Tutorial on Orthogonal Frequency Division Multiplexing (OFDM)

Discrete Fourier Transform (DFT)

- In practice, there is a huge demand on processing finite duration signals
- Given a finite duration signal $\{x[0], x[1], ..., x[N-1]\}$, its Fourier transform is

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

- Drawback: The spectrum of DTFT is continuous $\Rightarrow$ Cannot be handle by computer.
- Discrete Fourier Transfrom (DFT) is developed for digital processing of finite duration signals

$$\widetilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n} \quad k = < N >$$

- DFT: frequency sampling of DTFT

$$\widetilde{X}[k] = \frac{1}{N}X(e^{j2k\pi/N}) \quad k = < N >$$

Inverse DFT

- Equation system of DFT:

$$\begin{pmatrix} \widetilde{X}[0] \\ \widetilde{X}[1] \\ \cdots \\ \widetilde{X}[N-1] \end{pmatrix} = \underbrace{\frac{1}{N}\begin{pmatrix} e^{-j0} & e^{-j0} & \cdots & e^{-j0} \\ e^{-j0} & e^{-j2\pi/N} & \cdots & e^{-j2(N-1)\pi/N} \\ \cdots & \cdots & \cdots & \cdots \\ e^{-j0} & e^{-j2(N-1)\pi/N} & \cdots & e^{-j2(N-1)(N-1)\pi/N} \end{pmatrix}}_{\mathbf{F}} \begin{pmatrix} x[0] \\ x[1] \\ \cdots \\ x[N-1] \end{pmatrix}$$

- Transform matrix **F** is full rank.
- Observation: $\{\widetilde{X}[k] | k =< N >\}$ maintains all the information of $\{x[0], x[1], ..., x[N-1]\}$
- Inverse DFT is feasible:

$$x[n] = \sum_{k=0}^{N-1} \widetilde{X}[k] e^{jk(2\pi/N)n} \quad n = 0, 1, ..., N-1$$

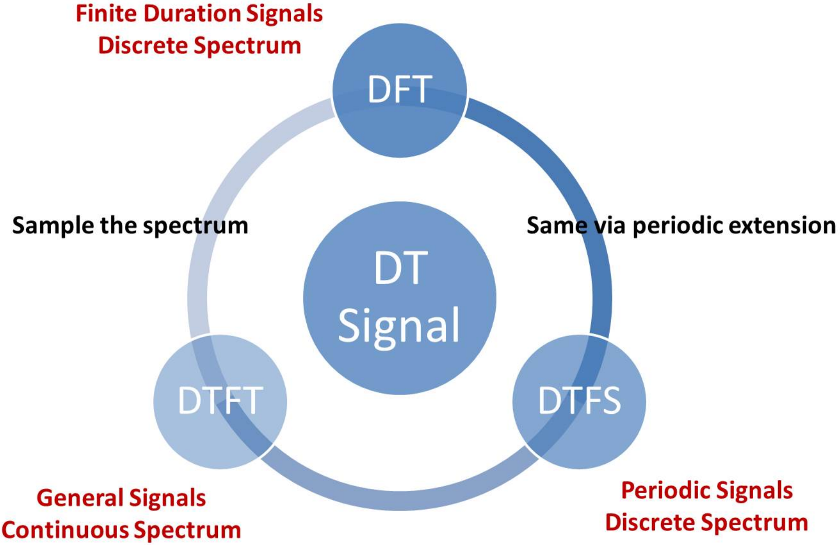
DFT and DTFS

- DFT is for finite duration signals; DTFS is for periodic signals
- Define $\tilde{x}[n]$ as the periodic extension of $x[n]$: Repeat $x[n]$ with period N
- Fourier series of $\tilde{x}[n]$:

$$\frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{-jk(2\pi/N)n} \Rightarrow \text{DFT of } x[n]$$

- DFT of a finite-duration signal = DTFS of its periodic extension
- Reference on DFT:
  - Textbook: Problem 5.53, 5.54
  - http://en.wikipedia.org/wiki/Discrete_Fourier_transform

Comparison



Finite Duration Signals
Discrete Spectrum
DFT
Sample the spectrum
Same via periodic extension
DT
Signal
DTFT
DTFS
General Signals
Continuous Spectrum
Periodic Signals
Discrete Spectrum

Periodic Convolution of Finite Duration Signals

Periodic Convolution

Let x and y be two finite duration signals with duration N, $\tilde{x}$ and $\tilde{y}$ be the associated periodic extension, then the periodic convolution of finite duration signals is defined as

