Notes

Assignments

4.50 & 4.51 attached → 附件上的两道题

Tutorial problems

7.41, 7.44, 7.47, 7.49

Outline

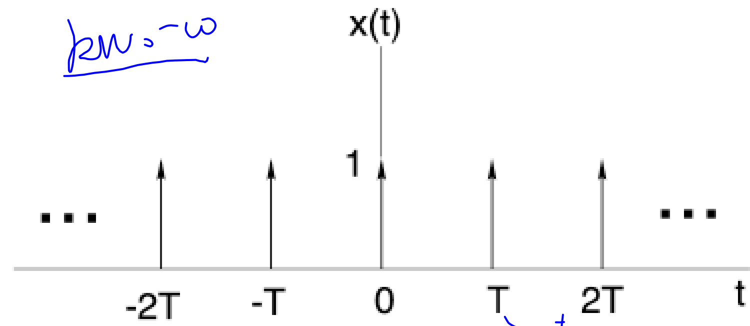
- Sampling is a general procedure to generate DT signals from CT signals, where information of the original signals can be kept
- Core sampling theory:
  - Impulse train, zero-order hold, 1st-order hold, etc.
  - Analysis in frequency domain
  - Nyquist rate
- Undersampling: Aliasing 混叠
- Application: process continuous-time signals discretely
- More sampling techniques: decimation, downsampling and upsampling

Review

Example 4.8

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$ — sampling function

$k\omega_0 - \omega$



$$x(t) \xleftrightarrow{FS} a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_o t}dt = \frac{1}{T}$$

$$\Downarrow\ x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_o t}$$

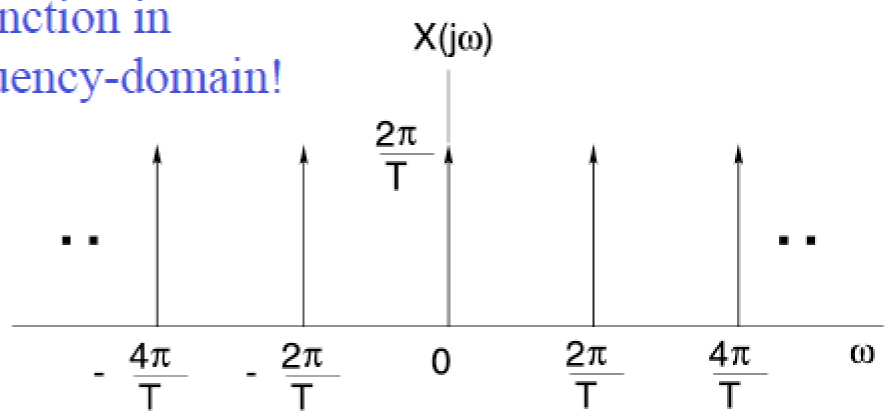
CTFT: $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

$= \int_{-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \frac{1}{T} e^{jk\omega_0 t} e^{-j\omega t} dt$

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T}\delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_o})$$ →

$\omega_s = 2\pi / T$: sampling frequency

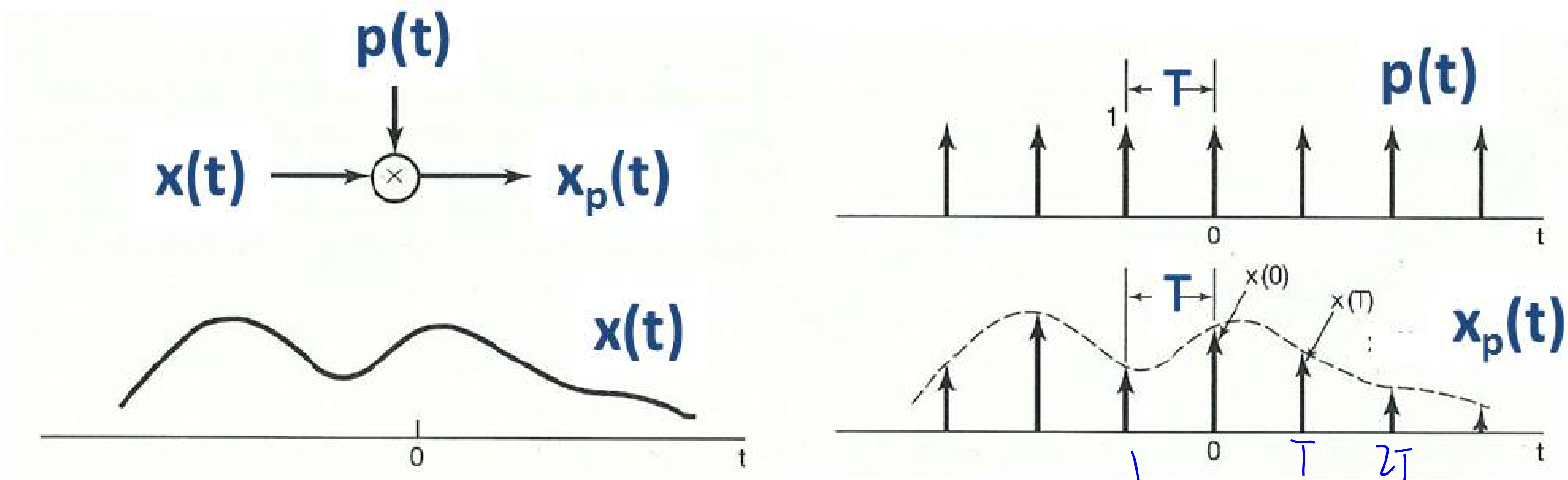
Same function in the frequency-domain!



Note in this case, periodic in both time domain (with a period T) and frequency domain (with a period $2\pi/T$)

Impulse-train sampling

- Mathematically, sampling can be represented by multiplication



频谱以ω_s为周期

采样频率

- Sampling function: $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$
- Sampling period: T
- Sampling:

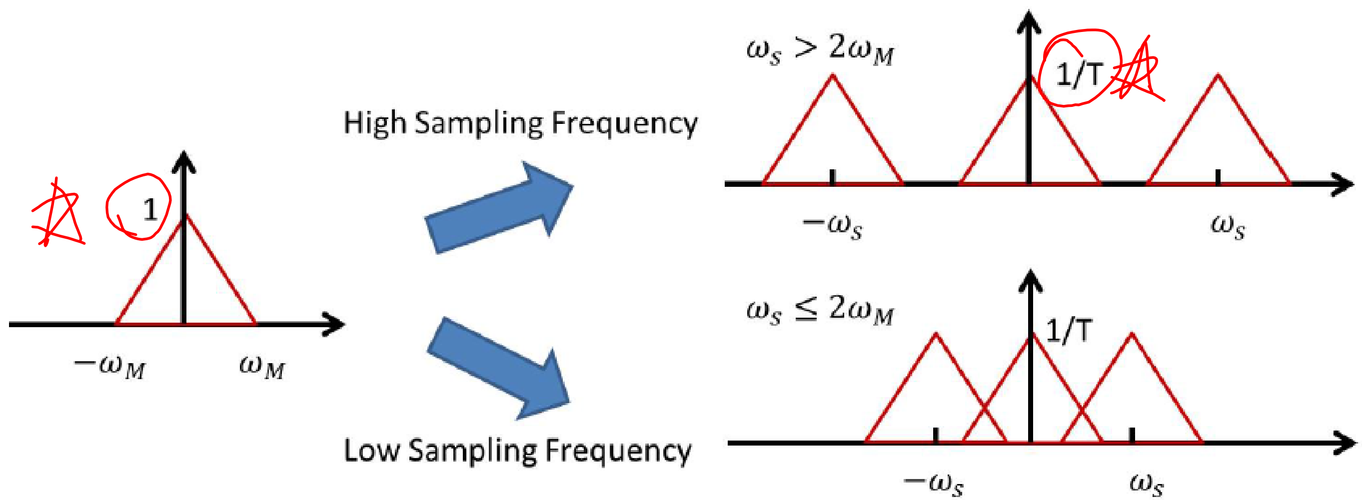
$$x_p(t) = x(t) \times p(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

