Tutorial Problems

- Basic Problems with Answers 4.10,4.16
- Basic problems 4.26.
- Advanced Problems 4.30.

Shuguang Xiao（肖书光），TA
12131043@mail.sustech.edu.cn

Problem 1

4.10 **(a)** Use Tables 4.1 and 4.2 to help determine the Fourier transform of the following signal:

$$x(t) = t\left(\frac{\sin t}{\pi t}\right)^2$$

(b) Use Parseval's relation and the result of the previous part to determine the numerical value of

$$A = \int_{-\infty}^{+\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt$$

Solution (a) We know from Table 4.2 that

$$\frac{\sin t}{\pi t} \xleftrightarrow{FT} \text{Rectangular function } Y(j\omega) \text{ [See Figure S4.10]}$$

Therefore

$$\left(\frac{\sin t}{\pi t}\right)^2 \xleftrightarrow{FT} (1/2\pi)\,[\text{Rectangular function } Y(j\omega) * \text{Rectangular function } Y(j\omega)]$$

This is a triangular function $Y_1(j\omega)$ as shown in the Figure S4.10.

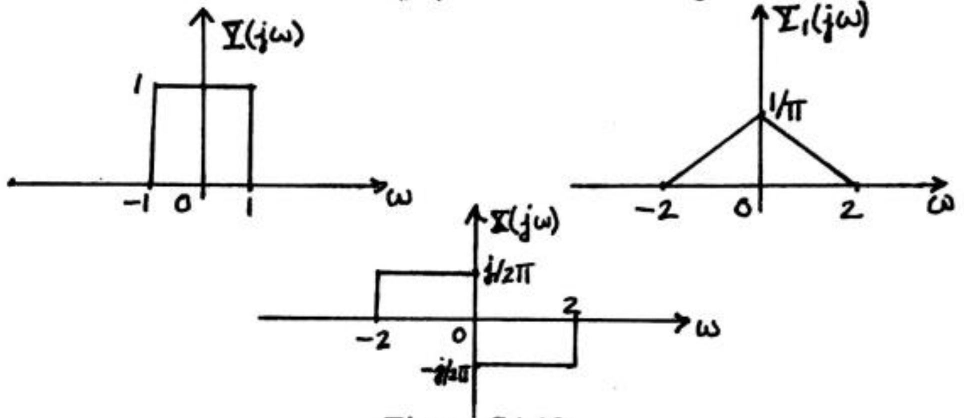


Figure S4.10

Using Table 4.1, we may write

$$t\left(\frac{\sin t}{\pi t}\right)^2 \xleftrightarrow{FT} X(j\omega) = j\frac{d}{d\omega}Y_1(j\omega)$$

This is as shown in the figure above. $X(j\omega)$ may be expressed mathematically as

$$X(j\omega) = \begin{cases} j/2\pi, & -2 \le \omega < 0 \\ -j/2\pi, & 0 \le \omega 2 \\ 0. & \text{otherwise} \end{cases}$$

(b) Using Parseval's relation,

$$\int_{-\infty}^{\infty} t^2 \left(\frac{\sin t}{\pi t}\right)^4 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega = \frac{1}{2\pi^3}$$

Problem 2

4.16 Consider the signal

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(k\frac{\pi}{4})}{(k\frac{\pi}{4})} \delta(t - k\frac{\pi}{4}).$$

(a) Determine $g(t)$ such that

$$x(t) = \left(\frac{\sin t}{\pi t}\right) g(t).$$

(b) Use the multiplication property of the Fourier transform to argue that $X(j\omega)$ is periodic. Specify $X(j\omega)$ over one period.

Solution (a) We may write

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/4)}{k\pi/4}\delta(t-k\pi/4) \\ &= \frac{\sin t}{\pi t}\sum_{k=-\infty}^{\infty} \pi\delta(t-k\pi/4) \end{aligned}$$

Therefore, $g(t) = \sum_{k=-\infty}^{\infty} \pi\delta(t-k\pi/4)$.

(b) Since $g(t)$ is an impulse train, its Fourier transform $G(j\omega)$ is also an impulse train. From Table 4.2,

$$\begin{aligned} G(j\omega) &= \pi\frac{2\pi}{\pi/4}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{\pi/4}\right) \\ &= 8\pi\sum_{k=-\infty}^{\infty} \delta\left(\omega - 8k\right) \end{aligned}$$

We see that $G(j\omega)$ is periodic with a period of 8. Using the multiplication property, we know that

$$X(j\omega) = \frac{1}{2\pi}\left[\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\} * G(j\omega)\right]$$

If we denote $\mathcal{FT}\left\{\frac{\sin t}{\pi t}\right\}$ by $A(j\omega)$, then

$$\begin{aligned} X(j\omega) &= (1/2\pi)[A(j\omega) * 8\pi\sum_{k=-\infty}^{\infty} \delta\left(\omega - 8k\right) \\ &= 4\sum_{k=-\infty}^{\infty} A\left(j\omega - 8k\right) \end{aligned}$$

$X(j\omega)$ may thus be viewed as a replication of $4A(j\omega)$ every 8 rad/sec. This is obviously periodic.

Using Table 4.2, we obtain

$$A(j\omega) = \begin{cases} 1, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, we may specify $X(j\omega)$ over one period as

$$X(j\omega) = \begin{cases} 4, & |\omega| \le 1 \\ 0, & 1 < |\omega \le 4 \end{cases}$$

Problem 3

4.26 (a) Compute the convolution of each of the following pairs of signals $x(t)$ and $h(t)$ by calculating $X(j\omega)$ and $H(j\omega)$, using the convolution property, and inverse transforming.

