南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Tutorial Questions （Week 6）

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Outline

- Review
- Quiz Problem
- Basic Problems with Answers 3.8
- Basic Problems 3.34
- Advanced Problems 3.40
- Q&A

Prior knowledge review on Chap 1

- Basic knowledge on signal computation
- Exponential signals: Euler's relation, periodic, integral
- CT/DT unit impulse/step function
- System Properties

  1. Memoryless or with memory
  2. Causality
  3. Invertibility
  4. Stability
  5. Time-invariance
  6. Linearity

Basic knowledge review on Chap 2

- CT/DT LTI systems
- Convolution operation procedure
    1. **Figure computation** based on "Flip-slide-multiply-sum/integral"
    2. Some known or **typical convolution results**
    3. **Properties** of convolution
- Unit impulse response and properties of LTI systems
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Example 1

- $x[n] = a^n u[n]$
- $h[n] = u[n]$
- $y[n] = x[n] * h[n]$?

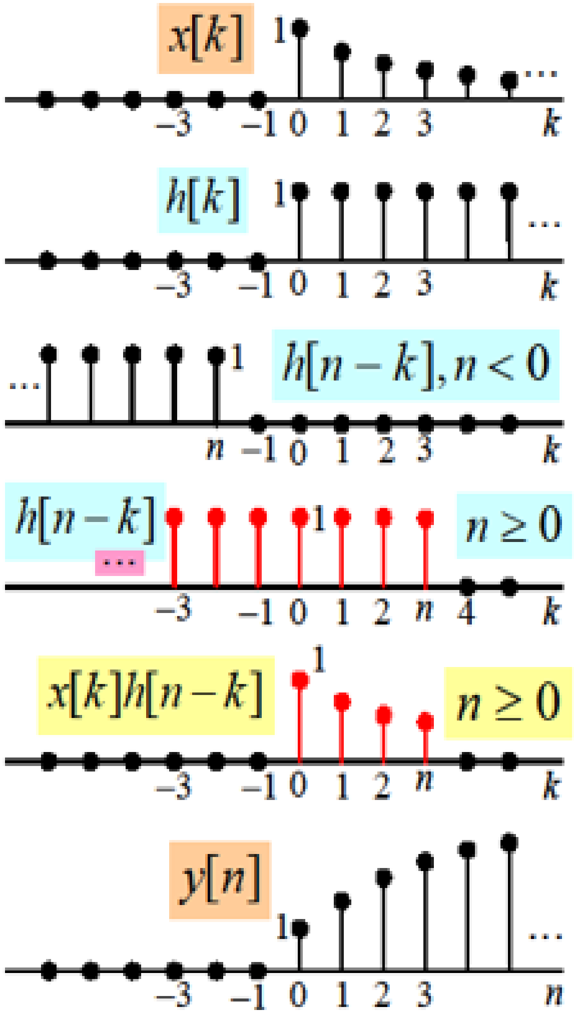
$$y[n] = \begin{cases} \dfrac{1-a^{n+1}}{1-a}, & n \geq 0 \\ 0, \quad n < 0 \end{cases}$$

Flip

Slide

Multiply

Sum

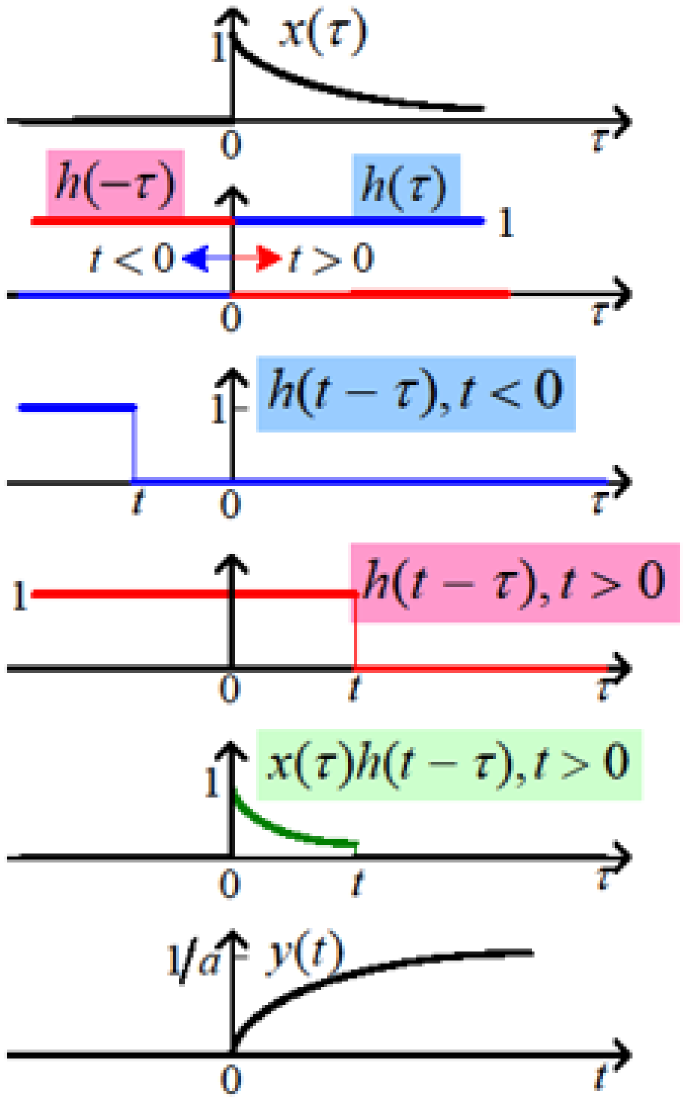


$$h[n-k]$$

$$x[k]h[n-k] = \begin{cases} a^k, & 0 \leq k \leq n \\ 0, & k < 0, k > n \end{cases}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$$

Example 2



$$x(t) = e^{-at}u(t)$$
$$h(t) = u(t)$$
$$y(t) = x(t) * h(t)$$

$$y(t) = \begin{cases} \dfrac{1-e^{-at}}{a}, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow \frac{1-e^{-at}}{a}u(t)$$

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表 3.4 基本信号的卷积表

连续时间卷积积分			离散时间卷积和		
$x(t)$	$h(t)$	$x(t)*h(t)$	$x[n]$	$h[n]$	$x[n]*h[n]$
$x(t)$	$\delta(t)$	$x(t)$	$x[n]$	$\delta[n]$	$x[n]$
$x(t)$	$u(t)$	$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$	$x[n]$	$u[n]$	$\sum_{k=-\infty}^{n} x[k]$
$x(t)$	$\delta'(t)$	$x'(t)$	$x[n]$	$\Delta\delta[n]$	$x[n]-x[n-1]$
$u(t)$	$u(t)$	$tu(t)$	$u[n]$	$u[n]$	$(n+1)u[n]$
$\mathrm{e}^{-at}u(t)$	$u(t)$	$\frac{1-\mathrm{e}^{-at}}{a}u(t)$	$a^n u[n]$	$u[n]$	$\frac{1-a^{n+1}}{1-a}u[n]$
$\sin(\omega t)u(t)$	$u(t)$	$\frac{1-\cos(\omega t)}{\omega}u(t)$	$\sin(\Omega n)u[n]$	$u[n]$	
$\cos(\omega t)u(t)$	$u(t)$	$\frac{\sin(\omega t)}{\omega}u(t)$	$\cos(\Omega n)u[n]$	$u[n]$	
$\mathrm{e}^{-at}u(t)$	$\mathrm{e}^{-at}u(t)$	$t\mathrm{e}^{-at}u(t)$	$a^n u[n]$	$a^n u[n]$	$(n+1)a^n u[n]$
$\mathrm{e}^{-at}u(t)$	$\mathrm{e}^{-bt}u(t)$	$\frac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{b-a}u(t)$	$a^n u[n]$	$b^n u[n]$	$\frac{b^{n+1}-a^{n+1}}{b-a}u[n]$

