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# Signals and Systems Tutorials

2021.11.22

Tutorial Problems: 5.24 5.36

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega, \tag{5.8}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}. \tag{5.9}$$

Equations (5.8) and (5.9) are the discrete-time counterparts of eqs. (4.8) and (4.9). The function $X(e^{j\omega})$ is referred to as the *discrete-time Fourier transform* and the pair of equations as the *discrete-time Fourier transform pair.* Equation (5.8) is the *synthesis equation*, eq. (5.9) the *analysis equation.* Our derivation of these equations indicates how an aperiodic sequence can be thought of as a linear combination of complex exponentials. In particular, the synthesis equation is in effect a representation of $x[n]$ as a linear combination of complex exponentials infinitesimally close in frequency and with amplitudes $X(e^{j\omega})(d\omega/2\pi)$. For this reason, as in continuous time, the Fourier transform $X(e^{j\omega})$ will often be referred to as the *spectrum* of $x[n]$, because it provides us with the information on how $x[n]$ is composed of complex exponentials at different frequencies.

**TABLE 5.1** PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

| Section | Property | Aperiodic Signal | Fourier Transform |
|---|---|---|---|
| | | $x[n]$<br>$y[n]$ | $\left.\begin{array}{l} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{array}\right\}$ periodic with period $2\pi$ |
| 5.3.2 | Linearity | $ax[n] + by[n]$ | $aX(e^{j\omega}) + bY(e^{j\omega})$ |
| 5.3.3 | Time Shifting | $x[n - n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| 5.3.3 | Frequency Shifting | $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 5.3.4 | Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 5.3.6 | Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| 5.3.7 | Time Expansion | $x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$ | $X(e^{jk\omega})$ |
| 5.4 | Convolution | $x[n] * y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 5.5 | Multiplication | $x[n]y[n]$ | $\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| 5.3.5 | Differencing in Time | $x[n] - x[n-1]$ | $(1 - e^{-j\omega})X(e^{j\omega})$ |
| 5.3.5 | Accumulation | $\sum_{k=-\infty}^{n} x[k]$ | $\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$<br>$+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$ |
| 5.3.8 | Differentiation in Frequency | $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| 5.3.4 | Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \Re e\{X(e^{j\omega})\} = \Re e\{X(e^{-j\omega})\} \\ \Im m\{X(e^{j\omega})\} = -\Im m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$ |
| 5.3.4 | Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| 5.3.4 | Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| 5.3.4 | Even-odd Decomposition of Real Signals | $x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$<br>$x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$ | $\Re e\{X(e^{j\omega})\}$<br>$j\Im m\{X(e^{j\omega})\}$ |
| 5.3.9 | Parseval's Relation for Aperiodic Signals | | |

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

**TABLE 5.2** BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS

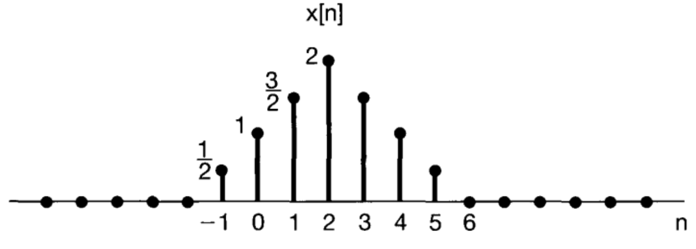
| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi r}{N}$<br>$a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| Periodic square wave<br>$x[n] = \begin{cases} 1, & \|n\| \le N_1 \\ 0, & N_1 < \|n\| \le N/2 \end{cases}$<br>and<br>$x[n + N] = x[n]$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$<br>$a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n - kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for all $k$ |
| $a^n u[n], \quad \|a\| < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | — |
| $x[n] \begin{cases} 1, & \|n\| \le N_1 \\ 0, & \|n\| > N_1 \end{cases}$ | $\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ | — |
| $\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$<br>$0 < W < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \le \|\omega\| \le W \\ 0, & W < \|\omega\| \le \pi \end{cases}$<br>$X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | 1 | — |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi \delta(\omega - 2\pi k)$ | — |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n + 1)a^n u[n], \quad \|a\| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |
| $\frac{(n + r - 1)!}{n!(r - 1)!} a^n u[n], \quad \|a\| < 1$ | $\frac{1}{(1 - ae^{-j\omega})^r}$ | — |

| | |
|---|---|
| $a^n u[n], \quad \|a\| < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ |