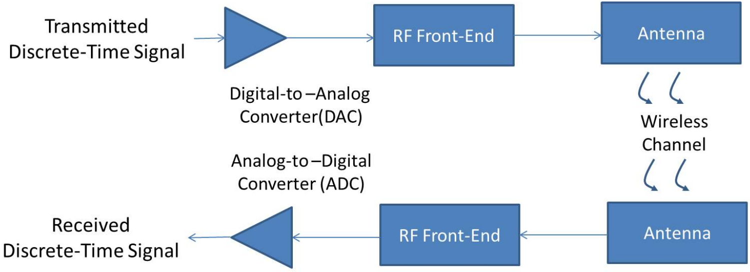
$$x[n] \circledast y[n] := \widetilde{x}[n] \circledast \widetilde{y}[n] = \sum_{k=<N>} \tilde{x}[k]\tilde{y}[n-k] \quad n = 0, 1, 2..., N-1$$

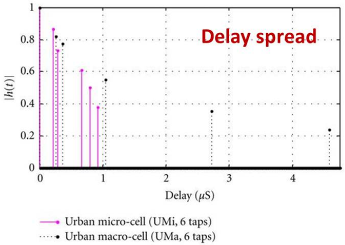
Convoluation Property of DFT

Time domain periodic convolution is equivalent to frequency domain multiplication, thus,

- $x[n] \circledast y[n] \longleftrightarrow N\widetilde{X}[k]\widetilde{Y}[k]$

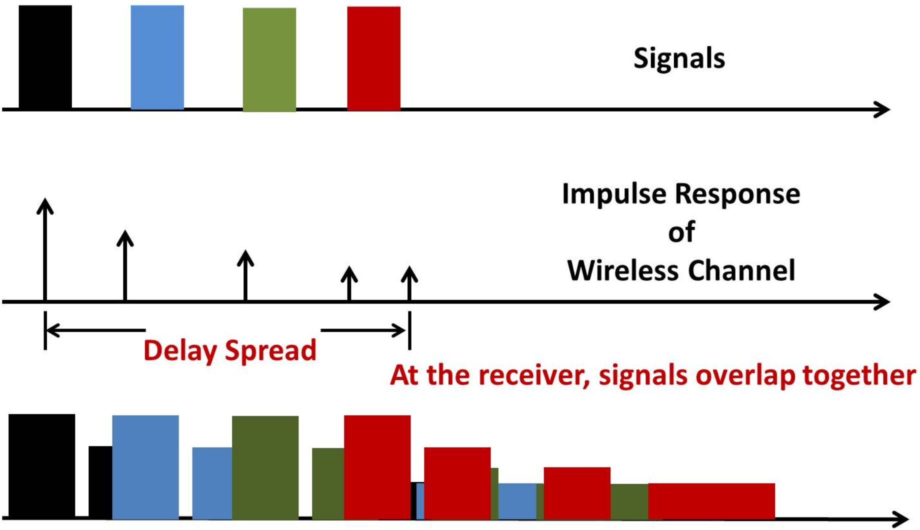
Wireless Systems





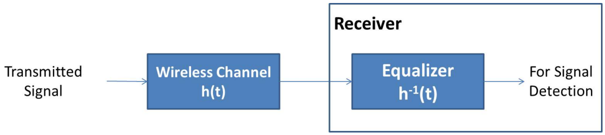
Delay spread causes inter-symbol interference

Inter-Symbol Interference (1/2)



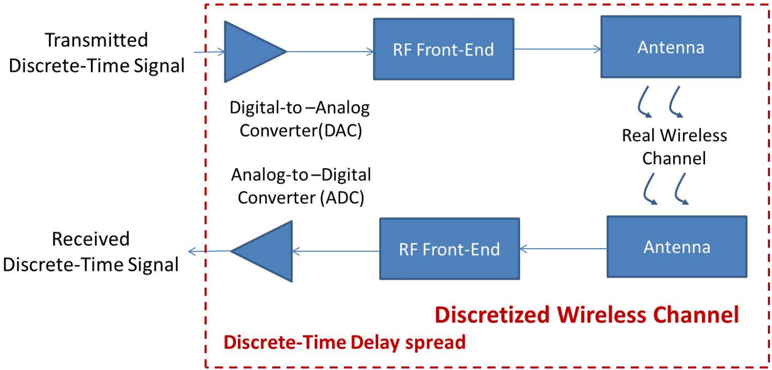
Inter-Symbol Interference (2/2)

