Tutorial12

1. The general form of the transfer function $H(z)$ of a linear-phase FIR filter with a real-valued impulse response is given by

$$H(z) = (1+z^{-1})^{N_1}(1-z^{-1})^{N_2}\prod_{i=1}^{N_3}(1+\alpha_i z^{-1}+z^{-2})\prod_{i=1}^{N_4}(1+\beta_i z^{-1}+\gamma_i z^{-2}+\beta_i z^{-3}+z^{-4})$$

What are the values of the constants N_1, N_2, N_3, and N_4 for the lowest-order Type I, Type II, Type III, and Type IV linear-phase FIR filters, respectively.

A: (1) Type I filter has a even filter order with a mirror image polynomial transfer function. The minimum even order filter is the second order filter. Thus,

(a) $N_1 = 2, N_2 = 0, N_3 = 0, N_4 = 0$, or

(b) $N_1 = 0, N_2 = 2, N_3 = 0, N_4 = 0$, or

(c) $N_1 = 0, N_2 = 0, N_3 = 1, N_4 = 0$

all may result a second order mirror image polynomial transfer function. $[1, \pm 2\ or\ a_1, 1]$

Example: $H_1(z) = -1 + 2z^{-1} - 3z^{-2} + 6z^{-3} - 3z^{-4} + 2z^{-5} - z^{-6}$

(2) Type II filter has an odd filter order with a mirror image polynomial transfer function. The minimum odd order filter is the first order filter. Only $N_1 = 1, N_2 = 0, N_3 = 0, N_4 = 0$ may result a first order mirror image polynomial transfer function. $[1, 1]$

(3) Type III filter has a even filter order with a anti-mirror image polynomial transfer function. The minimum even order filter is the second order filter. Thus, only $N_1 = 1, N_2 = 1, N_3 = 0, N_4 = 0$, may result a second order anti-mirror image polynomial transfer function. $[1, 0, -1]$ $0 * z^{-3}$

Example: $H_3(z) = 1 - 2z^{-1} + 3z^{-2} - 3z^{-4} + 2z^{-5} - z^{-6}$

(4) Type VI filter has an odd filter order with an anti-mirror image polynomial transfer function. The minimum odd order filter is the first order filter. Thus, only $N_1 = 0, N_2 = 1, N_3 = 0, N_4 = 0$ may result a first order anit-mirror image polynomial transfer function. $[1, -1]$

2. Design a first-order lowpass IIR digital filter with a normalized 3-dB cutoff frequency of 0.42 rad/samples.

A: The first-order lowpass IIR digital filters have a general form transfer function given by $H(z) = \frac{1-\alpha}{2}\frac{1+z^{-1}}{1-\alpha z^{-1}}$, for $0 < \alpha < 1$.

The 3-dB cutoff frequency is at $\omega_c = \cos^{-1}\frac{2\alpha}{1+\alpha^2}$. (P. 456)

Thus, for given ω_c, $\alpha = \frac{1-\sin\omega_c}{\cos\omega_c}$, i.e.,$\alpha = \frac{1-\sin 0.42}{\cos 0.42} = 0.6486$.

Therefore, the z-transform transfer function of the lowpass IIR digital filter is

$$H(z) = \frac{1-0.6486}{2}\frac{1+z^{-1}}{1-0.6486z^{-1}} = \frac{0.1757(1+z^{-1})}{1-0.6486z^{-1}}$$

3. A bandstop IIR digital filter can be generated by a second-order transfer functions given by

$$H_{BS}(z) = \frac{1+\alpha}{2} \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}}, |\alpha| < 1, |\beta| < 1$$

(a) Determine the squared-magnitude response of the bandstop IIR filter.

(b) Show that the notch frequency ω_0, at which the magnitude response is 0, is given by $\omega_0 = \cos^{-1}\beta$.

(c) Determine the magnitude response at $\omega = 0$ and $\omega = \pi$.

(d) It is known that the maximum magnitude response of the filter is 1. Show that the 3-dB notch bandwidth of the bandstop filter is given by $B_w = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$.

