Signal and Systems 2021: Homework

Probelm 1

4.50 Consider interpolating a signal x[n] by repeating each value q times, as depicted in Fig.P4.50. That is, we define $x_O = x\left[floor\left(\frac{n}{q}\right)\right]$, where floor(z) is the greatest integer less than or equal to z. Let $x_Z[n]$ be derived from $x[n]$ by inserting q-1 zeros between each value of $x[n]$; that is,

$$x_Z[n] = \begin{cases} x\left[\dfrac{n}{q}\right], & \dfrac{n}{q} \text{ integer} \\ 0, & \text{otherwise} \end{cases}$$

We may now write $x_O[n] = x_Z[n] \times h_O[n]$, where

$$h_O[n] = \begin{cases} 1, 0 \le n \le q-1 \\ 0, \text{ otherwise} \end{cases}$$

Note that this is the discrete-time analog of the zero-order hold. The interpolation process is completed by passing $x_O[n]$ through a filter with frequency response $H(e^{j\Omega})$.

a) Express $X_O(e^{j\Omega})$ in terms of $X(e^{j\Omega})$ and $H_O(e^{j\Omega})$. Sketch $|X_O(e^{j\Omega})|$ if $x[n] = \frac{sin\left(\frac{3\pi}{4}n\right)}{\pi n}$.

b) Assume that $X(e^{j\Omega})$ is as shown in Fig.P4.49. Specify the constrains on $H(e^{j\Omega})$ so that ideal interpolation is obtained for the following cases:

i. q=2,W=$\frac{3\pi}{4}$

ii. q=4,W=$\frac{3\pi}{4}$

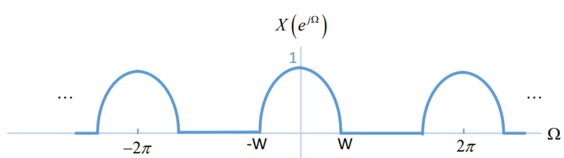

FIGURE P4.49

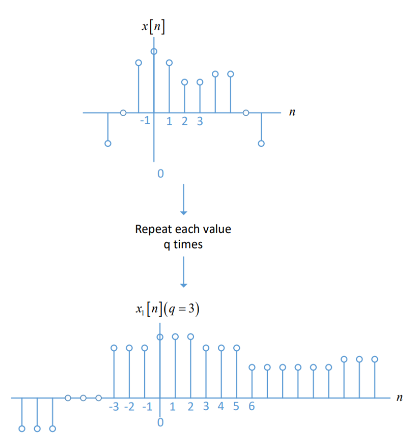

FIGURE P4.50

Probelm 1 continued on next page…

Solution:

a). From the problem, we can know that $x_O[n] = x_Z[n] * h_O[n]$ and $x_Z[n] = \begin{cases} x\left[\frac{n}{q}\right], & \frac{n}{q} \text{ integer} \\ 0, & \text{otherwise} \end{cases}$. According to the property Time Expansion and Convolution from the TABLE 5.1, we can know that: $X_O(e^{j\Omega}) = X_Z(e^{j\Omega})H_O(e^{j\Omega}) = X(e^{jq\Omega})H_O(e^{j\Omega})$.

In addition, as $x[n] = \frac{sin(\frac{3\pi}{4}n)}{\pi n}$, according to the TABLE 5.2, we can know that:

$$X(e^{j\Omega}) = \begin{cases} 1, & 0 \leq |\Omega| \leq \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} \leq |\Omega| \leq \pi \end{cases} \text{ periodic repeat with } T = 2\pi$$

.

Then we can know that:

$$X_Z(e^{j\Omega}) = X(e^{jq\Omega}) = \begin{cases} 1, & 0 \leq |\Omega| \leq \frac{3\pi}{4q} \\ 0, & \frac{3\pi}{4q} \leq |\Omega| \leq \frac{\pi}{q} \end{cases} \text{ periodic repeat with } T = \frac{2\pi}{q}$$

.

Besides, as $h_O[n] = \begin{cases} 1, 0 \leq n \leq q-1 \\ 0, & \text{otherwise} \end{cases}$, then we can know that: $H_O(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h_O[n]e^{-j\Omega n} = \frac{1-e^{-j\Omega q}}{1-e^{-j\Omega}} = \frac{e^{\frac{j\Omega q}{2}} - e^{-\frac{j\Omega q}{2}}}{e^{\frac{j\Omega(q-1)}{2}}(e^{\frac{j\Omega}{2}} - e^{-\frac{j\Omega}{2}})} = \frac{sin(\frac{\Omega q}{2})}{sin(\frac{\Omega}{2})} e^{\frac{j\Omega(1-q)}{2}}$.

