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CT Fourier Series Pairs

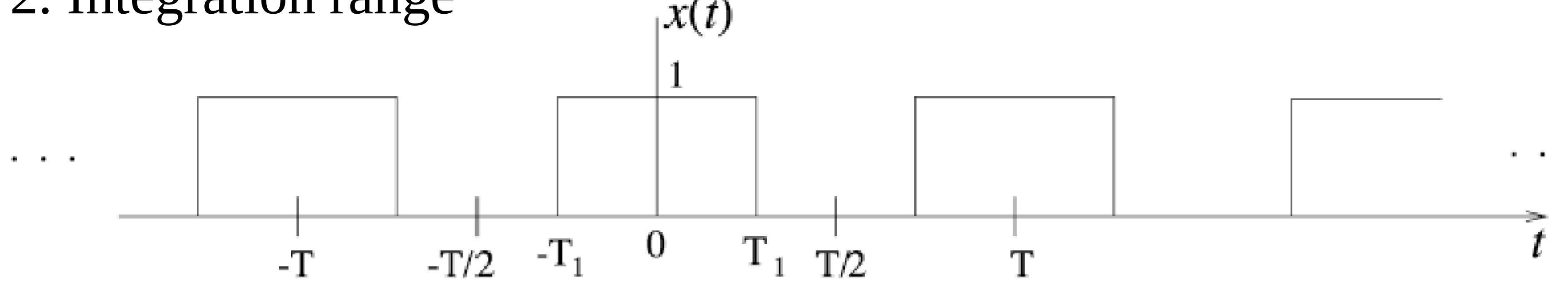
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{j2\pi kt/T}$$

Harmonically related

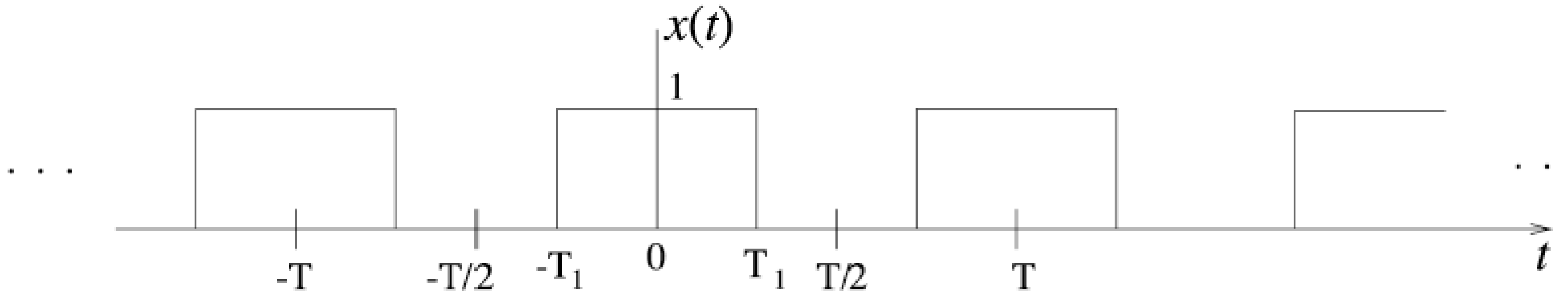
$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt$$

1. $\omega_0 = \dfrac{2\pi}{T}$?

2. Integration range



Example 3.5: Periodic Square Wave



$$a_o = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{2T_1}{T}$$

$k \neq 0$

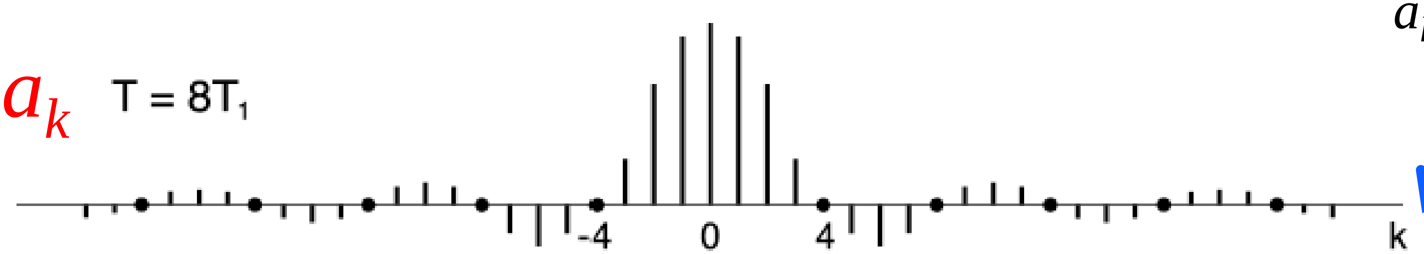
$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt$$

$(\omega_o = \frac{2\pi}{T})$:

$$= -\frac{1}{jk\omega_o T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{\sin(k\omega_o T_1)}{k\pi}$$

a_k: a function of k

a_k $T = 8T_1$



CT Fourier Series Property

- Linearity $x(t) \longleftrightarrow a_k,\ y(t) \longleftrightarrow b_k, \Rightarrow \alpha x(t) + \beta y(t) \longleftrightarrow \alpha a_k + \beta b_k$
- Conjugate Symmetry

$$x(t) \quad real \quad \Rightarrow \quad a_{-k} = a_k^*$$

Proof: $$a_{-k} = \frac{1}{T}\int_T x(t)e^{jk\omega_o t}dt = \left[\frac{1}{T}\int_T x^*(t)e^{-jk\omega_o t}dt\right]^* = a_k^*$$

$$\Downarrow \quad a_k = \mathrm{Re}\{a_k\} + j\,\mathrm{Im}\{a_k\}$$

$$\boxed{\mathrm{Re}\{a_{-k}\} + j\,\mathrm{Im}\{a_{-k}\}} \qquad = |a_k|e^{j\angle a_k} \qquad \boxed{\mathrm{Re}\{a_k\} - j\,\mathrm{Im}\{a_k\}}$$

