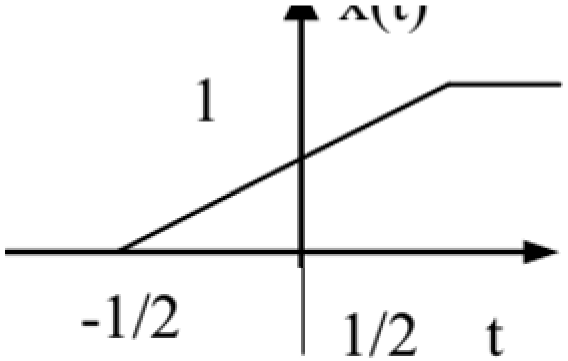
Tutorial Problems

- Basic Problems with Answers 4.8,4.9
- Basic problems 4.23
- Advanced Problems 4.39,4.40

Shuguang Xiao（肖书光），TA
12131043@mail.sustech.edu.cn

4.8. Consider the signal



$$x(t) = \begin{cases} 0, & t < -\frac{1}{2} \\ t + \frac{1}{2}, & -\frac{1}{2} \le t \le \frac{1}{2}. \\ 1, & t > \frac{1}{2} \end{cases}$$

(a) Use the differentiation and integration properties in Table 4.1 and the Fourier transform pair for the rectangular pulse in Table 4.2 to find a closed-form expression for $X(j\omega)$.

(b) What is the Fourier transform of $g(t) = x(t) - \frac{1}{2}$?

5) Differentiation/Integration

$$x(t)\begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \qquad \frac{2\sin\omega T_1}{\omega}$$

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$$

DC term

4.8 (a)The signal x(t) is as shown in Figure S4.8.

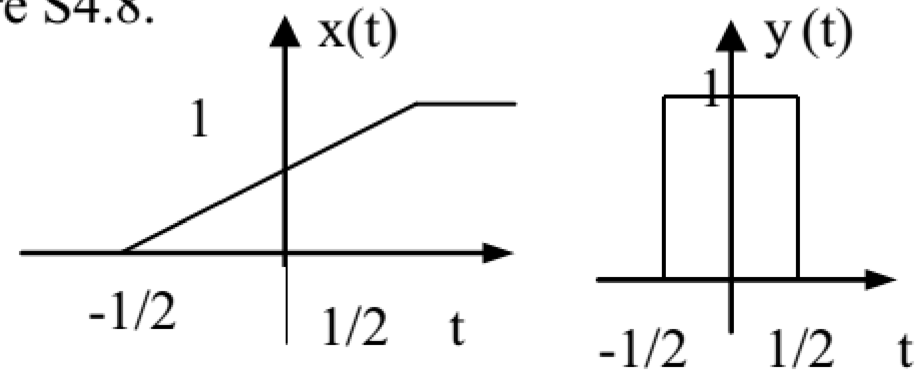


Figure S4.8

We may express this signal as

$$x(t) = \int_{-\infty}^{t} y(t)dt$$

Where y (t) is the rectangular pulse shown in S4.8 Using the integration property of FT we have

$$x(t) \xleftrightarrow{FT} X(j\omega) = \frac{1}{j\omega}Y(j\omega)+\pi Y(j0)\sigma(\omega)$$

we know from 4.2 that

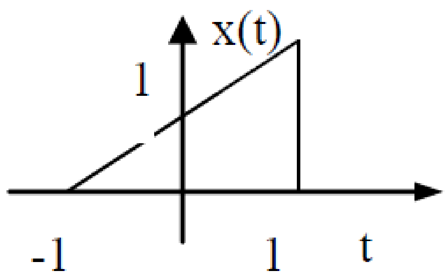
$$Y(j\omega)=\frac{2\sin(w/2)}{w}$$

Therefore $X(j\omega)= \frac{2\sin(w/2)}{jw^2}+\pi\sigma(\omega)$

(b) $Y(j\omega)=X(j\omega)-\frac{1}{2}(2\pi\sigma(\omega))= \frac{2\sin(w/2)}{jw^2}$

4.9. Consider the signal

$$x(t) = \begin{cases} 0, & |t| > 1 \\ (t+1)/2, & -1 \le t \le 1 \end{cases}.$$



(a) With the help of Tables 4.1 and 4.2, determine the closed-form expression for $X(j\omega)$.