Thus: $|X_O(e^{j\Omega})| = |X_Z(e^{j\Omega})||H_O(e^{j\Omega})| = \begin{cases} |\frac{sin(\frac{\Omega q}{2})}{sin(\frac{\Omega}{2})}|, & 0 \leq |\Omega| \leq \frac{3\pi}{4q} \\ 0, & \frac{3\pi}{4q} \leq |\Omega| \leq \frac{\pi}{q} \end{cases}$ periodic repeat with $T = 2\pi$.

When q=2, the plot of the $|X_O(e^{j\Omega})|$ is in the following:

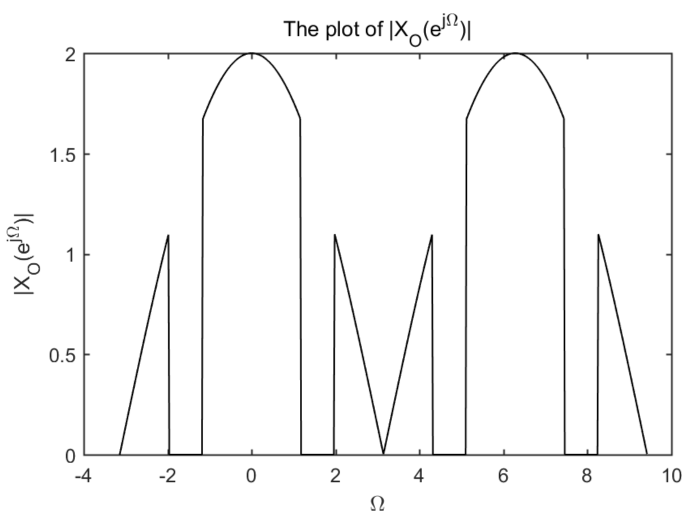

Figure.1 the plot of $X_O(e^{j\Omega})$ when q=2

b). As we have known that this is the discrete-time analog of the zero-order hold in fact, to realize the idel

interpolation which could reconstrute the original signal, a filter with freuqency response $H(e^{j\Omega})$ is need to discard the components other than those centered at multiples of 2π. By combining this filter with the rectangular impulse response from the Zero-Order hold, we can obtain an ideal Low-Pass filter which could realize the signal reconstruction in the Impulse-train sampling.
Thus, we can know that:

$$H(e^{j\Omega}) = \begin{cases} \dfrac{sin(\frac{\Omega}{2})}{sin(\frac{\Omega q}{2})} e^{\frac{j\Omega(q-1)}{2}},\ 0 \le |\Omega| \le \dfrac{W}{q} \\ 0,\ \dfrac{W}{q} \le |\Omega| \le 2\pi - \dfrac{W}{q} \end{cases} \text{periodic repeat with } T = 2\pi$$

i. q=2,W=$\frac{3\pi}{4}$

$$H(e^{j\Omega}) = \begin{cases} \dfrac{sin(\frac{\Omega}{2})}{sin(\Omega)} e^{\frac{j\Omega}{2}},\ 0 \le |\Omega| \le \dfrac{3\pi}{8} \\ 0,\ \dfrac{3\pi}{8} \le |\Omega| \le \dfrac{13\pi}{8} \end{cases} \text{periodic repeat with } T = 2\pi$$

ii. q=4,W=$\frac{3\pi}{4}$

$$H(e^{j\Omega}) = \begin{cases} \dfrac{sin(\frac{\Omega}{2})}{sin(2\Omega)} e^{\frac{3j\Omega}{2}},\ 0 \le |\Omega| \le \dfrac{3\pi}{16} \\ 0,\ \dfrac{3\pi}{16} \le |\Omega| \le \dfrac{29\pi}{16} \end{cases} \text{periodic repeat with } T = 2\pi$$

Probelm 2

4.51 The system shown in Fig.P4.51 is used to implement a band-pass filter. The frequency response of discrete-time filter is

$$H(e^{j\Omega}) = \begin{cases} 1,\ \Omega_a \le |\Omega| \le \Omega_b \\ 0,\ \text{otherwise} \end{cases}$$

on $-\pi \le \Omega \le \pi$. Find the sampling interval $T_s, \Omega_a, \Omega_b, W_1, W_2, W_3$, and W_4 so that the equivalent continuous-time frequency response $G(jw)$ satisifies

$$0.9 < |G(jw)| < 1.1,\ for\ 100\pi < w < 200\pi$$

$$G(jw) = 0 \text{ elsewhere}$$

In solving this problem, choose W_1 and W_3 as small as possible, and choose T_s, W_2 and W_4 as large as possible.

