Notes

Assignments

- 5.2, 5.5, 5.15, 5.21(a-f, h)

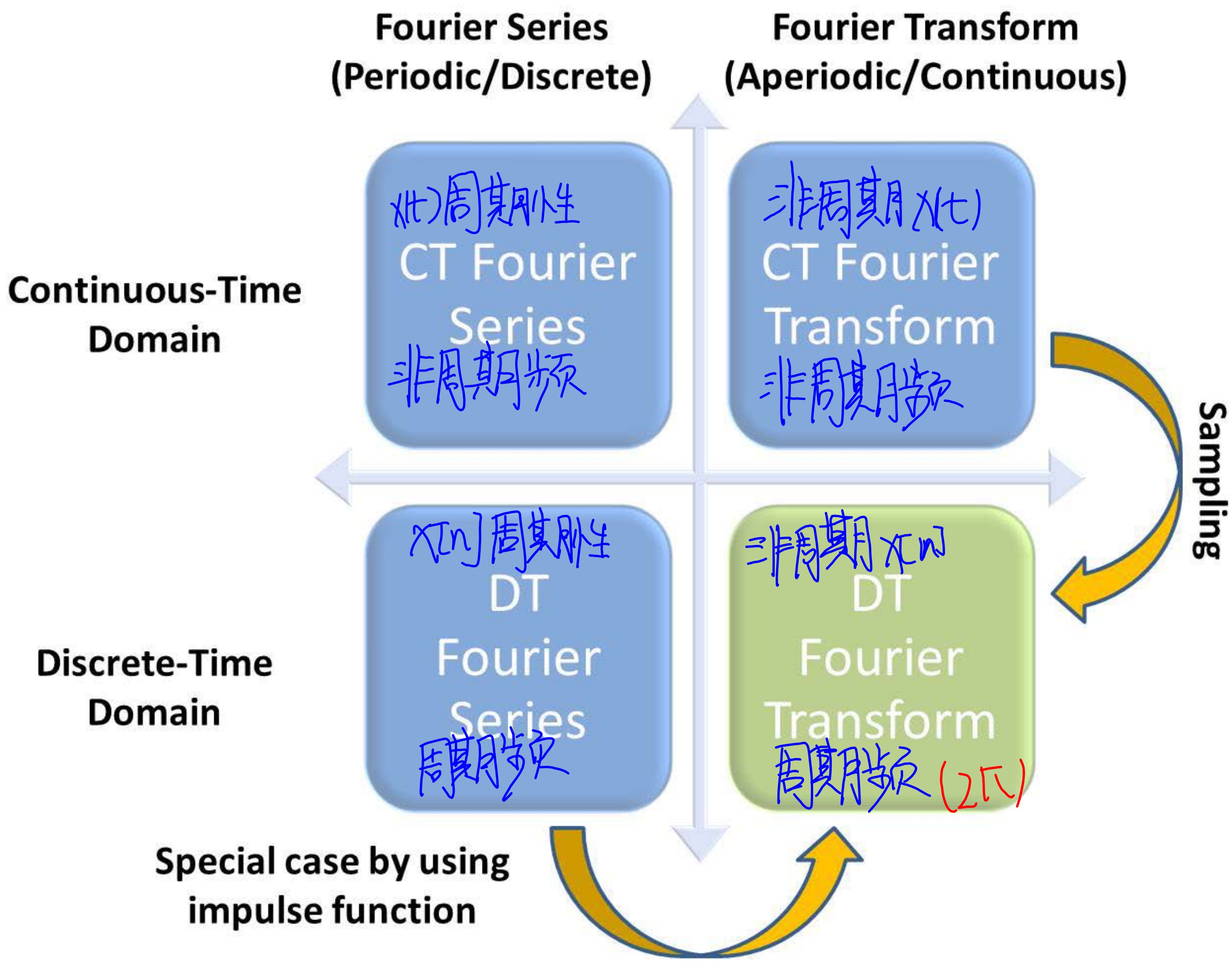
Tutorial problems

- 5.1, 5.3, 5.4, 5.41

Chapter 5
The Discrete-Time Fourier Transform

Frequency domain

Time domain



Thinking…

- **How to move from CT Fourier series to CT Fourier transform?**

Discrete-Time Fourier Transform (DTFT)

