Notes

Assignments

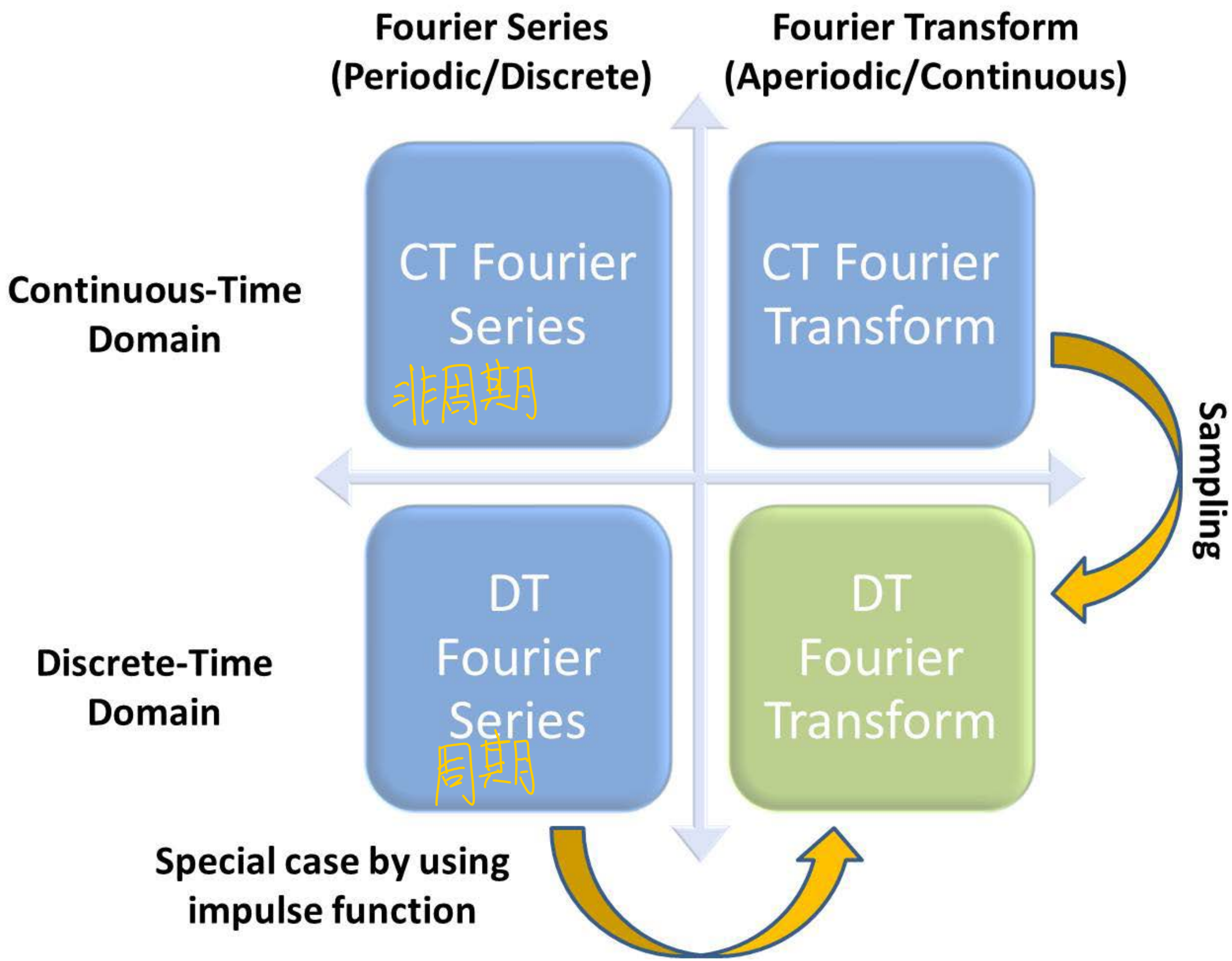
- 5.23, 5.29, 5.33

Tutorial problems

- Property of DTFT: 5.24
- Difference equation of LTI systems: 5.36

Frequency domain

Time domain



DTFT

- Therefore, we get the discrete-time Fourier transform pair

Discrete-Time Fourier Transform

$$\textit{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\textit{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Observations
  - Continuous spectrum: Similar to CTFT
  - Periodic with period 2π: Different from CTFT
  - Low frequency: close to 0 and 2π; high frequency: close to π

Discrete-Time Fourier Transform

$$\textit{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\textit{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

- If x (t) is periodical, then X(w) is computed by Fourier ________, and its shape is discrete?

  Further more,

  - If x (t) is CT, then a_k is ___________?
  - If x (t) is DT, then a_k is ___________?

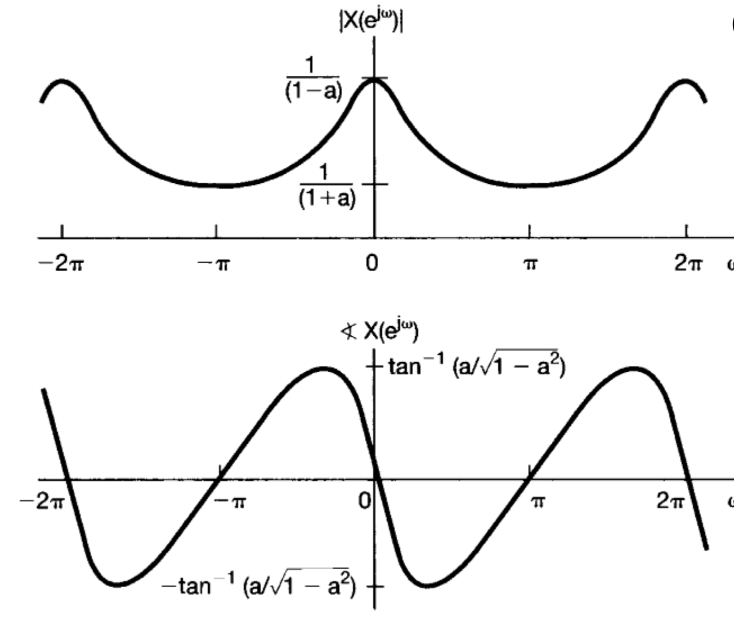
- If x (t) is aperiodical, then X(w) is computed by Fourier ________, and its shape is continuous?

