Lab 4. The Continuous-Time Fourier Transform

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Fall 2021

Overview

- CT Fourier Transform (CTFT):
  - How to calculate via Matlab?
- Frequency Response:
  - How to calculate via Matlab?
  - How to convert to impulse response?
- Application of CTFT
  - Analysis of amplitude modulation (AM)

In Matlab, fft() computes DTFS coefficients from signals

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}, \qquad \text{for } k = 0,1,\dots,N-1 \Rightarrow a_0, a_1, \dots, a_{N-1}$$

Because a_k is periodic, $a_k = a_{k+N} = a_{k-N}$

We can shift the zero-frequency component to the center of the array, using fftshift()

$$a_0, a_1, \dots, a_{N-1} \xrightarrow{\text{fftshift}} \begin{cases} a_{\frac{N}{2}}, a_{\frac{N}{2}+1}, \dots, a_{N-1}, a_0, a_1, \dots, a_{\frac{N}{2}-1} & \text{when } N \text{ is even} \\ a_{\frac{N+1}{2}}, \dots, a_{N-1}, a_0, a_1, \dots, a_{\frac{N-1}{2}} & \text{when } N \text{ is odd} \end{cases}$$

$$0,1,2,3,4,5,6,7 \xrightarrow{\text{fftshift}} 4,5,6,7,0,1,2,3$$

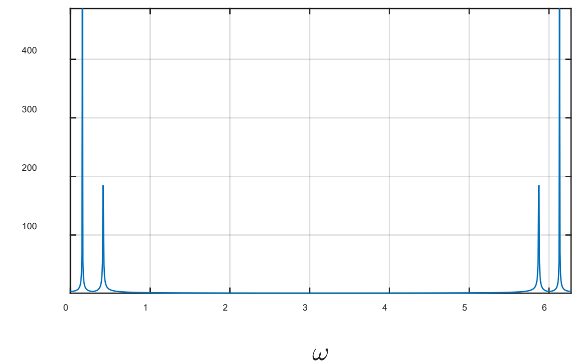
$$0,1,2,3,4,5,6,7,8 \xrightarrow{\text{fftshift}} 5,6,7,8,0,1,2,3,4,$$

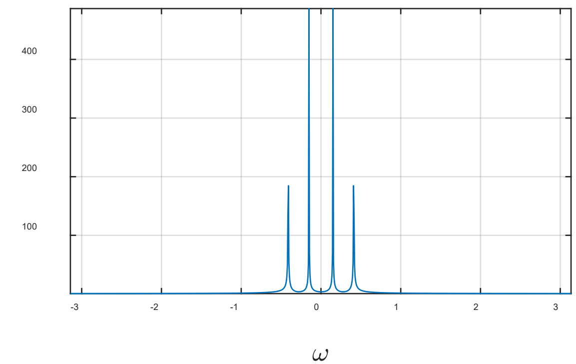
移回去用ifftshift

- fftshift is useful for visualizing the Fourier transform with the zero-frequency component in the middle of the spectrum.

```
N = 1000;
n = 1:N;
sig = sin(0.15*n)+0.5*sin(0.411*n+pi/5);
fftsig = fft(sig);
w = (0:N-1)*2*pi/N;
wshift = (-N/2:N/2-1)*2*pi/N;

figure; subplot(211);  plot(w,abs(fftsig));
xlabel('\omega');grid on; axis tight
subplot(212);  plot(wshift,abs(fftshift(fftsig)));
xlabel('\omega');grid on; axis tight
```





CT Fourier Transform (CTFT):

- How to calculate via Matlab?

CT Fourier Transform

- Definition of CT Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

  - Function of continuous frequency $\boldsymbol{\omega}$
  - Represent the "spectrum" of a signal
- Plot of CTFT
  - Choose a sequence of frequency $\boldsymbol{\omega}$ (sampling)
  - Calculate integration for each $\boldsymbol{\omega}$
  - Plot each sample and do interpolation (approximation)
  - Can we reduce the complexity?