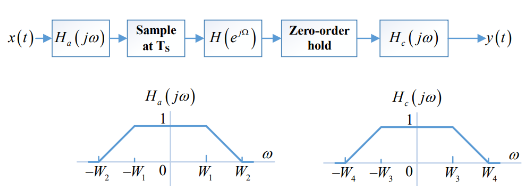

FIGURE P4.51

Solution:

Firstly, we label the signals appearing in the system. Then the figure of the system is in the following:

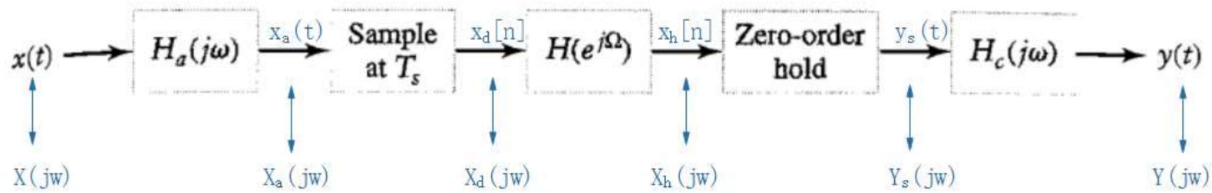

Figure.2 the label of the signal appearing in the system

In order to get the $G(jw)$ for this band pass filter, we just set $x(t) = \delta(t)$ as input signal, which $X(jw) = 1$. Then the plot of the signal of the signal appearing in the system is in the following:

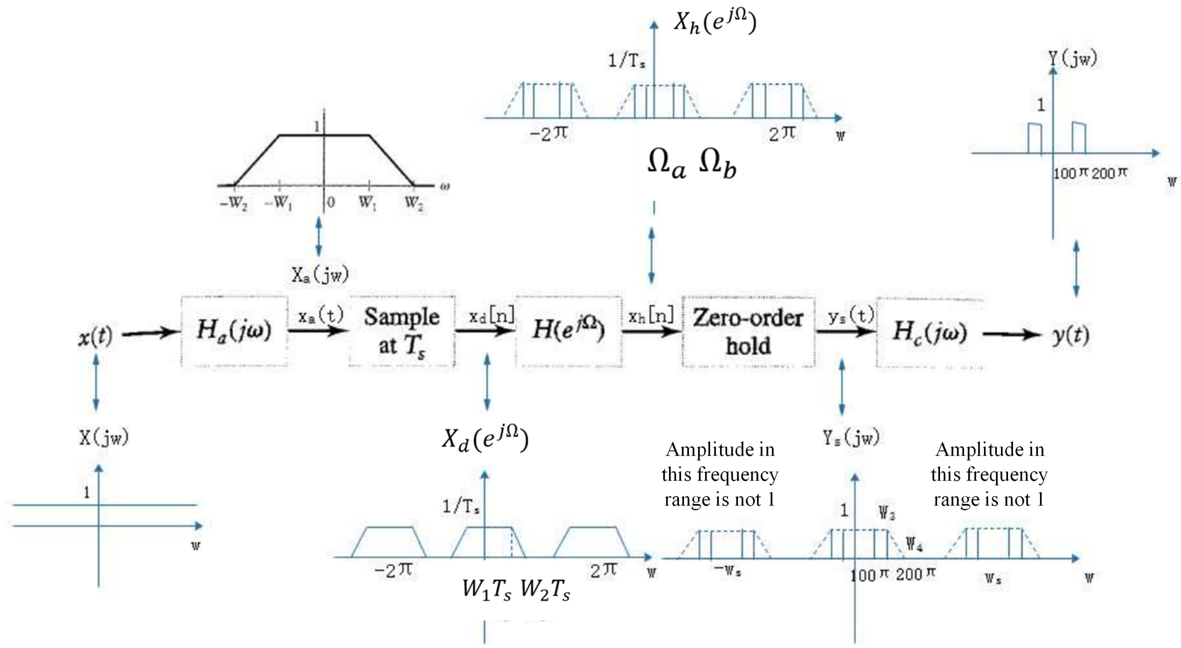

Figure.3 the plot of the signal appearing in the system

Let $H_O(jw)$ be the frequency response of rectangular impulse response from the zero-order hold, then we can know that: $|H_O(jw)| = |\frac{2sin(w\frac{T_s}{2})}{w}|$.

