Lecture 4
Frequency Domain Representation of Discrete Time Signal

- The frequency domain representation of discrete time sequence is the **discrete-time Fourier transform (DTFT).** DTFT is a frequency analysis tool for aperiodic discrete-time signals

- This transform maps a time-domain sequence into a continuous function of the frequency variable ω.

Review of Fourier Series and Continuous-Time Fourier Transform (CTFT)

- Fourier Series (for continuous periodic signal):

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \qquad x(t) \overset{\text{FS}}{\leftrightarrow} a_k$$

- Fourier Series Coefficients

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\omega_0 t}\,dt = \frac{1}{T}\int_T x(t)e^{-jk\left(\frac{2\pi}{T}\right)t}\,dt$$

- CTFT (for continuous aperiodic signal)

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}dt.$$

Fourier Spectrum
Or Spectrum

- Inverse CTFT:

$$x_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}d\Omega.$$

$$x_a(t) \overset{\text{CTFT}}{\longleftrightarrow} X_a(j\Omega)$$

Definition of Discrete-Time Fourier Transform (DTFT) (for discrete aperiodic signal)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{DTFT}$$

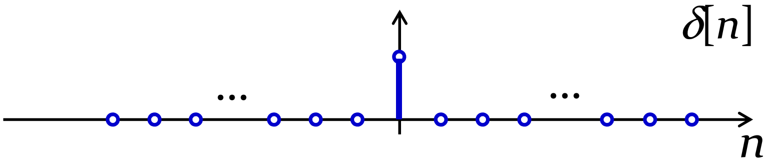
where, ω is a continuous variable in the range $-\infty < \omega < \infty$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{Inverse DTFT}$$

Why one is sum and the other integral?

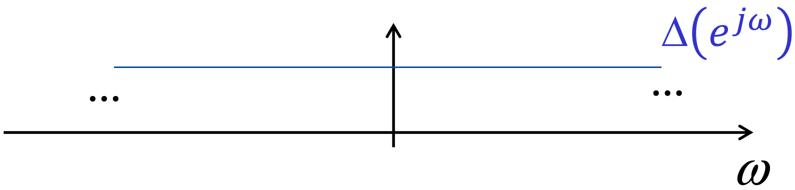
Example 1

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$



DTFT:

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]e^{-j\omega n} = \sum_{n=0} e^{-j\omega n} = 1$$



Example 2

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Causal sequence $x[n] = \alpha^n \mu[n]$, $|\alpha|<1$

  Its DTFT is given by

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{1}{1-\alpha e^{-j\omega}}$$

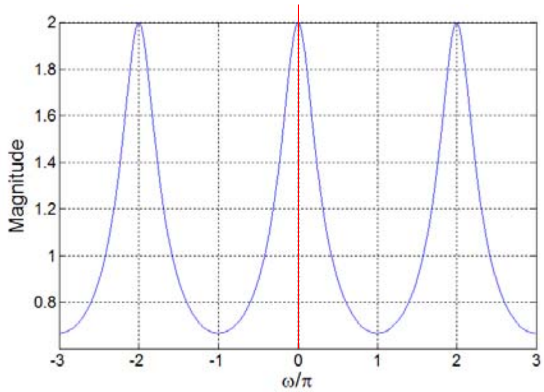
as $\left|\alpha e^{-j\omega}\right| = |\alpha|<1$

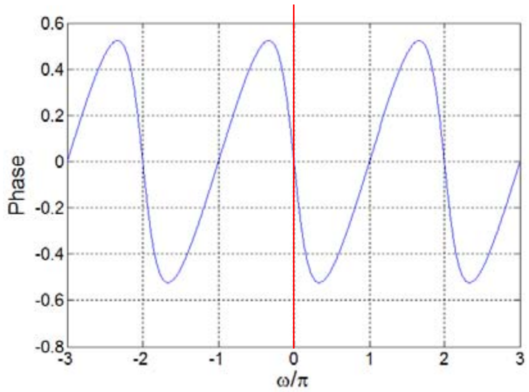
$$\boxed{\text{Recall: } 1 + q + q^2 + \cdots + q^\infty = \frac{1}{1-q} \qquad \text{for } |q| < 1}$$

Example 2 (Cont'd)

- The magnitude and phase of DTFT of

$$X(e^{j\omega}) = \frac{1}{1 - 0.5e^{-j\omega}}$$





$|X(e^{j\omega})| = |X(e^{-j\omega})|$ $\theta(\omega) = -\theta(-\omega)$

DTFT

- In general, $X(e^{j\omega})$ is a complex function of the real variable ω, and can be written as

$$X(e^{j\omega}) = X_{\text{re}}(e^{j\omega}) + jX_{\text{im}}(e^{j\omega})$$

- $X(e^{j\omega})$ can alternatively be expressed as

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\theta(\omega)}$$

  where, $\theta(\omega) = \arg\{X(e^{j\omega})\}$
  - $|X(e^{j\omega})|$ is called the magnitude function
  - $\theta(\omega)$ is called the phase function
  - In applications where DTFT is called Fourier spectrum, $|X(e^{j\omega})|$ and $\theta(\omega)$ are called magnitude and phase spectra

Symmetry of DTFT

- For **a real sequence $x[n]$**, $|X(e^{j\omega})|$ and $X_{\text{re}}(e^{j\omega})$ are even functions of ω, whereas $\theta(\omega)$ and $X_{\text{im}}(e^{j\omega})$ are odd functions of ω.
- Proof: $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}$$
$$= \left\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right\}^{*}, \text{for real } x[n]$$
$$= X^{*}(e^{j\omega})$$

- Therefore, $|X(e^{j\omega})| = |X(e^{-j\omega})|$ and $\theta(\omega) = -\theta(-\omega)$

Periodicity of DTFT

- $X(e^{j\omega}) = X(e^{j(\omega+2k\pi)})$, i. e. $|X(e^{j\omega})|e^{j[\theta(\omega)+2k\pi]} = |X(e^{j\omega})|e^{j\theta(\omega)}$ for any integer k.
- Proof:

$$X(e^{j(\omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$= X(e^{j\omega})$$

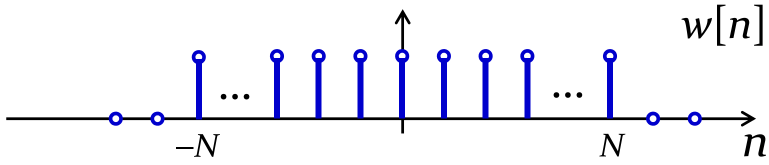
- In other words, the phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT.
- Unless otherwise stated, we assume that the phase function $\theta(\omega)$ is restricted to the range of

$$-\pi < \theta(\omega) < \pi$$

called the **principle value**.

Example 3

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$



$w[n]$ "Window"

DTFT:

$$W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} w[n]e^{-j\omega k} = \sum_{k=-N}^{N} e^{-j\omega k}$$

$$= e^{-j\omega N}\left(1 + e^{j\omega} + e^{j2\omega} + \dots + e^{j2N\omega}\right)$$

Recall: $1 + q + q^2 + \dots + q^M = \dfrac{1 - q^{M+1}}{1 - q}$ $\quad q = e^{j\omega}$, $M = 2N$

Example 3 Cont.

DTFT: $W(e^{j\omega}) = e^{-j\omega N}\left(1 + e^{j\omega} + e^{j2\omega} + \dots + e^{j2N\omega}\right)$

$$= e^{-j\omega N}\frac{1 - e^{j\omega(2N+1)}}{1 - e^{j\omega}}$$

$$= \frac{e^{-j\omega N} - e^{j\omega N}e^{j\omega}}{1 - e^{j\omega}} \qquad \times \frac{e^{-j\frac{\omega}{2}}}{e^{-j\frac{\omega}{2}}}$$

$$= \frac{e^{-j\omega\left(N+\frac{1}{2}\right)} - e^{j\omega\left(N+\frac{1}{2}\right)}}{e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}}}$$

$$= \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

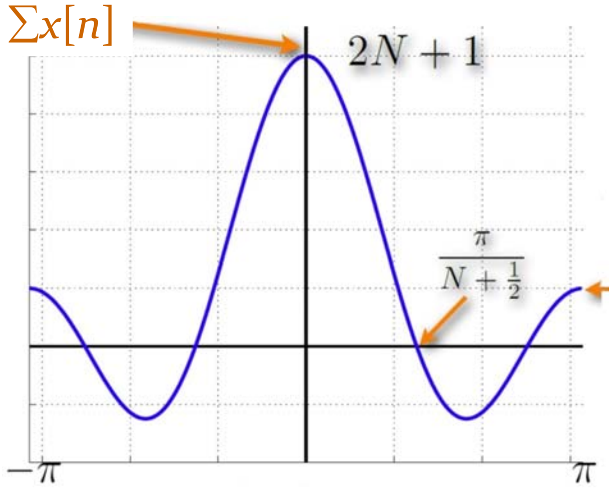
Periodic Sinc

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

Example 3 Cont.

$$W(e^{j\omega}) = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)} \longrightarrow 2N+1 \quad \text{as } \omega \longrightarrow 0$$

Also = $\sum x[n]$



=1 if N is even.
Why?

