	Fourier Series (FS)	Non-Periodic
Discrete	Discrete Time Fourier Transform (DTFT)	Discrete Fourier Transform (DFT)	Periodic
	Continuous	Discrete	Frequency Domain

DTFT vs. DFT

- Q: Can we reconstruct the DTFT spectrum (continuous in ω) from the DFT?

$$x[n] \xrightarrow{DFT} X[k] \xrightarrow{?} X(e^{j\omega})$$

- A: Since the N-length signal $x[n]$ can be exactly recovered from both the DFT coefficients and the DTFT spectrum, we expect that the DTFT spectrum (that is within $[0, 2\pi]$) can be exactly reconstructed by the DFT coefficients.

Reconstruct DTFT from DFT

- By substituting the inverse DFT into the $x[n]$, we have

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j2\pi kn/N} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \boxed{\sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j\omega n}}$$

Sum of a geometric sequence with $q = e^{-j(\omega - 2\pi k/N)}$

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N}\, e^{-j\omega n} = \frac{1-e^{-j(\omega N-2\pi k)}}{1-e^{-j(\omega-2\pi k/N)}}$$

$$= \frac{1-e^{-j(\omega N-2\pi k)}}{1-e^{-j(\omega N-2\pi k)/N}} \quad \boxed{\times \frac{e^{j\frac{\omega N-2\pi k}{2}}}{e^{j\frac{\omega N-2\pi k}{2N}}} \times \frac{e^{-j\frac{\omega N-2\pi k}{2}}}{e^{-j\frac{\omega N-2\pi k}{2N}}}}$$

$$= \frac{e^{-j\frac{\omega N-2\pi k}{2}}}{e^{-j\frac{\omega N-2\pi k}{2N}}} \times \frac{e^{j\frac{\omega N-2\pi k}{2}} - e^{-j\frac{\omega N-2\pi k}{2}}}{e^{j\frac{\omega N-2\pi k}{2N}} - e^{-j\frac{\omega N-2\pi k}{2N}}}$$

$$= \frac{\sin\left(\frac{\omega N-2\pi k}{2}\right)}{\sin\left(\frac{\omega N-2\pi k}{2N}\right)} e^{-j\left(\omega-\frac{2\pi k}{N}\right)\left(\frac{N-1}{2}\right)}$$

- Therefore,

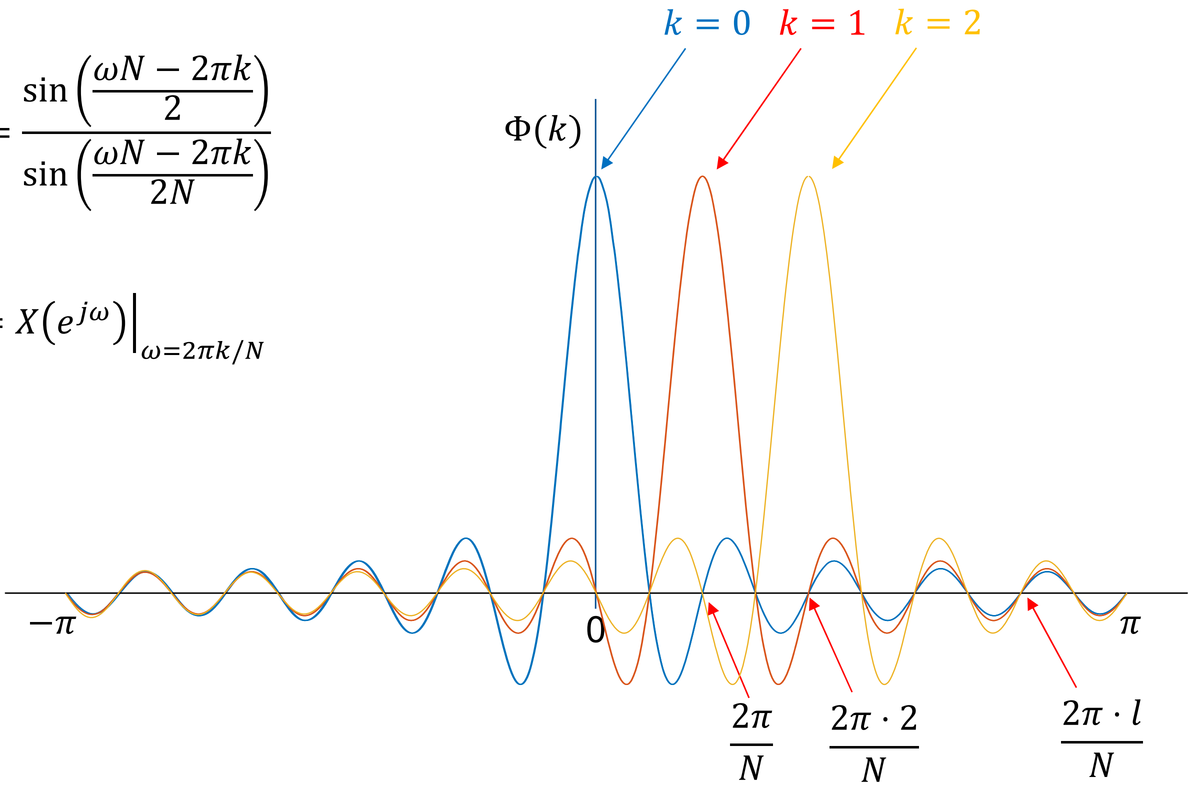
$$X\left(e^{j\omega}\right) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \underbrace{\boxed{\frac{\sin\left(\frac{\omega N-2\pi k}{2}\right)}{\sin\left(\frac{\omega N-2\pi k}{2N}\right)}}}_{\Phi} e^{-j\left(\omega-\frac{2\pi k}{N}\right)\left(\frac{N-1}{2}\right)}$$

- When $\omega = \frac{2\pi l}{N}$ for $0 \le l \le N-1$, if $l = k$, $\Phi = 1$, and if $l \ne k$, $\Phi = 0$. Therefor, $X\left(e^{j\omega}\right)\Big|_{\omega=\frac{2\pi l}{N}} = X[l]$.
- To recover the DTFT $X\left(e^{j\omega}\right)$ of a length-N sequence $x[n]$, for $n = 0, 1, \dots, N-1$ from a K-point DFT sequence $X[k]$, for $k = 0, 1, \dots, K-1$, K must be $\ge N$

$$X(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} e^{-j\left(\omega - \frac{2\pi k}{N}\right)\left(\frac{N-1}{2}\right)}$$

$$\Phi(k) = \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)}$$

$$X[k] = X(e^{j\omega})\Big|_{\omega = 2\pi k/N}$$



Sampling the DTFT

- Consider a length M sequence $x[n]$ $(0 \le n \le M-1)$ going through the following transforms and operations:

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}) \xrightarrow[\ ]{\substack{\text{Sample} \\ N \text{ points}}} Y[k] \xrightarrow{\substack{N\text{−point} \\ \text{IDFT}}} y[n]$$

Find the relation between $x[n]$ and $y[n]$.

- Since

$$Y[k] = X(e^{j\omega_k}) = X(e^{j(2\pi k/N)}) = \sum_{l=-\infty}^{\infty} x[l]\, W_N^{kl}$$

for $0 \le k \le N-1$

- And

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=-\infty}^{\infty} x[l]\, W_N^{kl}\, W_N^{-kn}$$

$$= \sum_{l=-\infty}^{\infty} x[l]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-l)}\right], \qquad \text{for } 0 \le n \le N-1$$

- Recalling from the identity that

$$\sum_{k=0}^{N-1} W_N^{-k(n-l)} = \begin{cases} N, & \text{for } l = n + mN, m \text{ is any integer} \\ 0, & \text{otherwise} \end{cases}$$

we have

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \qquad 0 \le n \le N-1$$

- This relation indicates:
  - $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$.
  - If $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$, and $x[n]$ can be recovered from $y[n]$ by extracting M samples of $y[n]$ for $0 \le n \le M-1$.
  - If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$.

Example

- Let $\{x[n]\} = \{0,1,2,3,4,5\}$ for $0 \le n \le 5$
- By sampling the DTFT of $x[n]$ by 8 samples at $\omega_k = 2\pi k/8,\ \ 0 \le k \le 7$, and then applying an 8-point IDFT to these samples, we arrive at the sequence

  $$y[n] = x[n] + x[n+8] + x[n-8] + \cdots, 0 \le n \le 7$$

  that is, $\{y[n]\} = \{0,1,2,3,4,5,0,0\}$, for $0 \le n \le 7$

  $x[n]$ can be recovered from $y[n]$
- By sampling the DTFT of $x[n]$ by 4 samples, we arrive

  $$y[n] = x[n] + x[n+4] + x[n-4] + \cdots, 0 \le n \le 3$$

  that is, $\{y[n]\} = \{4,6,2,3\}$, for $0 \le n \le 3$

  $x[n]$ cannot be recovered from $y[n]$

DFT and Inverse DFT

- Both computed similarly.....let's play

$$Nx^*[n] = N\left(\frac{1}{N}\sum_{k=0}^{N-1} X[k]W_N^{-kn}\right)^*$$

$$= \sum_{k=0}^{N-1} X^*[k]W_N^{kn} = \text{DFT}\{X^*[k]\}$$

- Also,

$$Nx^*[n] = N(\text{ IDFT }\{X[k]\})^*$$

DFT and Inverse DFT

- So,

$$\text{DFT}\{X^*[k]\} = N(\text{ IDFT }\{X[k]\})^*$$

- Or,

$$\text{IDFT }\{X[k]\} = \frac{1}{N}(\text{DFT }\{X^*[k]\})^*$$

- Implement IDFT by:
  - Take complex conjugate
  - Take DFT
  - Multiply by $1/N$
  - Take complex conjugate

Why useful?

DFT as Matrix Operator

Let

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi\frac{1}{N}} & e^{-j2\pi\frac{2}{N}} & e^{-j2\pi\frac{3}{N}} & \cdots & e^{-j2\pi\frac{N-1}{N}} \\ 1 & e^{-j2\pi\frac{2}{N}} & e^{-j2\pi\frac{4}{N}} & e^{-j2\pi\frac{6}{N}} & \cdots & e^{-j2\pi\frac{2(N-1)}{N}} \\ 1 & e^{-j2\pi\frac{3}{N}} & e^{-j2\pi\frac{6}{N}} & e^{-j2\pi\frac{9}{N}} & \cdots & e^{-j2\pi\frac{3(N-1)}{N}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j2\pi\frac{N-1}{N}} & e^{-j2\pi\frac{2(N-1)}{N}} & e^{-j2\pi\frac{3(N-1)}{N}} & \cdots & e^{-j2\pi\frac{(N-1)(N-1)}{N}} \end{bmatrix}$$

- The (n, k)-th element of $\mathbf{W}_N$ is W_N^{nk}

DFT

$$\mathbf{W}_N \cdot \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}$$

IDFT

$$\frac{1}{N} \cdot \mathbf{W}_N^* \cdot \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}$$

Straightforward implementation requires N^2 complex multiplies :-(

DFT as Matrix Operator Cont.

- Can write compactly as:

$$\boldsymbol{X} = \mathbf{W}_N \boldsymbol{x}$$

$$\boldsymbol{x} = \frac{1}{N}\mathbf{W}_N^* \boldsymbol{X}$$

- So

$$\boldsymbol{x} = \frac{1}{N}\mathbf{W}_N^* \boldsymbol{X} = \frac{1}{N}\mathbf{W}_N^* \mathbf{W}_N \boldsymbol{x} = \frac{1}{N}(N\mathbf{I}_N)\,\boldsymbol{x} = \boldsymbol{x}$$

Why

As expected

Circular Time-Reversal

- let $x[n]$ is defined for the range of $0 \le n \le N-1$
- Time-reversed sequence $x_1[n] = x[-n]$ is no longer defined for the range of $0 \le n \le N - 1$
- Define the **circular time-reversal** $y[n]$

$$y[n] = x[\langle -n \rangle_N]$$

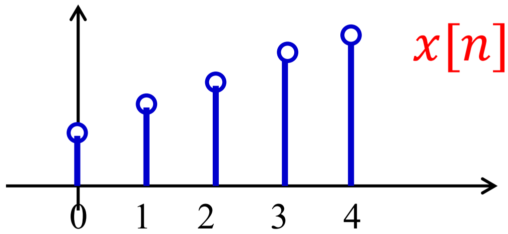
where $\langle k \rangle_N = k$ modulo N.

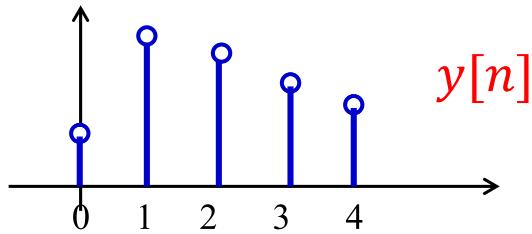
  - Mathematically, if we let $r = \langle k \rangle_N$, then $r = k + lN$, where l is an integer chosen to make $k + lN$ an integer between 0 and $N-1$.

Example

- $\{x[n]\} = \{x[0], x[1], x[2], x[3], x[4]\}$
- Then the circular reversed sequence $\{y[n]\}$ is given by:

$$y[n] = x[\langle -n \rangle_N]$$
$$y[0] = x[\langle -0 \rangle_5] = x[0],$$
$$y[1] = x[\langle -1 \rangle_5] = x[4],$$
$$y[2] = x[\langle -2 \rangle_5] = x[3],$$
$$y[3] = x[\langle -3 \rangle_5] = x[2],$$
$$y[4] = x[\langle -4 \rangle_5] = x[1],$$





- Hence, $\{y[n]\} = \{x[0], x[4], x[3], x[2], x[1]\}$

Symmetry of Finite-Length Sequence

- Circular Conjugate Symmetry
  - An N-point sequence is said to be **circular conjugate symmetric sequence** if

$$x[n] = x^*[\langle -n \rangle_N] = x^*[\langle N - n \rangle_N]$$

  - An N-point sequence is said to be **circular conjugate anti-symmetric sequence** if

$$x[n] = -x^*[\langle -n \rangle_N] = -x^*[\langle N - n \rangle_N]$$

- An arbitrary N-point complex sequence can be expressed as the sum of the circular Conjugate Symmetric and circular conjugate anti-symmetric parts, i.e.,

$$x[n] = x_{cs}[n] + x_{ca}[n], \qquad 0 \le n \le N-1$$

where,

$$x_{cs}[n] = \frac{1}{2}(x[n] + x^*[\langle -n \rangle_N]), \qquad 0 \le n \le N-1,$$
$$x_{ca}[n] = \frac{1}{2}(x[n] - x^*[\langle -n \rangle_N]), \qquad 0 \le n \le N-1,$$

are the **circular conjugate-symmetric** and **circular conjugate-antisymmetric** parts, respectively.

Example

- Q: Find the circular conjugate-symmetric and circular conjugate-antisymmetric parts of

$$\{u[n]\} = \{1 + j4, -2 + j3, 4 - j2, -5 - j6\}$$

- A: We first form its complex conjugate sequence

$$\{u^*[n]\} = \{1 - j4, -2 - j3, 4 + j2, -5 + j6\}$$

Then compute $\{u^*[\langle -n \rangle_4]\}$:

$$u^*[\langle -0 \rangle_4] = u^*[0], \qquad u^*[\langle -1 \rangle_4] = u^*[3]$$
$$u^*[\langle -2 \rangle_4] = u^*[2], \qquad u^*[\langle -3 \rangle_4] = u^*[1]$$

Hence, $\{u^*[\langle -n \rangle_4]\} = \{1 - j4, -5 + j6, 4 + j2, -2 - j3\}$

$$\{x_{cs}[n]\} = \{1, -3.5 + j4.5, 4, -3.5 - j4.5\}$$

$$\{x_{ca}[n]\} = \{j4, 1.5 - j1.5, -j2, -1.5 - j1.5\}$$

Symmetry of Finite-Length Sequence

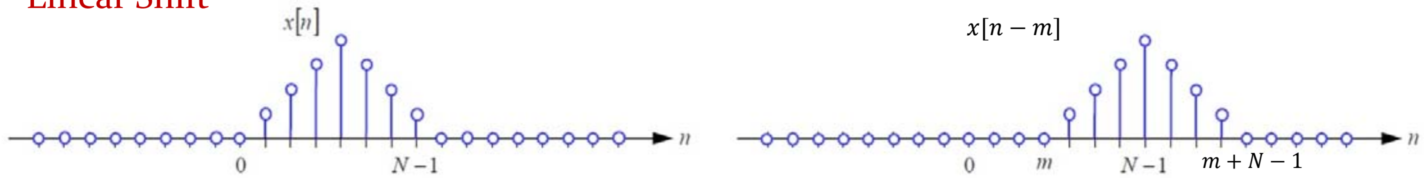
- Geometric Symmetry
  - A length-N symmetric sequence
    $$x[n] = x[N-1-n], \qquad 0 \le n \le N-1$$
  - A length-N antisymmetric sequence
    $$x[n] = -x[N-1-n], \qquad 0 \le n \le N-1$$

Circular Shift of A Sequence

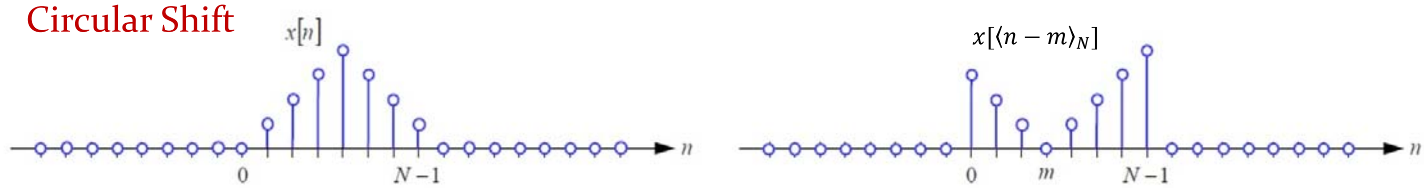
- let $x[n]$ is defined for the range of $0 \le n \le N-1$
- Linear shifted sequence $x_1[n] = x[n-m]$ is no longer defined for the range of $0 \le n \le N-1$
- Define
  $$x_c[n] = x[\langle n-m \rangle_N]$$
  where $\langle k \rangle_N = k$ modulo N.
- For $m > 0$ (right circular shift), the above equation implies:
  $$x_c[n] = \begin{cases} x[n-m], & \text{for } m \le n \le N-1, \\ x[n-m+N], & \text{for } 0 \le n < m \end{cases}$$

$$x_c[n] = x[\langle n-m\rangle_N] = \begin{cases} x[n-m], & \text{for } m \le n \le N-1, \\ x[n-m+N], & \text{for } 0 \le n \le m \end{cases}$$

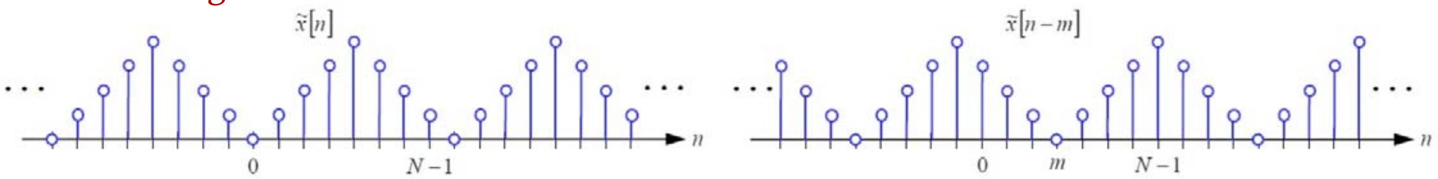
Linear Shift



Circular Shift



Periodic Signal



Circular Convolution

- Definition:

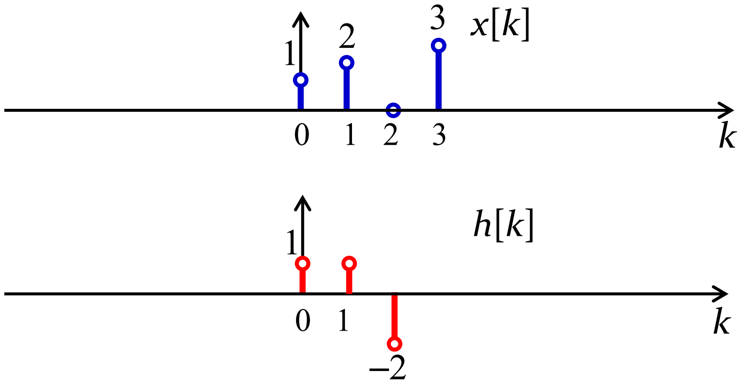
$$y_c[n] = x[n] \circledN h[n] \triangleq \sum_{m=0}^{N-1} x[m]h[\langle n-m\rangle_N]$$

for two signals of length N

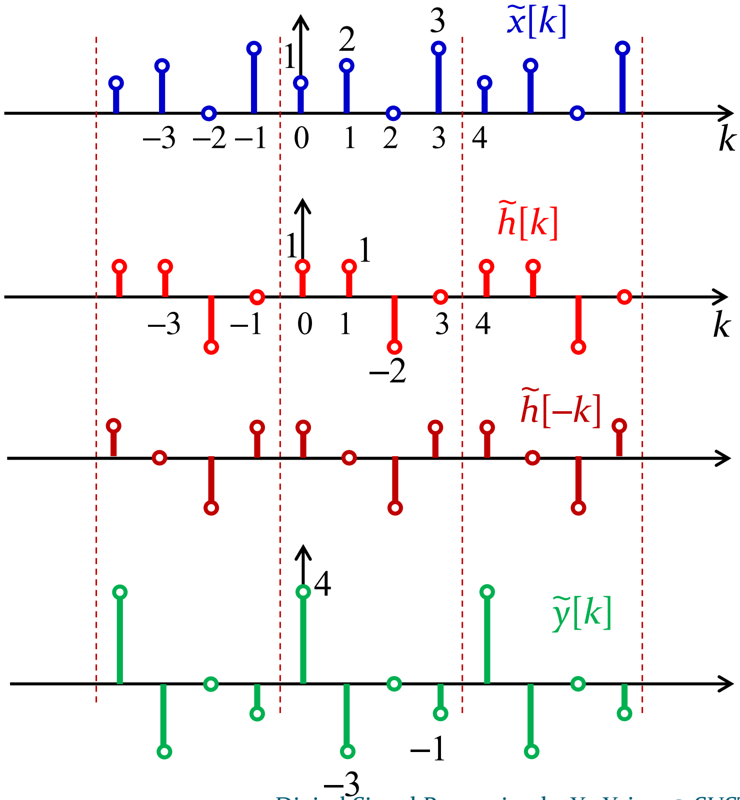
- Circular convolution is commutative:

$$x[n] \circledN h[n] = h[n] \circledN x[n]$$

Example 4



Example 4 (continued)



Properties of DFT

- Many are analogous to the properties of DTFT, but replacing shifting to circular shifting
- Inherited from Fourier Transform, so no need to be proved

General Properties of the DFT

Properties	Sequence	*N*-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time shifting	$g[\langle n - n_0 \rangle_N]$	$W_N^{kn_0} G[k]$
Circular frequency shifting	$W_N^{-k_0 n} g[n]$	$G[\langle k - k_0 \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
Circular Convolution	$g[n] \textcircled{N} h[n]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{l=0}^{N-1} G[l]H[\langle k - l \rangle_N]$
Parseval's Theorem	$\sum_{n=0}^{N-1} g[n]h^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} G[k]H^*[k]$	

Symmetry Properties of the DFT

an N-point Complex Sequence

Length-N Sequence	N-point DFT
$x[n] = x_{\text{re}}[n] + jx_{\text{im}}[n]$	$X[k] = X_{\text{re}}[k] + jX_{\text{im}}[k]$
$x[\langle -n \rangle_N]$	$X[\langle -k \rangle_N]$
$x^*[\langle -n \rangle_N]$	$X^*[k]$
$x^*[n]$	$X^*[\langle -k \rangle_N]$
$x_{\text{re}}[n]$	$X_{\text{cs}}[k] = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$
$jx_{\text{im}}[n]$	$X_{\text{ca}}[k] = \frac{1}{2}\{X[k] - X^*[\langle -k \rangle_N]\}$
$x_{\text{cs}}[n]$	$X_{\text{re}}[k]$
$x_{\text{ca}}[n]$	$jX_{\text{im}}[k]$

Symmetry Properties of the DFT

an N-point Real Sequence

Length-N Sequence	N-point DFT
$x[n] = x_{\text{ev}}[n] + x_{\text{od}}[n]$	$X[k] = X_{\text{re}}[k] + jX_{\text{im}}[k]$
$x_{\text{ev}}[n]$	$X_{\text{re}}[k]$
$x_{\text{od}}[n]$	$jX_{\text{im}}[k]$
Conjugate Symmetric	$X[k] = X^*[\langle -k \rangle_N]$
	$X_{\text{re}}[k] = X_{\text{re}}[\langle -k \rangle_N]$
	$X_{\text{im}}[k] = -X_{\text{im}}[\langle -k \rangle_N]$
	$\lvert X[k] \rvert = \lvert X[\langle -k \rangle_N] \rvert$
	$\arg\{X[k]\} = -\arg\{X[\langle -k \rangle_N]\}$

Example 5

- Q: if a complex sequence $x[n] \leftrightarrow X[k]$, then $x^*[n] \leftrightarrow X^*[\langle -k \rangle_N]$
- Proof: since $X^*[k] = \sum_{n=0}^{N-1} x^*[n]e^{j2\pi kn/N}$

  then $X^*[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x^*[n]e^{j2\pi(\langle -k \rangle_N)n/N}$

  For $k = 0$, we have $X^*[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x^*[n]$

  For $1 \leq k \leq N-1$, we have

  $$X^*[\langle -k \rangle_N] = X^*[N-k] = \sum_{n=0}^{N-1} x^*[n]e^{j2\pi(N-k)n/N}$$
  $$= \sum_{n=0}^{N-1} x^*[n]e^{j2\pi n}e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x^*[n]e^{-j2\pi kn/N}$$

  Combining the above two results, we get

  $X^*[\langle -k \rangle_N] = \sum_{n=0}^{N-1} x^*[n]e^{-j2\pi kn/N}$, i.e., the DFT of $x^*[n]$

Example 6

- Q: If real sequence $x[n] \leftrightarrow X[k]$, then $X[\langle -k \rangle_N] = X[\langle N-k \rangle_N] = X^*[k]$
- Proof:

$$X[\langle N-k \rangle_N] = \sum_{n=0}^{N-1} x[n]W_N^{(N-k)n} = \sum_{n=0}^{N-1} x[n]W_N^{Nn}W_N^{-kn}$$

$$= \sum_{n=0}^{N-1} x[n]W_N^{-kn} = X^*[k]$$

Example 7

- Q: Consider a length-N sequence $x[n], 0 \leq n \leq N-1$, with N even. Define two subsequences of length-$\left(\frac{N}{2}\right)$ each: $g[n] = x[2n]$ and $h[n] = x[2n+1], 0 \leq n \leq \frac{N}{2} - 1$. Denote $X[k], 0 \leq k \leq N-1$, the N-point DFT of $x[n]$, and $G[k]$ and $H[k], 0 \leq k \leq \frac{N}{2} - 1$, the $\left(\frac{N}{2}\right)$-point DFT of $g[n]$ and $h[n]$, respectively. Express $X[k]$ as a function of $G[k]$ and $H[k]$.

- A: Given the DFT of the original sequence, $X[k]$, we can express it in terms of even and odd parts.

$$
\begin{aligned}
X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} &= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_N^{2rk} + \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_N^{(2r+1)k} \\
&= \sum_{r=0}^{\frac{N}{2}-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} x[2r+1] W_{N/2}^{rk} \\
&= \sum_{r=0}^{\frac{N}{2}-1} g[r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{\frac{N}{2}-1} h[r] W_{N/2}^{rk} \\
&= G\left[\langle k \rangle_{\frac{N}{2}}\right] + W_N^k H\left[\langle k \rangle_{\frac{N}{2}}\right], \qquad 0 \leq k \leq N-1
\end{aligned}
$$

Properties of the DFT

- Circular Convolution: Let $x_1[n]$ and $x_2[n]$ be length N with DFT $X_1[k]$ and $X_2[k]$

$$x_1[n] \circledN x_2[n] \leftrightarrow X_1[k] \cdot X_2[k]$$

  - Very useful!!! (for linear convolutions with DFT)

- Multiplication (Modulation): Let $x_1[n]$ and $x_2[n]$ be length N with DFT $X_1[k]$ and $X_2[k]$

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{N} X_1[k] \circledN X_2[k]$$

Linear Convolution

- Next....
  - Using DFT, circular convolution is easy
  - But, **linear** convolution is useful, not circular
  - So, show how to perform linear convolution with circular convolution
  - Use DFT to do linear convolution

Linear Convolution

- We start with two non-periodic sequences:

$$x[n] \quad 0 \le n \le L-1$$
$$h[n] \quad 0 \le n \le P-1$$

for example, $x[n]$ is a signal, and $h[n]$ an impulse response of a system.

- • We want to compute the linear convolution:

$$h[n] \circledast x[n] = \sum_{k=0}^{L-1} x[k]h[n-k]$$

$y[n]$ is nonzero for $0 \le n \le L+P-2$ with length $M=L+P-1$

- • Requires $L \cdot P$ multiplications if computed directly

Linear Convolution via Circular Convolution

- Zero-pad $x[n]$ by $P-1$ zeros

$$x_{zp}[n] = \begin{cases} x[n] & 0 \le n \le L-1 \\ 0 & L \le n \le L+P-2 \end{cases}$$

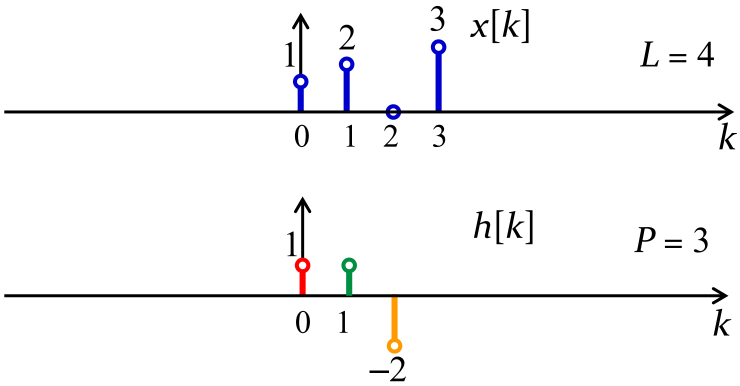
- Zero-pad $h[n]$ by $L-1$ zeros

$$h_{zp}[n] = \begin{cases} h[n] & 0 \le n \le P-1 \\ 0 & P \le n \le L+P-2 \end{cases}$$

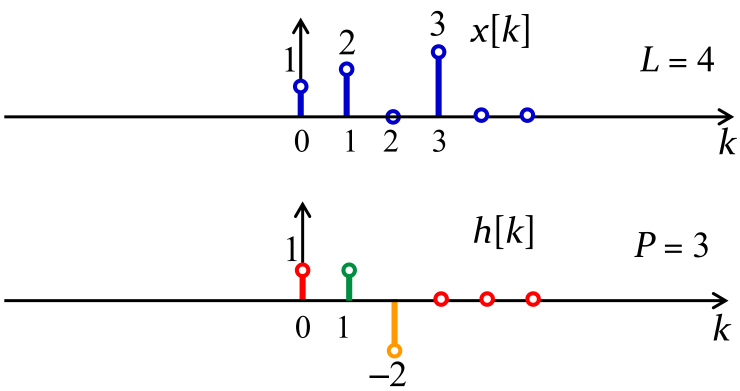
- Now, both sequences are of length $M=L+P-1$
- We can now compute the linear convolution using a circular one with length $M = L+P-1$

$$y[n] = h[n] * x[n] = x_{zp}[n] \, \textcircled{\scriptsize N} \, h_{zp}[n]$$

Example 8

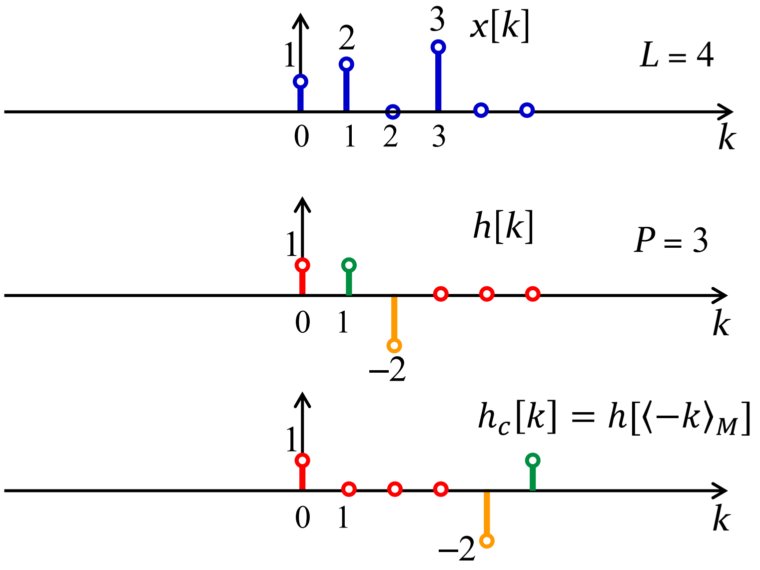


Example 8 (continued)



$M=L+P-1=6$

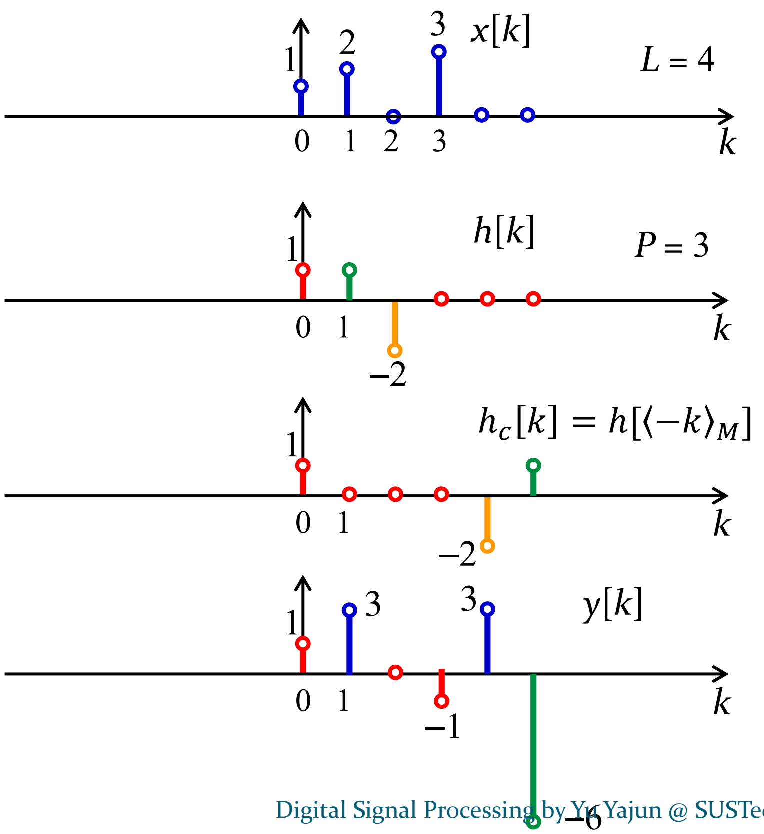
Example 8 (continued)



M=L+P−1=6

Circular Flip
(Circular time-reversal)

Example 8 (continued)



M=L+P−1=6

Circular Flip
(Circular time-reversal)

Linear Convolution using DFT

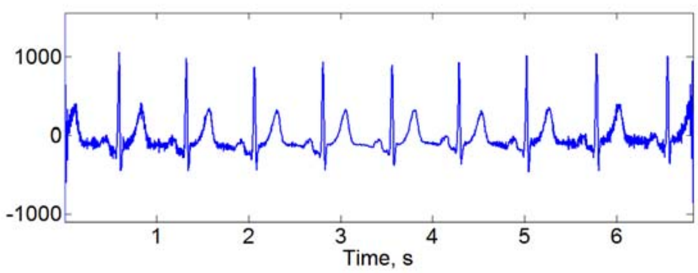
- In practice we can implement a circular convolution using the DFT property:

$$h[n] \circledast x[n] = x_{zp}[n] \, \text{Ⓝ} \, h_{zp}[n]$$
$$= \mathcal{IDFT}\{\mathcal{DFT}\{x_{zp}[n]\} \cdot \mathcal{DFT}\{h_{zp}[n]\}\}$$

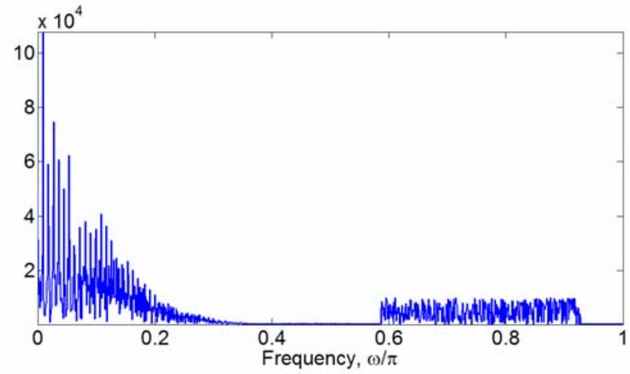
- Advantage: DFT can be computed with $N\log_2 N$ complexity (FFT algorithm!)
- Drawback: Must wait for all the samples -- huge delay -- incompatible with real-time implementation

Fourier-Domain Filtering

- Remove some frequency bands directly from Fourier-Domain, since $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$



ECG Signal + Noise



ECG Signal + Noise

- In this particular example, set

$$H(e^{j\omega}) = \begin{cases} 0, & 0.55\pi \le \omega \le 0.95\pi \\ 1, & \text{otherwise} \end{cases}$$

Fourier-Domain Filtering

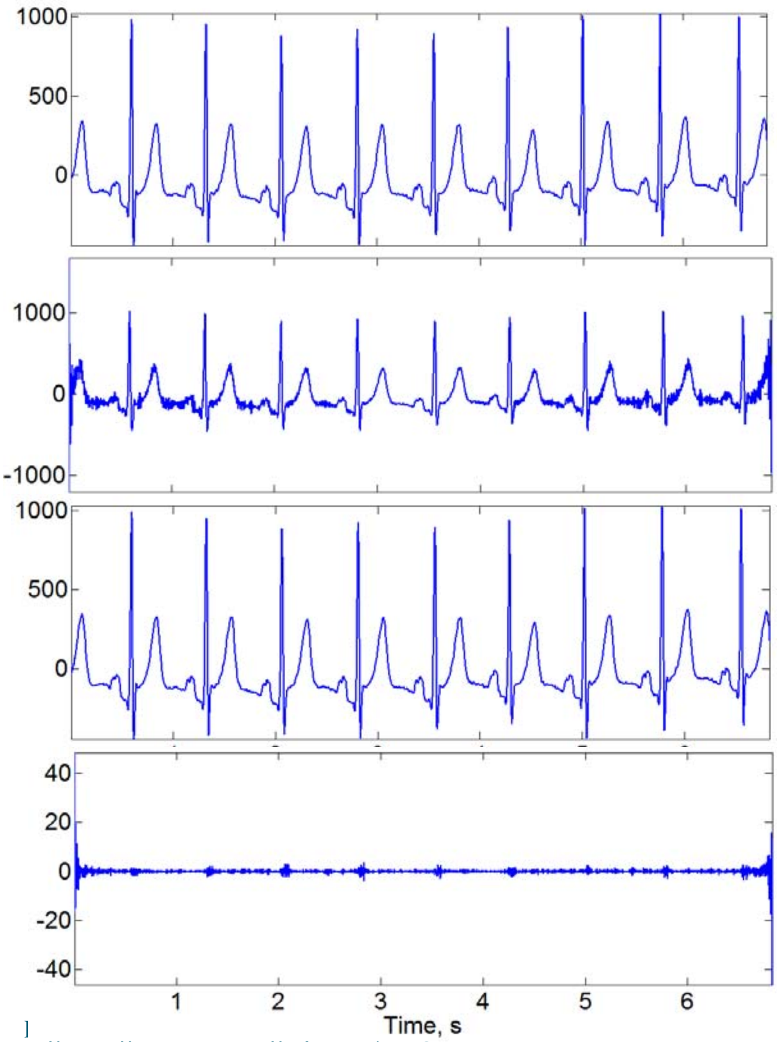
- Find the DTFT of the ECG signal to get $X(e^{j\omega})$, multiply with $H(e^{j\omega})$ to obtain $Y(e^{j\omega})$, and find the IDTFT of $Y(e^{j\omega})$
- We can use DFT to compute $X(e^{j\omega})$ and $Y(e^{j\omega})$ at frequency values of $\omega = 2\pi k/N$, for , $k = 0, 1, \dots, N-1$
- This approach is equivalent to the circular convolution of the finite-length signal $x[n]$ and the finite-length ideal filter $h[n]$.
- However, the ideal filter has an infinite length impulse response. Sampling the Fourier transform to create DFT samples leads to the time domain aliasing.

Original Signal without noise $s[n]$

$x[n]$: Original Signal with noise

$y[n]$: Filtered $x[n]$

$s[n] - y[n]$: time aliasing of the recovered signal



Computation of DFT

- The N-point DFT $X[k]$ of a length-N sequence $x[n]$, $0 \leq n \leq N-1$, is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \qquad k = 0, 1, \ldots, N-1$$

where $W_N = e^{-j2\pi/N}$.

- Direct computation of all N samples of $\{X[k]\}$ requires N^2 complex multiplications and $N(N-1)$ complex additions.

