Chapter 9
The Laplace Transform

Why Laplace Transform?

Name after
Pierre-Simon Laplace (1749–1827)

- Laplace transform is a generalization of continuous-time Fourier transform
- It provides additional tools and insights on signals and systems
  - E.g., poles and zeros
- It can be applied to the scenarios where Fourier transform does not exist
  - E.g., instable system

Laplace Transform

- e^{st} is the eigenfunction of LTI system:

$$e^{st} \longrightarrow H(s)e^{st} \text{ where } H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

- In Fourier transform, we let $s = j\omega$ (pure imaginary)

- In Laplace transform, s is general complex number $s = \sigma + j\omega$

$$\boxed{\text{Laplace Transform: } X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt}$$

$$\boxed{\text{Fourier transform: } X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt}$$

- Question: How to calculate Laplace transform from knowledge of Fourier transform.

with s expressed as $s = \sigma + j\omega$,

$$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t}\, dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\, dt.$$

The Laplace transform of x(t) can be interpreted as the Fourier transform of x(t) after multiplication by a real exponential signal.

Laplace Transform: Example

- $x(t) = e^{-at}u(t)$:

$$
\begin{aligned}
X(s) = X(\sigma + j\omega) &= \int_0^{+\infty} e^{-at} e^{(-\sigma - j\omega)t} dt = \int_0^{+\infty} e^{-(a+\sigma)t} e^{-j\omega t} dt \\
&= \frac{1}{s+a} \quad Re\{s\} > -a \quad (or \quad \sigma > -a)
\end{aligned}
$$

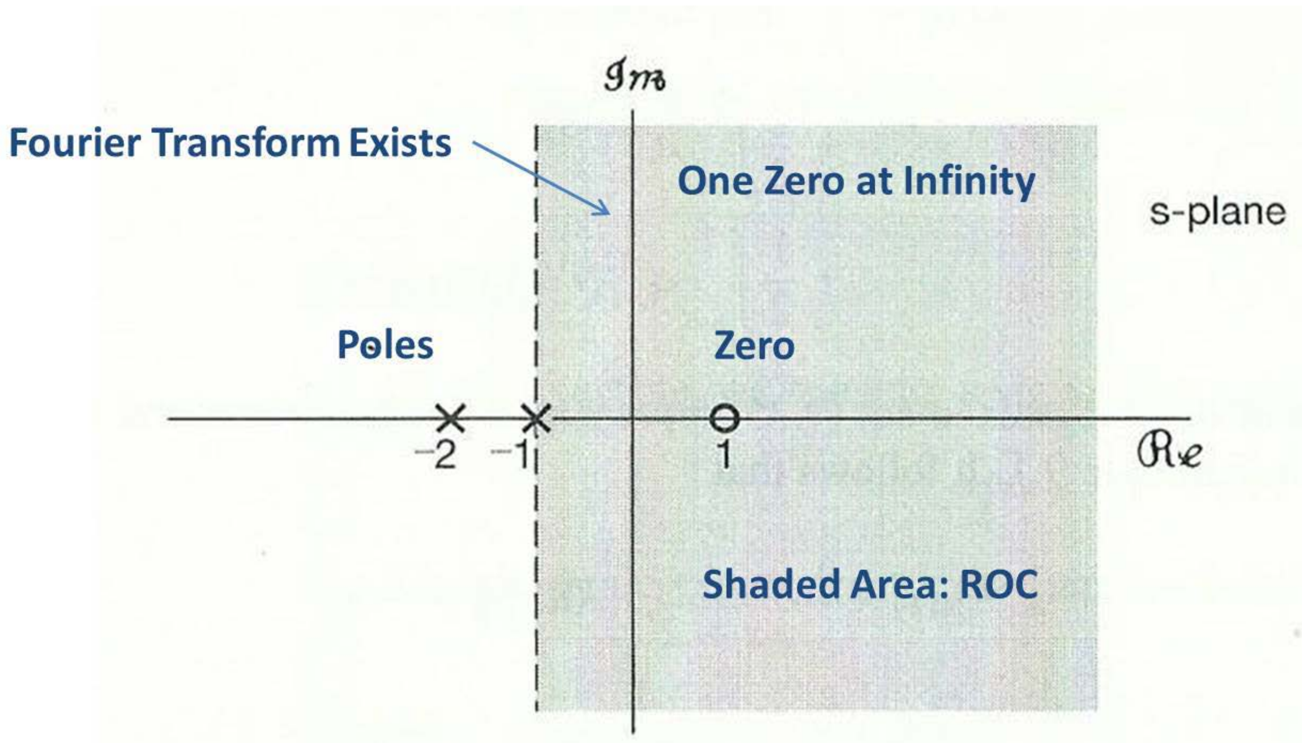
- $x(t) = -e^{-at}u(-t)$:

$$
\begin{aligned}
X(s) = X(\sigma + j\omega) &= -\int_{-\infty}^{0} e^{-at} e^{(-\sigma - j\omega)t} dt \\
&= \frac{1}{s+a} \quad Re\{s\} < -a \quad (or \quad \sigma < -a)
\end{aligned}
$$