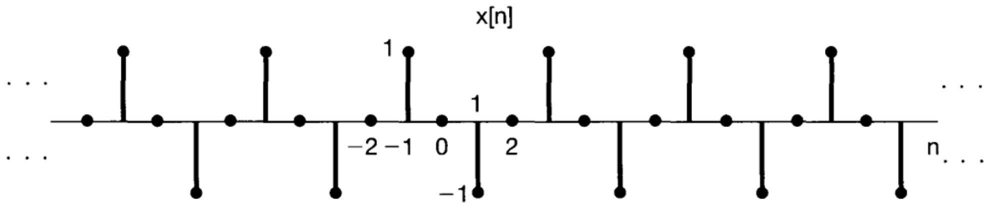
**5.24.** Determine which, if any, of the following signals have Fourier transforms that satisfy each of the following conditions:

1. $\mathcal{R}e\{X(e^{j\omega})\} = 0.$
2. $\mathcal{I}m\{X(e^{j\omega})\} = 0.$
3. There exists a real $\alpha$ such that $e^{j\alpha\omega}X(e^{j\omega})$ is real.
4. $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0.$
5. $X(e^{j\omega})$ periodic.
6. $X(e^{j0}) = 0.$

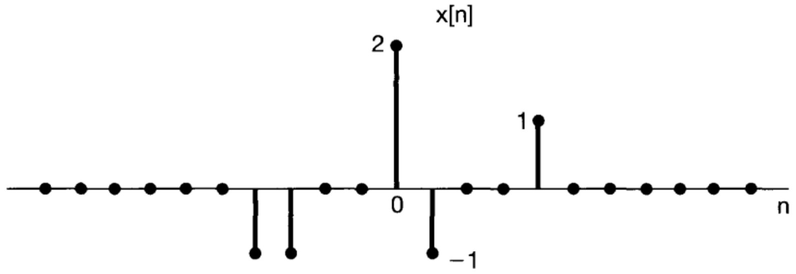
**(a)** $x[n]$ as in Figure P5.24(a)
**(b)** $x[n]$ as in Figure P5.24(b)
**(c)** $x[n] = (\frac{1}{2})^n u[n]$
**(d)** $x[n] = (\frac{1}{2})^{|n|}$
**(e)** $x[n] = \delta[n-1] + \delta[n+2]$
**(f)** $x[n] = \delta[n-1] + \delta[n+3]$
**(g)** $x[n]$ as in Figure P5.24(c)
**(h)** $x[n]$ as in Figure P5.24(d)
**(i)** $x[n] = \delta[n-1] - \delta[n+1]$



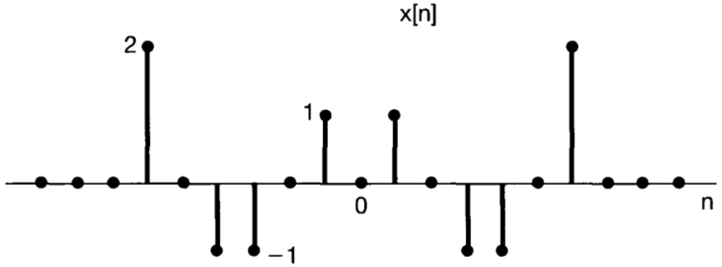
(a)



(b)



(c)



(d)

**5.24.** (1) For $\mathcal{R}e\{X(e^{j\omega})\}$ to be zero, the signal must be real and odd. Only signals (b) and (i) are real and odd.

(2) For $\mathcal{I}m\{X(e^{j\omega})\}$ to be zero, the signal must be real and even. Only signals (d) and (h) are real and even.

(3) Assume $Y(e^{j\omega}) = e^{j\alpha\omega}X(e^{j\omega})$. Using the time shifting property of the Fourier transform we have $y[n] = x[n+\alpha]$. If $Y(e^{j\omega})$ is real, then $y[n]$ is real and even (assuming that $x[n]$ is real). Therefore, $x[n]$ has to be symmetric about $\alpha$. This is true only for signals (a), (b), (d), (e), (f), and (h).

(4) Since $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi x[0]$, the given condition is satisfied only if $x[0] = 0$. This is true for signals (b), (e), (f), (h), and (i).

(5) $X(e^{j\omega})$ is always periodic with period $2\pi$. Therefore, all signals satisfy this condition.

(6) Since $X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$, the given condition is satisfied only if the samples of the signal add up to zero. This is true for signals (b), (g), and (i).

**5.36.** **(a)** Let $h[n]$ and $g[n]$ be the impulse responses of two stable discrete-time LTI systems that are inverses of each other. What is the relationship between the frequency responses of these two systems?

**(b)** Consider causal LTI systems described by the following difference equations. In each case, determine the impulse response of the inverse system and the difference equation that characterizes the inverse.