说明：表 3.4 中空着的卷积和运算结果，感兴趣的读者可自行补上。

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[1] 徐守时等著，中国科学技术大学出版社，信号与系统·第三版，第78页

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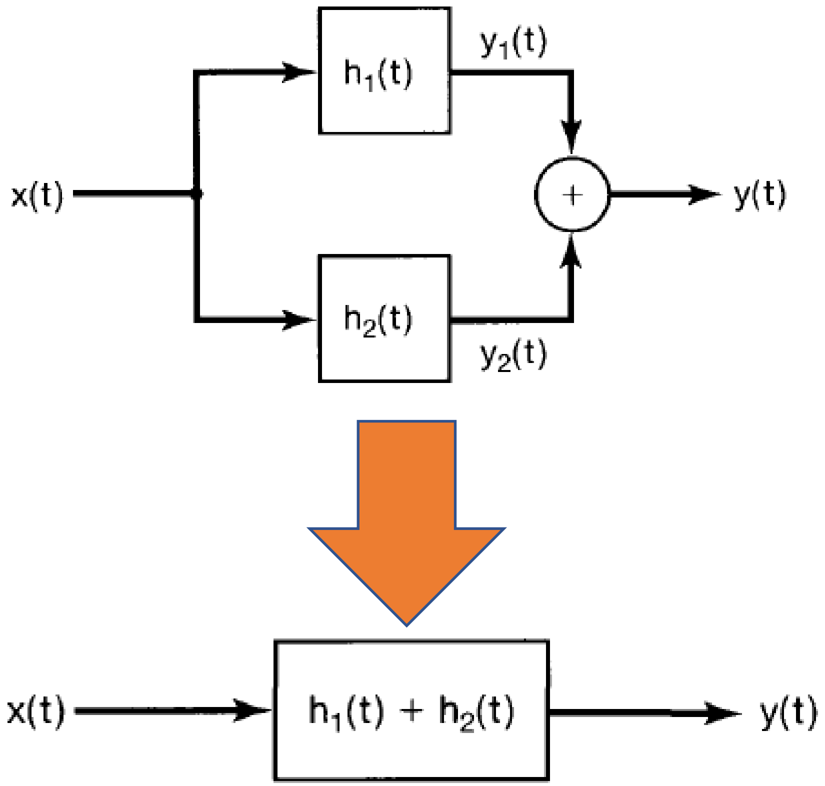
Commutative property

$$x(t)*h(t)=h(t)*x(t)$$

$$\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

$$x[n]*h[n]=h[n]*x[n]$$

$$\sum_{m=-\infty}^{\infty}x[m]h[n-m]=\sum_{m=-\infty}^{\infty}h[m]x[n-m]$$



Distributive property

$$x(t)*\left[h_1(t)+h_2(t)\right]=x(t)*h_1(t)+x(t)*h_2(t)$$

$$x[n]*\left\{h_1[n]+h_2[n]\right\}=x[n]*h_1[n]+x[n]*h_2[n]$$

Associative property

$$\left[x(t)*h_1(t)\right]*h_2(t)=x(t)*\left[h_1(t)*h_2(t)\right]$$

$$\left\{x[n]*h_1[n]\right\}*h_2[n]=x[n]*\left\{h_1[n]*h_2[n]\right\}$$

Time-invariant property (Collect the time shift)

$$y(t)=x(t)*h(t)$$

$$x(t)*h\left(t-t_0\right)=y\left(t-t_0\right)$$

$$\boxed{x\left(t-t_1\right)*h\left(t-t_2\right)=y\left(t-t_1-t_2\right)}$$

$$x[n]*h[n]=y[n]$$

$$x[n]*h[n-m]=y[n-m]$$

$$\boxed{x\left[n-m_1\right]*h\left[n-m_2\right]=y\left[n-m_1-m_2\right]}$$

Difference property

$$\boxed{\frac{d}{dt}\left[x(t)*h(t)\right]=x(t)*\frac{dh(t)}{dt}=\frac{dx(t)}{dt}*h(t)=\frac{dy(t)}{dt}}$$

$$\nabla\left\{x[n]*h[n]\right\}=\nabla x[n]*h[n]=x[n]*\nabla h[n]=\nabla y[n]$$

Integral property

$$\boxed{\int_{-\infty}^{t}\left[x(\tau)*h(\tau)\right]d\tau=x(t)*\int_{-\infty}^{t}h(\tau)d\tau=\int_{-\infty}^{t}x(\tau)d\tau*h(t)=\int_{-\infty}^{t}y(\tau)d\tau}$$

$$\sum_{k=-\infty}^{n}\left\{x[k]*h[k]\right\}=x[n]*\left\{\sum_{k=-\infty}^{n}h[k]\right\}=\left\{\sum_{k=-\infty}^{n}x[k]\right\}*h[n]=\sum_{k=-\infty}^{n}y[k]$$

- For unit impulse/step signal
- More unit impulse/step signals, more simple

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta\left(t - t_0\right) = x\left(t - t_0\right)$$

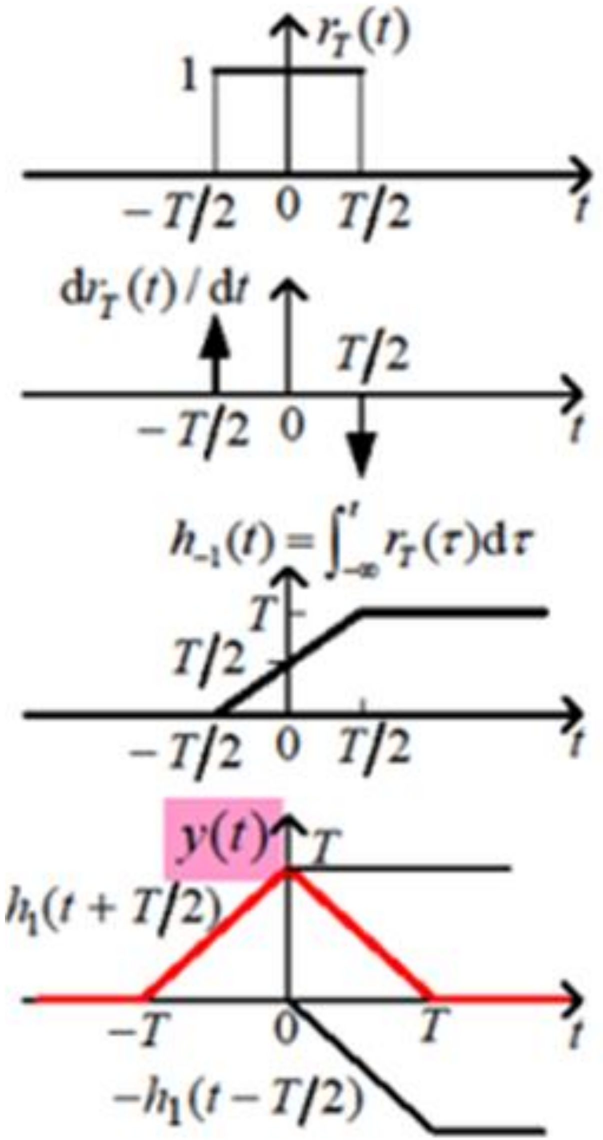
$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$x[n] * \delta[n] = x[n]$$