  Further more,

  - If x (t) is CT, then X(w) is ___________?
  - If x (t) is DT, then X(w) is ___________?

Example 5.1

a>0



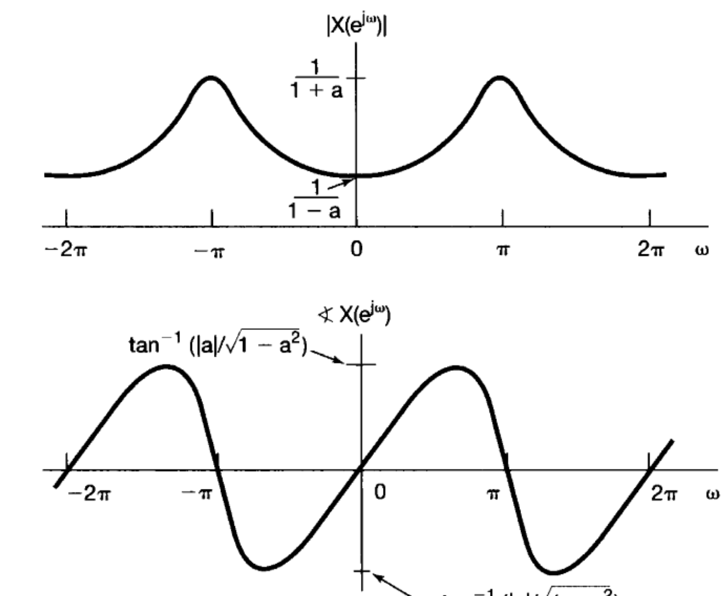
(a)

$$x[n] = a^n u[n], \qquad |a| < 1.$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}.$$

a<0



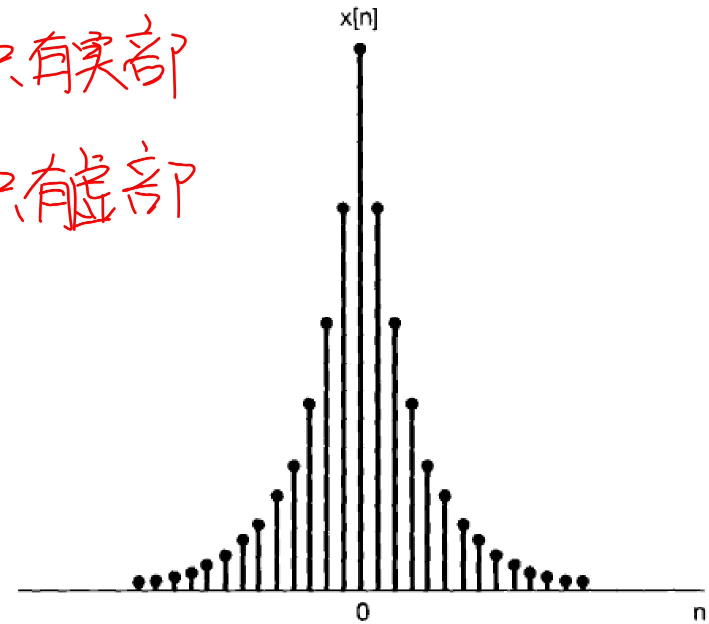
(b)

Example 5.2

偶函数 → 只有实部

虚函数 → 只有虚部

$$x[n] = a^{|n|}, \qquad |a| < 1.$$



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

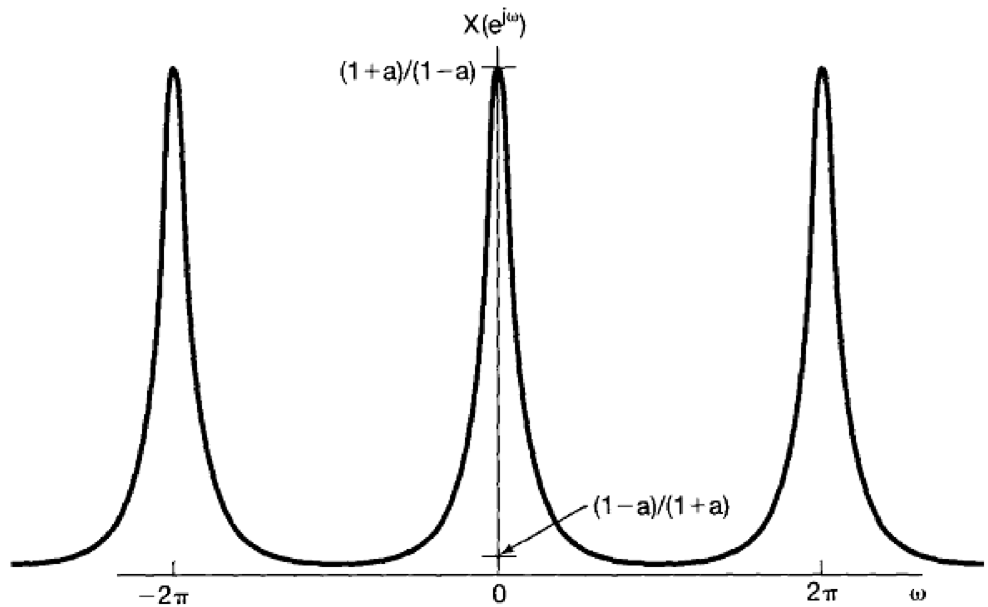
$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}.$$

m= -n

$$= \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{m=1}^{\infty} (ae^{j\omega})^m.$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}.$$



Example 5.4

$$x[n] = \delta[n]$$

$$\Downarrow$$

$$X(e^{j\omega}) = 1$$

Periodicity, Linearity and Shifting

DTFT

- Periodicity

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$ (2π为周期)