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

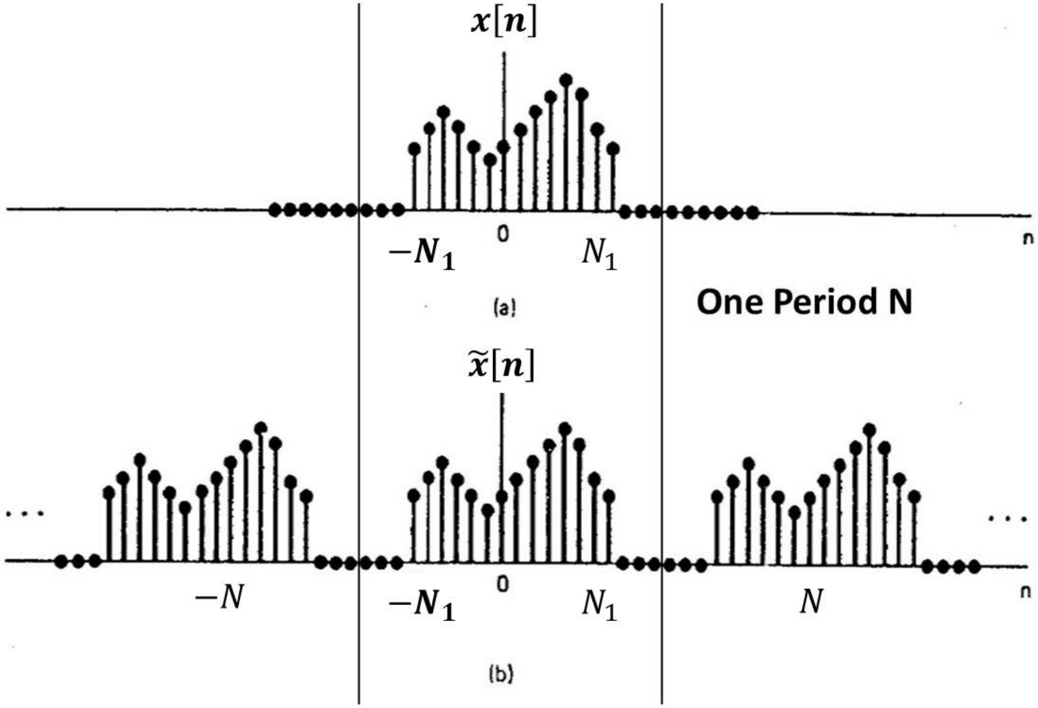
$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

Aperiodic signals can be treated as periodic signals with period $N \to \infty$

- $x[n]$ must be like $\sum_k b_k e^{jk(2\pi/N)n}$ or $\int_\omega b(\omega) e^{j\omega n} d\omega$
- b_k or $b(\omega)$ can be calculated from $x[n]$

DTFT Derivation (1/3)



Original signal: $x[n]$

Define new periodic signal with period N: $\tilde{x}[n]$, such that

$$\tilde{x}[n] = x[n], \;\; n = -N/2, ..., N/2 - 1$$

Notice: when $N \to \infty$, $\tilde{x}[n]$ becomes $x[n]$

DTFT Derivation (2/3)

- Look at the Fourier series of $\widetilde{x}[n]$:

$$a_k = \frac{1}{N} \sum_{n=-N/2}^{N/2+1} \widetilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

DTFT Derivation (3/3)

- Therefore, we get the discrete-time Fourier transform pair

Discrete-Time Fourier Transform

$$\text{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\text{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

1) continuous spectrum: Similar to CTFT

2) Periodic with period 2π: Different from CTFT

3) Low frequency: close to 0 and 2π;

high frequency: close to π

Periodicity Properties of DT Complex Exponentials

- $x[n]$ - periodic with fundamental period N, fundamental frequency

$$x[n+N] = x[n] \qquad and \qquad \omega_o = \frac{2\pi}{N}$$
$$n = \ldots, -1, 0, 1, 2, 3, \ldots$$

- For DT complex exponentials, signal are periodic only when

$$\omega_0 N = k \cdot 2\pi, \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$\boxed{e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1 \rightarrow \omega_0 N = k \cdot 2\pi}$$

- For DT complex exponentials, signals with frequencies ω_0 and ω_0+k•2π are identical.

$$\boxed{e^{j(\omega_0 + k\cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jk\cdot 2\pi n} = e^{j\omega_0 n}}$$

  - We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

Cont.