Another Approach

- CT Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$= \lim_{\tau\to 0} \tau \sum_{n=-\infty}^{\infty} x(n\tau)e^{-j\omega\tau n}$$

Look similar

Can we use fft() to calculate the CTFT?

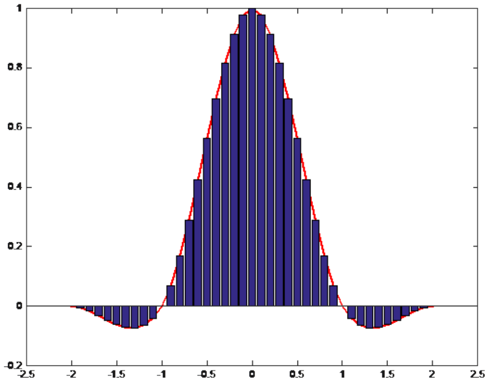
- DT Fourier Series:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Derivation

1. Suppose the dominant region of x(t) is in [0,T], then we can use the following approximation

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \approx \int_{0}^{T} x(t)e^{-j\omega t}dt \approx \sum_{n=0}^{N-1} x(n\tau)e^{-j\omega n\tau}\tau$$



Slice into N intervals, each with length τ=T/N

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt\approx\int_{0}^{T}x(t)e^{-j\omega t}dt\approx\sum_{n=0}^{N-1}x(n\tau)e^{\underline{-j\omega n\tau}}\tau$$

$$a_k=\frac{1}{N}\sum_{n=0}^{N-1}x[n]e^{\underline{-jk\left(\frac{2\pi}{N}\right)n}}$$

2. Sampling of ω

$$k\left(\frac{2\pi}{N}\right)=\omega_k\tau \quad \omega_k=k\left(\frac{2\pi}{N}\right)\frac{1}{\tau},k=0,1,\ldots,N-1$$

It could be observed that

$$X\left(j\frac{2\pi k}{N\tau}\right)\approx\tau\boxed{\sum_{n=0}^{N-1}x(n\tau)e^{-j\frac{2\pi kn}{N}}}$$

$\Downarrow$

fft() of [$x(0)$, $x(\tau)$, $x(2\tau)$, ..., $x((N-1)\tau)$]

- It could be observed that

$$X\left(j\frac{2\pi k}{N\tau}\right) \approx \tau \boxed{\sum_{n=0}^{N-1} x(n\tau)e^{-j\frac{2\pi kn}{N}}}$$

$$k = \frac{1-N}{2}, \dots, -1, 0, 1, \dots, \frac{N-1}{2}$$

When N is odd

$k = ?$ When N is even

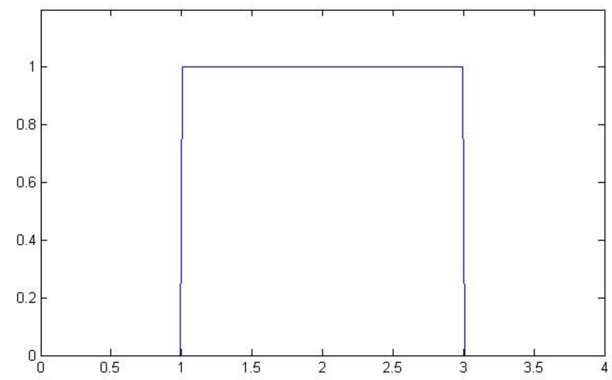
Assignment 4.2

fft() of [$x(0)$, $x(\tau)$, $x(2\tau)$, ..., $x((N-1)\tau)$]

- **Conclusion: we can use fft() to calculate CTFT approximately, which could reduce the computation complexity**