$$x[n] * \delta[n-m] = x[n-m]$$

$$x[n] * u[n] = \sum_{m=-\infty}^{n} x[m]$$

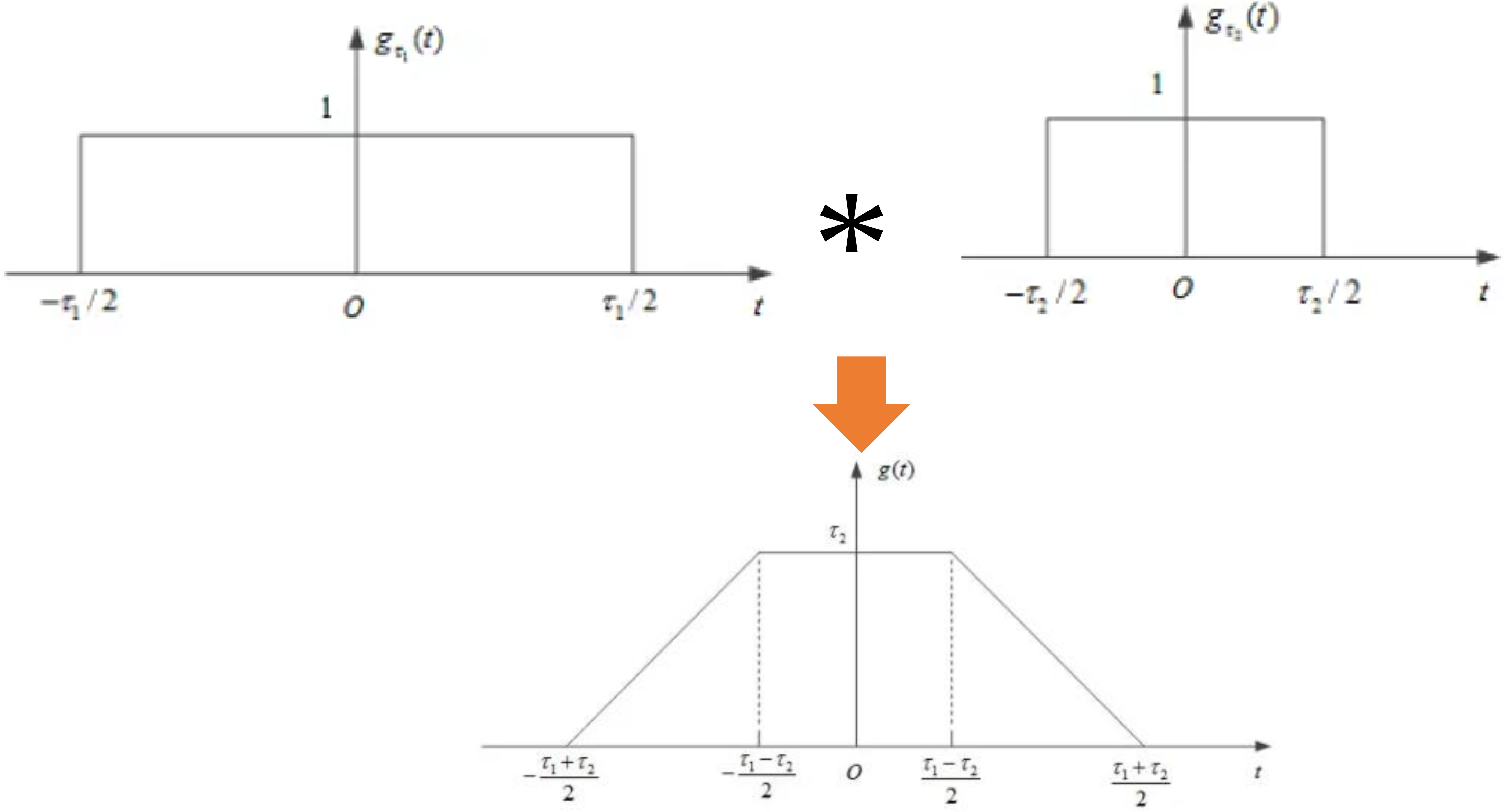


$$y(t)=r_T(t)^{*}r_T(t)=\frac{d}{dt}r_T(t)^{*}\int_{-\infty}^{t} r_T(\tau)d\tau$$

$$h_{-1}(t)=\int_{-\infty}^{t} r_T(\tau)d\tau$$

1. **Figure computation** based on "Flip-slide-multiply-sum/integral"
2. Some known or **typical convolution results**
3. **Properties** of convolution

$$y(t)=[\delta(t+T/2)-\delta(t-T/2)]*h_{-1}(t)$$
$$=h_{-1}(t+T/2)-h_{-1}(t-T/2)$$



$g_{\tau_1}(t)$
1
$-\tau_1/2$
O
$\tau_1/2$
t
*
$g_{\tau_2}(t)$
1
$-\tau_2/2$
O
$\tau_2/2$
t
$g(t)$
τ_2
$-\frac{\tau_1+\tau_2}{2}$
$-\frac{\tau_1-\tau_2}{2}$
O
$\frac{\tau_1-\tau_2}{2}$
$\frac{\tau_1+\tau_2}{2}$
t

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System properties:

- With memory or memoryless

  $y(n)=f\ (\ x(n)\ )$

- Invertible

  For a system $x \rightarrow y$, if $x1 \neq x2$, then $y1 \neq y2$

- Causal

  … up to that time n …

- Stable (BIBO)

  either prove the system is stable, or find a specific counterexample

LTI System properties:

- With memory or memoryless
  - A linear, time-invariant, causal system is memoryless only

    if $h[n] = K\delta[n]$ $h(t) = K\delta(t)$

    $y[n] = Kx[n]$ $y(t) = Kx(t)$

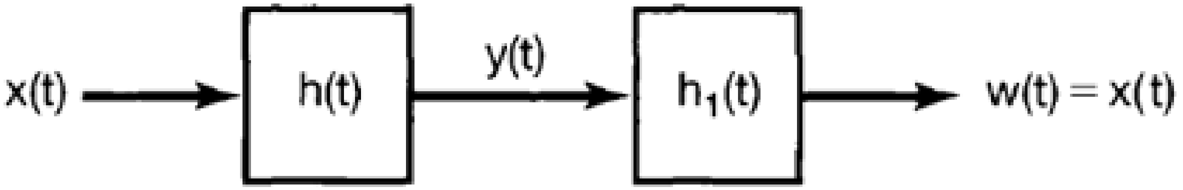
    if k=1 further, they are identity systems

$$y[n] = x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n]$$

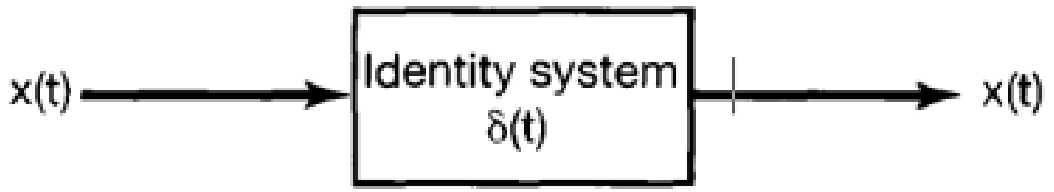
$$y(t) = x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t) * \delta(t)$$

LTI System properties:

- Invertible



(a)



(b)

LTI System properties:

- Causal

Causality: CT LTI system is causal $\leftrightarrow$ $h(t) = 0$, at $t < 0$

- This is because that the input unit impulse function $\delta(t)=0$ at $t<0$

$t - \tau \geq 0$, or $\tau \leq t$

As a result:

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

$y(t)$ only depends on $x(\tau < t)$.