- Laplace transform should be specified by both algebraic expression and region of convergence (ROC)

Pole-Zero Plot

- $3e^{-2t}u(t) - 2e^{-t}u(t) \longleftrightarrow \frac{s-1}{s^2+3s+2} \quad Re\{s\} > -1$



- To within a scale factor, a rational Laplace transform can be specified by the pole-zero plot and its ROC

Region of Convergence (ROC)

Property (1)

The ROC of $X(s)$ consist of strips parallel to the $j\omega$-axis in the s-plane

- $X(s) = \mathcal{F}\{x(t)e^{-\sigma t}\}$

Property (2)

For rational Laplace transform, the ROC does not contain any poles

- Poles: $X(s) \to \infty$

$$\boxed{X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt}$$

Property (3)

If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane

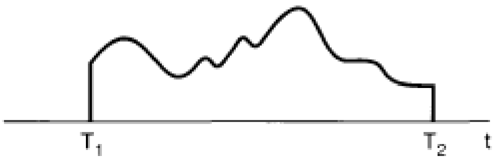


Figure 9.4 Finite-duration signal.

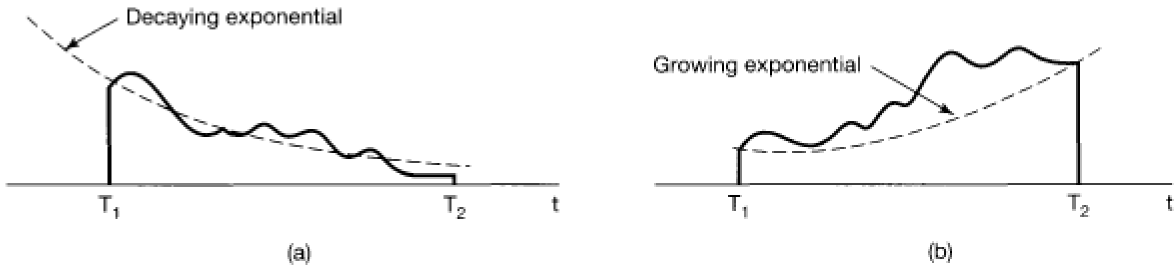


Figure 9.5 (a) Finite-duration signal of Figure 9.4 multiplied by a decaying exponential; (b) finite-duration signal of Figure 9.4 multiplied by a growing exponential.

$$\int_{T_1}^{T_2} |x(t)|\,dt < \infty. \xrightarrow{?} \int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < \infty.$$

For $\sigma > 0$, $$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < e^{-\sigma T_1}\int_{T_1}^{T_2} |x(t)|\,dt.$$

if $\sigma < 0$, $$\int_{T_1}^{T_2} |x(t)|e^{-\sigma t}\,dt < e^{-\sigma T_2}\int_{T_1}^{T_2} |x(t)|\,dt$$

Property (4,5)

If $x(t)$ is left sided (right sided), and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} < \sigma_0$ ($Re\{s\} > \sigma_0$) will also be in the ROC

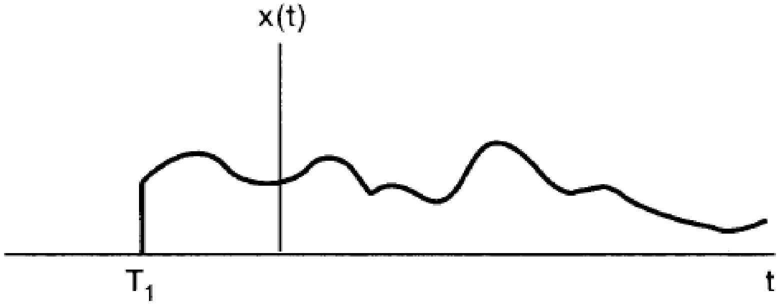


Figure 9.6 Right-sided signal.

$$\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t}\, dt < \infty.$$

Then if $\sigma_1 > \sigma_0$,

$$\int_{T_1}^{\infty} |x(t)| e^{-\sigma_1 t}\, dt = \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t}\, dt$$

$$\leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{\infty} |x(t)| e^{-\sigma_0 t}\, dt.$$

Property (6)

If $x(t)$ is two sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s-plane that include the line $Re\{s\} = \sigma_0$

- How to derive it from Property 4 and 5?