Decimation-in-time FFT algorithm

- Most conveniently illustrated by considering the special case of N an integer power of 2, i.e, $N = 2^v$.
- Since N is an even integer, we can consider computing $X[k]$ by separating $x[n]$ into two $(N/2)$-point sequences consisting of the even numbered point in $x[n]$ and the odd-numbered points in $x[n]$, with the substitution of variable $n = 2r$ for n even and $n = 2r + 1$ for n odd

- We have

$$X[k] = \sum_{r=0}^{(N/2)-1} x[2r]\, W_N^{2rk} + \sum_{r=0}^{(N/2)-1} x[2r+1]\, W_N^{(2r+1)k}$$

$$= \sum_{r=0}^{(N/2)-1} x[2r]\, (W_N^2)^{rk} + W_N^k \sum_{r=0}^{(N/2)-1} x[2r+1]\, (W_N^2)^{rk}$$

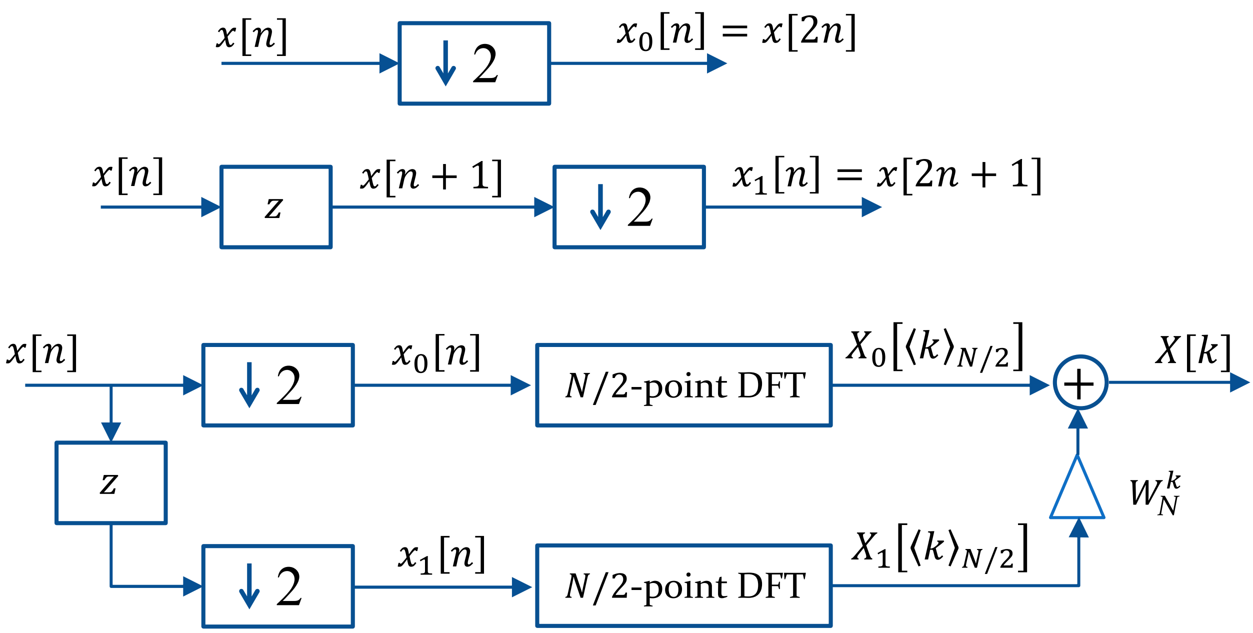
$$= X_0\big[\langle k\rangle_{N/2}\big] + W_N^k X_1\big[\langle k\rangle_{N/2}\big], \qquad 0 \le k \le N-1$$

where $X_0[k] = \sum_{r=0}^{\frac{N}{2}-1} x_0\,[r]\, W_{N/2}^{rk} = \sum_{r=0}^{\frac{N}{2}-1} x[2r]\, W_{N/2}^{rk}$ and

$X_1[k] = \sum_{r=0}^{\frac{N}{2}-1} x_1[r]\, W_{N/2}^{rk} = \sum_{r=0}^{\frac{N}{2}-1} x[2r+1]\, W_{N/2}^{rk}$, $0 \le k \le \frac{N}{2} - 1$

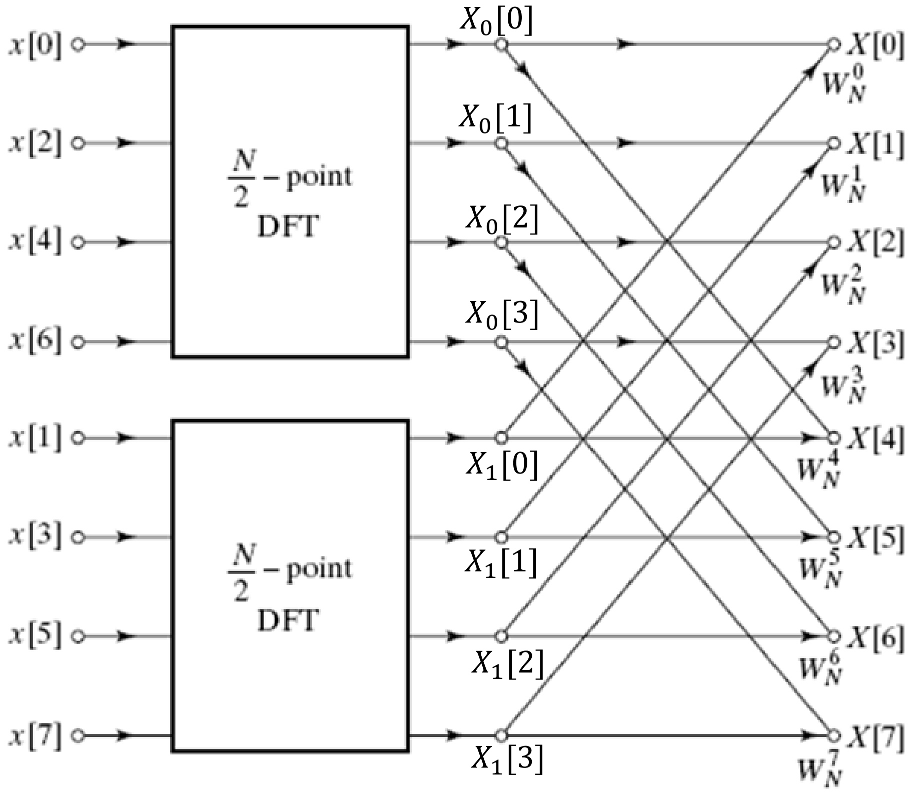
- Both $X_0[k]$ and $X_1[k]$ can be computed by $(N/2)$-point DFT

Structure Interpretation



- Decomposing N-point DFT into two $(N/2)$-point DFT for the case of N=8

$X[k] = X_0[\langle k \rangle_{N/2}] + W_N^k X_1[\langle k \rangle_{N/2}],$



Cost to compute an *N*-point DFT (1)

- A direct computation
  - N^2 complex multiplications
  - $N^2 - N \cong N^2$ complex additions
- Using a decomposition computing two $(N/2)$-point DFTs
  - $N + 2(N/2)^2 = N + N^2/2$ complex multiplications
  - Approximately $N + N^2/2$ complex additions
- For $N \geq 3$, $N + N^2/2 < N^2$

- We can further decompose the $(N/2)$-point DFT into two $(N/4)$-point DFTs.

$$X_0[k] = X_{00}\big[\langle k\rangle_{N/4}\big] + W_{N/2}^{k}\, X_{01}\big[\langle k\rangle_{N/4}\big], 0 \le k \le N/2-1$$

$$X_1[k] = X_{10}\big[\langle k\rangle_{N/4}\big] + W_{N/2}^{k} X_{11}\big[\langle k\rangle_{N/4}\big], 0 \le k \le N/2-1$$

where,

$$X_{00}[k] = \sum_{r=0}^{\frac{N}{4}-1} x_{00}\,[r]\, W_{N/4}^{rk} = \sum_{r=0}^{\frac{N}{4}-1} x_0[2r]\, W_{N/4}^{rk}$$

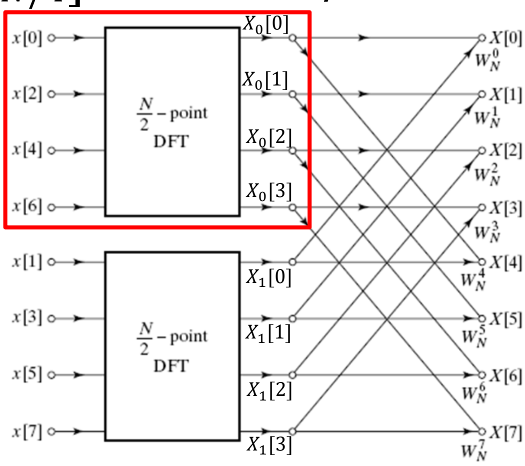
$$= \sum_{r=0}^{\frac{N}{4}-1} x[4r]\, W_{N/4}^{rk}$$

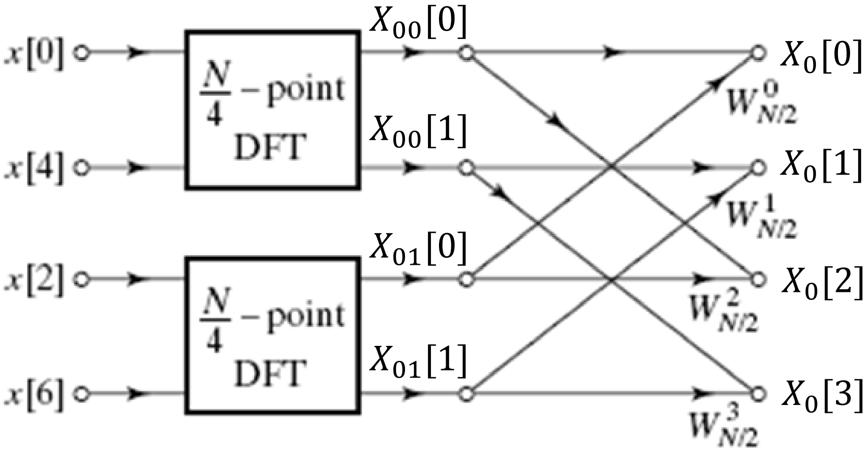
etc.

- For example, the upper half of the previous diagram, corresponding to

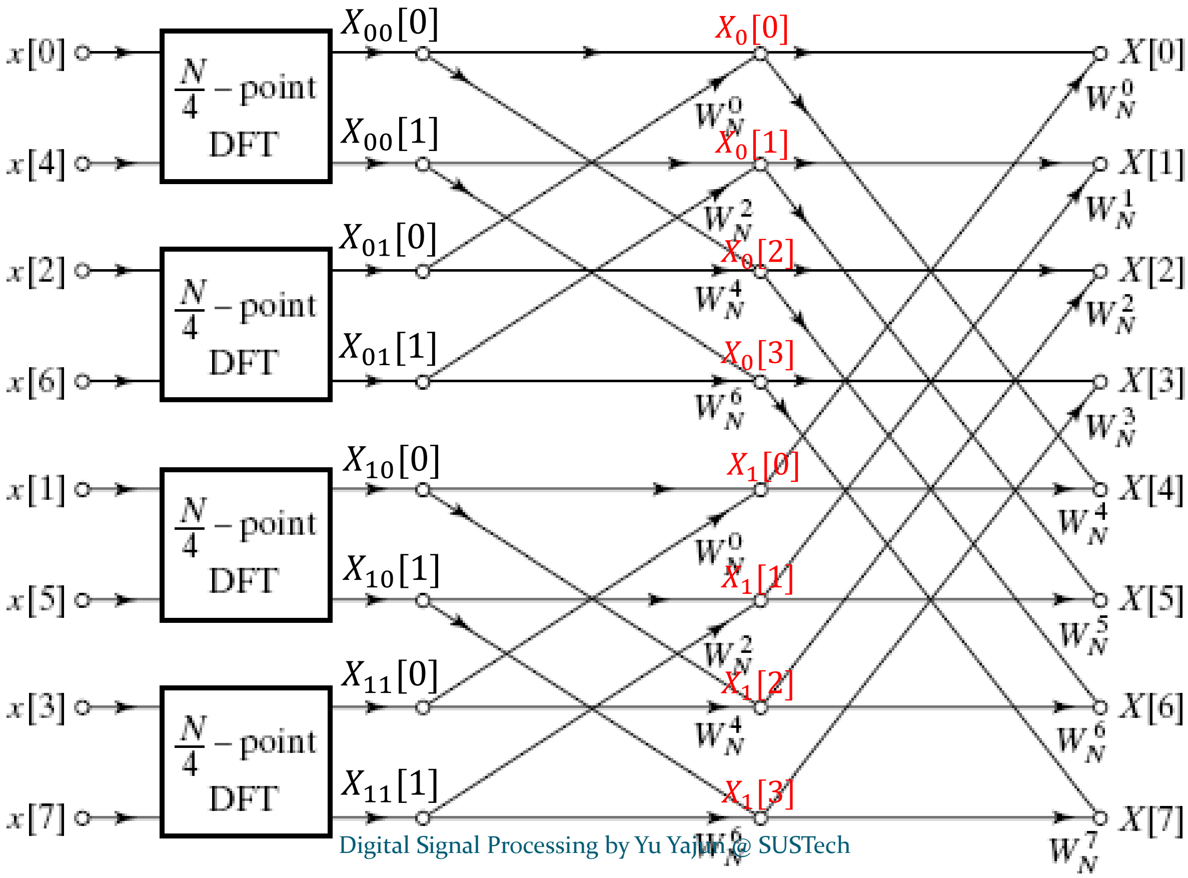
$$X_0[k] = X_{00}\big[\langle k\rangle_{N/4}\big] + W_{N/2}^{k} X_{01}\big[\langle k\rangle_{N/4}\big], 0 \le k \le N/2-1$$

can be de-composed as

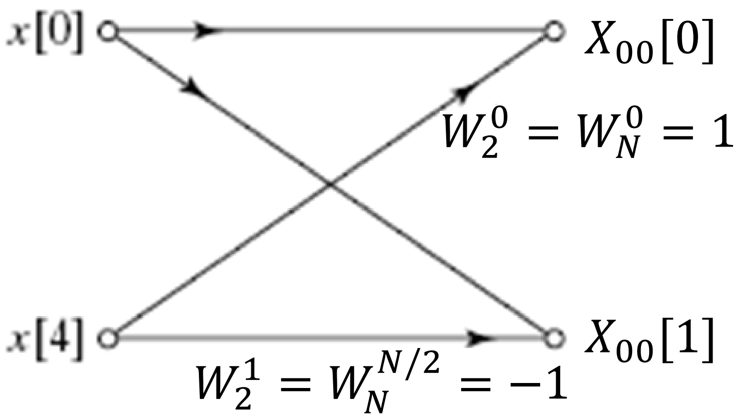




- Hence, the 8-point DFT can be obtained by the following diagram with four 2-point DFTs.

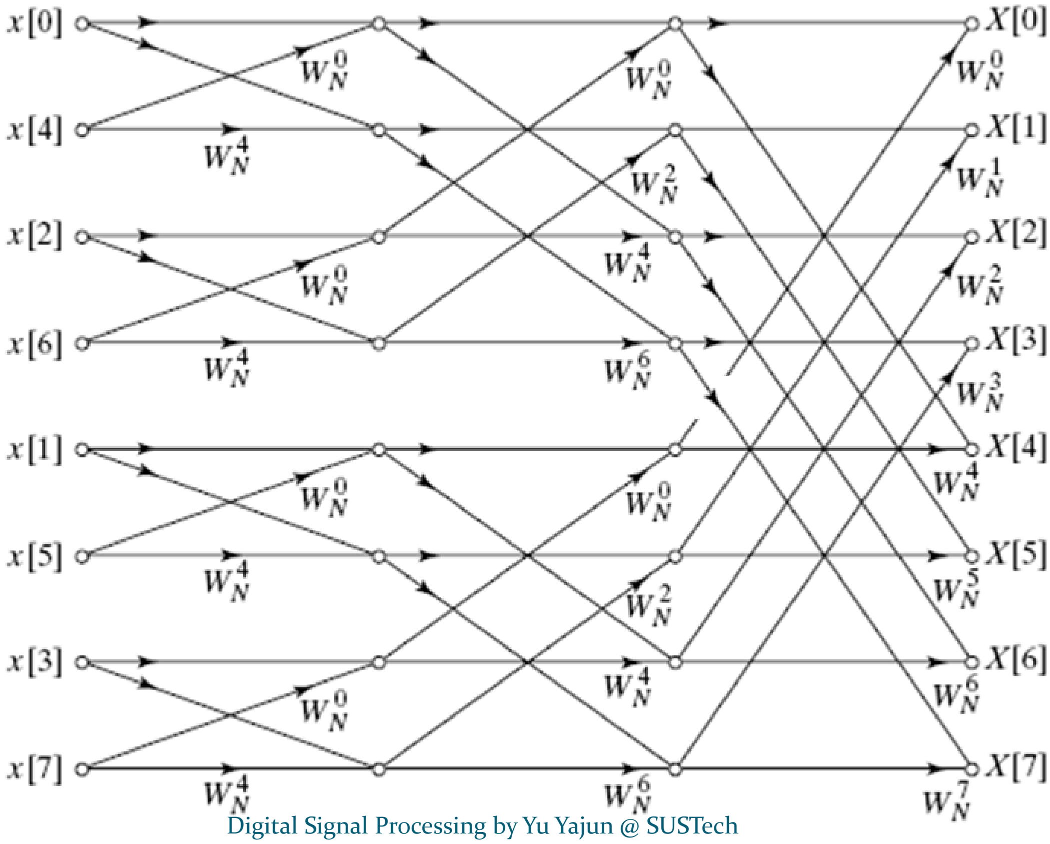


- Finally, each 2-point DFT can be implemented by the following signal-flow graph, where no multiplications are needed.



$$X_{00}[0] = x[0]W_2^0 + x[4]W_2^0$$
$$X_{00}[1] = x[0]W_2^0 + x[4]W_2^1$$

- Complete flow graph of decimation-in-time decomposition of an 8-point DFT.



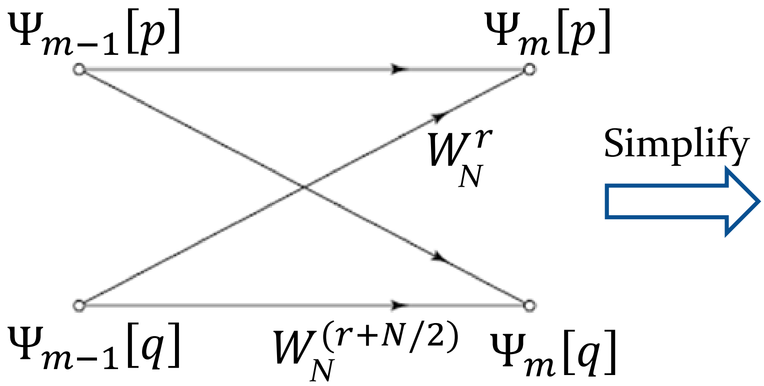
Cost to compute an *N*-point DFT (2)

- Eight-point FFT consists of three stages.
  - The 1st stage computes the four 2-point DFTs; the 2nd stage computes the two 4-point DFTs; the 3rd stage computes the desired 8-point DFT.
  - The number of complex multiplications and additions performed at each stage is 8, the size of transformation. In total 24 complex multiplications and additions are required.
- When N is the power of 2, $N = 2^v$, it requires $v = \log_2 N$ stages of computation. The number of complex multiplications and additions required is therefore $Nv = N \log_2 N$.

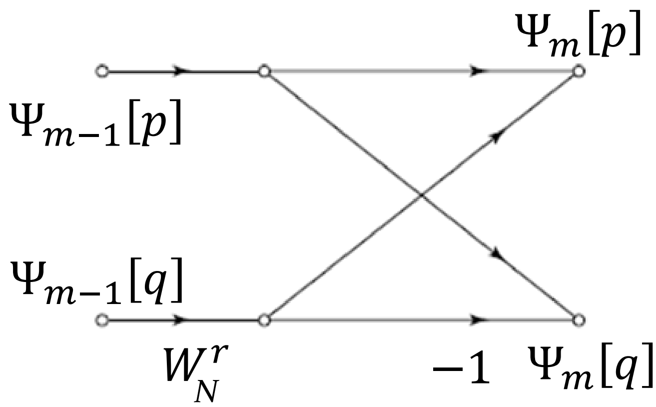
- In each stage of the decimation-in-time FFT algorithm, there are a basic structure called the butterfly computation:

$$\Psi_m[p] = \Psi_{m-1}[p] + W_N^r \Psi_{m-1}[q]$$

$$\Psi_m[q] = \Psi_{m-1}[p] + W_N^{r+(N/2)} \Psi_{m-1}[q]$$

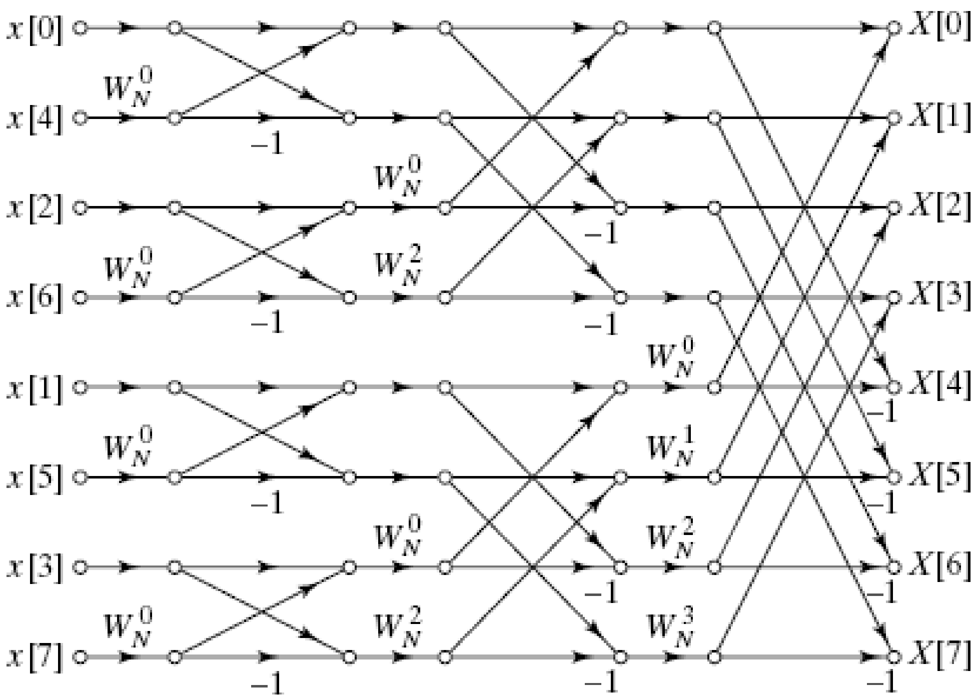


Flow graph of a basic butter-fly Computation in FFT.



Simplified butterfly computation.

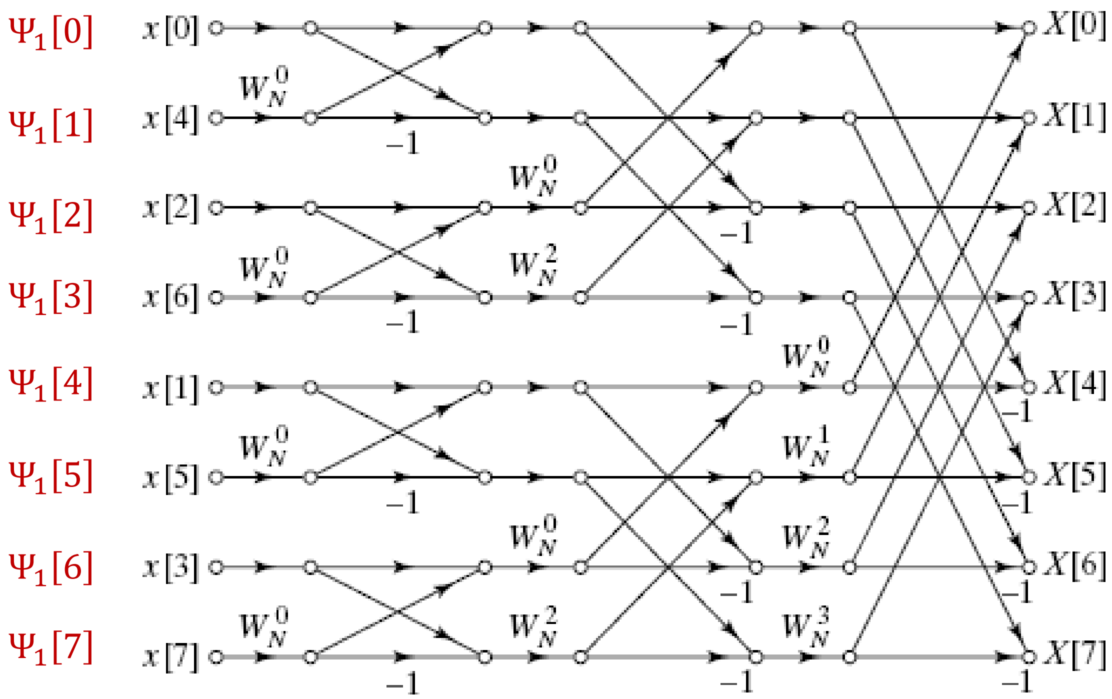
- Flow graph of 8-point FFT using the simplified butterfly computation



Cost to compute an N-point DFT (3)

- Using the simplified butterfly computation, the number of complex multiplications ~~and additions~~ performed at each stage is reduced to $N/2$. Thus the total numbers become $N\upsilon/2 = \frac{N}{2}\log_2 N$
- By excluding trivial complex multiplications with $W_N^0 = 1$ and $W_N^{N/2} = -1$, the exact count of non-trivial complex multiplications are even less, given by $\frac{N}{2}(\log_2 N - 2) + 1$

- Note the ordering of the input sequence $x[n]$, while the DFT samples $X[k]$ appear at the output in a sequential order.



- Let $\Psi_1[m]$ be the sequentially ordered new representation of the input sample $x[n]$.

$\Psi_1[m]$	$x[n]$
$\Psi_1[0]$	$x[0]$
$\Psi_1[1]$	$x[4]$
$\Psi_1[2]$	$x[2]$
$\Psi_1[3]$	$x[6]$
$\Psi_1[4]$	$x[1]$
$\Psi_1[5]$	$x[5]$
$\Psi_1[6]$	$x[3]$
$\Psi_1[7]$	$x[7]$

m is a bit-reversed version of n

When *N* is not the power of 2

- By zero-padding a sequence into an N-point sequence with $N = 2^v$, we can choose the nearest power-of-two FFT algorithm for implementing a DFT.
  - ➢ The FFT algorithm of power-of-two is also called the Cooley-Tukey algorithm since it was first proposed by them.

Decimation-in-frequency FFT algorithm

The decimation-in-time FFT algorithms are all based on structuring the DFT computation by forming smaller and smaller subsequences of the input sequence $x[n]$. Alternatively, we can consider dividing the output sequence $X[k]$ into smaller and smaller subsequences in the same manner.

$$X[k]=\sum_{n=0}^{N-1} x[n]W_N^{nk} \qquad k=0,1,\ldots,N-1$$

$$X[k]=\sum_{n=0}^{N-1} x[n]W_N^{nk} \qquad k=0,1,\ldots,N-1$$

The even-numbered frequency samples are

$$X[2r]=\sum_{n=0}^{N-1} x[n]W_N^{n(2r)}=\sum_{n=0}^{(N/2)-1} x[n]W_N^{n(2r)}+\sum_{n=(N/2)}^{N-1} x[n]W_N^{n(2r)}$$

$$X[2r]=\sum_{n=0}^{(N/2)-1} x[n]W_N^{2nr}+\sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{2r(n+(N/2))}$$

Since $W_N^{2r[n+(N/2)]}=W_N^{2rn}W_N^{rN}=W_N^{2rn}$

$$X[2r]=\sum_{n=0}^{(N/2)-1}(x[n]+x[n+(N/2)])W_{N/2}^{rn} \qquad r=0,1,\ldots,(N/2)-1$$

$$X[2r] = \sum_{n=0}^{(N/2)-1} (x[n] + x[n+(N/2)])W_{N/2}^{rn} \qquad r = 0,1,\ldots,(N/2)-1$$

The above equation is the $(N/2)$-point DFT of the $(N/2)$-point sequence obtained by adding the first and the last half of the input sequence.

Adding the two halves of the input sequence represents time aliasing, consistent with the fact that in computing only the even-number frequency samples, we are sub-sampling the Fourier transform of $x[n]$.

We now consider obtaining the odd-numbered frequency points:

$$X[2r+1] = \sum_{n=0}^{N-1} x[n]W_N^{n(2r+1)} = \sum_{n=0}^{(N/2)-1} x[n]W_N^{n(2r+1)} + \sum_{n=(N/2)}^{N-1} x[n]W_N^{n(2r+1)}$$

Since $$\sum_{n=N/2}^{N-1} x[n]W_N^{n(2r+1)} = \sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{(n+N/2)(2r+1)}$$

$$= W_N^{(N/2)(2r+1)} \sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{n(2r+1)}$$

$$= -\sum_{n=0}^{(N/2)-1} x[n+(N/2)]W_N^{n(2r+1)}$$

We obtain

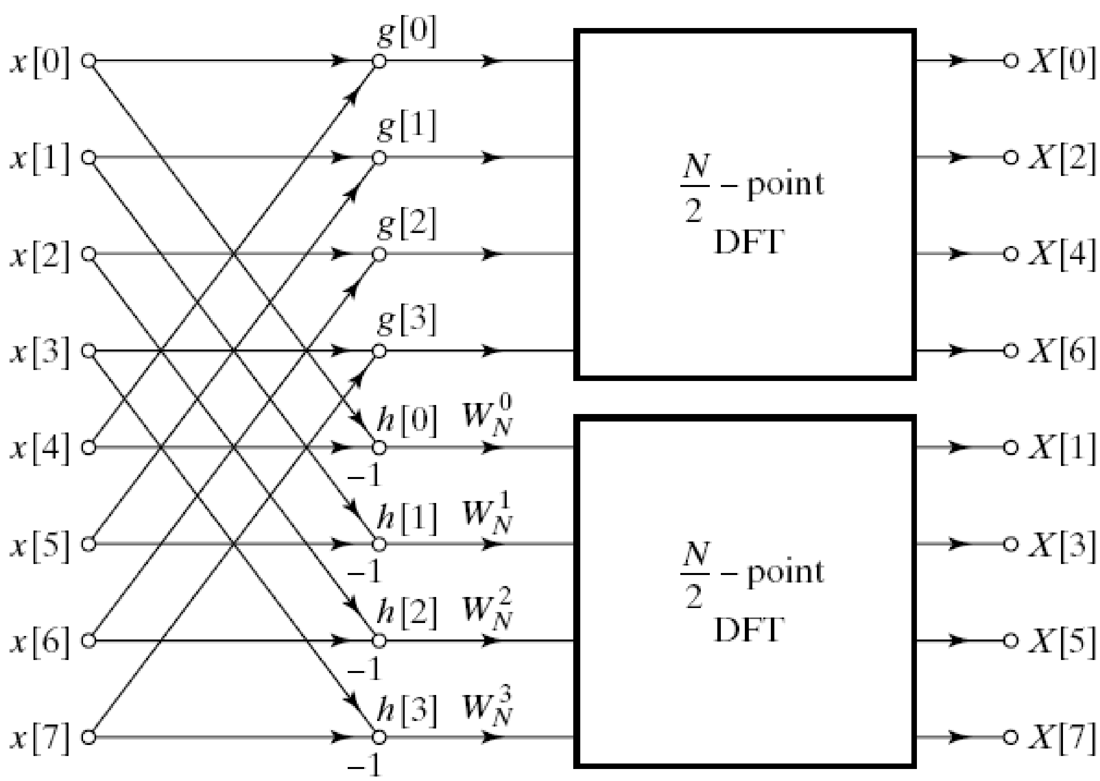
$$X[2r+1]=\sum_{n=0}^{(N/2)-1}(x[n]-x[n+N/2])W_N^{n(2r+1)}$$

$$=\sum_{n=0}^{(N/2)-1}\boxed{(x[n]-x[n+N/2])W_N^{n}}W_{N/2}^{nr} \qquad r=0,1,\ldots,(N/2)-1$$

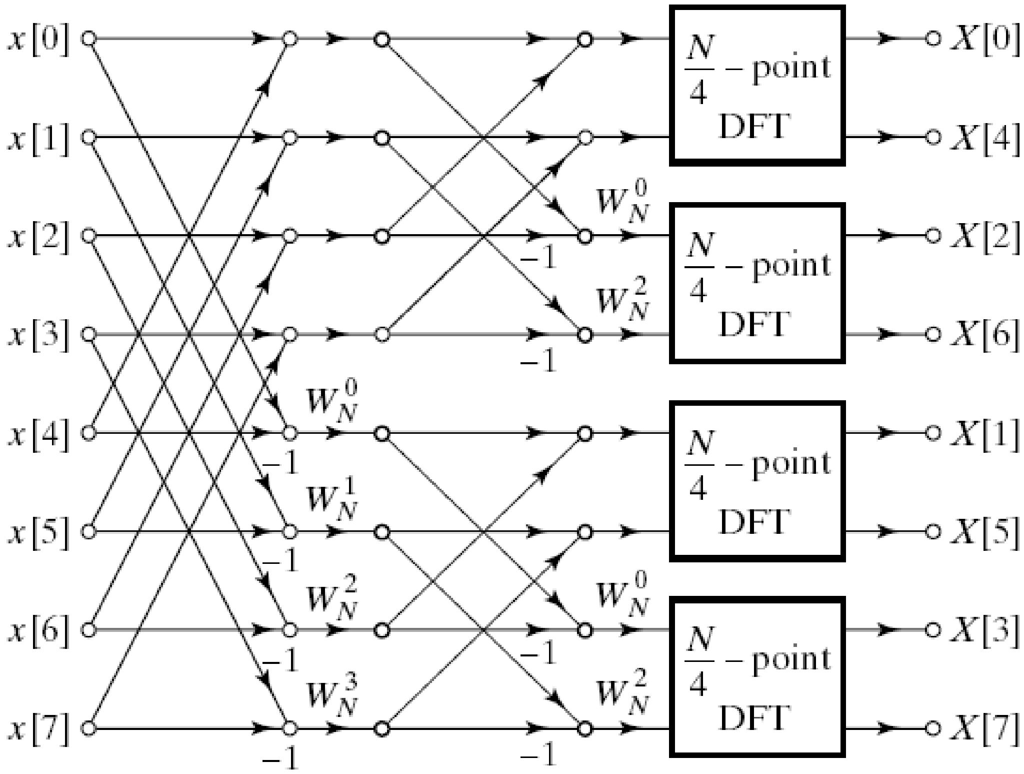
The above equation is the $(N/2)$-point DFT of the sequence obtained by subtracting the second half of the input sequence from the first half and multiplying the resulting sequence by W_N^n.

Let $g[n] = x[n]+x[n+N/2]$ and $h[n] = x[n]-x[n+N/2]$, the DFT can be computed by forming the sequences $g[n]$ and $h[n]$, then computing $h[n]W_N^n$, and finally computing the $(N/2)$-point DFTs of these two sequences.

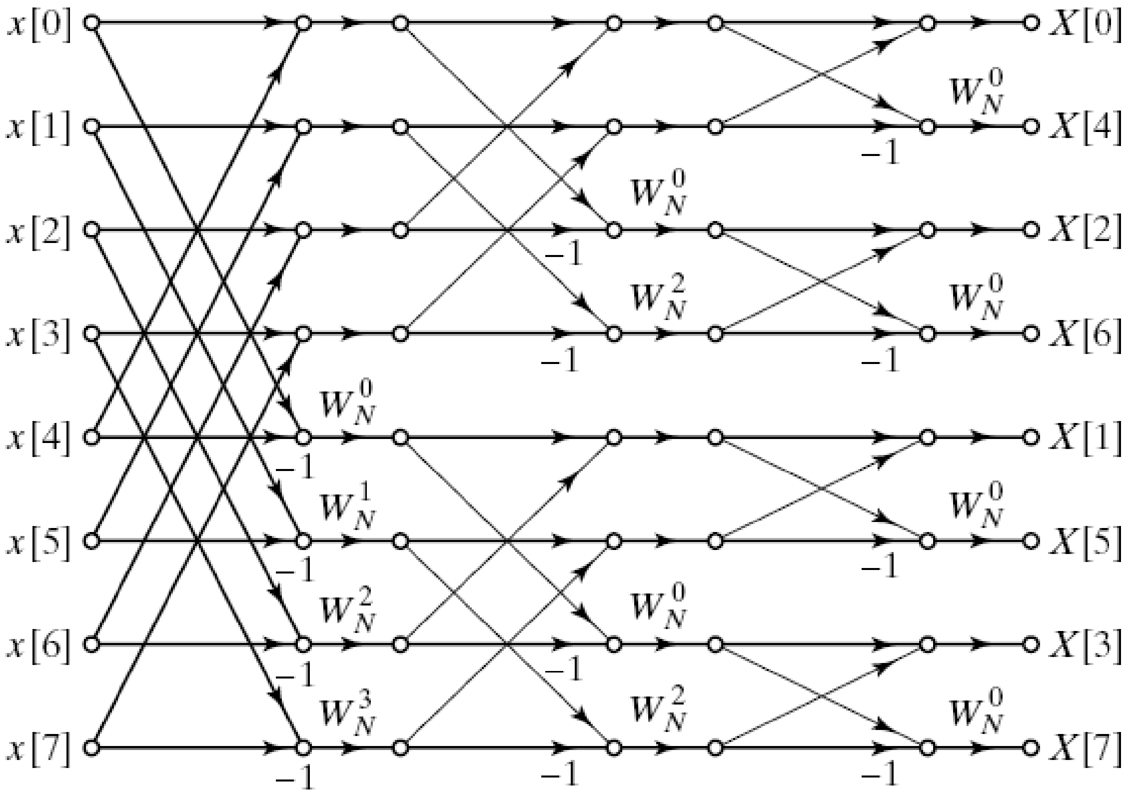
Flow graph of decimation-in-frequency decomposition of an N-point DFT ($N = 8$).



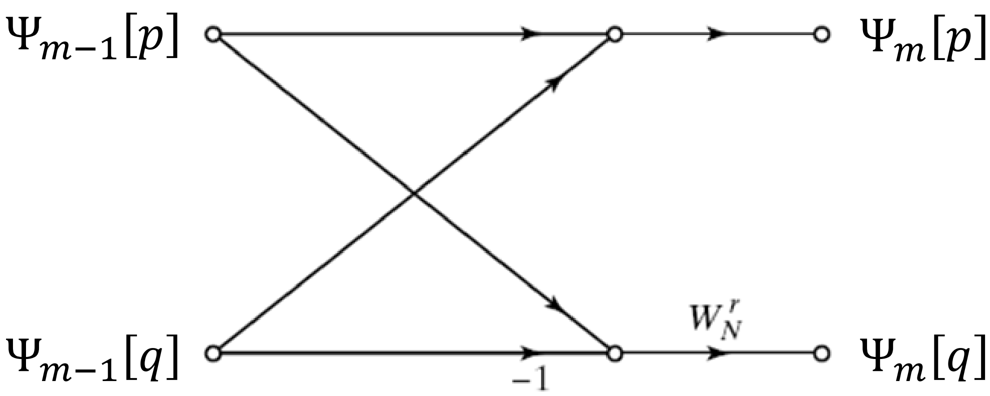
Recursively, we can further decompose the $(N/2)$-point DFT into smaller substructures:



Finally, we have



Butterfly structure for decimation-in-frequency FFT algorithm:



The decimation-in-frequency FFT algorithm also has the computation complexity of $O(N\log_2 N)$

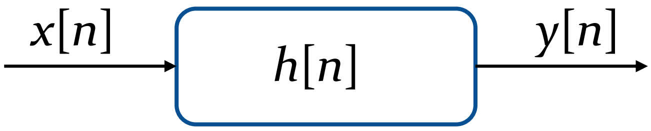
Lecture 7
z-Transform

Motivation

- Fourier Transform provides a frequency domain representation of discrete-time signal, but it may not exist for some sequences. (Reason?)
- Not easy for algebraic manipulations.
- z-transform used for:
  - Analysis of LTI systems
  - Solving difference equations
  - Determining system stability
  - Finding frequency response of stable systems

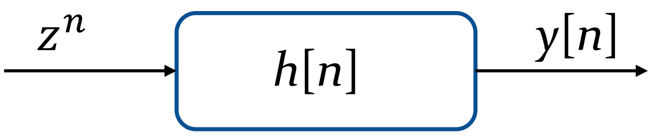
Eigen Functions of LTI Systems

- Consider an LTI system with impulse response $h[n]$:



- We already showed that $x[n] = e^{j\omega n}$ are eigen-functions
- What if $x[n] = z^n$, where z is a continuous complex variable $z = \mathrm{Re}(z) + j\mathrm{Im}(z)$?

Eigen Functions of LTI Systems



$$y[n] = \sum_{k=-\infty}^{\infty} h[k]z^{n-k} = \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right)z^n = H(z)z^n$$

- $x[n] = z^n$ are also eigen-functions of LTI Systems
- $H(z)$ is called a z-transform transfer function
- $H(z)$ exists for larger class of $h[n]$ than $H(e^{j\omega})$

Definition

- z-Transform:

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

where, z is a complex variable.

- **Example**

n	$n \leq -1$	0	1	2	3	4	5	$n > 5$
$x[n]$	0	2	4	6	4	2	1	0

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5}$$

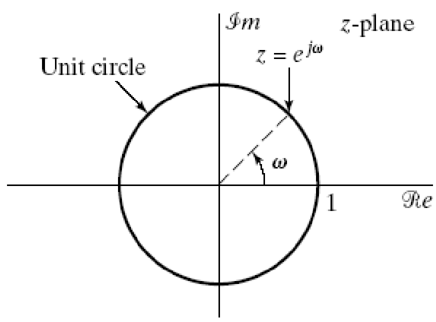
z-Transform vs. DTFT

- Let $z = re^{j\omega}$, then the expression reduces to

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n},$$

This can be interpreted as the Fourier Transform of the modified sequence $x[n]r^{-n}$.