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Example of Inverse DTFT

- **Find the inverse DTFT of**

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- A:

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \boxed{\frac{\sin \omega_c n}{\pi n}}, \text{for } n \neq 0$$

$$h_{LP}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{LP}(e^{j\omega})d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi}$$

Properties of the DTFT

- **Linearity:**

Let $g[n] \leftrightarrow G(e^{j\omega})$ and $h[n] \leftrightarrow H(e^{j\omega})$

Then $\alpha g[n] + \beta h[n] \leftrightarrow \alpha G(e^{j\omega}) + \beta H(e^{j\omega})$

- **Periodicity:** $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$

Properties of the DTFT Cont.

- **Time Reversal:**

Let $x[n] \leftrightarrow X(e^{j\omega})$

Then $x[-n] \leftrightarrow X(e^{-j\omega})$

$= X^*(e^{j\omega})$ if $x[n]$ is real

If $x[n]$=$x[-n]$ and $x[n]$ is real, then

$X(e^{j\omega}) = X^*(e^{j\omega}) \rightarrow X(e^{j\omega})$ is real

- Q: Suppose $x[n] \leftrightarrow X(e^{j\omega})$, $x[n] \in \mathcal{R}eal$

$$? \leftrightarrow \mathcal{R}e\{X(e^{j\omega})\}$$

- A: Decompose $x[n]$ to even and odd functions

$$x[n]=x_e[n]+x_o[n],$$

where
$$x_e[n] = \frac{1}{2}(x[n] + x[-n])$$
$$x_o[n] = \frac{1}{2}(x[n] - x[-n])$$

$$\boxed{x_e[n]} \leftrightarrow X_e(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) + X(e^{-j\omega})\right)$$
$$= \frac{1}{2}\left(X(e^{j\omega}) + X^*(e^{j\omega})\right) = \mathcal{R}e\{X(e^{j\omega})\}$$

$$x_o[n] \leftrightarrow X_o(e^{j\omega}) = \frac{1}{2}\left(X(e^{j\omega}) - X(e^{-j\omega})\right) = j\mathcal{I}m\{X(e^{j\omega})\}$$

Properties of the DTFT Cont.

- **Time and Frequency Shifting**

Let
$$x[n] \leftrightarrow X(e^{j\omega})$$
Then
$$x[n-n_d] \leftrightarrow e^{-j\omega n_d}X(e^{j\omega})$$
$$e^{j\omega_0 n}x[n] \leftrightarrow X\left(e^{j(\omega-\omega_0)}\right)$$

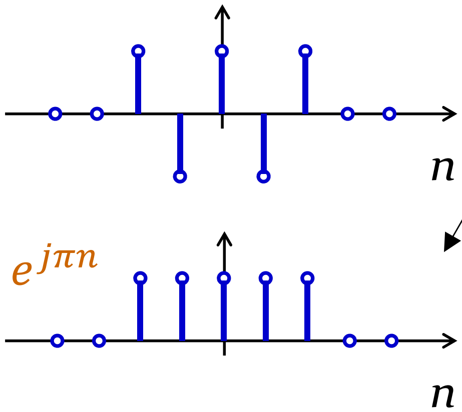
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

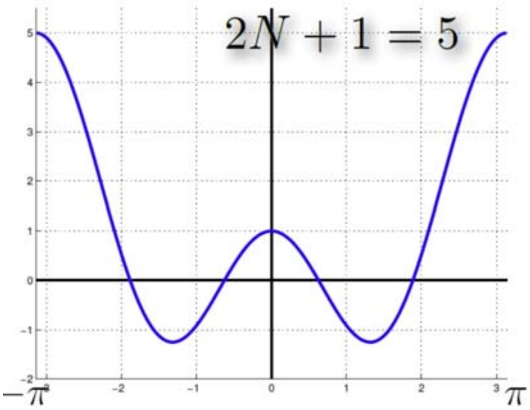
Example 4

$$W(e^{j\omega}) = \frac{\sin\left(\omega\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega}{2}\right)}$$

What is the DTFT of:

$\times\ e^{j\pi n}$





$$W(e^{j\omega}) = \frac{\sin\left((\omega-\pi)\left(N+\frac{1}{2}\right)\right)}{\sin\left(\frac{\omega-\pi}{2}\right)}$$

Properties of the DTFT Cont.

- **Differentiation in frequency**

Let $\quad x[n] \leftrightarrow X(e^{j\omega})$

Then $\quad nx[n] \leftrightarrow j\dfrac{dX(e^{j\omega})}{d\omega}$

Proof: $\quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

Differentiate both side to get $\dfrac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$

Multiply both side by j, we get $j\dfrac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$

Example 5

- Determine DTFT $Y(e^{j\omega})$ of
$$y[n] = (n+1)\alpha^n \mu[n], \qquad |\alpha| < 1$$
- Let $x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$
- We can therefore write
$$y[n] = nx[n] + x[n]$$
- From example 2, we have known that the DTFT of $x[n]$ is given by
$$X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}$$

- Using the differentiation in frequency, we observe that DTFT of $nx[n]$ is given by,
$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2}$$
- Next, using linear property of DTFT, we arrive at
$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1 - \alpha e^{-j\omega})^2} + \frac{1}{1 - \alpha e^{-j\omega}} = \frac{1}{(1 - \alpha e^{-j\omega})^2}$$

Properties of the DTFT Cont.

- **Convolution**

  Let $x[n] \leftrightarrow X(e^{j\omega})$ and $h[n] \leftrightarrow H(e^{j\omega})$

  - DTFT convolution theorem

  If $y[n] = x[n] \circledast h[n]$

  Then $y[n] \leftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$

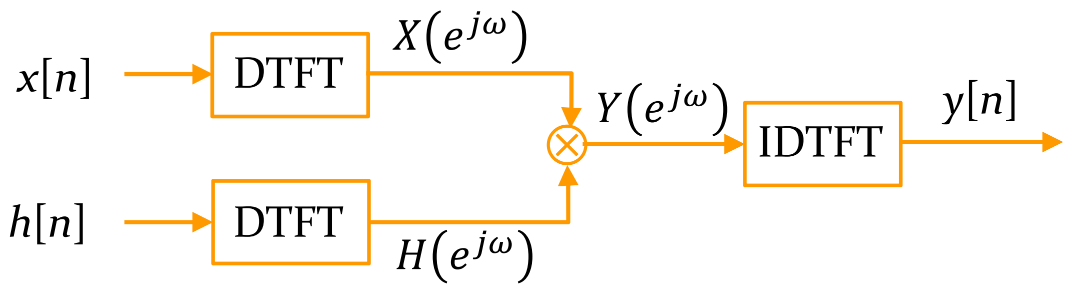
  - **DTFT convolution modulation**

  If $y[n] = x[n]h[n]$

  Then $y[n] \leftrightarrow Y(e^{j\omega}) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X(e^{j\theta})H(e^{j(\omega-\theta)})d\theta$

Linear Convolution Using DTFT

- Linear convolution $y[n]$ of the sequence $x[n]$ and $h[n]$ can be performed as follows:
  - Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively.
  - Form DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
  - Compute the IDTFT $y[n]$ of $Y(e^{j\omega})$



Properties of the DTFT Cont.

- **Parseval's theorem**

Let $x[n] \leftrightarrow X(e^{j\omega})$ $\quad h[n] \leftrightarrow H(e^{j\omega})$

Then $$\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H^*(e^{j\omega})d\omega$$

Proof: $$\sum_{n=-\infty}^{\infty} x[n]h^*[n] = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})e^{-j\omega n}d\omega\right)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H^*(e^{j\omega})\,X(e^{j\omega})d\omega$$

Energy & Energy Density Spectrum

- Energy: $E_g = \sum_{-\infty}^{\infty}|x[n]|^2$
- According to Parseval's theorem, when $h[n] = x[n]$,

$$E_g = \sum_{n=-\infty}^{\infty}|x[n]|^2 = \sum_{n=-\infty}^{\infty} x[n]x^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\boxed{\left|X(e^{j\omega})\right|^2}d\omega$$

- Energy spectral density:

$$S_{xx}(\omega) = \left|X(e^{j\omega})\right|^2$$

Energy Spectral Density

- Example – Compute the energy of the sequence

$$h_{LP}[n] = \frac{\sin(\omega_c n)}{\pi n}, \qquad -\infty < n < \infty$$

- Here

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|H_{LP}(e^{j\omega t})\right|^2 d\omega$$

where

$$H_{LP}(e^{j\omega t}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

- Therefore:

$$\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{\omega_c}{\pi} < \infty$$

Symmetry Relations

If $x[n]$ is real

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x_{\text{ev}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{od}}[n]$	$jX_{\text{im}}(e^{j\omega})$
Conjugate Symmetric	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{\text{re}}(e^{j\omega}) = X_{\text{re}}(e^{-j\omega})$
	$X_{\text{im}}(e^{j\omega}) = -X_{\text{im}}(e^{-j\omega})$
	$\left\|X(e^{j\omega})\right\| = \left\|X(e^{-j\omega})\right\|$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Symmetry Relations

If $x[n]$ is complex

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$x^*[n]$	$X^*(e^{-j\omega})$
$x_{\text{re}}[n]$	$X_{\text{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$jx_{\text{im}}[n]$	$X_{\text{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\text{cs}}[n]$	$X_{\text{re}}(e^{j\omega})$
$x_{\text{ca}}[n]$	$jX_{\text{im}}(e^{j\omega})$