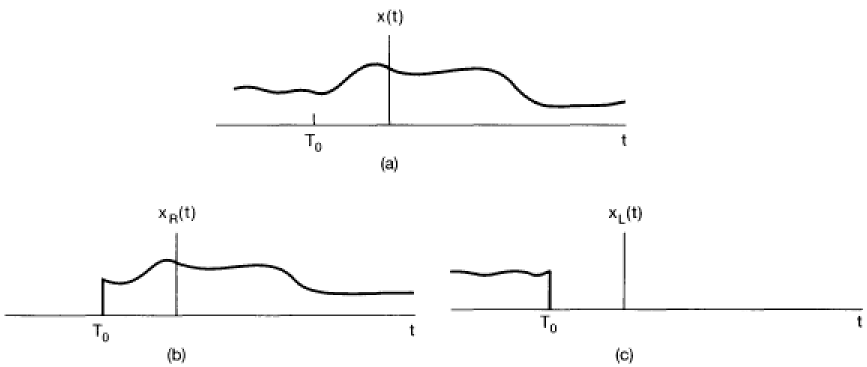


Figure 9.9 Two-sided signal divided into the sum of a right-sided and left-sided signal: (a) two-sided signal $x(t)$; (b) the right-sided signal equal to $x(t)$ for $t > T_0$ and equal to 0 for $t < T_0$; (c) the left-sided signal equal to $x(t)$ for $t < T_0$ and equal to 0 for $t > T_0$.

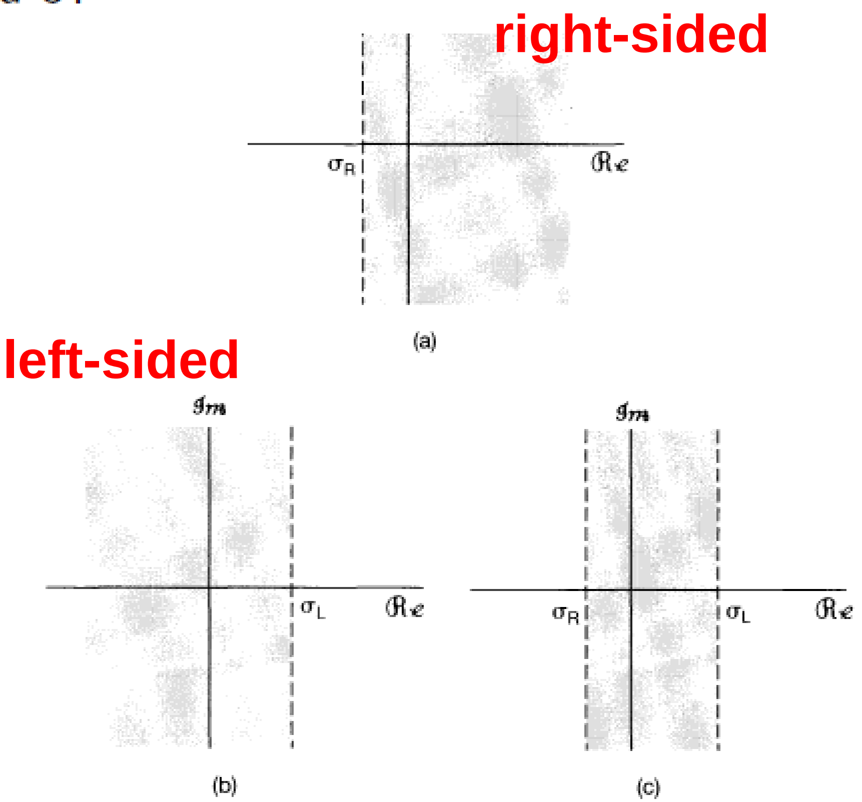


Figure 9.10 (a) ROC for $x_R(t)$ in Figure 9.9; (b) ROC for $x_L(t)$ in Figure 9.9; (c) the ROC for $x(t) = x_R(t) + x_L(t)$, assuming that the ROCs in (a) and (b) overlap.

- ROC: null, left-half plane, right-half plane, single strip, s-plane

Property (7)