- If $r = 1(\text{i.e.}, |z| = 1)$, the z-transform reduces to DTFT.
- The contour $|z| = 1$ is a circle in the z plan of unity radius, called **unit circle.**



z-Transform and LTI system

- Consider a system of an unit delay system

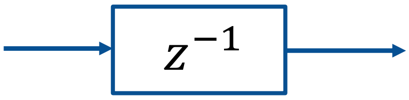
$$y[n] = x[n-1]$$

- The impulse response of the unit delay is

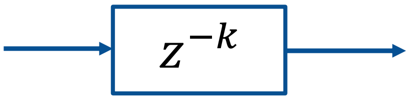
$$h[n] = \delta[n-1]$$

- Its z-transform is

$$H(z) = z^{-1}$$



- Similarly, delay of k samples: $h[n] = \delta[n-k]$



$$H(z) = z^{-k}$$

z-Transform of FIR System

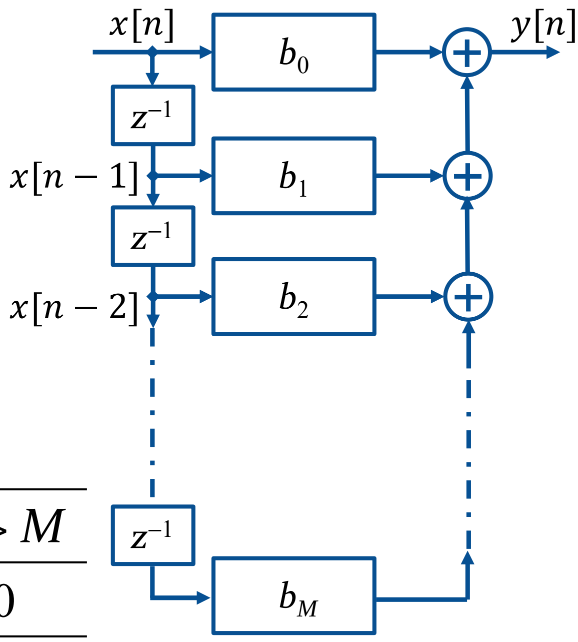
- Consider a causal FIR LTI system

$$y[n] = \sum_{m=0}^{M} b_m x[n-m]$$

- Its impulse response is

$$h[n] = \sum_{m=0}^{M} b_m \delta[n-m]$$

n	$n<0$	0	1	2	…	M	$n>M$
$h[n]$	0	b_0	b_1	b_2	…	b_M	0



System Diagram of an FIR system

z-Transform of FIR System

- Take z-transform on both side of the input-output relation

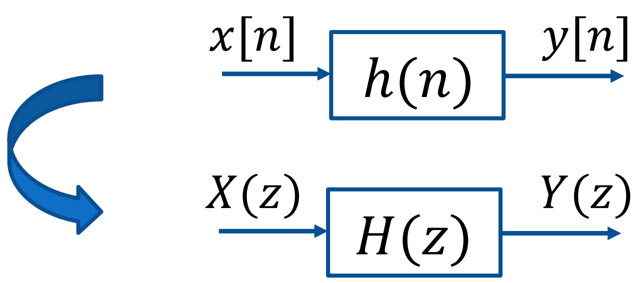
$$Y(z) = Z\{y[n]\} = Z\left\{\sum_{m=0}^{M} b_m x[n-m]\right\} = \sum_{n=-\infty}^{\infty}\sum_{m=0}^{M} b_m x[n-m]\, z^{-n}$$

$$= \sum_{m=0}^{M} b_m \sum_{n=-\infty}^{\infty} x[n-m]z^{-n} = \sum_{m=0}^{M} b_m\, z^{-m} \sum_{n=-\infty}^{\infty} x[n-m]z^{-(n-m)}$$

$$= \sum_{m=0}^{M} b_m\, z^{-m} Z\{x[n]\} = Z\{h[n]\}X(z) = H(z)X(z)$$

- The z-transform of the output of a FIR system is the product of the z-transform of the input signal and the z-transform of the impulse response.

Transfer Function



$$y[n] = x[n] \circledast h[n]$$

$$Y(z) = X(z)H(z)$$

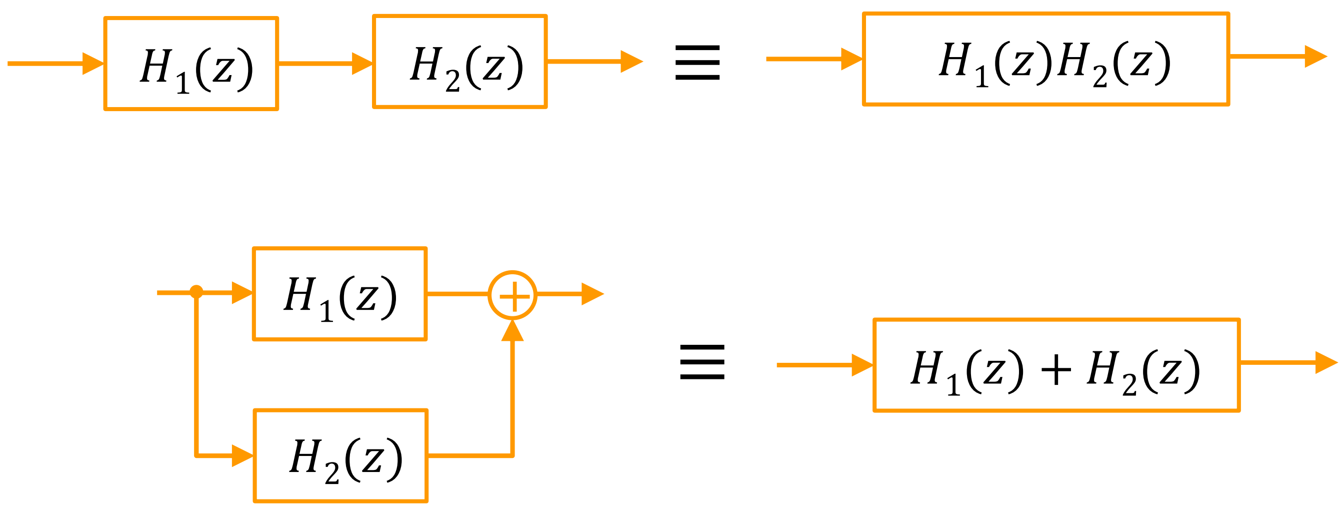
$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{M} b_m\, z^{-m}$$

is called the **z-transform transfer function** (or system function) of a LTI FIR system

Transfer Function and Impulse Response

- When the input $x[n] = \delta[n]$, the z-transform of the impulse response satisfies :
$$Z\{h[n]\} = H(z)Z\{\delta[n]\}.$$
- Since the z-transform of the unit impulse $\delta[n]$ is equal to one, we have
$$Z\{h[n]\} = H(z)$$
- That is, the z-transform transfer function $H(z)$ is the z-transform of the impulse response $h[n]$.

Cascade & Parallel Connection



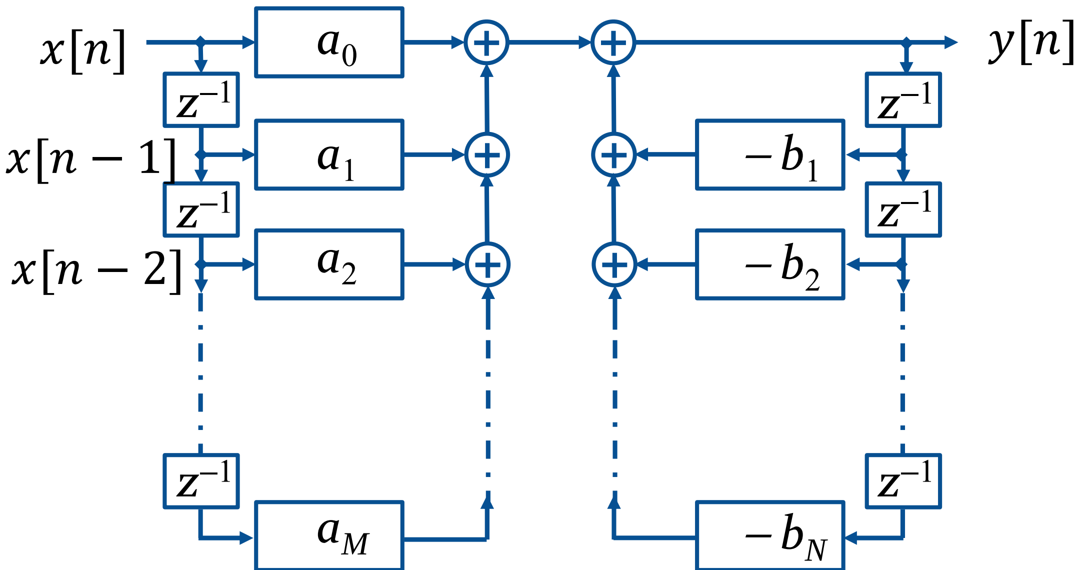
Example

- Consider an FIR system
$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$
- So, the impulse response is $h[n] = \{6, -5, 1\}, 0 \le n \le 2$
- The z-transform transfer function is:
$$H(z) = 6 - 5z^{-1} + z^{-2}$$
$$= (3 - z^{-1})(2 - z^{-1}) = 6\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}{z^2}$$

z-transform of Difference Equation

$$\sum_{m=0}^{N} b_m y[n-m] = \sum_{m=0}^{M} a_m x[n-m]$$

- Revisit system diagram for normalized $b_0 = 1$



- Take z-transform on both sides of the input-output relation

$$\sum_{m=0}^{N} b_m Y(z) z^{-m} = \sum_{m=0}^{M} a_m X(z) z^{-m}$$

- We have:

$$\frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{\sum_{m=0}^{N} b_m z^{-m}} \triangleq H(z)$$

- $H(z)$ is the z-transform transfer function of the LTI system defined by the linear constant-coefficient difference equation.
- The multiplication rule still holds: $Y(z) = H(z)X(z)$, i.e.,

$$Z\{y[n]\} = H(z)Z\{x[n]\}$$

Rational z-transform

- The transfer function of a difference equation (or a generally infinite impulse response (IIR) system) is a **rational form** $H(z) = P(z)/D(z)$.
- Since LTI systems are often realized by difference equations, the rational form is the most common and useful form of z-transforms.
- LTI system with z-transforms represented as a rational function of z^{-1}

$$H(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

where the degree of $P(z)$ is M, and that of $D(z)$ is N. The degree of the system is the larger one of M and N.

Alternate representations:

- A ratio of two polynomials in z,

$$H(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

- A product of second order rational z-transforms,

$$= \frac{p_0}{d_0} \cdot \frac{\prod_{l=1}^{M/2} (1 + p_{1l} z^{-1} + p_{2l} z^{-2})}{\prod_{l=1}^{N/2} (1 + d_{1l} z^{-1} + d_{2l} z^{-2})}$$

- Factorized form，

$$= \frac{p_0}{d_0} \cdot \frac{\prod_{l=1}^{M} (1 - \xi_l z^{-1})}{\prod_{l=1}^{N} (1 - \lambda_l z^{-1})} = z^{(N-M)} \frac{p_0}{d_0} \cdot \frac{\prod_{l=1}^{M} (z - \xi_l)}{\prod_{l=1}^{N} (z - \lambda_l)}$$

- For the $\boldsymbol{z}$-transform of General Difference Equation

$$\sum_{m=0}^{N} b_m Y(z) z^{-m} = \sum_{m=0}^{M} a_m X(z) z^{-m}$$

- When b_0 is normalized to 1, and $b_m = 0$ for $m = 1 \ \ldots \ N$, the difference equation degenerates to an FIR system we have investigated before.

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{M} a_m z^{-m}$$

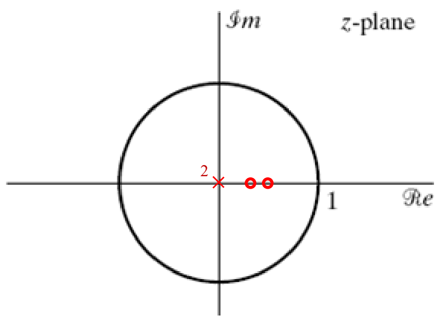
- It can still be represented by a rational form of the variable z as

$$H(z) = \frac{\sum_{m=0}^{M} a_m z^{(M-m)}}{z^M}$$

Poles and Zeros

- The **pole** of a z-transform $X(z)$ are the values of z for which $X(z) = \infty$.
- The **zero** of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.
- When $X(z) = P(z)/D(z)$ is a rational form, and both $P(z)$ and $D(z)$ are polynomials of z, the poles of $X(z)$ are the roots of $D(z)$, and the zeros are the roots of $P(z)$, respectively.

Examples



- Zeros of a system function
  - The system function of the FIR system $y[n] = 6x[n] - 5x[n-1] + x[n-2]$ has been shown as

$$H(z) = 6\frac{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}{z^2}$$

- The zeros of this system are $1/3$ and $1/2$, and the pole is 0.
- Since 0 and 0 are double roots of $D(z)$, the pole is a second-order pole.

- In most practical cases, the complex poles and zeros of z-transforms occur as complex conjugate pairs, and simple poles and zeros (i.e., poles or zeros of order 1) are real.
- In such cases, rational z-transform are ratios of polynomials with real coefficients.
- For example, let $z = a_i \pm jb_i$ be a pair of complex conjugate poles of the rational z-transform $H(z)$, where a_i and b_i are real, i.e.,

$$H(z) = \frac{Y(z)}{(z - a_i - jb_i)(z - a_i + jb_i)}$$
$$= \frac{Y(z)}{(z - a_i)^2 + {b_i}^2} = \frac{Y(z)}{z^2 - 2a_i z + ({a_i}^2 + {b_i}^2)}$$

Region of Convergence (ROC)

- **ROC:** the set $\mathcal{R}$ of values of z for which a sequence's z-transform converges, i.e., $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges.
- Since z-transform of $x[n]$ is equivalent to DTFT of $x[n]r^{-n}$, if $x[n]r^{-n}$ is absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty,$$

the z-transform of $x[n]$ uniformly converges.

ROC examples

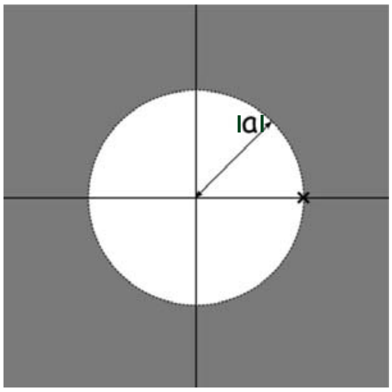
- Example 1: Right-sided sequence $x[n] = a^n \mu[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

recall: $1 + x + x^2 + \cdots = \frac{1}{1-x}$, if $|x| < 1$

- So, $X(z) = \frac{1}{1-az^{-1}}$, for $|az^{-1}| < 1$

- ROC $= \{z\colon |z| > |a|\}$



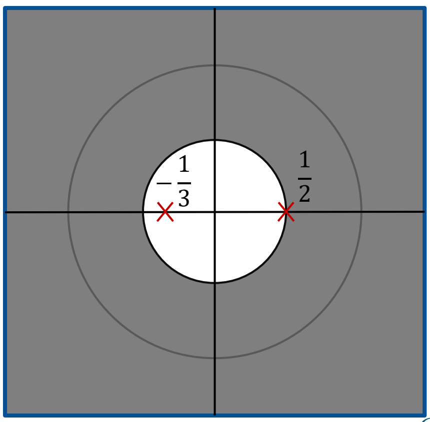
ROC examples

- Example 2: $x[n] = \left(\frac{1}{2}\right)^n \mu[n] + \left(-\frac{1}{3}\right)^n \mu[n]$

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}},$$

- ROC $= \left\{z\colon |z| > \frac{1}{2}\right\} \cap \left\{z\colon |z| > \frac{1}{3}\right\}$

$$= \left\{z\colon |z| > \frac{1}{2}\right\}$$



ROC examples

- Example 3: Left sided sequence $x[n] = -a^n \mu[-n-1]$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n}$$

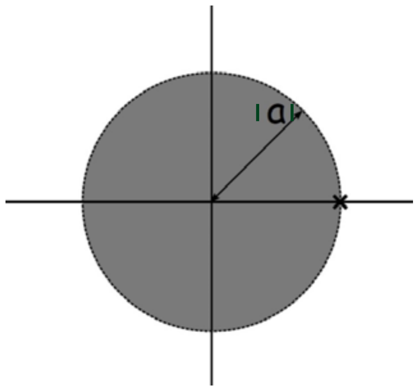
$$= \sum_{m=1}^{\infty} -a^{-m} z^m = 1 - \sum_{m=0}^{\infty} (a^{-1}z)^m$$

- If $|a^{-1}z| < 1$, i.e., $|z| < |a|$,

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$

ROC examples

- Example 3 continued.
- Expression is the same as that of Example 1!
- ROC $= \{z: \ |z| < |a|\}$ is different



- Different sequences may have the same z-transform expression.
- The z-transform without ROC does not uniquely define a sequence!

ROC examples

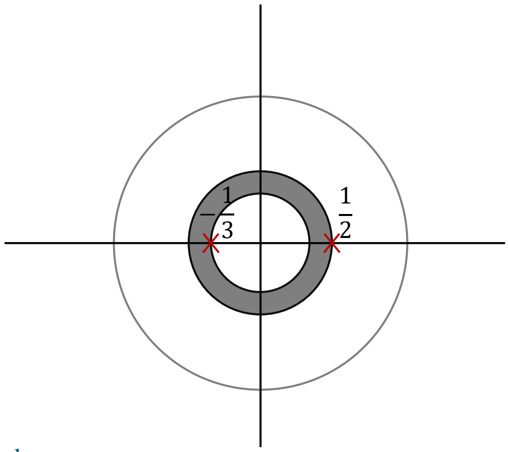
- Example 4: $x[n] = -\left(\frac{1}{2}\right)^n \mu[-n-1] + \left(-\frac{1}{3}\right)^n \mu[n]$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1+\frac{1}{3}z^{-1}},$$

Expression Same as that of Example 2

- $\mathrm{ROC} = \left\{z:\ |z| < \frac{1}{2}\right\} \cap \left\{z:\ |z| > \frac{1}{3}\right\}$

$$= \left\{z:\ \frac{1}{3} < |z| < \frac{1}{2}\right\}$$



ROC examples

- Example 5: $x[n] = \left(\frac{1}{2}\right)^n \mu[n] - \left(-\frac{1}{3}\right)^n \mu[-n-1]$

$$\mathrm{ROC} = \left\{z:\ |z| > \frac{1}{2}\right\} \cap \left\{z:\ |z| < \frac{1}{3}\right\} = \emptyset$$

- Example 6: $x[n] = a^n$, two sided $a \neq 0$

$$\mathrm{ROC} = \{z:\ |z| > a\} \cap \{z:\ |z| < a\} = \emptyset$$

ROC Examples

- Example 7: Finite sequence $x[n] = a^n \mu[n]\mu[-n+M-1]$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n}$$

Finite, always converges

$$= \frac{1-a^M z^{-M}}{1-az^{-1}} = \frac{1}{z^{M-1}} \cdot \frac{z^M - a^M}{z-a}$$

Zero cancels pole

There are M roots of $z^M = a^M, z_k = ae^{j\frac{2\pi k}{M}}$. The root of $k = 0$ cancels the pole at $z = a$. Thus there are M–1 zeros, $z_k = ae^{j\frac{2\pi k}{M}}, k = 1, \ldots, M$, and a ($M$–1)th order pole at zero.

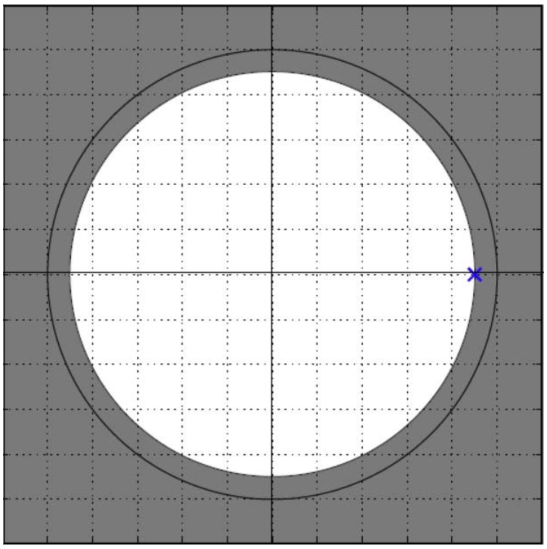
$$X(z) = \prod_{k=1}^{M-1} \left(1 - ae^{j\frac{2\pi k}{M}} z^{-1}\right)$$

- ROC = $\{z\colon |z| > 0\}$

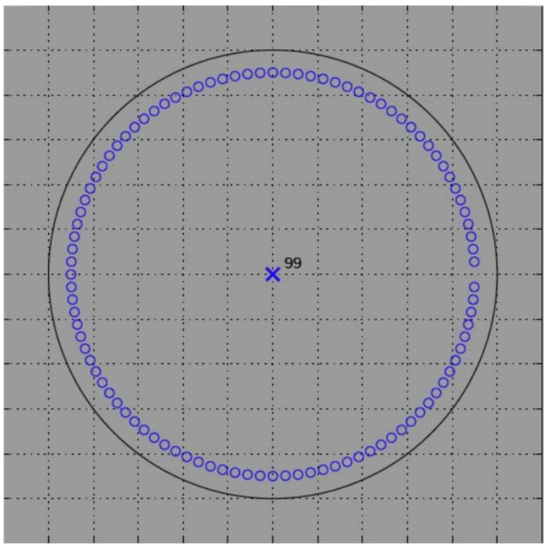
ROC examples

- Example 7 continued:

Infinite Sequence



Finite Sequence

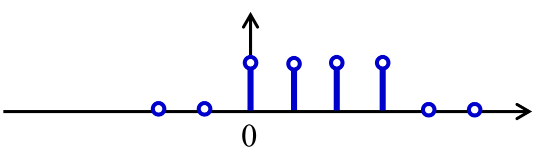


Properties of ROC

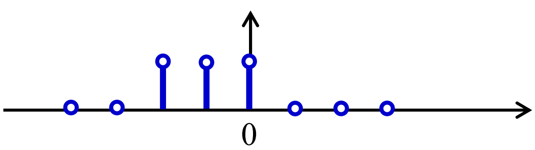
- For right-sided sequences: ROC extends outward from the outermost pole to infinity
  - Examples 1, 2
- For left-sided: ROC inwards from the inner most pole to the original point.
  - Example 3
- For two-sided: ROC either is a ring - or do not exist
  - Examples 4, 5, 6

Properties of ROC

- For finite duration sequences, ROC is the entire *z*-plane, except possibly z=0, z=∞



$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$ ROC excludes z = 0



$X(z) = 1 + z^{1} + z^{2}$ ROC excludes z = ∞

Properties of ROC

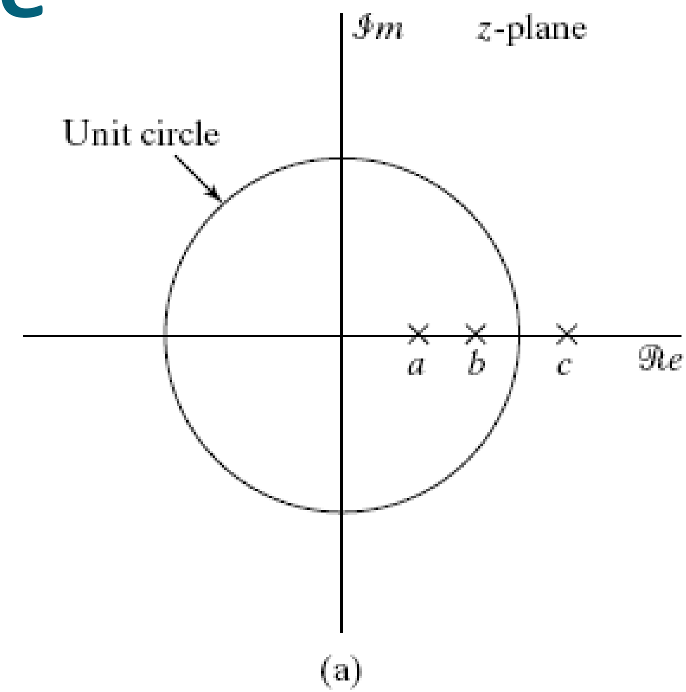
- In general, ROC of a z-transform is in a form:

  $R_{x^-} < |z| < R_{x^+}$, an annular region

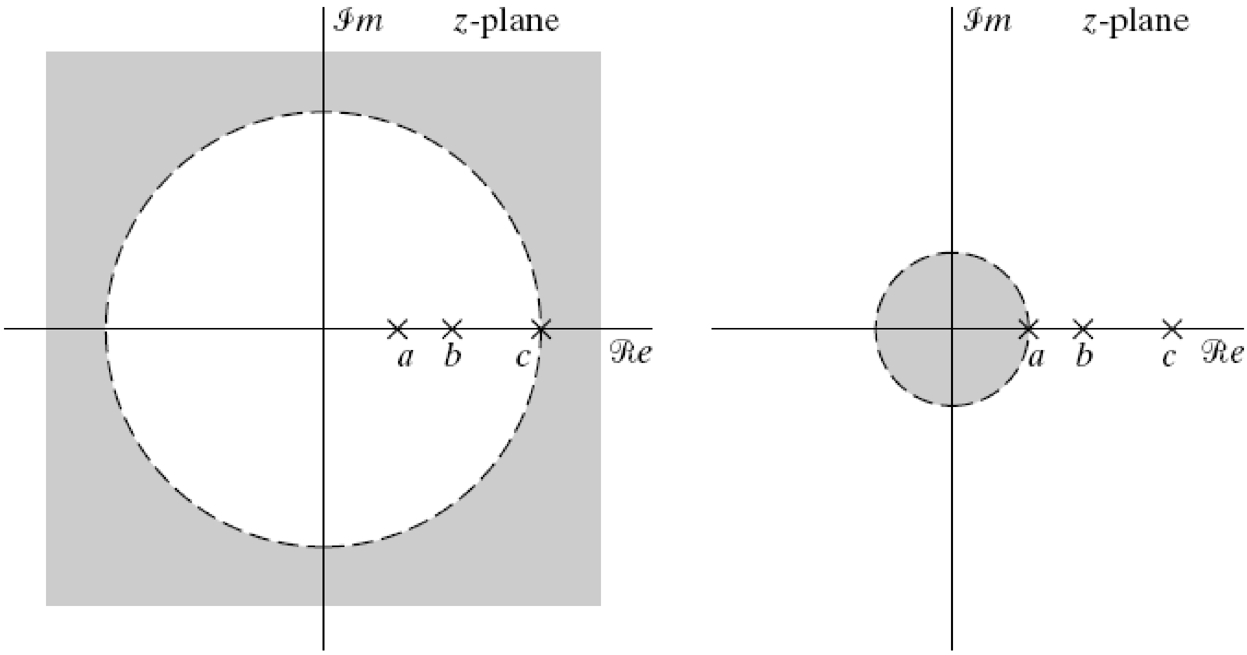
- The DTFT $X(e^{j\omega})$ of $x[n]$ absolutely convergent iff the ROC of the z-transform $X(z)$ of $x[n]$ includes the unit circle.

- ROC can't contain poles

Example

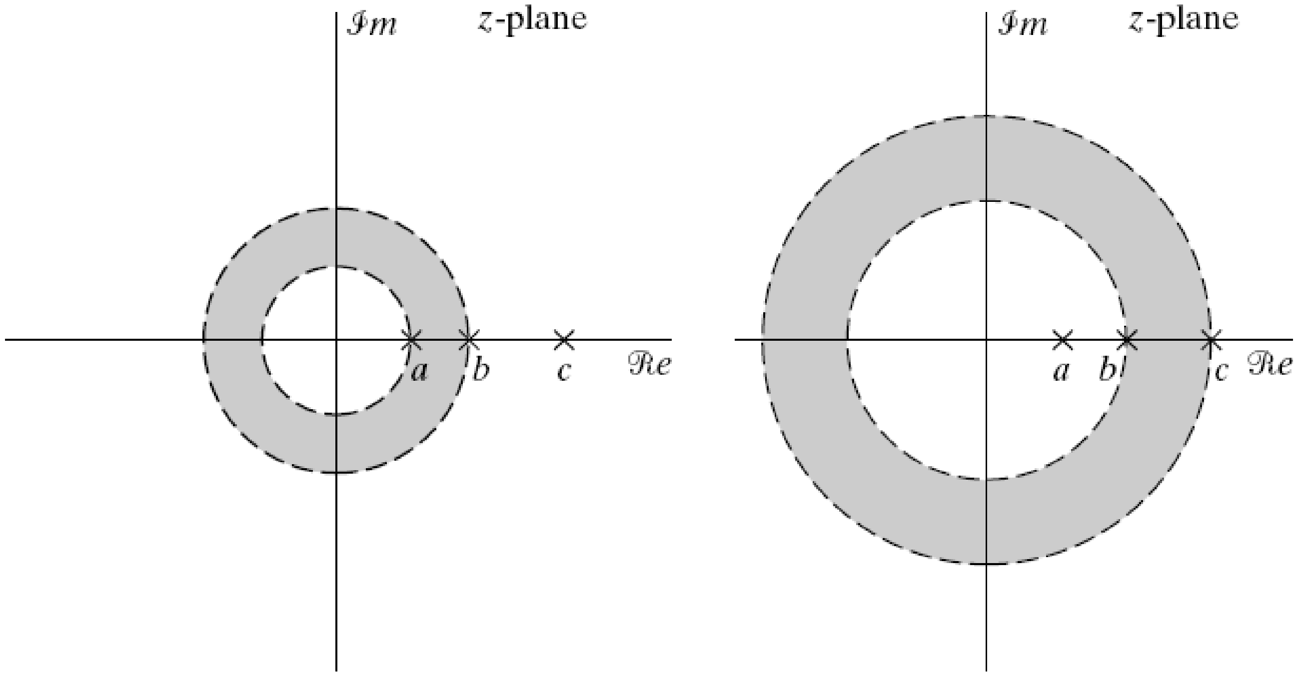


(a) A system with three poles.



(b) (c)

Different possibilities of the ROC. (b) ROC to a right-sided sequence. (c) ROC to a left-sided sequence.



(d) (e)

Different possibilities of the ROC. (d) ROC to a two-sided sequence. (e) ROC to another two-sided sequence.

ROC for LTI System

- Consider the transfer function $H(z)$ of a linear system:
  - If the system is stable, the impulse response $h(n)$ is absolutely summable and therefore has a Fourier transform, then the ROC must include the unit circle.
  - If the system is causal, then the impulse response $h(n)$ is right-sided, and thus the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $H(z)$ to (and possibly include) $z = \infty$.
  - Therefore, **a stable causal LTI system has all poles inside unit circle**.

Properties of the *z*-transform

Property	Sequence		z-Transform	ROC
	$x[n]$	$\leftrightarrow$	$X(z)$	$\mathcal{R}_x$
Conjugate	$x^*[n]$	$\leftrightarrow$	$X^*(z^*)$	$\mathcal{R}_x$
Time shifting	$x[n-n_d]$	$\leftrightarrow$	$z^{-n_d}X(z)$	$\mathcal{R}_x$ except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$r^n x[n]$	$\leftrightarrow$	$X\left(\frac{z}{r}\right)$	$\lvert r\rvert\mathcal{R}_x$
Differentiation of $X(z)$	$nx[n]$	$\leftrightarrow$	$-z\frac{dX(z)}{dz}$	$\mathcal{R}_x$ except possibly the point $z = 0$ or ∞
Time-reversal	$x[-n]$	$\leftrightarrow$	$X(z^{-1})$	$1/\mathcal{R}_x$
Convolution	$x[n] \circledast y[n]$	$\leftrightarrow$	$X(z)Y(z)$	Includes $\mathcal{R}_x \cap \mathcal{R}_y$

Commonly Used z-transform Pairs

Sequence		z-Transform	ROC
$\delta[n]$	$\leftrightarrow$	1	All values of z
$\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-\mu[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	$\leftrightarrow$	z^{-m}	All z, except 0 (if m>0) or ∞ (if m<0)
$\alpha^n\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n\mu[-n-1]$	$\leftrightarrow$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n\mu[n]$	$\leftrightarrow$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$

Commonly Used z-transform Pairs

Sequence		z-Transform	ROC
$-n\alpha^n\mu[-n-1]$	$\leftrightarrow$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$(n+1)\alpha^n\mu[n]$	$\leftrightarrow$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-(n+1)\alpha^n\mu[-n-1]$	$\leftrightarrow$	$\dfrac{1}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$(r^n\cos\omega_0 n)\mu[n]$	$\leftrightarrow$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$(r^n\sin\omega_0 n)\mu[n]$	$\leftrightarrow$	$\dfrac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$
$\begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\leftrightarrow$	$\dfrac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Example

- Determine the z-transform and its ROC of the causal sequence
$$x[n] = (r^n \cos\omega_0 n)\mu[n]$$
- We can express $x[n] = v[n] + v^*[n]$, where
$$v[n] = \frac{1}{2} r^n e^{j\omega_0 n}\mu[n] = \frac{1}{2}\alpha^n \mu[n]$$
- The z-transform of $v[n]$ is given by
$$V(z) = \frac{1}{2}\cdot\frac{1}{1-\alpha z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-re^{j\omega_0}z^{-1}}, |z| > |\alpha| = |r|$$

- Using the conjugate property, we obtain the z-transform of $v^*[n]$ as
$$V^*(z^*) = \frac{1}{2}\cdot\frac{1}{1-\alpha^* z^{-1}} = \frac{1}{2}\cdot\frac{1}{1-re^{-j\omega_0}z^{-1}}, |z| > |r|$$
- Finally, using the linear property, we get
$$X(z) = \frac{1}{2}\cdot\frac{1}{1-re^{j\omega_0}z^{-1}} + \frac{1}{2}\cdot\frac{1}{1-re^{-j\omega_0}z^{-1}}$$
$$= \frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}, |z| > |r|$$

Inversion of the z-Transform

- In general, by contour integral

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}\,dz$$

where C is any counterclockwise contour encircling the point $z = 0$ in the ROC.

- Ways to avoid it:
  - Inspection (known transforms)
  - Properties of the z-transform
  - Partial fraction expansion
  - Power series expansion
  - Residue theorem

By Inspection

- Eg. If we need to find the inverse z-transform of

$$X(z) = \frac{1}{1 - 0.5z^{-1}}, \qquad |z| < 0.5$$

- From the transform pair we see that

$$x[n] = 0.5^n\mu[n] \text{ or } x[n] = -0.5^n\mu[-n-1]$$

- Since ROC is $|z| < 0.5$, the sequence is left-sided. Therefore,

$$x[n] = -0.5^n\mu[-n-1]$$

By Partial Fraction Expansion

- If $X(z)$ is the rational form with

$$X(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i\, z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$, then $X(z)$ can be expressed as

$$X(z) = \sum_{l=0}^{M-N} \eta_l z^{-l} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N.

- The rational function $\frac{P_1(z)}{D(z)}$ is called a called a proper fraction.

- To develop the proper fraction of $\frac{P_1(z)}{D(z)}$ from $X(z)$, a long division of $P(z)$ by $D(z)$ should be carried out in a reversed order until the remainder polynomial $P_1(z)$ is of lower degree than that of the denominator $D(z)$.
- Example: consider

$$X(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division in a revered order, we arrive at

$$X(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

Proper fraction

- **Simple pole:** in most practical cases, the rational z-transform of interest $X(z)$ is a proper fraction with simple poles.
- Let the poles of $X(z)$ be at $z = \lambda_k, 1 \leq k \leq N$
- A **partial-fraction** expansion of $X(z)$ is of the form

$$X(z) = \sum_{l=1}^{N} \left(\frac{\rho_l}{1 - \lambda_l z^{-1}} \right)$$

- The constants ρ_l in the partial-fraction expansion are called the **residue**, and are given by

$$\rho_l = (1 - \lambda_l z^{-1})\, X(z)\Big|_{z=\lambda_l}$$

- Assume that each term of the sum in partial-fraction expansion has an ROC given by $|z| > |\lambda_l|$, and thus has an inverse transform of the form $\rho_l(\lambda_l)^n\mu[n]$.
- Therefore, the inverse transform $x[n]$ of $X(z)$ is given by

$$x[n] = \sum_{l=1}^{N} \rho_l(\lambda_l)^n \mu[n]$$

Example

- Let the z-transform $H(z)$ of a causal system $h[n]$ is given by

$$H(z) = 1 + \frac{z(z+2)}{(z-0.2)(z+0.6)} = 1 + \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- The second term is a proper fraction. A partial-fraction expansion of $H(z)$ is then of form

$$H(z) = 1 + \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

- And

$$\rho_1 = (1-0.2z^{-1})\frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}\Bigg|_{z=0.2} = 2.75$$

$$\rho_2 = (1+0.6z^{-1})\frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}\Bigg|_{z=-0.6} = -1.75$$

- Thus, we have

$$H(z) = 1 + \frac{2.75}{1-0.2z^{-1}} + \frac{-1.75}{1+0.6z^{-1}}$$

- Since it is given that $h[n]$ is causal, the inverse transform of the above is given by

$$h[n] = \delta[n] + 2.75(0.2)^n\mu[n] - 1.75(-0.6)^n\mu[n]$$

Another example

- Find the inverse z-transform of

$$X(z) = \frac{(1+z^{-1})^2}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})}, |z| > 1$$

- Since both the numerator and denominator are of degree 2, a constant term exists.

$$X(z) = B_0 + \frac{A_1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{A_2}{(1-z^{-1})}$$

- B_0 can be determined by the fraction of the coefficients of z^{-2}. $B_0 = \frac{1}{\frac{1}{2}} = 2$.

- Therefore, $X(z) = 2 + \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} = 2 + \frac{A_1}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{A_2}{(1-z^{-1})}$

$$A_1 = \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} \cdot \left(1-\frac{1}{2}z^{-1}\right)\Bigg|_{z=\frac{1}{2}} = -9$$

$$A_2 = \frac{-1+5z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)(1-z^{-1})} \cdot (1-z^{-1})\Big|_{z=1} = 8$$

$$X(z) = 2 - \frac{9}{\left(1-\frac{1}{2}z^{-1}\right)} + \frac{8}{(1-z^{-1})}$$

- From the ROC, the solution is right-handed. So

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n \mu[n] + 8\mu[n]$$

By Power Series Expansion

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$
$$= \cdots + x[-2]z^2 + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

- We can determine any particular value of the sequence by finding the coefficient of the appropriate power of z^{-1}.

Example: Finite-length Sequence

- Find the inverse z-transform of
$$X(z) = z^2(1 - 0.5z^{-1})(1 + z^{-1})(1 - z^{-1})$$

- By directly expand $X(z)$, we have
$$X(z) = z^2 - 0.5z - 1 + 0.5z^{-1}$$

- Thus,
$$x[n] = \delta[n+2] - 0.5\delta[n+1] - \delta[n] + 0.5\delta[n-1]$$

Example: Rational z-Transform

- If a rational z-transform is expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division.
- Consider

$$H(z) = \frac{1 + 2z^{-1}}{1 + 0.4z^{-1} - 0.12z^{-2}}$$

- The long division of the numerator by the denominator yields

$$H(z) = 1 + 1.6z^{-1} - 0.52z^{-2} + 0.4z^{-3} - 0.2224z^{-4} + \cdots$$

- Thus, $h[n] = \delta[n] + 1.6\delta[n-1] - 0.52\delta[n-2] + 0.4\delta[n-3] - 0.2224\delta[n-4] + \cdots$

Frequency Response from Transfer Function

- z-transform transfer function

$$H(z) = H_{\text{re}}(z) + jH_{\text{im}}(z) = |H(z)|e^{j\arg H(z)}$$

where $\arg H(z) = \tan^{-1}\frac{H_{\text{im}}(z)}{H_{\text{re}}(z)}$

- If the ROC of $H(z)$ includes the unit circle, the frequency response $H(e^{j\omega})$ of the LTI digital system can be obtained by evaluating $H(z)$ on the unit circle:

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(e^{j\omega})H(e^{-j\omega})$$
$$= H(z)H(z^{-1})\Big|_{z=e^{j\omega}}$$

Stability Condition in Terms of Pole Locations

- **A stable causal LTI system has all poles inside unit circle**.
  - A causal LTI FIR digital filter with bounded impulse response coefficients is always stable, as all its poles are at the origin in the z-plane.
  - A causal LTI IIR digital filter may or may not be stable.
  - An originally stable IIR filter characterized by infinite precision coefficients and with all poles inside the unit circle may become unstable after implementation due to the unavoidable quantization of all coefficients.

Example

- Analyze the stability of the causal system

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850586z^{-2}}$$

and the system implemented by keeping 2 digits after the decimal points of the coefficients.

A: the poles of the systems are the roots of

$$\begin{aligned} &1 - 1.845z^{-1} + 0.850586z^{-2} \\ &= z^{-2}(z^2 - 1.845z^1 + 0.850586) \end{aligned}$$

We have, $z_p = \frac{1.845 \pm \sqrt{1.845^2 - 4 \times 0.850586}}{2} = 0.943$, or 0.902

Both poles are in the unit circle, and the system is stable.

- If the system is implemented by keeping 2 digits after the decimal points of the coefficients, the transfer function becomes

$$H(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

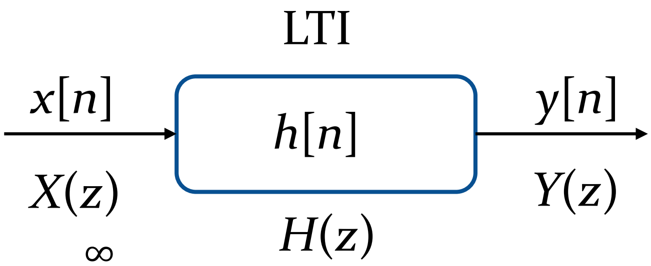
The root of the denominator is

$$z_p = \frac{1.85 \pm \sqrt{1.85^2 - 4 \times 0.85}}{2} = 1, \text{or } 0.85$$

i.e., one pole is on the unit circle. So the system becomes unstable.