- How to deliver signals without inter-symbol interference?
  - Suppose the duration of delay spread is ΔH seconds
- Approach 1: Send data on every ΔH seconds
- Approach 2: Channel equalizer



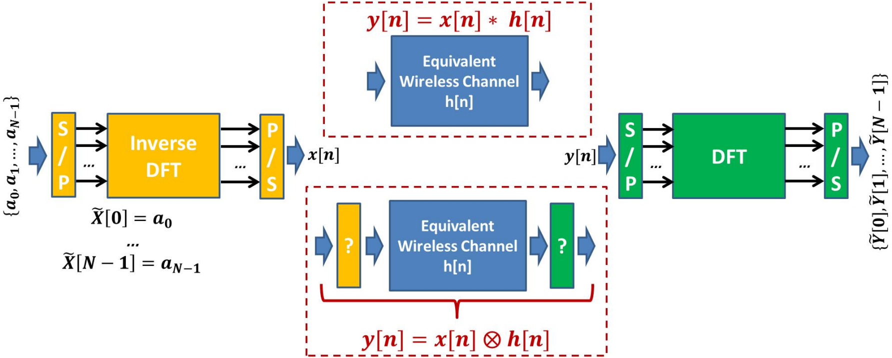
- Approach 3: Orthogonal Frequency Division Multiplexing
  - Pre-processing at the transmitter + post-processing at the receiver

Discretetize Wireless Channel



- Tx front-end + wireless channel + Rx front-end: approximately discrete-time LTI system
- Denote the impulse response as $h[n]$

OFDM at First Glance

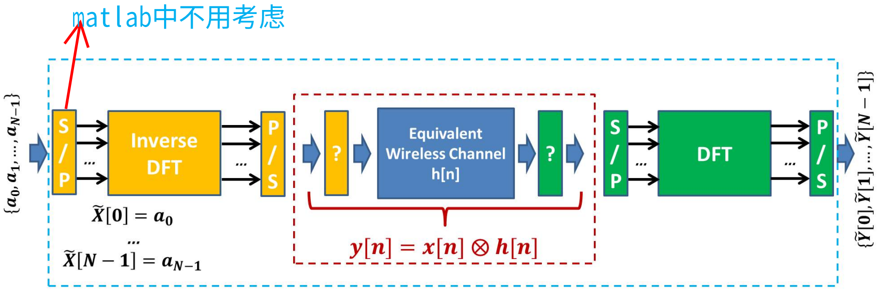


- Signals are loaded in frequency domain
- Some mechanism is necessary to generate the effect of periodic convolution

OFDM Analysis

- Let $\{a_0, a_1, ..., a_{N-1}\}$ be the sequence of bits to be delivered from the transmitter to the receiver.
- At the transmitter
  - we let
$$\widetilde{X}[0] = a_0, \widetilde{X}[1] = a_1, ..., \widetilde{X}[N-1] = a_{N-1}$$
  - Take inverse DFT:
$$\{x[0], x[1], ..., x[N-1]\} = DFT^{-1}\{\widetilde{X}[0], \widetilde{X}[1], ..., \widetilde{X}[N-1]\}$$
- With certain mechanism, the received signal becomes
$$y[n] = x[n] \circledast h[n]$$
- At the receiver, take DFT on $y[n]$:
$$\widetilde{Y}[k] = N\widetilde{X}[k]\widetilde{H}[k] = Na_k\widetilde{H}[k] \quad k = 0, 1, ..., N-1$$

So, ...



$\widetilde{Y}[k] = N a_k \widetilde{H}[k], \qquad k = 0, 1, \ldots, N-1$

- How to detect $\{a_k | \forall k\}$ from $\{\widetilde{Y}[k] | \forall k\}$?
- How to design the blocks "?" ?

Channel Estimation & Signal Detection

- Channel estimation: Estimate the channel gain $\widetilde{H}[k]$ $(k = 0, 1, ..., N-1)$
  - The transmitter sends the common information to the receiver $\{a_0^c, a_1^c, ..., a_{N-1}^c\}$:

$$\widetilde{Y}^c[k] = Na_k^c \widetilde{H}[k] \quad k = 0, 1, ..., N-1$$

  - Evaluate channel via

$$\widetilde{H}[k] = \widetilde{Y}^c[k]/(Na_k^c) \quad k = 0, 1, ..., N-1$$

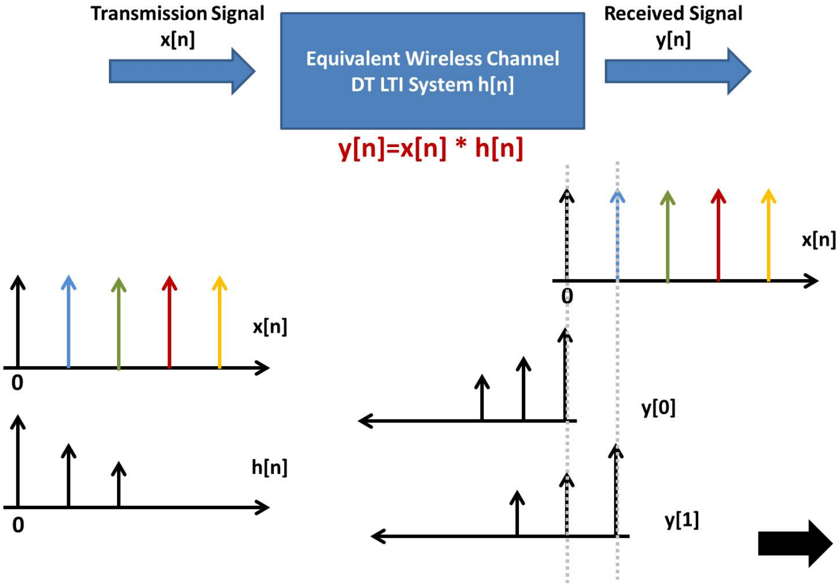
- Signal detection: Detect the transmitter's signal via the knowledge of $\widetilde{H}[k]$ $(k = 0, 1, ..., N-1)$:
  - The transmitter sends the information to the receiver $\{a_0, a_1, ..., a_{N-1}\}$:

$$\widetilde{Y}[k] = Na_k \widetilde{H}[k] \quad k = 0, 1, ..., N-1$$

  - Detect signal via

$$a_k = \widetilde{Y}[k]/(N\widetilde{H}[k]) \quad k = 0, 1, ..., N-1$$

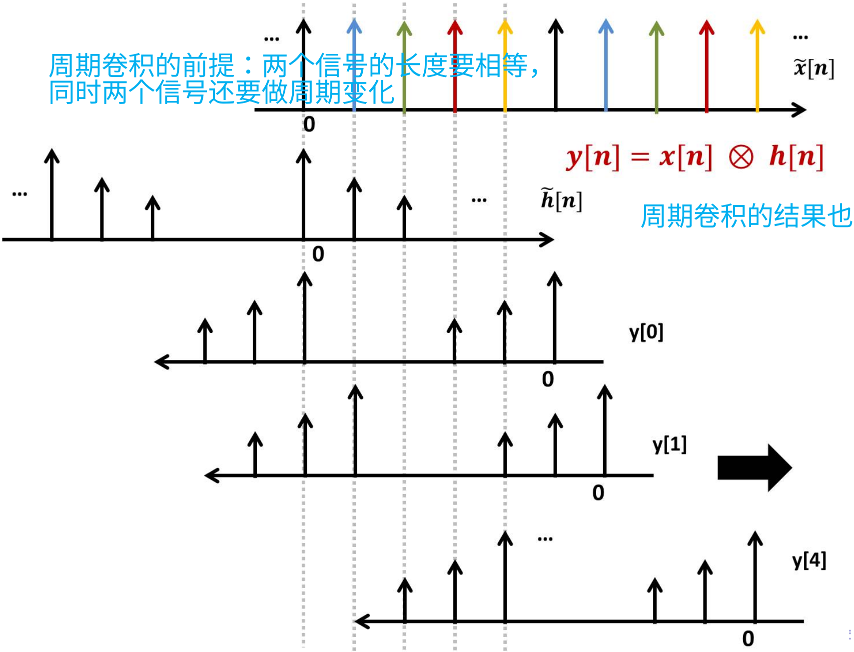
Convolution vs. Periodic Convolution



周期卷积的前提：两个信号的长度要相等，
同时两个信号还要做周期变化

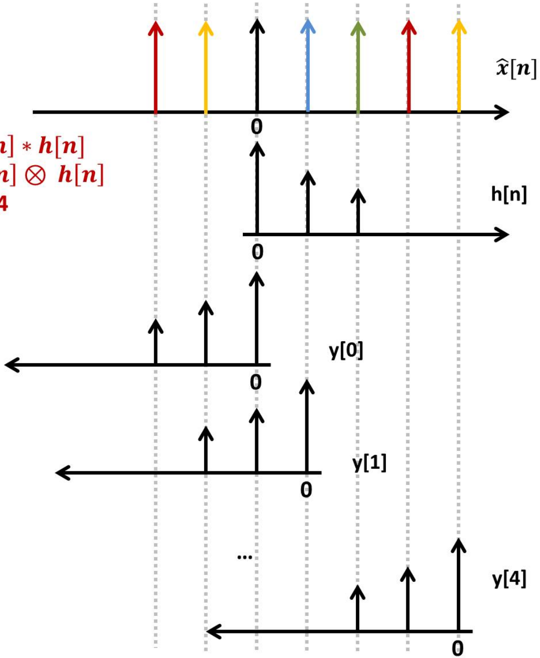
$$y[n] = x[n] \otimes h[n]$$

周期卷积的结果也是周期



$$y[n] = \hat{x}[n] * h[n]$$
$$= x[n] \otimes h[n]$$

n=0,1,2,3,4



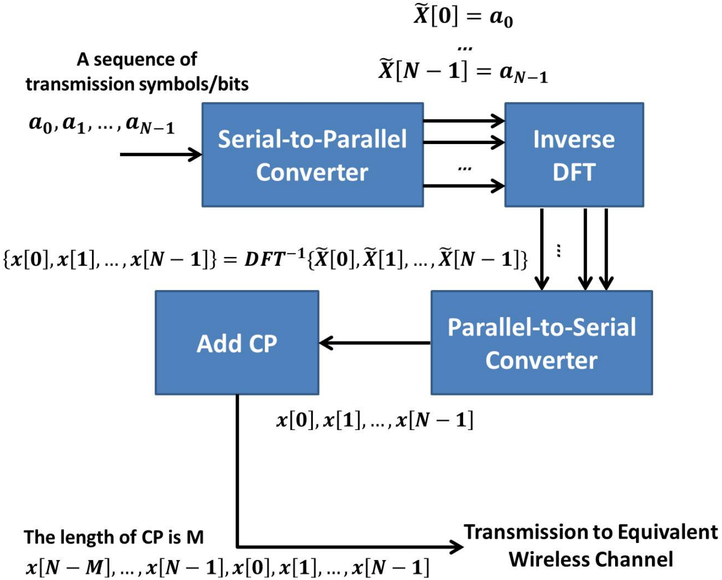
Observations

- Periodic convolution can be different from convolution
- In order to guarantee that

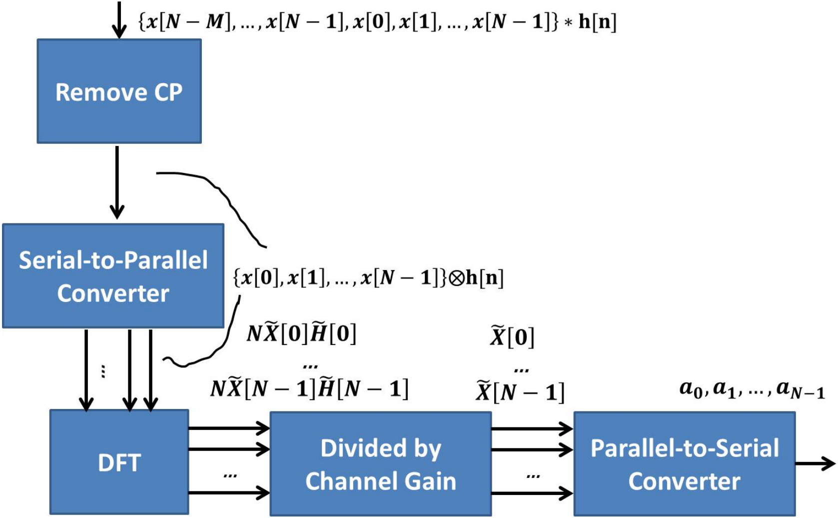
$$y[n] = x[n] \circledast h[n], \text{ for } n = 0, 1, 2, ..., N-1,$$

  we should copy the last few samples of $x[n]$ to the beginning, which is named as cyclic prefix
  - How many samples should we copy?

Transmitter Structure of OFDM



Receiver Structure of OFDM



Reference

- Reference
  - www.gaussianwaves.com/2011/05/introduction-to-ofdm-orthogonal-frequency-division-multiplexing-2/
  - www.wirelesscommunication.nl/reference/chaptr05/ofdm/ofdmmath.htm