频域 $\omega_s > 2\omega_M$

时域 $T < \frac{2\pi}{2\omega_M}$

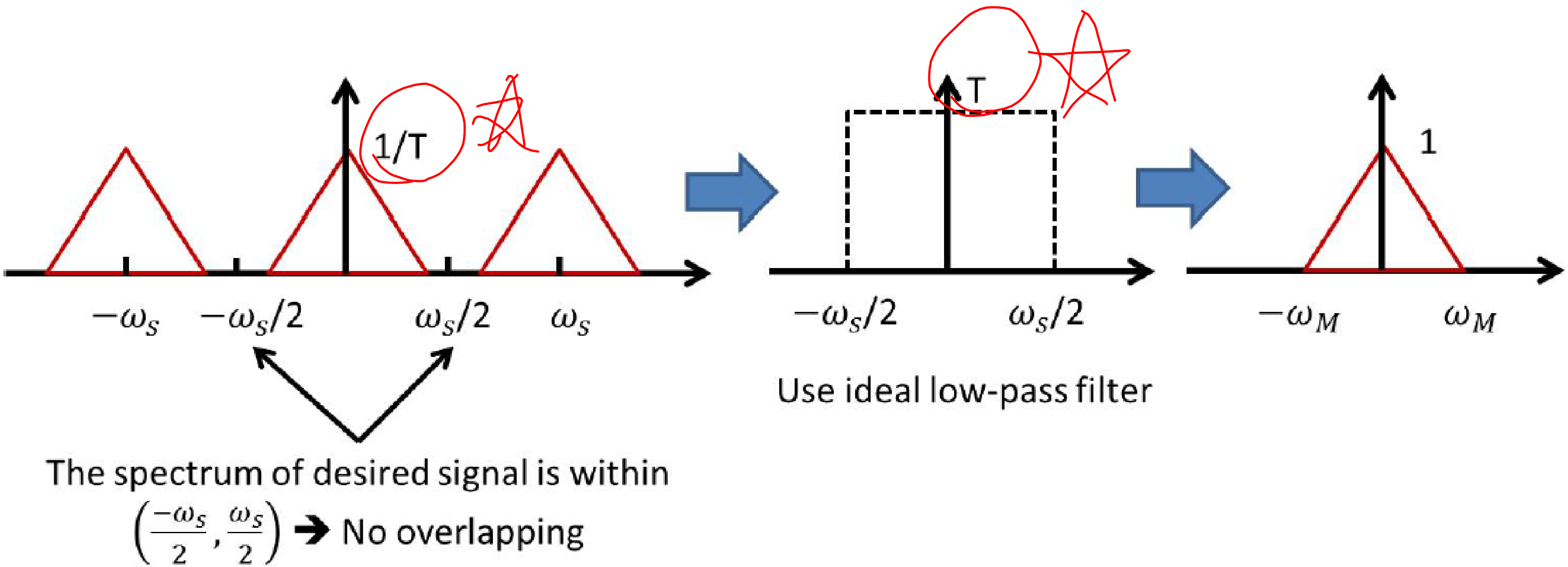
Frequency analysis

- Sampling: the Fourier transform of input signal is repeated with period ω_s



Reconstruction

- Scenario of $\omega_s > 2\omega_M$



Sampling theorem

Sampling Theorem

Let $x(t)$ be a band-limited signal with

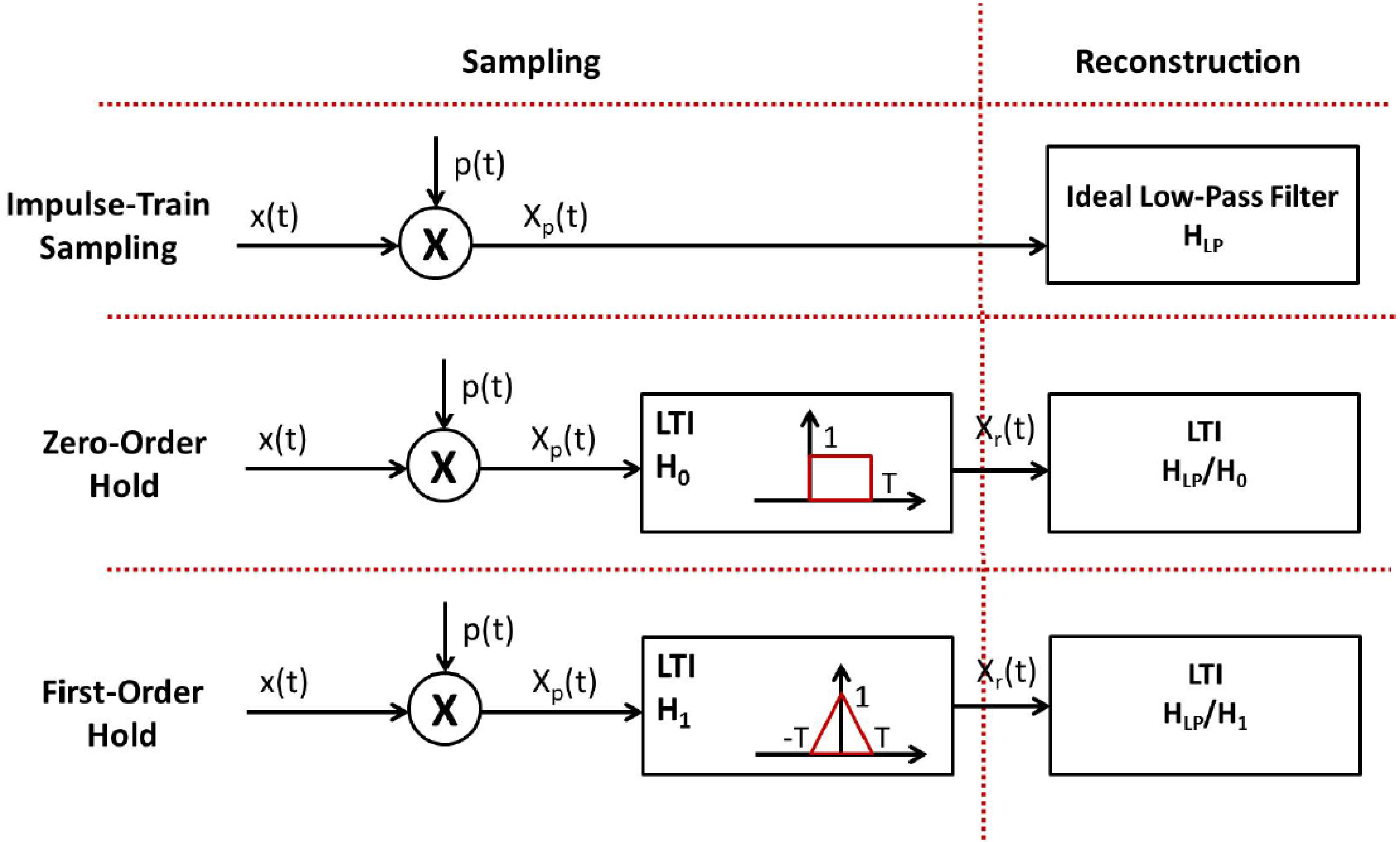
$$X(j\omega) = 0 \text{ for } |\omega| > \omega_M.$$

Then, $x(t)$ is uniquely determined by its samples $x(nT)$ or $x_p(t)$ if

$$\omega_s = \frac{2\pi}{T} > 2\omega_M,$$

where $2\omega_M$ is referred to as the Nyquist rate.

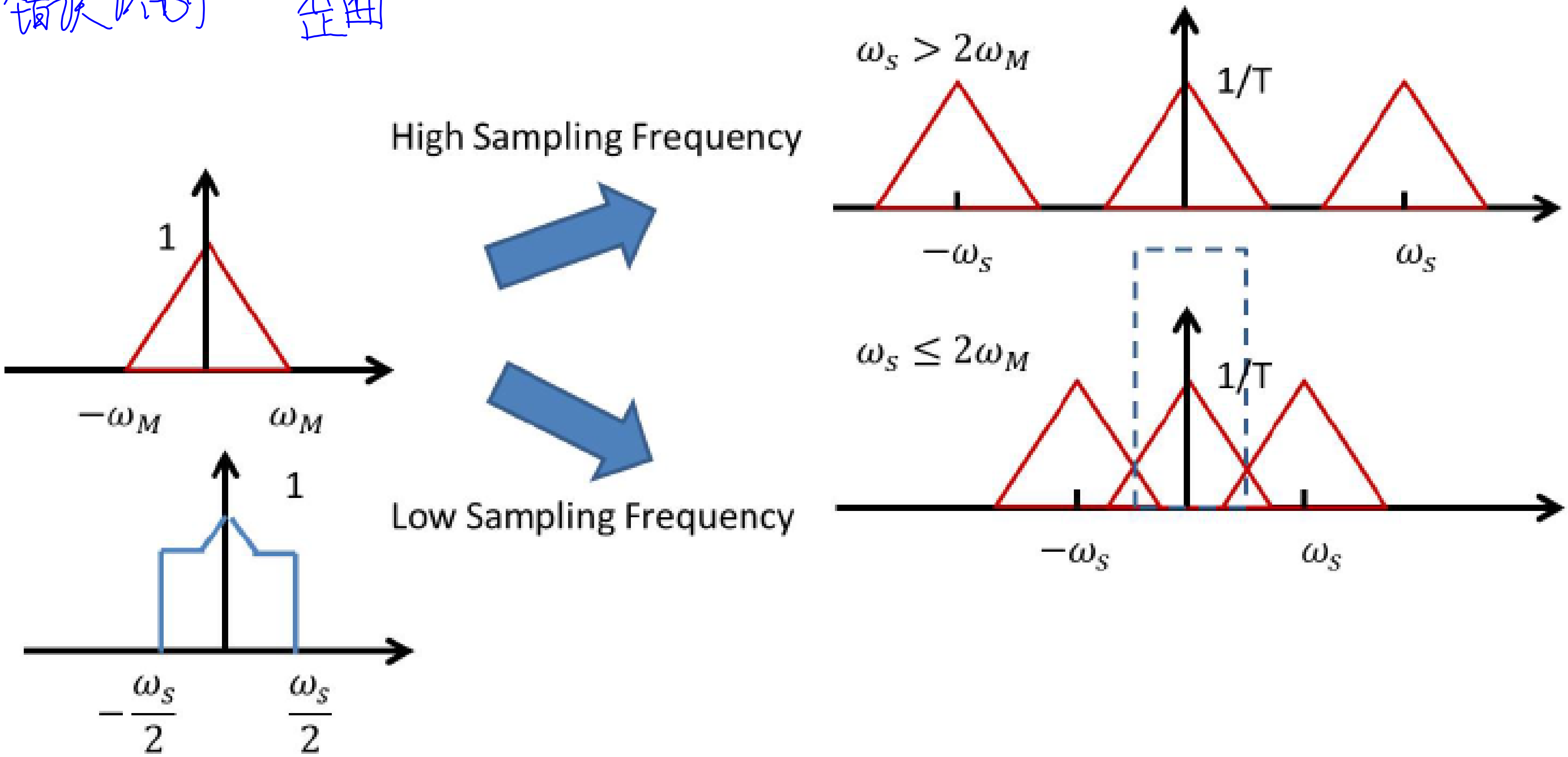
Summary: Sampling approaches



Undersampling & Aliasing

- Undersampling: insufficient sampling frequency $\omega_s < 2\omega_M$
- Perfect reconstruction is impossible with undersampling.
- Aliasing: distortion due to undersampling

错误识别 歪曲

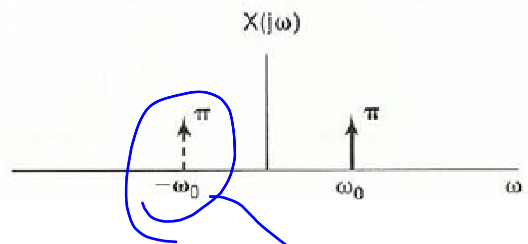


Aliasing: Example

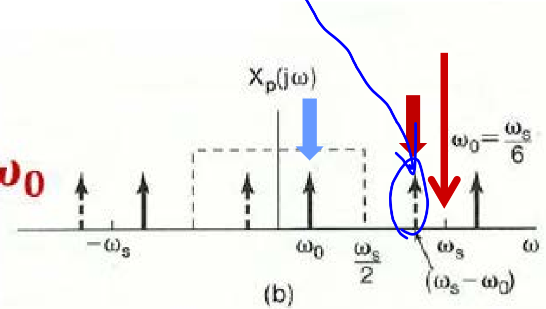
Signal before sampling: $\cos \omega_0 t$

Sampling rate: ω_s

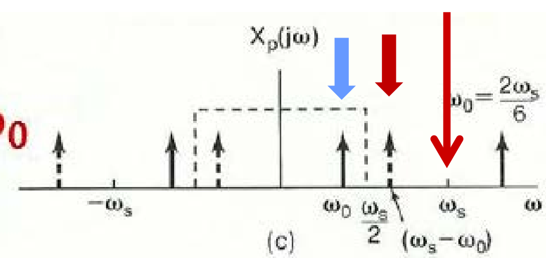
Lowpass Filter: $\frac{-\omega_s}{2} \sim \frac{\omega_s}{2}$



$\omega_s = 6\omega_0$



$\omega_s = 3\omega_0$

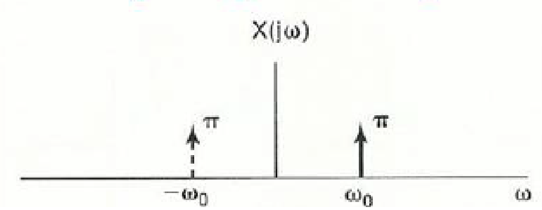


$\cos \omega_0 t$

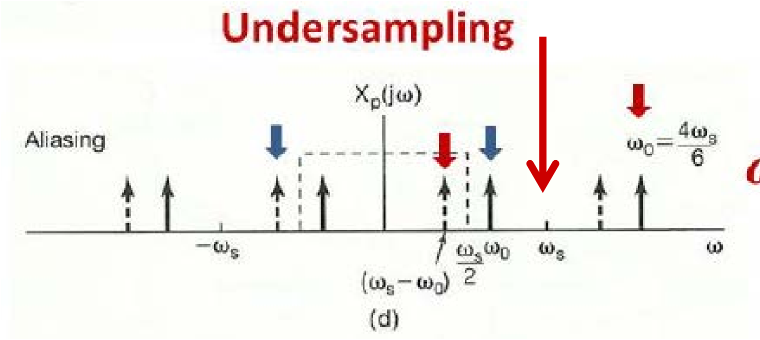
Aliasing: Example

Signal before sampling: $\cos \omega_0 t$

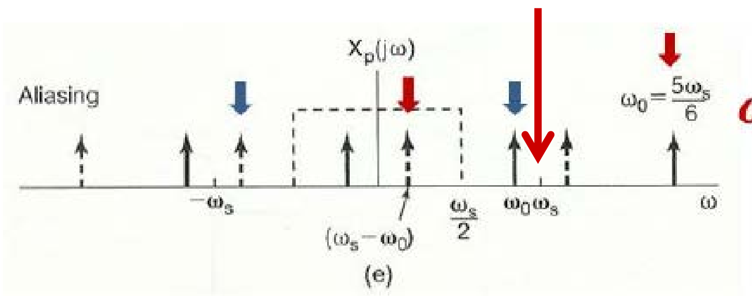
Sampling rate: ω_s



Lowpass Filter: $\frac{-\omega_s}{2} \sim \frac{\omega_s}{2}$



$$\omega_s = \frac{3}{2}\omega_0$$



$$\omega_s = \frac{6}{5}\omega_0$$

Aliasing: $\cos(\omega_s - \omega_0)t$

Under fixed w_s, when w_0 is increasing, what is the frequency of recovered wave?