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LTI System properties:

- Stable (BIBO)

BIBO Stability: CT LTI system is stable $\leftrightarrow \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$

→ Sufficient condition:

For $|x(t)| \le x_{\max} < \infty$.

Cauchy-Schwarz Inequation

$$|y(t)| = \left|\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right| \le x_{\max}\left|\int_{-\infty}^{+\infty} h(t-\tau)d\tau\right| < \infty.$$

→ Necessary condition:

Suppose $\int_{-\infty}^{+\infty} |h(\tau)|d\tau = \infty$

Contradiction Case

Let $x(t) = h^*(-t)/|h^*(-t)|$, then $|x(t)| \equiv 1$ bounded

But $y(0) = \int_{-\infty}^{+\infty} x(\tau)h(-\tau)d\tau = \int_{-\infty}^{+\infty} \frac{h^*(-\tau)h(-\tau)}{|h(-\tau)|}d\tau = \int_{-\infty}^{+\infty} |h(-\tau)|d\tau = \infty$

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$f_1(t) * f_2(t)$

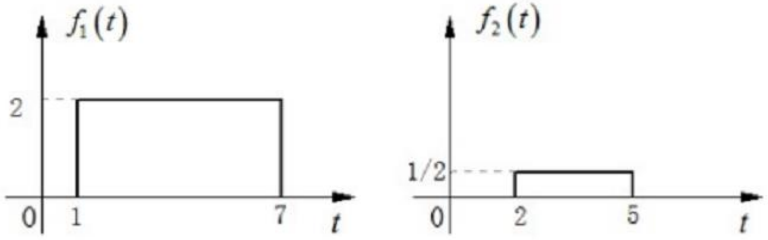
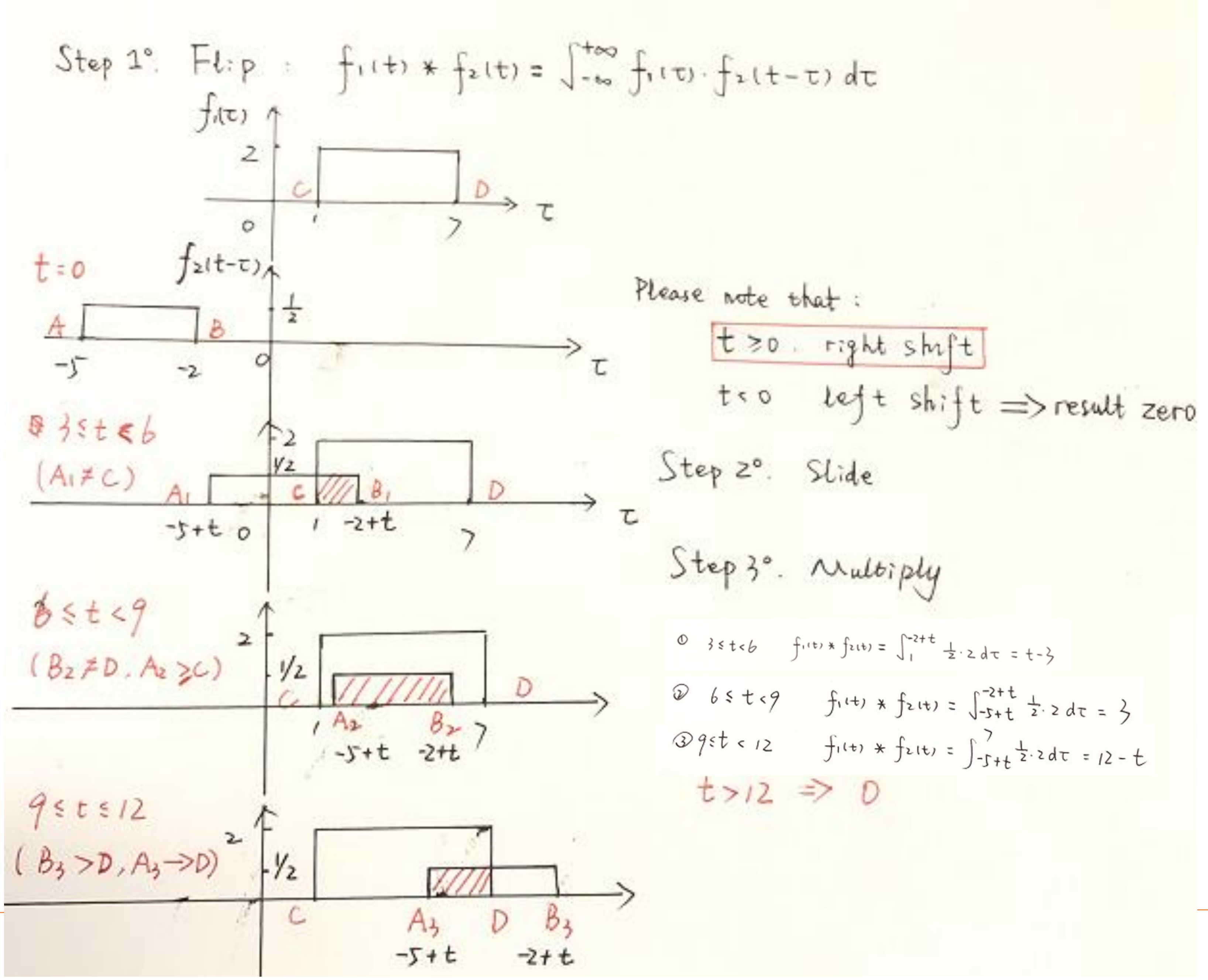


Figure computation based on "Flip-slide-multiply-sum/integral"



Step 4°. integral.

① $3 \le t < 6$ $\quad f_1(t) * f_2(t) = \int_{1}^{-2+t} \frac{1}{2} \cdot 2 \, d\tau = t-3$

② $6 \le t < 9$ $\quad f_1(t) * f_2(t) = \int_{-5+t}^{-2+t} \frac{1}{2} \cdot 2 \, d\tau = 3$

③ $9 \le t < 12$ $\quad f_1(t) * f_2(t) = \int_{-5+t}^{7} \frac{1}{2} \cdot 2 \, d\tau = 12-t$

$$\therefore \quad f_1(t) * f_2(t) = \begin{cases} 0, & t < 3 \\ t-3, & 3 \le t < 6 \\ 3, & 6 \le t < 9 \\ 12-t, & 9 \le t \le 12 \\ 0, & t > 12 \end{cases}$$