$\mathrm{Re}\{a_k\}$ is even , $\mathrm{Im}\{a_k\}$ is odd

or $|a_k|$ is even, $\angle a_k$ is odd

- Time shift $x(t) \longleftrightarrow a_k$

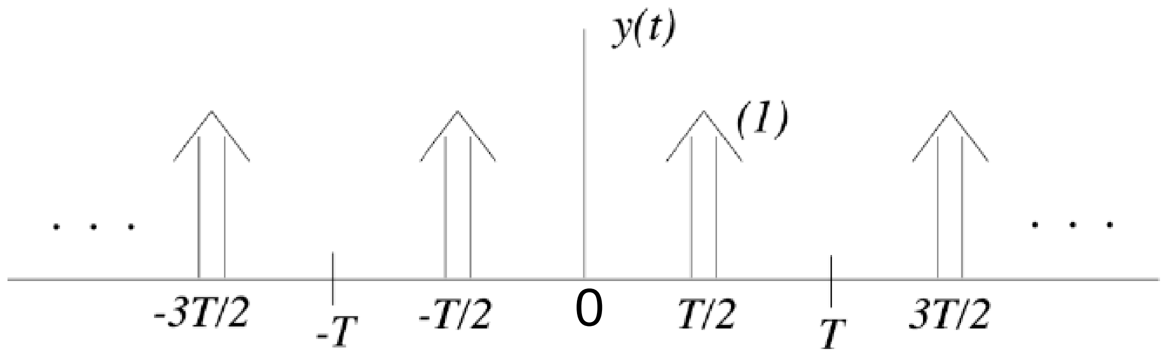
$$x(t - t_o) \longleftrightarrow a_k e^{-jk\omega_o t_o} = a_k e^{-jk2\pi t_o/T}$$

Introduce a linear phase shift $\propto t_o$

Example: Shift by Half Period

what is the Fourier Series of $\sum_{n=-\infty}^{+\infty} \delta(t - nT))$?

$$y(t) = x(t - T/2) \xleftrightarrow{e^{-jk\omega_0 T/2} = e^{-jk\pi}} a_k e^{-jk\pi} = (-1)^k a_k$$



$$y(t) \longleftrightarrow (-1)^k a_k \qquad \left(a_k = \frac{1}{T} = \text{F.C. of } \sum_{n=-\infty}^{+\infty} \delta(t - nT)\right)$$

$$\|$$

$$\frac{(-1)^k}{T}$$

Review

- Time Reversal

$$x(-t) \xleftrightarrow{FS} a_{-k}$$

the effect of sign change for $x(t)$ and a_k are identical

Example: $x(t)$: … a_{-2} a_{-1} a_0 a_1 a_2 …

$x(-t)$: … a_2 a_1 a_0 a_{-1} a_{-2} …

- Time Scaling

α : positive real number

$x(\alpha t)$: periodic with period T/α and fundamental frequency αw_0

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha w_0)t}$$

a_k unchanged, but $x(\alpha t)$ and each harmonic component are different

Cont.

- **How about *x*(t) is real, and even?**

$$x(t) = x(-t)$$

$$x(-t) \xleftrightarrow{FS} a_{-k} \Rightarrow a_k = a_{-k}$$

x(t) is real $\Rightarrow$ $a_{-k} = a_k^*$, $\text{Re}\{a_k\}$ is even, $\text{Im}\{a_k\}$ is odd $\Rightarrow$ $\text{Im}\{a_k\}$ is 0

or a_k is real, even

- **How about *x*(t) is real, and odd?**

$a_k = -a_k^*$

$$x(t) = -x(-t)$$

$$x(-t) \xleftrightarrow{FS} a_{-k} \Rightarrow a_k = -a_{-k} \Rightarrow$$

$\text{Re}\{a_k\}$ is 0

or a_k is imaginary, odd

- **Multiplication Property**

$$x(t) \longleftrightarrow a_k\ ,\ y(t) \longleftrightarrow b_k$$ (Both $x(t)$ and $y(t)$ are periodic with the same period T)

$$\Downarrow$$

$$x(t) \cdot y(t) \quad \longleftrightarrow \quad c_k = \sum_{l=-\infty}^{+\infty} a_l b_{k-l} = a_k * b_k$$

Proof: $$\underbrace{\sum_l a_l e^{jl\omega_o t}}_{x(t)} \cdot \underbrace{\sum_m b_m e^{jm\omega_o t}}_{y(t)} = \sum_{l,m} a_l b_m e^{j(l+m)\omega_o t} \xrightarrow{l+m=k} \sum_k \underbrace{\left[\sum_l a_l b_{k-l}\right]}_{c_k} e^{jk\omega_o t}$$

- **Parseval Relation**

$$\underbrace{\frac{1}{T}\int_T |x(t)|^2\,dt}_{\text{Average signal power}} = \sum_{k=-\infty}^{+\infty} \underbrace{|a_k|^2}_{\substack{\text{Power in the} \\ k_{th}\text{ harmonic}}}$$

Power is the same whether measured in the time-domain or the frequency-domain

More

Note:
$x(t)$ is periodic with fundamental frequency ω_0

$$x(t) \longleftrightarrow a_k$$

- **Frequency shifting**

$$e^{jM\omega_0 t} x(t) \longleftrightarrow a_{k-M}$$

- **Differentiation**

$$\frac{dx}{dt} \longleftrightarrow jk\omega_0 a_k$$

- **Integration**

$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow \left(\frac{1}{jk\omega_0}\right)a_k$$