Band-Limited DT Signals

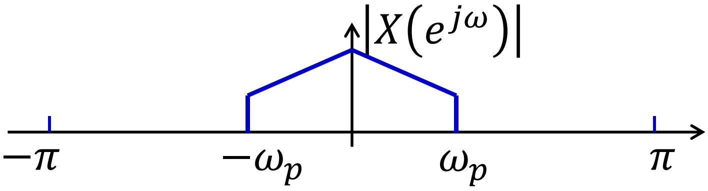
- The spectrum of a DT signal is a periodic function of ω with a period of 2π. Thus a **full-band** signal has a spectrum occupying the frequency range $-\pi < \omega \le \pi$.
- A **band-limited** DT signal has a spectrum that is limited to a portion of the frequency range $-\pi < \omega \le \pi$.
- An ideal real band-limited signal:

$$|X(e^{j\omega})| = \begin{cases} 0, & 0 \le |\omega| < \omega_a \\ \text{non zero}, & \omega_a \le |\omega| \le \omega_b \\ 0, & \omega_b \le |\omega| \le \pi \end{cases}$$

Classification of Band-Limited Signal

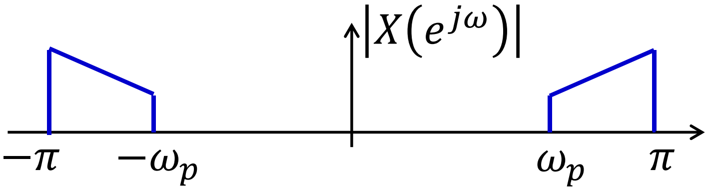
- Lowpass real signal

  Bandwidth: ω_p



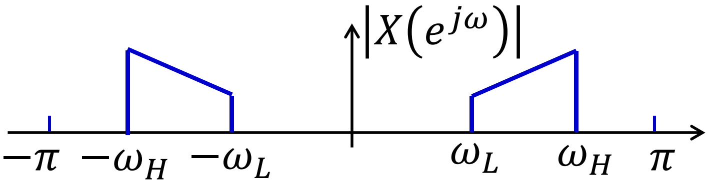
- Highpass real signal

  Bandwidth: $\pi - \omega_p$



- Bandpass real signal

  Bandwidth: $\omega_H - \omega_L$



DTFT Convergence Condition

- The infinite series

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

  may or may not converge.

- If it converges for all value of ω, then the DTFT $X(e^{j\omega})$ exists.
- Consider the finite sum:

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

- Strong convergence: $X(e^{j\omega})$ **converge uniformly, i.e.,**

$$\lim_{K\to\infty} X_K(e^{j\omega}) = X(e^{j\omega})$$

  - $x[n]$ absolutely summable $\Longrightarrow$ $X(e^{j\omega})$ **exist and converge uniformly**

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \Longrightarrow |X(e^{j\omega})| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} |x[n]||e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  - This is a **sufficient condition**, not necessary.

Example

$x[n] = \alpha^n \mu[n]$, $|\alpha|<1$ is absolutely summable, as

$$\sum_{n=-\infty}^{\infty} |\alpha^n \mu[n]| = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and therefore is DTFT $X(e^{j\omega})$ converge to $\frac{1}{1-\alpha e^{-j\omega}}$ uniformly.

Since $$\sum_{n=-\infty}^{\infty} |x[n]|^2 \le \left(\sum_{-\infty}^{\infty} |x[n]|\right)^2$$

- an absolutely summable sequence has always a finite energy,
- However, a finite energy sequence is not necessary absolutely summable.

Example: a sequence $x[n] = \frac{1}{n}\mu[n-1]$ has a finite energy, as $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$, however, it is not absolutely summable as $\sum_{n=1}^{\infty}\left|\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ does not converge.

- Weak convergence: $X(e^{j\omega})$ converge mean square

$$\lim_{K\to\infty}\int_{-\pi}^{\pi}\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|^2 d\omega = 0$$

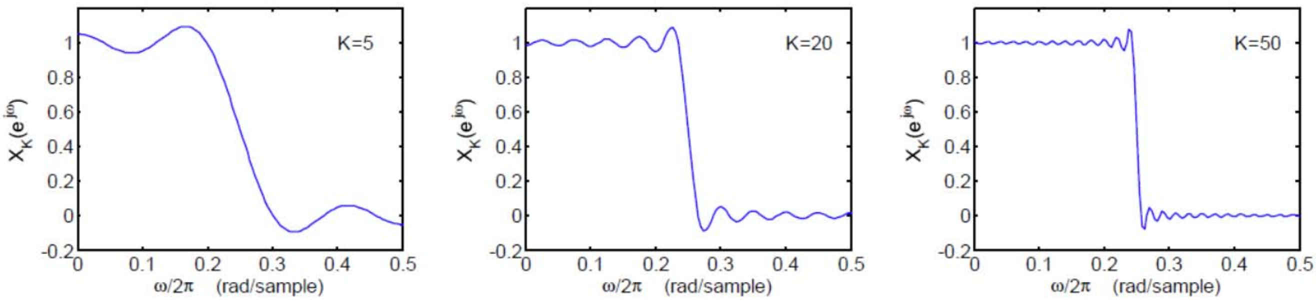
  - $x[n]$ finite energy $\Longrightarrow$ $X(e^{j\omega})$ converge mean square.

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 < \infty \Longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 d\omega < \infty$$

  - The absolute value of the error $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ may not go to zero when K goes to infinite.

Example

- $h_{LP}[n] = \frac{\sin 0.5\pi n}{\pi n} \leftrightarrow H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le 0.5\pi \\ 0, & 0.5\pi \le |\omega| \le \pi \end{cases}$
- $\sum_{n=-\infty}^{\infty} |h_{LP}[n]|^2 = \frac{\omega_c}{\pi} < \infty$, but not absolutely summable.

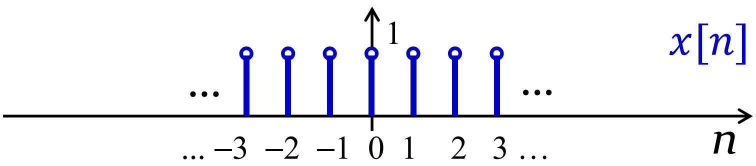


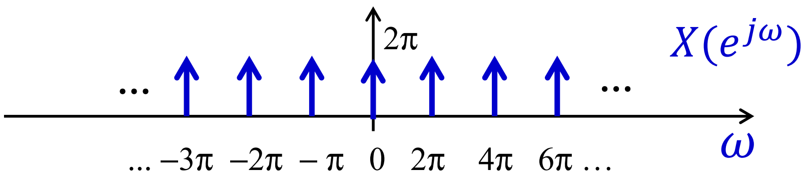
- **Gibbs phenomenon**: mean square converges ~~at each ω~~ as $K \to \infty$, but peak error does not get smaller.

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$ and the sinusoidal sequence $\cos(\omega_0 n + \varphi)$.
- For this type of sequences, a DTFT representation is possible using the Dirac delta function $\delta(\omega)$.

Example 5: DTFT of *x*[*n*]=1 for all *n*

- $x[n] = 1 = \sum_{k=-\infty}^{\infty} \delta[n-k]$
- It is more convenient to prove that the inverse DTFT of $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ is 1





- *Proof*: The inverse DTFT of $\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ is evaluated as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega-2\pi k)\right)e^{j\omega n}d\omega$$

$$= \int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k)\right)e^{j\omega n}d\omega$$

- From the sifting property, we have

$$\left(\sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k)\right)e^{j\omega n} = \left(\sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k)\right)e^{j2\pi kn}$$

$$= \sum_{k=-\infty}^{\infty} \delta(\omega-2\pi k)$$

We have used $e^{j2\pi kn} = 1$ for all n here

- When we integrate the sequence of impulse from $-\pi$ to π, we have only the impulse at $\omega = 0$.
- Hence

$$\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)\right)e^{j\omega n}d\omega$$
$$=\int_{-\pi}^{\pi}\left(\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)\right)d\omega$$
$$=\int_{-\pi}^{\pi}\delta(\omega)\,d\omega=\int_{-\infty}^{-\pi}\delta(\omega)\,d\omega+\int_{-\pi}^{\pi}\delta(\omega)\,d\omega+\int_{\pi}^{\infty}\delta(\omega)\,d\omega$$
$$=\int_{-\infty}^{\infty}\delta(\omega)\,d\omega=1\quad\text{for all } n$$

Example 6: DTFT of $\mu[n]$

- Let $\mu[n] = u_1[n] + u_2[n]$, where

$$u_1[n]=\frac{1}{2},\qquad \text{for } -\infty<n<\infty$$

and

$$u_2[n]=\begin{cases}\dfrac{1}{2} & \text{for } n\geq 0\\ -\dfrac{1}{2} & \text{for } n<0\end{cases}$$

Therefore, we have

$$\delta[n]=u_2[n]-u_2[n-1]$$

- Using $\delta[n] \leftrightarrow 1$ and $u_2[n] - u_2[n-1] \leftrightarrow U_2(e^{j\omega}) - e^{-j\omega}U_2(e^{j\omega}) = U_2(e^{j\omega})(1 - e^{-j\omega})$, we have

$$1 = U_2(e^{j\omega})(1 - e^{-j\omega})$$

i.e.

$$U_2(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}} \quad \text{for } \omega \neq 0$$

