Tutorial Questions （Week 4）

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Outline

- Review
- Basic Problems with Answers 2.3, 2.7, 2.13
- Basic Problems 2.24, 2.26
- Q&A

Prior knowledge review

- Basic knowledge on signal computation
- Exponential signals: Euler's relation, periodic, integral
- CT/DT unit impulse/step function
- System Properties
  1. Memoryless or with memory
  2. Causality
  3. Invertibility
  4. Stability
  5. Time-invariance
  6. Linearity

Basic knowledge review

- CT/DT LTI systems
- Convolution operation procedure
  1. **Figure computation** based on “Flip-slide-multiply-sum/integral”
  2. Some known or **typical convolution results**
  3. **Properties** of convolution
- Unit impulse response and properties of LTI systems
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Example 1

- $x[n]=a^n u[n]$
- $h[n]=u[n]$
- $y[n]=x[n]*h[n]$?

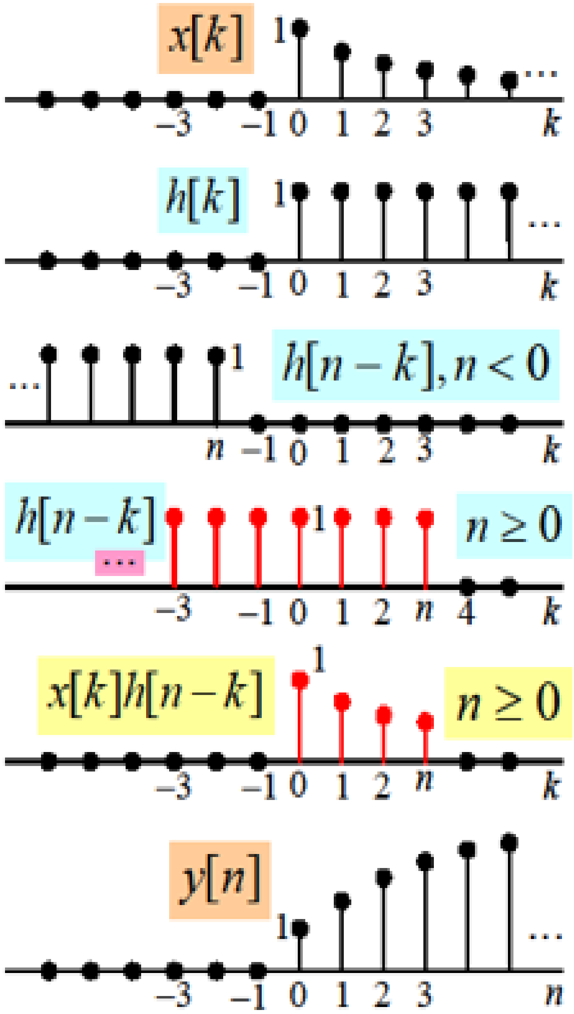
$$y[n]=\begin{cases}\dfrac{1-a^{n+1}}{1-a}, & n\geq 0\\ 0, & n<0\end{cases}$$

Flip

Slide

Multiply

Sum



$$h[n-k]$$

$$x[k]h[n-k]=\begin{cases}a^k, & 0\leq k\leq n\\ 0, & k<0, k>n\end{cases}$$

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=0}^{n}a^k=\frac{1-a^{n+1}}{1-a}$$

作业题 2.21 (c) 参考解答.

$x[n] = (\frac{1}{2})^n u[n-4] \qquad h[n] = 4^n u[2-n]$

注意这里!

$$x[n] * h[n] = \sum_{k=-\infty}^{+\infty} (\tfrac{1}{2})^k u[k-4] \cdot 4^{n-k} u[k-n+2]$$

$u[n]$ 这个信号具有筛选区间的作用，但不会改变信号的值.