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\Re e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example 3.9

Suppose we are given the following facts about a signal x(t)

1. x(t) is a real signal
2. x(t) is periodic with period T=4, and it has Fourier series coefficients a_k
3. a_k=0 for |k|>1
4. The signal with Fourier coefficients

$$b_k = e^{-j\pi k/2} a_{-k} \quad \text{is odd}$$

5. $$\frac{1}{4}\int_4 |x(t)|^2 \, dt = 1/2$$

1. $x(t)$ is a real signal
2. $x(t)$ is periodic with period T=4, and it has Fourier series coefficients a_k
3. a_k =0 for |k|>1
4. The signal with Fourier coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$

From 2), $\omega_0=2\pi/T=\pi/2$
From 3), $x(t)=a_0+a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}$

From 1), $a_{-k} = a_k^*$

From 4) and 1), suppose $y(t) \longleftrightarrow b_k$, $y(t)$ is a real and odd signal

∴ b_k is imaginary, and odd

∴ b_0 =0, b_1=-b_{-1} , and b_k =0 for |k|>1 (why?)

∴ a_0 =0, $a_1 = e^{-j\pi/2} b_{-1} = -jb_{-1} = jb_1$

$a_{-1}=(a_1)^* = -jb_1$

From 5), $|a_1|^2 + |a_{-1}|^2 = 1/2$

∴ b_1=j/2 or –j/2

Review

Periodicity Properties of DT Complex Exponentials

- $x[n]$ - periodic with fundamental period N, fundamental frequency

$$x[n+N] = x[n] \qquad and \qquad \omega_o = \frac{2\pi}{N}$$

$$n = \ldots, -1, 0, 1, 2, 3, \ldots$$

- For DT complex exponentials, signal are periodic only when

$$\omega_0 N = k \cdot 2\pi, \qquad k = 0, \pm 1, \pm 2, \cdots$$

$$\boxed{e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1 \rightarrow \omega_0 N = k \cdot 2\pi}$$

- For DT complex exponentials, signals with frequencies ω_0 and $\omega_0 + k \cdot 2\pi$ are identical.

$$\boxed{e^{j(\omega_0 + k \cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jk \cdot 2\pi n} = e^{j\omega_0 n}}$$

  - We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

Review

Cont.

- $e^{j\omega_0 n}$ does ***not*** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

  low-frequency (slowly varying): ω_0 near 0, 2π, …, or 2k • π

  high-frequency (rapid variation): ω_0 near ± π, …, or (2k+1) • π

$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

Cont.

- Harmonically-related **DT** complex exponential set

$$\{\phi_k[n] = e^{jk(\frac{2\pi}{N})n} \quad , \qquad \text{k} = 0, \pm 1, \pm 2, \dots\dots\}$$

with common period N, i.e. $\phi_{k+N}[n] = \phi_k[n]$

$$\boxed{\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} \cdot e^{j2\pi n} = e^{jk(\frac{2\pi}{N})n} = \phi_k[n]}$$

or there are only N distinct signals in the set

- So we *could* just use $e^{j0\cdot\omega_0 n}, e^{j1\cdot\omega_0 n}, e^{j2\cdot\omega_0 n}, \dots, e^{j(N-1)\cdot\omega_0 n}$,k=0, …, N-1
- However, it is often useful to allow the choice of N consecutive values of k to be *arbitrary* (*E.g.* choose {-(N-1)/2, (N-1)/2} if N is odd and $x[n]$ has definite parity).

DT Fourier Series Representation

$$x[n] = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$\sum_{k=<N>}$ = Sum over *any* N consecutive values of k

— This is a *finite* series

$\{a_k\}$ - Fourier (series) coefficients

Questions:

1) What DT periodic signals have such a representation?

2) How do we find a_k?

Answer to Question #1:

Any DT periodic signal has a Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow$$

n=0 $$x[0] = \sum_{k=<N>} a_k$$

n=1 $$x[1] = \sum_{k=<N>} a_k e^{jk\omega_0}$$

n=2 $$x[2] = \sum_{k=<N>} a_k e^{j2k\omega_0}$$

.

.

$$\vdots$$

n=N-1 $$x[N-1] = \sum_{k=<N>} a_k e^{j(N-1)k\omega_0}$$

N equations for N unknowns, a_0, a_1, … a_{N-1}

Answer to Question #2: A More Direct Way to Solve for a_k

Finite geometric series

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & , \quad \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & , \quad \alpha \neq 1 \end{cases}$$

$$\Downarrow \boxed{\alpha = e^{jk\omega_0} = e^{jk2\pi/N}}$$

$$\sum_{n=\langle N\rangle} e^{jk\omega_0 n} = \sum_{n=0}^{N-1} (e^{jk\omega_0})^n = \sum_{n=0}^{N-1} (e^{jk2\pi/N})^n$$

$$= \begin{cases} N & , \quad k = 0, \pm N, \pm 2N \\ \dfrac{1-e^{j\overset{k2\pi}{k\omega_o N}}}{1-e^{jk\omega_o}} = 0 & , \quad \text{otherwise} \end{cases}$$

$$\omega_o = \frac{2\pi}{N}$$

$$= N\delta[k - mN]$$

So, from

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}$$

$$\Downarrow \sum_{n=\langle N \rangle} e^{-jm\omega_0 n} \text{ to both sides}$$

— Taking an "inner product" of $x[n]$ and $e^{-jm\omega_0 n}$

$$\sum_{n=\langle N \rangle} x[n] e^{-jm\omega_0 n} = \sum_{n=<N>} \left(\sum_{k=<N>} a_k e^{jk\omega_0 n} \right) e^{-jm\omega_0 n}$$

$$= \sum_{k=<N>} a_k \underbrace{\left(\sum_{n=<N>} e^{j(k-m)\omega_0 n} \right)}_{\Vert}$$

$N\delta[k-m]$ — orthogonality

$k = m \rightarrow N$

$k \neq m \rightarrow 0$

$$= N a_m$$

$$a_m = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jm\omega_0 n}$$

DT Fourier Series Pair $\left(\omega_o = \frac{2\pi}{N}\right)$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} \qquad \text{(Analysis equation)}$$

Different from CT Fourier series

Cont.