(b) Take the real part of your answer to part (a), and verify that it is the Fourier transform of the even part of $x(t)$. $\mathcal{E}v\{x(t)\}=(x(t)+x(-t))/2$

(c) What is the Fourier transform of the odd part of $x(t)$?

$\boxed{\text{5) Differentiation/Integration}}$

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi \underbrace{X(j0)}_{\text{DC term}} \delta(\omega)$$

$$x(t)\begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \qquad \frac{2\sin\omega T_1}{\omega}$$

$$u(t) \qquad \frac{1}{j\omega} + \pi\delta(\omega)$$

Even-Odd Decomposition for Real Signals

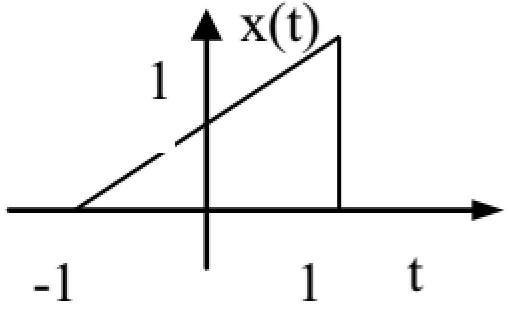
$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $\quad \mathcal{R}e\{X(j\omega)\}$

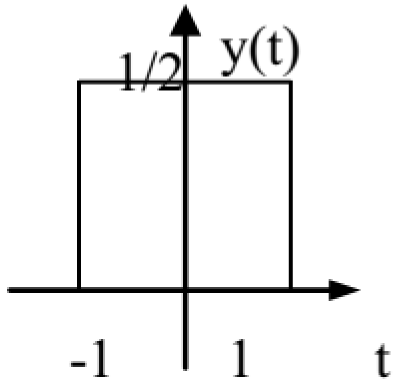
$x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real] $\quad j\mathcal{I}m\{X(j\omega)\}$

4.9 (a) the signal x(t) is plotted in figure S4.9

$$x(t)= \int_{-\infty}^{t} y(t)dt - u(t-1)$$

$$X(j\omega)= \frac{\sin\omega}{j\omega^2} - \frac{e^{-j\omega}}{j\omega}$$





(b) the even part of x(t) is given by

$\varepsilon v\{x(t)\}$=(x(t)+x(-t))/2

This is as shown in the 4.9

Therefore

$$FT\{\varepsilon v\{x(t)\}\} = \frac{\sin\omega}{\omega}$$

Now the real part of answer to part (a) is

$$\mathrm{Re}\{-\frac{e^{j\omega}}{j\omega}\} = \frac{1}{\omega}\mathrm{Re}\{j(\cos\omega - j\sin \cancel{j}\omega)\} = \frac{\sin\omega}{\omega}$$

(c) the FT of the odd part of x(t) is same as j times imaginary part of the answer to part (a),we have

$$\mathrm{Im}\{\frac{\sin\omega}{j\omega^2} - \frac{e^{-j\omega}}{j\omega}\} = -\frac{\sin\omega}{\omega^2} + \frac{\cos\omega}{\omega}$$

Therefore ,the desired result is FT{Odd part of x(t)}= $\frac{\sin\omega}{j\omega^2}$ - $\frac{\cos\omega}{j\omega}$