- $e^{j\omega_0 n}$ does ***not*** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

  low-frequency (slowly varying): ω_0 near 0, 2π, …, or 2k • π

  high-frequency (rapid variation): ω_0 near ± π, …, or (2k+1) • π

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

对比不同的地方，并解释原因

Discrete-Time Fourier Transform

$$\textit{Synthesis Equation:} \quad x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$\textit{Analysis Equation:} \quad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

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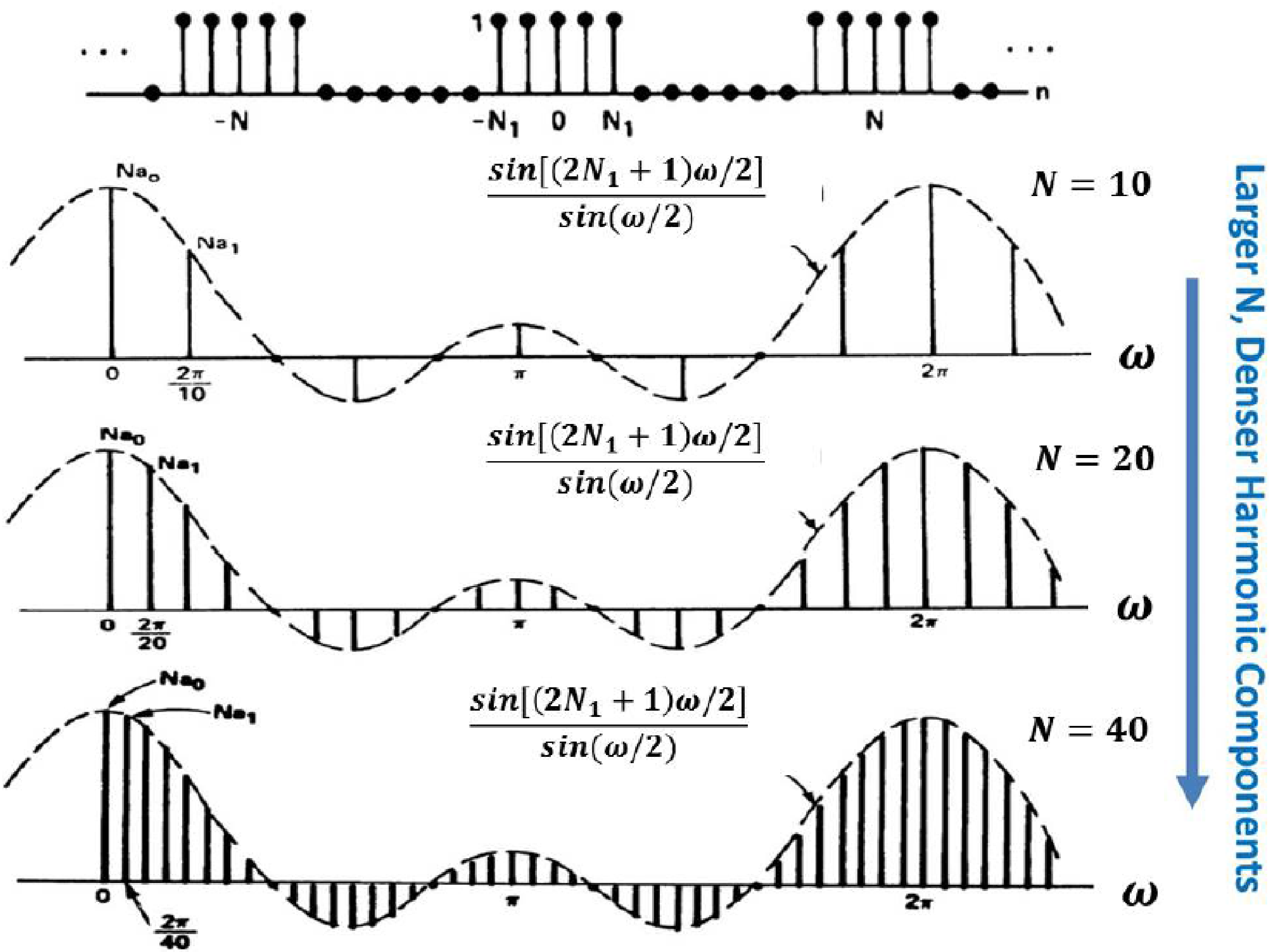
$k=\langle N\rangle$