Since

$$u_1[n] \leftrightarrow \sum_{k=-\infty}^{\infty} \pi\delta(\omega - 2\pi k) = U_1(e^{j\omega})$$

we have

$$\mu[n] \leftrightarrow \frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k) \quad \text{for } \omega \neq 0$$

DTFT Convergence

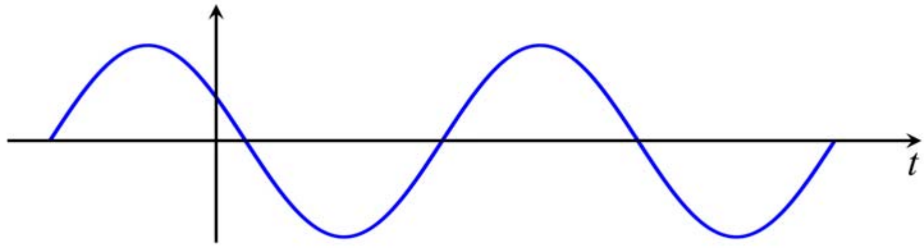
Sequence		DTFT
$\alpha^n\mu[n], (\lvert\alpha\rvert < 1)$ Absolutely Summable	Sufficient $\longrightarrow$	$\frac{1}{1 - \alpha e^{-j\omega}}$ Exist for all ω
$\mu[n]$ Neither absolutely summable, nor finite energy	Not necessary $\longrightarrow$	$\frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$ Not exist for $\omega = 0$
1 (for all n) Neither absolutely summable, nor finite energy		$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k)$ Exist for all ω
$h_{LP}[n] = \frac{\sin\omega_c n}{\pi n}$ Finite energy	Sufficient $\longrightarrow$	$H_{LP}(e^{j\omega}) = \begin{cases} 1, 0 \leq \lvert\omega\rvert \leq \omega_c\pi \\ 0, \omega_c\pi \leq \lvert\omega\rvert \leq \pi \end{cases}$ Exist for all ω

Commonly used DTFT pairs

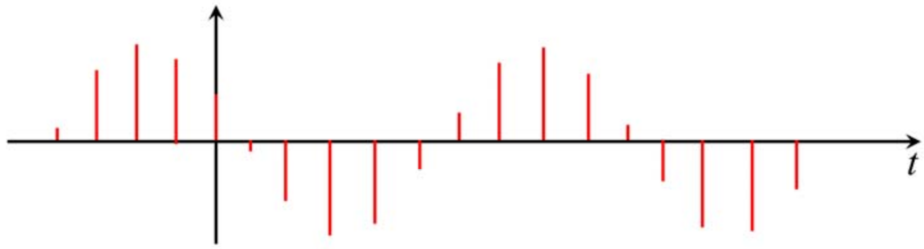
Sequence	DTFT
$\delta[n]$	1
$\mu[n]$	$\dfrac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega+2\pi k)$
1 (for all n)	$\sum_{k=-\infty}^{\infty}2\pi\delta(\omega-2\pi k)$
$\alpha^n\mu[n], (\lvert\alpha\rvert<1)$	$\dfrac{1}{1-\alpha e^{-j\omega}}$

Effect of Time-Domain Sampling in Frequency Domain

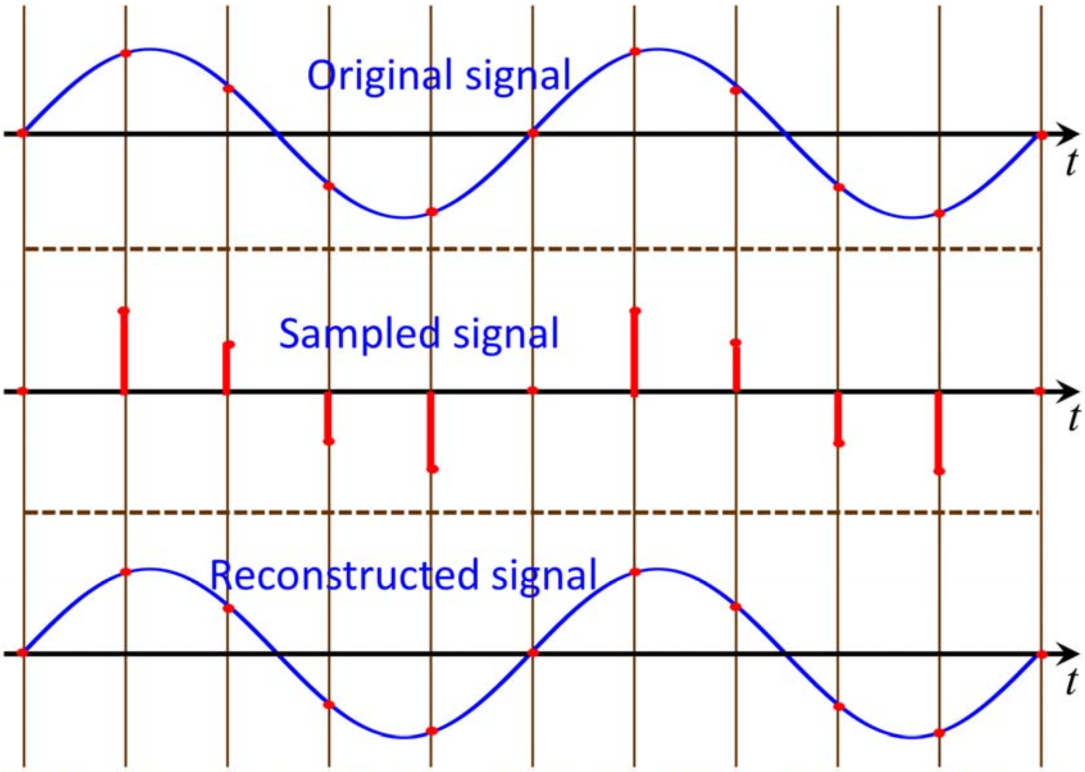
- Questions to be answered
  - When discrete signal is obtained by sampling, can we recover the original continuous signal from the discrete signal?
  - What is the condition that we can recover the continuous signal?
  - Relation Between Continuous and Discrete Signal Spectrum



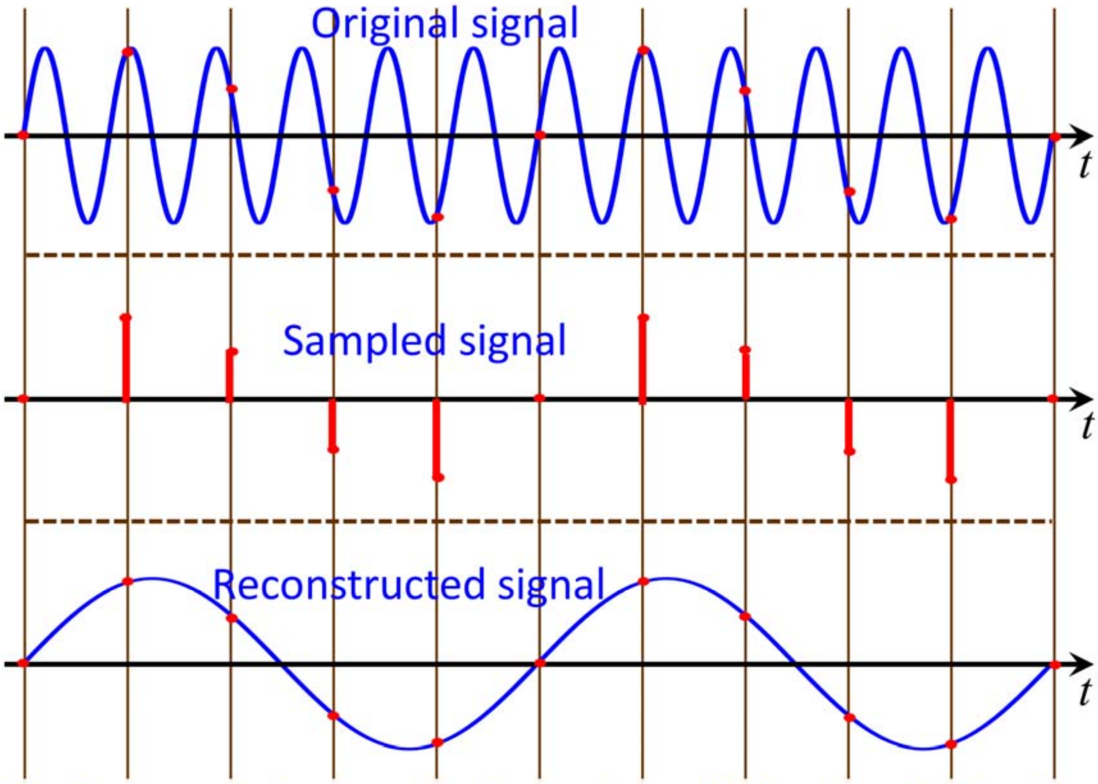
A continuous time signal may be sampled to produce a discrete time signal.



If the rate of sampling is very high, it is obvious by inspection that the sampled result will resemble that of the original continuous time signal.