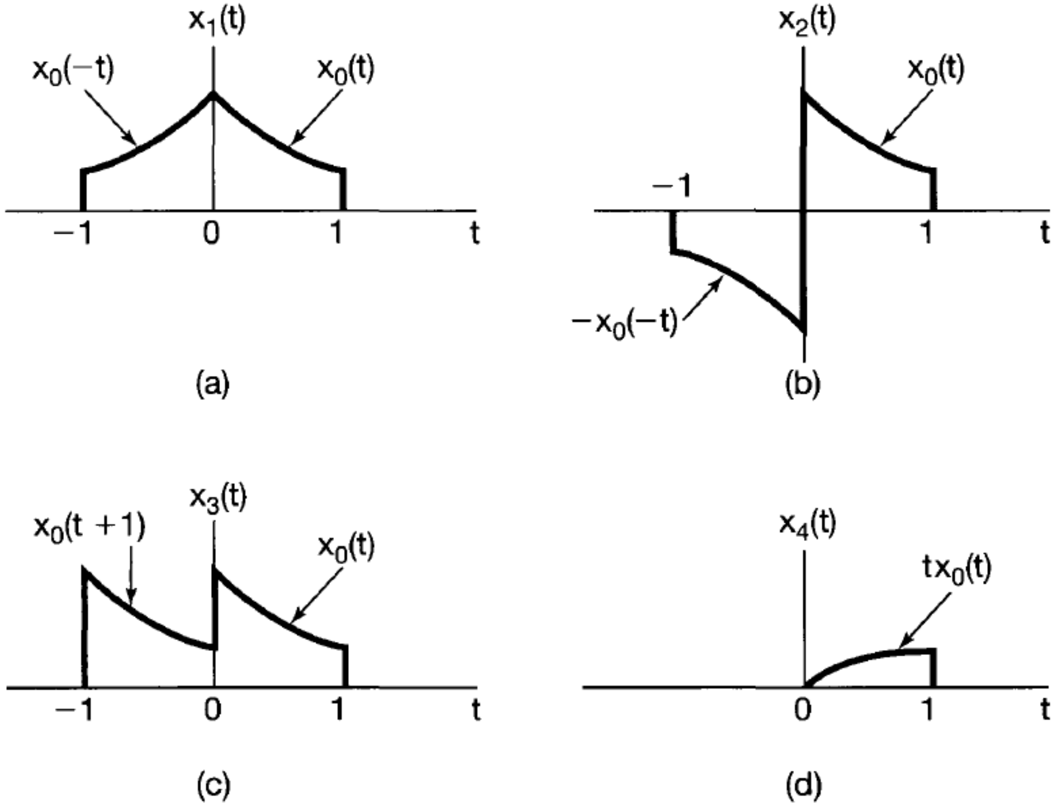
Note: $e^{ix} = \cos x + i\sin x$

4.23. Consider the signal

$$e^{-at}u(t),\ \Re e\{a\} > 0 \qquad \frac{1}{a+j\omega}$$

$$x_0(t) = \begin{cases} e^{-t}, & 0 \le t \le 1 \\ 0, & \text{elsewhere} \end{cases}. \qquad X_0(j\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$$

Determine the Fourier transform of each of the signals shown in Figure P4.23. You should be able to do this by explicitly evaluating *only* the transform of $x_0(t)$ and then using properties of the Fourier transform.



Time Reversal	$x(-t)$	$X(-j\omega)$
Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

Figure P4.23

4.23. For the given signal $x_0(t)$, we use the Fourier transform analysis eq.(4.8) to evaluate the corresponding Fourier transform

$$X_0(j\omega) = \frac{1-e^{-(1+j\omega)}}{1+j\omega}$$

we know that

$$x_1(t) = x_0(t) + x_0(-t)$$

Using the linearity and time reversal properties of the Fourier transform we have

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega) = \frac{2-2e^{-1}\cos\omega + 2\omega e^{-1}\sin\omega}{1+\omega^2}$$

(ii) we know that

$$x_2(t) = x_0(t) - x_0(-t)$$

Using the linearity and time reversal properties of Fourier transform we have

$$X_2(j\omega) = X_0(j\omega) - X_0(-j\omega) = j\frac{-2\omega + 2e^{-1}\sin\omega + 2\omega e^{-1}\cos\omega}{1+\omega^2}$$