- We only use N consecutive values of a_k in the synthesis equation.

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} \qquad \text{(Synthesis equation)}$$

- a_k can be defined for all integers k, and we have $\boxed{a_{k+N} = a_k}$

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \cdots + a_{N-1}\phi_{N-1}[n]$$

$$x[n] = a_1\phi_1[n] + \cdots + a_{N-1}\phi_{N-1}[n] + a_N\phi_N[n]$$

  - Since $x[n] = x[n+N]$ and $a_{k+N} = a_k$, there are only N pieces of information, whether in the time-domain or the coefficient-domain.
  - Unique for DT

Example #1: Sum of a pair of sinusoids

$$x[n] = \cos(\pi n/8) + \cos(\pi n/4 + \pi/4)$$

— periodic with period N = ?

$$x[n] = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\pi/4}e^{j2\omega_0 n} + e^{-j\pi/4}e^{-j2\omega_0 n}\right]$$

$$\Downarrow$$

$a_0 = 0$

$a_1 = 1/2$

$a_{-1} = 1/2$

$a_2 = e^{j\pi/4}/2$

$a_{-2} = e^{-j\pi/4}/2$

$a_3 = 0$

$a_{-3} = 0$

$\vdots$

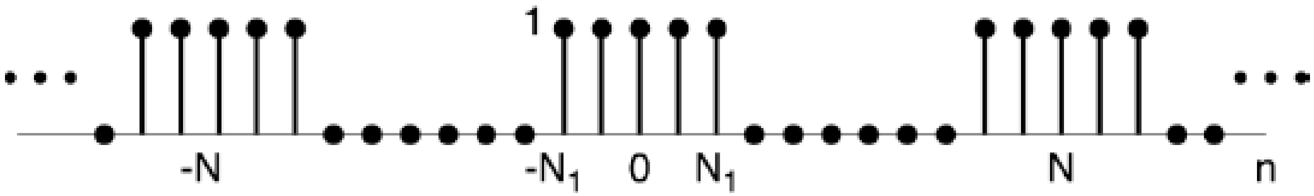
$a_{15} = a_{-1+16} = a_{-1} = 1/2$

$a_{66} = a_{2+4\times 16} = a_2 = e^{j\pi/4}/2$

$$\boxed{\begin{aligned}\cos(x) &= \mathrm{Re}(e^{jx}) = \frac{1}{2}(e^{jx} + e^{-jx})\\ \sin(x) &= \mathrm{Im}(e^{jx}) = \frac{1}{2}(e^{jx} - e^{-jx})\end{aligned}}$$

Example 3.12 DT Square wave

Period=?



$$a_0 = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n] = \frac{(2N_1+1)}{N} = a_N = a_{-N} = a_{6N} = \cdots$$

For $k \neq$ multiple of N:

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\omega_0 n} \overset{n=m-N_1}{=} \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk\omega_0 (m-N_1)}$$

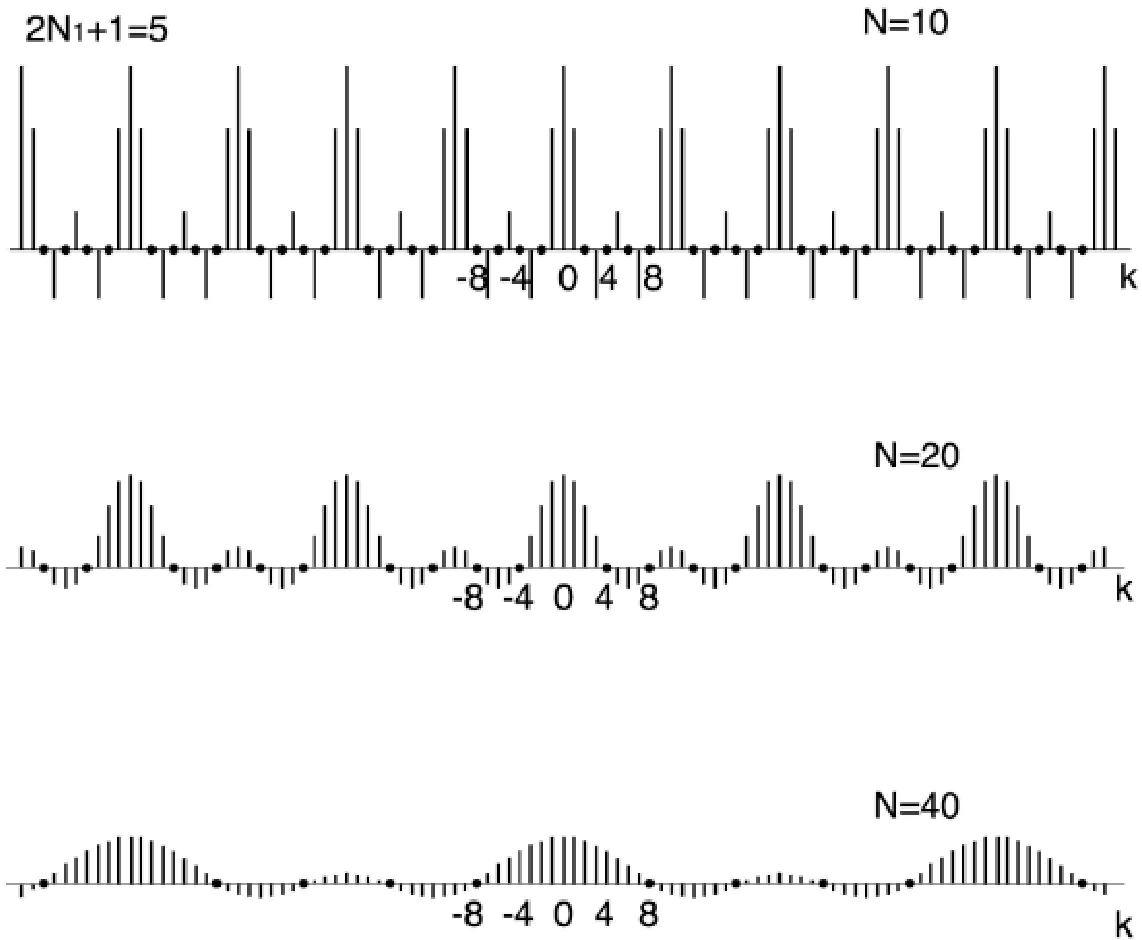
$$e^{jk\omega_0 N_1} = \cos(k\omega_0 N_1) + j\sin(k\omega_0 N_1)$$