Example

- Single rectangular wave with width=2
  - T should be larger than 3



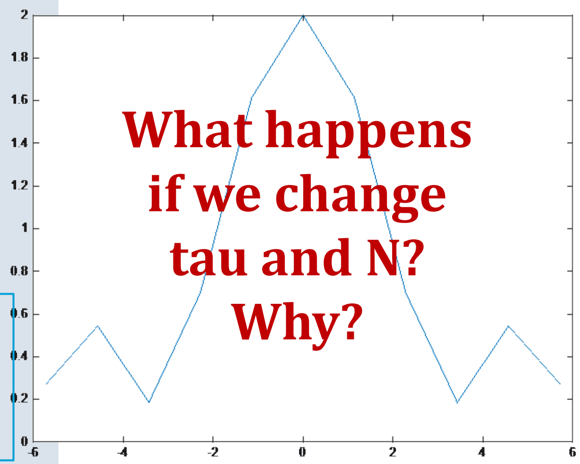

```
tau=0.5;  % interval of time sampling
N=11; % let N be odd to get symmetrical result, N*tau=T>3
x=[zeros(1,1/tau),ones(1,2/tau),zeros(1,N-3/tau)];
X=tau*fft(x);

lb = (1-N)*pi/N/tau;
ub = (N-1)*pi/N/tau;
step = 2*pi/N/tau;
plot(lb:step:ub, abs(fftshift(X)));
```

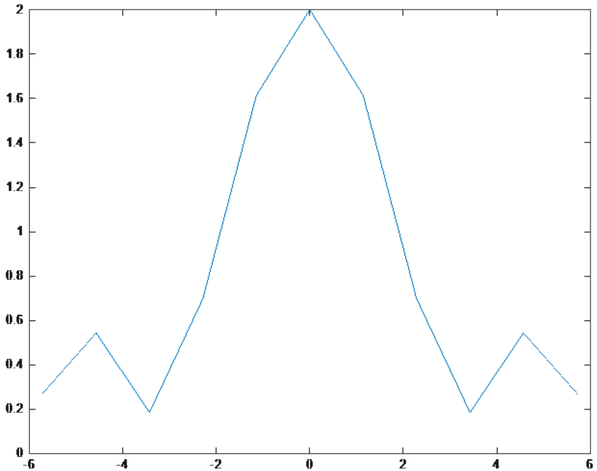
$$\omega = \frac{2\pi k}{N\tau}$$

$$k = \frac{1-N}{2}, \dots, -1, 0, 1, \dots, \frac{N-1}{2}$$

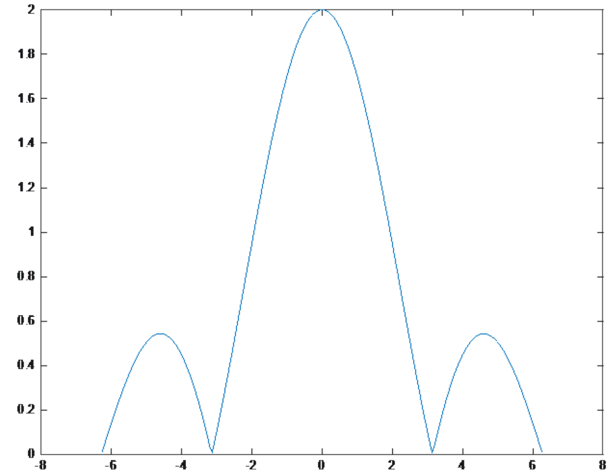
When N is odd



What happens if we change tau and N? Why?