  - How about CTFT? Why? → 无

- Linearity

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

- Time Shifting and Frequency Shifting

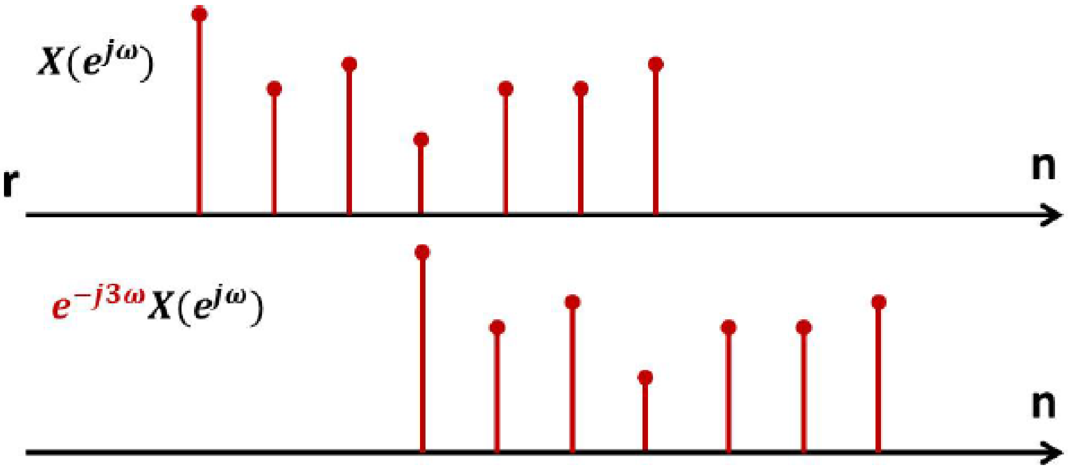
$$x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$$
$$e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

Duality!

Illustration on Shifting

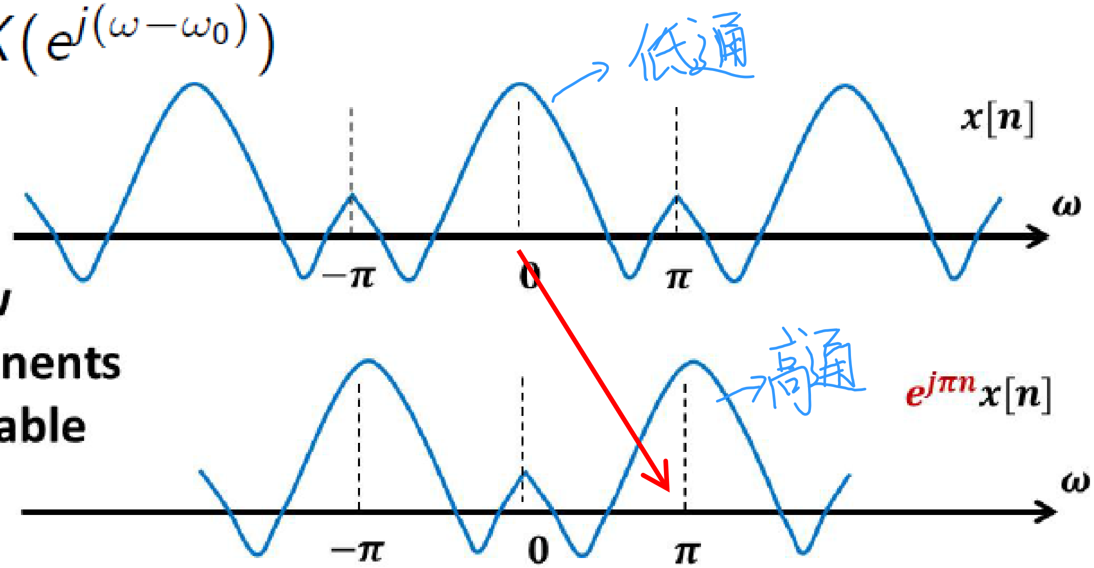
- $x[n-n_0] \longleftrightarrow e^{-j\omega n_0}X(e^{j\omega})$

Delay raises linear phase shifting



- $e^{j\omega_0 n}x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$

High and low frequency components are interchangeable



Conjugation, Differencing and Accumulation

- Conjugation

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

  - $X(e^{j\omega}) = X^*(e^{-j\omega}) \Leftrightarrow x[n]$ is real
  - If $x[n]$ is real, then $\Re\{X(e^{j\omega})\}$ is even, $\Im\{X(e^{j\omega})\}$ is odd

- Differencing

$$x[n] - x[n-1] \quad \longleftrightarrow \quad (1 - e^{-j\omega})X(e^{j\omega})$$

high-pass

  - High-pass or low-pass?

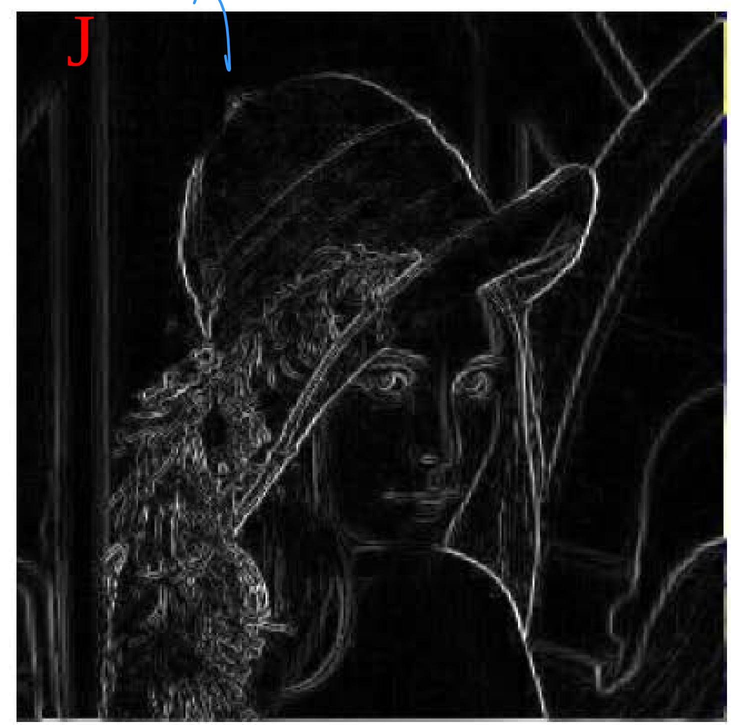
- Accumulation

$$\sum_{m=-\infty}^{n} x[m] \longleftrightarrow \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

  - How to derive it via differencing?