A: (a) The squared-magnitude response is:

$$\left|H_{BS}(e^{j\omega})\right|^2 = H_{BS}(z)H_{BS}(z^{-1})\Big|_{z=e^{j\omega}}$$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{1 - 2\beta z^{-1} + z^{-2}}{1 - \beta(1+\alpha)z^{-1} + \alpha z^{-2}} \frac{1 - 2\beta z + z^2}{1 - \beta(1+\alpha)z + \alpha z^2}\Bigg|_{z=e^{j\omega}}$$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{(z^{-2} + z^2) - 4\beta(z^{-1}+z) + 2 + 4\beta^2}{\alpha(z^{-2} + z^2) - \beta(1+\alpha)^2(z^{-1}+z) + 1 + \alpha^2 + \beta^2(1+\alpha)^2}\Bigg|_{z=e^{j\omega}}$$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{2\cos 2\omega - 8\beta\cos\omega + 2 + 4\beta^2}{2\alpha\cos 2\omega - 2\beta(1+\alpha)^2\cos\omega + 1 + \alpha^2 + \beta^2(1+\alpha)^2}$$

(b) $\left|H_{BS}(e^{j\omega})\right|^2 = 0 \Longrightarrow 2\cos 2\omega - 8\beta\cos\omega + 2 + 4\beta^2 = 0$

$$\Longrightarrow (\cos\omega)^2 - 2\beta\cos\omega + \beta^2 = 0$$

$$\Longrightarrow \cos\omega = \beta \Longrightarrow \omega_0 = \cos^{-1}\beta$$

Recall that $\left|H_{BS}(e^{j\omega})\right|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{2\cos 2\omega - 8\beta\cos\omega + 2 + 4\beta^2}{2\alpha\cos 2\omega - 2\beta(1+\alpha)^2\cos\omega + 1 + \alpha^2 + \beta^2(1+\alpha)^2}$

(c) $H_{BS}(e^{j0}) = \left(\frac{1+\alpha}{2}\right)^2 \frac{2-8\beta+2+4\beta^2}{2\alpha-2\beta(1+\alpha)^2+1+\alpha^2+\beta^2(1+\alpha)^2}$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{4(1-\beta)^2}{(1-\beta)^2(1+\alpha)^2} = 1$$

$H_{BS}(e^{j\pi})$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{2+8\beta+2+4\beta^2}{2\alpha+2\beta(1+\alpha)^2+1+\alpha^2+\beta^2(1+\alpha)^2}$$

$$= \left(\frac{1+\alpha}{2}\right)^2 \frac{4(1+\beta)^2}{(1+\beta)^2(1+\alpha)^2} = 1$$

(d) $\left|H_{BS}(e^{j\omega})\right|^2 = \left(\frac{1+\alpha}{2}\right)^2 \frac{4(\beta-\cos\omega)^2}{(1+\alpha)^2(\beta-\cos\omega)^2+(1-\alpha)^2(\sin\omega)^2} = \frac{1}{2}$

$\Longrightarrow (1+\alpha)^2(\beta-\cos\omega)^2 = (1-\alpha)^2(\sin\omega)^2$ (Eq. 1)

$\Longrightarrow \sin\omega_i = \pm\left(\frac{1+\alpha}{1-\alpha}\right)(\cos\omega_i - \beta), i = 1, 2$ (Eq. 2)

since $\omega_1 < \omega_0 < \omega_2$, $\sin\omega_1$ must have positive sign and $\sin\omega_2$ must have negative sign; otherwise, $\sin\omega_2 < 0$ for $0 < \omega_2 < \pi$.

Now (Eq. 1) can be written as:

$$(1+\alpha)^2(\beta-\cos\omega_i)^2 - (1-\alpha)^2(\sin\omega_i)^2 = 0$$

$$\Longrightarrow 2(1+\alpha^2)(\cos\omega_i)^2 - 2\beta(1+\alpha)^2\cos\omega_i$$
$$+ (1+\alpha)^2\beta^2 - (1-\alpha)^2 * 1 = 0$$

$$2(1+\alpha^2)(\cos\omega_i)^2 - 2\beta(1+\alpha)^2\cos\omega_i + (1+\alpha)^2\beta^2 - (1-\alpha)^2 = 0$$

Hence, the sum of 2 roots is

$$\cos\omega_1 + \cos\omega_2 = \beta\frac{(1+\alpha)^2}{1+\alpha^2}$$

And the multiplication of 2 roots is

$$\cos\omega_1\cos\omega_2 = \frac{\beta^2(1+\alpha)^2 - (1-\alpha)^2}{2(1+\alpha^2)}$$