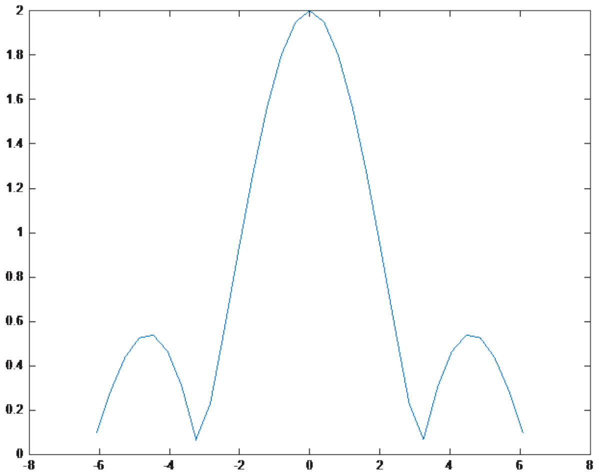


$\tau = 0.5$
$N = 11$

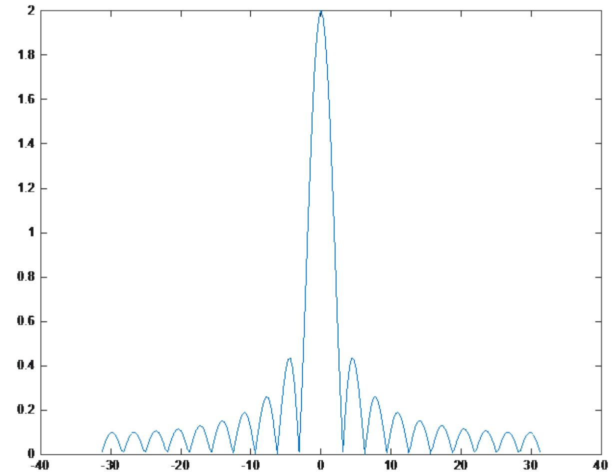


$\tau = 0.5$
$N = 201$

Any problem in the approximation with fft()?