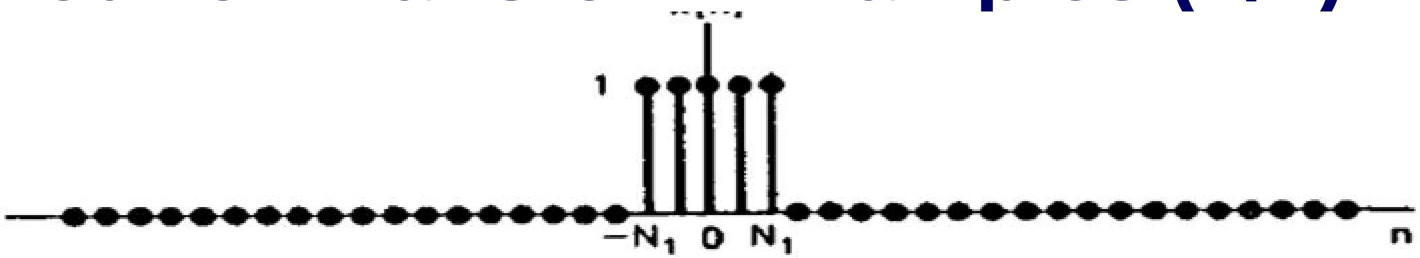
Example: From Periodic To Aperiodic



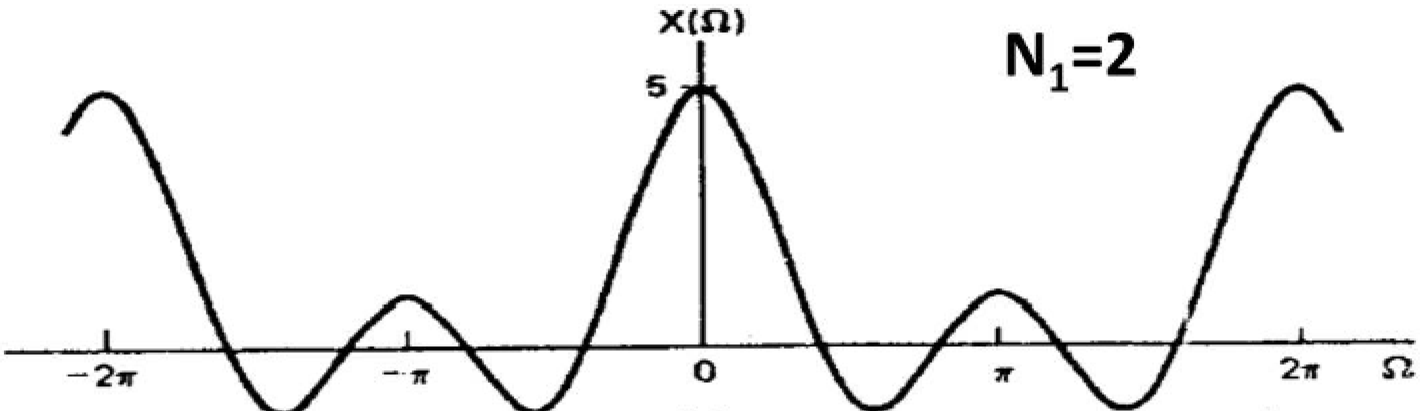
$x[n]=\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$

$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

Fourier Transform Examples (1/2)



(a)