Some known or **typical convolution results**

$$f_1(t) * f_2(t)$$

$$= 2 \cdot [u(t-1) - u(t-7)] * \frac{1}{2}[(t-2) - u(t-5)]$$

$$= 2 \cdot \frac{1}{2} \cdot u(t) * [\delta(t-1) - \delta(t-7)] * u(t) * [\delta(t-2) - \delta(t-5)]$$

$$\boxed{= u(t) * u(t)} * [\delta(t-1) - \delta(t-7)] * [\delta(t-2) - \delta(t-5)]$$

$$= tu * [\delta(t-3) - \delta(t-6) - \delta(t-9) + \delta(t-12)]$$

$$= (t-3)u(t-3) - (t-6)u(t-6) - (t-9)u(t-9) + (t-12)u(t-12)$$

$$(1) t < 3,\ f_1(t) * f_2(t) = 0$$

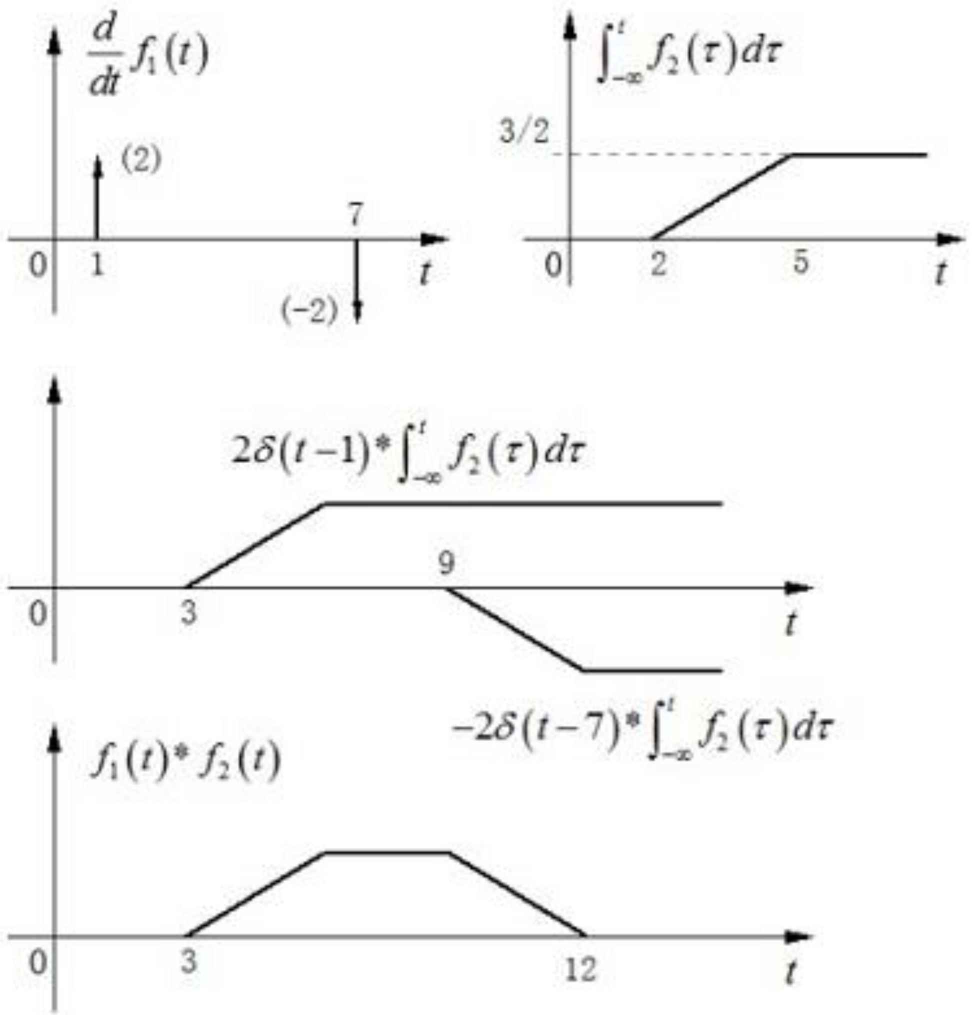
$$(2) 3 \le t < 6,\ f_1(t) * f_2(t) = t - 3$$

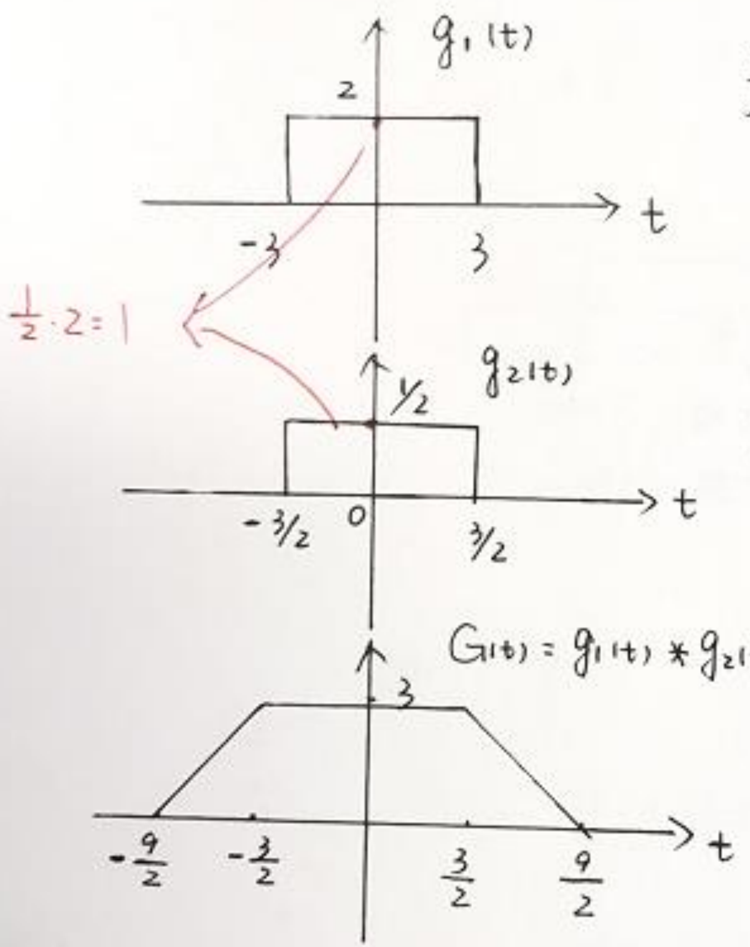
$$(3) 6 \leqslant t < 9,\ f_1(t) * f_2(t) = (t-3) - (t-6) = 3$$