$\tau = 0.5$
$N = 31$

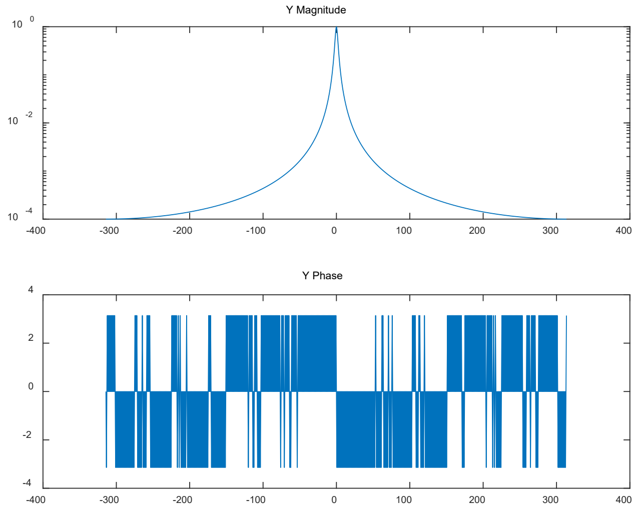


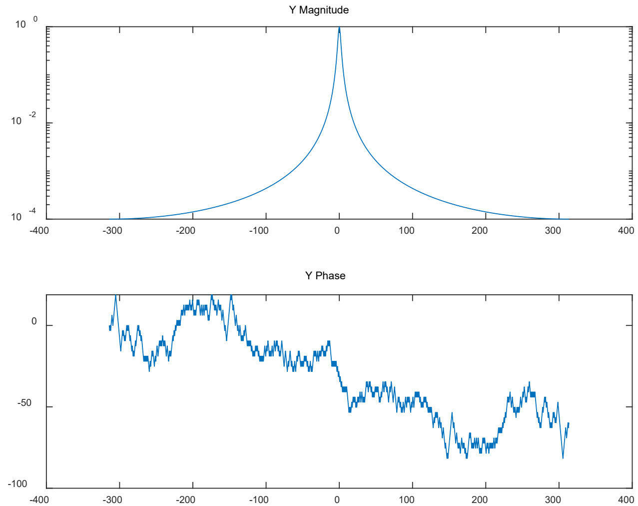
$\tau = 0.1$
$N = 201$

Lab Assignment 4

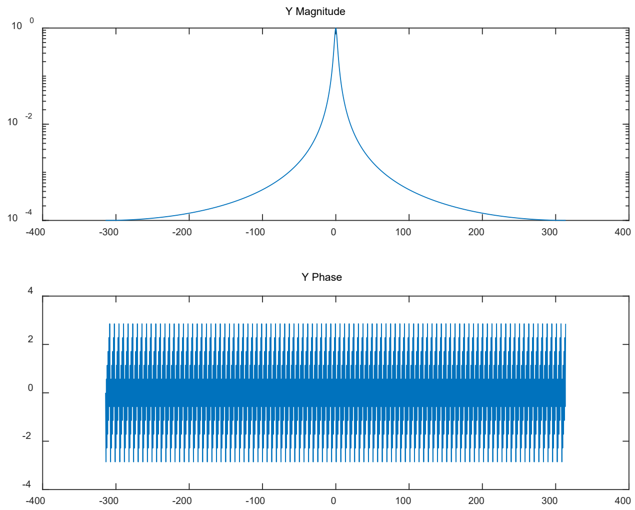
- Read tutorial 4.1 by yourself
- 4.2, 4.5 & 4.6
  - You need to download ctftmod.mat for 4.6
- In 4.2(g), you may encounter strange phase of Y, please read the next 2 pages and try Matlab function unwrap()
  - Have a try by yourself
  - We'll talk about it in the next week

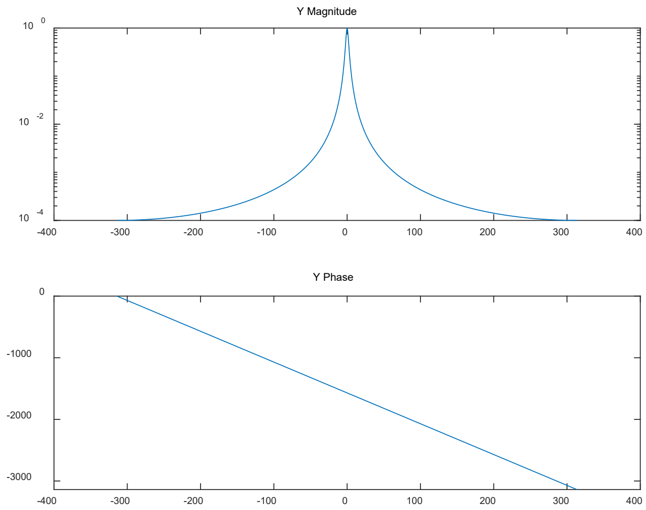
- unwrap(angle(X))
  - Correct phase angles to produce smoother phase plots: angle(Y) vs. unwrap(angle(Y))
  - Find more details in MATLAB HELP DOCUMENT





- If let $T = 11$


Y Magnitude
10 0
10 -2
10 -4
-400
-300
-200
-100
0
100
200
300
400
Y Phase
4
2
0
-2
-4


Y Magnitude
10 0
10 -2
10 -4
-400
-300
-200
-100
0
100
200
300
400
Y Phase
0
-1000
-2000
-3000

Example

```
t = linspace(0,6*pi,201);
x = t/pi.*cos(t);
y = t/pi.*sin(t);
z = x + 1i*y;
figure; plot(z); title('z');
xlabel('real part') ; ylabel('imaginary part');
figure;subplot(211);plot(t, angle(z)); title('wrapped')
subplot(212);plot(t, unwrap(angle(z))); title('unwrapped')
```

- Any questions?

Frequency Response

- How to calculate via Matlab?
- How to convert to impulse response?

What's Frequency Response

- Frequency response
  - A function of frequency
  - System gain in frequency domain

$$e^{j\omega_0 t} \Rightarrow \boxed{\begin{array}{c}\text{CT LTI System:}\\ H(j\omega)\end{array}} \Rightarrow H(j\omega_0)e^{j\omega_0 t}$$

- Math definition
  - Fourier transform of impulse response

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt = Y(j\omega)/X(j\omega)$$

Differential Equation and Frequency Response

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$H(j\omega) = \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) + b_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

Proof:

$$\mathcal{F}\left[\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{N} \mathcal{F}\left[a_k \frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega)$$