（iii） we know that

$$x_3(t) = x_0(t) + x_0(t+1)$$

Using the linearity and time shifting properties of Fourier transform we have

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega} X_0(j\omega)$$

（iv） we know that

Using the differentiation frequency property $X_4(j\omega) = j\dfrac{d}{d\omega} X_0(j\omega)$

Therefore,

$$X_4(j\omega) = \frac{1 - j\omega e^{-1-j\omega} - 2e^{-(1+j\omega)}}{(1+j\omega)^2}$$

4.39. Suppose that a signal $x(t)$ has Fourier transform $X(j\omega)$. Now consider another signal $g(t)$ whose shape is the same as the shape of $X(j\omega)$; that is,

$$g(t) = X(jt).$$

(a) Show that the Fourier transform $G(j\omega)$ of $g(t)$ has the same shape as $2\pi x(-t)$; that is, show that

$$G(j\omega) = 2\pi x(-\omega).$$

(b) Using the fact that

$$\mathfrak{F}\{\delta(t + B)\} = e^{jB\omega}$$

in conjunction with the result from part (a), show that

$$\mathfrak{F}\{e^{jBt}\} = 2\pi\,\delta(\omega - B).$$

4.39. (a) From the Fourier analyses equation. We have

$$G(j\omega)=\int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt \qquad \text{(S4.39-1)}$$

Also from the Fourier transform equation, we have

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Switching the variables t and ω, we have

$$x(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(jt)e^{j\omega t}dt$$

We may also write this equation as

$$2\pi x(-\omega)=\int_{-\infty}^{\infty} X(jt)e^{-j\omega t}dt$$

Substituting this equation in eq. (S4.39-1), we obtain

$$G(j\omega)=2\pi x(-\omega)$$

(b) If in part (a) we have $x(t)=\delta(t+B)$, then we would have $g(t)=X(jt)=e^{jBt}$ and

$$G(j\omega)=2\pi x(-\omega)=2\pi\delta(-\omega+B)=2\pi\delta(\omega-B)$$

4.40. Use properties of the Fourier transform to show by induction that the Fourier transform of

$$x(t) = \frac{t^{n-1}}{(n-1)!}e^{-at}u(t), \quad a > 0,$$

is

$$\frac{1}{(a+j\omega)^n}.$$

$e^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{a+j\omega}$
$te^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{(a+j\omega)^2}$

Differentiation in Frequency	$tx(t)$	$j\dfrac{d}{d\omega}X(j\omega)$

4.40. When n=1, $x_1(t)=e^{-at}u(t)$ and $X_1(j\omega)=1/(a+j\omega)$

When n=2, $x_2(t)=te^{-at}u(t)$ and $X_2(j\omega)=1/(a+j\omega)^2$

Now, let us assume that the given statement is true when n=m, that is,

$$x_m(t)=\frac{t^{m-1}}{(m-1)!}e^{-at}u(t)\xleftrightarrow{FS}X_m(jw)=\frac{1}{(a+j\omega)^m}$$

For n=m+1 we may use the differentiation in frequency property to write,

$$x_{m+1}(t)=\frac{t}{m}x_m(t)\xleftrightarrow{FS}X_{m+1}(j\omega)=\frac{1}{m}j\frac{dX_m(j\omega)}{d\omega}=\frac{1}{(a+j\omega)^{m+1}}$$

This shows that if we assume that the given statement is true for n=m, then it is true for n=m+1 .Since we also shown that the given statement is true for n=2, we may argue that it is true for n=2+1=3, n=3+1=4, and so on. Therefore, the given statement is true for any n.

导数表内容

编号	原函数	导函数
1	$y=c$	$y'=0$
2	$y=n^x$	$y'=n^x\ln n$
3	$y=\log_a x$	$y'=\frac{1}{x\ln a}$
4	$y=\ln x$	$y'=\frac{1}{x}$
5	$y=x^n$	$y'=nx^{n-1}$
6	$y=\sqrt[n]{x}$	$y'=\frac{x^{-\frac{n-1}{n}}}{n}$
7	$y=\frac{1}{x^n}$	$y'=-\frac{n}{x^{n+1}}$