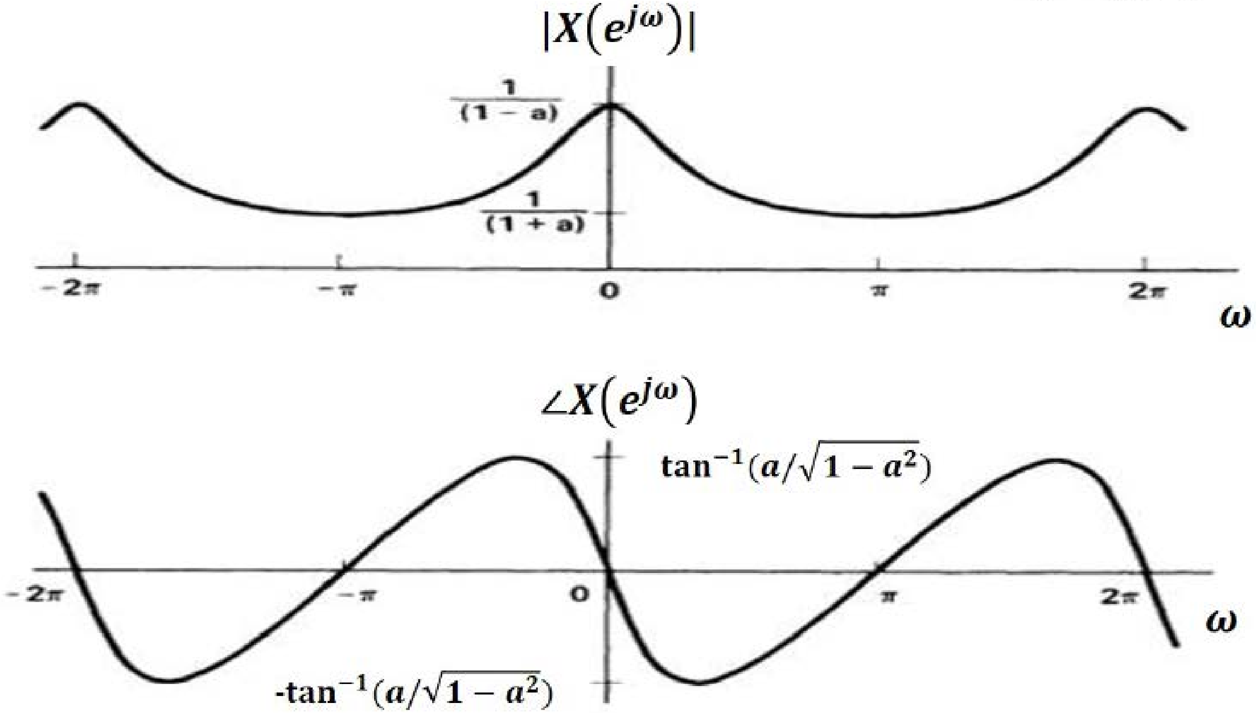


(b)

- $x[n]=1\ (n=-N_1,...,0,...,N_1)$
- $X(e^{j\omega})=\frac{sin\omega(N_1+1/2)}{sin(\omega/2)}$
- Width of $x[n]$: $W_t=2N_1+1$; width of $X(e^{j\omega})$: $W_f=\frac{4\pi}{2N_1+1}$
- $W_t\times W_f=4\pi$, which is a constant

Fourier Transform Examples (2/2)

$$x[n] = a^n u[n] \quad 0 < a < 1 \quad \Leftrightarrow \quad X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$



- See textbook, Example 5.1
- What's the shape of magnitude when $a \to 1$ or $a \to 0$?

Convergence Issue of Analysis Equation

Sufficient Condition of Convergence

The analysis equation $X(e^{j\omega}) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$ *will converge either if* $x[n]$ *is absolutely summable or if the sequence has finite energy, thus,*

$$\sum_{-\infty}^{\infty} |x[n]| < \infty \text{ or } \sum_{-\infty}^{\infty} |x[n]|^2 < \infty$$

Cont.

- Do the following signals have Fourier transform:
  - $a^n u[n]$ $(0 < a < 1)$
  - $\delta[n] \longleftrightarrow 1$
  - $u[n]$ 无
  - $e^{j\frac{2}{5}\pi n}$, $cos(\frac{2}{5}\pi n)$
  - $a^n u[n]$ $(a > 1)$

Can Periodic Signals Have DTFT?

- Definition of DTFT:

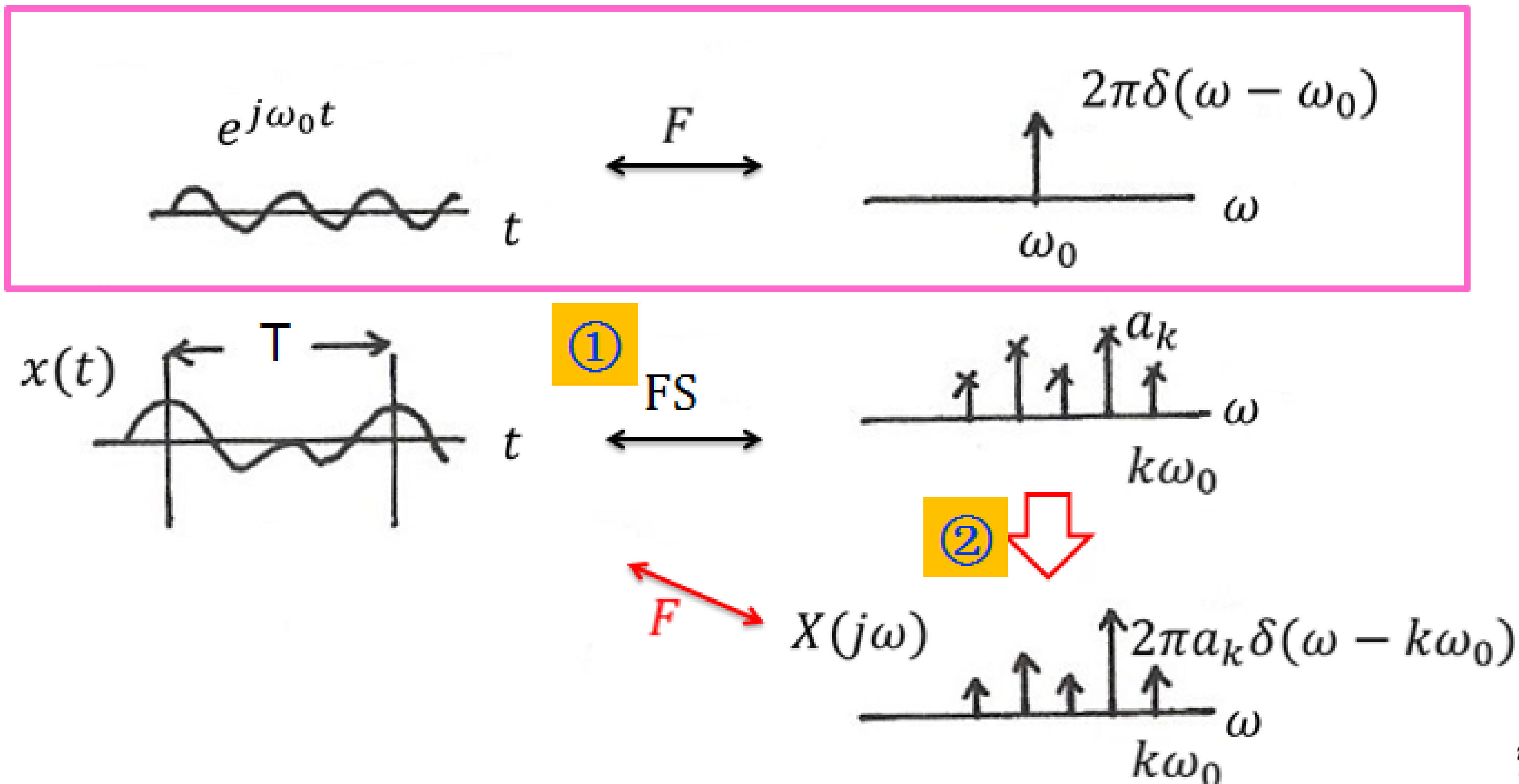
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- Justification of divergence:
  - Let $\omega = 2k\pi$, we have $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]$
  - Since $x[n]$ is periodic, the summation $\sum_{n=-\infty}^{\infty} x[n]$ will never converge unless $x[n] = 0$
- Conclusion: Most of periodic signals do NOT have DTFT according to the definition
- However, it's of significant engineering importance to extend Fourier transform to periodic signals

Can Periodic Signals Have DTFT?

Review

Fourier Transform for Periodic Signals – Unified Framework



DTFT with Periodic Signals (1/2)

Fourier Transform of $e^{j\omega n}$

The following transform pair is actually NOT rigorously defined:

$$x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

- Synthesis:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

- Analysis:

$$\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega_0 - \omega)n} \quad \textit{converge??}$$

DTFT with Periodic Signals (2/2)

- According to the Fourier series, for a periodic signal with period N:

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + \ldots + a_k e^{jk(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$