故 $u[k-4] \longrightarrow k \geq 4$ ， $u[k-n+2] \longrightarrow k \geq n-2$

综上： $k \geq \max\{4, n-2\}$. k取其它值，卷积未遂 结果为0

$$\therefore x[n] * h[n] = \begin{cases} \sum_{k=4}^{+\infty} (\frac{1}{2})^k 4^{n-k}, & 4 \geq n-2 \Rightarrow n \leq 6 \\ \sum_{k=n-2}^{+\infty} (-\frac{1}{2})^k 4^{n-k}, & 4 < n-2 \Rightarrow n > 6 \end{cases}$$

① $\sum_{k=4}^{+\infty} \left(-\frac{1}{2}\right)^k 4^{n-k}$, $n \le 6$

$= \sum_{k=4}^{+\infty} \left(-\frac{1}{8}\right)^k \cdot 4^n$ （关于k求和，与n无关）

$= \left\{ \underbrace{\sum_{k=0}^{+\infty} \left(-\frac{1}{8}\right)^k - \sum_{k=0}^{3} \left(-\frac{1}{8}\right)^k}_{\text{等比数列求和}} \right\} \cdot 4^n$ （将4~+∞拆成两个区间 即(0~+∞)−(0,3)）

$= \left\{ \frac{1}{1+\frac{1}{8}} - \frac{1-\left(-\frac{1}{8}\right)^4}{1+\frac{1}{8}} \right\} \cdot 4^n = \boxed{\frac{8}{9}\left(-\frac{1}{8}\right)^4 \cdot 4^n, \ n \le 6}$

② $\sum_{k=n-2}^{+\infty} \left(-\frac{1}{2}\right)^k 4^{n-k}$, $n > 6$

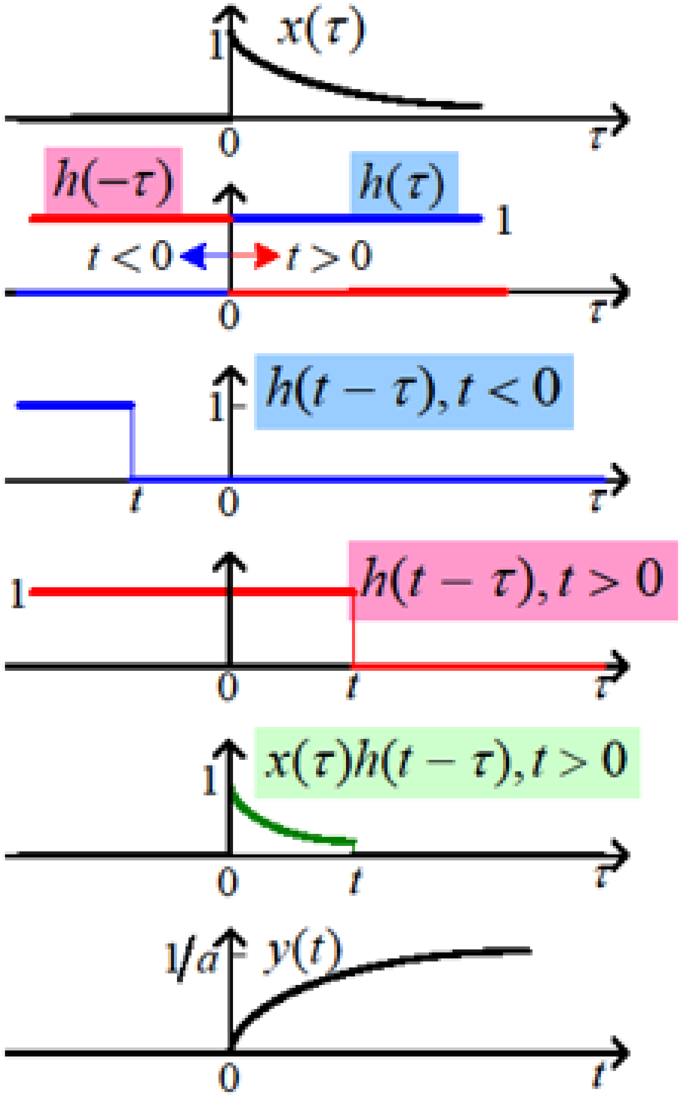
$= \left\{ \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k \cdot 4^{-k} - \sum_{k=0}^{n-1} \left(-\frac{1}{2}\right)^k 4^{-k} \right\} 4^n$