Aliasing: Example

翻译

Low-pass filtering: Interpret the samples by cosine function with frequency lower than $\omega_s/2$

Original:

$\omega_s = \frac{3}{2}\omega_0$

Reconstructed:

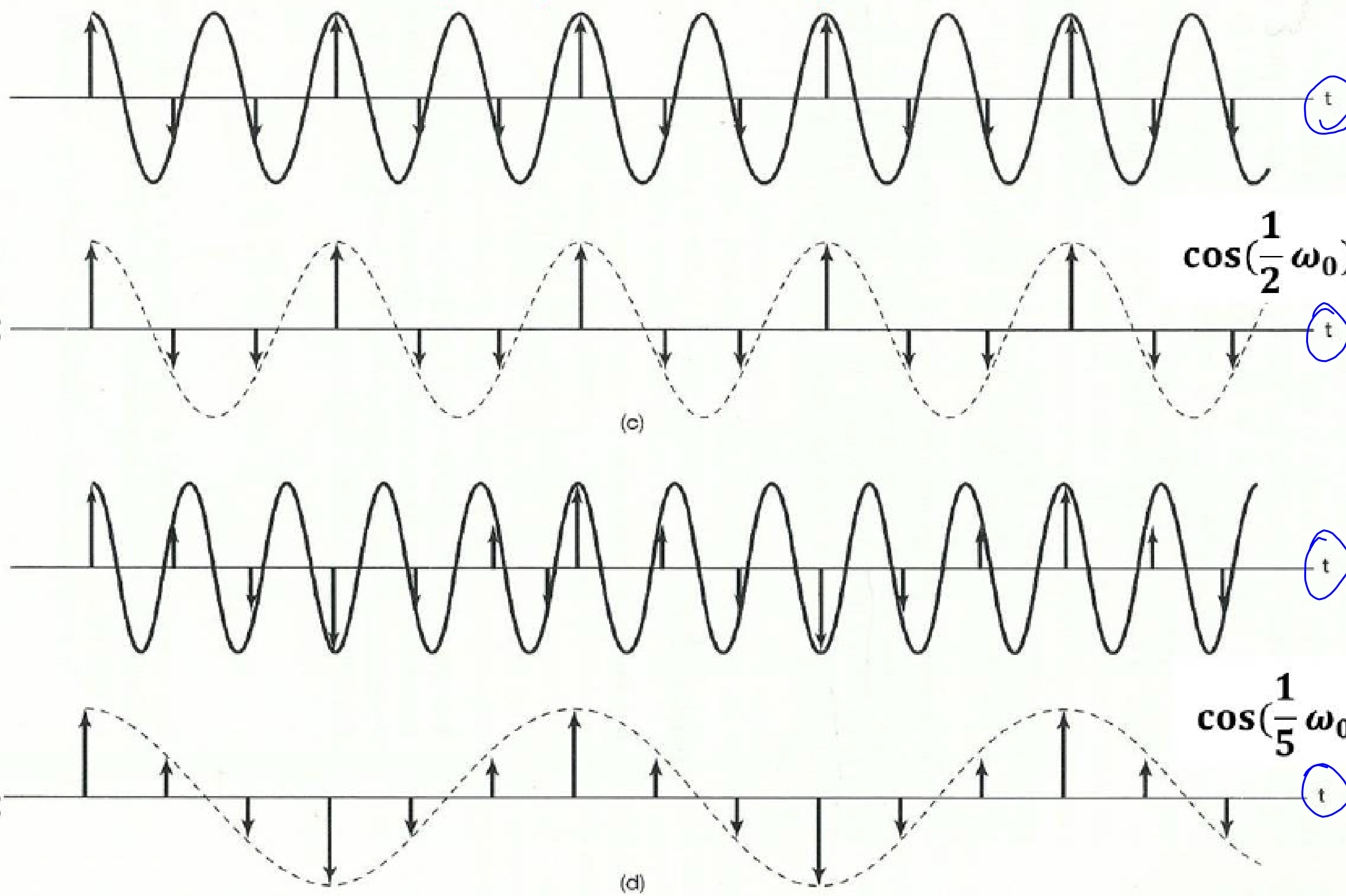
$\cos(\frac{1}{2}\omega_0)t$

Original:

$\omega_s = \frac{6}{5}\omega_0$

Reconstructed:

$\cos(\frac{1}{5}\omega_0)t$

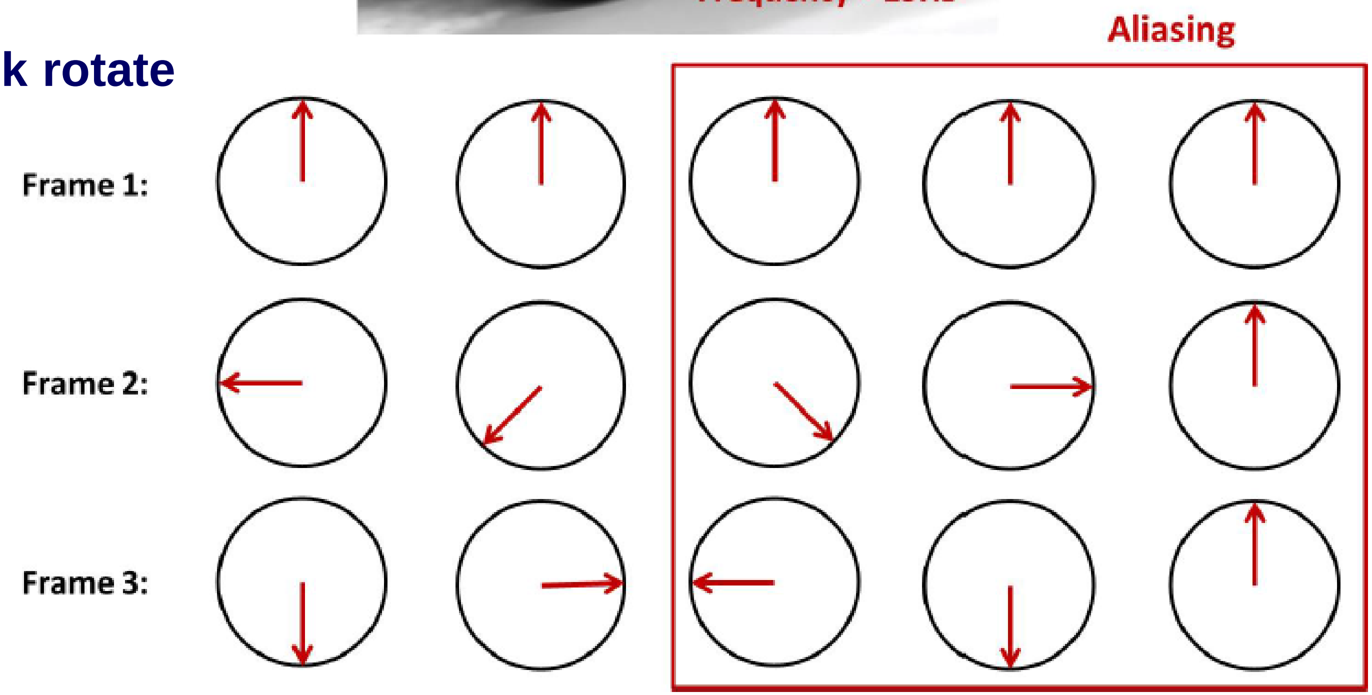


Aliasing: $\cos(\omega_s - \omega_0)t$

Aliasing in Movies

- Wheel's rotation in movies

Anti-clock rotate

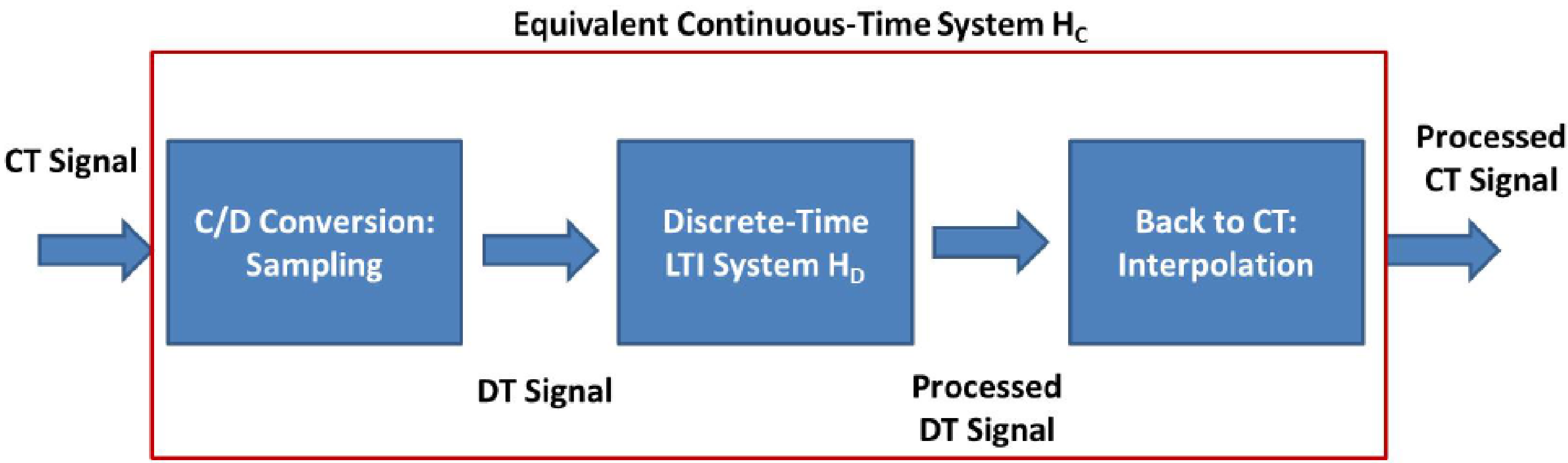


90^0 / frame 135^0 225^0 270^0 360^0

Process Continuous-Time Signals Discretely

- People would like to process continuous-time signal in discrete-time (digital) domain

Block Diagram



- It is much easier to design DT system.
- What's the relation between H_C and H_D?

or

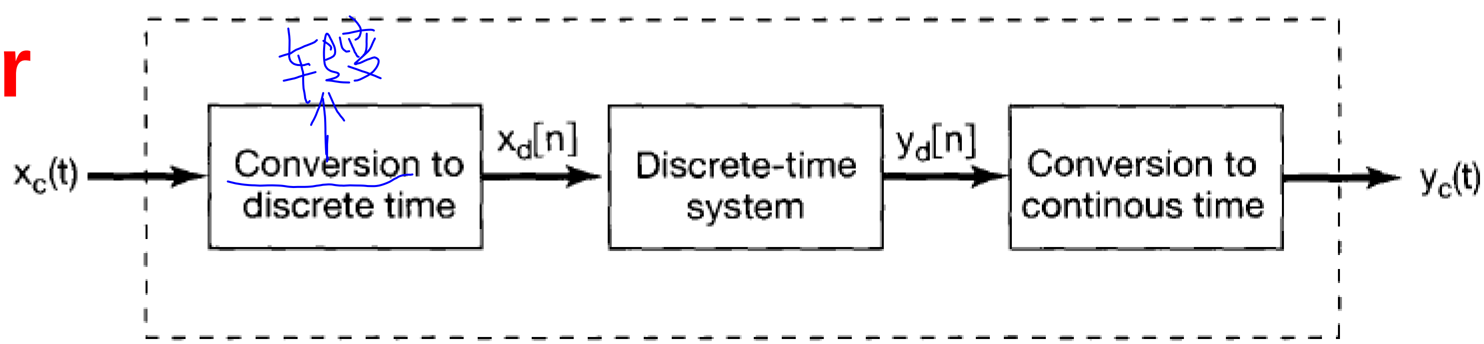


Figure 7.19 Discrete-time processing of continuous-time signals.