Lecture 8
LTI Discrete-Time Systems in the Transform Domain

Types of Transfer Functions



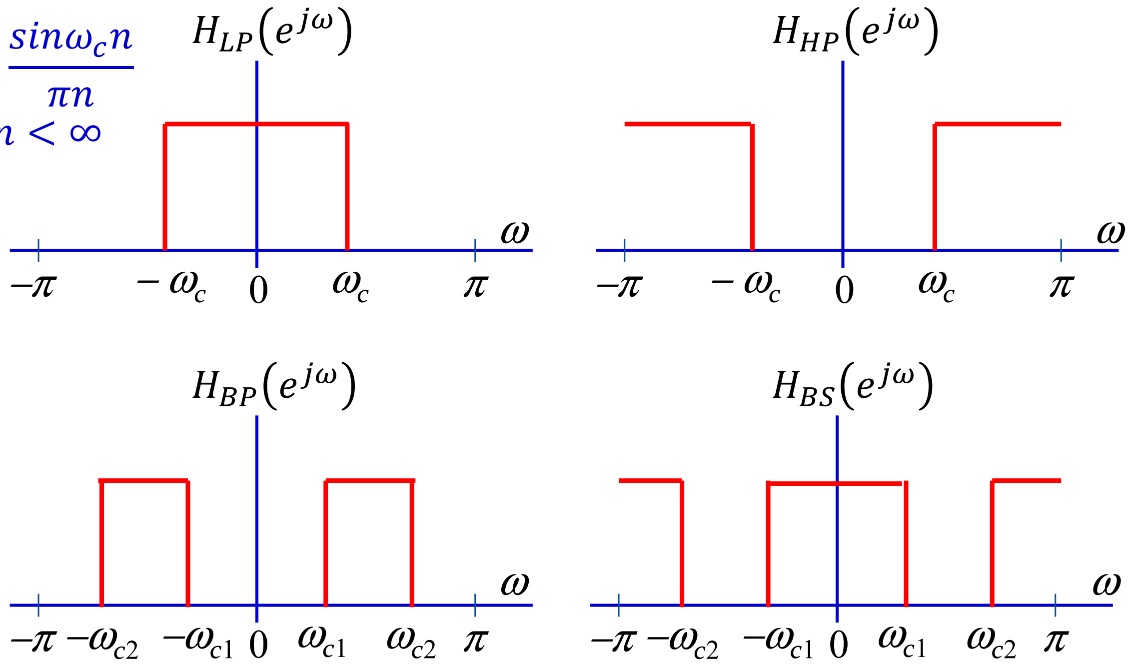
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \qquad Y(z) = X(z)H(z)$$

- For digital transfer function with frequency selective frequency response, there are two types of classifications.
  - Based on the shape of magnitude function $|H(e^{j\omega})|$
  - Based on the form of phase function $\theta(\omega)$

Magnitude Characteristics

- Digital Filter with Ideal Magnitude Responses:

$$h_{LP}[n] = \frac{\sin\omega_c n}{\pi n}$$
$$-\infty < n < \infty$$



- The range of frequencies where the magnitude response takes the value of one is called the **passband**
- The range of frequencies where the magnitude response takes the value of zero is called the **stopband**
- The frequencies , ω_c, ω_{c1} and ω_{c2} are called the **cutoff frequencies**
- An ideal filter has a magnitude response equal to one in the passband and zero in the stopband, and has a **zero phase** everywhere

Phase Characteristics

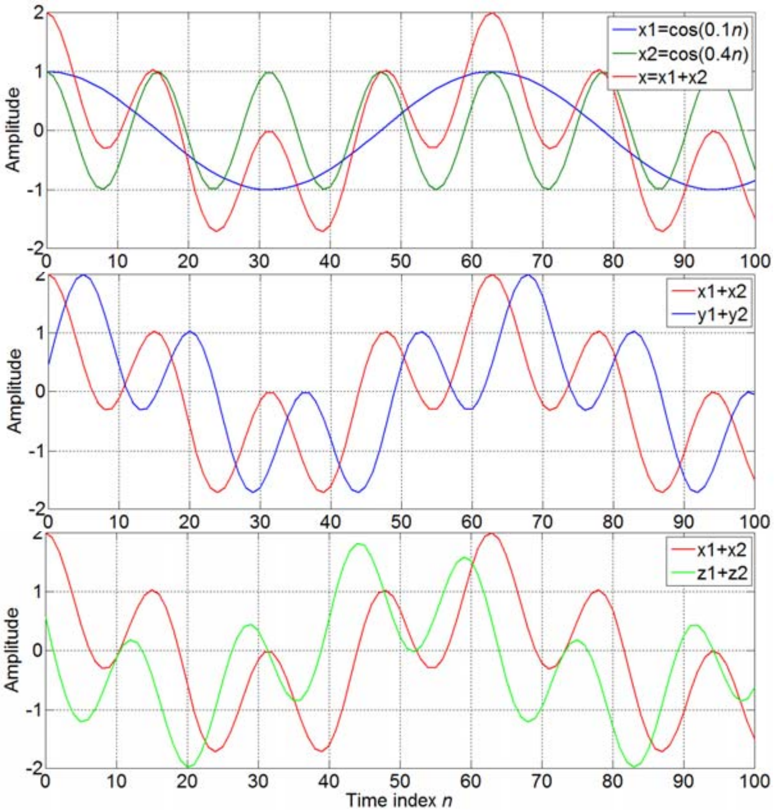
- Linear Phase Transfer Function
  - In many application, it is necessary that the digital filter designed does not distort the phase of the input signal components with frequencies in passband.
- The most general type of a filter with a linear phase has a frequency response given by

$$H(e^{j\omega}) = Ae^{-j\omega D}$$

which has a linear phase from $\omega = 0$ to $\omega = 2\pi$.

- Note: $\theta(\omega) = -\omega D$.

Phase Distortion



- $x = x_1 + x_2$

$$x_1 = \cos(0.1n)$$
$$x_2 = \cos(0.4n)$$

- $Y(e^{j\omega}) = X(e^{j\omega})H_1(e^{j\omega})$

$\theta_1(\omega) = \angle H_1(e^{j\omega}) = -5\omega$

- $y = y_1 + y_2$

$$y_1 = \cos(0.1(n-5))$$
$$y_2 = \cos(0.4(n-5))$$

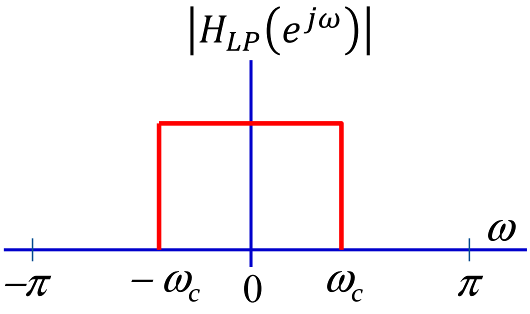
- $Z(e^{j\omega}) = X(e^{j\omega})H_2(e^{j\omega})$

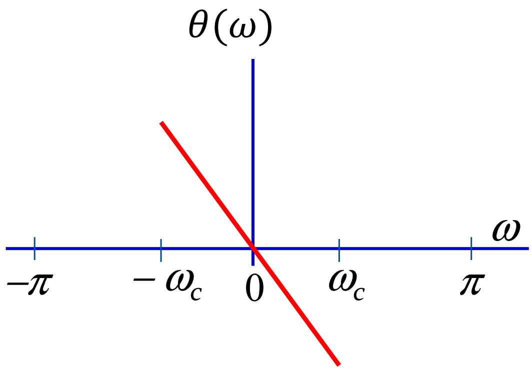
$\theta_2(\omega) = \angle H_2(e^{j\omega}) = -5\mathrm{sign}(\omega)$

- $z = z_1 + z_2$

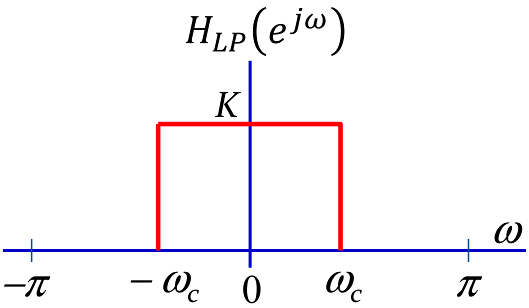
$$z_1 = \cos(0.1n - 5)$$
$$z_2 = \cos(0.4n - 5)$$

- It is desired to pass input signal components in a certain frequency range undistorted in both magnitude and phase
- The transfer function should exhibit a unity magnitude response and a linear phase response in the band of interest.





Design of Idea Lowpass FIR filter

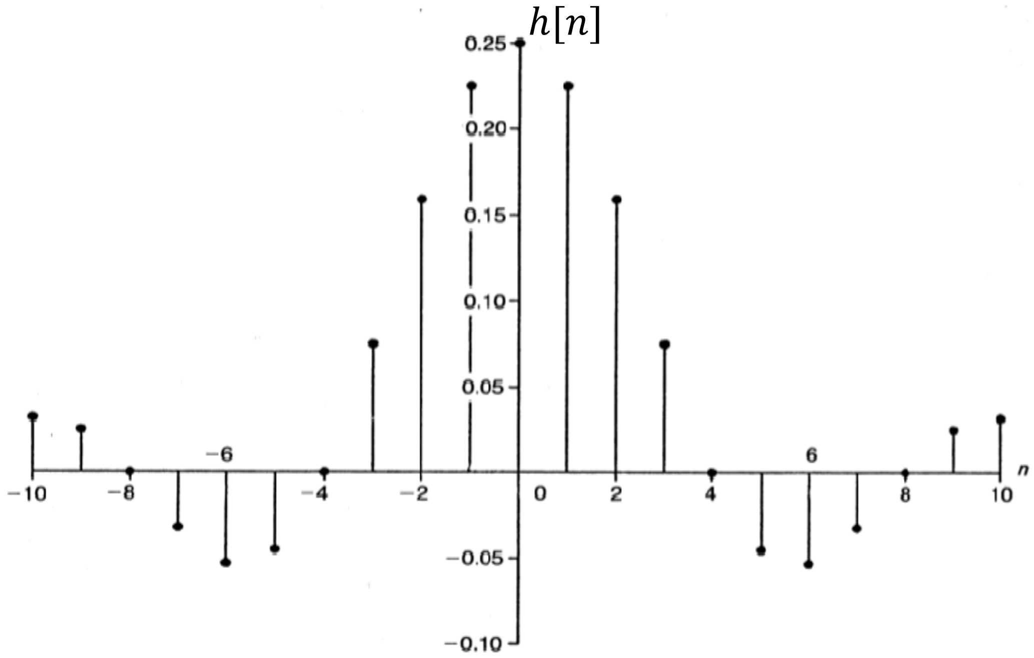


$$h_{LP}[n] = \frac{K sin \omega_c n}{\pi n}$$
$$-\infty < n < \infty$$

Let $K = 1$, and $\omega_c = \pi/4$, $n = 0, \pm 1, \dots, \pm 10$,

$h[0] = 0.25$,
$h[\pm 1] = 0.225$,
$h[\pm 2] = 0.159$,
$h[\pm 3] = 0.075$
$h[\pm 4] = 0$,
$h[\pm 5] = -0.043$,
$h[\pm 6] = -0.053$,
$h[\pm 7] = -0.032$
$h[\pm 8] = 0$,
$h[\pm 9] = 0.025$,
$h[\pm 10] = 0.032$

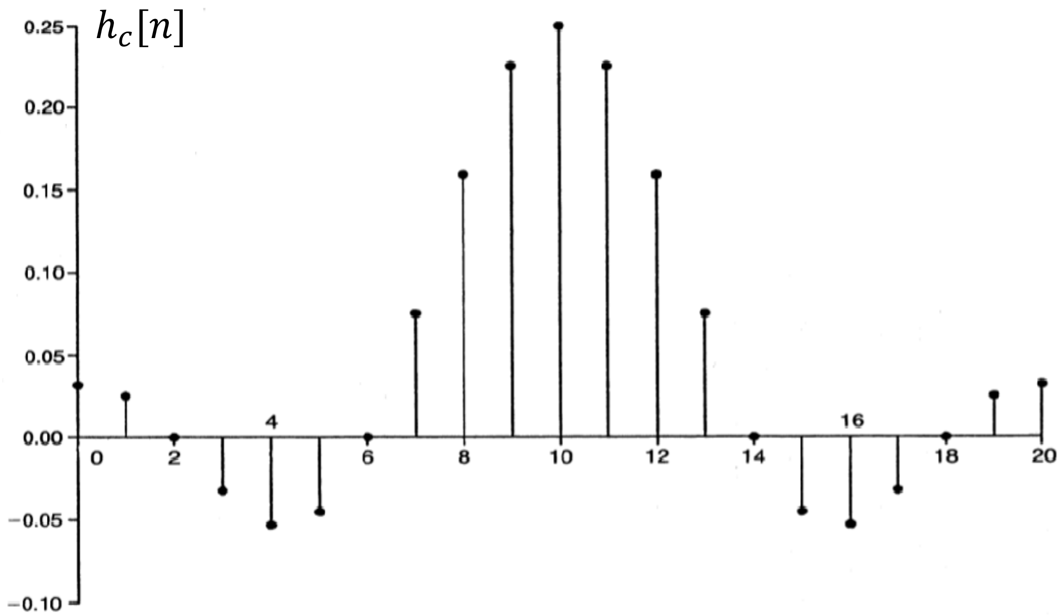
Non-causal FIR Impulse Response



We can make it causal if we shift $h_{LP}[n]$ by 10 units to the right:

$$h_c[n] = \frac{K}{\pi(n-10)} sin\omega_c(n-10), \qquad n = 0, 1, \ldots, 20$$

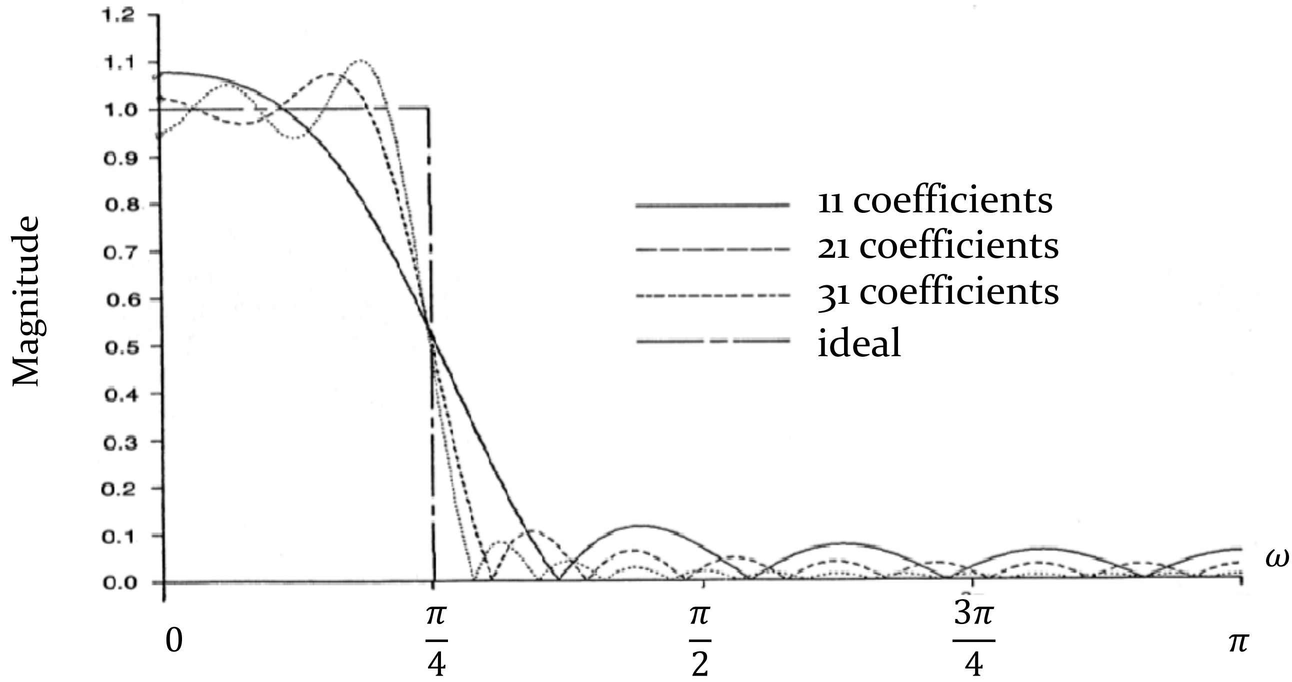
Causal FIR (N=21) Impulse Response



Notice the symmetry: $h_c[n] = h_c[N-1-n]$ which satisfies the linear phase condition.
Frequency response: $H_c\left(e^{j\omega}\right) = \sum_{n=0}^{20} h_c[n] e^{-j\omega n}$

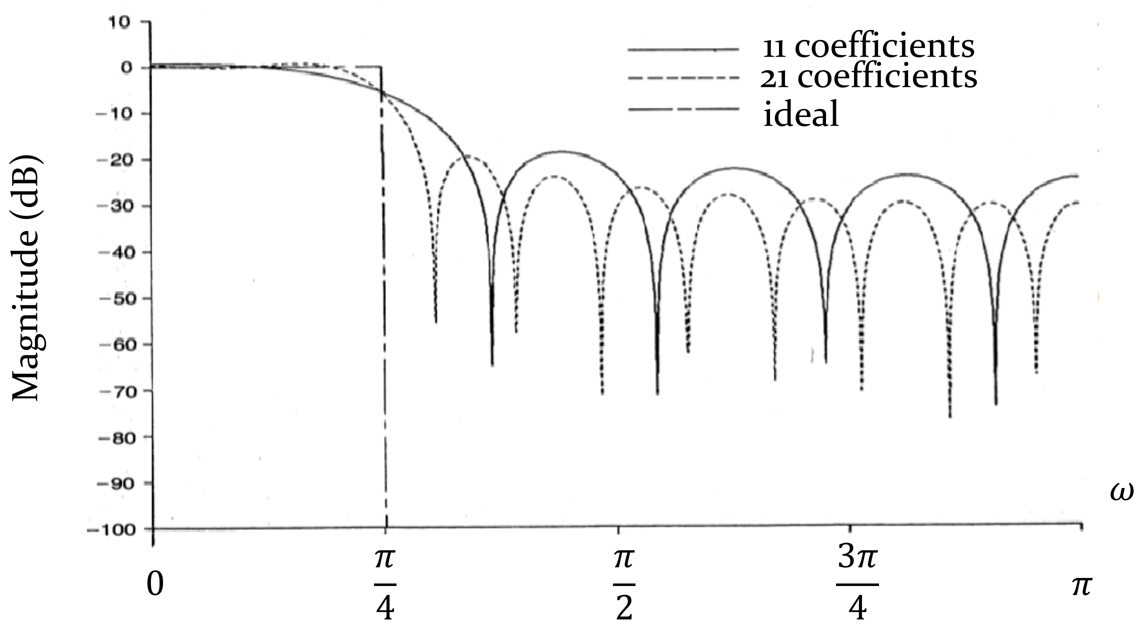
Magnitude of filter frequency response

for filter length 11, 21, and 31

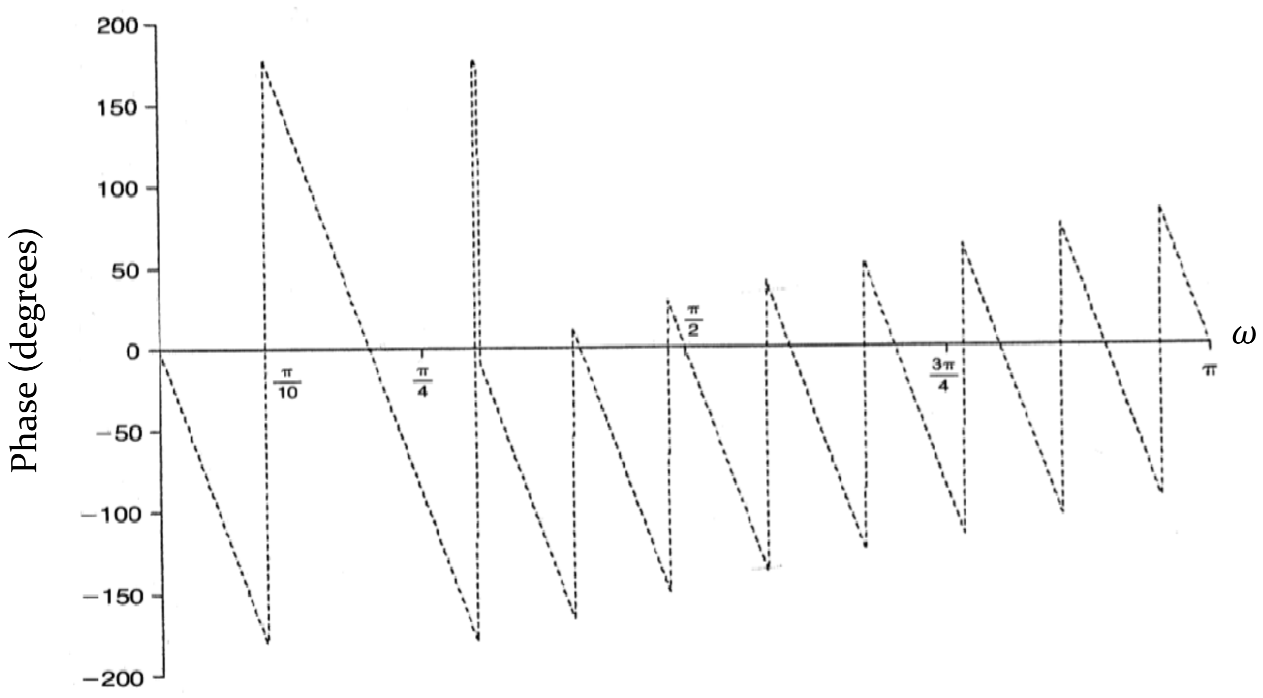


Lowpass filter frequency response in dB

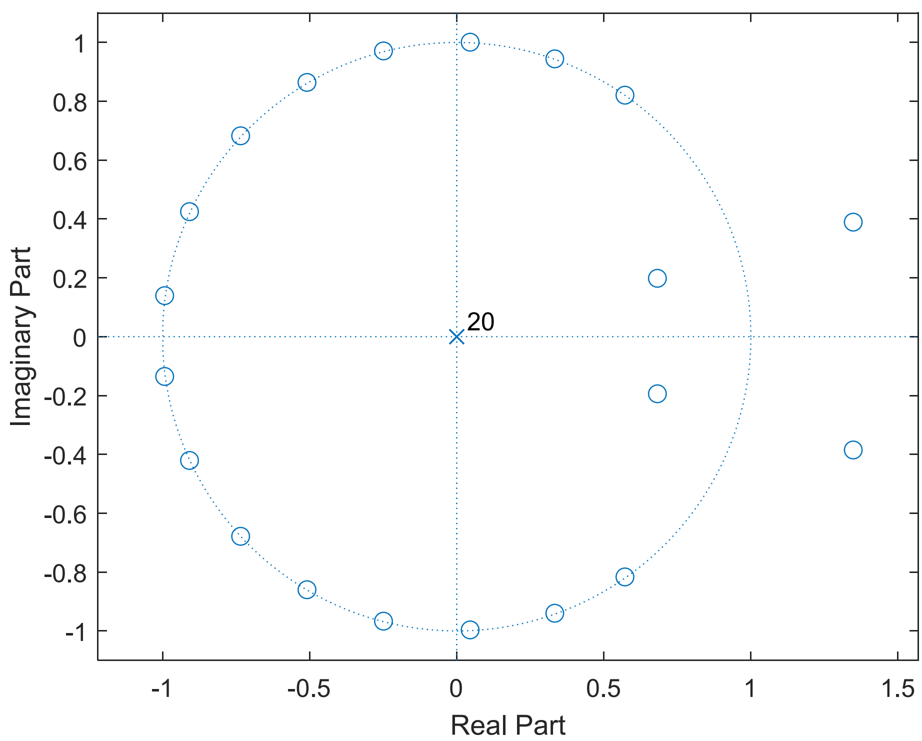
for filter length 11 and 21



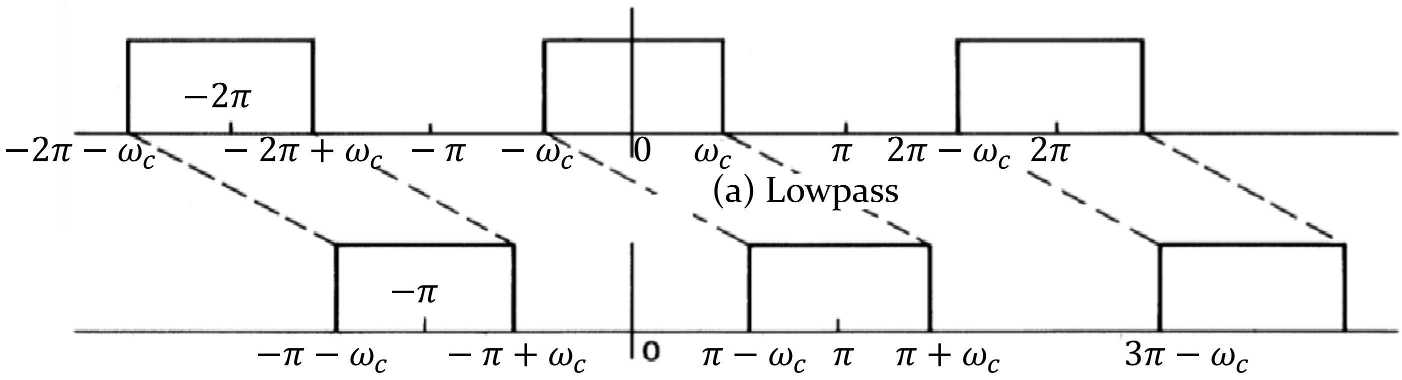
Lowpass filter phase response



Pole-zero plot for $H(z)$



Magnitude Characteristics of Idea Lowpass and Highpass Filters



(a) Lowpass

(b) Highpass

The above figure implies

$$H_{HP}(e^{j\omega}) = H_{LP}(e^{j(\omega-\pi)})$$

$$h_{HP}[n] = h_{LP}[n] \cdot e^{j\pi n} = h_{LP}[n] \cdot (-1)^n$$

Linear-Phase FIR Transfer Functions

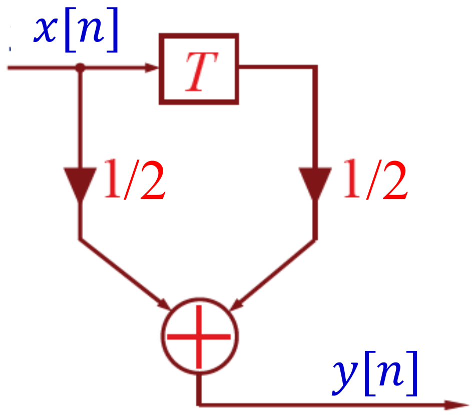
- It is impossible to design an IIR transfer function with an exact linear-phase
- It is always possible to design an FIR transfer function with an exact linear-phase response
- We consider real impulse response $h[n]$

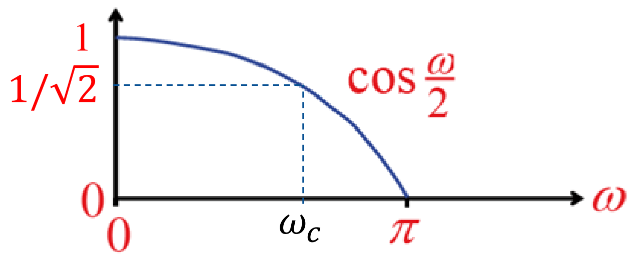
Simple Examples

- $H(z) = \frac{1+z^{-1}}{2} \leftrightarrow \{h[n]\} = \left\{\frac{1}{2}, \frac{1}{2}\right\}$

$$\uparrow$$

$$H(e^{j\omega}) = (1 + e^{-j\omega})/2$$
$$= e^{-j\omega/2} \frac{e^{j\omega/2} + e^{-j\omega/2}}{2}$$
$$= e^{-j\omega/2} \cos(\omega/2)$$





A lowpass filter

ω_c is the 3db cutoff frequency

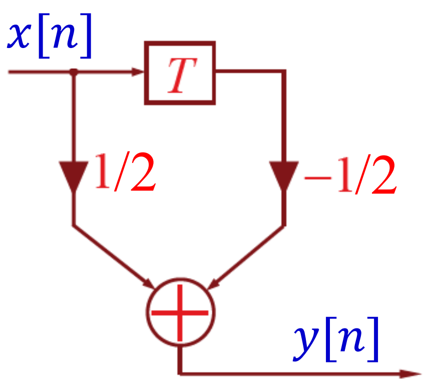
$$\mathcal{G}(\omega_c) = 20\log_{10}\left|\frac{H(e^{j\omega_c})}{H(e^{j0})}\right| = 20\log_{10}\frac{1}{\sqrt{2}} \cong -3\text{dB}$$

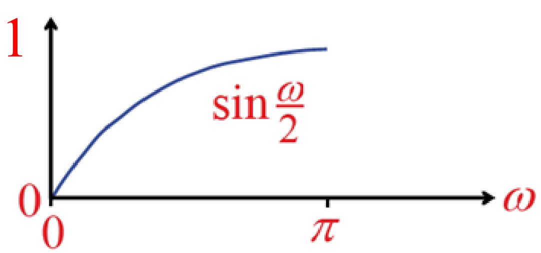
Simple Examples

- $H(z) = \frac{1-z^{-1}}{2} \leftrightarrow \{h[n]\} = \left\{\frac{1}{2}, -\frac{1}{2}\right\}$

$$\uparrow$$

$$H(e^{j\omega}) = (1 - e^{-j\omega})/2$$
$$= e^{j\pi/2} e^{-j\omega/2} \frac{e^{j\omega/2} - e^{-j\omega/2}}{2j}$$
$$= e^{j(\pi/2-\omega/2)} \sin(\omega/2)$$





A highpass filter

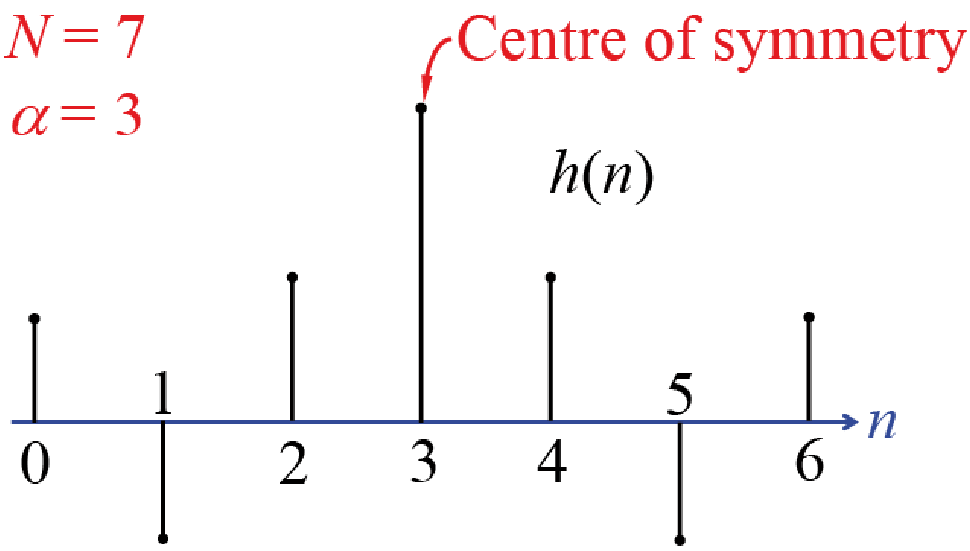
Linear Phase FIR Filter

- An FIR filter may be designed to have linear phase characteristics. The phase response, $\theta(\omega)$, of a linear phase FIR filter is $\beta - \alpha\omega$, where $\alpha = \frac{N-1}{2}$, ω is the frequency, $\beta = 0$ or $\pm 0.5\pi$ and N is the filter length.
- Its frequency response is given by $e^{-j\left(\frac{N-1}{2}\omega-\beta\right)}R(\omega)$, where $R(\omega)$ is a real function.
- The group delay is $-d\{\theta(\omega)\} = \alpha$.

- Its impulse response is either symmetrical or anti-symmetrical.
- If its impulse response is symmetrical, its phase response is $-\alpha\omega$.
- If its impulse response is anti-symmetrical, its phase response is $\pm 0.5\pi - \alpha\omega$.

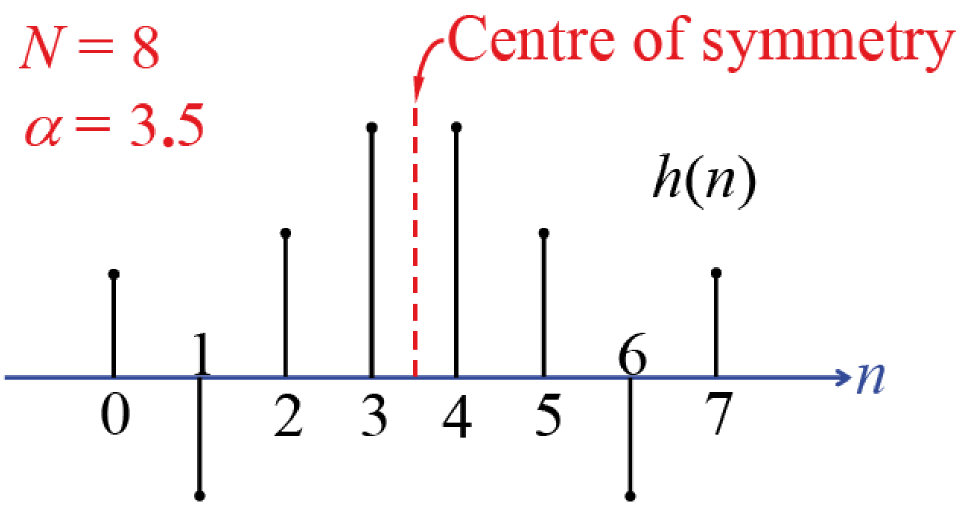
Example: Symmetrical impulse response,

N odd, where N is the length of the impulse response

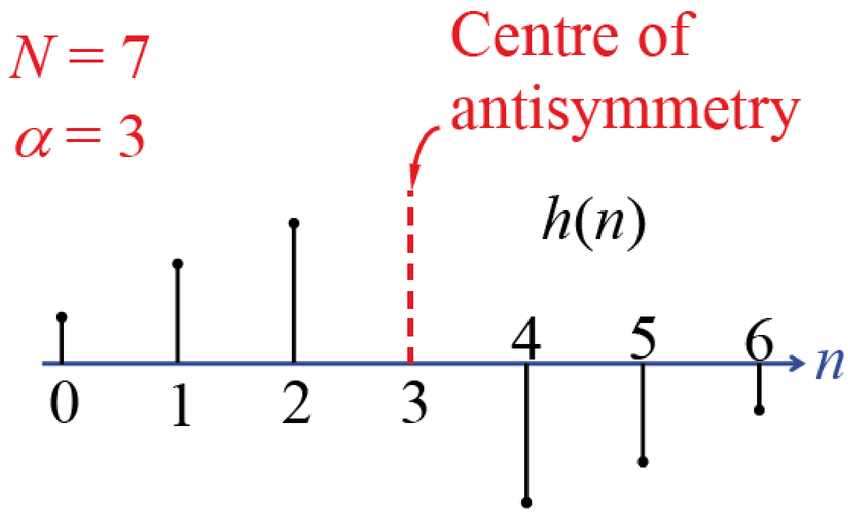


Example: Symmetrical impulse response,

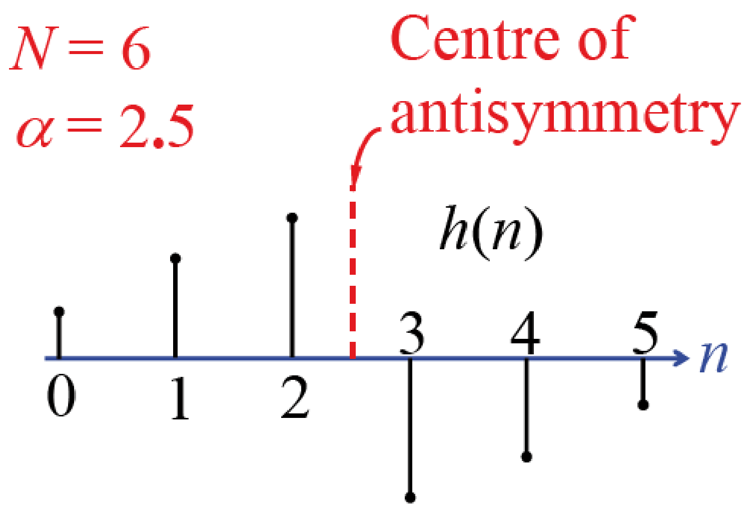
N even.



Example: Anti-symmetrical impulse response, N odd.



Example: Anti-symmetrical impulse response, N even.



Frequency response of linear phase FIR filter

- Four cases, depending on whether N is odd or even and whether the impulse response is symmetrical or anti-symmetrical.
- Type I: Symmetrical impulse response, N odd.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h[n]e^{-j\omega n} + h\left[\frac{N-1}{2}\right]e^{-j\omega\frac{N-1}{2}} + \sum_{n=\frac{N+1}{2}}^{N-1} h[n]e^{-j\omega n}$$

$n' = N-1-n$

$$= e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=0}^{\frac{N-3}{2}} h[n]\left(e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)}\right) + h\left[\frac{N-1}{2}\right]\right]$$

$$= e^{-j\omega\frac{N-1}{2}}\left[\sum_{n=0}^{(N-3)/2} 2h[n]\cos\left(\omega\left(\frac{N-1}{2}-n\right)\right) + h\left[\frac{N-1}{2}\right]\right]$$

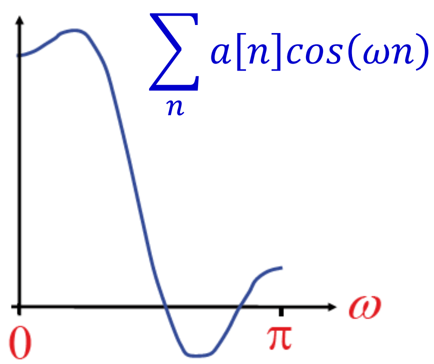
$$\xRightarrow{m=\frac{N-1}{2}-n} e^{-j\omega\frac{N-1}{2}}\left[\sum_{m=1}^{(N-1)/2} 2h\left[\frac{N-1}{2}-m\right]\cos(\omega m) + h\left[\frac{N-1}{2}\right]\right]$$

$$\therefore H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}}\sum_{n=0}^{(N-1)/2} a[n]\cos(\omega n)$$

$$a[0] = h\left[\frac{N-1}{2}\right]$$

$$a[n] = 2h\left[\frac{N-1}{2}-n\right],$$

$$\text{for } n = 1, 2, \ldots, \frac{N-1}{2}$$

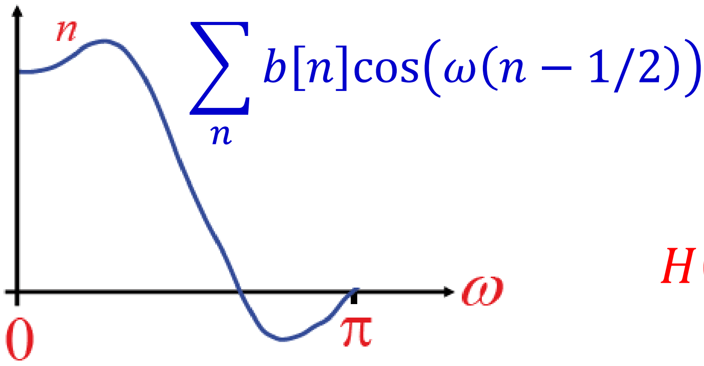


Type II:

- Symmetrical impulse response, N even

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \sum_{n=1}^{N/2} b[n]\cos(\omega(n-1/2))$$

$$b[n] = 2h[N/2-n], \text{for } n = 1, 2, \ldots, N/2$$



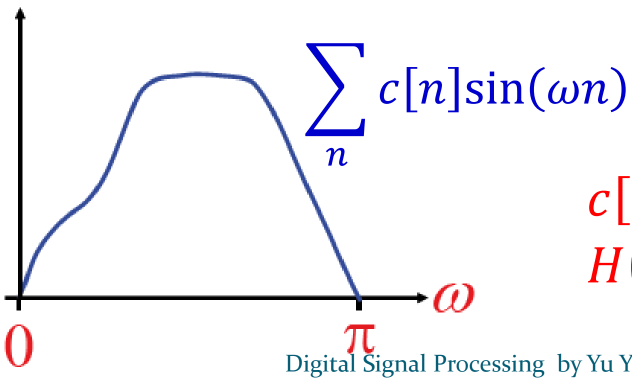
$H(e^{j\omega}) = 0$ at $\omega = \pi$

Type III:

- Anti-symmetrical impulse response, N odd

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=1}^{(N-1)/2} c[n]\sin(\omega n)$$

$$c[n] = 2h\left[\frac{N-1}{2} - n\right], \text{for } n = 1, 2, \ldots, \frac{N-1}{2}$$



$c[0] = 0$

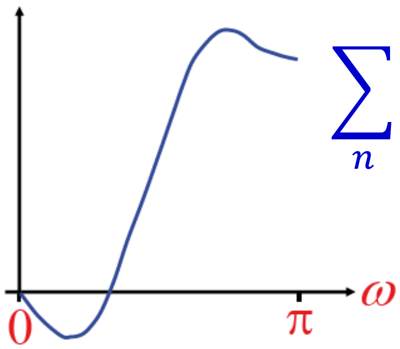
$H(e^{j\omega}) = 0$ at $\omega = 0$ and π

Type IV:

- Anti-symmetrical impulse response, N even

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} e^{j\frac{\pi}{2}} \sum_{n=1}^{N/2} d[n]\sin(\omega(n - 1/2))$$

$$d[n] = 2h[N/2 - n], \text{for } n = 1, 2, \dots, N/2$$



$$\sum_{n} d[n]\sin(\omega(n - 1/2))$$

$H(e^{j\omega}) = 0$ at $\omega = 0$

Mirror Image Polynomial

- For an FIR filter with a symmetric impulse response, its transfer function $H(z)$ can be written as:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} h[N-1-n]z^{-n}$$

$$\xRightarrow{m=N-1-n} H(z) = \sum_{m=0}^{N-1} h[m]z^{-N+1+m} = z^{-(N-1)} \sum_{m=0}^{N-1} h[m]z^{m} = z^{-(N-1)}H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called a **mirror-image polynomial**.

Example: $H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6}$

Amplitude response: $\breve{H}_1(\omega) = 6 - 6\cos(\omega) + 4\cos(2\omega) - 2\cos(3\omega)$

Phase response: $\theta_1(\omega) = -3\omega$

- If $H_2(z) = -H_1(z)$, then $\breve{H}_2(\omega) = \breve{H}_1(\omega)$, and $\theta_2(\omega) = -3\omega + \pi$

Antimirror-Image Polynomial

- Similarly, for an FIR filter with an anti-symmetric impulse response, its transfer function $H(z)$ can be written as:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} = \sum_{n=0}^{N-1} -h[N-1-n]z^{-n} = -z^{-(N-1)}H(z^{-1})$$

- A real-coefficient polynomial $H(z)$ satisfying the above condition is called an **antimirror-image polynomial**.