$= \left\{ \frac{1}{1+\frac{1}{8}} - \frac{1-\left(-\frac{1}{8}\right)^n}{1+\frac{1}{8}} \right\} \cdot 4^n = \boxed{\frac{8}{9}\left(-\frac{1}{8}\right)^n \cdot 4^n, \ n > 6}$

另外，大家学习完Z变换后可以重做一下这题！

如解答有误，请联系lush2021@mail.sustech.edu.cn by TA 卢仕航

Example 2



$$x(t) = e^{-at}u(t)$$
$$h(t) = u(t)$$
$$y(t) = x(t) * h(t)$$

$$y(t) = \begin{cases} \dfrac{1-e^{-at}}{a}, & t \geq 0 \\ 0, & t < 0 \end{cases} \Rightarrow \frac{1-e^{-at}}{a}u(t)$$

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表 3.4 基本信号的卷积表

连续时间卷积积分			离散时间卷积和		
$x(t)$	$h(t)$	$x(t)*h(t)$	$x[n]$	$h[n]$	$x[n]*h[n]$
$x(t)$	$\delta(t)$	$x(t)$	$x[n]$	$\delta[n]$	$x[n]$
$x(t)$	$u(t)$	$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau$	$x[n]$	$u[n]$	$\sum_{k=-\infty}^{n} x[k]$
$x(t)$	$\delta'(t)$	$x'(t)$	$x[n]$	$\Delta\delta[n]$	$x[n]-x[n-1]$
$u(t)$	$u(t)$	$tu(t)$	$u[n]$	$u[n]$	$(n+1)u[n]$
$\mathrm{e}^{-at}u(t)$	$u(t)$	$\frac{1-\mathrm{e}^{-at}}{a}u(t)$	$a^n u[n]$	$u[n]$	$\frac{1-a^{n+1}}{1-a}u[n]$
$\sin(\omega t)u(t)$	$u(t)$	$\frac{1-\cos(\omega t)}{\omega}u(t)$	$\sin(\Omega n)u[n]$	$u[n]$	
$\cos(\omega t)u(t)$	$u(t)$	$\frac{\sin(\omega t)}{\omega}u(t)$	$\cos(\Omega n)u[n]$	$u[n]$	
$\mathrm{e}^{-at}u(t)$	$\mathrm{e}^{-at}u(t)$	$t\mathrm{e}^{-at}u(t)$	$a^n u[n]$	$a^n u[n]$	$(n+1)a^n u[n]$
$\mathrm{e}^{-at}u(t)$	$\mathrm{e}^{-bt}u(t)$	$\frac{\mathrm{e}^{-at}-\mathrm{e}^{-bt}}{b-a}u(t)$	$a^n u[n]$	$b^n u[n]$	$\frac{b^{n+1}-a^{n+1}}{b-a}u[n]$