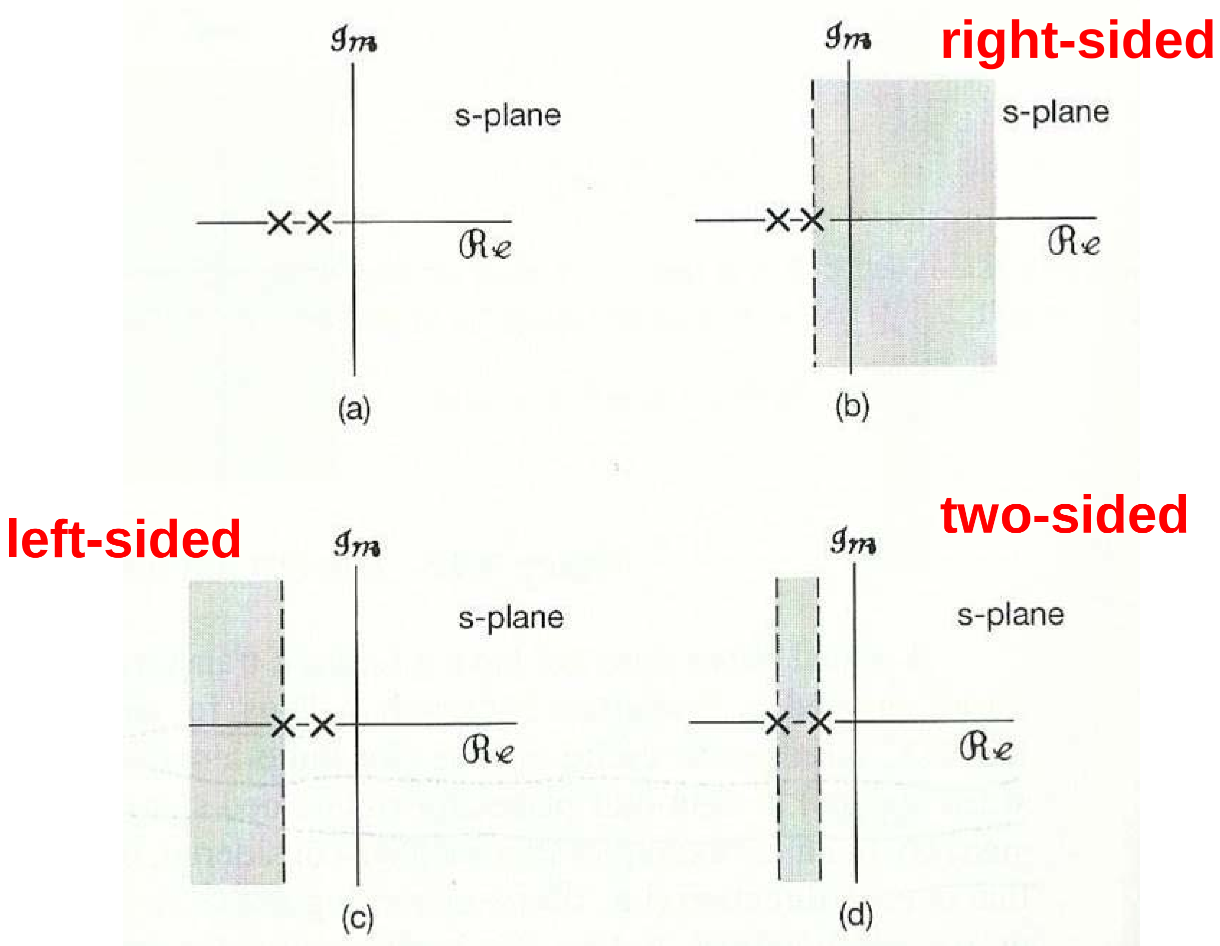
If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity.

Property (8)

If $X(s)$ is rational, then if $x(t)$ is right sided (left sided), the ROC is the region in the s-plane to the right (left) of the rightmost (leftmost) pole

ROC: Example

- $X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$ has two poles $s = -1, -2$
  - $X(s) = \frac{1}{s+1}$: $e^{-t}u(t)$ or $-e^{-t}u(-t)$?
  - $X(s) = \frac{1}{s+2}$: $e^{-2t}u(t)$ or $-e^{-2t}u(-t)$?



Inverse Laplace Transform

$$\begin{aligned} X(\sigma + j\omega) &= \mathcal{F}\{x(t)e^{-\sigma t}\} \\ &\Downarrow \\ x(t) &= e^{\sigma t}\mathcal{F}^{-1}\{X(\sigma + j\omega)\} \end{aligned}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}\, d\omega$$

- Inverse Laplace transform:
  $s = \sigma + j\omega$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

- Another approach: partial fraction expansion
  - Algebraic expression:

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s + a_i}$$

  - ROC: ROC of each term should contain the ROC of $X(s)$

LTI Systems

$$\text{Time Domain: } y(t) = h(t) * x(t) \longleftrightarrow \text{ S-Domain: } Y(s) = H(s)X(s)$$

- $H(s)$: system function or transfer function
- Causality: impulse response is right sided
  - The ROC of causal system is a right-half plane (How about the inverse?)
  - For rational system function, causality is equivalent to the ROC being the right-half plane to the right of rightmost pole
  - Question: How about anticausal ($h(t) = 0, t > 0$)?
- Stability: An LTI system is stable if and only if the ROC of its system function $H(s)$ include the $j\omega$-axis
  - Example: $H(s) = \frac{s-1}{(s+1)(s-2)}$
  - Unstable systems have Laplace transform
- Question:What's the ROC of causal stable LTI system with rational system function?

LTI System: Differential Equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\Downarrow$$

$$\left(\sum_{k=0}^{N} a_k s^k\right) Y(s) = \left(\sum_{k=0}^{M} b_k s^k\right) X(s)$$

$$\Downarrow$$

$$H(s) = \left(\sum_{k=0}^{M} b_k s^k\right) \Big/ \left(\sum_{k=0}^{N} a_k s^k\right)$$

- ROC: stable, causal ...
- Reading Assignment: Necessary knowledge of Laplace transform in Section 9.0-9.3 9.4.1 9.5-9.7

Chapter 10
The z Transform

Z-Transform

- Discrete-time system:

$$z^n \rightarrow H(z)z^n \text{ where } H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}$$

- In Fourier transform, $z = e^{j\omega}$ (unit magnitude)

- In z-transform, z is general complex number $z = re^{j\omega}$

$$\boxed{\text{z-transform:} \quad X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}}$$

- Question: What's the relation between z-transform and discrete-time Fourier transform?