Example: $H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6}$

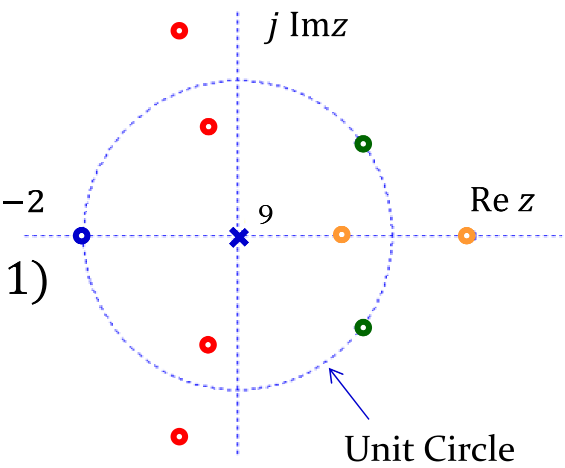
Amplitude response: $\breve{H}_3(\omega) = 6\sin(\omega) - 4\sin(2\omega) + 2\sin(3\omega)$

Phase response: $\theta_3(\omega) = -3\omega + \pi/2$

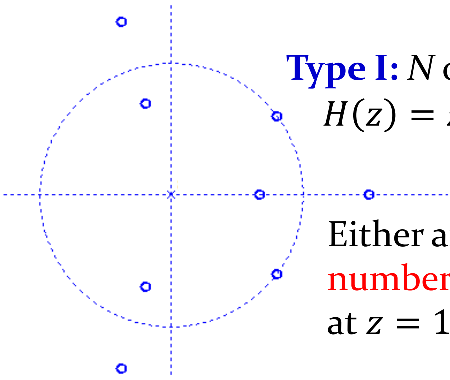
- If $H_4(z) = -H_3(z)$, then $\breve{H}_4(\omega) = \breve{H}_3(\omega)$, and $\theta_4(\omega) = -3\omega - \pi/2$

Mirror Image Symmetry of Zeros

- The zeros of linear phase FIR filter with real coefficients exhibit **mirror image symmetry** with respect to unit circle.
- $H(z) = z^{-(N-1)}H(z^{-1})$ or $H(z) = -z^{-(N-1)}H(z^{-1})$
- Zeros are in forms of:
  - Real zeros
    - $z = r$ and $z = \frac{1}{r}$, $r \neq \pm 1$
      $\Longrightarrow (z-r)(z-r^{-1}) \Longrightarrow 1 + az^{-1} + z^{-2}$
    - $z = -1$ or $z = 1 \Longrightarrow (z+1)$ or $(z-1)$
  - Complex zeros
    - $z = re^{\pm j\varphi}$ and $z = \frac{1}{r}e^{\pm j\varphi}$, $r \neq \pm 1$
      $\Longrightarrow 1 + az^{-1} + cz^{-2} + az^{-3} + z^{-4}$
    - $z = e^{\pm j\varphi} \Longrightarrow 1 + az^{-1} + z^{-2}$



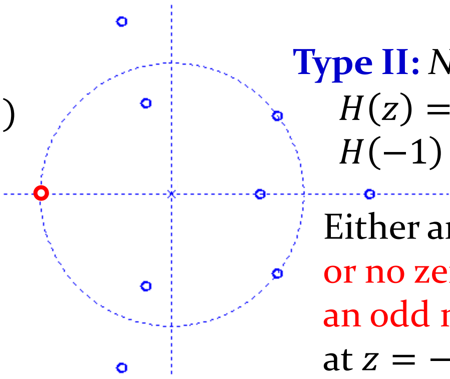
Zero-Locations of FIR Filters



Type I: N odd

$H(z) = z^{-(N-1)}H(z^{-1})$

Either an even number or no zeros at $z = 1$ and $z = -1$

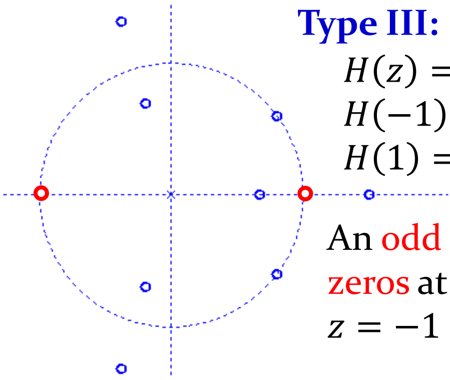


Type II: N even

$H(z) = z^{-(N-1)}H(z^{-1})$

$H(-1) = -H(-1)$

Either an even number or no zeros at $z = 1$ and an odd number of zeros at $z = -1$



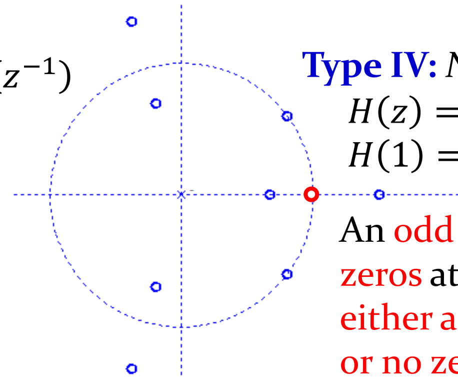
Type III: N odd

$H(z) = -z^{-(N-1)}H(z^{-1})$

$H(-1) = -H(-1)$

$H(1) = -H(1)$

An odd number of zeros at $z = 1$ and $z = -1$



Type IV: N even

$H(z) = -z^{-(N-1)}H(z^{-1})$

$H(1) = -H(1)$

An odd number of zeros at $z = 1$ and either an even number or no zeros at $z = -1$

Zero-Phase FIR Filters

- The impulse response $h[n]$ is symmetric around $h[0]$, i.e., $h[n] = h[-n]$.
- Thus, the frequency response $H(e^{j\omega})$ is real, but not necessary positive (unlike $|H(e^{j\omega})|$)
- For example, the type I FIR filter:

$$H(e^{j\omega}) = \sum_{n=-\frac{N-1}{2}}^{-1} h[n]e^{-j\omega n} + h[0] + \sum_{n=1}^{\frac{N-1}{2}} h[n]e^{-j\omega n}$$

$$= h[0] + 2\sum_{n=1}^{\frac{N-1}{2}} h[n]\cos(\omega n)$$

- Zero-phase FIR filter is not causal, but the causality may be recovered by delaying the impulse response

Simple IIR Filters

- General Form

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1 + \sum_{m=1}^{N} b_m z^{-m}}$$

Lowpass & Highpass IIR Filter

$$H(z) = \frac{K}{1 - \alpha z^{-1}}, \qquad 0 < |\alpha| < 1, \qquad \alpha, K \text{ are real}$$

- Its squared-magnitude function is given by

$$\left|H(e^{j\omega})\right|^2 = H(z)H(z^{-1})\Big|_{z=e^{j\omega}} = \frac{K^2}{(1+\alpha^2) - 2\alpha\cos\omega}$$

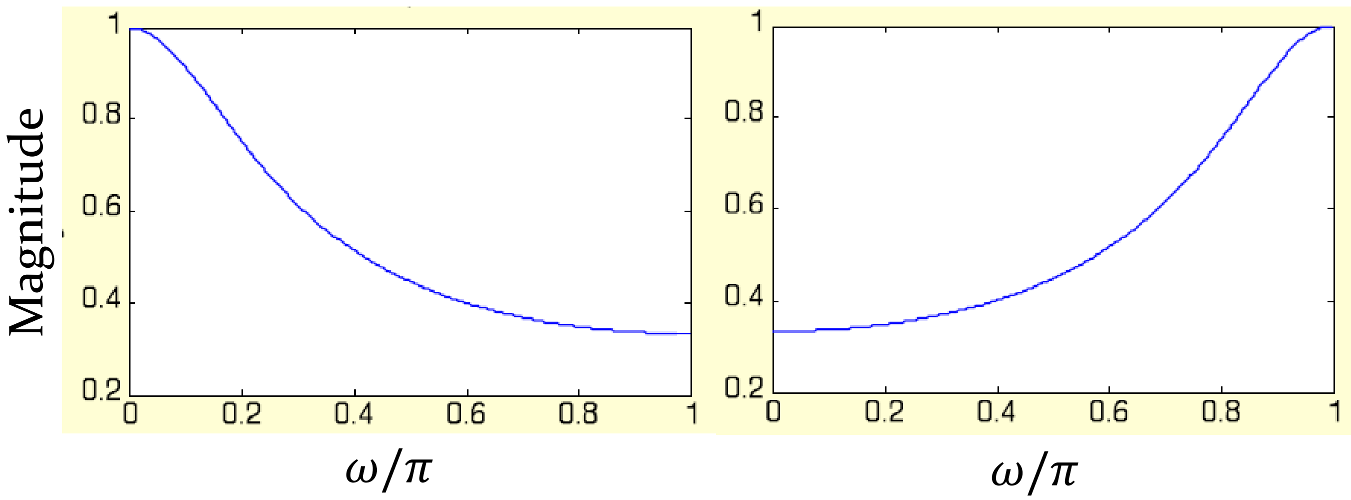
- When $\alpha > 0$, $\left|H(e^{j\omega})\right|^2_{\max} = \frac{K^2}{(1+\alpha^2)-2\alpha}$, at $\omega = 0$,

$$\left|H(e^{j\omega})\right|^2_{\min} = \frac{K^2}{(1+\alpha^2)+2\alpha}, \text{ at } \omega = \pi.$$

- When $\alpha < 0$, $\left|H\left(e^{j\omega}\right)\right|^2_{\max} = \frac{K^2}{(1+\alpha^2)+2\alpha}$, at $\omega = \pi$,

$$\left|H\left(e^{j\omega}\right)\right|^2_{\min} = \frac{K^2}{(1+\alpha^2)-2\alpha}, \text{ at } \omega = 0.$$

$\alpha > 0$ $\quad K = \pm(1 - |\alpha|)$ $\quad \alpha < 0$



Improved Lowpass IIR Filters

$$H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

- A factor $1 + z^{-1}$ added to the numerator to force the magnitude function to have a zero at $\omega = \pi$

$$\left|H_{LP}\left(e^{j\omega}\right)\right|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

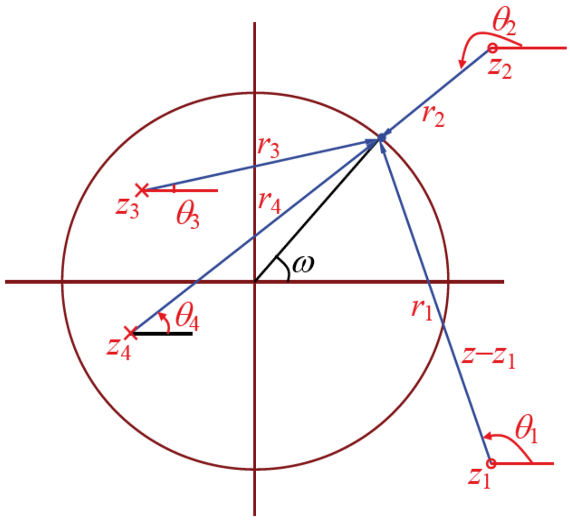
$$\left|H_{LP}\left(e^{j0}\right)\right| = 1, \text{and } \left|H_{LP}\left(e^{j\pi}\right)\right| = 0$$

Zeros, Poles and Geometric Interpolation of Frequency Response

$$H(z) = \frac{(z-z_1)(z-z_2)}{(z-z_3)(z-z_4)}$$

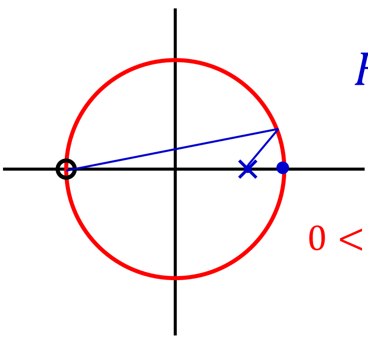
$$H(e^{j\omega}) = \frac{(e^{j\omega}-z_1)(e^{j\omega}-z_2)}{(e^{j\omega}-z_3)(e^{j\omega}-z_4)}$$

$$= \frac{r_1 r_2}{r_3 r_4} \angle \theta_1 + \theta_2 - \theta_3 - \theta_4$$

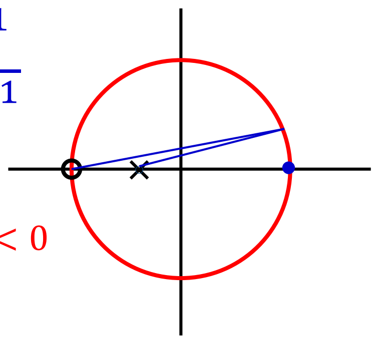


Pole, Zero, and Response

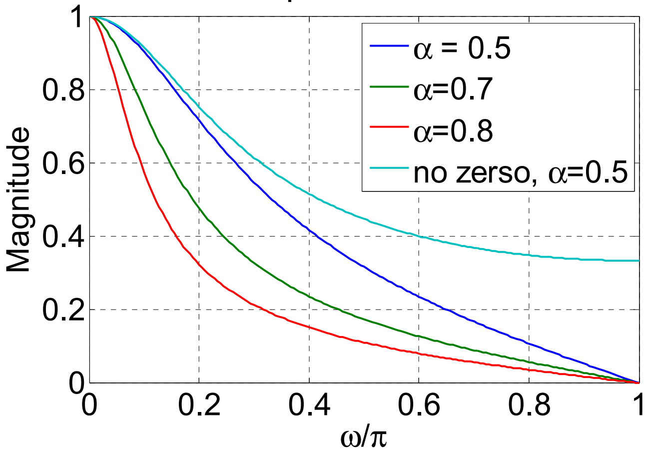
$$H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}$$

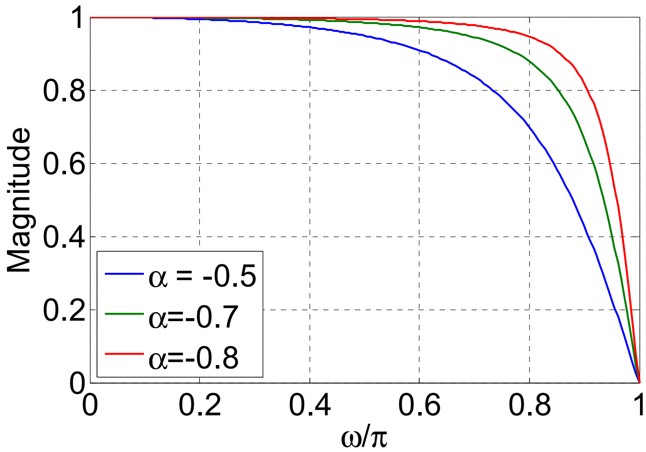


$0 < \alpha < 1$



$-1 < \alpha < 0$





Improved Highpass IIR Filters

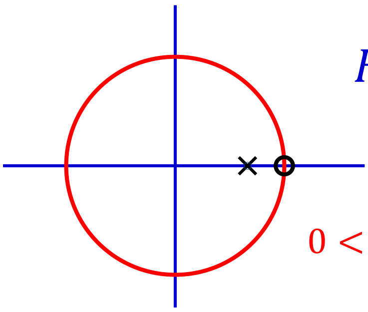
$$H(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

- A factor $1 - z^{-1}$ added to the numerator to force the magnitude function to have a zero at $\omega = 0$

$$\left|H_{LP}(e^{j\omega})\right|^2 = \frac{(1+\alpha)^2(1-\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)}$$

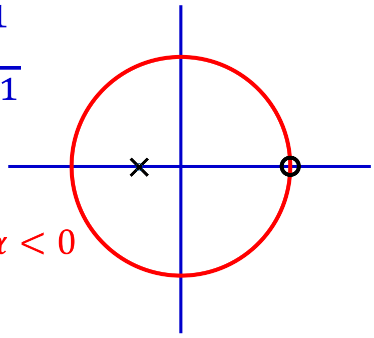
$$\left|H_{LP}(e^{j0})\right| = 0, \text{and } \left|H_{LP}(e^{j\pi})\right| = 1$$

Pole, Zero, and Response

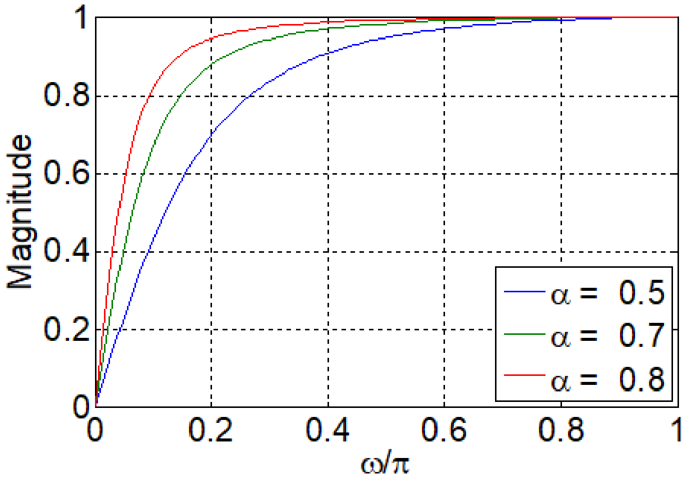


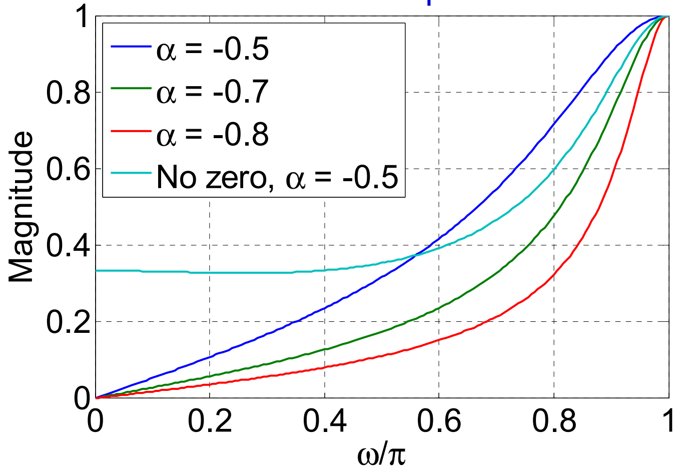
$0 < \alpha < 1$

$$H(z) = \frac{1+\alpha}{2}\frac{1-z^{-1}}{1-\alpha z^{-1}}$$



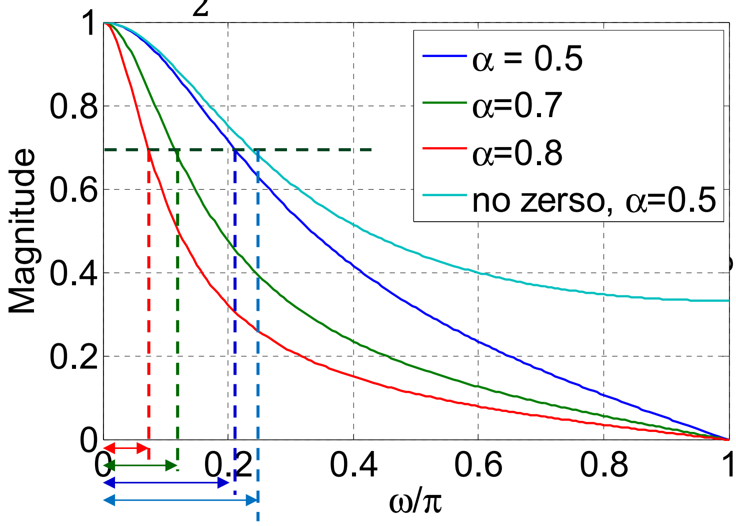
$-1 < \alpha < 0$



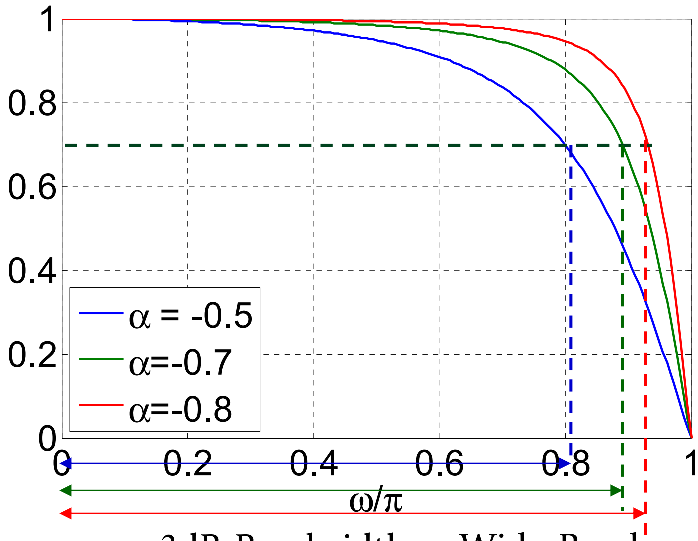


3-dB Cutoff Frequency

- The frequency where the magnitude is reduced to $\frac{1}{\sqrt{2}}$ of the ideal passband gain, or the squared magnitude is reduced to $\frac{1}{2}$ of the ideal one, is the 3-dB cutoff frequency.



3dB-Bandwidth Narrow-Band



3dB-Bandwidth Wide-Band

Compute the 3-dB cutoff frequency

$$H(z) = \frac{1-\alpha}{2}\,\frac{1+z^{-1}}{1-\alpha z^{-1}}, \qquad 0 < |\alpha| < 1$$

- Given α, compute ω_c. Squared magnitude function

$$\left|H_{LP}\left(e^{j\omega}\right)\right|^2 = \frac{(1-\alpha)^2(1+\cos\omega)}{2(1+\alpha^2-2\alpha\cos\omega)} = \frac{1}{2}$$

which, when solved, yields

$$\cos\omega_c = \frac{2\alpha}{1+\alpha^2} \Longrightarrow \omega_c = \cos^{-1}\frac{2\alpha}{1+\alpha^2}$$

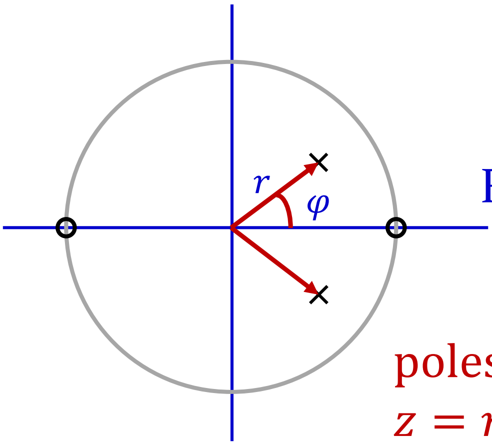
- Given ω_c, determine α. A solution resulting in a stable transfer function is

$$\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$$

Bandpass IIR Digital Filter

- A 2nd-order general form

$$H_{BP}(z) = \frac{K(1 - z^{-2})}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}$$



$$r = \sqrt{\alpha}$$

$$\varphi = \cos^{-1}\left(\frac{\beta(1+\alpha)}{2\sqrt{\alpha}}\right)$$

For stability, we have $r < 1 \;\rightarrow\; |\alpha| < 1$

poles:
$z = re^{\pm j\varphi}$

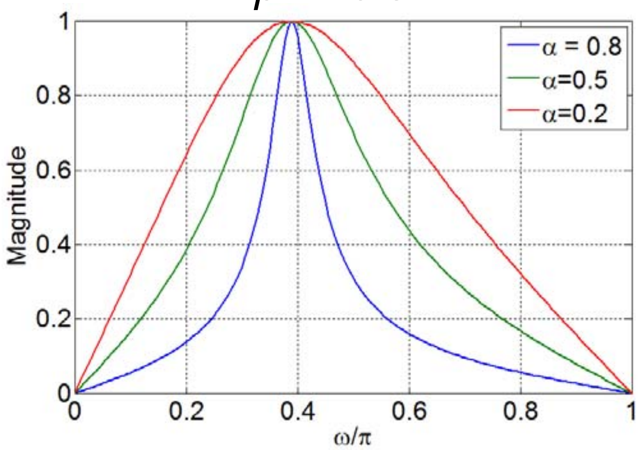
- The squared magnitude function:

$$\left|H_{BP}(e^{j\omega})\right|^2 = \frac{4K^2\sin^2\omega}{(1+\alpha)^2(\beta - \cos\omega)^2 + (1-\alpha)^2\sin^2\omega}$$

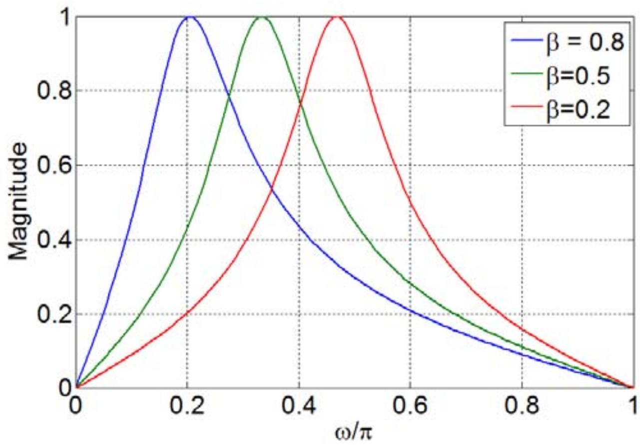
- $\left|H_{BP}(e^{j\omega})\right|^2 = 0$ at $\omega = 0$ and $\omega = \pi$
- $\left|H_{BP}(e^{j\omega})\right|^2 = \frac{2K}{1-\alpha}$, the maximum, at $\omega_0 = \cos^{-1}\beta$
  - ω_0 is called the **center frequency** of the bandpass filter
  - Choose $K = \frac{1-\alpha}{2}$ to make the maximum magnitude to be 1.
- The frequencies, ω_{c1} and ω_{c2}, where $\left|H_{BP}(e^{j\omega})\right|^2 = \frac{1}{2}$ are the 3-dB cutoff frequencies.

- **3-dB Bandwidth** $B_w = \omega_{c2} - \omega_{c1} = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$
- **Quality factor** $Q = \frac{\omega_0}{B_w}$

$\beta = 0.34$



$\alpha = 0.6$



$$\omega_0 = \cos^{-1}\beta \qquad B_w = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$$

Example

- Design a second-order bandpass digital filter with center frequency at 0.4π and a 3dB bandwidth of 0.1π.
- A: $\beta = \cos\omega_0 = \cos(0.4\pi) = 0.309016994$

$$\frac{2\alpha}{1+\alpha^2} = \cos B_w = \cos(0.1\pi) = 0.951056516$$

$\Longrightarrow \alpha = 1.37638192$ (not stable) or $\alpha = 0.726542528$

So, the transfer function of the second-order bandpass filter is:

$$H_{BP}(e^{j\omega}) = \frac{0.136728736(1 - z^{-2})}{1 - 0.53353098z^{-1} + 0.726542528z^{-2}}$$

Allpass Filter

- **Allpass Transfer Function**: an IIR transfer function $A(z)$ with unity magnitude response for all frequencies, i.e.,

$$\left|A(e^{j\omega})\right|^2 = 1, \qquad \text{for all } \omega$$

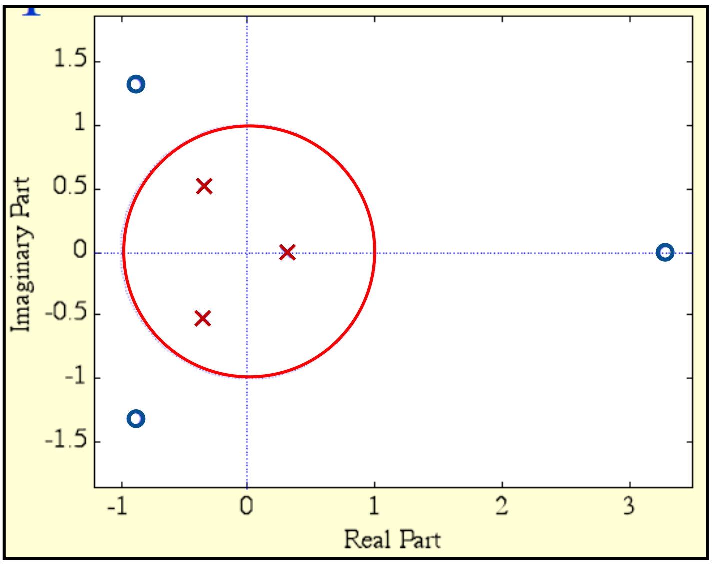
- An M-th order causal **real-coefficient** allpass transfer function is of form

$$A_M(z) = \frac{d_M + d_{M-1}z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1}z^{-M+1} + d_M z^{-M}}$$

$$= \frac{z^{-M} D_M(z^{-1})}{D_M(z)}$$

- Implying that the poles and zeros exhibits mirror-image symmetry in z-plane.
- Example:

$$A_3(z) = \frac{-0.2 + 0.18z^{-1} + 0.4z^{-2} + z^{-3}}{1 + 04z^{-1} + 0.18z^{-2} - 0.2z^{-3}}$$

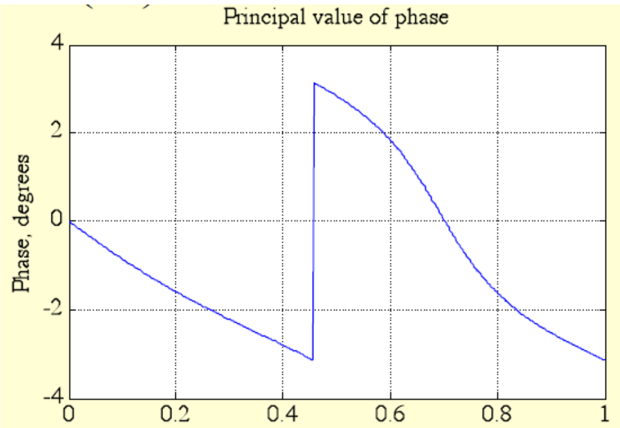


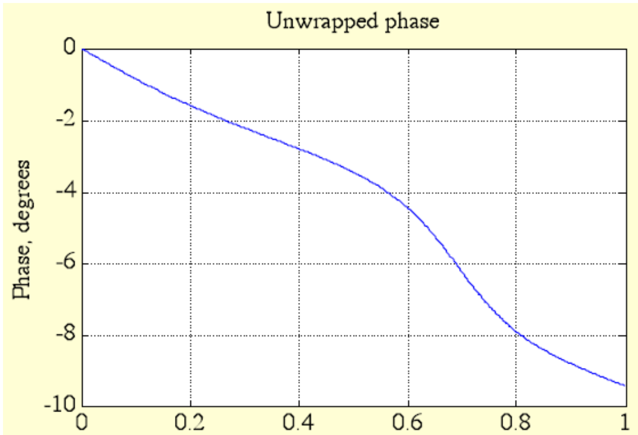
- The poles of a causal stable transfer function must lie inside the unit circle in z-plane
- Pairs of conjugated poles and zeros for real coefficient allpass filter, unless real pole or zeros.

- It can be shown that

$$\left|A_M(e^{j\omega})\right|^2 = A_M(z)A_M(z^{-1})\Big|_{z=e^{j\omega}}$$
$$= \frac{z^{-M}D_M(z^{-1})}{D_M(z)} \cdot \frac{z^{M}D_M(z)}{D_M(z^{-1})} = 1$$

- Q: what's the use of allpass filters?
- A: Its phase plays the role



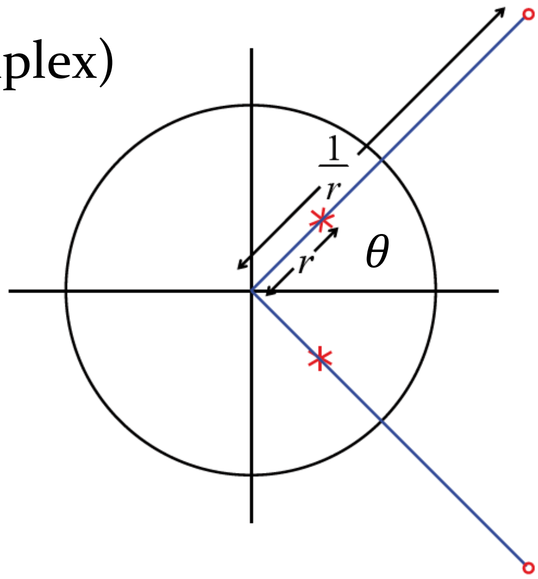


- Factorized form of real (or complex) coefficient allpass filter

$$A_M(z) = \pm \prod_{i=1}^{M} \left(\frac{-\lambda_i^* + z^{-1}}{1 - \lambda_i z^{-1}} \right)$$

Pole: $z_{pi} = \lambda_i = r_i e^{j\theta_i}$

Zero: $z_{zi} = \frac{1}{\lambda_i^*} = \frac{1}{r_i} e^{j\theta_i}$



- Hence, all zeros of a causal stable allpass transfer function must lie outside the unit circle in a mirror-image symmetry with its poles suited inside the unit circle.

A First-Order Allpass Filter

- A first-order complex coefficient allpass transfer function

$$A(z) = \frac{-\lambda^* + z^{-1}}{1 - \lambda z^{-1}}, |\lambda| < 1, \lambda = re^{j\varphi}$$

- Its frequency response is given by

$$A(e^{j\omega}) = \frac{-\lambda^* + e^{-j\omega}}{1 - \lambda e^{-j\omega}} = e^{-j\omega} \frac{1 - re^{j(\omega-\varphi)}}{1 - re^{-j(\omega-\varphi)}}$$

So the phase function is

$$\theta(\omega) = -\omega - 2\tan^{-1}\frac{r\sin(\omega-\varphi)}{1 - r\cos(\omega-\varphi)}$$

$$\frac{d\theta(\omega)}{d\omega} = \frac{-(1-r^2)}{(1 - r\cos(\omega-\varphi))^2 + r^2\sin^2(\omega-\varphi)} < 0$$

The phase of the first-order allpass filter decreases monotonically.

Minimum Phase System

- **Definition:** A causal system with all zeros located inside the unit circles in z-plane is a minimum phase system, denoted as $H_{\min}(z)$.
- **Property:** Any real coefficient causal system can be represented as

$$H(z) = H_{\min}(z)A_m(z)$$

where, $H_{\min}(z)$ has the same magnitude response as $H(z)$, and $A_m(z)$ is an allpass system.

Minimum Phase System Property

- **Proof:** Assume that $H(z)$ has only one zero $z = \frac{1}{a^*}$, $|a| < 1$, located outside the unit circle.
  Thus, $H(z)$ can be represented as
$$H(z) = H_1(z)(z^{-1} - a^*).$$
- According to definition, $H_1(z)$ is a minimum system.
- And $H(z)$ can be equivalent to
$$H(z) = H_1(z)(z^{-1} - a^*)\frac{1 - az^{-1}}{1 - az^{-1}}$$
$$= H_1(z)(1 - az^{-1})\frac{z^{-1} - a^*}{1 - az^{-1}}$$

- i.e., $H(z) = H_{\min}(z)A_m(z)$, where
$$H_{\min}(z) = H_1(z)(1 - az^{-1})$$

and
$$A_m(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

- Why it is called a minimum-phase system?
  - **The minimum phase-lag property**: the phase delay of the minimum phase system is always less than the phase delays of the other systems with the same magnitude response.

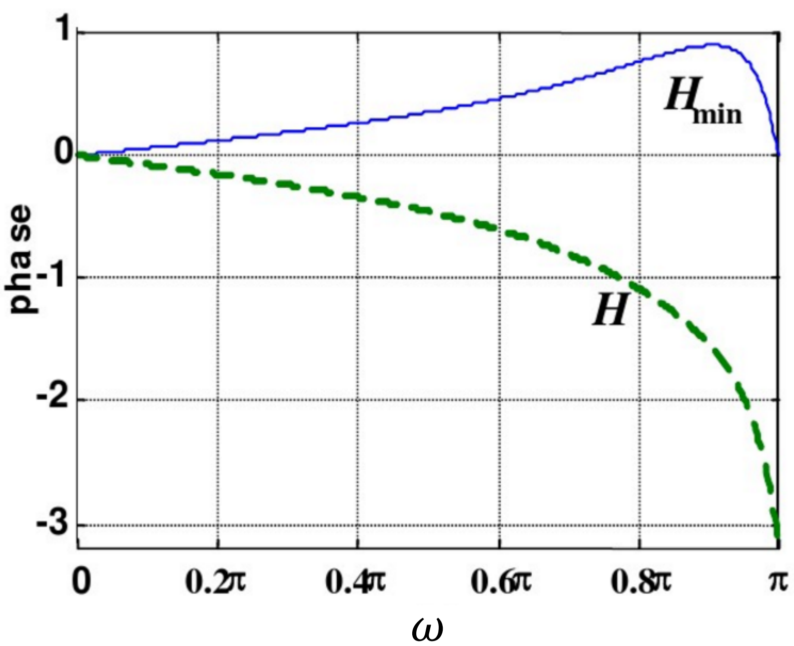
Example

- Q: Given the transfer function of a real coefficient causal stable LTI system
$$H(z) = \frac{b + z^{-1}}{1 + az^{-1}}, |a| < 1 \text{ and } |b| < 1$$
Find the minimum phase system having the same magnitude response as that of $H(z)$

- A: Since the zero of $H(z)$ is $-\frac{1}{b}$ and $|b| < 1$, it is not a minimum phase system.
$$H(z) = \frac{b + z^{-1}}{1 + az^{-1}} \frac{1 + bz^{-1}}{1 + bz^{-1}} = \frac{1 + bz^{-1}}{1 + az^{-1}} \frac{b + z^{-1}}{1 + bz^{-1}}$$

- So the minimum phase system is

$$H_{\min}(z) = \frac{1 + bz^{-1}}{1 + az^{-1}}$$



The phase response of $H(e^{j\omega})$ and $H_{\min}(e^{j\omega})$ when $a = 0.9$, and $b = 0.4$

Maximum Phase System

- A causal system with all zeros located outside the unit circles in z-plane is a maximum phase system, denoted as $H_{\max}(z)$.
- Example: The transfer function of a real coefficient causal stable LTI system is give by

$$H(z) = \frac{b + z^{-1}}{1 + az^{-1}}, |a| < 1 \text{ and } |b| < 1$$

- Obviously, this is a maximum phase system.

Inverse System

- Inverse system has $h_1[n] \circledast h_2[n] = \delta[n]$
- Then, in z-Domain

$$H_1(z)H_2(z) = 1$$

- If we have $H_1(z)$, then

$$H_2(z) = \frac{1}{H_1(z)}$$

Lecture 9
Digital Filter Structure

Block Diagram Representation

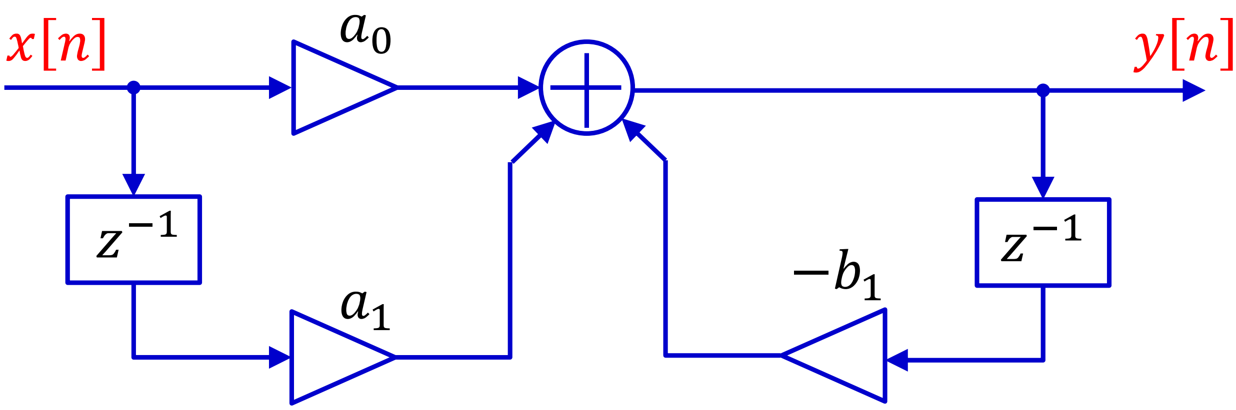
- It has advantages to represent time domain input-output relation, for example the convolution, or the difference equations, as block diagrams.

$$y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$

$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

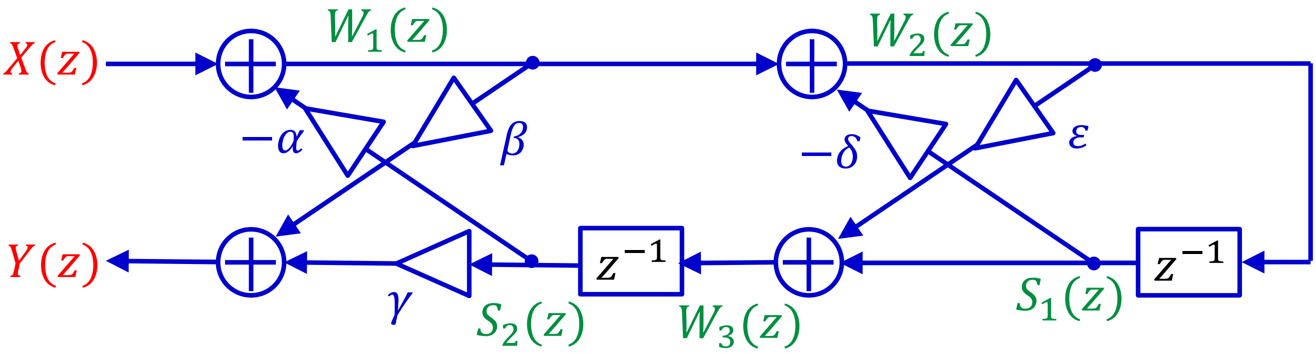
A First Order LTI Digital Filter

$$y[n] = -b_1 y[n-1] + a_0 x[n] + a_1 x[n-1]$$



$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

Analysis of Block Diagrams



$$W_1 = X - \alpha S_2$$
$$W_2 = W_1 - \delta S_1$$
$$W_3 = \varepsilon W_2 + S_1$$
$$Y = \beta W_1 + \gamma S_2$$

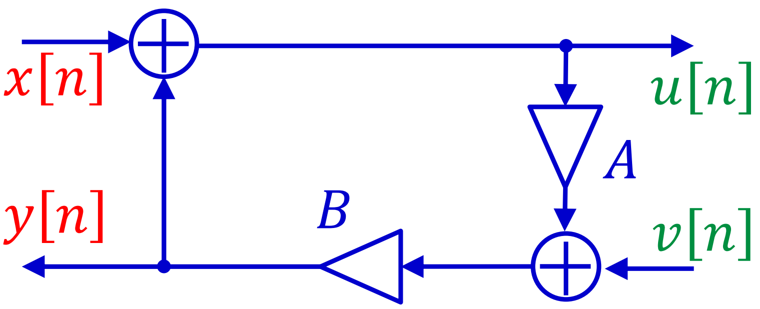
$$S_2 = z^{-1} W_3$$
$$S_1 = z^{-1} W_2$$

$$W_1 = X - \alpha z^{-1} W_3$$
$$W_2 = W_1 - \delta z^{-1} W_2$$
$$W_3 = \varepsilon W_2 + z^{-1} W_2$$
$$Y = \beta W_1 + \gamma z^{-1} W_3$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\beta + (\beta\delta + \gamma\varepsilon) z^{-1} + \gamma z^{-2}}{1 + (\delta + \alpha\varepsilon) z^{-1} + \alpha z^{-2}}$$

Delays in Block Diagram

- A block diagram containing **delay-free loop**, i.e., a feedback loop without any delay element, is physically **IMPOSSIBLE** to achieve.