Effect of Differencing

点的亮度用来表示幅度



- $J(i,j) = |M(i,j) - M(i+1,j+1)| + |M(i+1,j) - M(i,j+1)|$

Fourier Transform of $u[n]$

$\frac{1}{(e^{-j\omega})^{+\infty}}$ $\quad$ $\frac{1-(e^{-j\omega})^{+\infty}}{1-e^{-j\omega}}$

有疑惑，与P6对比

- How to derive the Fourier transform of u[n]?
- Option 1: From definition of Fourier transform

$$\mathcal{F}\{u[n]\} = \sum_{n=-\infty}^{+\infty} u[n]e^{-j\omega n} = \sum_{n=0}^{+\infty} e^{-j\omega n} \quad \textit{Converge??}$$

don't converge

- Option 2: Since $u[n] = \sum_{m=-\infty}^{n} \delta[m]$, according to the property of accumulation

$$\mathcal{F}\{u[n]\} = \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

- Observation: Fourier transform of $u[n]$ does not exist according to the definition; however, it can be expressed in terms of $\delta(\cdot)$

Time Reversal and Expansion

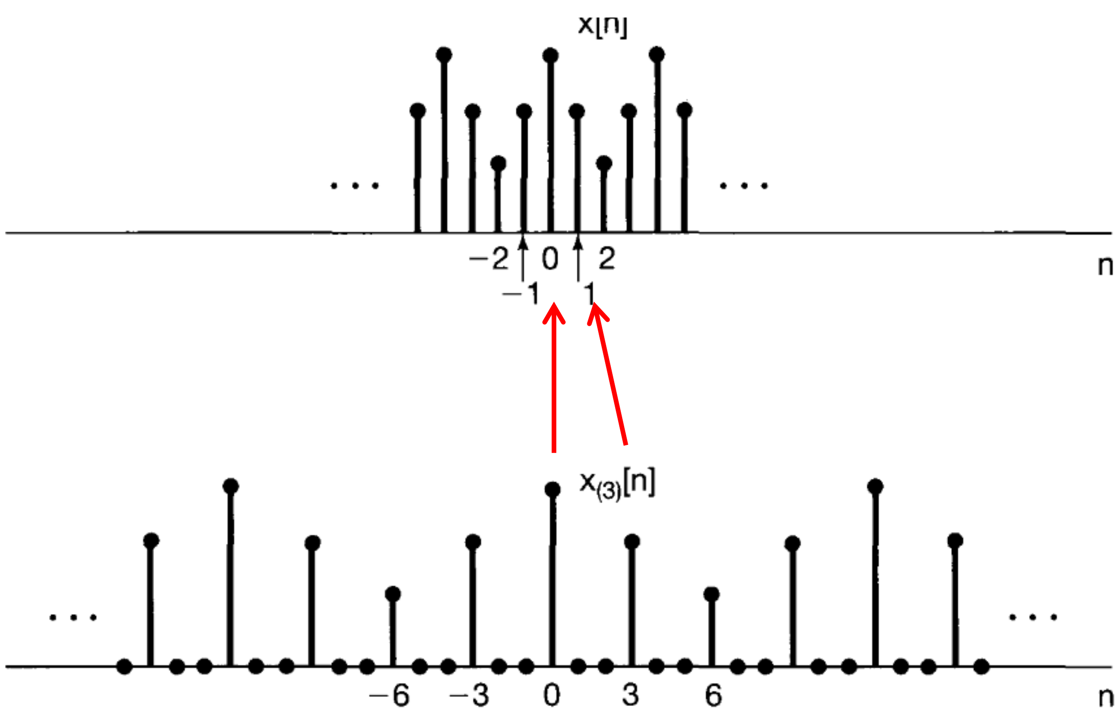
- Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

- Time Expansion
  - Define $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of k} \\ 0, & \text{Otherwise} \end{cases}$, then