Z-Transform: Example

- $x[n] = a^n u[n]$:

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

- $x[n] = -a^n u[-n-1]$:

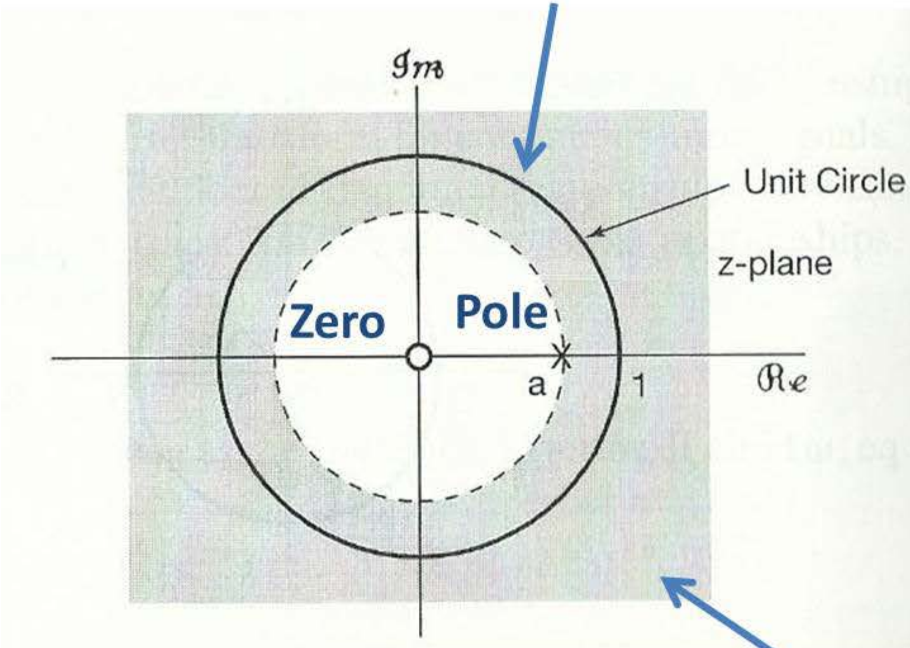
$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{+\infty} a^n u[-n-1] z^{-n} = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1 - a^{-1}z} \\ &= \frac{z}{z-a}, \quad |z| < |a| \end{aligned}$$

- Z-transform should be specified by both algebraic expression and region of convergence (ROC)

Pole-Zero Plot

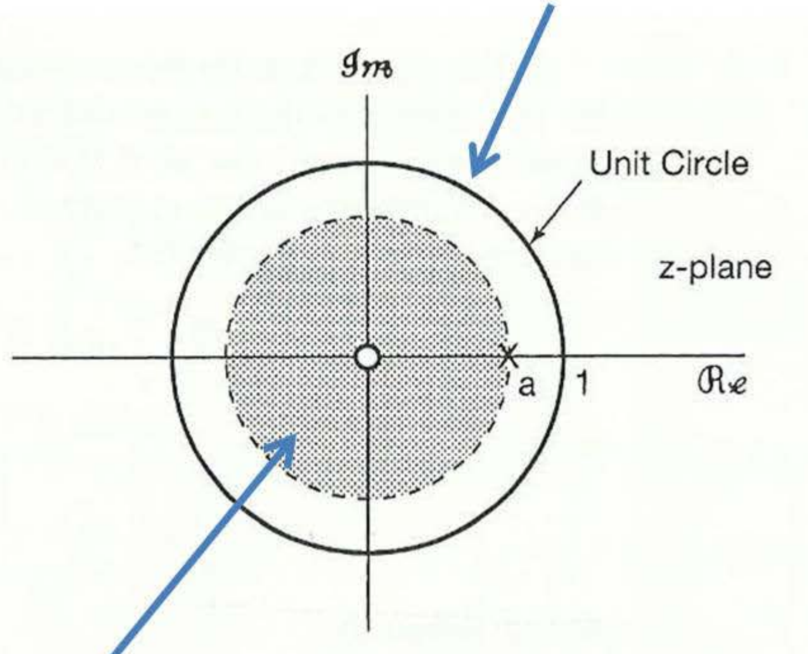
$a^n u[n]$ ($|a|<1$)

Fourier Transform Exists



$-a^n u[-n-1]$ ($|a|<1$)

Fourier Transform Diverge



Shaded Area: ROC --- Ring

Region of Convergence (1/3)

Property (1)

The ROC of $X(z)$ consists of a ring in the z-transform centered about the origin

- $X(z) = \mathcal{F}\{x[n]r^{-n}\}$

Property (2)