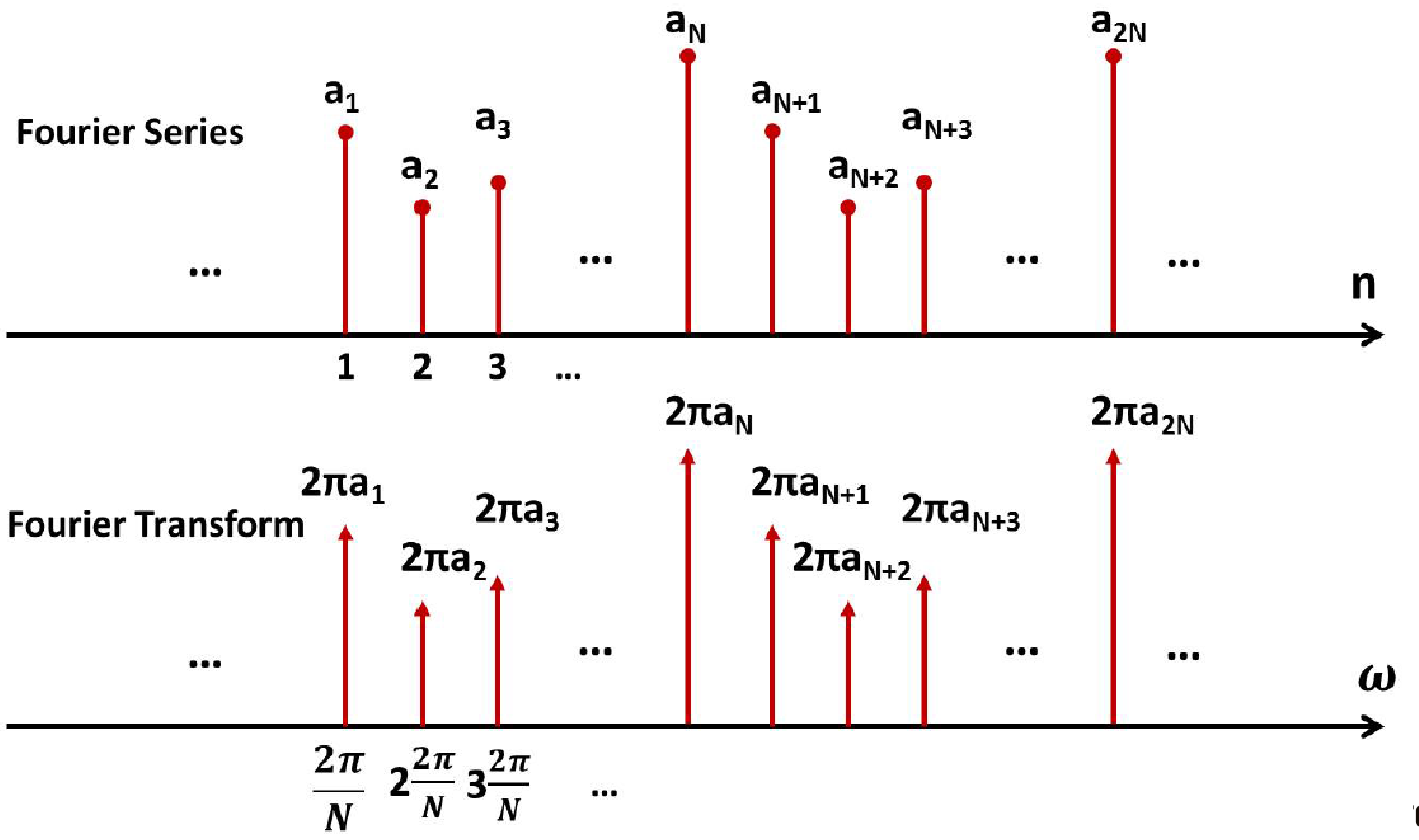
- $e^{jk(2\pi/N)n} \longleftrightarrow \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)$
- Then, due to the linearity of Fourier transform

$$\begin{aligned}\mathcal{F}\left\{x[n]\right\} = \sum_{k=0}^{N-1} a_k \mathcal{F}\left\{e^{jk(2\pi/N)n}\right\} &= \sum_{k=0}^{N-1} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - k(2\pi/N) - 2\pi l) \\ &= \sum_{l=-\infty}^{\infty} \sum_{k=0}^{N-1} a_k 2\pi\delta(\omega - k(2\pi/N) - 2\pi l)\end{aligned}$$

- Fourier transform of a periodic signal is a periodic sequence of impulses
  - What's the period? How many impulses within one period?

F

Fourier Series v.s. Fourier Transform



Example: Discrete-Time Impulse Chain

- What's the Fourier transform of $x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN]$?
- First of all, we calculate the Fourier series:

$$
\begin{aligned}
a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \sum_{k=-\infty}^{+\infty} \delta[n - kN] e^{-jk\frac{2\pi}{N}n} \\
&= \frac{1}{N} \sum_{n=<N>} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}
\end{aligned}
$$

Frequency domain