$$x_{(k)}[n] \longleftrightarrow X(e^{jk\omega})$$

对 $X(e^{j\omega})$ 压缩



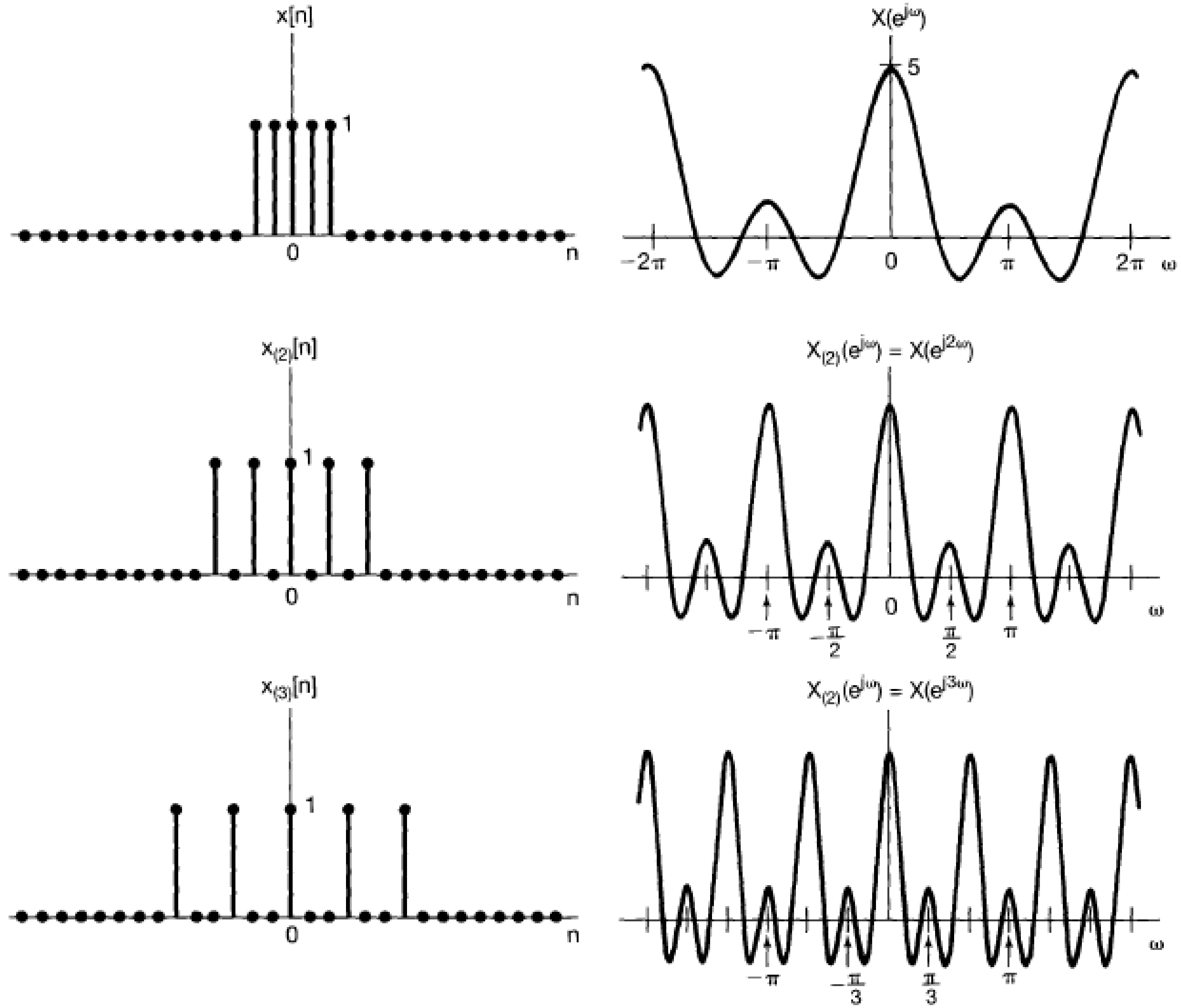


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Differentiation and Parseval

- Differentiation in Frequency

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

- Parseval's Relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

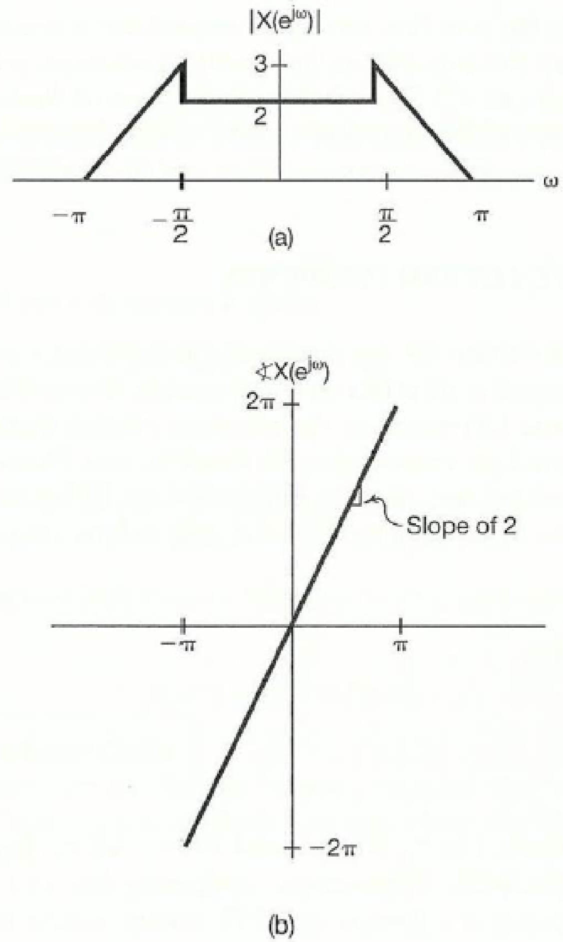
  - Energy density spectrum — $\frac{|X(e^{j\omega})|^2}{2\pi}$

Example

$x[n] \longleftrightarrow X(e^{j\omega})$

非周期性 $\longleftrightarrow$ 连续

周期 $\longleftrightarrow$ 离散



x[n] is real

- See textbook, Example 5.10
- Spectrum within $[-\pi, \pi]$
- Is it periodic, real, even, of finite energy?

Convolution Property & LTI Systems (1/2)

- Let $h[n]$ be the impulse response of certain LTI system
- The output $y[n]$ of input $x[n]$ is given by $y[n] = x[n] * h[n]$
- For input signal $x[n] = e^{j\omega n}$,

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}_{\text{Frequency Response } H(e^{j\omega})}$$

- For periodic input signal $x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$:

$$y[n] = \sum_{k=<N>} a_k H(e^{jk(\frac{2\pi}{N})})e^{jk(\frac{2\pi}{N})n}$$

$$b_k = a_k H(e^{jk(\frac{2\pi}{N})})$$

Convolution Property

If $y[n] = x_1[n] * x_2[n]$, *then*

$$Y(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$$

- For general input signal $x[n]$:

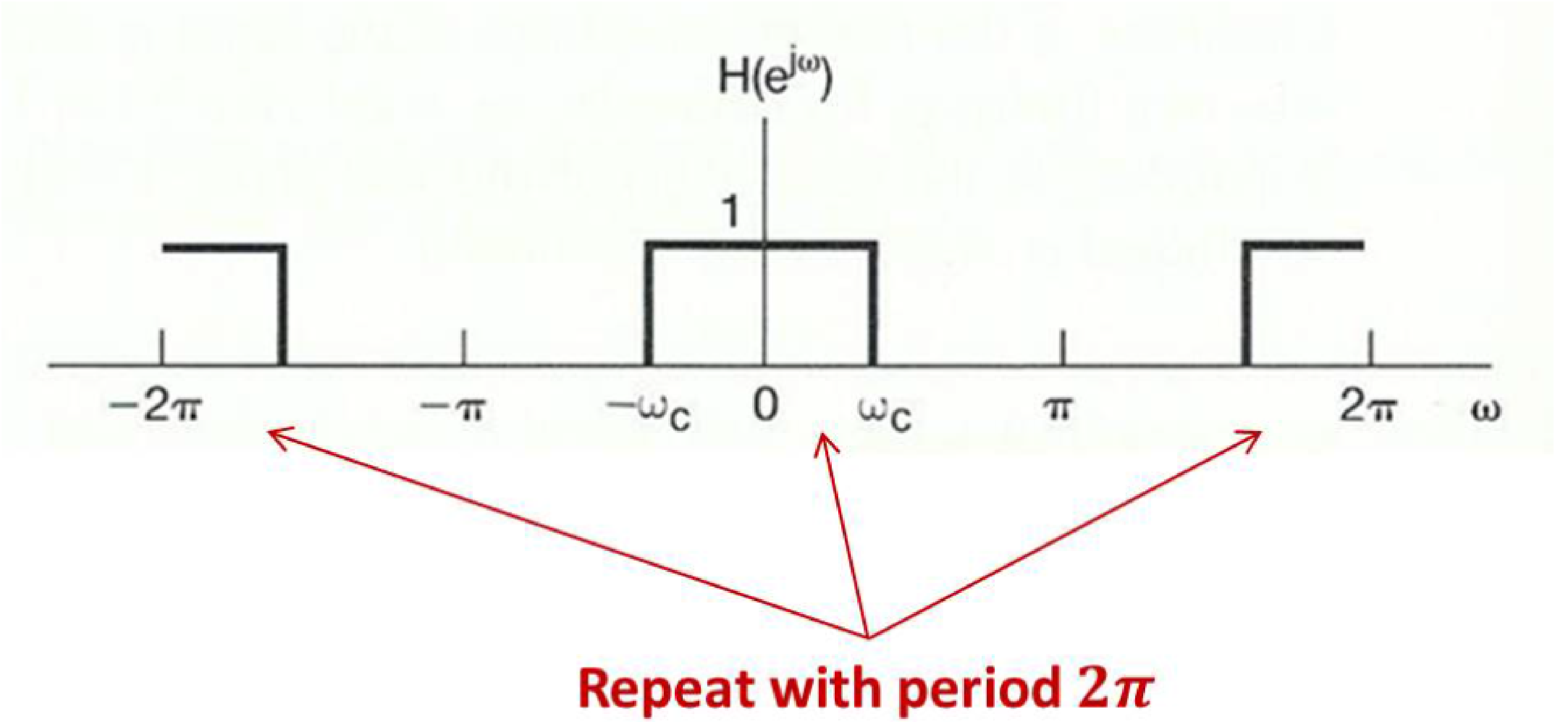
$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- Observation: It's easier to evaluate LTI systems in frequency domain
- Drawback: Not every LTI system has frequency response
  - $h[n] = a^n u[n]\ (a > 1)$ → no frequency response
  - $\boxed{\text{Stable}}$ LTI system has frequency response, because

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Example: Ideal Low-Pass Filter (1/2)

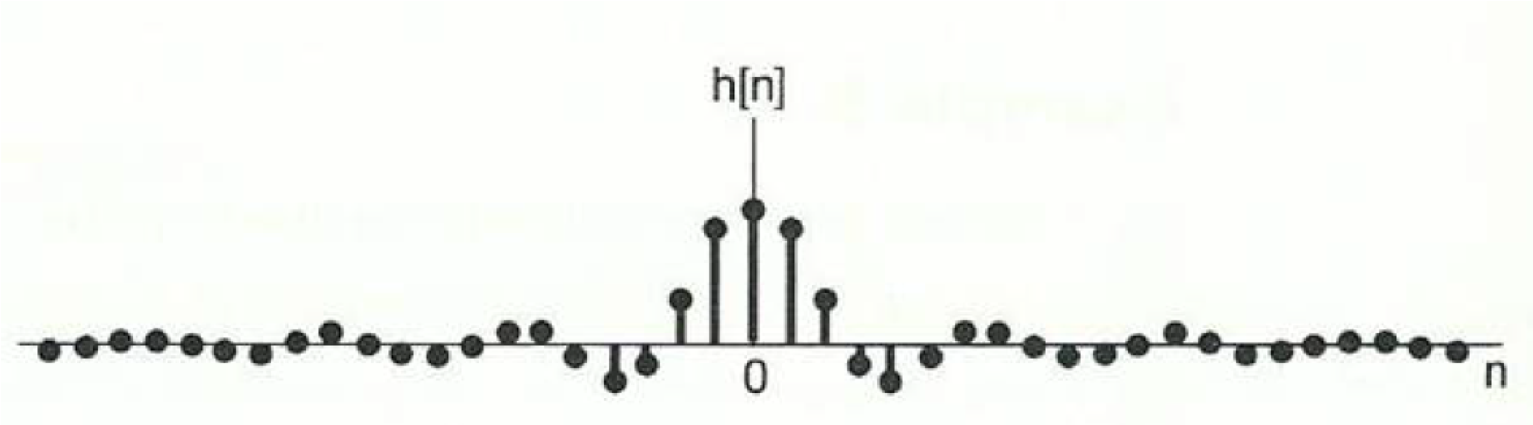
- What's ideal low-pass filter?
- Perfectly maintain the low-frequency component
- Perfectly cancel the high-frequency component



Example: Ideal Low-Pass Filter (2/2)

- Impulse response:

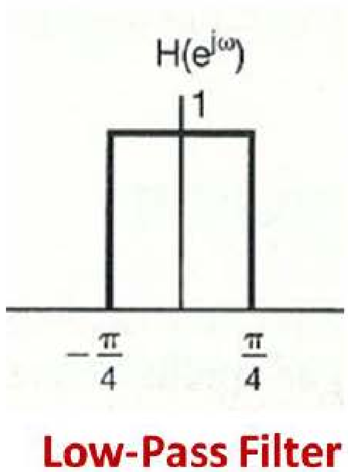
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega = \frac{sin\omega_c n}{\pi n}$$