The ROC does not contain any pole

Property (3)

If $x[n]$ is of finite duration, then the ROC is the entire z-plane, except possible $z = 0$ and/or $z = \infty$

$$X(z) = \sum_{n=N_1}^{N_2} x[n]z^{-n}.$$

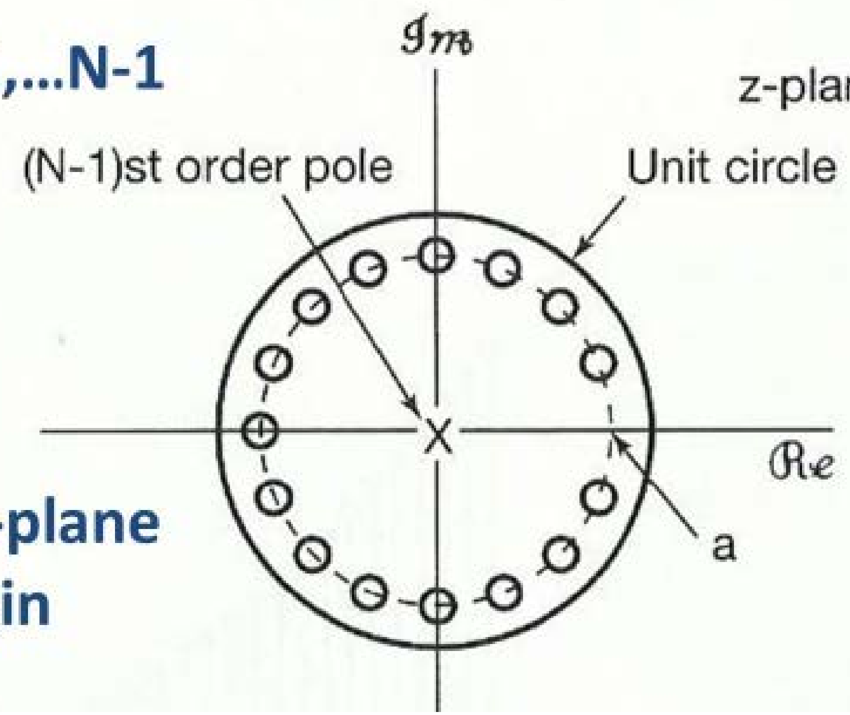
- Question: When is $z = 0$ or $z = \infty$ in ROC?

ROC: Example

$$x[n] = a^n \quad 0 \leq n \leq N-1, a > 0$$

$$X(z) \;=\; \sum_{n=0}^{N-1} a^n z^{-n} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}$$

Zeros: $z_k = ae^{j2k\pi/N}$, k=1,…N-1



ROC is the entire z-plane except the origin

Region of Convergence (2/3)

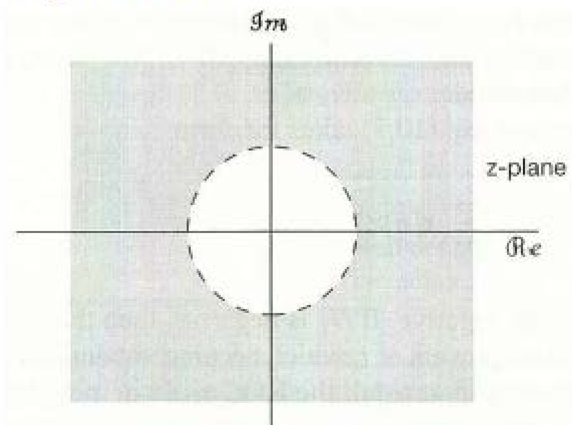
Property (4,5)

If $x[n]$ is a right-sided (left-sided) sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite value of z for which $|z| > r_0$ ($0 < |z| < r_0$) will also be in the ROC

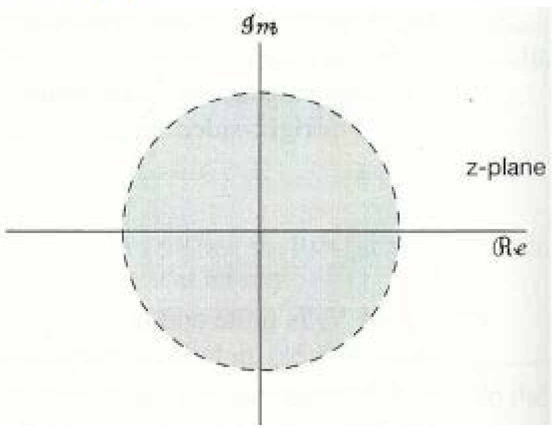
Property (6)

If $x[n]$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will consist of a ring including the circle $|z| = r_0$