Since $\cos(\omega_2 - \omega_1) = \cos\omega_2\cos\omega_1 + \sin\omega_2\sin\omega_1$

$$\boxed{\sin\omega_i = \pm\left(\frac{1+\alpha}{1-\alpha}\right)(\cos\omega_i - \beta), i = 1, 2 \qquad \text{(Eq. 2)}}$$

$$\xRightarrow{Eq.2} \cos(\omega_2 - \omega_1) = \cos\omega_2\cos\omega_1 - \left(\frac{1+\alpha}{1-\alpha}\right)^2\left(\cos\omega_2\cos\omega_1 + \beta^2 - \beta(\cos\omega_1 + \cos\omega_2)\right) = \frac{2\alpha}{1+\alpha^2}$$

Therefore, $B_w = \omega_2 - \omega_1 = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right)$

4. Based on the results obtained in Question 3, design a bandstop filter with notch frequency at 0.35 π, and a 3-dB notch bandwidth of 0.15 π.

A: The transfer function of the bandstop filter is given by

$$H_{BS}(z) = \frac{1+\alpha}{2}\frac{1-2\beta z^{-1}+z^{-2}}{1-\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

$$\omega_0 = \cos^{-1}\beta \Longrightarrow \beta = \cos\omega_0 = \cos 0.35\pi = 0.4540$$

$$B_w = \cos^{-1}\left(\frac{2\alpha}{1+\alpha^2}\right) \Longrightarrow \frac{2\alpha}{1+\alpha^2} = \cos B_w = \cos 0.15\,\pi$$

$$= \ 0.8910 \Longrightarrow \alpha^2 - \frac{2\alpha}{0.8910} + 1 = 0$$

$$\Longrightarrow \alpha_1 = 1.6319 \text{ (not stable)}, \qquad \alpha_2 \ = 0.6128$$

$$\Longrightarrow H_{BS}(z) = \frac{0.8064(1-0.908z^{-1}+z^{-2})}{1-0.7322z^{-1}+0.6128z^{-2}}$$

5. Show that the following M^{th}-order complex coefficien transfer function is that of a causal allpass filter.

$$A_M(z) = \frac{d_M^* + d_{M-1}^* z^{-1} + \cdots + d_1^* z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}}$$

A: $|A_M(z)|^2 = A_M(z) A_M^*(z)$

$$= \frac{d_M^* + d_{M-1}^* z^{-1} + \cdots + d_1^* z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}} \times \frac{d_M + d_{M-1} z + \cdots + d_1 z^{M-1} + z^M}{1 + d_1^* z + \cdots + d_{M-1}^* z^{M-1} + d_M^* z^M}$$

$$= \frac{d_M^* + d_{M-1}^* z^{-1} + \cdots + d_1^* z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}} \times \frac{z^M (d_M z^{-M} + d_{M-1} z^{-M+1} + \cdots + d_1 z^{-1} + 1)}{z^M (z^{-M} + d_1^* z^{-M+1} + \cdots + d_{M-1}^* z^{-1} + d_M^*)} = 1$$

Thus, it's an allpass filter.

$$\boxed{\left|A\left(e^{j\omega}\right)\right|^2 = 1, \qquad \text{for all } \omega}$$

Since $A_M(z) = \frac{Y(z)}{X(z)}$

$$= \frac{d_M^* + d_{M-1}^* z^{-1} + \cdots + d_1^* z^{-M+1} + z^{-M}}{1 + d_1 z^{-1} + \cdots + d_{M-1} z^{-M+1} + d_M z^{-M}},$$

i.e.,

$$\sum_{m=0}^{M} d_m Y(z) z^{-m} = \sum_{m=0}^{M} d_{M-m}^* X(z) z^{-m}$$

where $d_0 = 1$. Thus, the corresponding difference equation relating the input and output is given by

$$y[n] = \sum_{m=0}^{M} d_{M-m}^* x[n-m] - \sum_{m=1}^{M} d_m y[n-m]$$

i.e., $y[n]$ depends only on the input and output signal up to time instant *n*, and thus is a causal system.