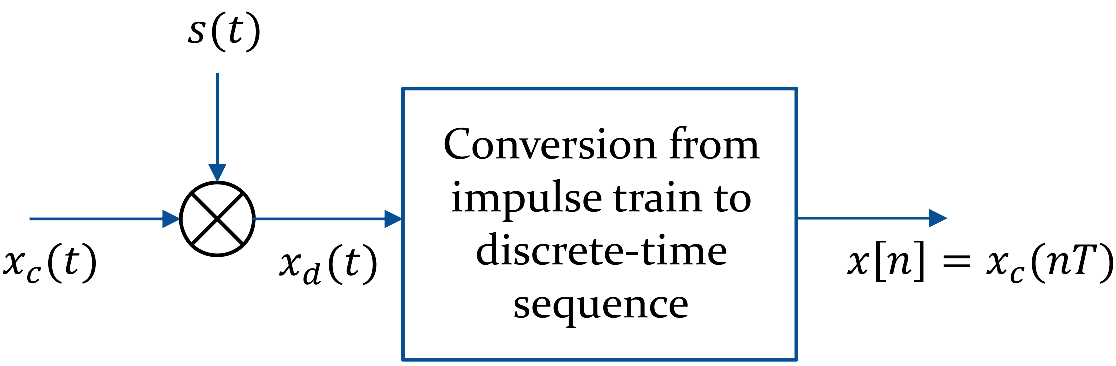


If the sampling rate is sufficiently high, it is possible to reconstruct the original signal from the sampled signal.



If the sampling rate is not sufficiently high, the reconstructed signal is different from the original signal.

Sampling with a periodic impulse train



$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ where } \delta(t) = \infty \text{ for } t = 0$$
$$= 0 \text{ for } t \neq 0$$

$$\text{and } \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

Frequency-Domain Representation of Sampling

Let $x_c(t)$ be continuous time signal

Let the sampled version of $x_c(t)$ be denoted by $x_d(t)$.

Let the sampling interval be T.

Let $\Omega_s = 2\pi/T$.

Let $X_c(j\Omega)$ be the Fourier transform of $x_c(t)$.

Let $X_d(j\Omega)$ be the Fourier transform of $x_d(t)$.

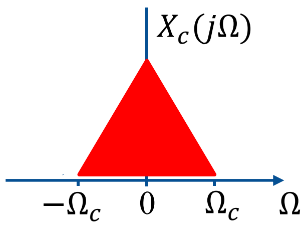
It can be shown that

$$X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

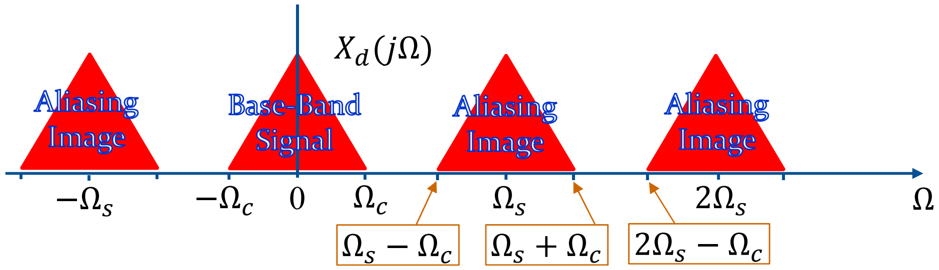
$1/T$ indicates that the magnitude of $X_d(j\omega)$ increases with sampling density $1/T$

$X_c(j(\Omega - n\Omega_s))$ is $X_c(j\Omega)$ shifted along the Ω-axis by $n\Omega_s$

$$X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

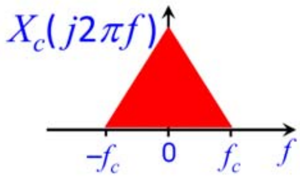


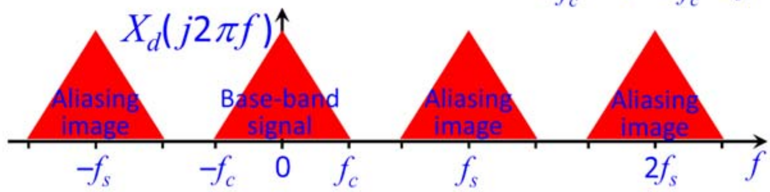
$X_c(j\Omega)$, frequency spectrum of the original continuous time signal, the base-band signal



"Hz" is often used as frequency unit in communication systems. Hence, replacing Ω by $2\pi f$ we have

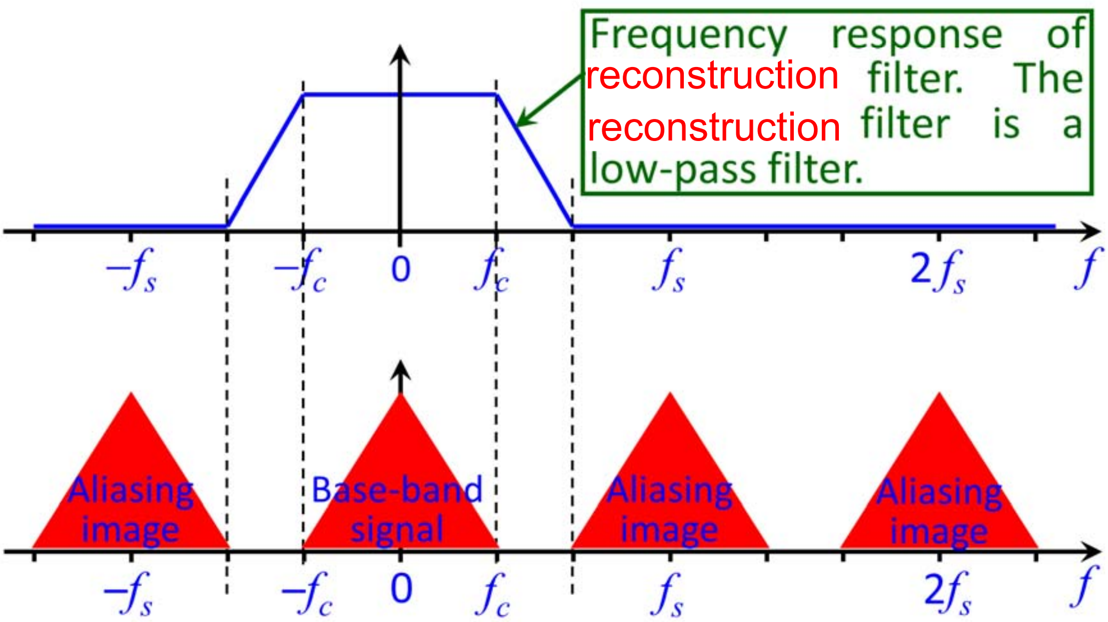
$$X_d(j2\pi f) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(j2\pi(f - nf_s)\right)$$

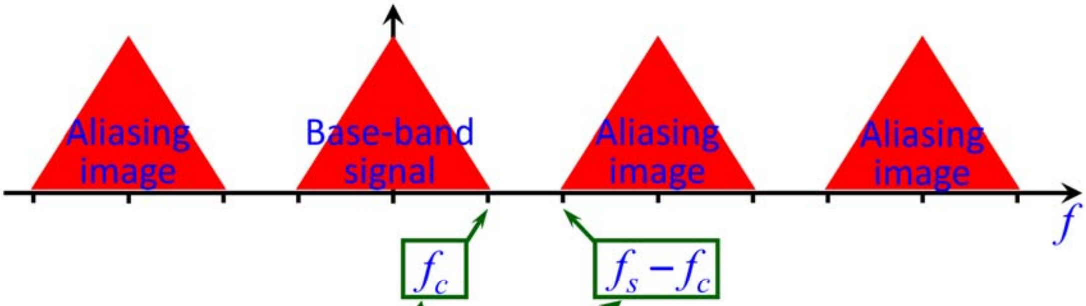




When a continuous time signal (the base-band signal) is sampled at a rate of f_s samples per second, the frequency spectrum of the sampled signal is that of the base-band signal plus duplicates (aliasing) of that of the base-band signal centred at kf_s where k = …, –1, 0, 1, 2, 3, ….

The original continuous time signal can be recovered by removing the aliasing images by low-pass filtering. The low-pass filter is called a reconstruction filter.





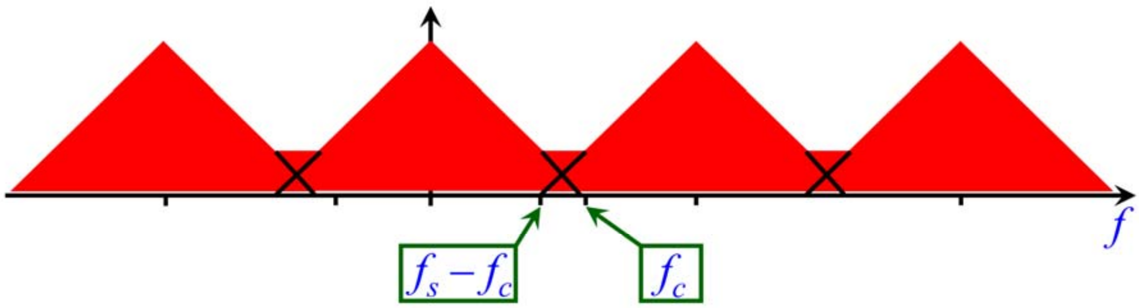
If the original continuous time signal can be completely recovered, the aliasing is not a serious problem. Recovering of the original continuous time signal is possible if and only if there is no overlap between the base-band signal and the aliasing images.

$f_s - f_c > f_c$, i.e. $f_s > 2f_c$.