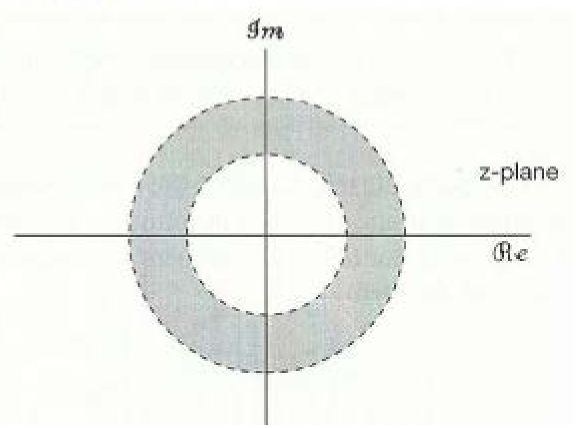
Right Sided



Left Sided



Two Sided



Region of Convergence (3/3)

Property (7)

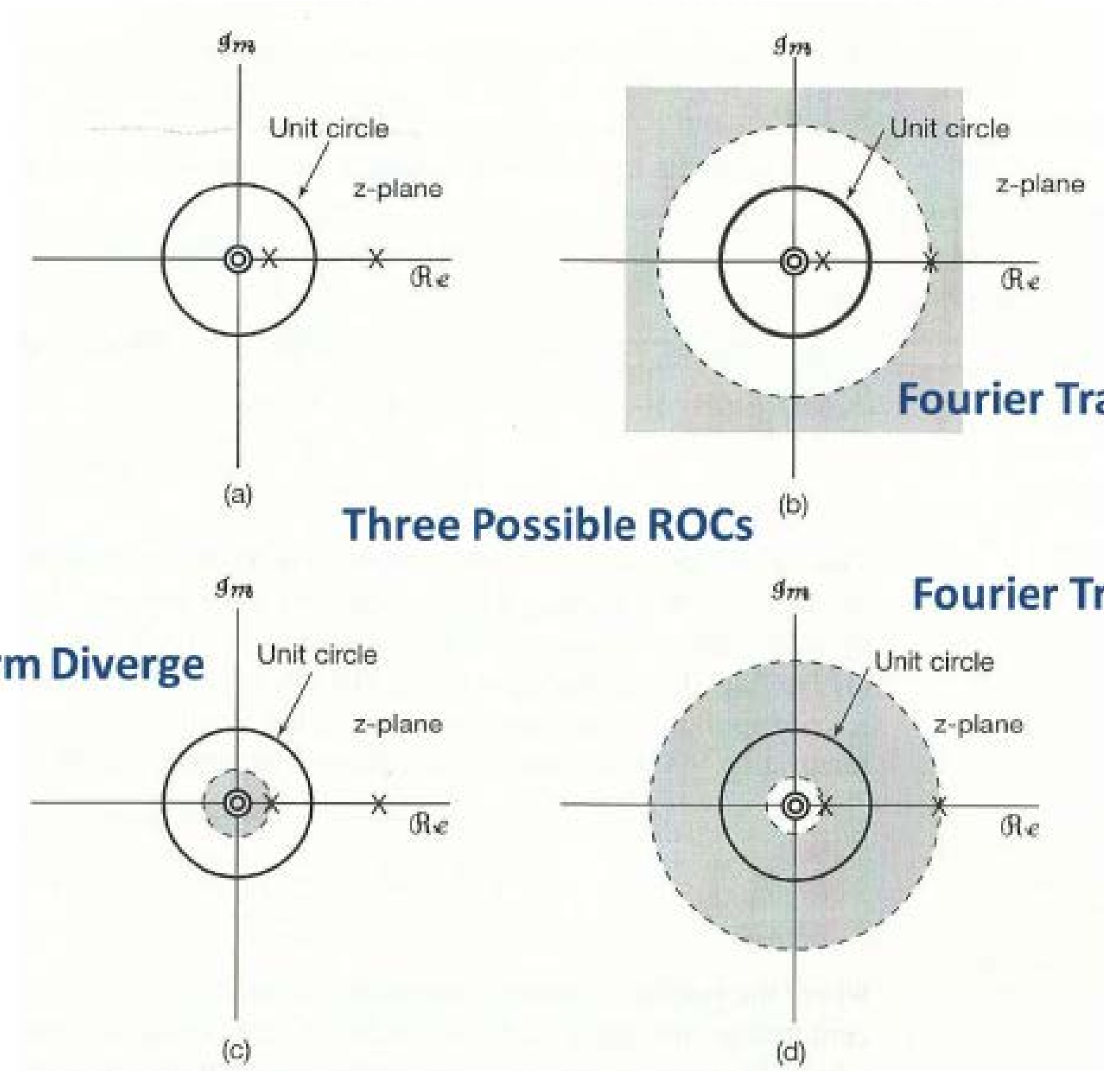
If $X(z)$ is rational, then its ROC is bounded by poles or extends to infinity

Property (8,9)

If $X(z)$ is rational and $x[n]$ is right sided (left sided), then the ROC is the region outside the outmost pole (inside the innermost nonzero pole, possibly including $z = 0$)

ROC: Example

$$X(z) = \frac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})} = \frac{z^2}{(z-\frac{1}{3})(z-2)}$$