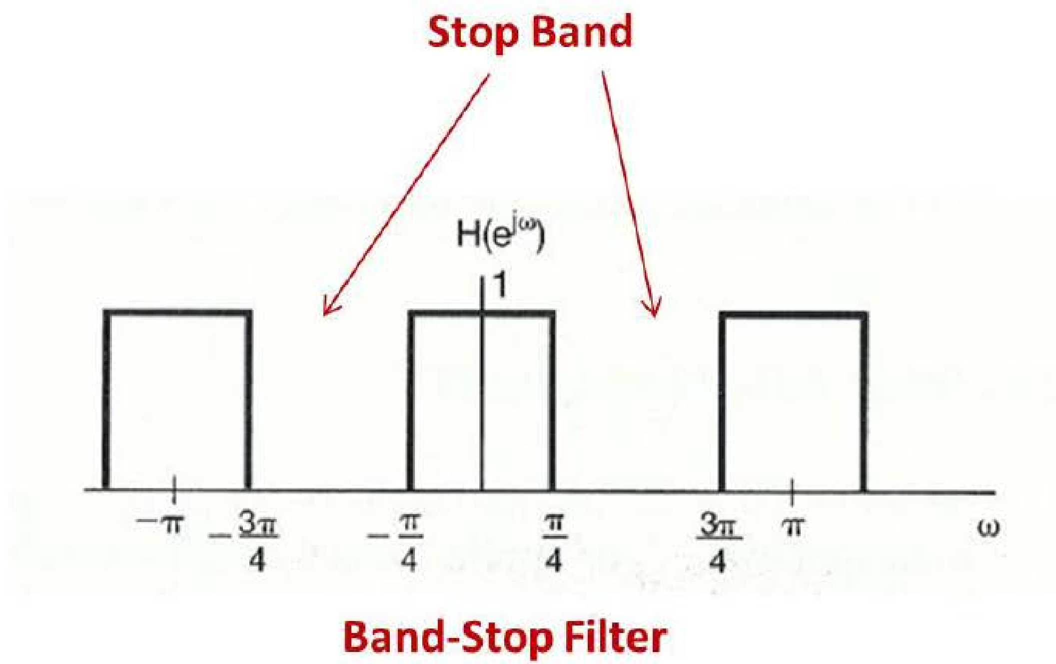


- Pros: no distortion in frequency domain
- Cons: non-causal 非因果
- See textbook, Example 5.12

Example: Band-Stop Filter (1/2)

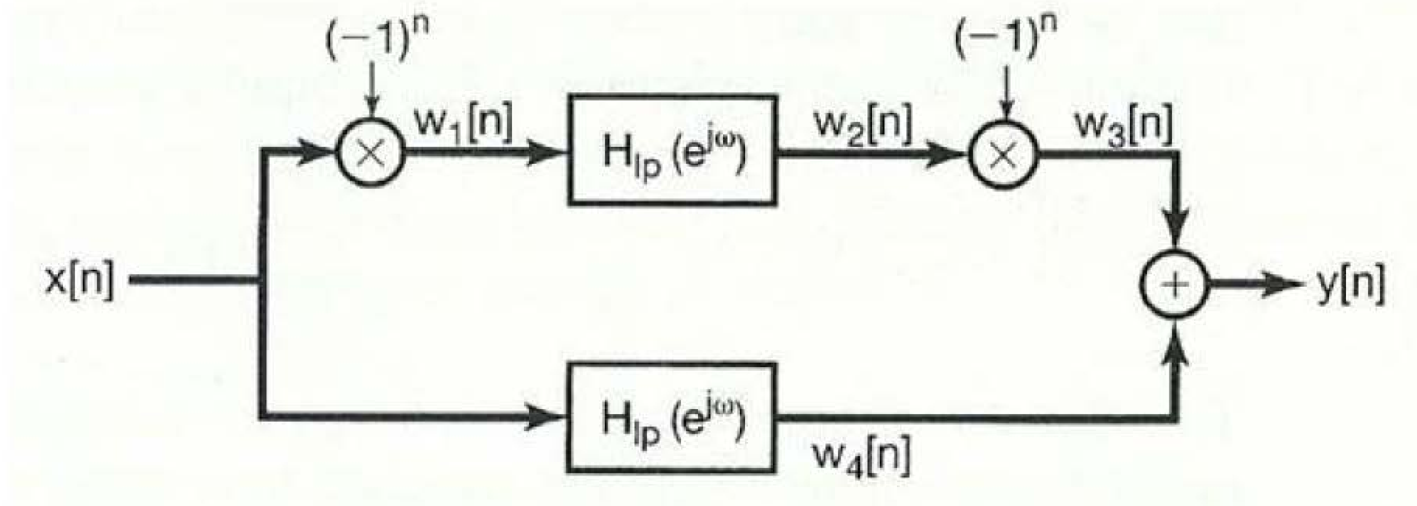
- Given low-pass filter $H_{lp}(e^{j\omega})$, how to generate band-stop effect from low-pass filter?





Example: Band-Stop Filter (2/2)

- Two branches: low-pass + high-pass
- See textbook, Example 5.14



- $(-1)^n = e^{j\pi n} \Rightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

$$\Rightarrow W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

$$\Rightarrow W_3(e^{j\omega}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) = H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega})$$

Multiplication Property

Multiplication Property

Let $y[n] = x_1[n]x_2[n]$, *then* DTFT is periodic

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta,$$

which is periodic convolution of $X_1(e^{j\omega})$ *and* $X_2(e^{j\omega})$

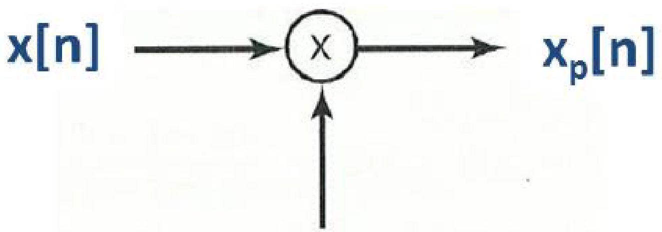
- Comparison: Multiplication of periodic signals

$$x_1[n] \longleftrightarrow a_k \quad \text{and} \quad x_2[n] \longleftrightarrow b_k$$ DTFS

$$\Rightarrow x_1[n]x_2[n] \longleftrightarrow \sum_{k=<N>} a_k b_{n-k}$$ discrete-time periodic convolution