$$= \frac{1}{N} e^{jk\omega_0 N_1}\sum_{m=0}^{2N_1} (e^{-jk\omega_0})^m = \frac{1}{N} e^{jk\omega_0 N_1} \frac{1-e^{-jk\omega_0(2N_1+1)}}{1-e^{jk\omega_0}}$$

$$= \frac{1}{N}\frac{\sin\left[k(N_1+1/2)\omega_0\right]}{\sin(k\omega_0/2)} = \frac{1}{N}\frac{\sin\left[2\pi k(N_1+1/2)/N\right]}{\sin(\pi k/N)}$$

DT Square wave (continued)

$$a_k = \frac{1}{N} \frac{\sin\left[2\pi k(N_1 + 1/2)/N\right]}{\sin(\pi k/N)}$$



DT Fourier Series - Convergence

- Not an issue, since all series are finite sums.

– $x[n]$ has only N parameters, represented by N coefficients

sum of N terms gives the exact value

– N odd

$$x[n]_M = \sum_{k=-M}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n]_M = x[n], \text{ if } \; M = \frac{(N-1)}{2}$$

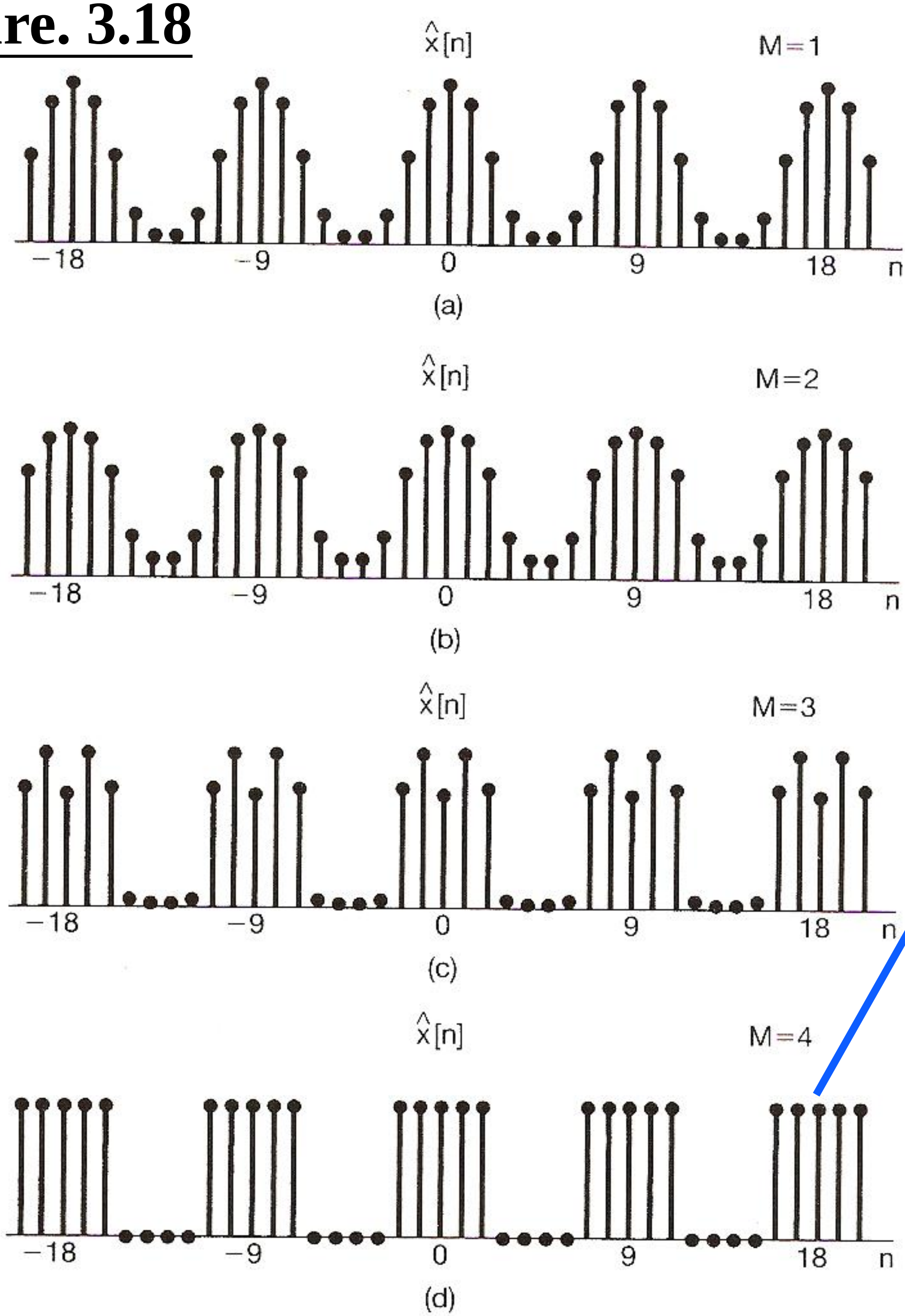
– N even

$$x[n]_M = \sum_{k=-M+1}^{M} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$x[n]_M = x[n], \text{ if } \; M = \frac{N}{2}$$

See Fig. 3.18

Figure. 3.18



N=9, 2N$_1$+1=5

1) The same as original DT square wave

2) No Gibbs phenomenon, and no discontinuity

Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

DT Fourier Series - Property

- Strong similarities between the properties of DT and CT Fourier series [Comparing Table 3.2 to Table 3.1]

Example: $x[n] \longleftrightarrow a_k$

$e^{jM\omega_0 n}x[n] \longleftrightarrow b_k = ?$

$$x[n]e^{jM\omega_0 n} = \sum_{r=<N>} a_r e^{jr\omega_0 n} e^{jM\omega_0 n}$$

$$\overset{k=r+M}{=} \sum_{k=<N>} a_{k-M} e^{jk\omega_0 n}$$

$$\Downarrow$$

$$b_k = a_{k-M}$$

Frequency shift

$jM\omega_0, \rightarrow j(k\text{-}M)\omega_o$

Review Complex Exponentials - The *Only* Eigenfunctions of *Any* LTI Systems

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= \left[\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau\right] e^{st}$$

$$= \underbrace{H(s)}_{\text{eigenvalue}} \underbrace{e^{st}}_{\text{eigenfunction}}$$

$$\boxed{y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau) x(t-\tau) d\tau}$$

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{m=-\infty}^{\infty} h[m] z^{n-m}$$