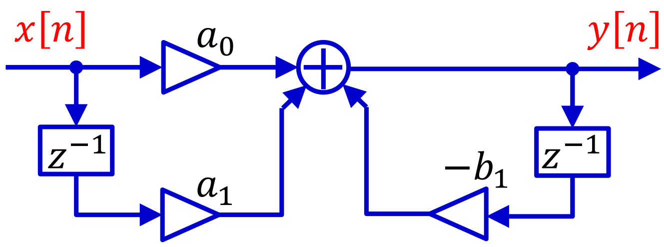
$$y[n] = B(A(x[n] + y[n]) + v[n])$$

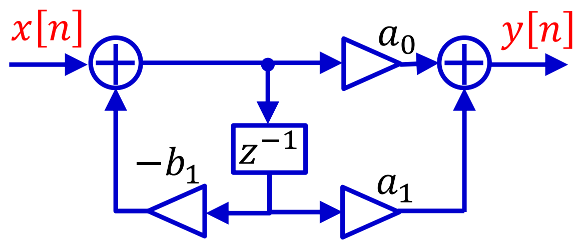
- The number of delays in a **canonic structure** is equal to the order of the transfer function (or the order of the difference equation).

Equivalent Structure

$$H(z) = \frac{a_0 + a_1 z^{-1}}{1 + b_1 z^{-1}}$$

- Two digital filter structures are defined to be equivalent if they have the same transfer function.





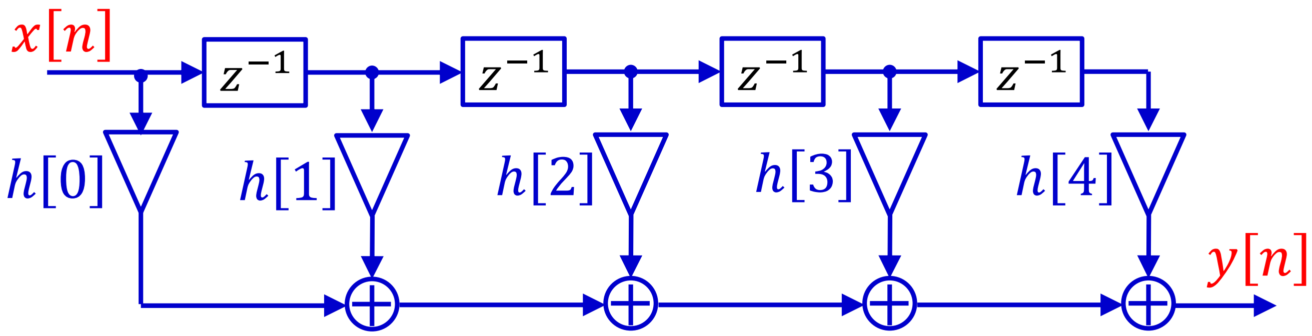
- Transpose Operation to obtain an equivalent structure
  - Reverse all paths
  - Replace branching nodes with adders, and vice versa,
  - Interchange the input and output nodes.

Basic FIR Digital Filter Structures

- Direct-Form Structures

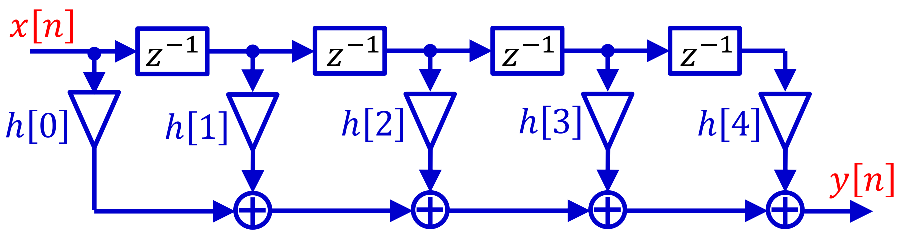
$$Y(z) = H(z)X(z) = \sum_{k=0}^{N-1} h[k]z^{-k}X(z)$$

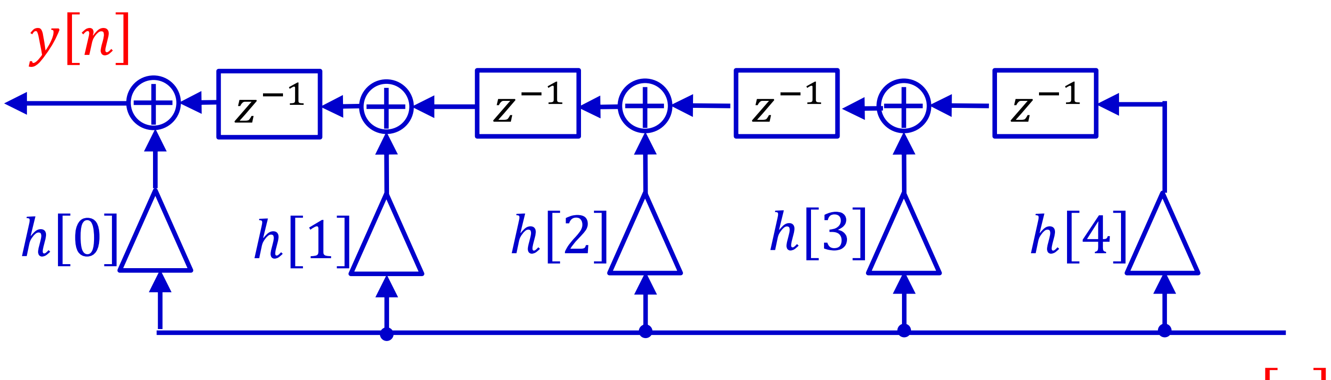
$$y[n] = \sum_{k=0}^{N-1} h[k]x[n-k]$$



- Transposed Direct Form Structures

Direct Form





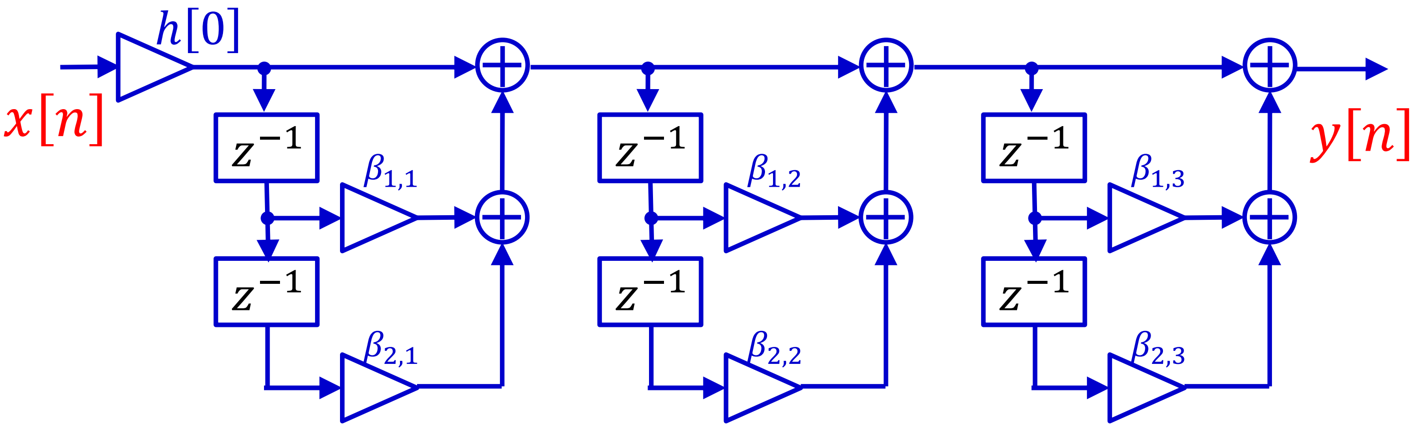
Transposed Direct Form

Cascade-Form Structures

$$H(z) = h[0]\prod_{k=1}^{K}\left(1 + \beta_{1,k}z^{-1} + \beta_{2,k}z^{-2}\right)$$

where $K = (N-1)/2$ if N is odd,

and $K = N/2$ if N is even, with $\beta_{2,K} = 0$

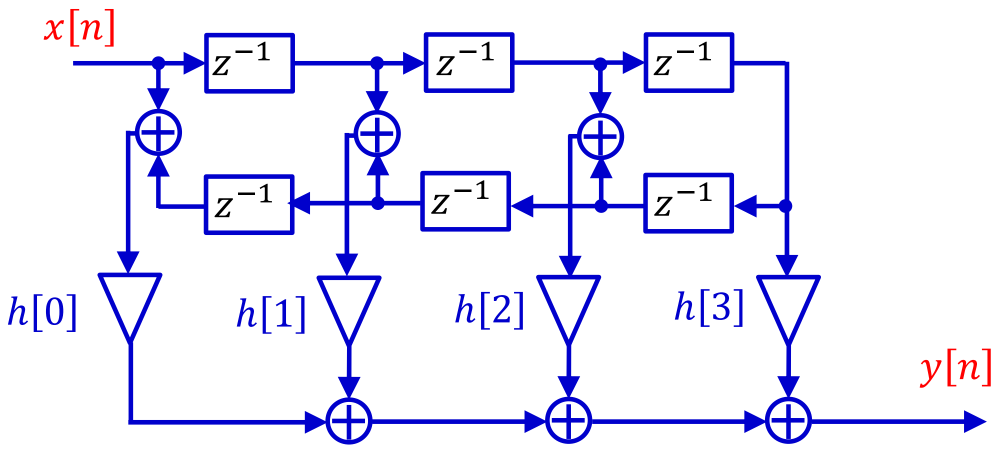


Linear-Phase FIR Filter Structure

$$h[n] = h[N-1-n], \text{or } h[n] = -h[N-1-n]$$

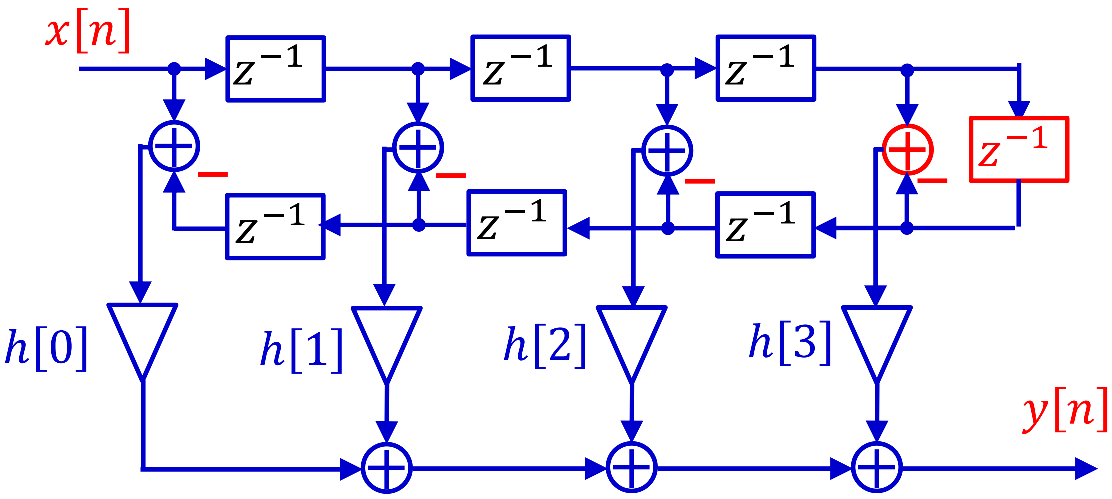
- For example, a length-7 Type I filter

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$
$$+h[2]z^{-4} + h[1]z^{-5} + h[0]z^{-6}$$
$$= h[0](1 + z^{-6}) + h[1]\left(z^{-1} + z^{-5}\right)$$
$$+h[2](z^{-2} + z^{-4}) + h[3]z^{-3}$$



- a length-8 Type IV filter

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$
$$-h[3]z^{-4} - h[2]z^{-5} - h[1]z^{-6} - h[0]z^{-7}$$
$$= h[0](1 - z^{-7}) + h[1](z^{-1} - z^{-6})$$
$$+h[2](z^{-2} - z^{-5}) + h[3](z^{-3} - z^{-4})$$



Basic IIR Digital Filter Structure

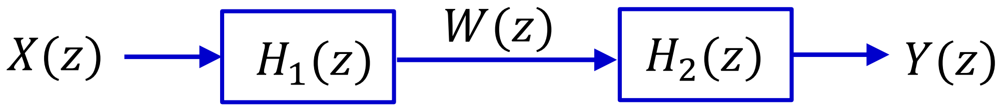
$$y[n] = \sum_{m=0}^{M} a_m x[n-m] - \sum_{m=1}^{N} b_m y[n-m]$$

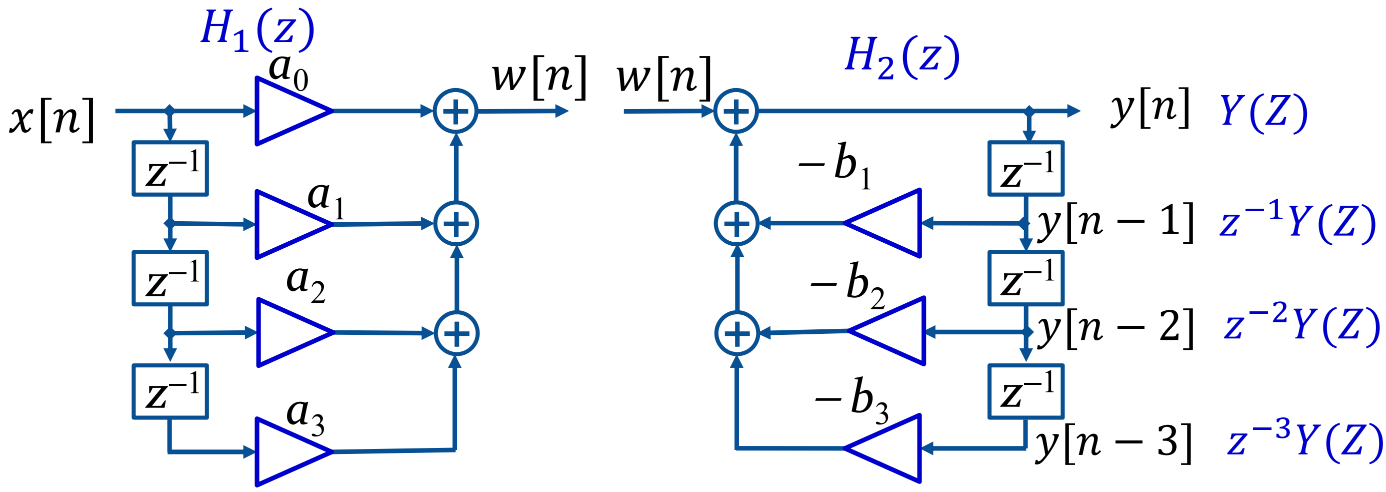
$$H(z) = \frac{A(z)}{B(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1 + \sum_{m=1}^{N} b_m z^{-m}} = H_1(z)H_2(z)$$

For example, when $M = N = 3$, we have

$$H_1(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$$
$$H_2(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}$$

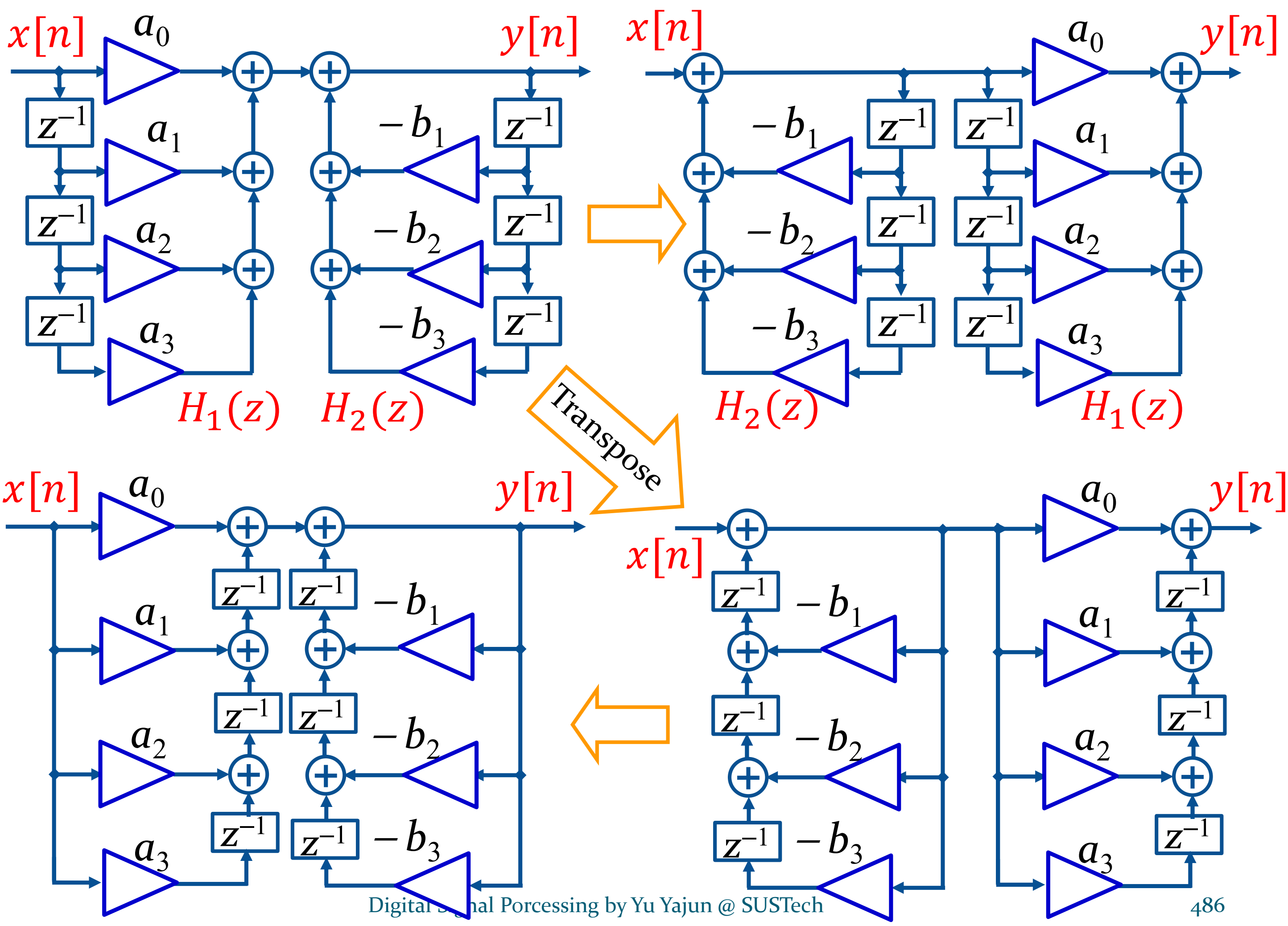
- Direct-Form Structure

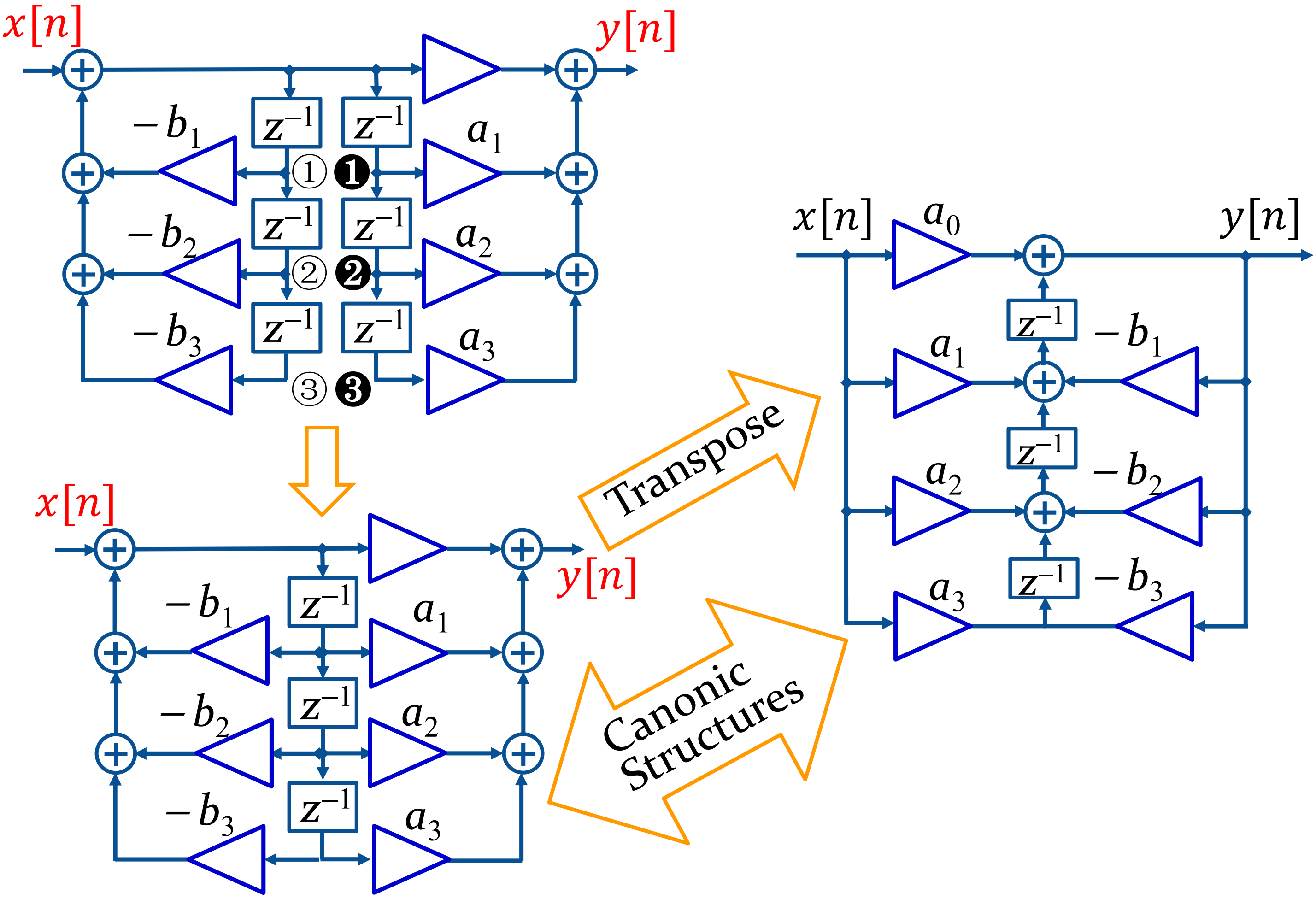




$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}$$

$$Y(z) = W(z) - b_1 Y(z) z^{-1} - b_2 Y(z) z^{-2} - b_3 Y(z) z^{-3}$$

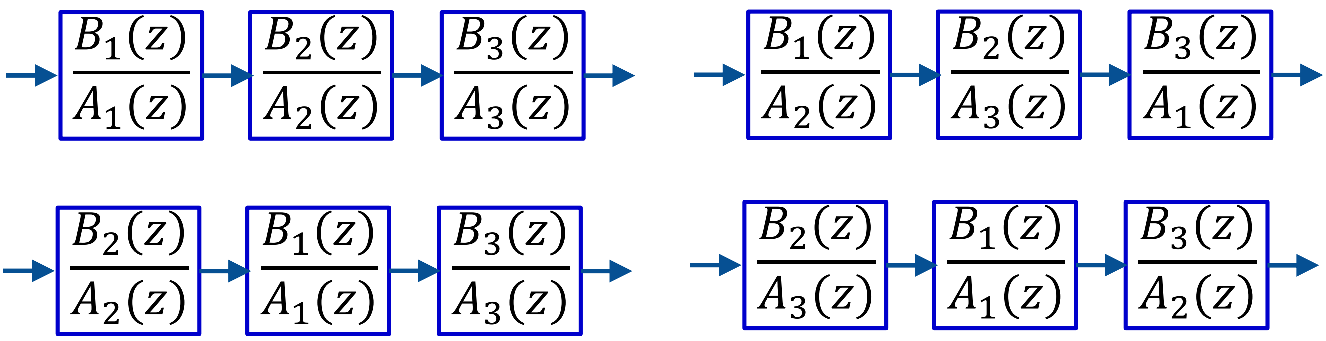




Cascade Realization

$$H(z)=\frac{\prod_{k=1}^{M_1}(1-f_k z^{-1})\prod_{k=1}^{M_2}(1-g_k z^{-1})(1-g_k{}^{*}z^{-1})}{\prod_{k=1}^{N_1}(1-c_k z^{-1})\prod_{k=1}^{N_2}(1-d_k z^{-1})(1-d_k{}^{*}z^{-1})}$$

$$H(z)=\frac{B(z)}{A(z)}=\frac{B_1(Z)B_2(Z)B_3(Z)}{A_1(Z)A_2(Z)A_3(Z)}$$

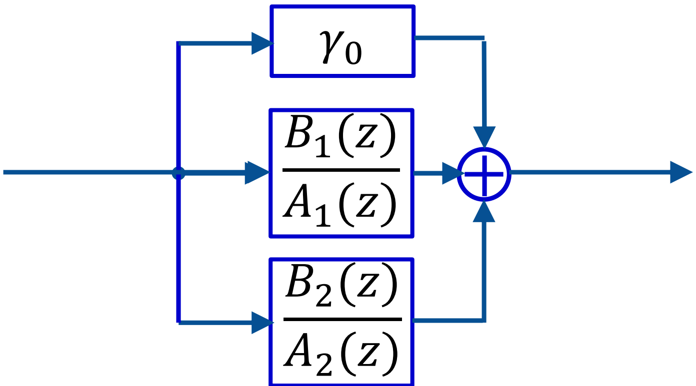


Parallel Realization

$$H(z) = \gamma_0 + \sum_k \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

For example: $H(z) = \gamma_0 + \frac{B_1(z)}{A_1(z)} + \frac{B_2(z)}{A_2(z)}$

- Can be obtained by partial-fraction expansion



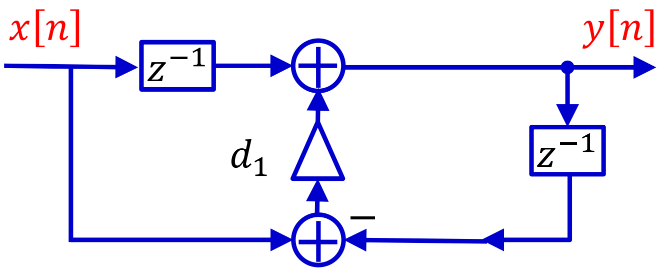
Allpass Filter Structure

- Transfer function of **real** coefficient allpass filter

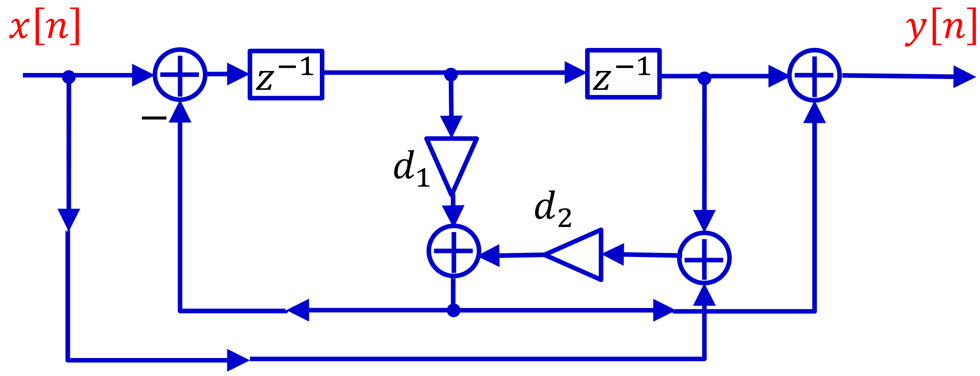
$$A_M(z) = \frac{d_M + d_{M-1} z^{-1} + \cdots + d_1 z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

- Objective: efficient structure using N multipliers to implement N-th order allpass filter, for example:

- First order: $A_M(z) = \frac{d_1 + z^{-1}}{1 + d_1 z^{-1}}$



- Second order: $A_M(z) = \frac{d_2+d_1z^{-1}+z^{-2}}{1+d_1z^{-1}+d_2z^{-2}}$



- Allpass filters with this structure have a magnitude gain of 1 even with coefficient errors

*Allpass with Lattice Structure

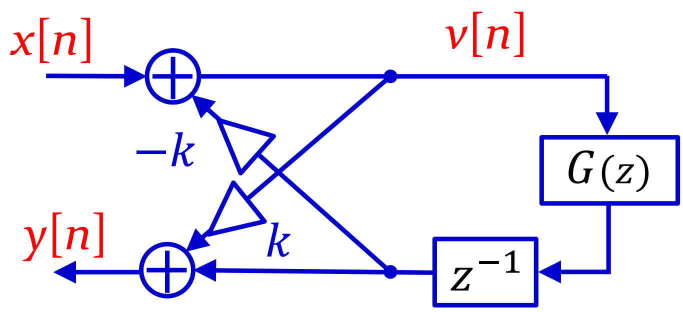
- Lattice Stage:
- Suppose $G(z)$ is allpass: $G(z) = \frac{z^{-N}A(z^{-1})}{A(z)}$

$$V(z) = X(z) - kV(z)G(z)z^{-1}$$

$$V(z) = \frac{1}{1+kG(z)z^{-1}}X(z)$$

$$Y(z) = kV(z) + V(z)G(z)z^{-1}$$

$$= \frac{k+G(z)z^{-1}}{1+kG(z)z^{-1}}X(z)$$



$$\frac{Y(z)}{X(z)} = \frac{k+G(z)z^{-1}}{1+kG(z)z^{-1}} = \frac{kA(z)+z^{-N-1}A(z^{-1})}{A(z)+kz^{-N-1}A(z^{-1})} \triangleq \frac{z^{-(N+1)}D(z^{-1})}{D(z)}$$

- Obtaining $\{d[n]\}$ from $\{a[n]\}$:

$$d[n] = \begin{cases} 1, & n = 0 \\ a[n] + ka[N-n+1], & 1 \le n \le N \\ k, & n = N+1 \end{cases}$$

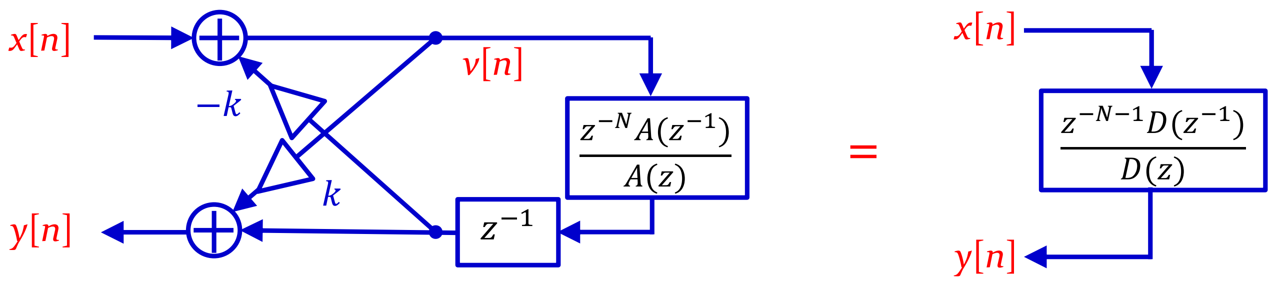
- Obtaining $\{a[n]\}$ from $\{d[n]\}$:

$$k = d[N+1], \qquad a[n] = \frac{d[n] - kd[N+1-n]}{1-k^2}$$

- If $G(z)$ is stable then $\frac{Y(z)}{X(z)}$ is stable if and only if $|k| < 1$

Example $A(z) \leftrightarrow D(z)$

- Suppose $N = 3, k = 0.5$ and $A(z) = 1 + 4z^{-1} - 6z^{-2} + 10z^{-3}$



- $A(z) \rightarrow D(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$A(z)$	1	4	−6	10	
$z^{-4}A(z^{-1})$		10	−6	4	1
$D(z)$	1	9	−9	12	0.5

$$D(z) = A(z) + kz^{-4}A(z^{-1})$$

$D(z) \rightarrow A(z)$

	z^0	z^{-1}	z^{-2}	z^{-3}	z^{-4}
$D(z)$	1	9	−9	12	0.5
$k = d[N+1]$					0.5
$z^{-4}D(z^{-1})$	0.5	12	−9	9	1
$A(z)$	1	4	−6	10	

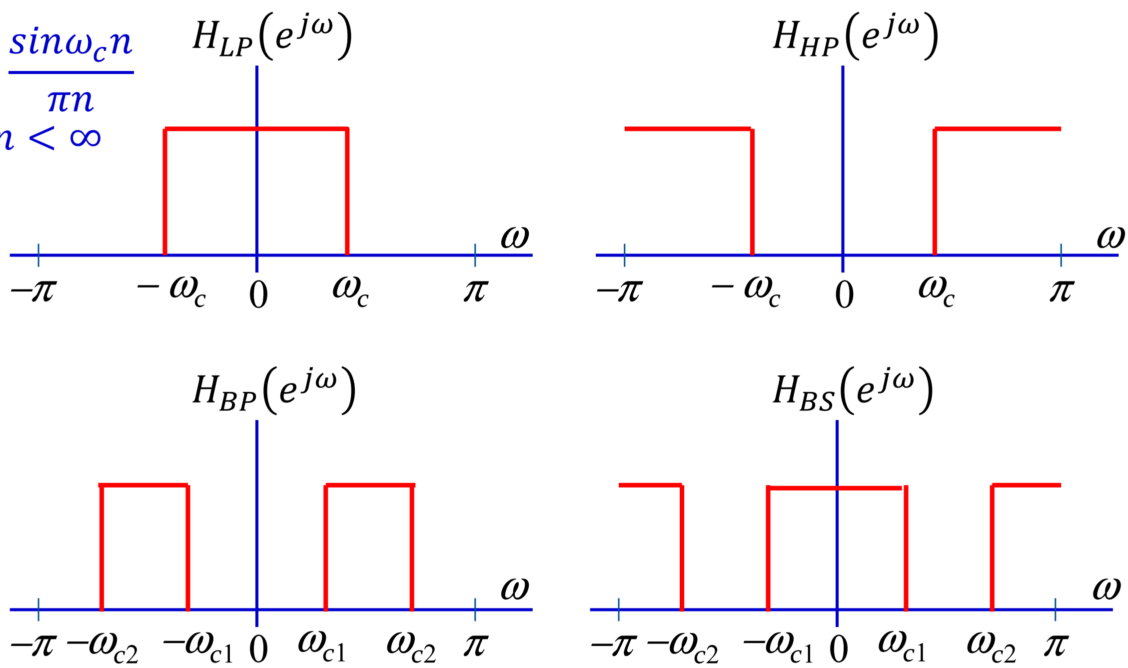
$$A(z) = \frac{D(z) - kz^{-4}D(z^{-1})}{1-k^2}$$

Lecture 10
Digital Filter Design

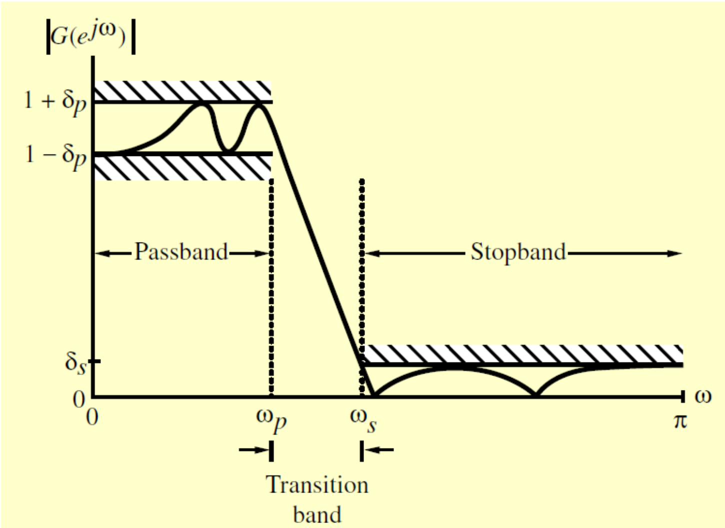
Filter Specifications

- Ideal but not practical specifications:

$$h_{LP}[n] = \frac{sin\omega_c n}{\pi n}$$
$$-\infty < n < \infty$$



Typical magnitude Specifications



- Passband edge: ω_p
- Stopband edge: ω_s
- Peak ripple value in passband: δ_p
- Peak ripple value in stopband: δ_s

- **Passband:** $\omega \le \omega_p,\ \ 1-\delta_p \le |G(e^{j\omega})| \le 1+\delta_p$
- **Stopband:** $\omega_s \le \omega \le \pi,\ \ |G(e^{j\omega})| \le \delta_s$
- **Transition band:** $\omega_p < \omega < \omega_s$, arbitrary response

Specifications Given as Loss function

- Loss Function
$$\mathcal{A}(\omega) = -20\log_{10}|G(e^{j\omega})|$$
- Peak passband ripple:
$$\alpha_p = -20\log_{10}(1-\delta_p), \text{ in dB}$$
- Minimum stopband attenuation
$$\alpha_s = -20\log_{10}(\delta_s), \qquad \text{in dB}$$

- **Example of ripples:** the peak passband ripple α_p and the minimum stopband attenuation α_s of a digital filter are, respectively, 0. 1 dB and 35dB. Determine their corresponding peak ripple values δ_p and δ_s.
- A: $\delta_p = 1 - 10^{-\frac{\alpha_p}{20}} = 1 - 10^{-0.005} = 0.0144690$

$$\delta_s = 10^{-\frac{\alpha_s}{20}} = 10^{-1.75} = 0.01778279$$

Obtain Band Edge Frequencies

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz, along with sampling frequencies
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_{\text{sampling}}} = \frac{2\pi F_p}{F_{\text{sampling}}} = 2\pi F_p T_{\text{sampling}}$$

$$\omega_s = \frac{\Omega_s}{F_{\text{sampling}}} = \frac{2\pi F_s}{F_{\text{sampling}}} = 2\pi F_p T_{\text{sampling}}$$

- **Example** – ECG signal typically exhibits frequencies in the range from 0.01 Hz to 150 Hz. Some studies are interested to low frequency range 0.03 Hz to 0.12 Hz, and high frequency range 0.12 Hz to 0.488 Hz. If the ECG signal is sampled at 300 Hz, what are the passband edges for filters to extract the corresponding signal? How if the sampling frequency is 200 Hz?
- A: Low frequency part:

$$\omega_{p1} = \frac{0.03\times 2\pi}{300} = 0.0002\pi,\ \omega_{p2} = \frac{0.12\times 2\pi}{300} = 0.0008\pi.$$

B: High frequency part:

$$\omega_{p1} = \frac{0.12\times 2\pi}{300} = 0.0008\pi,\ \omega_{p2} = \frac{0.488\times 2\pi}{300} = 0.00325\pi.$$

Selection of Filter Type

- **Considerations:**
  - Certainly, filters must be causal and stable.
  - Complexity is proportional to the filter length. The lower order the filter, the better.
  - For FIR filter, if linear phase is required, the coefficients must symmetric.
- **Advantages in using FIR filters:**
  - Always stable
  - Can be designed with exact linear phase
- **Disadvantages in using FIR filters:**
  - Usually need a higher order than IIR

IIR Filter Design

- Most common approach to IIR filter design -

  (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications

  (2) Determine the analog lowpass filter transfer function $H(S)$

  (3) Transform $H(S)$ into the desired digital transfer function $G(z)$

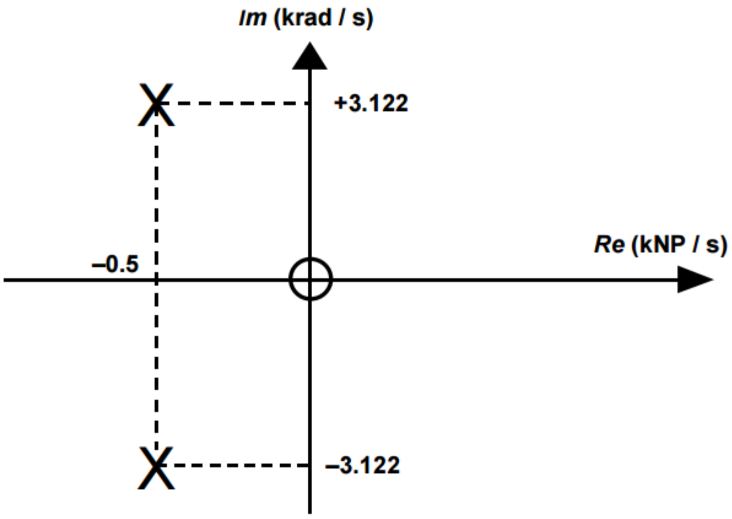
Analog Filter and *s*-plane

- The transfer function of analog filters are given by $H(s)$. The frequency response of $H(s)$ is evaluated at $s = j\Omega$.
- Example:

$$H(s) = \frac{10^3 s}{s^2 + 10^3 s + 10^7}$$

Factorizing the equation gives:

$$H(s) = \frac{10^3 s}{[s - (-0.5 + j3.122) \times 10^3][s - (-0.5 - j3.122) \times 10^3]}$$

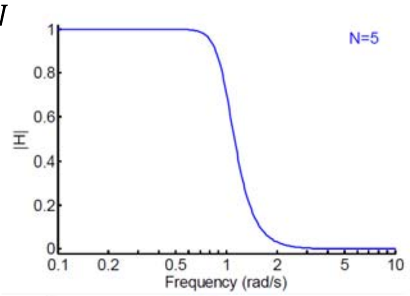


- A zero at origin and two poles
- $H(j\Omega) = \dfrac{10^3 j\Omega}{[j\Omega - (-0.5 + j3.122)\times 10^3][j\Omega - (-0.5 - j3.122)\times 10^3]}$

Analog Filters

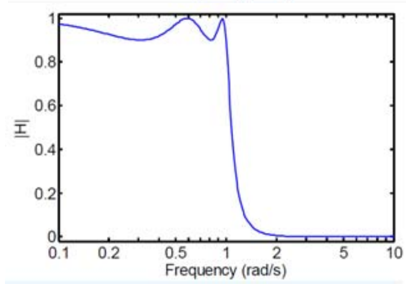
Assume 3-dB cutoff frequency $\Omega_c = 1$

- Classical continuous-time filters optimize tradeoff:
  - passband ripple vs. stopband ripple vs. transition width
- Butterworth: $G_a^2(\Omega) = |H_a\,(\Omega)|^2 = \frac{1}{1+\Omega^{2N}}$
  - Monotonic for all Ω
  - $G_a(\Omega) = 1 - \frac{1}{2}\Omega^{2N} + \frac{3}{8}\Omega^{4N} + \cdots$
  - "Maximally flat", $2N - 1$ derivatives are 0



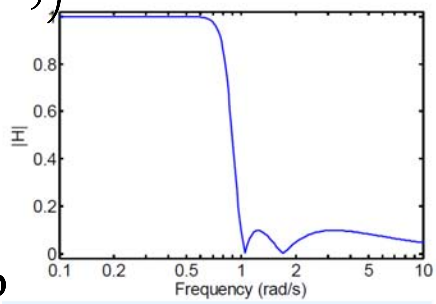
- Chebyshev: $G_a^2(\Omega) = \frac{1}{1+\epsilon^2 T_N^2(\Omega)}$

  where polynomial $T_N\ (\cos x) = \cos Nx$
  - Passband equiripple + very flat at ∞

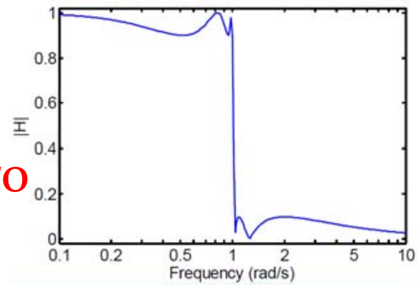


Analog Filters

- Inverse Chebyshev: $G_a^2(\Omega) = \frac{1}{1+\left(\epsilon^2 T_N^2(\Omega^{-1})\right)^{-1}}$
  - Stopband equiripple + very flat at 0



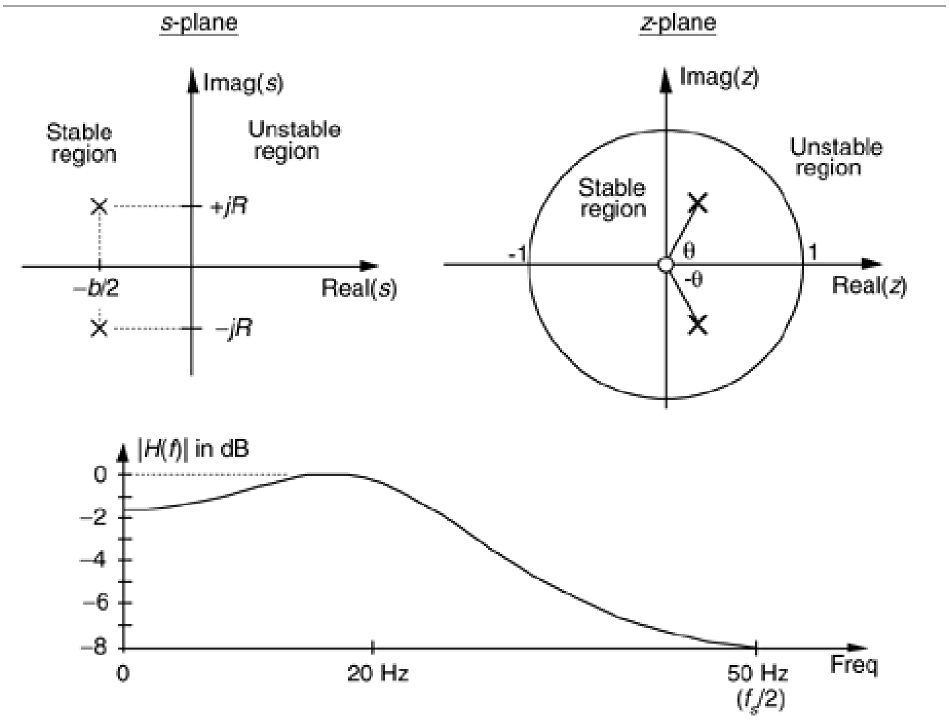
- Elliptic (No nice formula)
  - Very steep + equiripple in pass and stop bands



- There are explicit formulae for pole/zero positions.