Inverse Z-Transform

- Approach I: via Fourier Transform

$$X(re^{j\omega}) = \mathcal{F}\{x[n]r^{-n}\} \Rightarrow x[n] = r^n\mathcal{F}^{-1}\{X(re^{j\omega})\}$$

- Approach II: via definition

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Integration around a counter-clockwise circle

- Approach III: partial fraction expansion

$$X(z) = \sum_{i=1}^{m}\frac{A_i}{1-a_iz^{-1}}$$

- Approach IV: power-series expansion

$$E.g.,\quad X(z) = 4z^2 + 2 + 3z^{-1}$$

LTI Systems

$$y[n] = h[n] * x[n] \longleftrightarrow Y(z) = H(z)X(z)$$

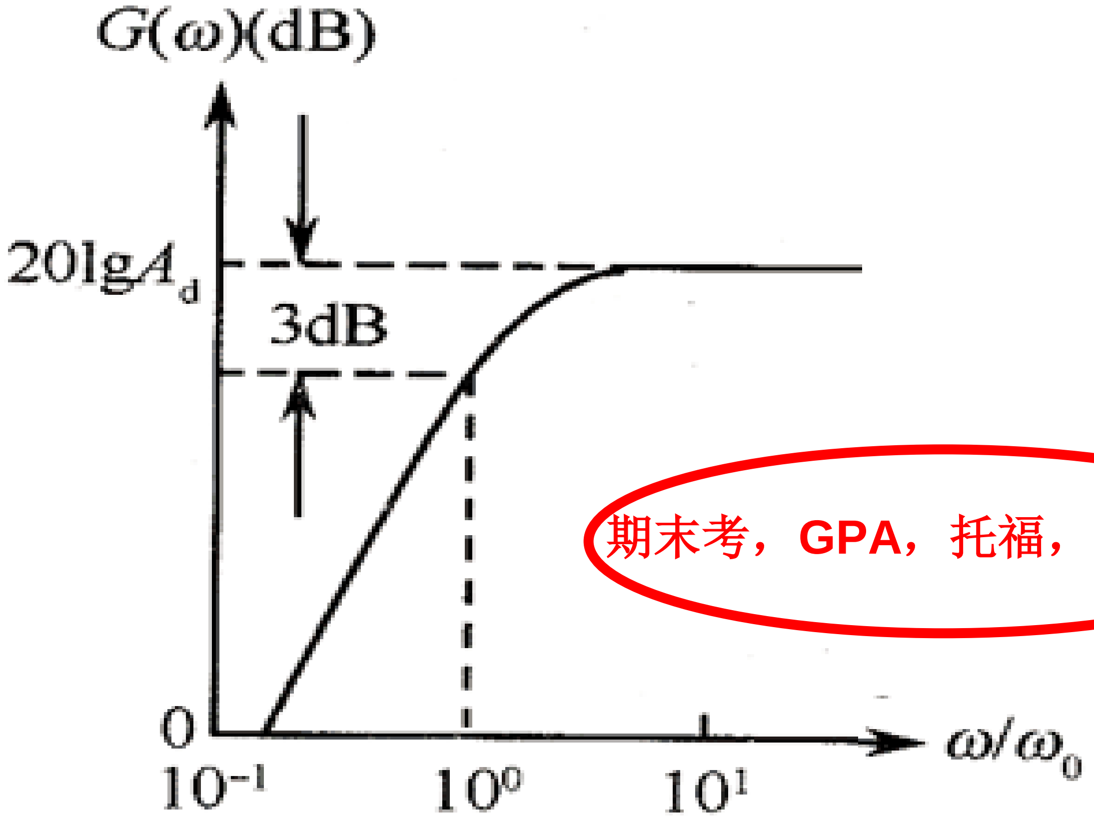
- $H(z)$ is the system function or transfer function
- Causality:
  - the ROC is the exterior of a circle, including infinity
  - Rational: (a) the ROC is the exterior of a circle outside the outermost pole; (b) order of numerator cannot be greater than the order of denominator
- Stability:
  - The ROC includes $|z| = 1$
  - Causal & Rational: all the poles lie inside the unit circle

LTI Systems: Difference Equation

$$
\begin{aligned}
\sum_{k=0}^{N} a_k y[n-k] &= \sum_{k=0}^{M} b_k x[n-k] \\
&\Downarrow \\
\sum_{k=0}^{N} a_k z^{-k} Y(z) &= \sum_{k=0}^{M} b_k z^{-k} X(z) \\
&\Downarrow \\
H(z) &= \left(\sum_{k=0}^{M} b_k z^{-k} \right) \Big/ \left(\sum_{k=0}^{N} a_k z^{-k} \right)
\end{aligned}
$$

- ROC: causal, stable, ...
- Reading Assignment: Necessary knowledge of z-transform in Section 10.0-10.3 10.4.1 10.5-10.7

Wish all of you **'high-pass'！**



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