**(i)** $y[n] = x[n] - \frac{1}{4}x[n-1]$

**(ii)** $y[n] + \frac{1}{2}y[n-1] = x[n]$

**(iii)** $y[n] + \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

**(iv)** $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{4}x[n-1] - \frac{1}{8}x[n-2]$

**(v)** $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-1]$

**(vi)** $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$

**(c)** Consider the causal, discrete-time LTI system described by the difference equation

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n-1] - \frac{1}{2}x[n-2]. \qquad \text{(P5.36–1)}$$

What is the inverse of this system? Show that the inverse is not causal. Find another causal LTI system that is an "inverse with delay" of the system described by eq. (P5.36–1). Specifically, find a causal LTI system such that the output $w[n]$ in Figure P5.36 equals $x[n-1]$.

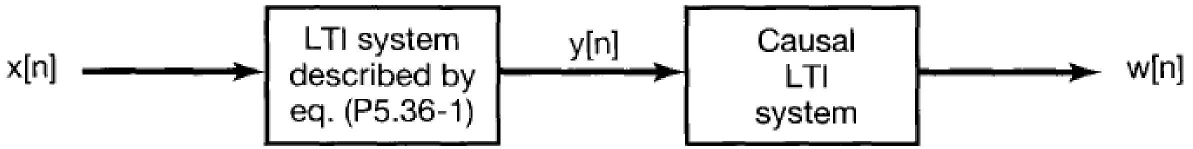



**Fig P5.36**

This is as sketched in Figure S5.35.

**5.36.** (a) The frequency responses are related by the following expression:

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})}.$$

(b) (i) Here, $H(e^{j\omega}) = 1 - \frac{1}{4}e^{-j\omega}$. Therefore, $G(e^{j\omega}) = 1/(1-\frac{1}{4}e^{-j\omega})$ and $g[n] = (\frac{1}{4})^n u[n]$. Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-\frac{1}{4}e^{-j\omega}},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{4}y[n-1] = x[n].$$

(ii) Here, $H(e^{j\omega}) = 1/(1+\frac{1}{2}e^{-j\omega})$. Therefore, $G(e^{j\omega}) = 1 + \frac{1}{2}e^{-j\omega}$ and $g[n] = \delta[n] + \frac{1}{2}\delta[n-1]$. Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 1 + \frac{1}{2}e^{-j\omega},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] = x[n] + \frac{1}{2}x[n-1].$$

(iii) Here, $H(e^{j\omega}) = (1-\frac{1}{4}e^{-j\omega})/(1+\frac{1}{2}e^{-j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{1}{2}e^{-j\omega})/(1-\frac{1}{4}e^{-j\omega})$ and $g[n] = (\frac{1}{4})^n u[n] + \frac{1}{2}(\frac{1}{4})^{n-1}u[n-1]$. Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{4}y[n-1] = x[n] + \frac{1}{2}x[n-1].$$

(iv) Here, $H(e^{j\omega}) = (1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})/(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})/(1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$. Therefore,

$$G(e^{j\omega}) = 1 + \frac{2}{1-(1/2)e^{-j\omega}} - \frac{2}{1+(1/4)e^{-j\omega}}$$

and

$$g[n] = \delta[n] + 2\left(\frac{1}{2}\right)^n u[n] - 2\left(-\frac{1}{4}\right)^n u[n].$$

Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})}{(1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{4}y[n-1] - \frac{1}{8}y[n-1] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$



(v) Here, $H(e^{j\omega}) = (1-\frac{1}{2}e^{-j\omega})/(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{5}{4}e^{-j\omega} - \frac{1}{8}e^{-2j\omega})/(1-\frac{1}{2}e^{-j\omega})$ Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})}{(1-\frac{1}{2}e^{-j\omega})},$$

the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$

(vi) Here, $H(e^{j\omega}) = 1/(1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$. Therefore, $G(e^{j\omega}) = (1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$ Since

$$G(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = (1+\frac{5}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega})$$

we have

$$g[n] = \delta[n] + \frac{5}{4}\delta[n-1] - \frac{1}{8}\delta[n-2]$$

and the difference equation relating the input $x[n]$ and output $y[n]$ is

$$y[n] = x[n] + \frac{5}{4}x[n-1] - \frac{1}{8}x[n-2].$$

(c) The frequency response of the given system is

$$H(e^{j\omega}) = \frac{e^{-j\omega}-\frac{1}{2}e^{-2j\omega}}{1+e^{-j\omega}+\frac{1}{4}e^{-2j\omega}}.$$

The frequency response of the inverse system is

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})} = \frac{e^{j\omega}+1+\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}}.$$

Therefore,

$$g[n] = \left(\frac{1}{2}\right)^{n+1} u[n+1] + \left(\frac{1}{2}\right)^n u[n] + \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1].$$

Clearly, $g[n]$ is not a causal impulse response.

If we delay this impulse response by 1 sample, then it becomes causal. Furthermore, the output of the inverse system will then be $x[n-1]$. The impulse response of this causal system is

$$g_1[n] = g[n-1] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{2}\right)^{n-1} u[n-1] + \frac{1}{4}\left(\frac{1}{2}\right)^{n-2} u[n-2].$$