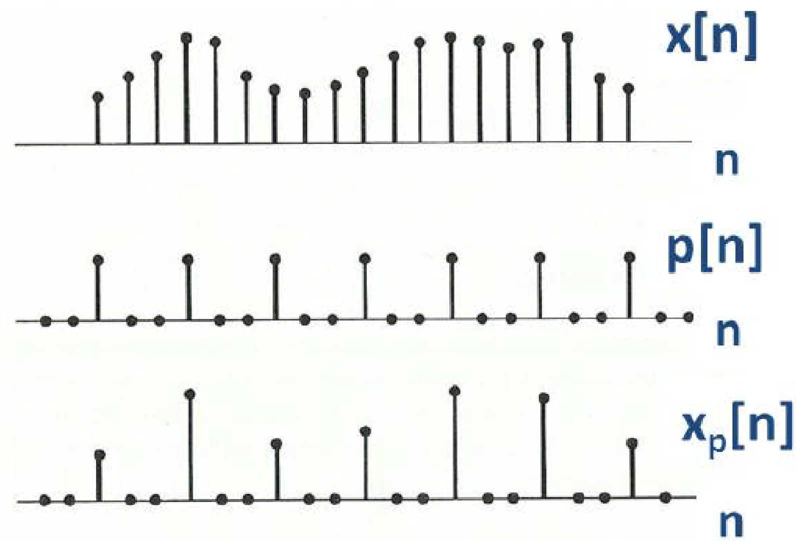
Example: Sampling on Discrete-Time Signals

- In digital signal processing, we may sample the discrete-time signals
- Example: image, audio, video compression
- Its mathematical model is give below:



$$\mathbf{p}[\mathbf{n}] = \sum_{\mathbf{k}=-\infty}^{+\infty} \delta[\mathbf{n} - \mathbf{kN}]$$

$$\mathbf{x_p}[\mathbf{n}] = \sum_{\mathbf{k}=-\infty}^{+\infty} x[kN]\delta[\mathbf{n} - \mathbf{kN}]$$



Example: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n-kN]$?
- First of all, we calculate the Fourier series:

n=kN

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{n=<N>} \sum_{k=-\infty}^{+\infty} \delta[n-kN] e^{-jk\frac{2\pi}{N}n} \\ &= \frac{1}{N} \sum_{n=<N>} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N} \end{aligned}$$

- Then, we have

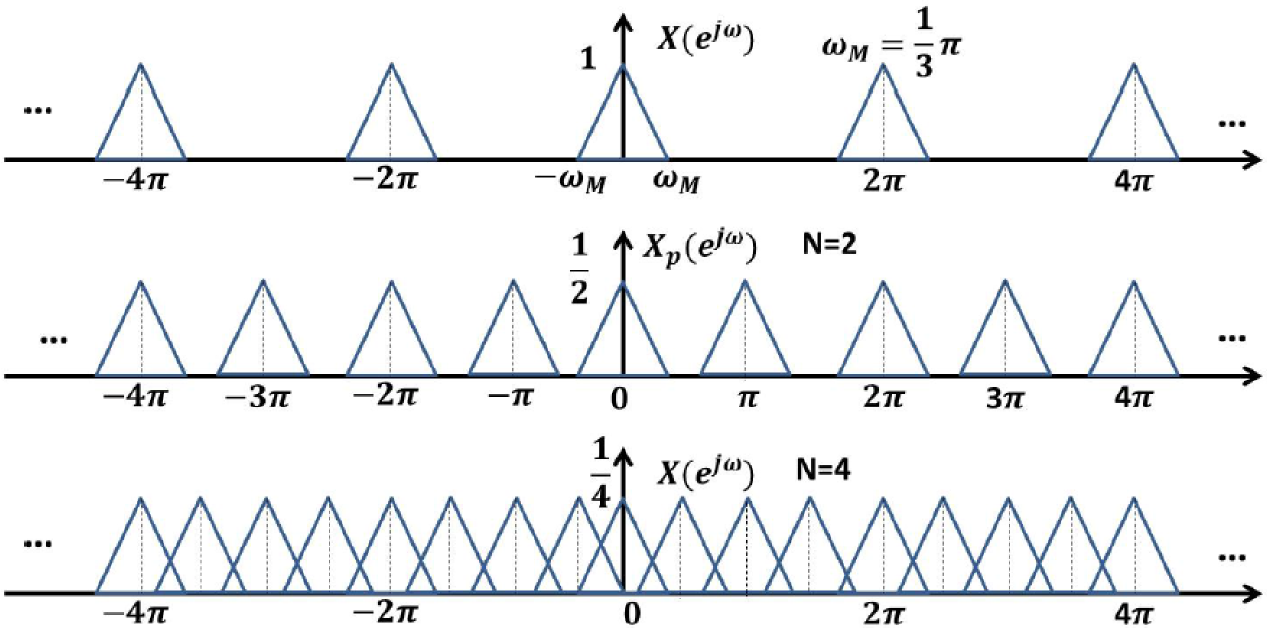
$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} \frac{2\pi}{N} \delta(\omega - k(2\pi/N) - 2\pi l) = \boxed{\frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2\pi k}{N})}$$

STC

Example: Frequency Analysis

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s) \quad \textit{where } \omega_s = 2\pi/N$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta = \boxed{\frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})}$$



Example #8: LTI Systems Described by LCCDE's

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

$$\boxed{H(j\omega) = \left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}$$

Linear constant-coefficient difference equations (LCCDE)

$$nx[n] \overset{\mathfrak{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$\Downarrow$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\Downarrow$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

Example 5.19

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n].$$

$$Y(e^{j\omega}) - \frac{3}{4}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{8}e^{-j2\omega}Y(e^{j\omega}) = 2X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j2\omega}}.$$

$$H(e^{j\omega}) = \frac{2}{(1 - \frac{1}{2}e^{-j\omega})(1 - \frac{1}{4}e^{-j\omega})} = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}$$

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$