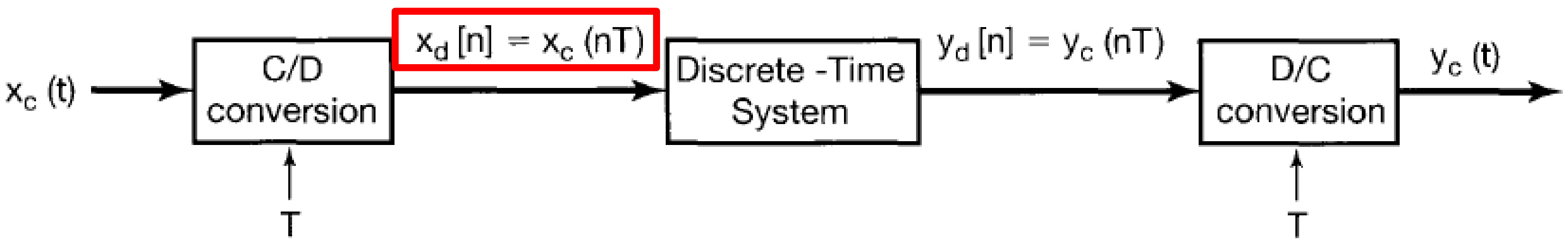
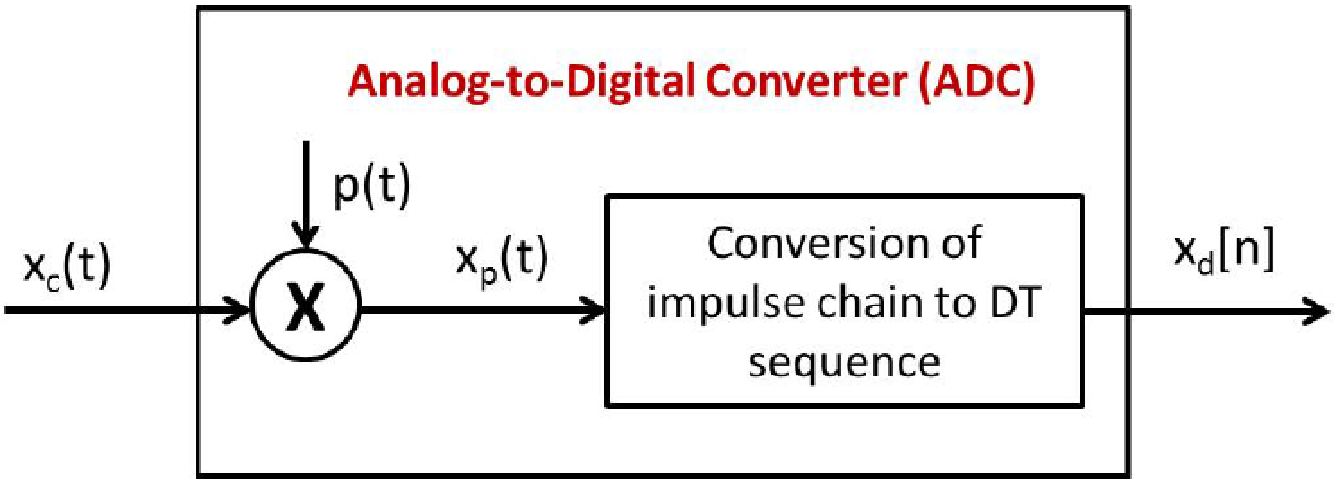


Figure 7.20 Notation for continuous-to-discrete-time conversion and discrete-to-continuous-time conversion. T represents the sampling period.

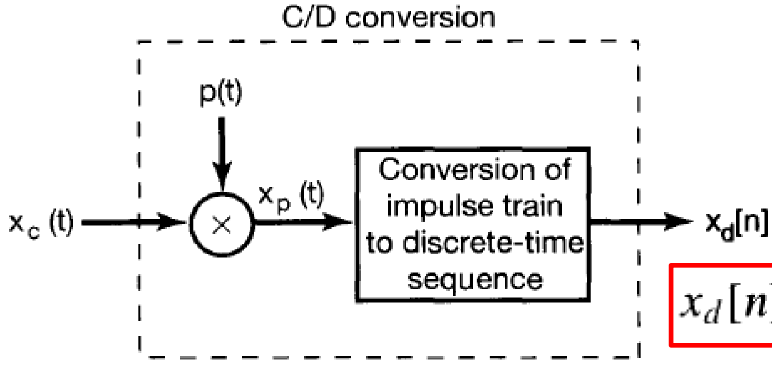
Discretization: C/D Conversion



- Mathematical Interpretation (Fourier Transform)

$$x_c(t) \quad \longleftrightarrow \quad X_c(j\omega)$$

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x_c(nT)\delta(t-nT) \quad \longleftrightarrow \quad X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$



(a)

$$\boxed{x_d[n] \;=\; x_c(nT)}$$

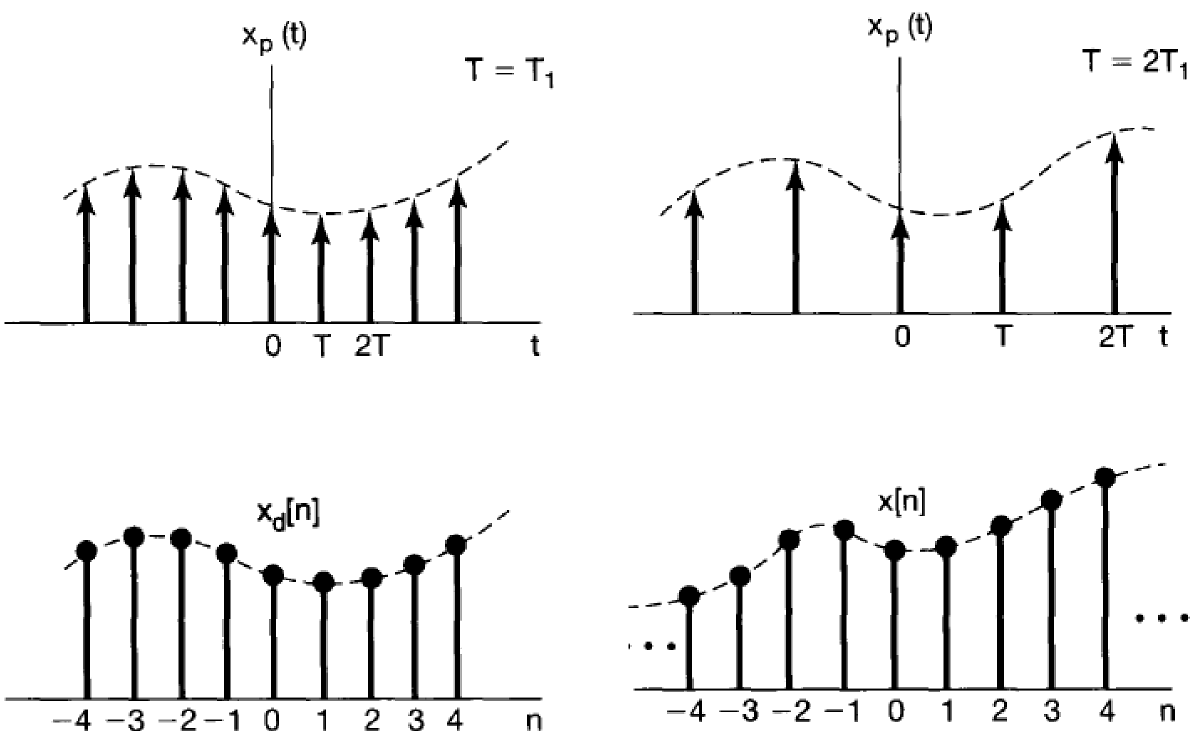
DTFT ←

$$X_d(e^{j\Omega}) \;=\; \sum_{n=-\infty}^{+\infty} x_d[n]e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\Omega n}$$

CTFT ←

$$X_p(j\omega) \;=\; \sum_{n=-\infty}^{+\infty} x_c(nT)e^{-j\omega nT}$$



$$X_d(e^{j\Omega}) \;=\; X_p(j\Omega/T)$$

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\Omega}) = X_p(j\Omega/T)$$

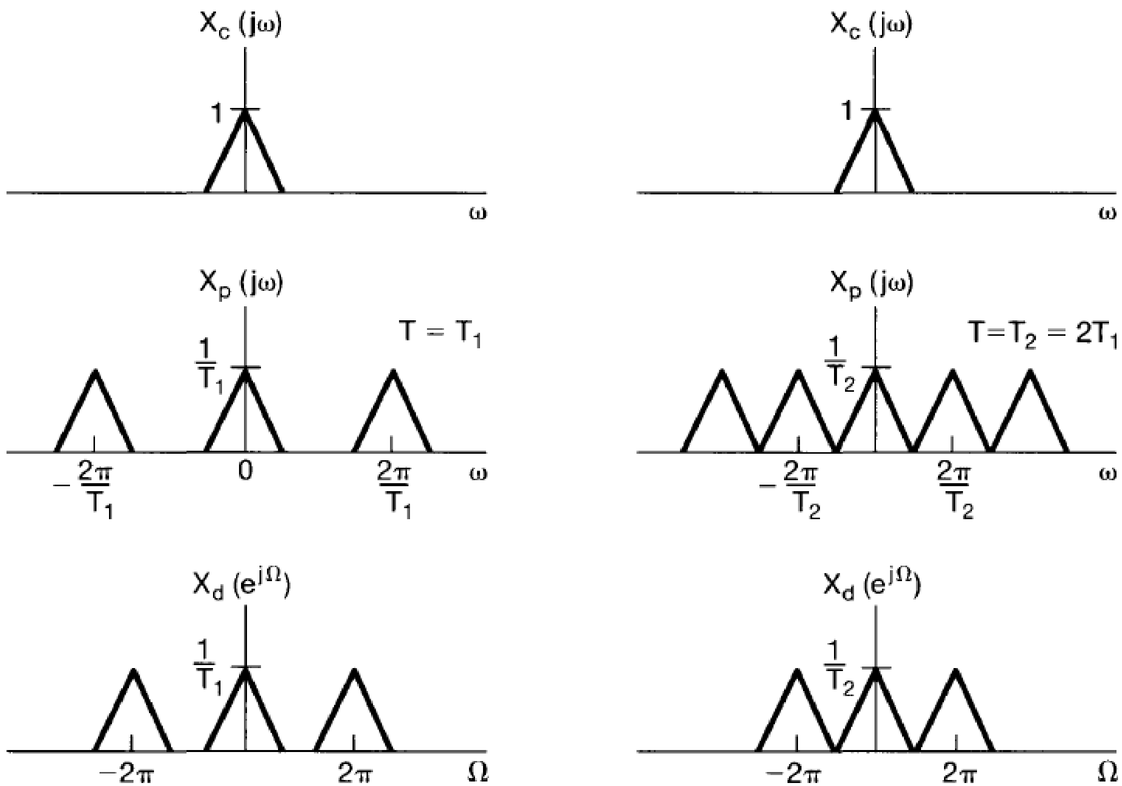


Figure 7.22 Relationship between $X_c(j\omega)$, $X_p(j\omega)$, and $X_d(e^{j\Omega})$ for two different sampling rates.

1. pulse-train sampling

2. scaling (expansion)

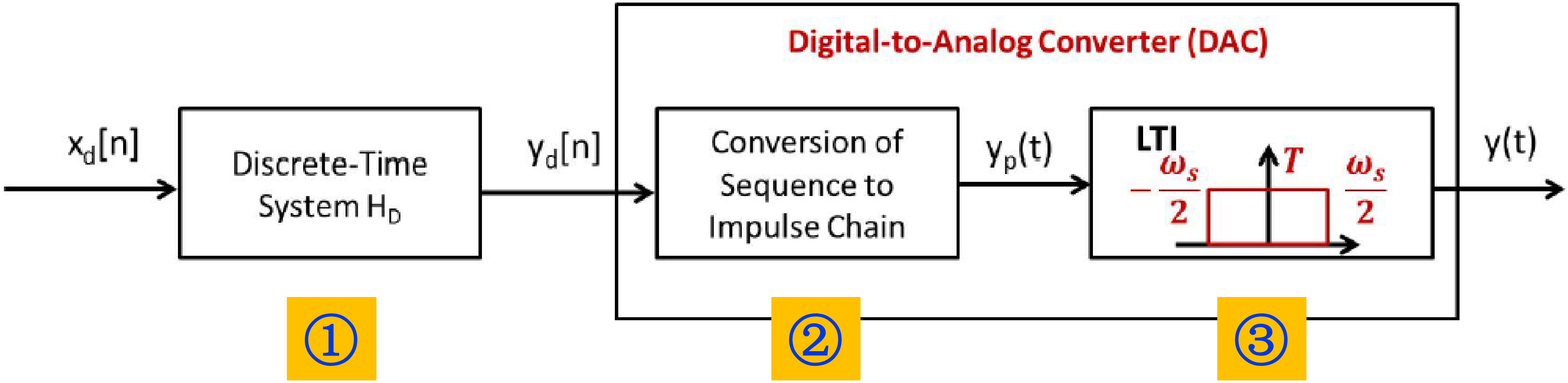
$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$X_d(e^{j\omega}) = X_p(j\omega / T)$$

$$X_d(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - k\omega_s))$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega - 2\pi k}{T}))$$