说明：表 3.4 中空着的卷积和运算结果，感兴趣的读者可自行补上。

[1] 徐守时等著，中国科学技术大学出版社，信号与系统·第三版，第78页

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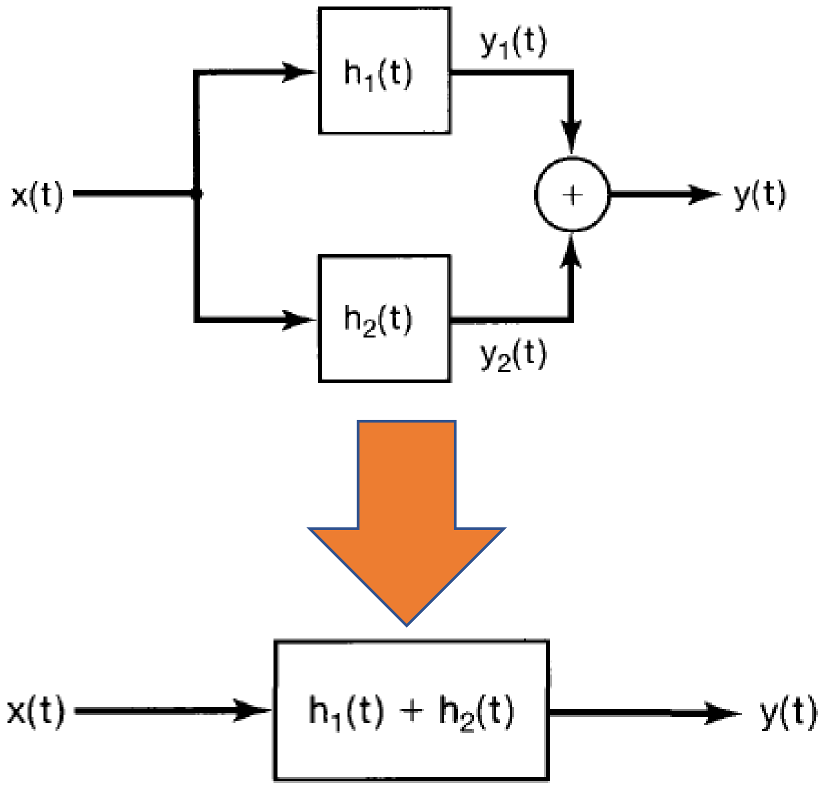
Commutative property

$$x(t)*h(t)=h(t)*x(t)$$

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$x[n]*h[n]=h[n]*x[n]$$

$$\sum_{m=-\infty}^{\infty} x[m]h[n-m]=\sum_{m=-\infty}^{\infty} h[m]x[n-m]$$



Distributive property

$$x(t)*\left[h_1(t)+h_2(t)\right]=x(t)*h_1(t)+x(t)*h_2(t)$$

$$x[n]*\left\{h_1[n]+h_2[n]\right\}=x[n]*h_1[n]+x[n]*h_2[n]$$

Properties of convolution

☐ Associative property

$$\left[x(t)*h_1(t)\right]*h_2(t)=x(t)*\left[h_1(t)*h_2(t)\right]$$

$$\left\{x[n]*h_1[n]\right\}*h_2[n]=x[n]*\left\{h_1[n]*h_2[n]\right\}$$

☐ Time-invariant property （Collect time shift）

$$y(t)=x(t)*h(t)$$

$$x(t)*h\left(t-t_0\right)=y\left(t-t_0\right)$$

$$\boxed{x\left(t-t_1\right)*h\left(t-t_2\right)=y\left(t-t_1-t_2\right)}$$

$$x[n]*h[n]=y[n]$$

$$x[n]*h[n-m]=y[n-m]$$

$$\boxed{x\left[n-m_1\right]*h\left[n-m_2\right]=y\left[n-m_1-m_2\right]}$$

Difference property

$$\boxed{\frac{d}{dt}\left[x(t)*h(t)\right]=x(t)*\frac{dh(t)}{dt}=\frac{dx(t)}{dt}*h(t)=\frac{dy(t)}{dt}}$$

$$\nabla\left\{x[n]*h[n]\right\}=\nabla x[n]*h[n]=x[n]*\nabla h[n]=\nabla y[n]$$

Integral property

$$\boxed{\int_{-\infty}^{t}\left[x(\tau)*h(\tau)\right]d\tau=x(t)*\int_{-\infty}^{t}h(\tau)d\tau=\int_{-\infty}^{t}x(\tau)d\tau*h(t)=\int_{-\infty}^{t}y(\tau)d\tau}$$

$$\sum_{k=-\infty}^{n}\left\{x[k]*h[k]\right\}=x[n]*\left\{\sum_{k=-\infty}^{n}h[k]\right\}=\left\{\sum_{k=-\infty}^{n}x[k]\right\}*h[n]=\sum_{k=-\infty}^{n}y[k]$$