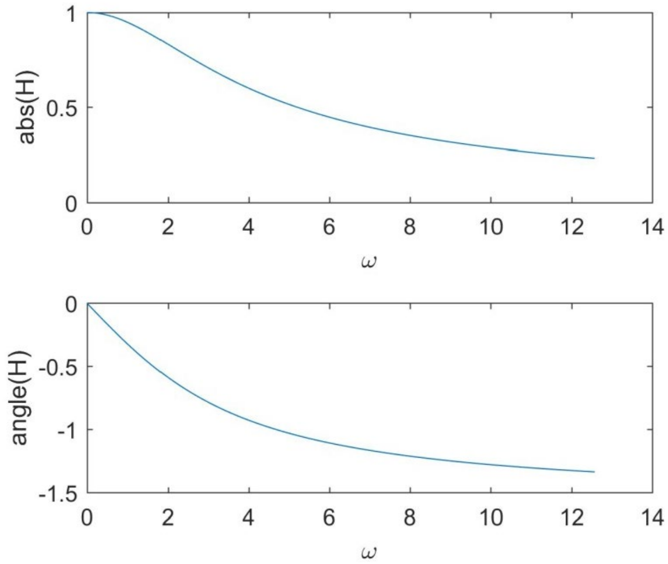
$$\mathcal{F}\left[\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right] = \sum_{k=0}^{M} \mathcal{F}\left[b_k \frac{d^k x(t)}{dt^k}\right] = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

Matlab Function: freqs() 连续时间的频率响应

- Description: generate the frequency response of LTI system
- Syntax: freqs(b,a,w), w - angular frequencies in rad/s 与连续的进行区别
- Example: $\frac{dy(t)}{dt} + 3y(t) = 3x(t)$

High-pass or Low-pass?

```
a=[1 3];
b=3;
w=linspace(0,4*pi);
H=freqs(b,a,w);
subplot(2,1,1), plot(w,abs(H));
xlabel('\omega'); ylabel('abs(H)')
subplot(2,1,2), plot(w,angle(H));
xlabel('\omega'); ylabel('angle(H)')
```



Frequency Response and Impulse Response

Impulse Response $\xleftrightarrow{\text{Fourier Transform}}$ **Frequency Response**

- Transform pair
  - $e^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{j\omega+a}$
- Example
  - $\frac{3}{j\omega+3} \Leftrightarrow \quad 3e^{-3t}u(t)$
- How about general fraction?

$$H(j\omega) = \frac{b_M(j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_1(j\omega) + b_0}{a_N(j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_1(j\omega) + a_0}$$

Partial Fraction Expansion

- Partial Fraction Expansion (No identical poles):

$$H(j\omega) = \frac{b_M(j\omega - z_1)(j\omega - z_2)\ldots(j\omega - z_M)}{a_N(j\omega - p_1)(j\omega - p_2)\ldots(j\omega - p_N)}$$
$$= \frac{A_1}{j\omega - p_1} + \frac{A_2}{j\omega - p_2} + \cdots + \frac{A_N}{j\omega - p_N}$$

- Partial Fraction Expansion (with identical poles):

$$H(j\omega) = \frac{b_M(j\omega - z_1)(j\omega - z_2)\ldots(j\omega - z_M)}{a_N(j\omega - p_1)^{k_1}(j\omega - p_2)^{k_2}\ldots(j\omega - p_n)^{k_n}}$$
$$= \frac{A_{1,1}}{(j\omega - p_1)^{k_1}} + \frac{A_{1,2}}{(j\omega - p_1)^{k_1-1}} + \cdots \frac{A_{1,k_1}}{(j\omega - p_1)}$$
$$+ \cdots + \frac{A_{n,1}}{(j\omega - p_n)^{k_n}} + \frac{A_{n,2}}{(j\omega - p_n)^{k_n-1}} + \cdots \frac{A_{n,k_n}}{(j\omega - p_n)}$$

- Therefore, we should know the FT of

$$\frac{1}{(j\omega + a)^k}$$

We've already known

$$e^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{j\omega + a}$$

According to the FT property

$$tx(t) \quad \Leftrightarrow \quad j\frac{d}{d\omega}X(j\omega)$$

We have

$$te^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{(j\omega + a)^2}$$

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \quad Re\{a\} > 0 \quad \Leftrightarrow \quad \frac{1}{(j\omega + a)^n}$$

Matlab Function: residue()