$$= \left[\sum_{m=-\infty}^{\infty} h[m] z^{-m}\right] z^n$$

$$= \underbrace{H(z)}_{\text{eigenvalue}} \underbrace{z^n}_{\text{eigenfunction}}$$

Eigenfunction Property of Complex Exponentials

CT: $e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)e^{st}$

$$\boxed{H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st}dt}$$

system function

DT: $z^{n} \longrightarrow \boxed{h[n]} \longrightarrow H(z)z^{n}$

$$\boxed{H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n}}$$

system function

Frequency Response of an LTI System

$$(s = j\omega)$$

$$e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow H(j\omega)e^{j\omega t}$$

CT Frequency response: $\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt}$

$$e^{j\omega n} \longrightarrow \boxed{H(e^{j\omega})} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

$$(z = e^{j\omega})$$

DT Frequency response: $\boxed{H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n}}$

Fourier Series and LTI Systems

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow y(t)=\sum_{k=-\infty}^{\infty} H(jk\omega_0)a_k e^{jk\omega_0 t}$$

$$a_k \longrightarrow \underbrace{H(jk\omega_0)}_{\text{"gain"}}a_k$$

$$H(jk\omega_o)=\left|H(jk\omega_o)\right| e^{j\angle H(jk\omega_o)},$$

includes both amplitude & phase

$$x[n]=\sum_{k=<N>} a_k e^{jk\omega_0 n} \longrightarrow \boxed{h[n]} \longrightarrow y[n]=\sum_{k=-\infty}^{\infty} H(e^{jk\omega_o})a_k e^{jk\omega_0 n}$$

$$a_k \longrightarrow \underbrace{H(e^{jk\omega_o})}_{\text{"gain"}}a_k$$

$$H(e^{jk\omega_o})=\left|H(e^{jk\omega_o})\right| e^{j\angle H(e^{jk\omega_o})},$$

includes both amplitude & phase

The effect of the LTI system is to modify each a_k through multiplication by the value of the frequency response at the corresponding frequency.

Example 3.17

$$h[n]=\alpha^n u[n] \quad , \quad |\alpha|<1$$

$$x[n]=\cos\left(\frac{2\pi n}{N}\right)=\frac{1}{2}e^{j(\frac{2\pi}{N})n}+\frac{1}{2}e^{-j(\frac{2\pi}{N})n}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{\infty}\alpha^n e^{-j\omega n}=\frac{1}{1-\alpha e^{-j\omega}}$$