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- For unit impulse/step signal
- More unit impulse/step signals, more simple

$$x(t)*\delta(t)=x(t)$$

$$x(t)*\delta\left(t-t_0\right)=x\left(t-t_0\right)$$

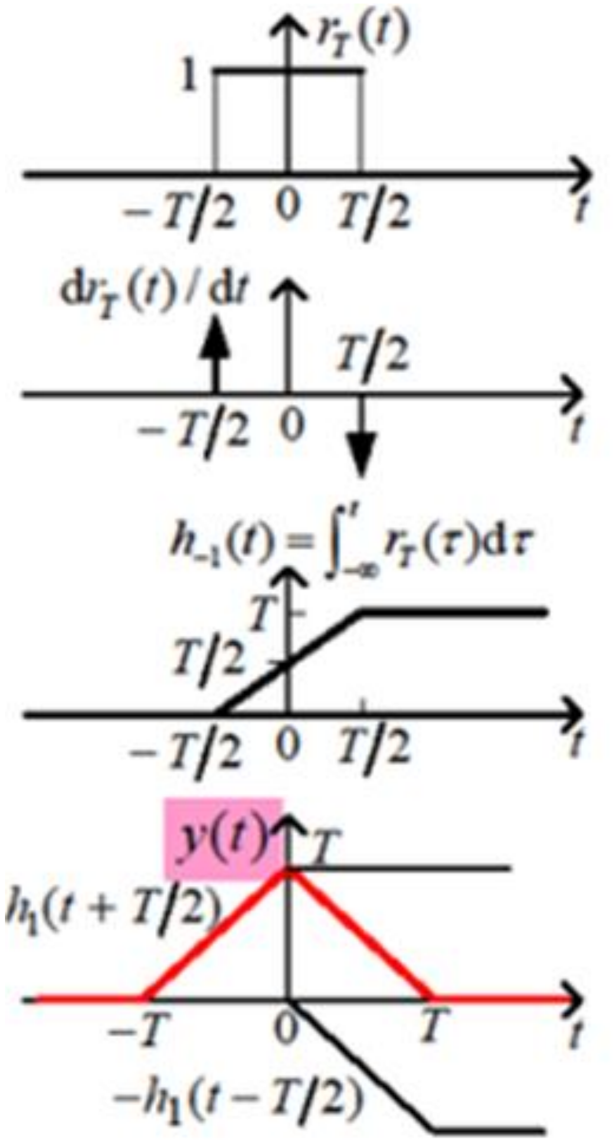
$$x(t)*u(t)=\int_{-\infty}^{t}x(\tau)d\tau$$

$$u(t)=\int_{-\infty}^{t}\delta(\tau)d\tau$$

$$x[n]*\delta[n]=x[n]$$

$$x[n]*\delta[n-m]=x[n-m]$$

$$x[n]*u[n]=\sum_{m=-\infty}^{n}x[m]$$



$$y(t)=r_T(t)^* r_T(t)=\frac{d}{dt}r_T(t)^*\int_{-\infty}^{t} r_T(\tau)d\tau$$

1. **Figure computation** based on "Flip-slide-multiply-sum/integral"
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$$h_{-1}(t)=\int_{-\infty}^{t} r_T(\tau)d\tau$$

$$\begin{aligned}y(t)&=[\delta(t+T/2)-\delta(t-T/2)]*h_{-1}(t)\\&=h_{-1}(t+T/2)-h_{-1}(t-T/2)\end{aligned}$$

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2.3. Consider an input $x[n]$ and a unit impulse response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2],$$

$$h[n] = u[n+2].$$

Determine and plot the output $y[n] = x[n] * h[n]$.