Nyquist Frequency & Nyquist Rate

- The highest frequency (f_c) contained in a continuous signal $x_c(t)$ is usually called **Nyquist frequency**, which determines the minimum sampling frequency ($f_s = 2f_c$) that must be used to fully recover $x_c(t)$ from its sampled version.
- The minimum sampling frequency (or sampling rate), $f_s = 2f_c$, required to avoid aliasing from irrecoverable problem is called **Nyquist rate**.

If $f_s - f_c < f_c$, i.e. $f_s < 2f_c$,

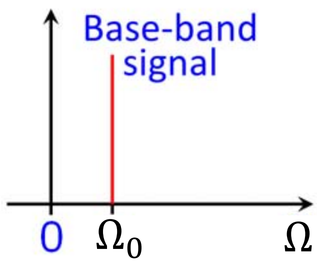


Example:

The output of an ideal low-pass filter with band-edges at $\pm\pi$ radians per second is sampled at a rate of f_s samples per second. What is the minimum f_s in order to avoid aliasing from causing irrecoverable problem?

Since the band-edges of the ideal low-pass filter is $\pm\pi$ radians per second, the highest frequency component of the low-pass filter output is π radians per second. We have $\Omega = 2\pi f$. Thus, the highest frequency component of the low-pass filter output is $\pi/(2\pi)$ = ½ Hz. The sampling rate must be greater than twice the maximum frequency. Hence, minimum sampling rate is 1 sample per second.

Spectral lines of a sampled complex sinusoid.



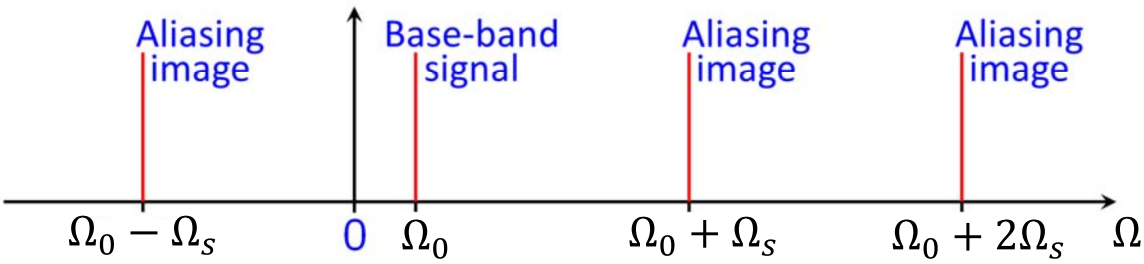
$x_c(t) = e^{j\Omega_0 t} \Leftrightarrow X_c(j\Omega)$

$X_c(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$

Sampling interval = T

$\Omega_s = \frac{2\pi}{T}$

$$X_d(j\Omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\Omega - \Omega_0 - n\Omega_s)$$



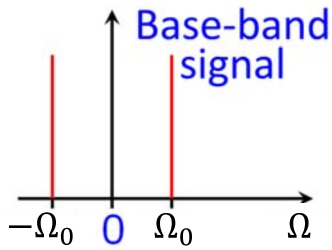
Spectral lines of a sampled sinusoid.

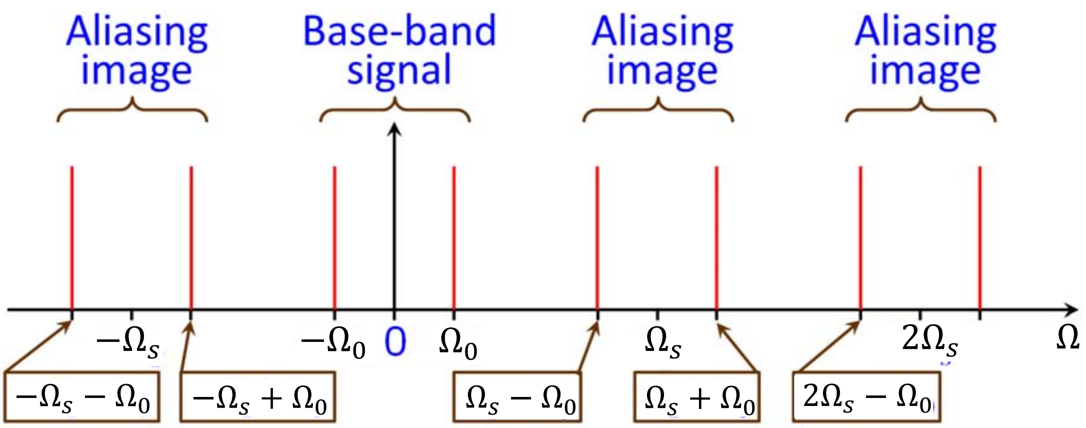
$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$

Sampling interval = T

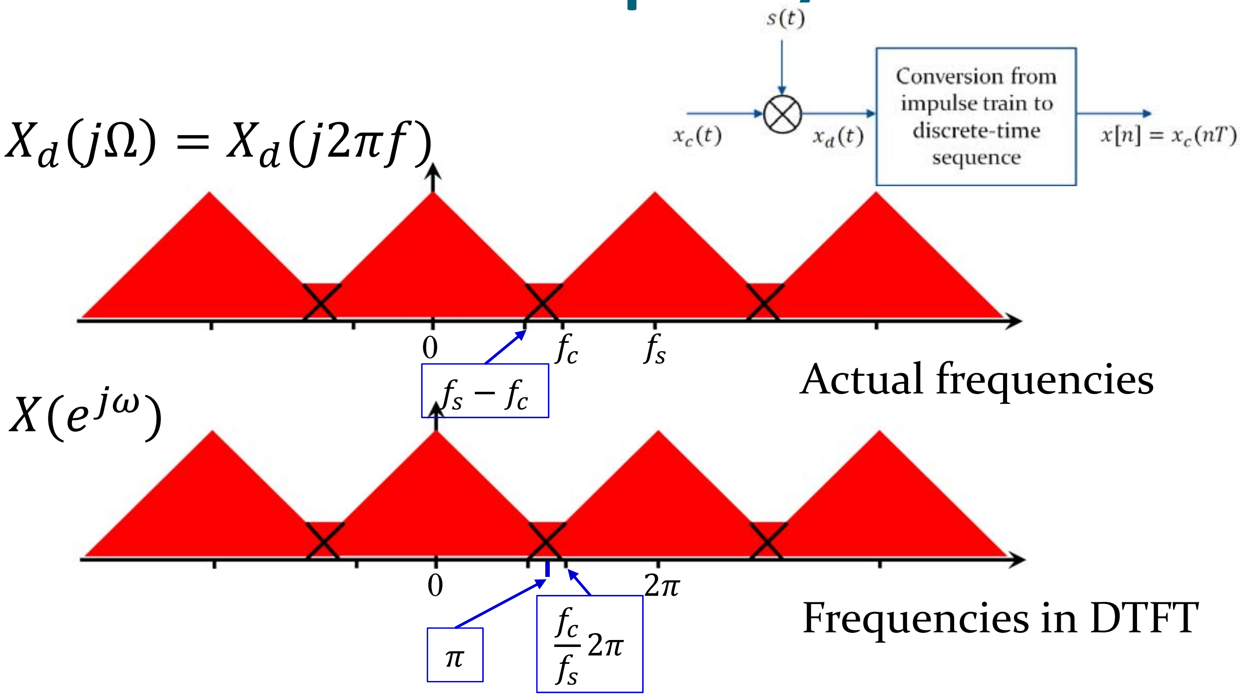
$\Omega_s = \frac{2\pi}{T}$

$$X_d(j\Omega) = \frac{\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\Omega \pm \Omega_0 - n\Omega_s)$$



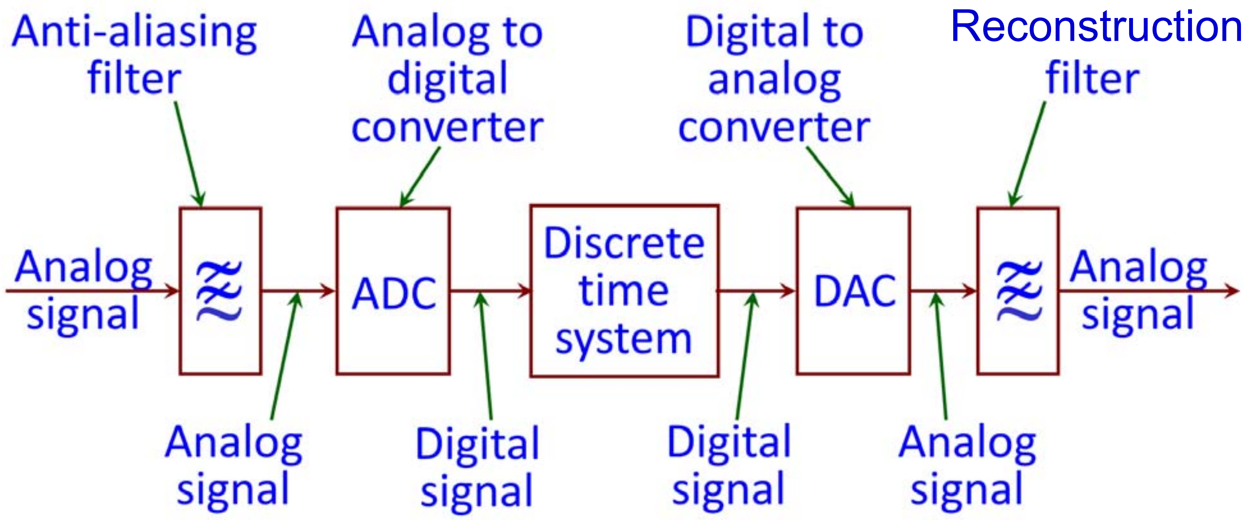


Normalization of frequency in DTFT



- Actual sampling frequency $\Omega_s = 2\pi f_s \leftrightarrow 2\pi$ in DTFT
- Normalization of frequency: $\frac{\omega}{\Omega} = \frac{\omega}{2\pi f} = \frac{2\pi}{\Omega_s}, \omega \leftrightarrow 2\pi\frac{\Omega}{\Omega_s} = 2\pi\frac{f}{f_s}$