DT Processing and Conversion



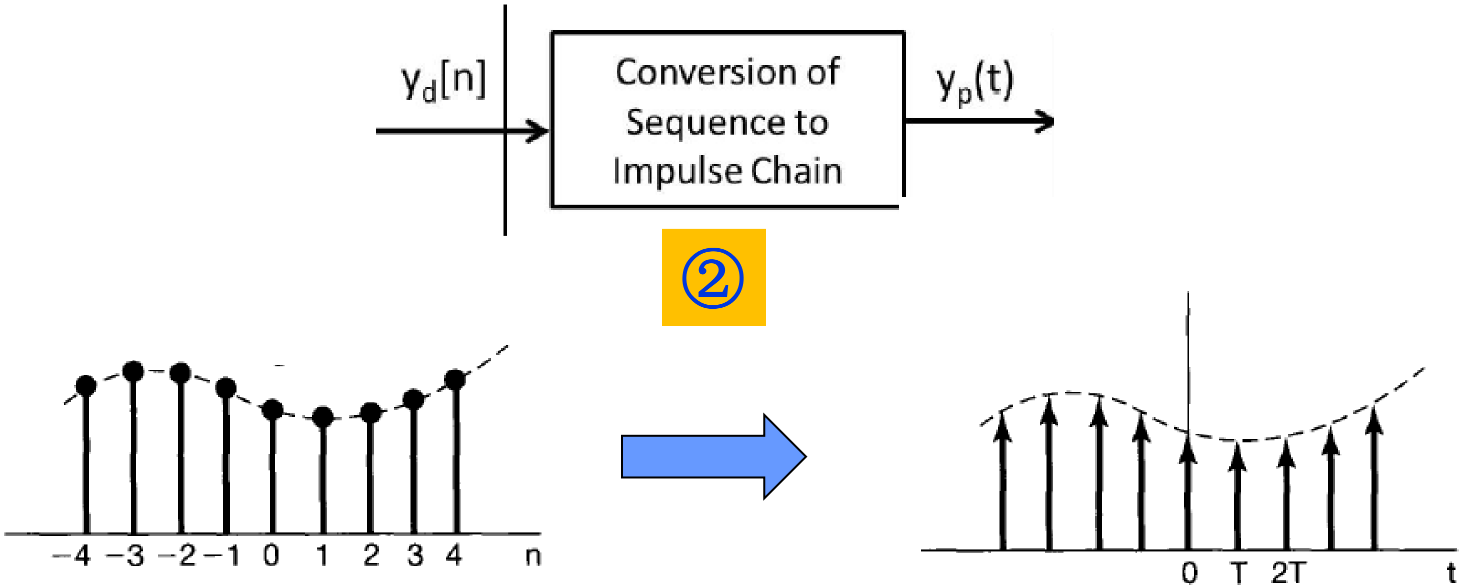
- Mathematical Interpretation (Fourier Transform)

① $$y_d[n] = x_d[n] * h_D[n] \quad \longleftrightarrow \quad Y_d(e^{j\omega}) = X_d(e^{j\omega})H_D(e^{j\omega})$$

② $$y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT) \quad \longleftrightarrow \quad Y_p(j\omega) = Y_d(e^{j\omega T})$$

③ $$y(t) = y_p(t) * h_{LP}(t) \quad \longleftrightarrow \quad Y(j\omega) = Y_p(j\omega)H_{LP}(j\omega)$$

$y_d[n] = y_c(nT)$

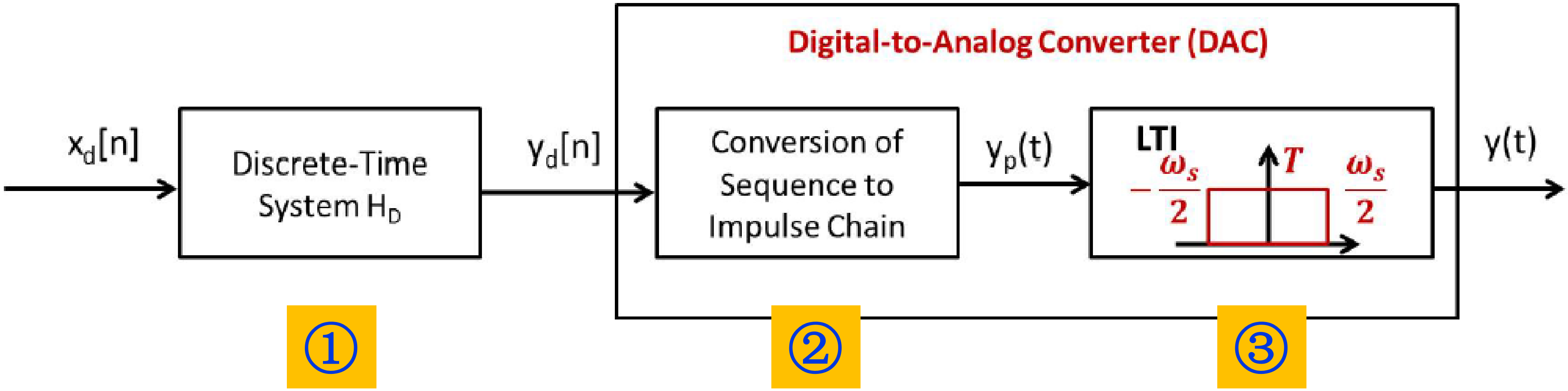


1. **n → t**
2. **time-scaling**

$$y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT) \quad \longleftrightarrow \quad Y_p(j\omega) = Y_d(e^{j\omega T})$$

Prove this !

DT Processing and Conversion



- Mathematical Interpretation (Fourier Transform)

① $$y_d[n] = x_d[n] * h_D[n] \longleftrightarrow Y_d(e^{j\omega}) = X_d(e^{j\omega})H_D(e^{j\omega})$$

② $$y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\delta(t - nT) \longleftrightarrow Y_p(j\omega) = Y_d(e^{j\omega T})$$

③ $$y(t) = y_p(t) * h_{LP}(t) \longleftrightarrow Y(j\omega) = Y_p(j\omega)H_{LP}(j\omega)$$

$$Y(j\omega) = X_d(e^{j\omega T})H_D(e^{j\omega T})H_{LP}(j\omega)$$

$$Y(j\omega) = X_d(e^{j\omega T})H_D(e^{j\omega T})H_{LP}(j\omega)$$

$$X_d(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - k\omega_s))$$

$H_c(j\omega) \to$ 非周期

$H_D(e^{j\omega T}) \to$ 周期

$$\begin{aligned} Y(j\omega) &= \left[\frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))\right] H_D(e^{j\omega T})H_{LP}(j\omega) \\ &= X_c(j\omega)H_D(e^{j\omega T}) \\ &= X_c(j\omega)\widetilde{H}_D(e^{j\omega T}) \end{aligned} \tag{1}$$

where

$$\widetilde{H}_D(e^{j\omega T}) = \begin{cases} H_D(e^{j\omega T}) & |\omega| < \omega_s/2 \\ 0 & otherwise \end{cases} \tag{2}$$

- It is equivalent to a continuous-time LTI system $H_C(j\omega) = \widetilde{H}_D(e^{j\omega T})$
- $H_D(e^{j\omega T})$ is a periodic extension of $\widetilde{H}_D(e^{j\omega T})$ with period $\omega_s = 2\pi/T$

System Design

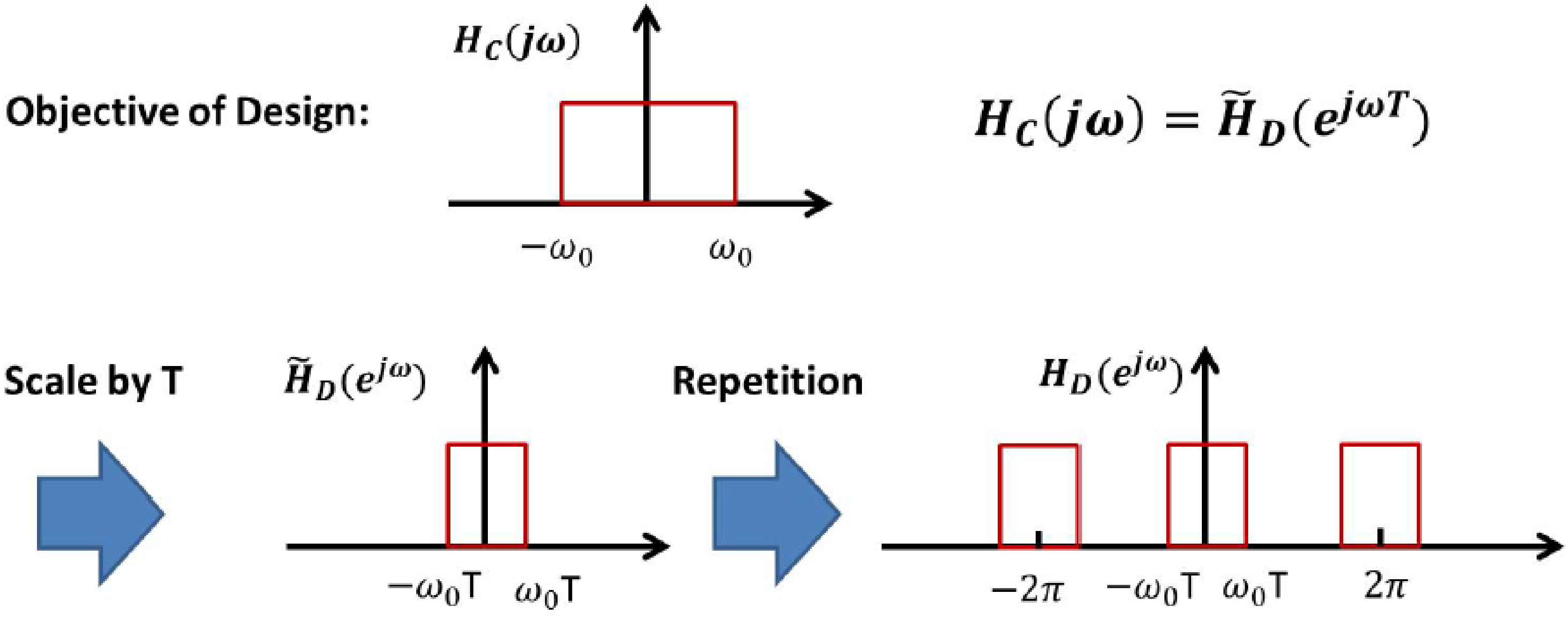
$H_D \xrightarrow{①}$ 压缩 $\xrightarrow{②} \tilde{H}_D \to H_c(j\omega)$

$$H_C(j\omega) = \widetilde{H}_D(e^{j\omega T})$$

- How can we design a CT LTI system with frequency response H_C via DT LTI system?
- Step 1: Sampling frequency ω_s or $2\pi/T$ should be larger than Nyquist rate
- Step 2: $\widetilde{H}_D(e^{j\omega T}) = H_C(j\omega)$
- Step 3: Frequency response of DT LTI system
  $H_D(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} \widetilde{H}_D(e^{j(\omega - k\omega_s)T})$ or
  $H_D(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \widetilde{H}_D(e^{j(\omega - k\omega_s T)}) = \sum_{k=-\infty}^{\infty} H_C(j\frac{\omega - 2k\pi}{T})$

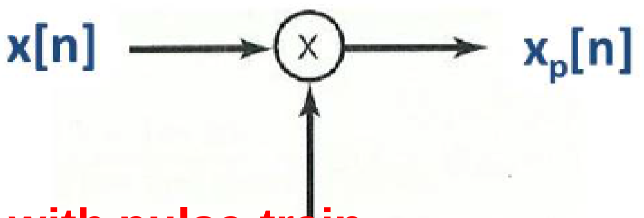
System Design Example

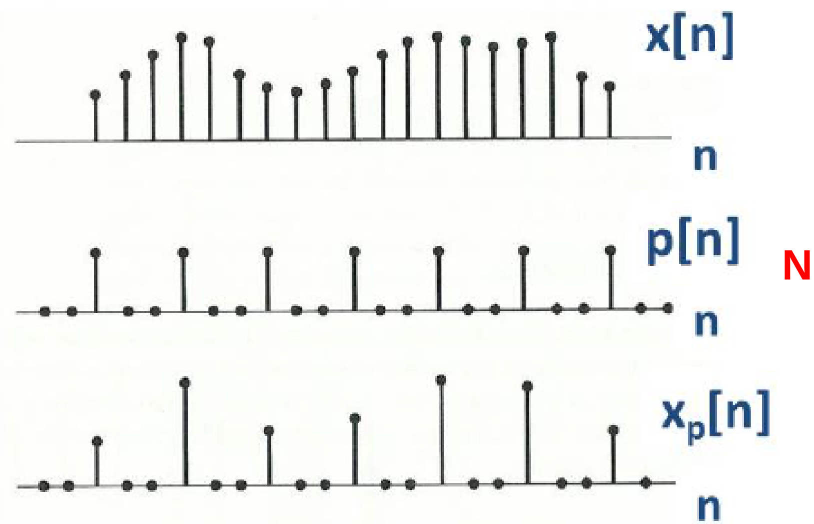
- How to implement an ideal CT lowpass filter?