Mapping *s*-plane to *z*-plane

- Basic idea behind the conversion of $H(s)$ into $G(z)$ is to apply a mapping from the s-domain to the z-domain so that essential properties of the analog frequency response are preserved
- Thus mapping function should be such that
  - Imaginary ($j\Omega$) axis in the s-plane be mapped onto the unit circle of the z-plane (to **preserve the frequency selective properties**)
  - Left-half of the s-plane be mapped inside the unit circle (to **ensure a stable digital transfer function**).



Bilinear Transformation

$$s = k\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right), k > 0, \qquad \text{or } z = \frac{k + s}{k - s}$$

- Thus, relation between $G(z)$ and $H(s)$ is then given by

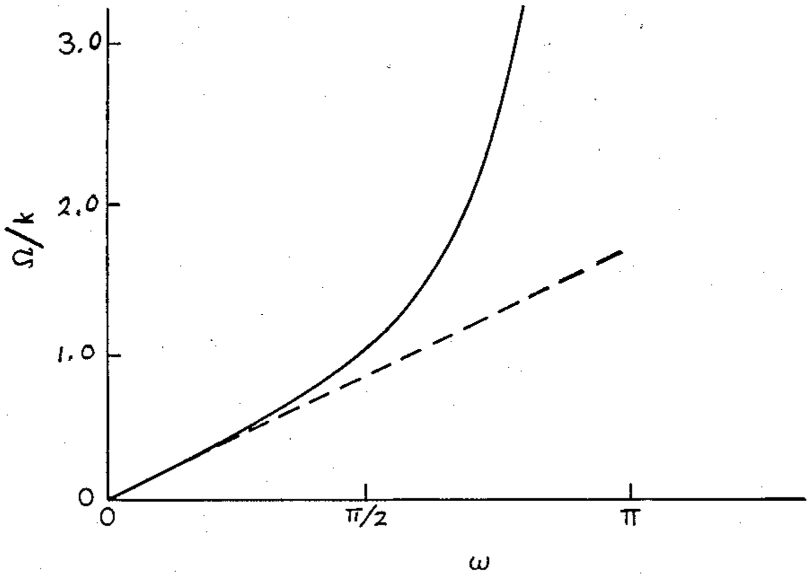
$$G(z) = H(s)\Big|_{s=k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

- When $s = j\Omega$, $z = \frac{k+j\Omega}{k-j\Omega} = \frac{\sqrt{k^2+\Omega^2}e^{j\tan^{-1}\frac{\Omega}{k}}}{\sqrt{k^2+\Omega^2}e^{-j\tan^{-1}\frac{\Omega}{k}}} = e^{2j\tan^{-1}\frac{\Omega}{k}}$

$\therefore |z| = 1 \Longrightarrow z = e^{j\omega}$, and

$$\omega = 2\tan^{-1}\frac{\Omega}{k}, \text{ or } \Omega = k\tan\left(\frac{\omega}{2}\right)$$

$$\omega = 2\tan^{-1}\frac{\Omega}{k}, \text{ or } \Omega = k\tan\left(\frac{\omega}{2}\right)$$



The relation between analog and digital frequency scales for the bilinear transformation.

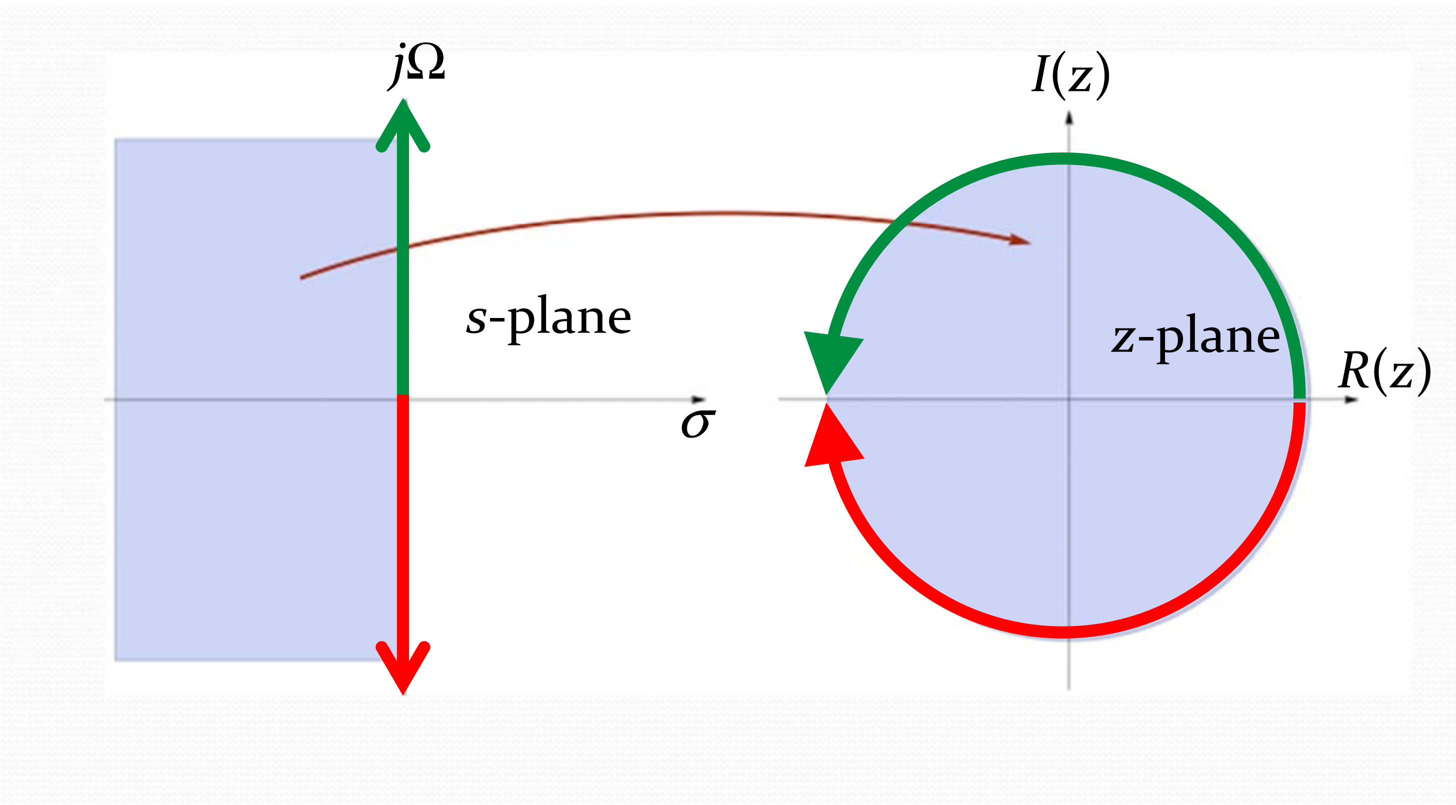
- When $\Omega = 0$, $\omega = 0$. When $\Omega \to \infty$, $\omega \to \pi$.
- Hence, the $j\Omega$-axis is mapped into the unit circle.

- For $s = \sigma_0 + j\Omega_0$, $\sigma_0 < 0$

$$z = \frac{(k+\sigma_0) + j\Omega_0}{(k-\sigma_0) - j\Omega_0} \Longrightarrow |z| = \sqrt{\frac{(k+\sigma_0)^2 + {\Omega_0}^2}{(k-\sigma_0)^2 + {\Omega_0}^2}} < 1$$

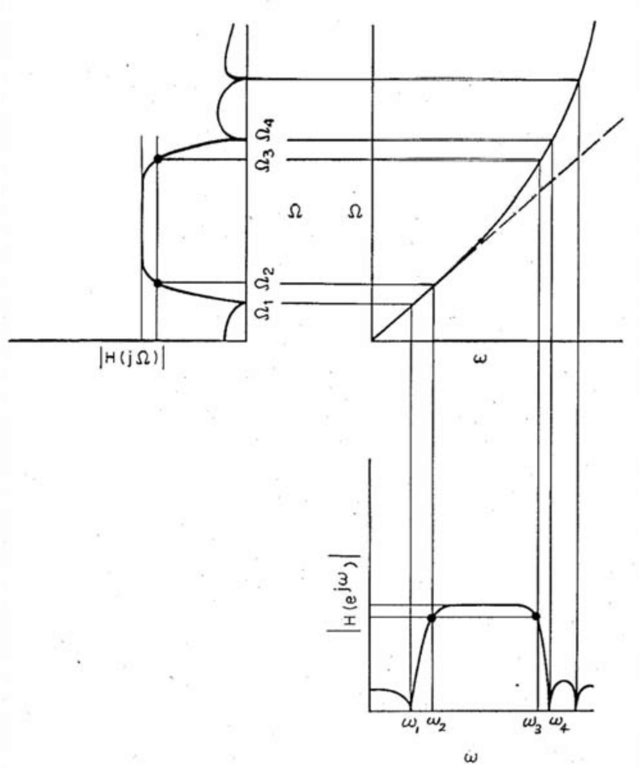
i.e., the left-half s-plane is mapped inside the unit circle.

- Mapping is highly nonlinear
  - Complete negative imaginary axis in the s-plane from $\Omega = 0$ to $\Omega = -\infty$ is mapped into the lower half of the unit circle in the z-plane from $z = 1$ to $z = -1$
  - Complete positive imaginary axis in the s-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the z-plane from $z = 1$ to $z = -1$



Frequency Warping

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping**
- • Effect of warping



Design Steps

- **Step 1:** Develop the specifications of $H(s)$ by applying the inverse bilinear transformation to specifications of $G(z)$
- **Step 2**: Design $H(s)$
- **Step 3:** Determine $G(z)$ by applying bilinear transformation to $H(s)$

Example: (About Step 3)

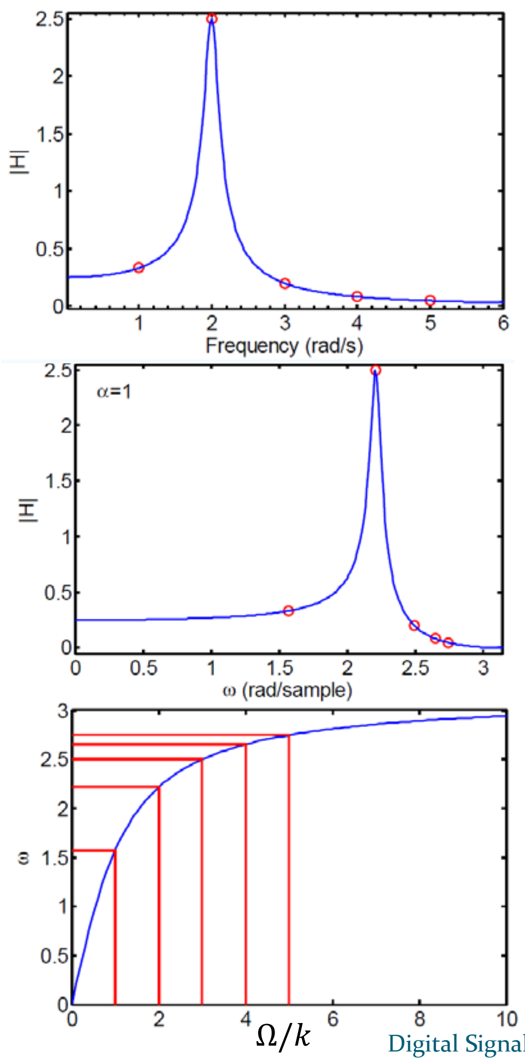
- Transform $H(s) = \frac{1}{s^2+0.2s+4}$ to digital filter
- A: Choose $k = 1$. Substitute $s = k\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ to $H(s)$

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.2\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 4}$$

$$= \frac{(1+z^{-1})^2}{(1-z^{-1})^2 + 0.2(1-z^{-1}) + 4(1+z^{-1})^2}$$

$$= 0.19\frac{1+2z^{-1}+z^{-2}}{1+1.15z^{-1}+0.92z^{-2}}$$

Extra zeros at z=−1



- Frequency response is identical (both magnitude and phase) but with a distorted frequency axis
- Frequency mapping:

$$\omega = 2\tan^{-1}\frac{\Omega}{k}$$

  - $\Omega = [k, 2k, 3k, 4k, 5k]$ $\rightarrow \omega = [1.6, 2.2, 2.5, 2.65, 2.75]$
- Choices of k
  - Set $k = \frac{\Omega_0}{\tan\frac{\omega_0}{2}}$ to map $\Omega_0 \rightarrow \omega_0$
  - Set $k = 2f_s = \frac{2}{T}$ to map low frequencies to themselves.

Example: Butterworth Lowpass

- Transform $H_a(s) = \frac{1}{s+1}$ into a lowpass digital filter transfer function with 32 kHz sampling frequency and 4kHz 3dB frequency. (General form: $H_a(s) = \frac{\Omega_c}{s+\Omega_c}$)
- A: $H_a(j\Omega) = \frac{1}{j\Omega+1}$,

  $\therefore$ the 3dB frequency of the analog filter is $\Omega = 1$

  It is given that the 3dB frequency of digital filter is at

$$\omega = 2\pi \frac{4k\text{Hz}}{32k\text{Hz}} = \frac{\pi}{4}$$

- Use the bilinear transformation

$$s \to k\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right),$$

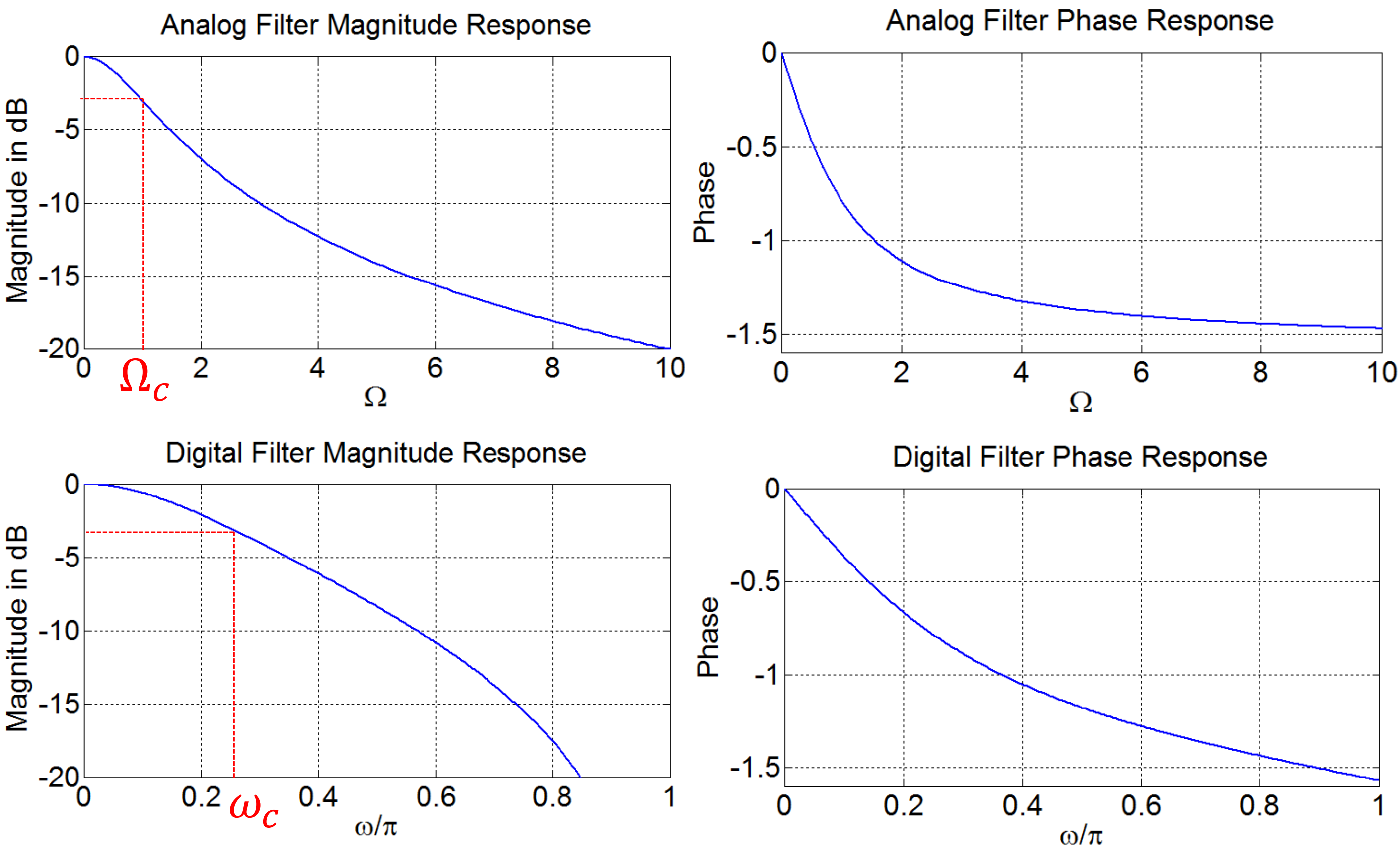
  where k is given by

$$\Omega = k \tan\left(\frac{\omega}{2}\right)$$

- Hence, we have $1 = k \tan\left(\frac{\pi}{8}\right)$

  $\therefore k = 2.414$ and

$$H(z) = \frac{1}{1 + 2.414\frac{1 - z^{-1}}{1 + z^{-1}}}$$



Determine Analog Filter Specifications (Step 1)

- **Objective:**
  - from $\{\omega_p, \omega_s, \delta_p, \delta_s\}$ to determine $\{\Omega_p, \Omega_s, \delta_p, \delta_s\}$
- **Solution 1:**
  - We can arbitrary choose Ω_p, for example to be Ω_0, because by choosing $k = \frac{\Omega_0}{\tan\frac{\omega_p}{2}}$ we can map ω_p to any chosen Ω_0.
  - Using the chosen k to determine Ω_s, i.e.,

$$\Omega_s = \frac{\Omega_0}{\tan\frac{\omega_p}{2}} \tan\left(\frac{\omega_s}{2}\right)$$

- **Solution 2:**
  - We can arbitrary choose k, for example choose $k = 1$.
  - Using the chosen k to determine Ω_p and Ω_s, i.e.,

$$\Omega_p = k \tan\left(\frac{\omega_p}{2}\right)$$
$$\Omega_s = k \tan\left(\frac{\omega_s}{2}\right)$$

Example (About Step 1)

- A lowpass digital filter is supposed to pass the frequency components with frequencies lower than 4kHz, and block the frequency components with frequencies higher than 8kHz, assuming that the sampling frequency is 32 kHz. Determine the frequency bandedges for an analog prototype filter when bilinear transform is used for the digital filter design. (Using solution 1)
- A: $\omega_p = \frac{4k}{32k} 2\pi = 0.25\pi$, $\omega_s = \frac{8k}{32k} 2\pi = 0.5\pi$

  We arbitrary choose $\Omega_p = 1$ radian/second $\Longrightarrow k = 2.414$

  Thus, $\Omega_s = 2.414 \tan\left(\frac{0.5\pi}{2}\right) = 2.414$ radian/second

Spectral Transformation

- **Lowpass to lowpass transformation:**
- We can transform z-plane to change the cutoff frequency by substituting

$$z^{-1} = \frac{\hat{z}^{-1} - \alpha}{1 - \alpha \hat{z}^{-1}} \Leftrightarrow \hat{z}^{-1} = \frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$$

  - where z^{-1} and $\hat{z}^{-1}$ denote the unit delay in the prototype lowpass digital filter and the transformed filter.
- Frequency mapping
  - If $z = e^{j\omega}$, $\hat{z} = \frac{1+\alpha e^{-j\omega}}{e^{-j\omega}+\alpha} \Longrightarrow |\hat{z}| = 1$. Hence, the unit circle is preserved.

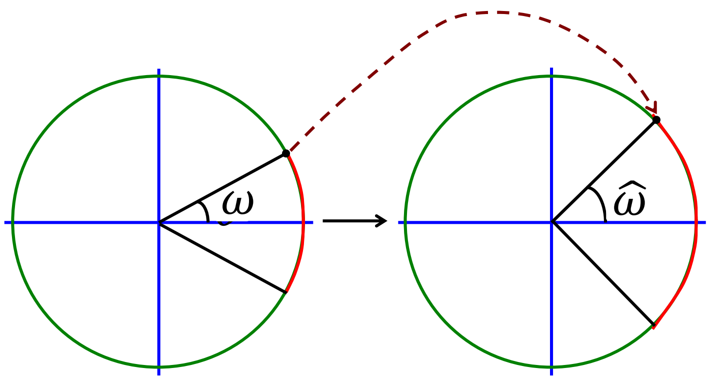
- On unit circle

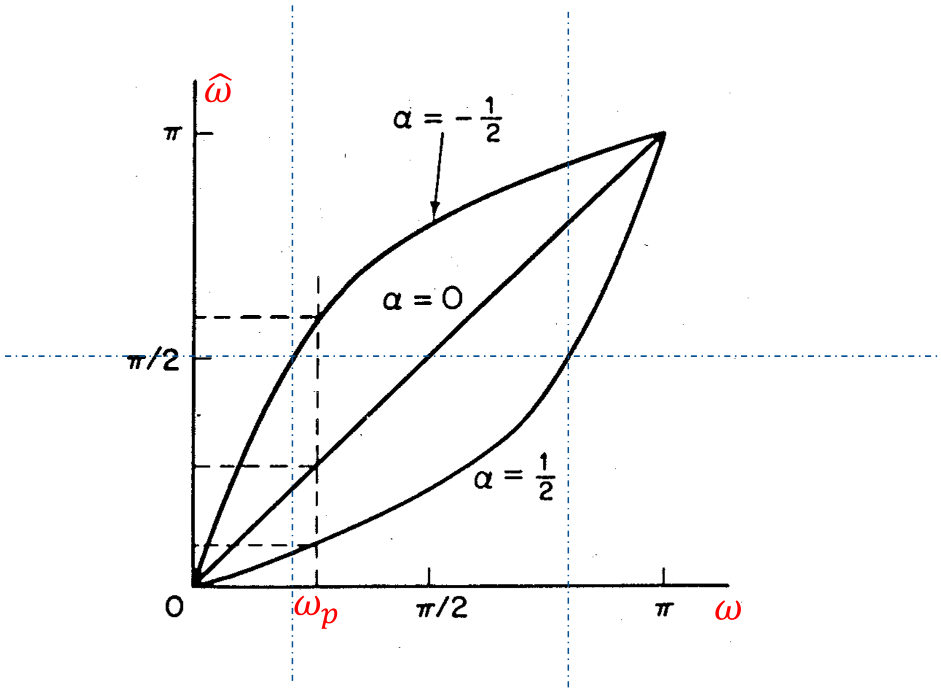
$$e^{-j\omega} = \frac{e^{-j\widehat{\omega}} - \alpha}{1 - \alpha e^{-j\widehat{\omega}}}$$

from which we arrive at

$$\tan\left(\frac{\omega}{2}\right) = \left(\frac{1+\alpha}{1-\alpha}\right)\tan\left(\frac{\widehat{\omega}}{2}\right)$$

$$\alpha = \frac{\sin\left(\frac{\omega - \widehat{\omega}}{2}\right)}{\sin\left(\frac{\omega + \widehat{\omega}}{2}\right)}$$

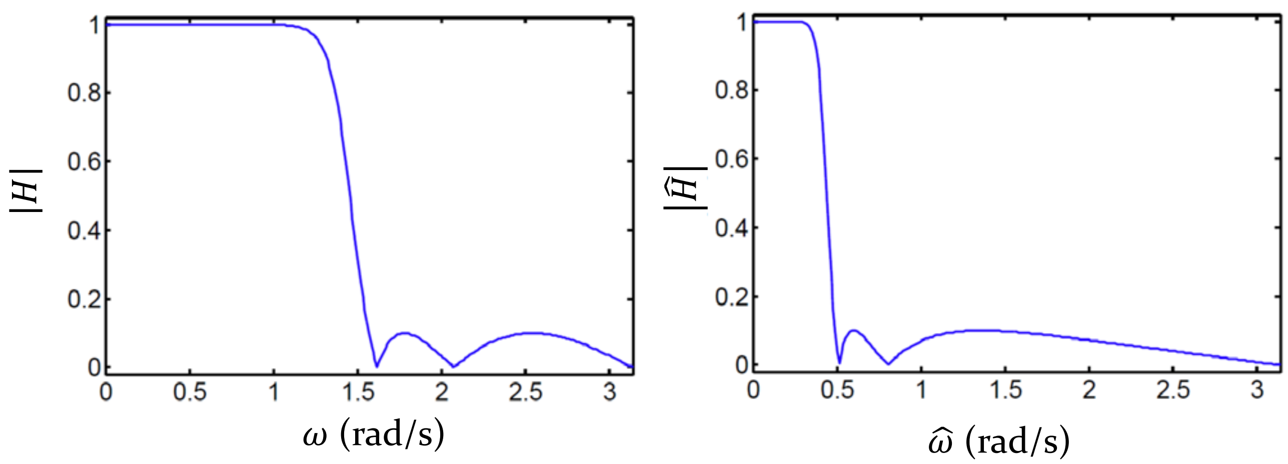




Warping of the frequency scale in
Low-pass – low-pass transformation.

Example

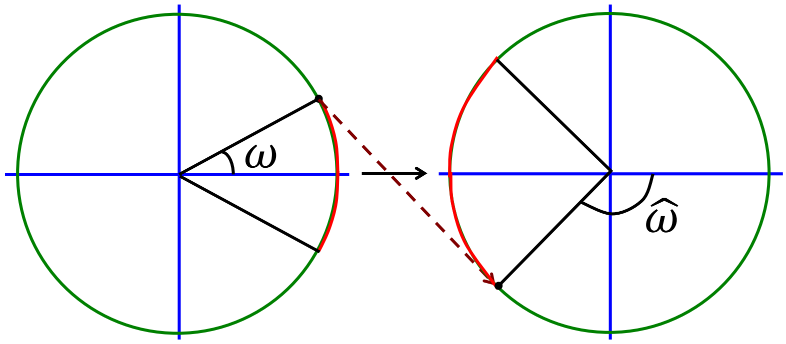
- A 5-th order inverse Chebyshev:
- $\omega_0 = \frac{\pi}{2} = 1.57 \xrightarrow{\alpha=0.6} \widehat{\omega}_0 = 0.49$



- **Lowpass to Highpass Transformation**

$$z^{-1} = -\frac{\hat{z}^{-1} + \alpha}{1 + \alpha\hat{z}^{-1}} \quad \Longrightarrow \quad \alpha = -\frac{\cos\left(\dfrac{\omega - \widehat{\omega}}{2}\right)}{\cos\left(\dfrac{\omega + \widehat{\omega}}{2}\right)}$$

$$\operatorname{ctan}\left(\frac{\omega}{2}\right) = \left(\frac{-1 + \alpha}{1 + \alpha}\right)\tan\left(\frac{\widehat{\omega}}{2}\right)$$



FIR Digital Filter Design

- A causal FIR transfer function $H(z)$ of length N is a polynomial in z^{-1} of degree $N-1$:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$$

- The corresponding frequency response is given by

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} h[n]e^{-j\omega n}$$

- $h[n] = \pm h[N-1-n]$ is enforced to ensure a linear phase design.

Basic Approaches

- Windowed Fourier series approach
- Frequency sampling approach
- Computer based digital filter design method

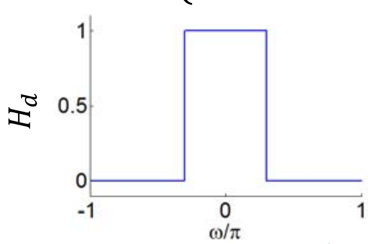
Window Method

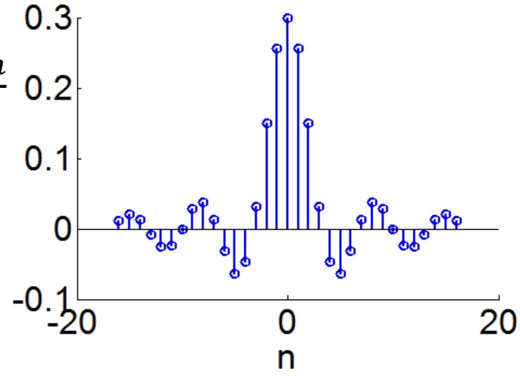
- Inverse DTFT
  - For any BIBO stable filter, $H(e^{j\omega})$ is the DTFT of $h[n]$

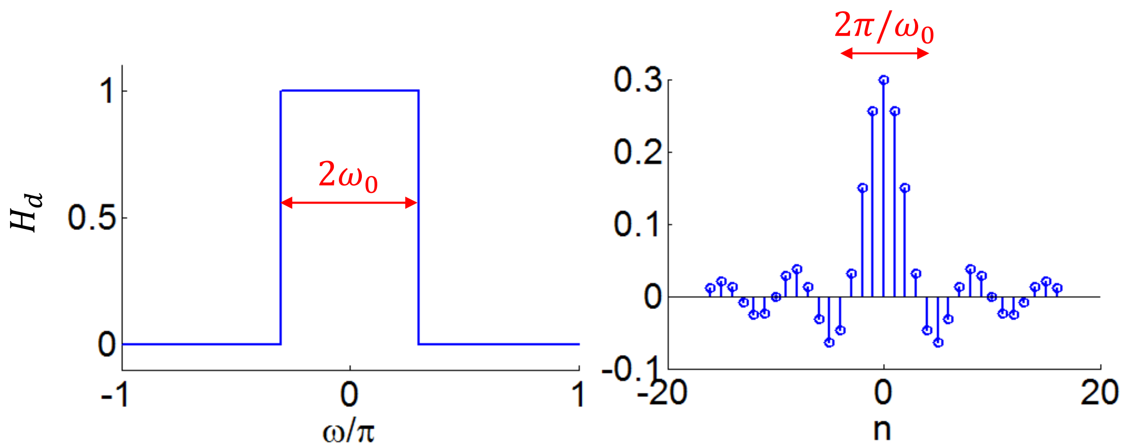
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \Leftrightarrow h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

  - If we know $H(e^{j\omega})$ exactly, the IDTFT gives the ideal $h[n]$
- Example: ideal lowpass filter

$$H_d(e^{j\omega}) = \begin{cases} 1 & |\omega| \le \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases} \Leftrightarrow h_d[n] = \frac{\sin \omega_0 n}{\pi n}$$







- **Note:** Width in ω is $2\omega_0$, and width in n is $\frac{2\pi}{\omega_0}$
  - Product is 4π always
- Sadly $h_d[n]$ is infinite and non-causal.
- **Solution:** Multiply $h_d[n]$ by a window

Rectangular Window

- Truncate to $\pm M$ to make finite; $h[n]$ is now of length $2M+1$.
- Mean square error (MSE) Optimality:
  - Define MSE in frequency domain

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega$$

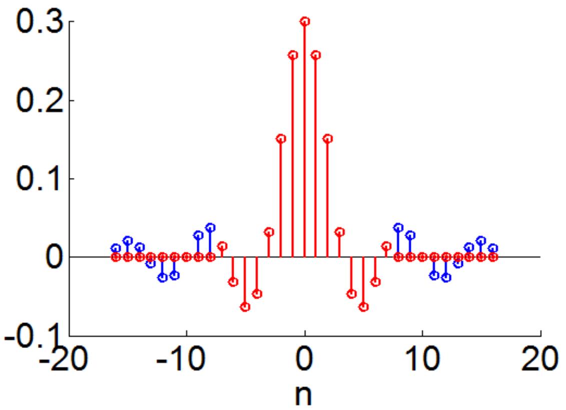
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H_d(e^{j\omega}) - \sum_{n=-M}^{M} h[n]e^{-j\omega n}\right|^2 d\omega$$

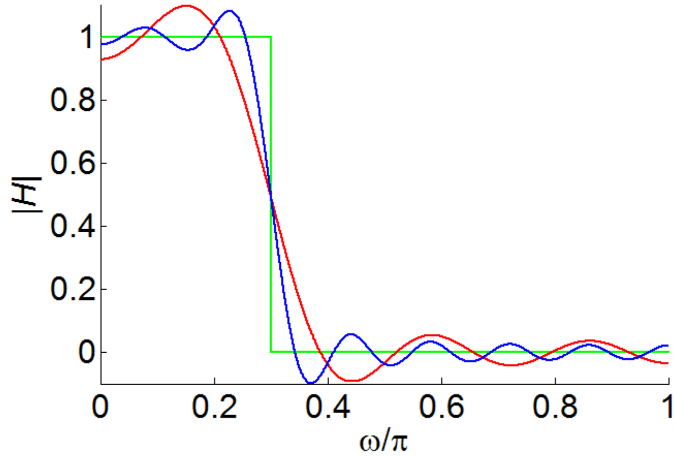
  - Minimum E is when $h[n] = h_d[n]$ for $-M \le n \le M$

- Proof: From Parseval:

$$E = \sum_{n=-M}^{M} |h_d[n] - h[n]|^2 + \sum_{|n|>M} |h_d[n]|^2$$

- However, 9% overshoot at a discontinuity even for large M.

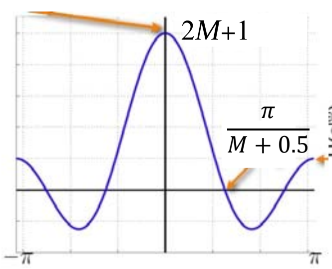


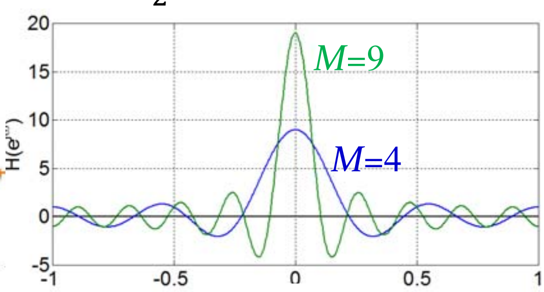


- Normal to delay by M to make causal. Multiply $H(e^{j\omega})$ by $e^{-jM\omega}$

Gibbs Phenomenon

- Truncation
  - $\Leftrightarrow$ Multiply $h_d[n]$ by a rectangular window $w_R[n] = \sum_{k=-M}^{M} \delta[n-k]$ (in time domain)
  - $\Leftrightarrow$ Convolution $H_{2M+1}(e^{j\omega}) = \frac{1}{2\pi} H_d(e^{j\omega}) \circledast W_R(e^{j\omega})$ (in frequency domain)
  - $W_R(e^{j\omega}) = \frac{\sin\frac{(2M+1)\omega}{2}}{\sin\frac{\omega}{2}}$, Width of mainlobe is $\frac{4\pi}{2M+1}$

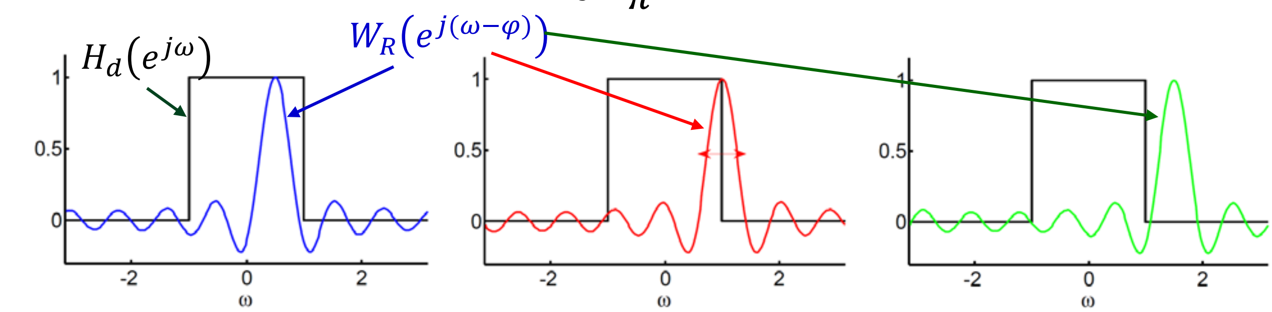


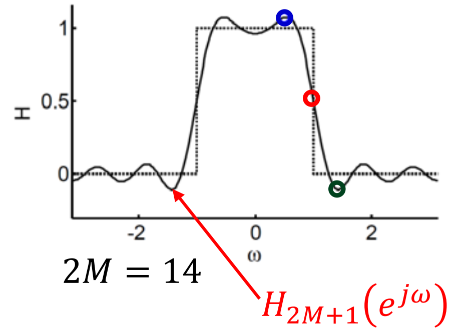


When M increase, the width of the main lobe decreases, but the height increases.

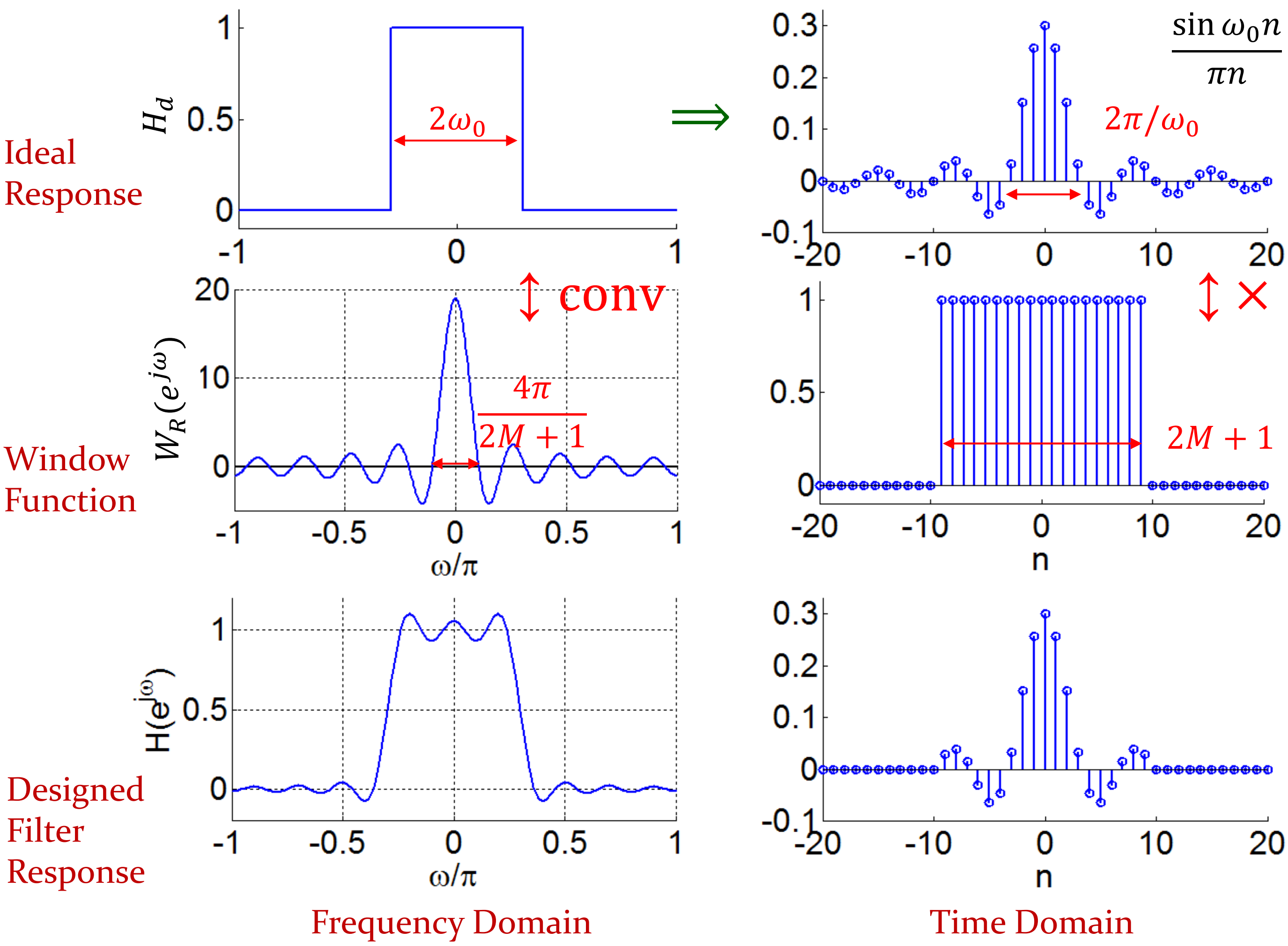
- Effects: convolve ideal frequency response with periodic sinc function.

$$H_{2M+1}\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d\left(e^{j\varphi}\right)W_R\left(e^{j(\omega-\varphi)}\right)d\varphi$$



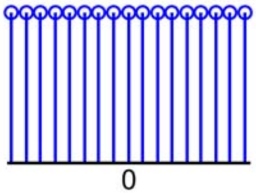


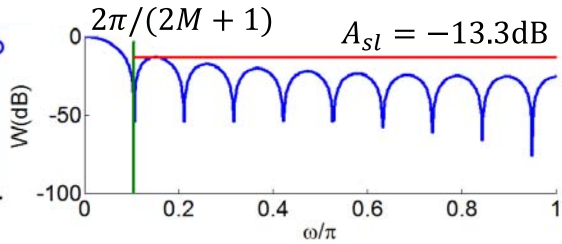
- The convolution is the area under the curve of the product of the two waveforms.
- When M increases, ripples in $H_{2M+1}\left(e^{j\omega}\right)$ around the cutoff frequency occur closely but with no decrease in amplitude.