A typical discrete time system configuration



Examples of sampling rate:

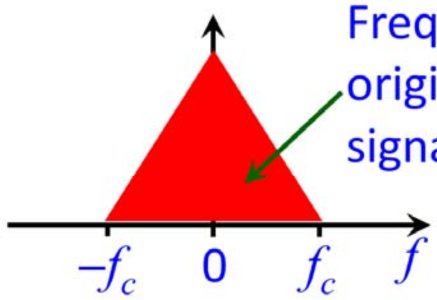
(a) CD : 44.1 kHz.

(b) Digital audio tape : 48 kHz.

(c) Telephone system : 8 KHz.

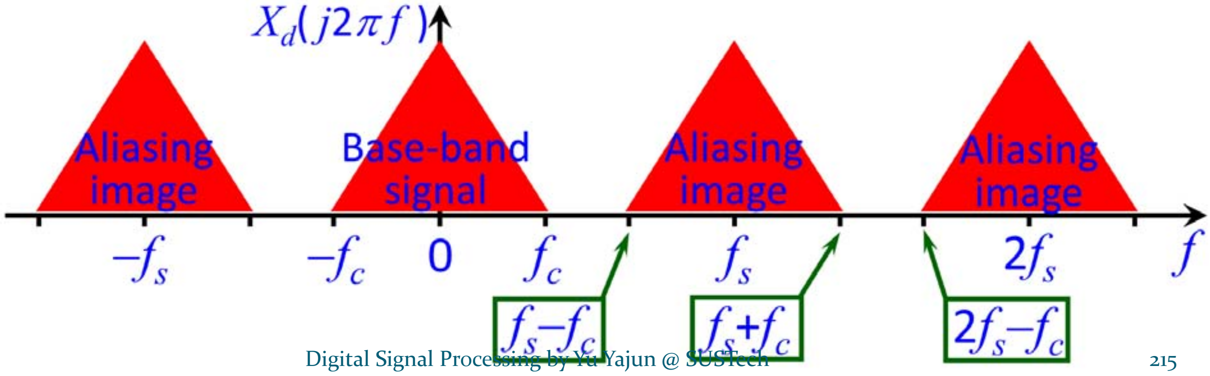
Summary:

1.

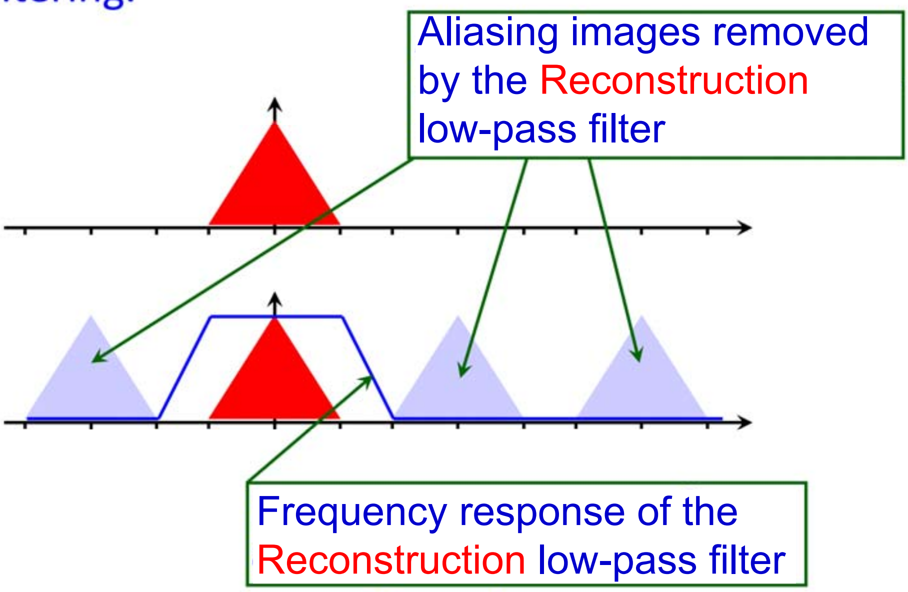


Frequency spectrum of the original continuous time signal.

Frequency spectrum of the sampled discrete time signal.

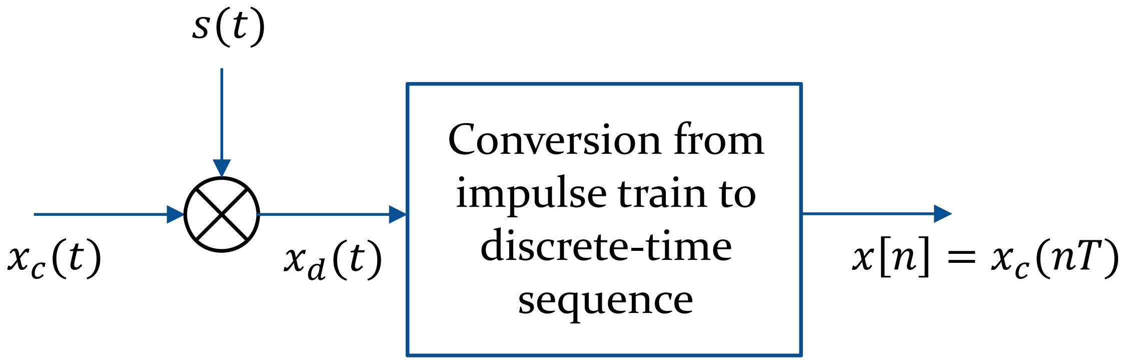


2. Information in the original continuous time signal can be recovered from the sampled signal by low-pass filtering.



3. Nyquist rate = 2 × maximum frequency.

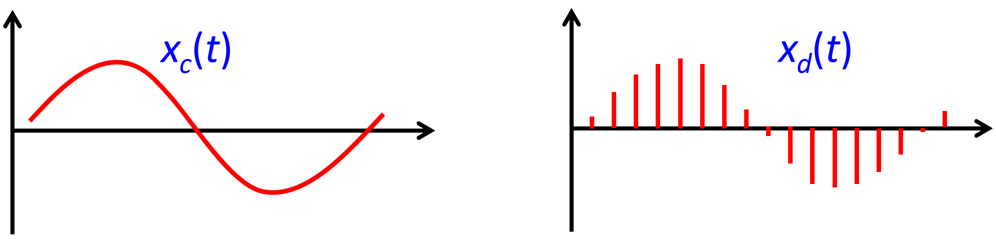
Mathematical Derivation



To prove $X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\big(j(\Omega - n\Omega_s)\big)$

$$x_d(t) = x_c(t)s(t) = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Approach 1:



Recall, the Fourier Series of

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t}, \quad \text{where } \Omega_s = \frac{2\pi}{T}$$

Thus,

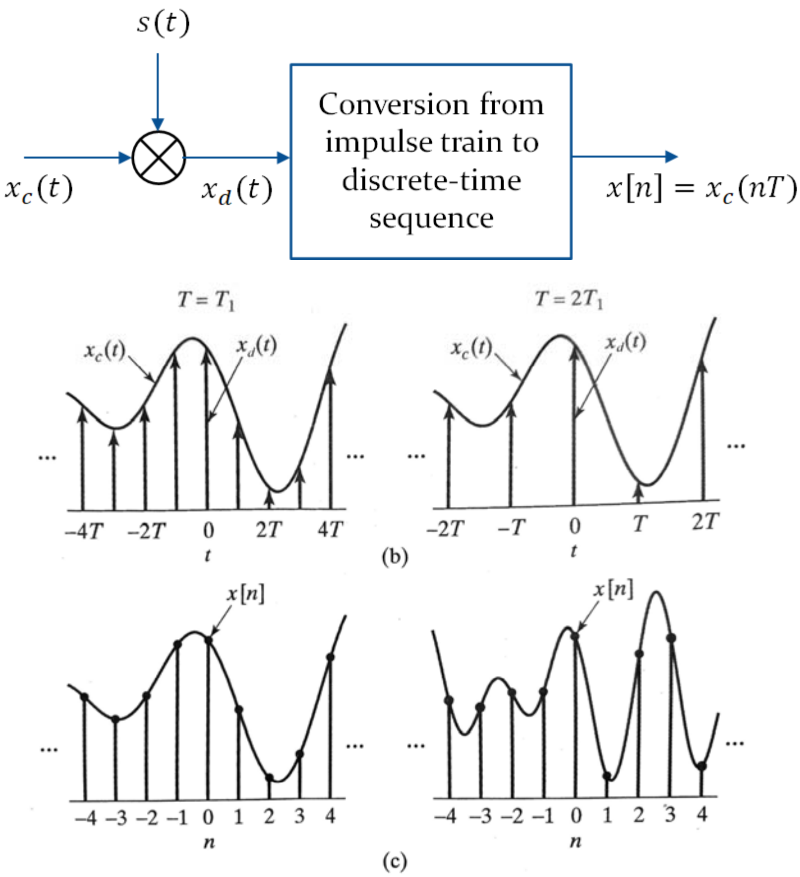
$$x_d(t) = x_c(t)\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\Omega_s t},$$