$$(4) 9 \leqslant t < 12,\ f_1(t) * f_2(t) = (t-3) - (t-6) - (t-9) = 12 - t$$

$$(5) 12 \leqslant t,\ f_1(t) * f_2(t) = (t-3) - (t-6) - (t-9) + (t-12) = 0$$

Properties of convolution





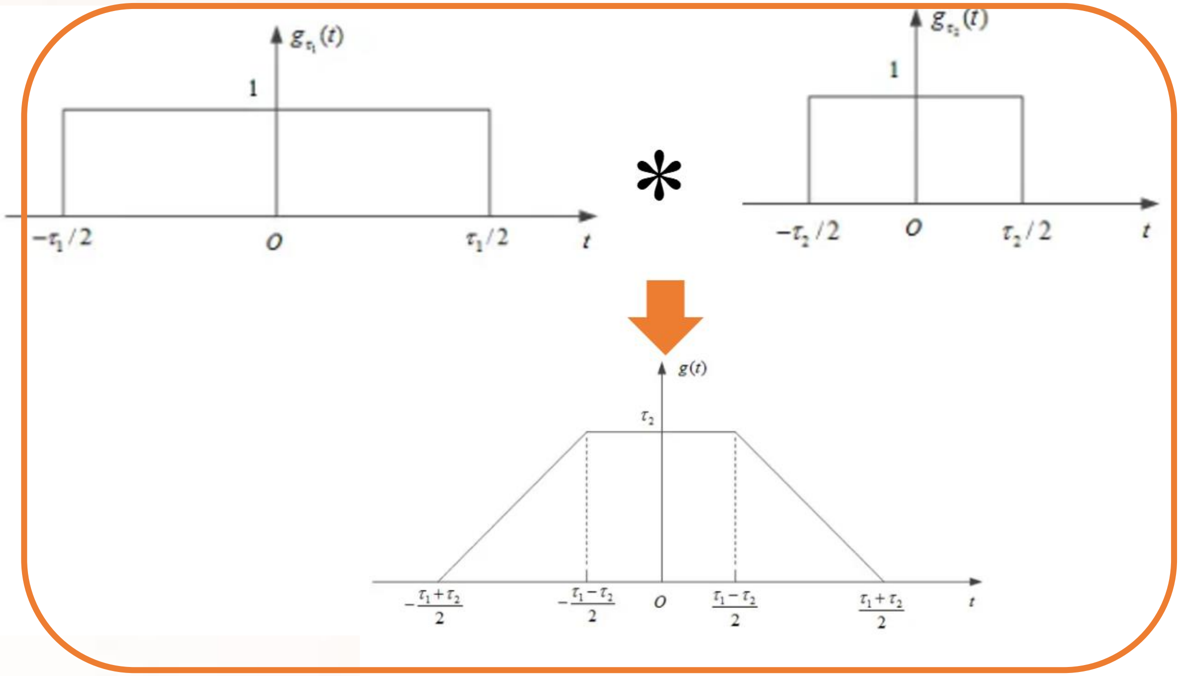
$$f_1(t) = g_1(t) * \delta(t-4)$$

$$f_2(t) = g_2(t) * \delta(t-\tfrac{7}{2})$$

$$\therefore f_1(t) * f_2(t) = g_1(t) * g_2(t) * \delta(t-\tfrac{15}{2})$$

$$= G(t) * \delta(t-\tfrac{15}{2})$$

$$= G(t-\tfrac{15}{2})$$



Basic knowledge Review on Chap 3a

- Fourier Analysis
  1. Response for Complex Exponentials
  2. Fourier Series: Synthesis Equation & Analysis Equation
  3. Properties
  4. Computation on ak
     ① When x(t) are sin, cos or complex exponential signals (Euler Equation)
     ② By the signal Fourier transform in [-T/2,T/2]
     ③ Formula

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- Fourier Analysis

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- CT LTI systems

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau = \left[\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau\right] e^{st} = \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}$$

$$\boxed{y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau}$$

$e^{st} \rightarrow H(s) \cdot e^{st}$

A(常数) $\rightarrow H(0) \cdot A$

- DT LTI systems

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow$$

$$y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m} = \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m}\right] z^n = \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}}$$

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CT Fourier Series Pair $\quad (\omega_o = 2\pi/T)$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_o t} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt \qquad \text{(Analysis equation)}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \xrightarrow{\times e^{-jn\omega_0 t}} x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} \xrightarrow{\text{Integral}} \int_0^T x(t)e^{-jn\omega t}dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t}dt$$

Since $\int_0^T e^{jk\omega_0 t} e^{-jn\omega_0 t}dt = \begin{cases} T, k = n \\ 0, k \neq n \end{cases}$

$$\longrightarrow \quad a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega t}dt \quad (k = n)$$

Please note that (0,T) can be replaced by any interval of length T

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- Fourier Analysis
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- Linearity $x(t) \longleftrightarrow a_k,\ y(t) \longleftrightarrow b_k, \Rightarrow \alpha x(t) + \beta y(t) \longleftrightarrow \alpha a_k + \beta b_k$

Based on the same period among x and y!

- Conjugate Symmetry

$$x(t) \quad \Rightarrow \quad a_{-k} = a_k^*$$

Proof: $$a_{-k} = \frac{1}{T}\int_T x(t)e^{jk\omega_0 t}dt = \left[\frac{1}{T}\int_T x^*(t)e^{-jk\omega_0 t}dt\right]^* = a_k^*$$

$$\Downarrow \quad a_k = \mathrm{Re}\{a_k\} + j\,\mathrm{Im}\{a_k\}$$

$$= |a_k| e^{j\angle a_k}$$

$\mathrm{Re}\{a_k\}$ is even , $\mathrm{Im}\{a_k\}$ is odd

or $|a_k|$ is even, $\angle a_k$ is odd

- Time shift

$$x(t) \longleftrightarrow a_k$$

$$x(t-t_o) \longleftrightarrow a_k e^{-jk\omega_0 t_o} = a_k e^{-jk2\pi t_o/T}$$

Introduce a linear phase shift $\propto t_o$

- Time Reversal

$$x(-t) \stackrel{FS}{\longleftrightarrow} a_{-k}$$

the effect of sign change for $x(t)$ and a_k are identical

Example: $x(t)$: $\ldots\ a_{-2}\ a_{-1}\ a_0\ a_1\ a_2\ \ldots$

$x(-t)$: $\ldots\ a_2\ a_1\ a_0\ a_{-1}\ a_{-2}\ \ldots$

- Time Scaling

α : positive real number

$x(\alpha t)$: periodic with period T/α and fundamental frequency αw_0

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha w_0)t}$$

a_k unchanged, but $x(\alpha t)$ and each harmonic component are different

- **Multiplication Property**

$x(t) \longleftrightarrow a_k$, $y(t) \longleftrightarrow b_k$ (Both $x(t)$ and $y(t)$ are

$\Downarrow$ periodic with the same period T)

$$x(t)\cdot y(t) \longleftrightarrow c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l} = a_k * b_k$$

Proof

$$x(t)y(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \cdot \sum_{l=-\infty}^{+\infty} b_l e^{jl\omega_0 t} = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} a_k b_l e^{j(l+k)\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} a_k b_{p-k} e^{jp\omega_0 t} \qquad (l+k=p, l=p-k)$$

$$= \sum_{p=-\infty}^{+\infty} \left(\sum_{k=-\infty}^{+\infty} a_k b_{p-k} \right) e^{jp\omega_0 t}$$

$$= \sum_{k=-\infty}^{+\infty} \boxed{\left(\sum_{l=-\infty}^{+\infty} a_l b_{k-l} \right)} e^{jk\omega_0 t} \qquad (p=k, k=l)$$

- Parseval Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Observation: power is the same whether measured in the time-domain or the frequency-domain

Proof

$$|x(t)|^2 = x(t)x^*(t)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, x^*(t) = \sum_{l=-\infty}^{+\infty} a_{-l}^* e^{jk\omega_0 t} \overset{l=-l}{=} \sum_{l=-\infty}^{+\infty} a_l^* e^{-jk\omega_0 t}$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_k a_l^* e^{j(k-l)\omega_0 t} dt = \frac{1}{T}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty} a_k a_l^* \int_T e^{j(k-l)\omega_0 t} dt$$

$$\int_T e^{jk\omega_0 t} e^{-jl\omega_0 t} dt = \begin{cases} T, k = l \\ 0, k \neq l \end{cases}$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

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- Fourier Analysis

1. Response for Complex Exponentials
2. Fourier Series: Synthesis Equation & Analysis Equation
3. Properties
4. Computation on ak

① When x(t) are sin, cos or complex exponential signals (Euler Equation)

② By the signal Fourier transform in [-T/2,T/2]

③ Formula

Example 1

$$x(t) = \sin\left(\omega_0 t\right) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

$$a_1 = \frac{1}{2j}, a_{-1} = -\frac{1}{2j}$$

Conjugate Symmetry

Example 2

$$x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$x(t) = 1 + \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) + \frac{2}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right) + \frac{1}{2}\left(e^{j\left(2\omega_0 t + \frac{\pi}{4}\right)} + e^{-j\left(2\omega_0 t + \frac{\pi}{4}\right)}\right)$$

$$= 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{j\frac{\pi}{4}}e^{j2\omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{4}}e^{-j2\omega_0 t}$$

$$a_0 = 1, a_1 = \left(1 + \frac{1}{2j}\right), a_2 = \frac{\sqrt{2}}{4}(1 + j), a_{-1} = \left(1 - \frac{1}{2j}\right), a_{-2} = \frac{\sqrt{2}}{4}(1 - j)$$