6. The transfer function of a Type 2 linear phase FIR filter is given by

$$H_1(z) = 2.5(1 - 1.6z^{-1} + 2z^{-2})(1 + 1.6z^{-1} + z^{-2}) \times (1 + z^{-1})(1 - 0.8z^{-1} + 0.5z^{-2})$$

(a) Determine the transfer function $H_2(z)$ of a minimum-phase FIR filter having the same magnitude as that of $H_1(z)$.

(b) Determine the transfer function $H_3(z)$ of a maximum-phase FIR filter having the same magnitude as that of $H_1(z)$.

(c) How many other length-8 FIR filter exist that have the same magnitude response as that of $H_1(z)$?

A: The root of the first factor are $0.8 \pm j1.1662$, and its magnitude is larger 1, i.e., the zeros are <u>outside unit circle</u>.

The root of the second factor are $0.8 \pm j0.6$, and its magnitude is equal to 1, i.e., the zeros are <u>on unit circle</u>.

The root of the third factor is -1, i.e., the zero is <u>on unit circle</u>.

The root of the forth factor is $0.4 \pm j0.5831$, and its magnitude is smaller than 1, i.e., zeros are <u>inside the unit circle</u>. Therefore, $\boxed{H(z) = H_{\min}(z)A_m(z)}$

(a) $$H_1(z) = H_2(z)\frac{(1-1.6z^{-1}+2z^{-2})}{(2-1.6z^{-1}+z^{-2})}$$

$$H_2(z) = H_1(z)\frac{(2-1.6z^{-1}+z^{-2})}{(1-1.6z^{-1}+2z^{-2})}$$
$$= 2.5(2-1.6z^{-1}+z^{-2})(1+1.6z^{-1}+z^{-2})$$
$$(1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})$$

(b) $$H_1(z) = H_3(z)\frac{(1-0.8z^{-1}+0.5z^{-2})}{(0.5-0.8z^{-1}+z^{-2})}$$

$$H_3(z) = H_1(z)\frac{(0.5-0.8z^{-1}+z^{-2})}{(1-0.8z^{-1}+0.5z^{-2})}$$
$$= 2.5(1-1.6z^{-1}+2z^{-2})(1+1.6z^{-1}+z^{-2})$$
$$\times (1+z^{-1})(0.5-0.8z^{-1}+z^{-2})$$

$$H_1(z) = 2.5(1-1.6z^{-1}+2z^{-2})(1+1.6z^{-1}+z^{-2})$$
$$\times (1+z^{-1})(1-0.8z^{-1}+0.5z^{-2})$$

(c) There are in total 3 length-8 FIR filter that have the same magnitude response, i.e., $H_1(z), H_2(z),$ and $H_3(z)$. (original, minimum/maximum pahse)

There are no other length-8 FIR filter that have the same magnitude response as $H_1(z)$.

7. A typical transmission channel is characterized by a causal transfer function

$$H(z) = \frac{(2.2 + 5z^{-1})(1 - 3.1z^{-1})}{(1 + 0.81z^{-1})(1 - 0.62z^{-1})}$$

In order to correct for the magnitude distortion introduced by the channel on a signal passing through it, we wish to connect a causal stable digital filter characterized by a transfer function $G(z)$ at the receiving end. Determine $G(z)$.

A: In order to correct for the magnitude distortion, we require that the transfer function $G(z)$ satisfies:

$$\left|G\left(e^{j\omega}\right)\right| = \frac{1}{|H(e^{j\omega})|}$$

Hence, a possible solution is

$$G_d(z) = \frac{1}{H(z)} = \frac{(1 + 0.81z^{-1})(1 - 0.62z^{-1})}{(2.2 + 5z^{-1})(1 - 3.1z^{-1})}$$

Note that both poles are outside unit circle, making $G_d(z)$ unstable.

Multiply it with an allpass filter $\frac{(2.2+5z^{-1})(1-3.1z^{-1})}{(5+2.2z^{-1})(-3.1+1z^{-1})}$, resulting in the minimum phase transfer function

$$G(z) = \frac{(1 + 0.81z^{-1})(1 - 0.62z^{-1})}{(5 + 2.2z^{-1})(-3.1 + 1z^{-1})}$$

which is the desired stable solution satisfying $|G(e^{j\omega})||H(e^{j\omega})| = 1$

8. Figure 1 shows a typical closed-loop discrete-time feedback control system in which $G(z)$ is the plant and $C(z)$ is the compensator. If $G(z) = \frac{z^{-2}}{1+1.5z^{-1}+0.5z^{-2}}$ and $C(z) = K$, determine the range of values of K for which the overall structure is stable.