Sampling on Discrete-Time Signals

- In digital signal processing, we may sample the discrete-time signals
- Example: image, audio, video compression
- Its mathematical model is give below:





Different with pulse-train sampling

$$p[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$$

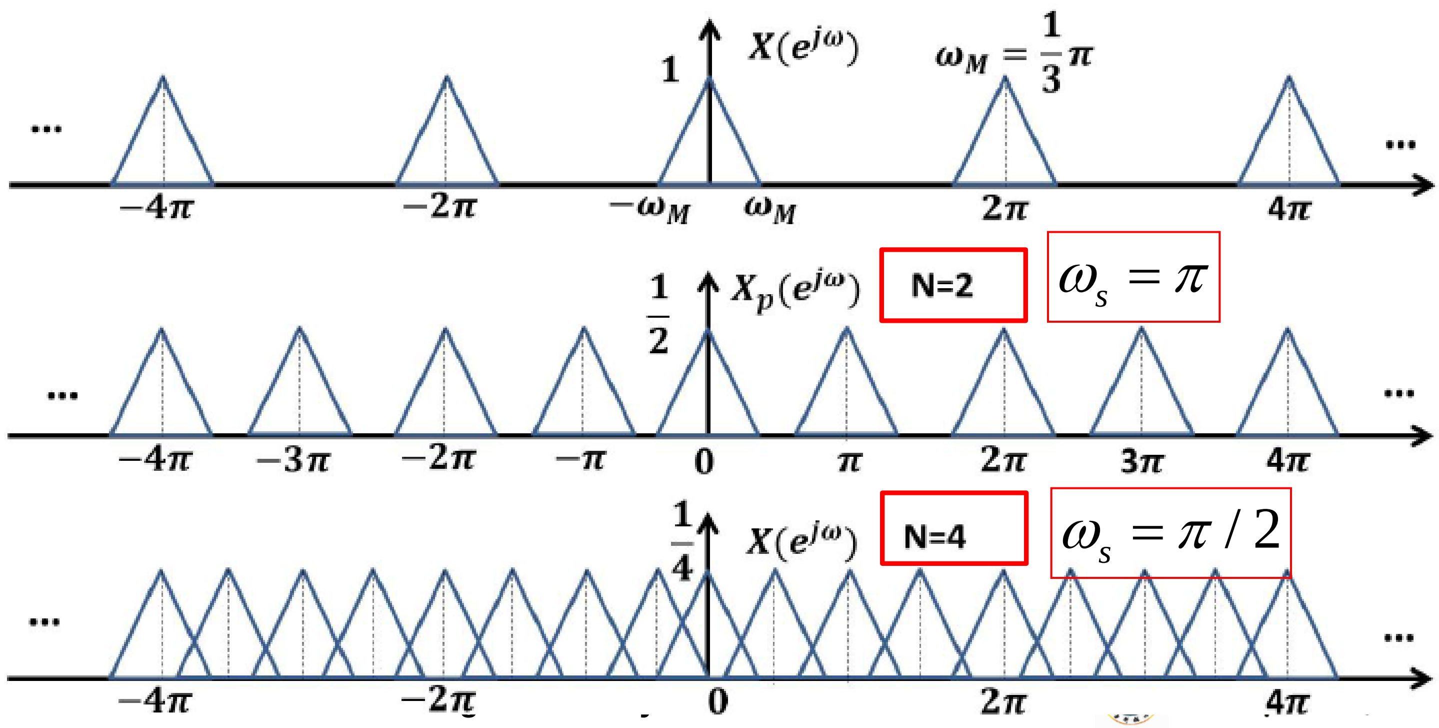
$$x_p[n] = \sum_{k=-\infty}^{+\infty} x[kN]\delta[n - kN]$$

Frequency Analysis

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s) \quad where \boxed{\omega_s = 2\pi/N}$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$

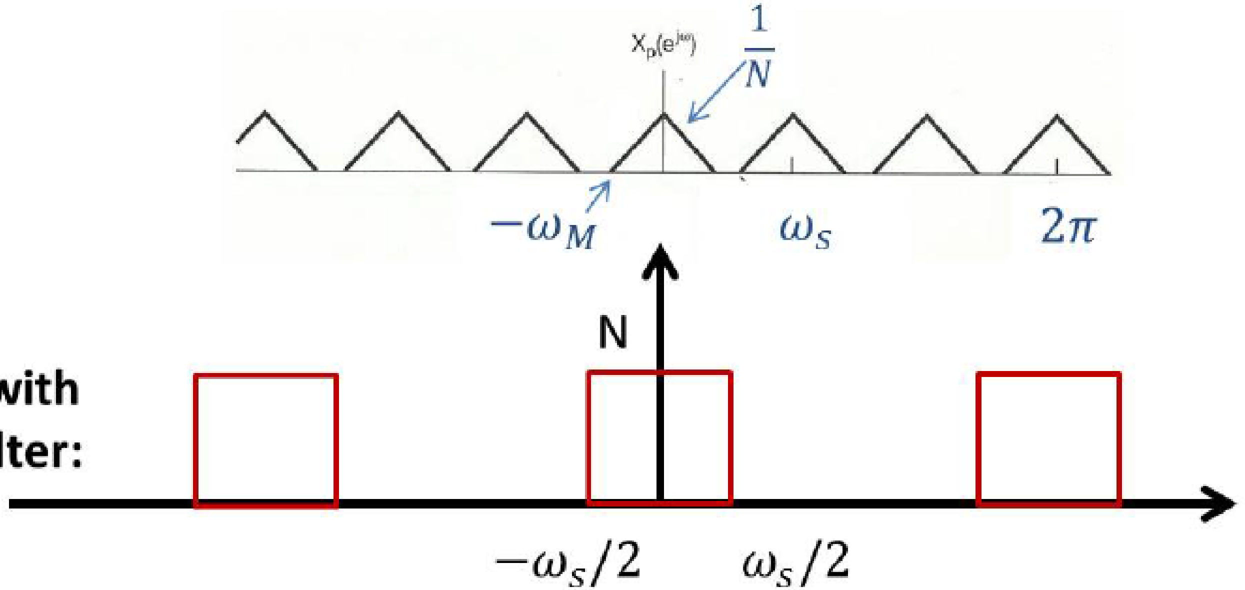
periodic convolution



Reconstruction

- Perfect reconstruction is applicable when $\omega_s > 2\omega_M \leftrightarrow N < \frac{\pi}{\omega_M}$

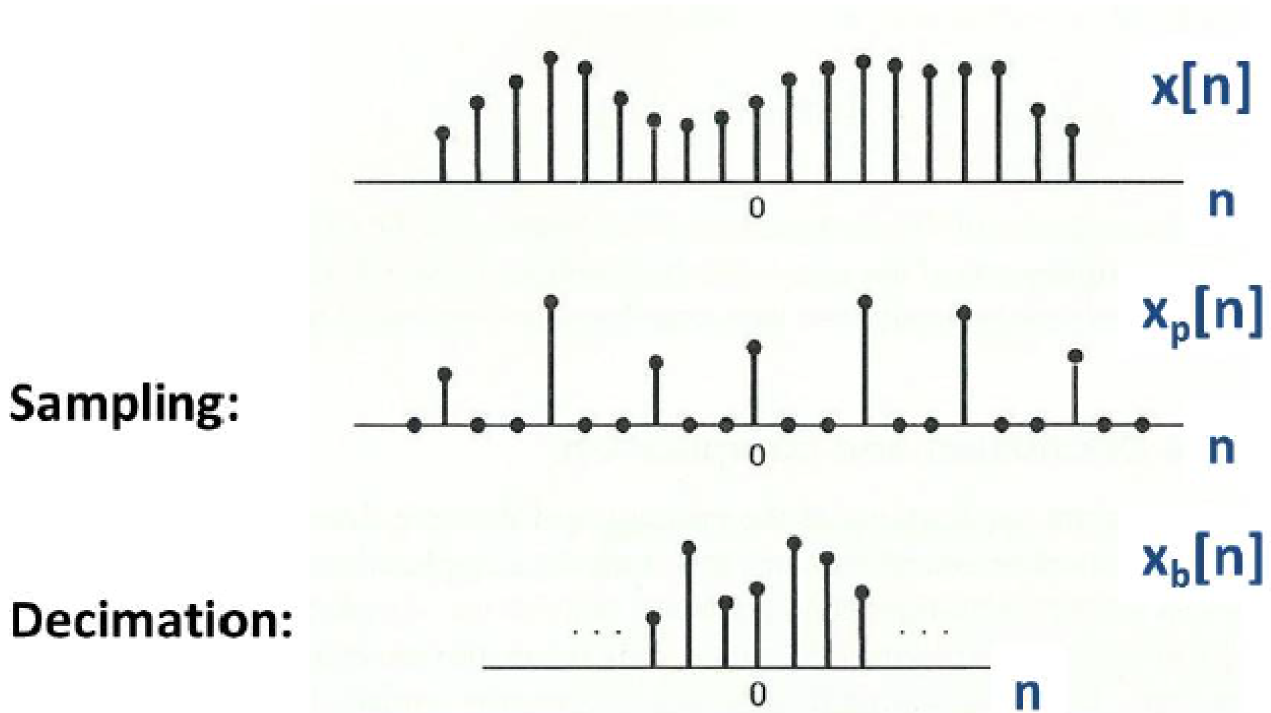
Reconstruction with ideal low-pass filter:



- Aliasing occurs when $\omega_s < 2\omega_M$

Decimation

- After sampling, there will be a great amount of redundancy
- Decimation: discrete-time sampling + remove zeros



$$x_b[n] = x_p[nN] = x[nN]$$

Frequency Analysis

$$X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_b[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N} = X_p(e^{j\omega/N})$$

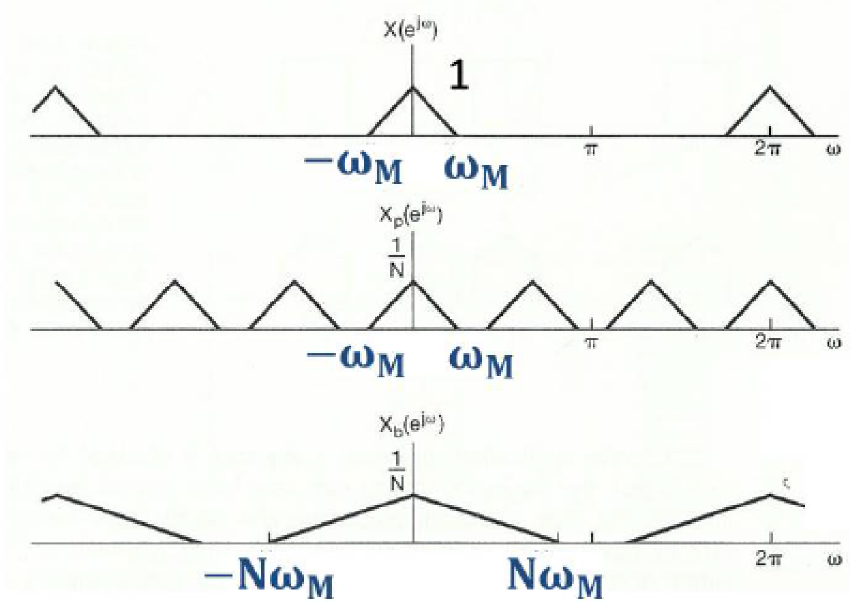
$$= \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j\omega/N - k\omega_s})$$

$$\boxed{X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})}$$

$$\boxed{\omega_s = 2\pi/N}$$

Sampling:

Decimation:

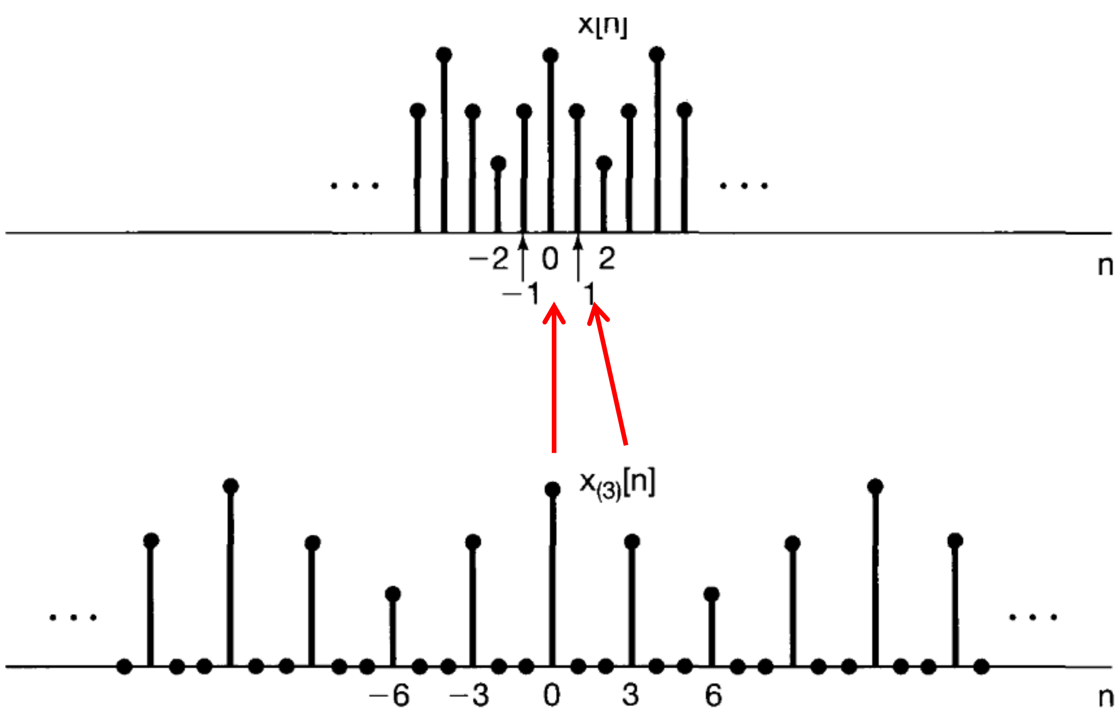


Condition for Perfect Reconstruction: $N\omega_M < \pi$

Review

- Time Expansion
  - Define $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of k} \\ 0, & \text{Otherwise} \end{cases}$, then

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega})$$



插零后，频谱压缩

时域变宽，频域变窄

Review

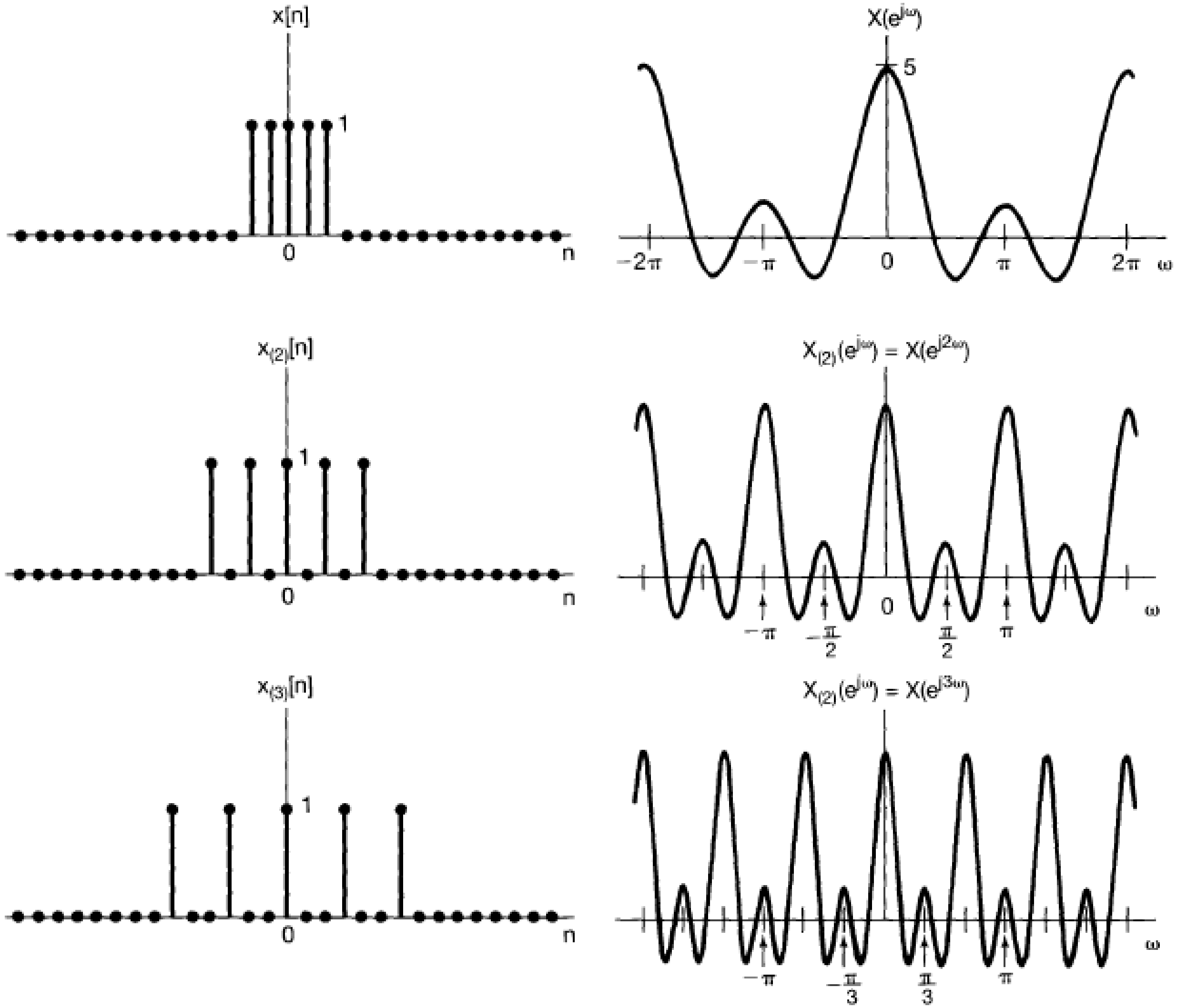
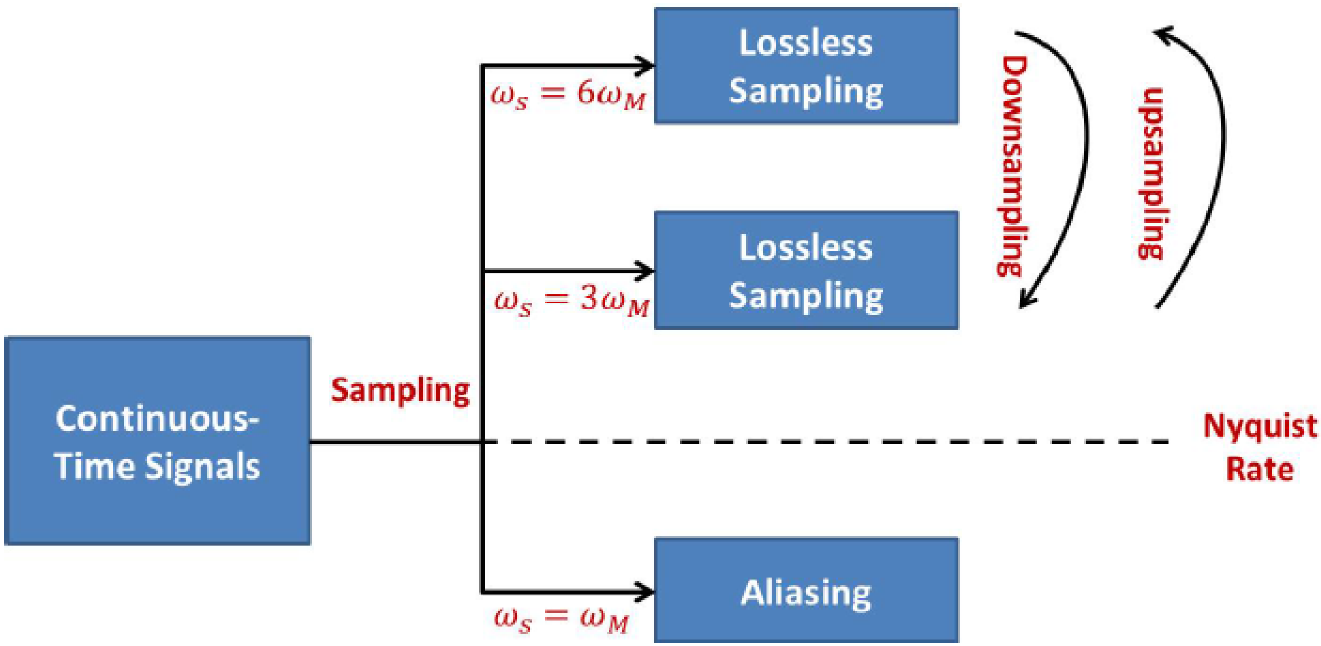


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

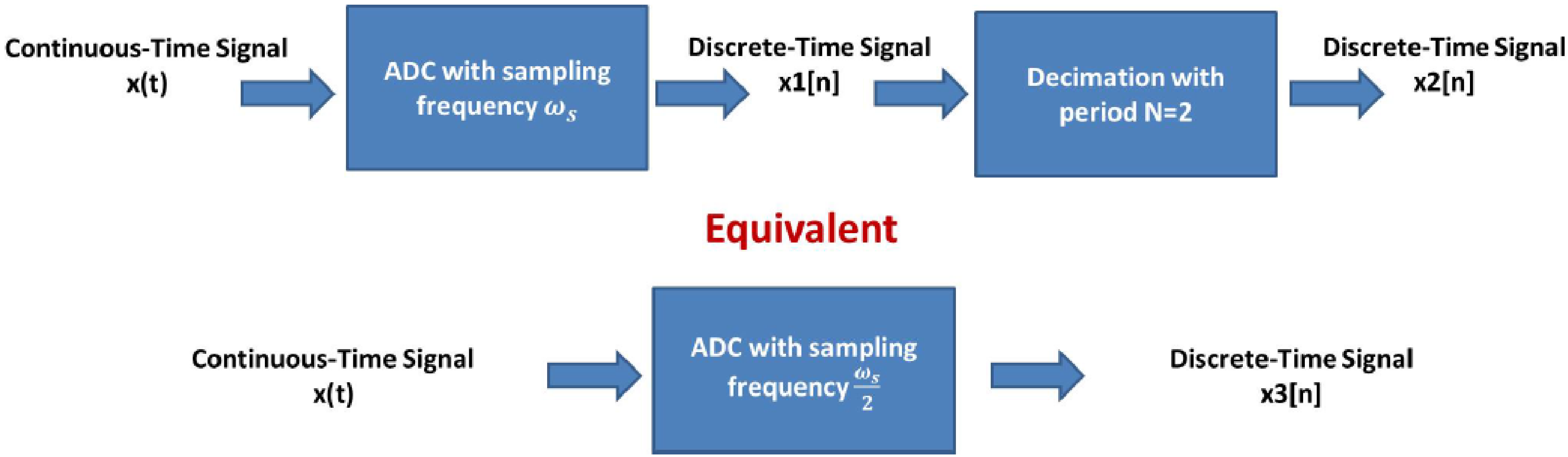
Anything Else?