Moreover, as the amplitude of $X_h(e^{j\Omega})$ is $\frac{1}{T_s}$ and $0.9 < |G(jw)| < 1.1,\ for\ 100\pi < w < 200\pi$, we can know that:

$$0.9 < |\frac{2sin(50\pi T_s)}{100\pi T_s}| < 1.1, 0.9 < |\frac{2sin(100\pi T_s)}{200\pi T_s}| < 1.1$$

Setting $x = w\frac{T_s}{2}$, we can get: $G(jw) = \frac{2sin(w\frac{T_s}{2})}{wT_s} = \frac{sin(w\frac{T_s}{2})}{w\frac{T_s}{2}} = \frac{sin(\pi\frac{wT_s}{2\pi})}{\pi\frac{wT_s}{2\pi}} = sinc(\frac{wT_s}{2\pi})$. Then the plot of the $|sinc(x)|$ is in the following:

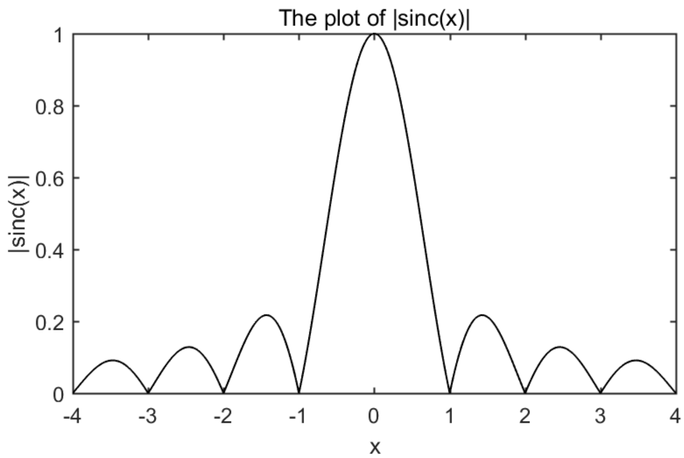

Figure.4 the plot of |sinc(x)|

Then the conditions are like the following:

$$0.9 < |sinc(50T_s)| < 1.1, 0.9 < |sinc(100T_s)| < 1.1$$

From the Figure.4, we can know that the first condition is always meet and for condition two, the value range of x should be between 0 and 1. Then by using the binary search in the [0,1] with tolerance error 0.00001, the result is in the following:

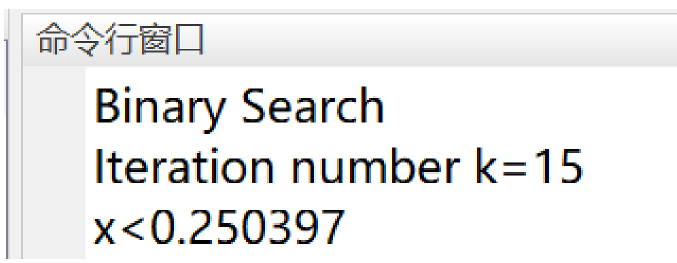

Figure.4 the result of the binary search

Thus, we can get that: $x < 0.2504 \to 100T_s < 0.2504 \to T_s < 0.002504 \to T_{smax} = 0.0025 \to \Omega_a = 100\pi T_s = 0.25\pi, \Omega_b = 200\pi T_s = 0.5\pi$.

In order to eliminate aliasing and choose W_1 and W_3 as small as possible(choose W_2 and W_4 as large as possible), we can get that:

$$W_1 \geq 200\pi$$

$$W_3 \geq 200\pi$$

$$W_2 \leq \frac{1}{2}\frac{2\pi}{T_s}\pi = 400\pi$$

$$W_4 \leq \frac{2\pi}{T_s} - 200\pi = 600\pi$$

In conclusion, we can get that: $W_{1min} = 200\pi$, $W_{3min} = 200\pi$, $W_{2max} = 400\pi$, $W_{4max} = 600\pi$, $T_{smax} = 0.0025$, $\Omega_a = 0.25\pi$ and $\Omega_b = 0.5\pi$.