- Convert between partial fraction expansion and polynomial coefficients
- [r,p,k] = residue(b,a) ; [r,p] = residue(b,a)

$$H(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^m + b_2 s^{m-1} + b_3 s^{m-2} + \cdots + b_{m+1}}{a_1 s^n + a_2 s^{n-1} + a_3 s^{n-2} + \cdots + a_{n+1}}$$

$$\frac{b(s)}{a(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} + k(s)$$

当分子的指数比分母的指数小时，k(S)为空，

$$\frac{b(s)}{a(s)} = \frac{5s^3 + 3s^2 - 2s + 7}{-4s^3 + 8s + 3}$$

```
b = [ 5 3 -2 7];
a = [-4 0 8 3];
[r, p, k] = residue(b,a)
r = -1.4167
     -0.6653
     1.3320
p = 1.5737
     -1.1644
     -0.4093
 k = -1.2500
```

```
[b, a] = residue(r,p,k)
b =   -1.2500   -0.7500    0.5000   -1.7500
a =    1.0000   -0.0000   -2.0000   -0.7500
```

matlab算回去的时候它定义a的第一项为1

$$\frac{b(s)}{a(s)} = \frac{-1.4167}{s - 1.5737} + \frac{-0.6653}{s + 1.1644} + \frac{1.3320}{s + 0.4093} - 1.2500$$

Have a try

$$\frac{b(s)}{a(s)} = \frac{s-2}{s^2 + \frac{3}{2}s + \frac{1}{2}}$$

b = ?

a = ?

[r, p, k] = residue(b,a)

$$\frac{b(s)}{a(s)} = \frac{r1}{s-p1} + \frac{r2}{s-p2} + k$$

$$s = j\omega$$

单选题 1分

设置

Partial fraction expansion of $\frac{b(s)}{a(s)} = \frac{s-2}{s^2+\frac{3}{2}s+\frac{1}{2}}$

A $\frac{b(s)}{a(s)} = \frac{6}{s+1} + \frac{-5}{s+0.5}$

B $\frac{b(s)}{a(s)} = \frac{6}{s-1} + \frac{-5}{s-0.5}$

C $\frac{b(s)}{a(s)} = \frac{6}{s+1} + \frac{-5}{s+0.5} + 1$

D $\frac{b(s)}{a(s)} = \frac{6}{s-1} + \frac{-5}{s-0.5} + 1$

b = ?
a = ?
[r, p, k] = residue(b,a)

$$\frac{b(s)}{a(s)} = \frac{r1}{s-p1} + \frac{r2}{s-p2} + k$$

提交

```
a = ...
b = ...
[r,p,k]=residue(b,a)

r =
    2.0000
    1.0000
   -3.0000

p =
   -3.0000
   -2.0000
   -2.0000

k =
    []
```

Partial fraction expansion of $\frac{b(s)}{a(s)}$ =?

If `p(j) = ... = p(j+m-1)` is a pole of multiplicity `m`, then the expansion includes terms of the form

$$\frac{r_j}{s-p_j}+\frac{r_{j+1}}{(s-p_j)^2}+\ldots+\frac{r_{j+m-1}}{(s-p_j)^m}.$$

$$\frac{2}{s+3}+\frac{1}{s+2}+\frac{-3}{(s+2)^2}$$

Lab Assignment 4

- Read tutorial 4.1 by yourself
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Advanced Problems

Consider the stable, continuous-time system whose inputs and outputs satisfy the differential equation

$$\frac{d^2y(t)}{dt^2} - 4y(t) = -4x(t).$$

(g). Define vectors `b3` and `a3` to represent the numerator and denominator polynomials of the system function $H_3(j\omega)$.

(h). Compute the partial fraction expansion of $H_3(j\omega)$ using `residue`. Analytically recombine the terms of your sum to verify you get back $H_3(j\omega)$.

(i). Determine the impulse response $h_3(t)$ for the system based on the partial fraction expansion. Remember that $h_3(t)$ must be absolutely integrable because you have assumed the system is stable. Is $h_3(t)$ causal?

You may ignore the last question of 4.5(i) in your lab report.

- Any questions?