- **Conjugate Symmetry**
- **Please think about** $x(t) = 1 + \sin(\omega_0 t) + 2\cos(\omega_0 t) + \cos\left(\frac{\omega_0}{2} t + \frac{\pi}{4}\right)$
- **Hint: You should find the period and the fundamental frequency!** $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{\omega_0}{2}t}$

$$a_0 = 1, a_2 = \left(1 + \frac{1}{2j}\right), a_1 = \frac{\sqrt{2}}{4}(1 + j), a_{-2} = \left(1 - \frac{1}{2j}\right), a_{-1} = \frac{\sqrt{2}}{4}(1 - j)$$

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- Fourier Analysis
  1. Response for Complex Exponentials
  2. Fourier Series: Synthesis Equation & Analysis Equation
  3. Properties
  4. Computation on ak
     ① When x(t) are sin, cos or complex exponential signals (Euler Equation)
     ② By the signal Fourier transform in [-T/2,T/2]
     ③ Formula

Lemma

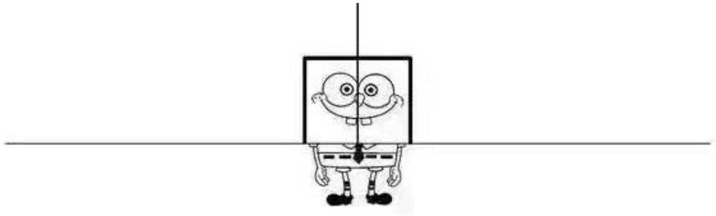
$$a_k = \frac{1}{T} X_0(j\omega)\Big|_{\omega = k\omega_0}$$

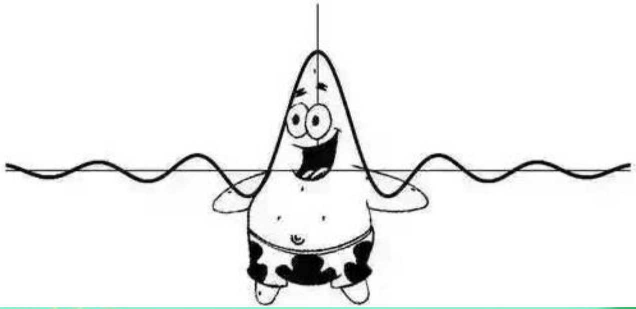
$$x_0(t) \xleftrightarrow{\mathcal{F}} X_0(j\omega) \qquad x_0(t) = x(t), -\frac{T}{2} \le t \le \frac{T}{2}$$

$\mathcal{F}$: *Fourier Transform*

Please refer to (4.10) on page 289 for more details

Typical FT Pair





Example 4.4

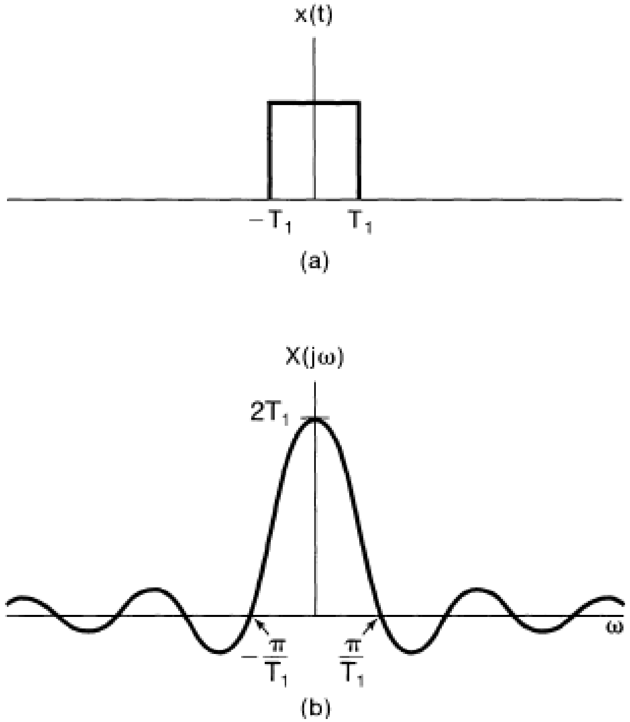
Consider the rectangular pulse signal

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}, \tag{4.16}$$

as shown in Figure 4.8(a). Applying eq. (4.9), we find that the Fourier transform of this signal is

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t}\,dt = 2\frac{\sin \omega T_1}{\omega}, \tag{4.17}$$

as sketched in Figure 4.8(b).



Recall Example 3.5 on Textbook

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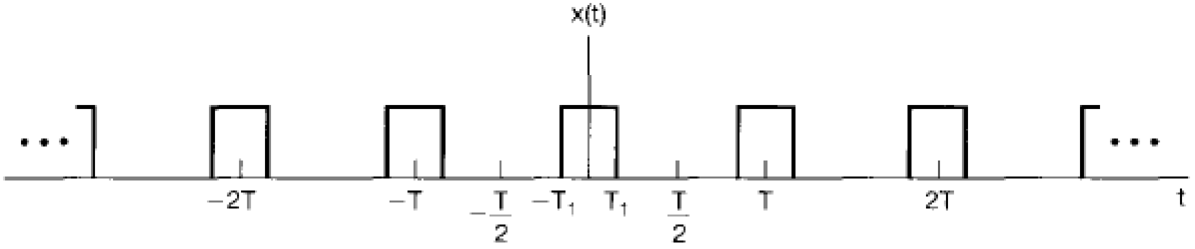

Rewrite

$$x(t)=\sum_{k=-\infty}^{+\infty} x_0(t-kT),\ x_0(t)=\begin{cases}1, |t|<T_1\\ 0, |t|>T_1\end{cases}$$

$$x_0(t)\xleftrightarrow{\mathcal{F}} X_0(j\omega)=\frac{2\sin(\omega T_1)}{\omega}$$

$$a_k=\frac{1}{T}X_0(j\omega)\Big|_{\omega=k\omega_0}=\frac{1}{T}\frac{2\sin(\omega T_1)}{\omega}\Big|_{\omega=k\omega_0}$$

$$=\frac{1}{T}\frac{2\sin(k\omega_0 T_1)}{k\omega_0}$$

$$=\frac{1}{T}\frac{2\sin(k\omega_0 T_1)}{k\frac{2\pi}{T}}$$

$$=\frac{\sin(k\omega_0 T_1)}{k\pi}$$

By the signal Fourier transform in [-T/2,T/2]

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t},\ \omega_0=\frac{2\pi}{T}$$

$$a_k=\frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt=\frac{\sin(k\omega_0 T_1)}{\pi k}, k\neq 0$$

Based on analysis equation

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3.8. Suppose we are given the following information about a signal $x(t)$:

1. $x(t)$ is real and odd.
2. $x(t)$ is periodic with period $T = 2$ and has Fourier coefficients a_k.
3. $a_k = 0$ for $|k| > 1$.
4. $\frac{1}{2}\int_0^2 |x(t)|^2\, dt = 1$.

Specify two different signals that satisfy these conditions.