(i) $x(t) = te^{-2t}u(t),\ h(t) = e^{-4t}u(t)$

(ii) $x(t) = te^{-2t}u(t),\ h(t) = te^{-4t}u(t)$

(iii) $x(t) = e^{-t}u(t),\ h(t) = e^{t}u(-t)$

(b) Suppose that $x(t) = e^{-(t-2)}u(t-2)$ and $h(t)$ is as depicted in Figure P4.26. Verify the convolution property for this pair of signals by showing that the Fourier transform of $y(t) = x(t) * h(t)$ equals $H(j\omega)X(j\omega)$.

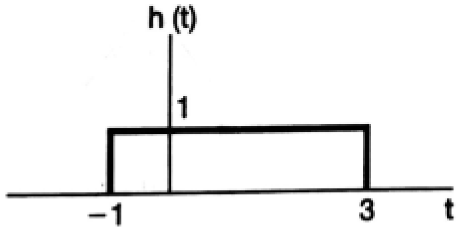


Figure P4.26

Solution

(a) (i) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{4+j\omega}\right] \\ &= \frac{(1/4)}{4+j\omega} - \frac{(1/4)}{2+j\omega} + \frac{(1/2)}{(2+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-4t}u(t) - \frac{1}{4}e^{-2t}u(t) + \frac{1}{2}te^{-2t}u(t).$$

(ii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) = \left[\frac{1}{(2+j\omega)^2}\right]\left[\frac{1}{(4+j\omega)^2}\right] \\ &= \frac{(1/4)}{2+j\omega} + \frac{(1/4)}{(2+j\omega)^2} - \frac{(1/4)}{4+j\omega} + \frac{(1/4)}{(4+j\omega)^2} \end{aligned}$$

Taking the inverse Fourier transform we obtain

$$y(t) = \frac{1}{4}e^{-2t}u(t) + \frac{1}{4}te^{-2t}u(t) - \frac{1}{4}e^{-4t}u(t) + \frac{1}{4}te^{-4t}u(t).$$

(iii) We have

$$\begin{aligned} Y(j\omega) &= X(j\omega)H(j\omega) \\ &= \left[\frac{1}{1+j\omega}\right]\left[\frac{1}{1-j\omega}\right] \\ &= \frac{1/2}{1+j\omega} + \frac{1/2}{1-j\omega} \end{aligned}$$

Taking the inverse Fourier transform, we obtain

$$y(t) = \frac{1}{2}e^{-|t|}.$$

(b) By direct convolution of $x(t)$ with $h(t)$ we obtain

$$y(t) = \begin{cases} 0, & t < 1 \\ 1 - e^{-(t-1)}, & 1 < t \le 5 \\ e^{-(t-5)} - e^{-(t-1)}, & t > 5 \end{cases}.$$

Taking the Fourier transform of $y(t)$,

$$\begin{aligned} Y(j\omega) &= \frac{2e^{-j3\omega}\sin(2\omega)}{\omega(1+j\omega)} \\ &= \left[\frac{e^{-j2\omega}}{1+j\omega}\right]\frac{e^{-j\omega}2\sin(2\omega)}{\omega} \\ &= X(j\omega)H(j\omega) \end{aligned}$$

Problem 4

4.30 Suppose $g(t) = x(t)\cos t$ and the Fourier transform of the $g(t)$ is

$$G(j\omega) = \begin{cases} 1, & |\omega| \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

Determine $x(t)$.

Solution

We know that

$$w(t) = \cos t \stackrel{FT}{\longleftrightarrow} W(j\omega) = \pi[\delta(\omega - 1) + \delta(\omega + 1)$$

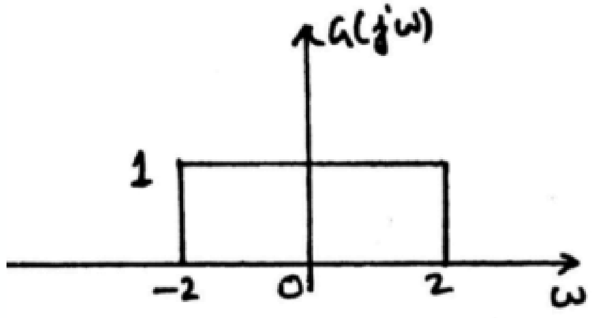
and

$$g(t) = x(t)\cos t \stackrel{FT}{\longleftrightarrow} G(j\omega) = \frac{1}{2\pi}\{X(j\omega) * W(j\omega)\}.$$

Therefore,

$$G(j\omega) = \frac{1}{2}X(j(\omega - 1)) + \frac{1}{2}X(j(\omega + 1)).$$

Since $G(j\omega)$ is as shown in Figure S4.30, it is clear from the above equation that $X(j\omega)$ is as shown in the Figure S4.30.



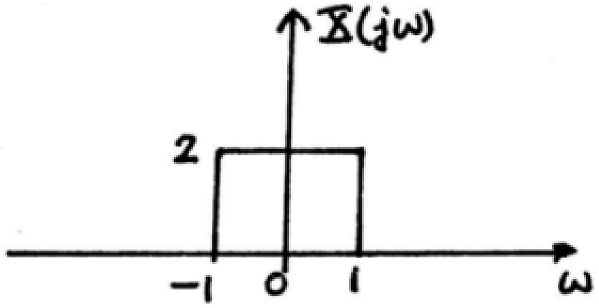


Figure S4.30

Therefore,

$$x(t) = \frac{2\sin t}{\pi t}.$$