2.3. Let us define the signals

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$$

and

$$h_1[n] = u[n].$$

We note that

$$x[n] = x_1[n-2] \quad \text{and} \quad h[n] = h_1[n+2]$$

Now,

$$\begin{aligned} y[n] &= x[n] * h[n] = x_1[n-2] * h_1[n+2] \\ &= \sum_{k=-\infty}^{\infty} x_1[k-2]h_1[n-k+2] \end{aligned}$$

By replacing k with $m+2$ in the abovr summation, we obtain

$$y[n] = \sum_{m=-\infty}^{\infty} x_1[m]h_1[n-m] = x_1[n] * h_1[n]$$

Using the results of Example 2.1 in the text book, we may write

$$y[n] = 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

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2.7. A linear system S has the relationship

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k]$$

between its input $x[n]$ and its output $y[n]$, where $g[n] = u[n] - u[n-4]$.

(a) Determine $y[n]$ when $x[n] = \delta[n-1]$.
(b) Determine $y[n]$ when $x[n] = \delta[n-2]$.
(c) Is S LTI?
(d) Determine $y[n]$ when $x[n] = u[n]$.

2.7. (a) Given that

$$x[n] = \delta[n-1],$$

we see that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k] = g[n-2] = u[n-2] - u[n-6]$$

(b) Given that

$$x[n] = \delta[n-2],$$

we see that

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]g[n-2k] = g[n-4] = u[n-4] - u[n-8]$$

(c) The input to the system in part (b) is the same as the input in part (a) shifted by 1 to the right. If S is time invariant then the system output obtained in part (b) has to the be the same as the system output obtained in part (a) shifted by 1 to the right. Clealry, this is not the case. Therefore, the system is not LTI.

(d) If $x[n] = u[n]$, then

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]g[n-2k] \\ &= \sum_{k=0}^{\infty} g[n-2k] \end{aligned}$$

The signal $g[n-2k]$ is plotted for $k = 0, 1, 2$ in Figure S2.7. From this figure it is clear that

$$y[n] = \begin{cases} 1, & n = 0, 1 \\ 2, & n > 1 \\ 0, & \text{otherwise} \end{cases} = 2u[n] - \delta[n] - \delta[n-1]$$

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2.13. Consider a discrete-time system S_1 with impulse response

$$h[n] = \left(\frac{1}{5}\right)^n u[n].$$

(a) Find the integer A such that $h[n] - Ah[n-1] = \delta[n]$.

(b) Using the result from part (a), determine the impulse response $g[n]$ of an LTI system S_2 which is the inverse system of S_1.

2.13. (a) We require that

$$\left(\frac{1}{5}\right)^{n} u[n] - A\left(\frac{1}{5}\right)^{(n-1)} u[n-1] = \delta[n]$$

Putting $n = 1$ and solving for A gives $A = \frac{1}{5}$.

(b) From part (a), we know that

$$\begin{aligned} h[n] - \frac{1}{5}h[n-1] &= \delta[n] \\ h[n] * \left(\delta[n] - \frac{1}{5}\delta[n-1]\right) &= \delta[n] \end{aligned}$$

From the definition of an inverse system, we may argue that

$$g[n] = \delta[n] - \frac{1}{5}\delta[n-1].$$

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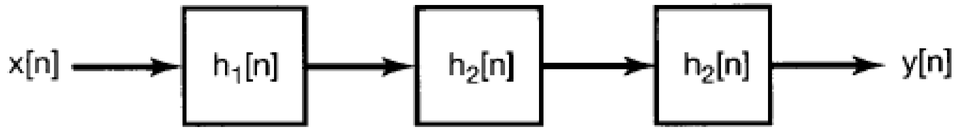
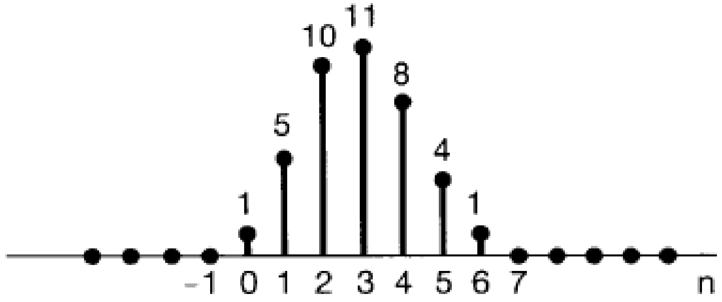
2.24. Consider the cascade interconnection of three causal LTI systems, illustrated in Figure P2.24(a). The impulse response $h_2[n]$ is

$$h_2[n] = u[n] - u[n-2], \longrightarrow \delta[n]+\delta[n-1]$$

and the overall impulse response is as shown in Figure P2.24(b).