3.8. Since $x(t)$ is real and odd (clue 1), its Fourier series coefficients a_k are purely imaginary and odd (See Table 3.1). Therefore, $a_k = -a_{-k}$ and $a_0 = 0$. Also, since it is given that $a_k = 0$ for $|k| > 1$, the only unknown Fourier series coefficients are a_1 and a_{-1}. Using Parseval's relation,

$$\frac{1}{T}\int_{<T>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2,$$

for the given signal we have

$$\frac{1}{2}\int_0^2 |x(t)|^2 dt = \sum_{k=-1}^{1} |a_k|^2.$$

Using the information given in clue (4) along with the above equation,

$$|a_1|^2 + |a_{-1}|^2 = 1 \qquad \Rightarrow \qquad 2|a_1|^2 = 1$$

Therefore,

$$a_1 = -a_{-1} = \frac{1}{\sqrt{2}j} \qquad \text{or} \qquad a_1 = -a_{-1} = -\frac{1}{\sqrt{2}j}$$

The two possible signals which satisfy the given information are

$$x_1(t) = \frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} - \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = -\sqrt{2}\sin(\pi t)$$

and

$$x_2(t) = -\frac{1}{\sqrt{2}j}e^{j(2\pi/2)t} + \frac{1}{\sqrt{2}j}e^{-j(2\pi/2)t} = \sqrt{2}\sin(\pi t)$$

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3.34. Consider a continuous-time LTI system with impulse response

$$h(t) = e^{-4|t|}.$$

Find the Fourier series representation of the output $y(t)$ for each of the following inputs:

(a) $x(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$

(b) $x(t) = \sum_{n=-\infty}^{+\infty} (-1)^n \delta(t-n)$

(c) $x(t)$ is the periodic wave depicted in Figure P3.34.

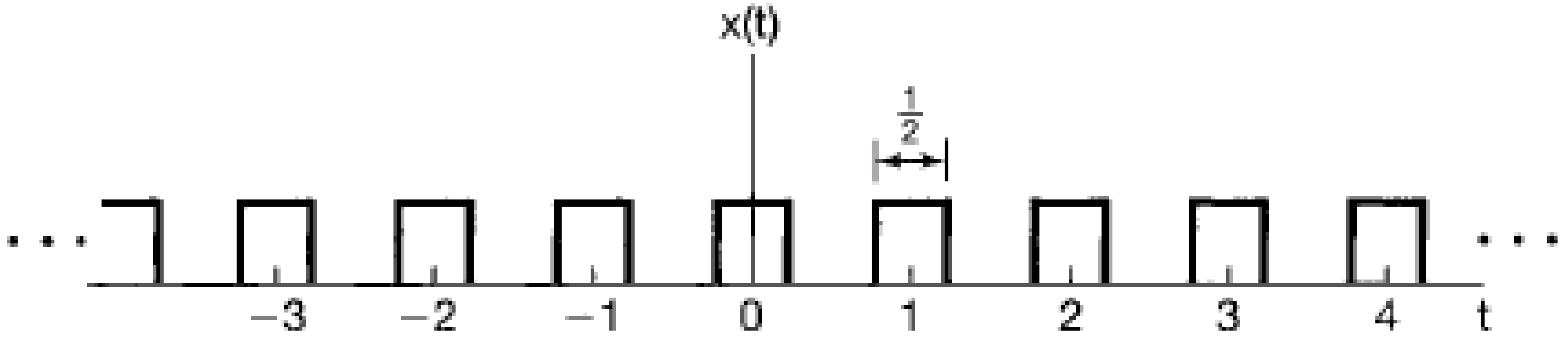

Figure P3.34

3.34. The frequency response of the system is given by

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\omega t} dt = \frac{1}{4+j\omega} + \frac{1}{4-j\omega}.$$

(a) Here, $T = 1$ and $\omega_0 = 2\pi$ and $a_k = 1$ for all k. The FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}.$$

(b) Here, $T = 2$ and $\omega_0 = \pi$ and

$$a_k = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 0, & k \text{ even} \\ \frac{1}{4+j\pi k} + \frac{1}{4-j\pi k}, & k \text{ odd} \end{cases}.$$

(c) Here, $T = 1$, $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1/2, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 1/4, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}\left[\frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}\right], & k \text{ odd} \end{cases}.$$

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3.40. Let $x(t)$ be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefficients of each of the following signals in terms of a_k:

(a) $x(t - t_0) + x(t + t_0)$

(b) $\mathcal{E}v\{x(t)\}$

(c) $\mathcal{R}e\{x(t)\}$

(d) $\frac{d^2 x(t)}{dt^2}$

(e) $x(3t - 1)$ [for this part, first determine the period of $x(3t - 1)$]

$$\text{Even}\{x(t)\} = \frac{x(t)+x(-t)}{2}$$

$$\text{Odd}\{x(t)\} = \frac{x(t)-x(-t)}{2}$$

(a) $x(t-t_0)$ is also periodic with period T. The Fourier series coefficients b_k of $x(t-t_0)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(t-t_0)e^{-jk(2\pi/T)t}dt \\ &= \frac{e^{-jk(2\pi/T)t_0}}{T}\int_T x(\tau)e^{-jk(2\pi/T)\tau}d\tau \\ &= e^{-jk(2\pi/T)t_0}a_k \end{aligned}$$

Similarly, the Fourier series coefficients of $x(t+t_0)$ are

$$c_k = e^{jk(2\pi/T)t_0}a_k.$$

Finally, the Fourier series coefficients of $x(t-t_0)+x(t+t_0)$ are

$$d_k = b_k + c_k = e^{-jk(2\pi/T)t_0}a_k + e^{jk(2\pi/T)t_0}a_k = 2\cos(k2\pi t_0/T)a_k.$$

(b) Note that $\mathcal{E}v\{x(t)\} = [x(t) + x(-t)]/2$. The FS coefficients of $x(-t)$ are

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(-t)e^{-jk(2\pi/T)t}dt \\ &= \frac{1}{T}\int_T x(\tau)e^{jk(2\pi/T)\tau}d\tau \\ &= a_{-k} \end{aligned}$$

Therefore, the FS coefficients of $\mathcal{E}v\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}}{2}.$$

(c) Note that $\mathcal{R}e\{x(t)\} = [x(t) + x^*(t)]/2$. The FS coefficients of $x^*(t)$ are

$$b_k = \frac{1}{T}\int_T x^*(t)e^{-jk(2\pi/T)t}dt.$$

Conjugating both sides, we get

$$b_k^* = \frac{1}{T}\int_T x(t)e^{jk(2\pi/T)t}dt = a_{-k}.$$

Therefore, the FS coefficients of $\mathcal{R}e\{x(t)\}$ are

$$c_k = \frac{a_k + b_k}{2} = \frac{a_k + a_{-k}^*}{2}.$$

(d) The Fourier series synthesis equation gives

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T)kt}.$$

Differentiating both sides wrt t twice, we get

$$\frac{d^2x(t)}{dt^2} = \sum_{k=-\infty}^{\infty} -k^2\frac{4\pi^2}{T^2}a_k e^{j(2\pi/T)kt}.$$

By inspection, we know that the Fourier series coefficients of $d^2x(t)/dt^2$ are $-k\frac{4\pi^2}{T^2}a_k$.

(e) The period of $x(3t)$ is a third of the period of $x(t)$. Therefore, the signal $x(3t-1)$ is periodic with period $T/3$. The Fourier series coefficents of $x(3t)$ are still a_k. Using the analysis of part (a), we know that the Fourier series coefficients of $x(3t-1)$ is $e^{-jk(6\pi/T)}a_k$.

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Thanks for Your Attendance

Q&A

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工学院南楼220（周五16:30-17:30）