- Downsampling: to reduce the sampling frequency (decimation)
- Upsampling: to generate a DT signal with higher sampling frequency
- As long as Nyquist rate is satisfied, the transform between low-sampling-frequency version and high-sampling-frequency versions is lossless.

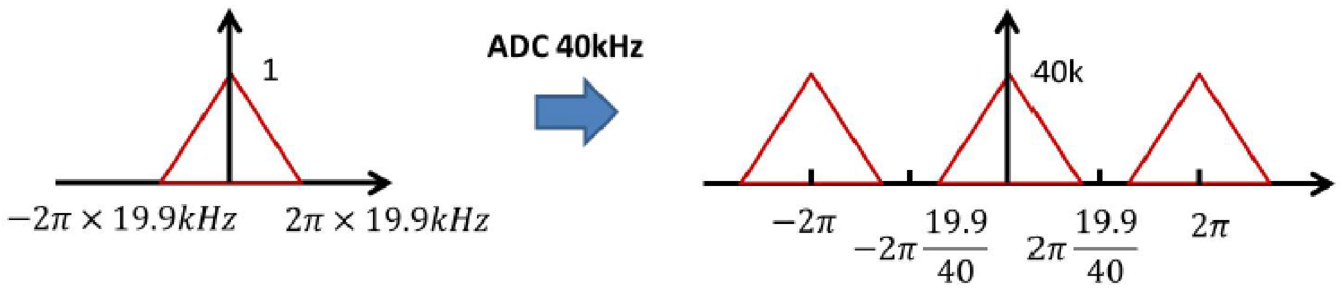
Downsampling

- Downsampling: a general procedure to reduce the sampling frequency



Downsampling Example (1/2)

- Suppose we have a clip of voice, $x(t)$, with bandwidth =19.9kHz
- It can be converted to DT signal with sampling frequency 40kHz, denoted as $x_1[n]$

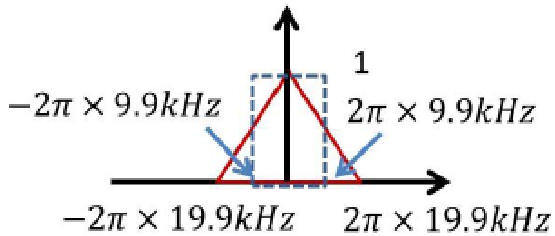


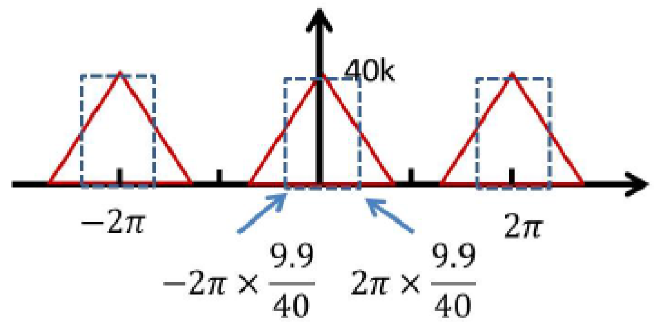
- Based on $x_1[n]$, if we want to save the voice information within 9.9kHz into another DT signal, what can we do?

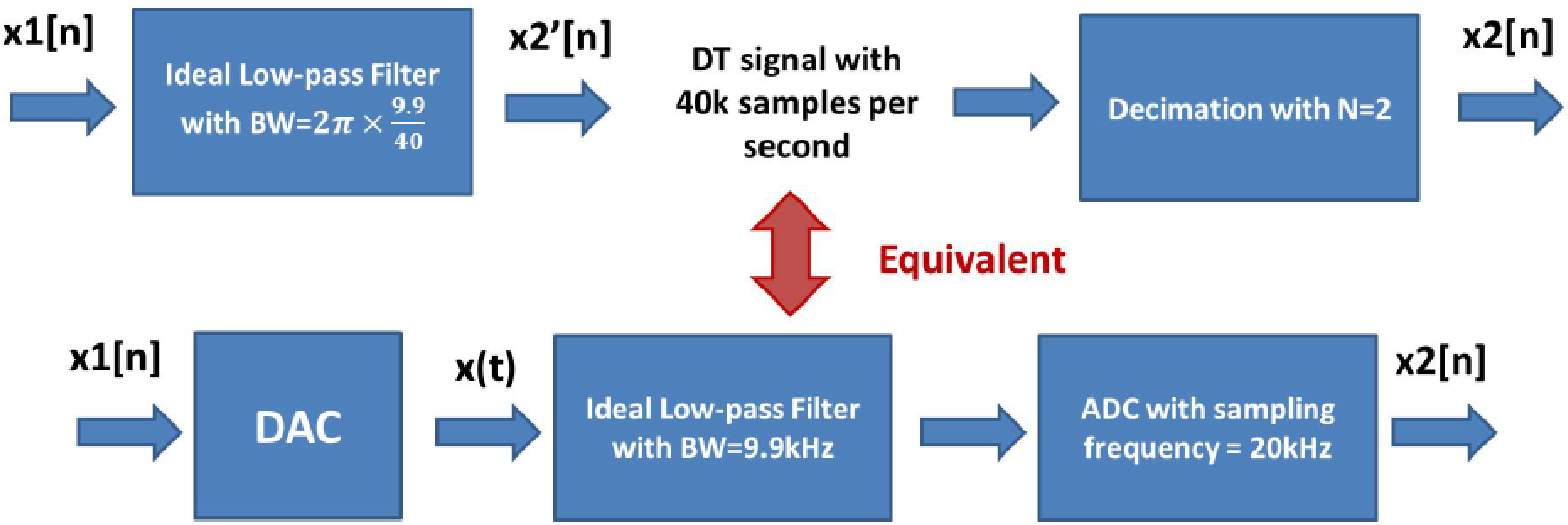
One Choice:

How can we generate x2[n] in discrete-time domain?

Downsampling Example (2/2)

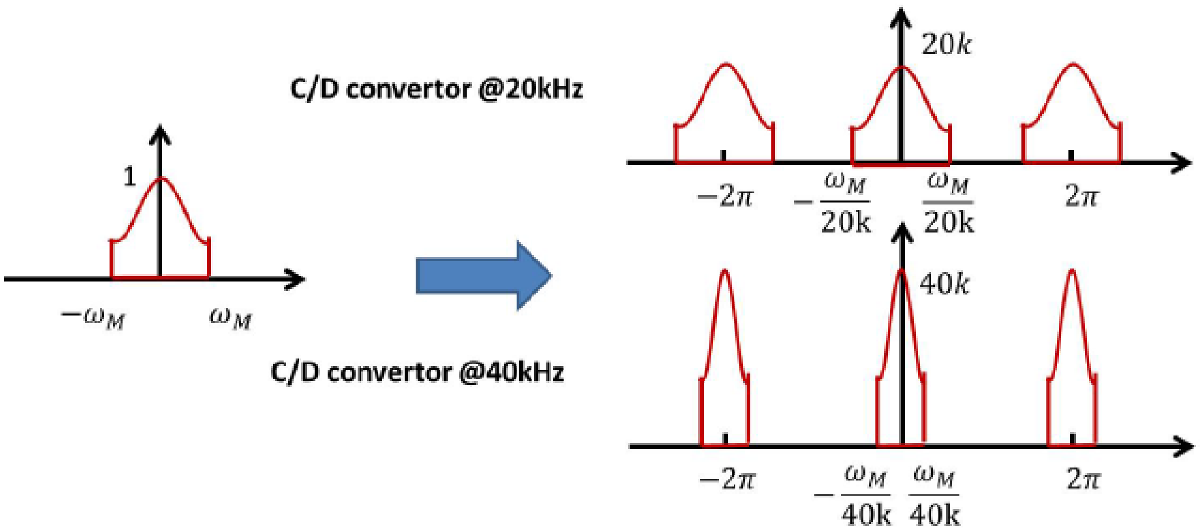






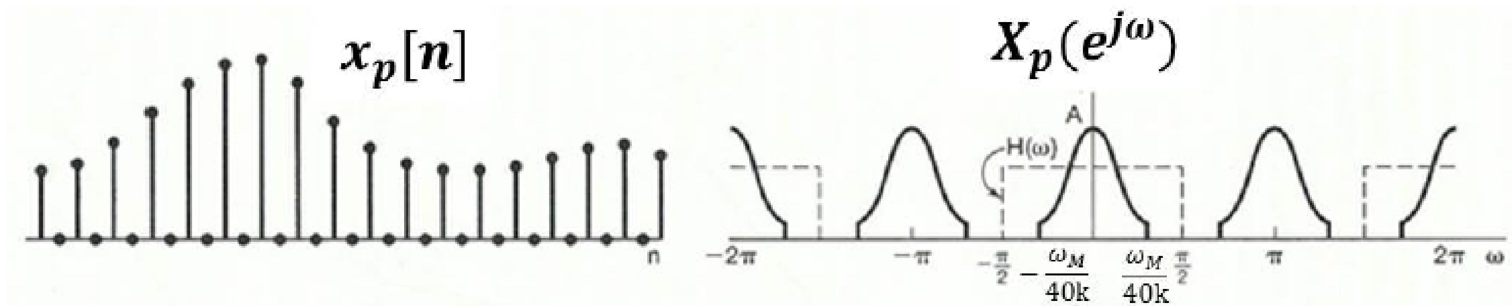
Upsampling

- Upsampling: a procedure to generate a sequence with higher sampling frequency
- Superpose the following two digital sound clips
  - Audio clip 1: Bandwidth= 19.9kHz, sampled at 40kHz
  - Audio clip 2: Bandwidth= 9.9kHz, sampled at 20kHz
- Double the sampling frequency of audio clip 2 (40kHz)
- How to do upsampling in discrete-time domain?



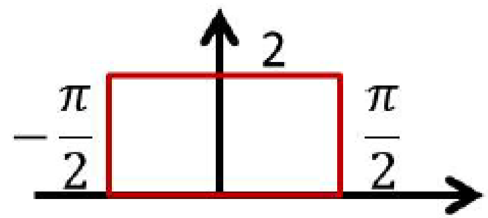
- Time expansion (Insert zeros):

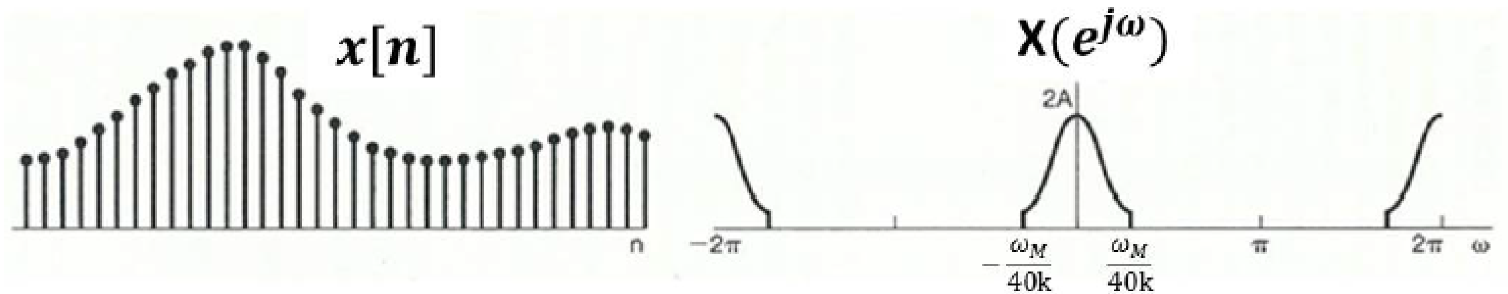
$$x_p[n] = x_{b(k)}[n] \longleftrightarrow X_p(e^{j\omega}) = X_b(e^{jk\omega})$$



- Low-pass filtering:

Ideal Low-Pass Filter:





Summary

- Undersampling
  - Aliasing
- How to process CT signal with DT system?

$$H_C(j\omega) = \widetilde{H}_D(e^{j\omega T})$$

- Sampling on DT signal
  - Downsampling, and upsampling