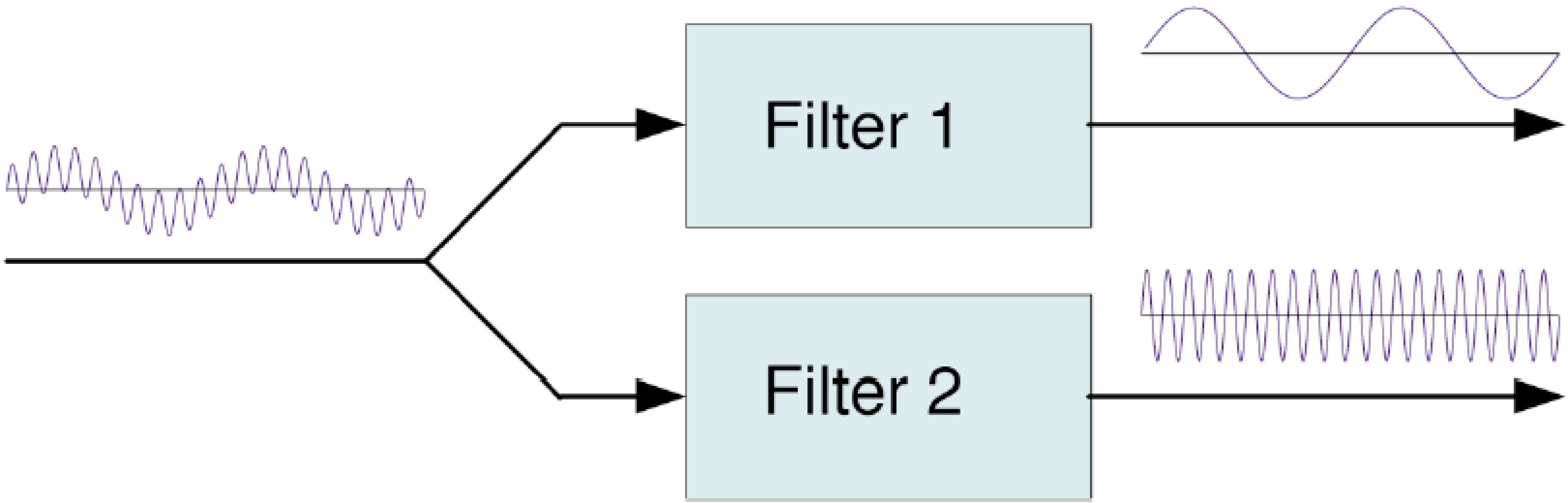
$$y[n]=\frac{1}{2}H\left(e^{j\frac{2\pi}{N}}\right)e^{j(\frac{2\pi}{N})n}+\frac{1}{2}H\left(e^{-j\frac{2\pi}{N}}\right)e^{-j(\frac{2\pi}{N})n}$$

$$=r\cos\left(\frac{2\pi n}{N}+\theta\right)$$

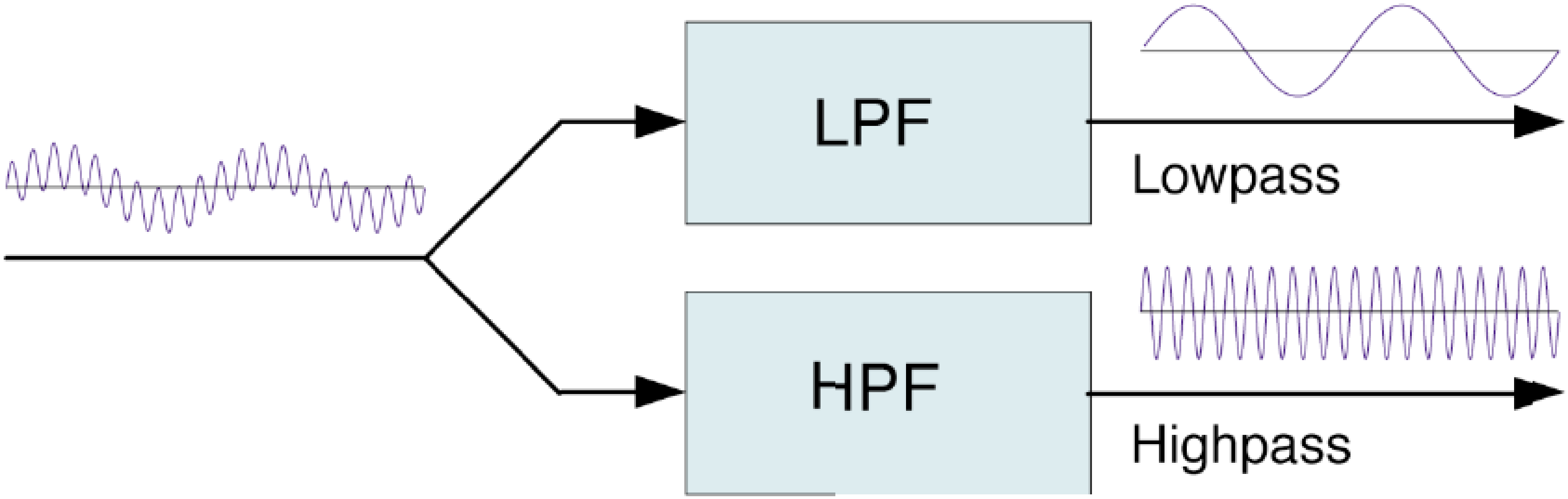
where $\quad re^{j\theta}=\dfrac{1}{1-\alpha e^{-j\frac{2\pi}{N}}}$

Signal Processing with Filters

- Separation of narrowband signals
  - Filter design — a main task in constructing communication systems.