Now, look at the spectrum of the transformed signal. Using the convolution property, we have

$$X_d(j\Omega) = \frac{1}{2\pi}\frac{1}{T}X_c(j\Omega) \circledast \sum_{n=-\infty}^{\infty} 2\pi\delta(\Omega - n\Omega_s)$$

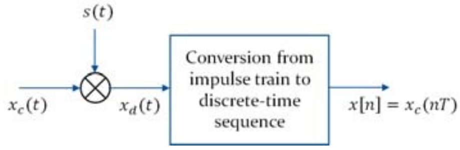
$$= \frac{1}{T}\int_{-\infty}^{\infty} X_c(j\varphi) \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_s - \varphi)\, d\varphi$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\big(j(\Omega - n\Omega_s)\big)$$



Since $x_d(t) = \sum_{n=-\infty}^{\infty} x_c(t)\delta(t-nT) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT)$

we have, $$X_d(j\Omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega Tn},$$



Furthermore, since $x[n] = x_c(nT)$, and

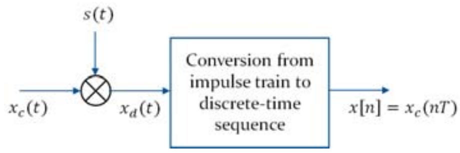
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n},$$

it follows that

$$X_d(j\Omega) = X(e^{j\omega})\Big|_{\omega=\Omega T} = X(e^{j\Omega T}).$$

Consequently,

$$X(e^{j\Omega T}) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

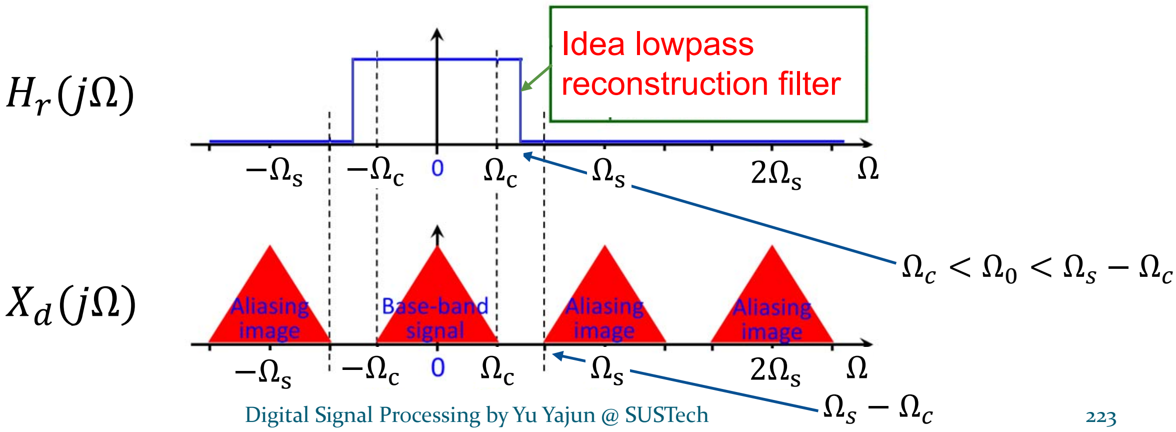


$$X(e^{j\Omega T}) = X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s))$$

- $X(e^{j\omega})$ is a frequency scaled version of $X_d(j\Omega)$ with the frequency scaling specified by $\omega = \Omega T$.
- This scaling can alternatively be thought of as a normalization of the frequency axis so that the frequency $\Omega = \Omega_s$ in $X_d(j\Omega)$ is normalized to $\omega = 2\pi$ for $X(e^{j\omega})$.
- The normalization in the transformation from $X_d(j\Omega)$ to $X(e^{j\omega})$ is directly a result of the time normalization in the transformation from $x_d(t)$ to $x[n]$.

Recovery of the Analog Signal

- If the discrete-time signal is obtained by sampling an analog signal and the sampling rate satisfies the Sampling Theory, the analog signal may be fully recovered.
- $x_c(t) \xrightarrow{sampling} x_d(t) \xrightarrow{normalization} x[n] \xrightarrow{reconstruction} \hat{x}_c(t)$



Reconstruction Filter

- The frequency response

$$H_r(j\Omega) = \begin{cases} T, |\Omega| \le \Omega_0 \\ 0, |\Omega| > \Omega_0 \end{cases}, \qquad \text{for } \Omega_0 = \frac{\Omega_s}{2}$$

- The impulse response

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_0}^{\Omega_0} e^{j\Omega t}d\Omega$$
$$= \frac{\sin(\Omega_0 t)}{\Omega_s t/2}, \qquad \text{for } -\infty < t < \infty$$

Recovering (i.e., filtering)

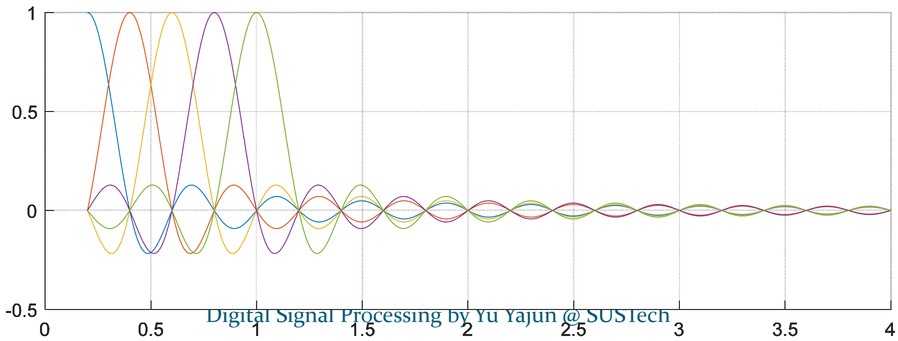
$$h_r(t) = \frac{\sin(\Omega_0 t)}{\Omega_s t/2}$$

- Convolution of the sampled signal and recovery filter

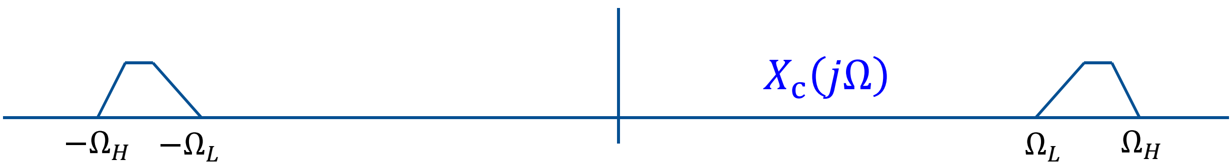
$$\hat{x}_c(t) = x[n] \circledast h_r(nT) = \sum_{n=-\infty}^{\infty} x[n] h_r(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\Omega_0(t - nT))}{\Omega_s(t - nT)/2} = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\Omega_0 T(t - nT)/T)}{\Omega_s T(t - nT)/2T}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}$$

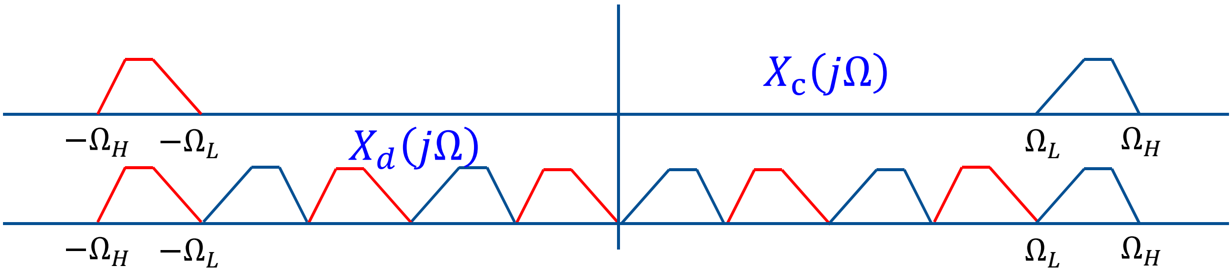


Sampling of Bandpass Signal



- We can, of course, sample such a continuous-time bandpass signal by a frequency rate higher than $2\Omega_H$ to prevent aliasing.
  - The spectrum of the discrete-time signal obtained by sampling will have spectral gaps.
  - If Ω_H is very large, the sampling frequency has to be very high, which may not be practical in some situation.

A more practical and efficient approach



- Let $\Omega_H - \Omega_L = \Delta\Omega$, defined to be the bandwidth of the bandpass signal.
- Assume that Ω_H is integer multiple of the bandwidth, i.e., $\Omega_H = M\Delta\Omega$.
- We choose $\Omega_s = 2\Delta\Omega = 2\Omega_H/M$. Thus, $X_d(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c(j(\Omega - n\Omega_s)) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\big(j(\Omega - 2n\Delta\Omega)\big)$