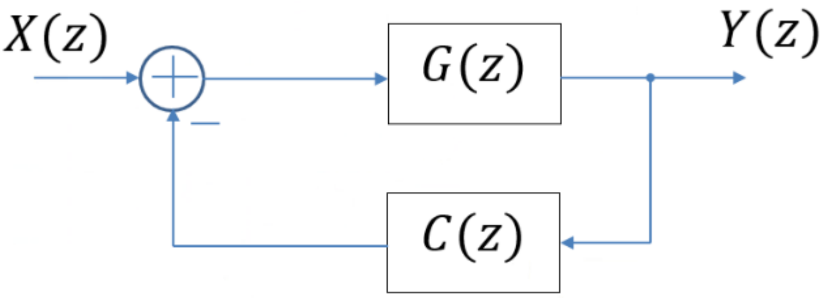

Figure 1

A: $Y(z) = (X(z) - KY(z))G(z)$

$$\Rightarrow Y(z)\big(1 + KG(z)\big) = X(z)G(z)$$

$$\Rightarrow H(z) = \frac{Y(Z)}{X(z)} = \frac{G(z)}{1 + KG(z)} = \frac{\frac{z^{-2}}{1 + 1.5z^{-1} + 0.5z^{-2}}}{1 + K\frac{z^{-2}}{1 + 1.5z^{-1} + 0.5z^{-2}}}$$

$$= \frac{z^{-2}}{1 + 1.5z^{-1} + (0.5 + K)z^{-2}}$$

The overall structure is stable if the poles of the system are inside the unit circle. The poles are located at:

$$z = \frac{-1.5 \pm \sqrt{2.25 - 4 \times 0.5 - 4K}}{2} = -0.75 \pm \frac{\sqrt{0.25 - 4K}}{2}$$

- If the poles are real poles, i.e., when $0.25 - 4K \geq 0$ $\Longrightarrow K \leq 0.0625$,

$$-0.75 + \frac{\sqrt{0.25 - 4K}}{2} < 1 \Longrightarrow K > -3$$

and

$$-0.75 - \frac{\sqrt{0.25 - 4K}}{2} > -1 \Longrightarrow K > 0$$

- If the poles are complex poles, i.e., when $0.25 - 4K < 0 \Longrightarrow K > 0.0625$

$$\frac{\sqrt{4K - 0.25}}{2} < \sqrt{1 - 0.75^2} \Longrightarrow K < 0.5$$

From the above, we obtain that when $0 < K < 0.5$, the structure is stable.

9. In the closed-loop discrete-time feedback control system of Figure 1, the plant transfer function is given by

$$G(z) = \frac{1.2 + 1.8z^{-1}}{1 + 0.7z^{-1} + 0.8z^{-2}}$$

Determine the transfer function $C(z)$ of the compensator so that the overall closed-loop transfer function of the feedback system is

$$H(z) = \frac{z^{-1} + 1.35z^{-2} + 0.9z^{-3} + 0.3375z^{-4}}{0.3 + 0.5z^{-1} + 0.505z^{-2} + 0.375z^{-3} + 0.21z^{-4}}$$

A:

$$Y(z) = \big(X(z) - C(z)Y(z)\big)G(z)$$
$$\Longrightarrow \; Y(z)\big(1 + C(z)G(z)\big) = X(z)G(z)$$

$$H(z) = \frac{Y(Z)}{X(z)} = \frac{G(z)}{1 + C(z)G(z)}$$
$$\Longrightarrow C(z) = \frac{G(z) - H(z)}{G(Z)H(z)} =$$

$$\frac{0.3 + 0.1167z^{-1} - 0.4533z^{-2} - 1.0717z^{-3} - 0.9338z^{-4} - 0.4819z^{-5} - 0.225z^{-6}}{z^{-1} + 2.85z^{-2} + 2.925z^{-3} + 1.6875 + 0.5063z^{-5}}$$