(a)



(b)

Figure P2.24

(a) Find the impulse response $h_1[n]$.

(b) Find the response of the overall system to the input

$$x[n] = \delta[n] - \delta[n-1].$$

2.24. (a) We are given that $h_2[n] = \delta[n] + \delta[n-1]$. Therefore,

$$h_2[n] * h_2[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$$

Since

$$h[n] = h_1[n] * [h_2[n] * h_2[n]],$$

we get

$$h[n] = h_1[n] + 2h_1[n-1] + h_1[n-2].$$

Therefore,

$$\begin{aligned}
h[0] &= h_1[0] & \Rightarrow \quad & h_1[0] = 1, \\
h[1] &= h_1[1] + 2h_1[0] & \Rightarrow \quad & h_1[1] = 3, \\
h[2] &= h_1[2] + 2h_1[1] + h_1[0] & \Rightarrow \quad & h_1[2] = 3, \\
h[3] &= h_1[3] + 2h_1[2] + h_1[1] & \Rightarrow \quad & h_1[3] = 2, \\
h[4] &= h_1[4] + 2h_1[3] + h_1[2] & \Rightarrow \quad & h_1[4] = 1 \\
h[5] &= h_1[5] + 2h_1[4] + h_1[3] & \Rightarrow \quad & h_1[5] = 0.
\end{aligned}$$

$h_1[n] = 0$ for $n < 0$ and $n \geq 5$.

(b) In this case,

$$y[n] = x[n] * h[n] = h[n] - h[n-1].$$

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2.26. Consider the evaluation of

$$y[n] = x_1[n] * x_2[n] * x_3[n],$$

where $x_1[n] = (0.5)^n u[n]$, $x_2[n] = u[n+3]$, and $x_3[n] = \delta[n] - \delta[n-1]$.

(a) Evaluate the convolution $x_1[n] * x_2[n]$.

(b) Convolve the result of part (a) with $x_3[n]$ in order to evaluate $y[n]$.

(c) Evaluate the convolution $x_2[n] * x_3[n]$.

(d) Convolve the result of part (c) with $x_1[n]$ in order to evaluate $y[n]$.

2.26. (a) We have

$$y_1[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$
$$= \sum_{k=0}^{\infty}(0.5)^k u[n+3-k].$$

This evaluates to

$$y_1[n] = x_1[n] * x_2[n] = \begin{cases} 2\left\{1-(1/2)^{n+4}\right\}, & n \geq -3 \\ 0, & \text{otherwise} \end{cases}.$$

(b) Now,

$$y[n] = x_3[n] * y_1[n] = y_1[n] - y_1[n-1].$$

Therefore,

$$y[n] = \begin{cases} 2\left\{1-(1/2)^{n+3}\right\} + 2\left\{1-(1/2)^{n+4}\right\} = (1/2)^{n+3}, & n \geq -2 \\ 1, & n = -3 \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, $y[n] = (1/2)^{n+3}u[n+3]$.

(c) We have

$$y_2[n] = x_2[n] * x_3[n] = u[n+3] - u[n+2] = \delta[n+3].$$

(d) From the result of part (c), we get

$$y[n] = y_2[n] * x_1[n] = x_1[n+3] = (1/2)^{n+3}u[n+3].$$

- **1教111，周一至周四**
- **21:00-22:00, in 11/12/13/14 Oct**
- Review
- Basic Problems with Answers 2.20
- Basic Problems 2.29
- Advanced Problems 2.40, 2.43, 2.47

Thanks for Your Attendance

Q&A

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