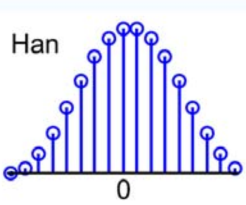
Fixed Windows

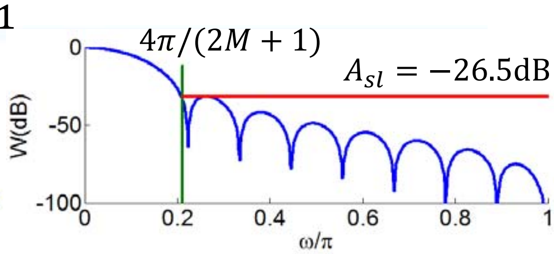
- Consider length $N = 2M + 1$ windows, for $-M \le n \le M$
- Rectangle: $w_R[n] \equiv 1$
  Don't Use





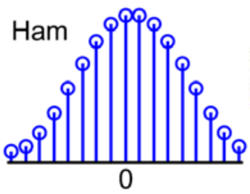
- Hanning: $w_{Hann}[n] = 0.5 + 0.5c_1$
  $c_k = \cos\dfrac{2k\pi n}{N}$
  Rapid sidelobe decay

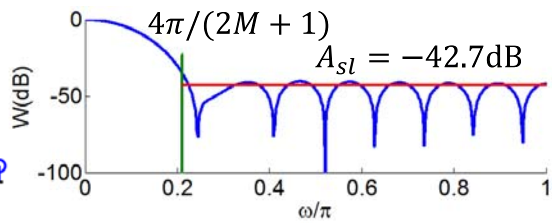




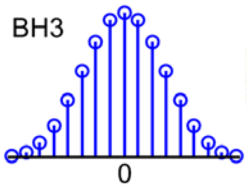
Fixed Windows (cont'd)

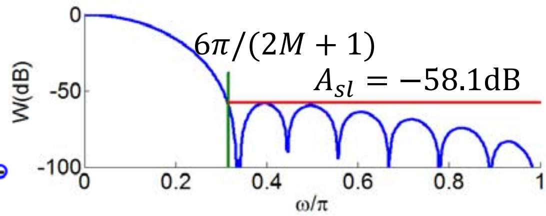
- Hamming: $w_{Hamm}[n] = 0.54 + 0.46c_1$
  Best peak sidelobe



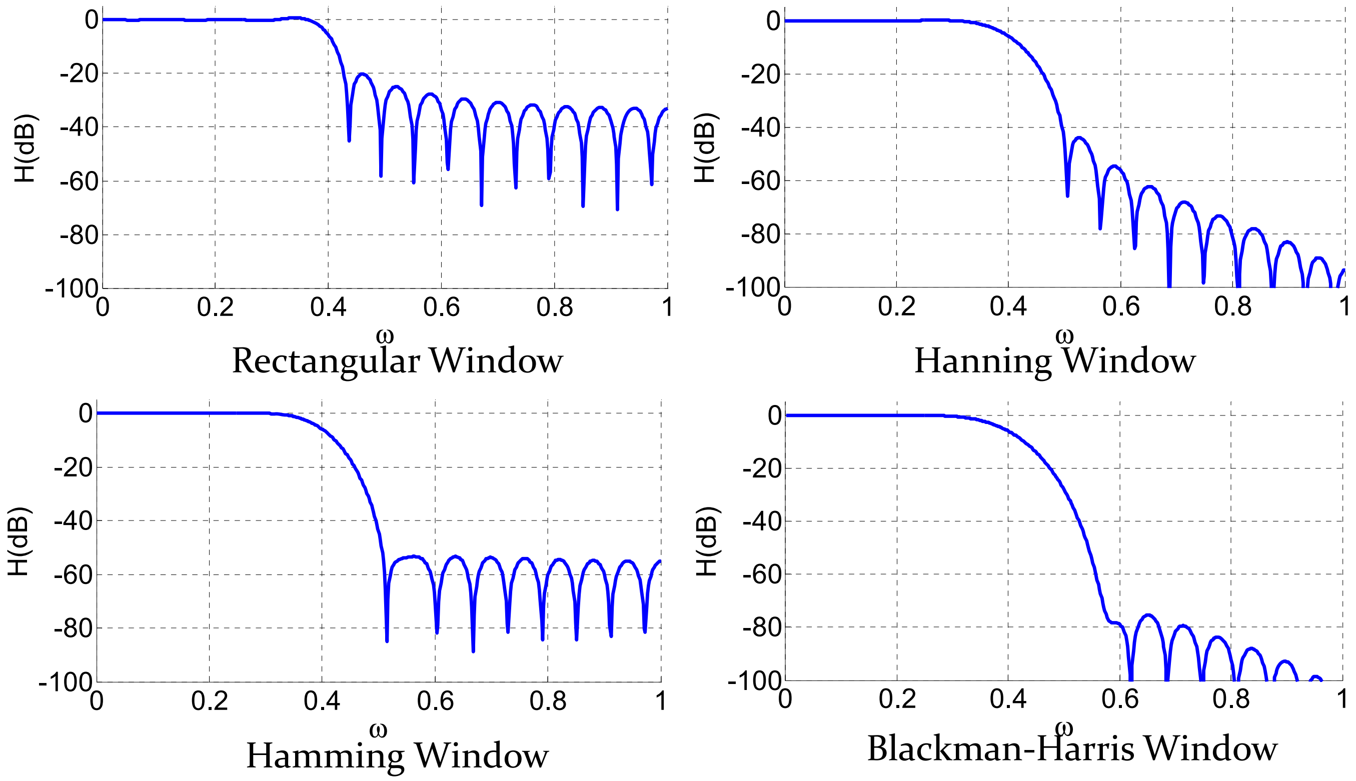


- Blackman-Harris 3 terms:
  $w_{BH}[n] = 0.42 + 0.5c_1 + 0.08c_2$
  Best peak sidelobe

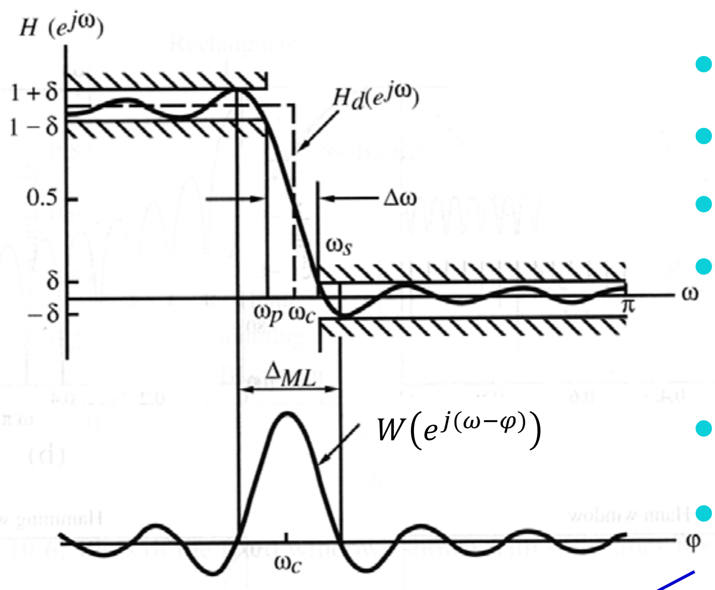




Lowpass Filters designed using



Frequency Domain Relation



- $H\left(e^{j(\omega_c+\omega)}\right) + H\left(e^{j(\omega_c-\omega)}\right) \cong 1$
- $H\left(e^{j\omega_c}\right) \cong 0.5$
- $\delta_p = \delta_s = \delta$
- Distance between the maximum passband deviation and the minimum value $\cong \Delta_{ML}$
- $\Delta\omega = \omega_s - \omega_p < \Delta_{ML}$
- A narrow transition band $\xrightarrow{\text{require}}$ small main lobe width
- Small $\delta \xrightarrow{\text{require}}$ small area under sidelobes

Contradictory requirements

Properties of Fixed Windows

Type of Window	Window function		Resultant Filter	
	Main Lobe Width Δ_{ML}	Relative Side-lobe Level A_{sl}	Minimum Stop-band Attenuation δ	Transition Bandwidth $\Delta\omega$
Rectangular	$\frac{4\pi}{2M+1}$	13.3dB	20.9dB	$\frac{0.92\pi}{M}$
Hanning	$\frac{8\pi}{2M+1}$	31.5dB	43.9dB	$\frac{3.11\pi}{M}$
Hamming	$\frac{8\pi}{2M+1}$	42.7dB	54.5dB	$\frac{3.32\pi}{M}$
Blackman-Harris	$\frac{12\pi}{2M+1}$	58.1dB	75.3dB	$\frac{5.56\pi}{M}$

- δ is independent from M, or ω_c, and is essentially constant.
- $\Delta\omega = \frac{c}{M}$

Summary of Window Method

- Factors determining the performance of the designed filters:
  - Ideal (lowpass) filter, $\frac{\sin \omega_0 n}{\pi n}$, determine the cutoff frequency.
  - Window type determines stopband band attenuation.
  - Window length determines transition width.
- Design Step:
  - Determine ideal impulse response $h[n]$
  - Select window type
  - Determine window length $N = 2M + 1$, $\Delta\omega = c/M$
  - Determine window function $w[n]$
  - Time domain multiplication $h[n] \cdot w[n]$
  - Recover causality of the filter $\boxed{h[n-M] \cdot w[n-M]}$

Design Example

- Using Window method to design a lowpass filter with passband edge $\omega_p = 0.3\pi$, stopband edge $\omega_s = 0.5\pi$, minimum stopband attenuation $\alpha_s = 40\text{dB}$.
- A: $\alpha_s = 40\text{dB}$. Hanning, Hamming and Blackman-Harris window meet the requirement.

$$\omega_c = \frac{\omega_p+\omega_s}{2} = 0.4\pi, \text{ and } \Delta\omega = \omega_s - \omega_p = 0.2\pi$$

Hanning has the minimum length:

$$M = \left\lceil \frac{3.11\pi}{0.2\pi} \right\rceil = 16. \text{ Window length } N = 2M+1 = 33$$

$$\text{So, } w_{Hann}[n] = 0.5 + 0.5\cos\frac{2\pi n}{33} \Longrightarrow h[n] = h_d[n]w_{Hann}[n]$$

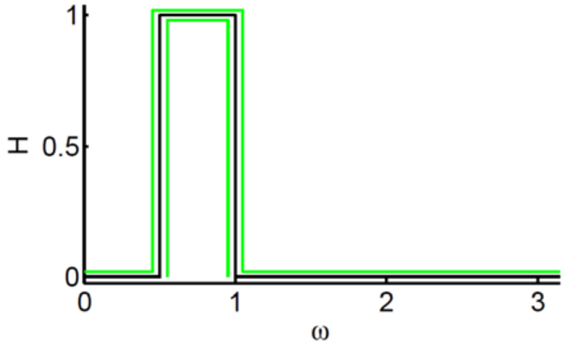
Adjustable Window

- Kaiser: $w_K[n] = \dfrac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)}$, where $I_0(\mu) = 1 + \sum_{r=1}^{\infty}\left[\dfrac{\left(\frac{\mu}{2}\right)^r}{r!}\right]^2$
  - β control minimum attenuation $\alpha_s = -20\log_{10}\delta_s$ in the stopband
  - Good compromise: width vs. sidelobe vs. decay
  - Estimation of β and filter length N from α_s and $\Delta\omega$:

$$\beta = \begin{cases} 0.1102(\alpha_s - 8.7), & \text{for } \alpha_s > 50 \\ 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21), & \text{for } 21 \le \alpha_s \le 50 \\ 0, & \text{for } \alpha_s < 21 \end{cases}$$

$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)} + 1$$

Design Example



- Bandpass: $\omega_{c1} = 0.5, \omega_{c2} = 1,$
  $\Delta\omega = 0.1, \delta_p = \delta_s = 0.02$
- A: $\alpha_s = -20\log_{10}\delta_s = 34\text{dB}$

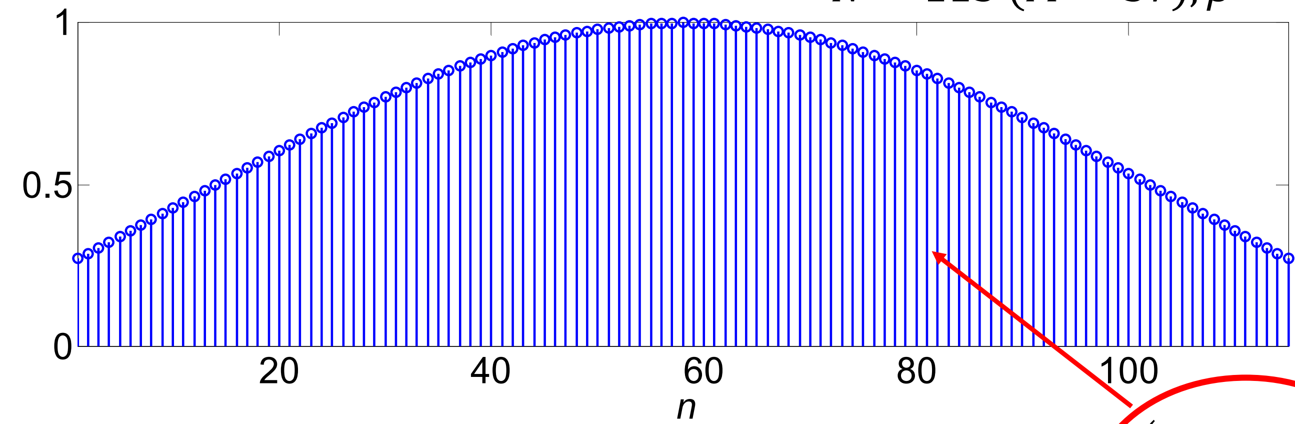
$$N = \frac{\alpha_s - 8}{2.285(\Delta\omega)} + 1 = 114 \cong 115$$

- Ideal impulse response: Difference of two lowpass filters

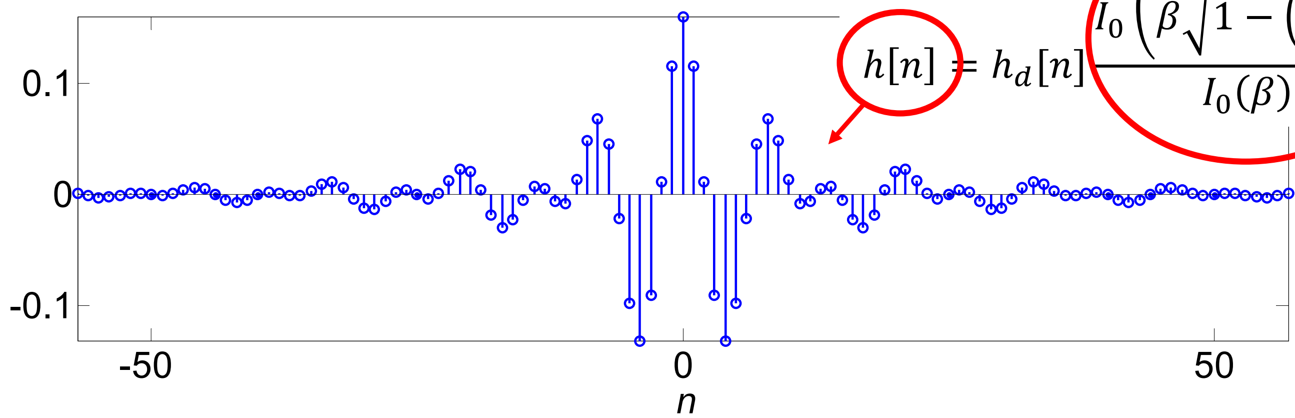
$$h_d[n] = \frac{\sin\omega_{c2}n}{\pi n} - \frac{\sin\omega_{c1}n}{\pi n}$$

- $\beta = 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) = 2.655$

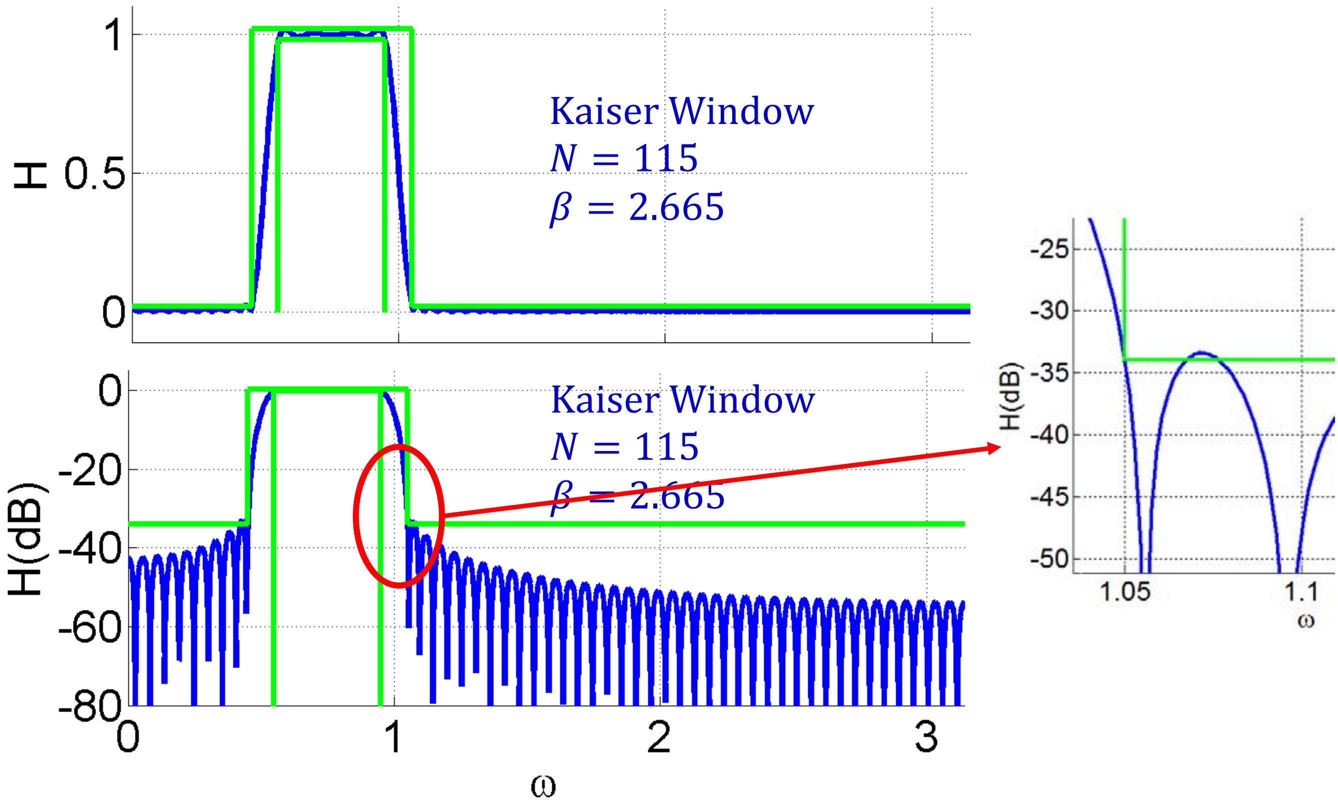
Kaiser Window $N = 115\ (M = 57), \beta = 2.665$



Impulse response



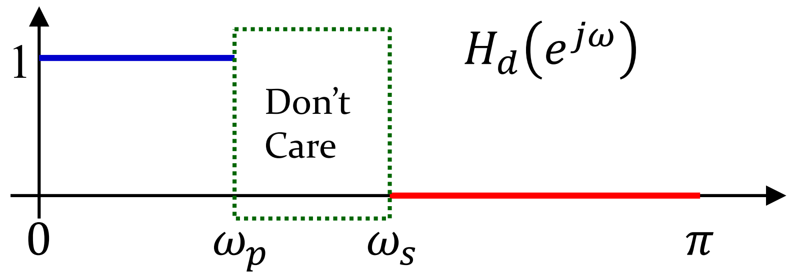
$$h[n] = h_d[n]\frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right)}{I_0(\beta)}$$



Optimal Filter Design

- Window method
  - Design Filters heuristically using windowed sinc functions
- Optimal design
  - Design a filter $h[n]$ with $H(e^{j\omega})$
  - Approximate $H_d(e^{j\omega})$ with some optimality criteria, or satisfies specs.

Optimality



- **Least Squares:**

$$\text{Minimize} \int_{\omega\in\text{care}} \left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

- A variation - **Weighted Least Squares:**

$$\text{Minimize} \int_{\omega\in\text{care}} W(\omega)\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

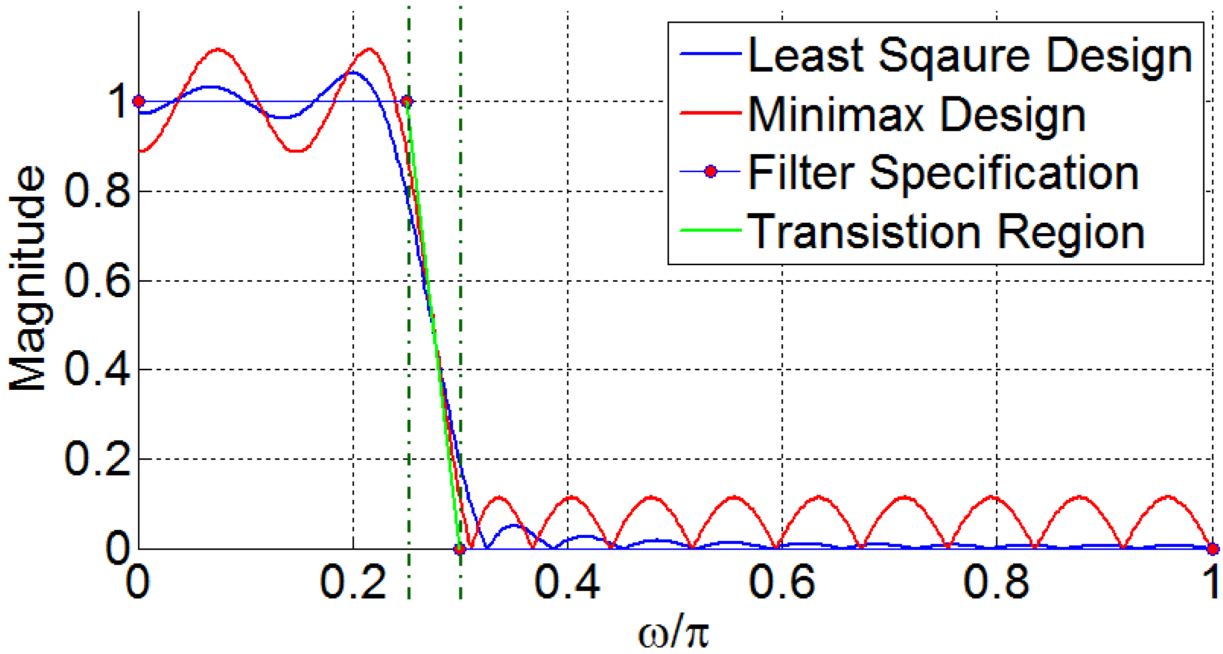
- **Chebychev Design (min-max)**

$$\text{Minimize}_{\omega\in\text{care}}\ \max\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|$$

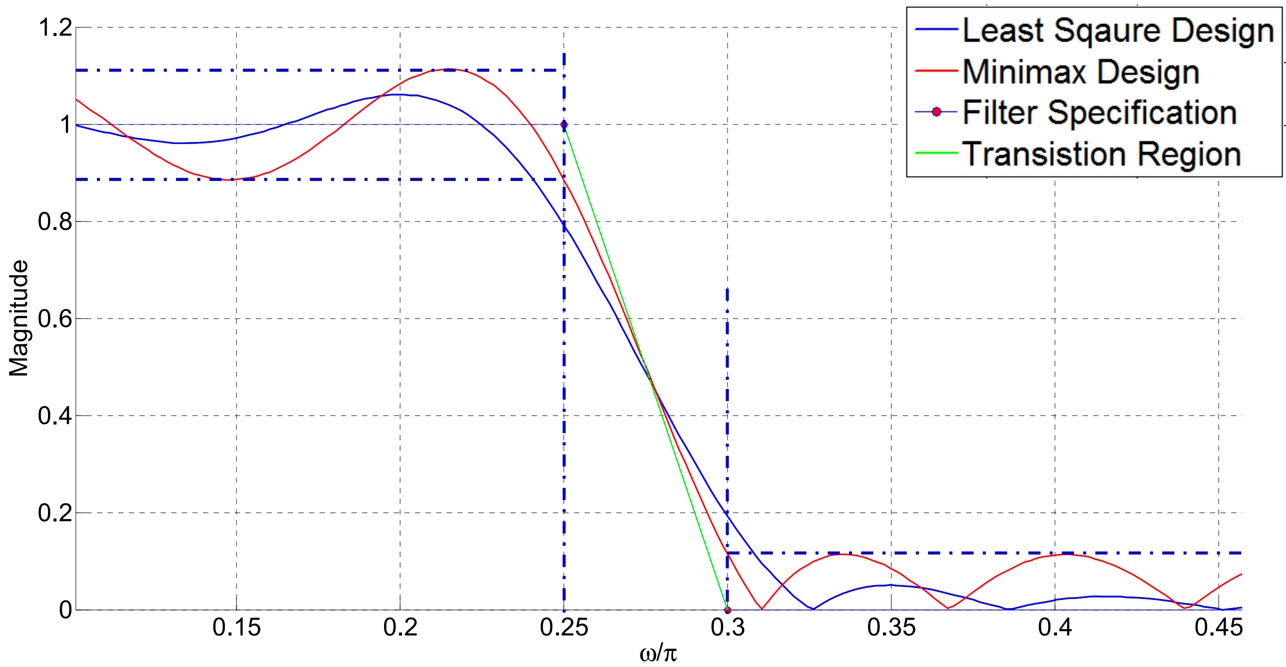
- Weighted Minimax Design

$$\text{Minimize}_{\omega\in\text{care}}\ \max\left|W(\omega)\left(H(e^{j\omega}) - H_d(e^{j\omega})\right)\right|$$

Least Square vs. Minimax



Zoom-in View



Design Through Optimization

- **Idea:** Sample/discretize the frequency response
$$H\left(e^{j\omega}\right) \Longrightarrow H\left(e^{j\omega_k}\right)$$
  - Sample points are fixed:
$$\omega_k = k\frac{\pi}{K}, \qquad -\pi < \omega_1 < \omega_2 \ldots < \omega_K \leq \pi$$
  - K has to be >> N, where N is the filter length. (Rule of thumb $K \geq 8N$)
  - Yields a (good) approximation of the original problem

Least Squares

- Target: Design a length $N = 2M + 1$ filter **Type I** filter
  - First design non-causal $\widetilde{H}(e^{j\omega})$ and hence $\tilde{h}[n]$.
  - Then, shift to make causal

$$h[n] = \tilde{h}[n - M]$$
$$H(e^{j\omega}) = e^{-jM\omega}\widetilde{H}(e^{j\omega})$$

- Frequency response of the filter:

$$\widetilde{H}(e^{j\omega}) = \sum_{n=-M}^{M} \tilde{h}[n]e^{j\omega n}$$

Least Squares Cont.

- **Matrix Formulation**

$$\tilde{\boldsymbol{h}} = \left[\tilde{h}[-M], \tilde{h}[-(M-1)], \dots \tilde{h}[0], \dots, \tilde{h}[M-1], \tilde{h}[M]\right]^T$$

$$\boldsymbol{b} = \left[\widetilde{H}_d(e^{j\omega_1}), \widetilde{H}_d(e^{j\omega_2}), \dots, \widetilde{H}_d(e^{j\omega_K})\right]^T$$

$$\mathbf{A} = \begin{bmatrix} e^{-j\omega_1(-M)} & e^{-j\omega_1(-M+1)} & \cdots & e^{-j\omega_1 M} \\ e^{-j\omega_2(-M)} & e^{-j\omega_2(-M+1)} & \cdots & e^{-j\omega_2 M} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\omega_K(-M)} & e^{-j\omega_K(-M+1)} & \cdots & e^{-j\omega_K M} \end{bmatrix}_{K\times(2M+1)}$$

Objective: $\min_{\tilde{\boldsymbol{h}}} \left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2$

Least Squares Cont.

Objective: $\min_{\tilde{\boldsymbol{h}}} \left\| \mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b} \right\|^2$

- Solution: $\frac{d\left\|\mathbf{A}\tilde{\boldsymbol{h}}-\boldsymbol{b}\right\|^2}{d\tilde{\boldsymbol{h}}} = 0$

$$\left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2 = \left(\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right)^*\left(\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right)$$
$$= \tilde{\boldsymbol{h}}^*\mathrm{A}^*\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}^*\mathbf{A}\tilde{\boldsymbol{h}} - \tilde{\boldsymbol{h}}^*\mathrm{A}^*\boldsymbol{b} + \boldsymbol{b}^*\boldsymbol{b}$$
$$\frac{d\left\|\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}\right\|^2}{d\tilde{\boldsymbol{h}}} = 2\mathrm{A}^*\mathbf{A}\tilde{\boldsymbol{h}} - 2\mathrm{A}^*\boldsymbol{b} = \mathbf{0}$$
$$\tilde{\boldsymbol{h}} = (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\boldsymbol{b}$$

- $\frac{d\boldsymbol{x}^T\boldsymbol{a}}{d\boldsymbol{x}} = \frac{d\boldsymbol{a}^T\boldsymbol{x}}{d\boldsymbol{x}} = \boldsymbol{a}$
- $\frac{d\boldsymbol{x}^T\mathbf{A}\boldsymbol{x}}{d\boldsymbol{x}} = \mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x}$
  - If A is symmetric $\mathbf{A}\boldsymbol{x} + \mathbf{A}^T\boldsymbol{x} = 2\mathbf{A}\boldsymbol{x}$

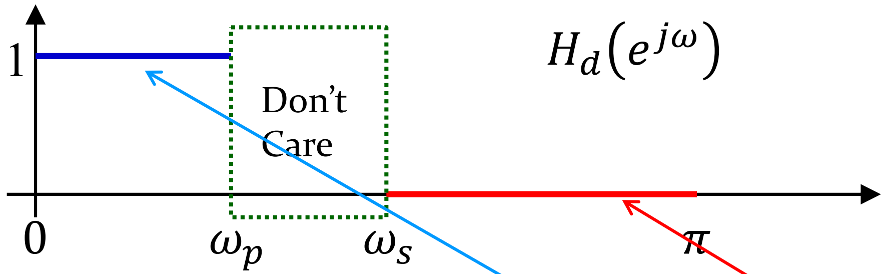
- Result will generally be non-symmetric and complex valued.
- However, if $\tilde{H}(e^{j\omega})$ is real, $\tilde{h}[n]$ should have symmetry!

Example: Design of Linear Phase LP Filter

- Suppose
  - $\tilde{H}(e^{j\omega})$ is real and symmetric, length N is odd
- Then，
  - $\tilde{h}[n]$ is real, and symmetric around $\tilde{h}[0]$
- So
  - $\tilde{H}(e^{j\omega}) = \tilde{h}[0] + \tilde{h}[1]e^{-j\omega} + \tilde{h}[-1]e^{j\omega} + \tilde{h}[2]e^{-j2\omega} + \tilde{h}[-1]e^{j2\omega} + \cdots$

$$= \tilde{h}[0] + 2\tilde{h}[1]\cos\omega + 2\tilde{h}[2]\cos 2\omega + \cdots$$
$$= \tilde{h}[0] + \sum_{n=1}^{M} \tilde{h}[n]\cos n\omega$$

Example: Cont.



- Given $N = 2M + 1, \omega_p,$ and ω_s, find the best LS filter:

$$\mathbf{A} = \begin{bmatrix} 1 & 2\cos\omega_1 & \cdots & 2\cos M\omega_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cos\omega_p & \cdots & 2\cos M\omega_p \\ 1 & 2\cos\omega_s & \cdots & 2\cos M\omega_s \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 2\cos\omega_K & \cdots & 2\cos M\omega_K \end{bmatrix}_{K\times(M+1)}$$

$$\boldsymbol{b} = [1\ 1\ \ldots 1\ 0\ 0\ \ldots 0]^T_{1\times K}$$

$$\tilde{\boldsymbol{h}}_+ = \left[\tilde{h}[0], \ldots, \tilde{h}[M]\right]^T_{1\times(M+1)}$$

$$= (\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*\boldsymbol{b}$$

*Extension: Weighted Least Squares

- LS has no preference for passband or stopband
- Use weighting of LS to change ratio

- Solve the discrete version of

$$\text{Minimize } \int_{-\pi}^{\pi} W(\omega)\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|^2 d\omega$$

  - where $W(\omega) = \frac{1}{\delta_p}$ in passband, $W(\omega) = \frac{1}{\delta_s}$ in stopband, and $W(\omega) = 0$ in transition band.
  - Equivalently, you may set $W(\omega) = 1$ in passband, $W(\omega) = \frac{\delta_p}{\delta_s}$ in stopband, and $W(\omega) = 0$ in transition band.

*Weighted Least Squares

- Written in matrix form:

$$\textbf{Objective:}\ \min_{\tilde{\boldsymbol{h}}} \left\| \boldsymbol{w} \cdot (\mathbf{A}\tilde{\boldsymbol{h}} - \boldsymbol{b}) \right\|^2$$

where, ‘ . ’ is inner product, and

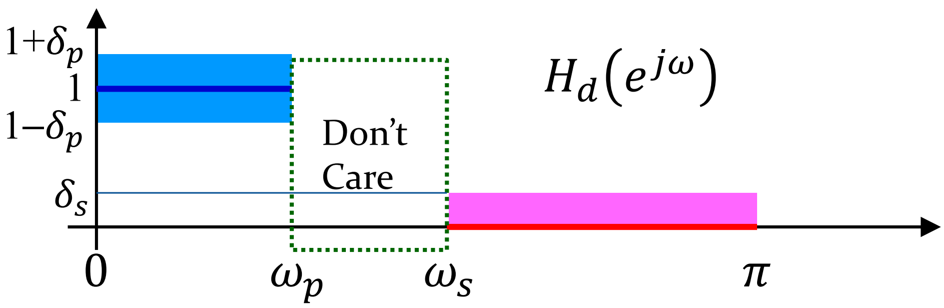
$$\boldsymbol{w} = \left[1\ 1\ \ldots 1\ \frac{\delta_p}{\delta_s} \frac{\delta_p}{\delta_s}\ \ldots\ \frac{\delta_p}{\delta_s} \right]^T_{1\times K}$$

- It can be written as $\min_{\tilde{\boldsymbol{h}}} \left(\mathbf{A}\tilde{\boldsymbol{h}}_+ - \boldsymbol{b}\right)^* \mathbf{W}^2 \left(\mathbf{A}\tilde{\boldsymbol{h}}_+ - \boldsymbol{b}\right)$, where

$$\mathbf{W} = \mathrm{diag}(w) = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \frac{\delta_p}{\delta_s} & & \\ & & & & \ddots & \\ 0 & & & & & \frac{\delta_p}{\delta_s} \end{bmatrix}_{K\times K}$$

Minimax Optimal Filters

- **Objective:** $\text{Minimize}_{\omega\in\text{care}}\ \max\left|H(e^{j\omega}) - H_d(e^{j\omega})\right|$
  - Parks-McClellan algorithm - equi-ripple
  - Also known as Remez exchange algorithms (signal.remez)
  - Linear Programming
  - Can also use convex optimization
- **Specifications:**



- Filter specifications are given in terms of boundaries

- More specifically, minimize
  - maximum passband ripple

$$1-\delta_p \leq \left|H\left(e^{j\omega}\right)\right| \leq 1+\delta_p, \qquad 0 \leq \omega \leq \omega_p$$

  - and, maximum stopband ripple

$$\left|H\left(e^{j\omega}\right)\right| \leq \delta_s, \qquad \omega_s \leq \omega \leq \pi$$

Estimation of the Filter Length

- Given $\omega_p, \omega_s, \delta_p, \delta_s$, estimate the filter length
- Kaiser's formula:

$$N \cong \frac{-20\log_{10}\left(\sqrt{\delta_p\delta_s}\right)-13}{\frac{14.6\left(\omega_s-\omega_p\right)}{2\pi}}+1$$

- Bellanger's formula:

$$N \cong -\frac{2\log_{10}\left(10\delta_p\delta_s\right)}{3\left(\omega_s-\omega_p\right)/2\pi}$$

Minimax Design via Remez exchange algorithm & Park-McClellan algorithm

- The Remez exchange algorithm is a generic iterative procedure to approximate any function optimally in the L_∞ sense (i.e., give the best worst-case approximation or in other words, minimize the maximum error or minmax).

- Parks-McClellan algorithm (PM) is a variation of the Remez exchange algorithm, applied specifically for FIR filters.

Formulation of the problem

- **Error function**

$$E(\omega) = W(\omega)\left(\tilde{H}(e^{j\omega}) - H_d(e^{j\omega})\right)$$

- **Objective:**

$$\text{Minimize}_{\omega\in\text{care}}\ \max\left|W(\omega)\left(\tilde{H}(e^{j\omega}) - H_d(e^{j\omega})\right)\right|$$

- For Low-pass filter design using Remez algorithm, we define:

$$H_d(e^{j\omega}) = \begin{cases} 1 & 0 \le \omega \le \omega_p \\ 0 & \omega_s \le \omega \le \pi \end{cases},$$

$$W(\omega) = \begin{cases} W_p & 0 \le \omega \le \omega_p \\ W_s & \omega_s \le \omega \le \pi \end{cases}$$

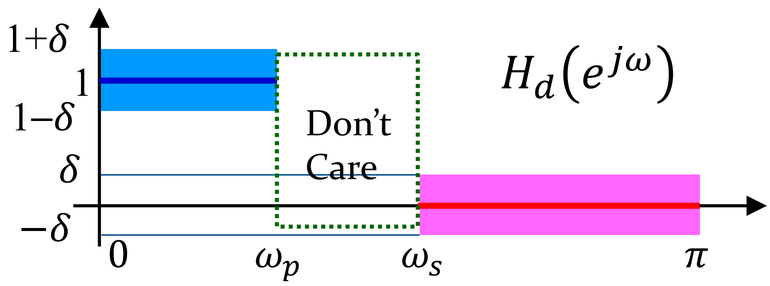
- **Using Alternate Theorem**

Matlab function

- [h, error, detail]=remez(N, F, A, W) (1934)
- [h, error, detail]=firpm(N, F, A, W) (1972)
  - N: filter order
  - F: a vector of frequency band edges in pairs, in ascending order between 0 and 1. 1 corresponds to the Nyquist frequency or half the sampling frequency. At least one frequency band must have a non-zero width.
  - A: a real vector the same size as F which specifies the desired amplitude of the frequency response of the resultant filter h.
  - W: a weight to weight the error. W has one entry per band which tells remez (or FIRPM) how much emphasis to put on minimizing the error in each band relative to the other bands.

Minimax Design via Linear Programming

- When $\tilde{H}(e^{j\omega})$ is real and symmetric
- Given $N = 2M + 1$, ω_p, ω_s, find $\tilde{\boldsymbol{h}}_+$ to



Minimize: δ

Subject to: $1 - \delta \le \tilde{H}(e^{j\omega_k}) \le 1 + \delta, 0 \le \omega \le \omega_p$

$$-\delta \le \tilde{H}(e^{j\omega_k}) \le \delta,\ \omega_s \le \omega \le \pi$$

- Both the objective function and the constrains are the linear functions of variables δ and $\tilde{\boldsymbol{h}}_+$
- A well studied class of problem

Linear Programming

- Linear Programming: ($\boldsymbol{x}$ is the vector to be optimized)

  Minimize: $\boldsymbol{c}^T\boldsymbol{x}$

  Subject to: $\mathbf{A}\boldsymbol{x} \le \boldsymbol{b}$

- Minimax Design Linear Phase FIR Filter

$$\boldsymbol{x} = \begin{bmatrix} \tilde{h}[0] \\ \vdots \\ \tilde{h}[M] \\ \delta \end{bmatrix}_{M+2} \qquad \boldsymbol{c} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{M+2}$$

$$\mathbf{A}\boldsymbol{x} = \begin{bmatrix} 1 & 2\cos\omega_1 & \cdots & 2\cos M\omega_1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2\cos\omega_p & \cdots & 2\cos M\omega_p & -1 \\ -1 & -2\cos\omega_1 & \cdots & -2\cos M\omega_1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2\cos\omega_p & \cdots & -2\cos M\omega_p & -1 \\ 1 & 2\cos\omega_s & \cdots & 2\cos M\omega_s & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2\cos\omega_K & \cdots & 2\cos M\omega_K & -1 \\ -1 & -2\cos\omega_s & \cdots & -2\cos M\omega_s & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2\cos\omega_K & \cdots & -2\cos M\omega_K & -1 \end{bmatrix}_{2K\times(M+2)} \times \begin{bmatrix} \tilde{h}[0] \\ \vdots \\ \tilde{h}[M] \\ \delta \end{bmatrix}_{M+2} = \boldsymbol{b} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{2K}$$

The End