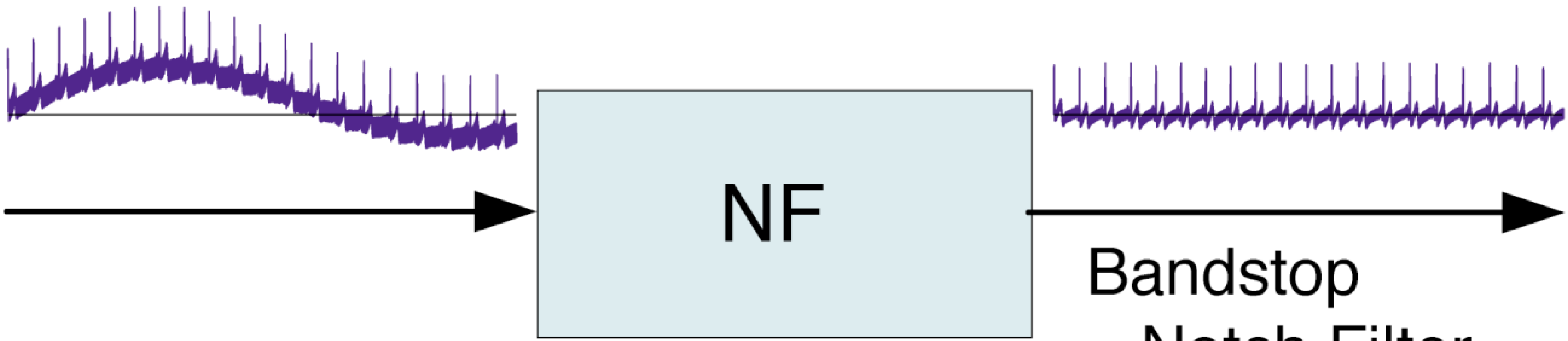
Example: Using LPF and HPF to Separate Signals



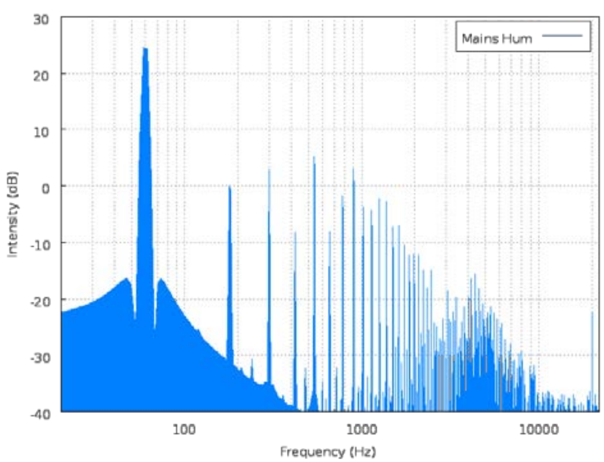
Filter design becomes more challenging if the two signals are close in frequency range.

Extraction of Narrowband Signals from Wideband Noise – BPF

Filtering out unwanted narrowband signal, for example, 60-Hz power lines –– Notch Filter



Bandstop
– Notch Filter

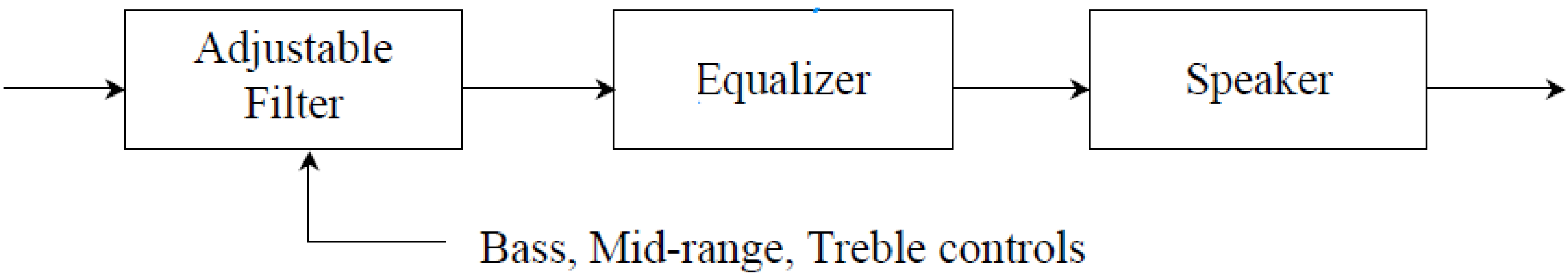


Frequency Shaping and Filtering

- By choice of $H(j\omega)$ (or $H(e^{j\omega})$) as a function of ω, we can *shape* the frequency composition of the output

  — Preferential amplification
  — Selective filtering of some frequencies

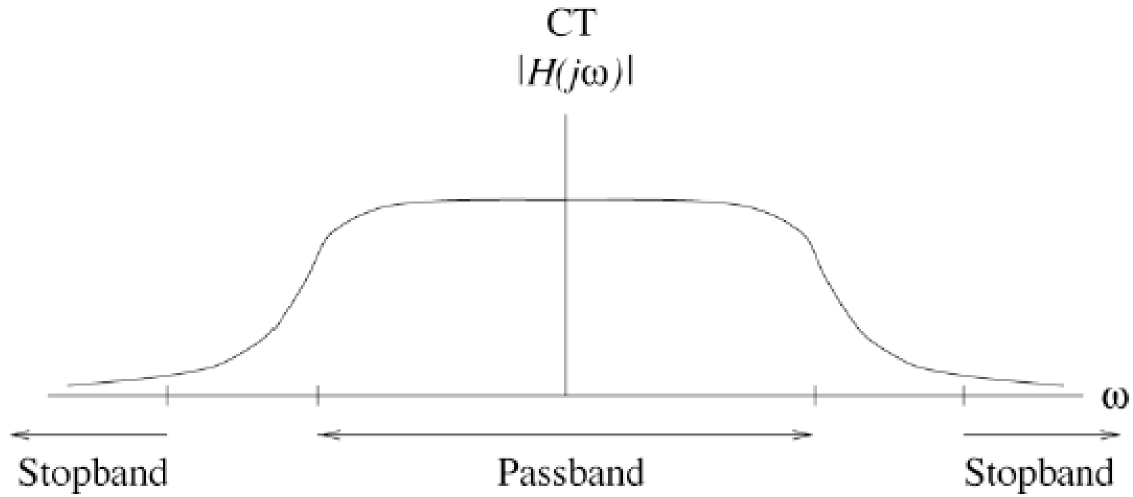
Example #1: Audio System



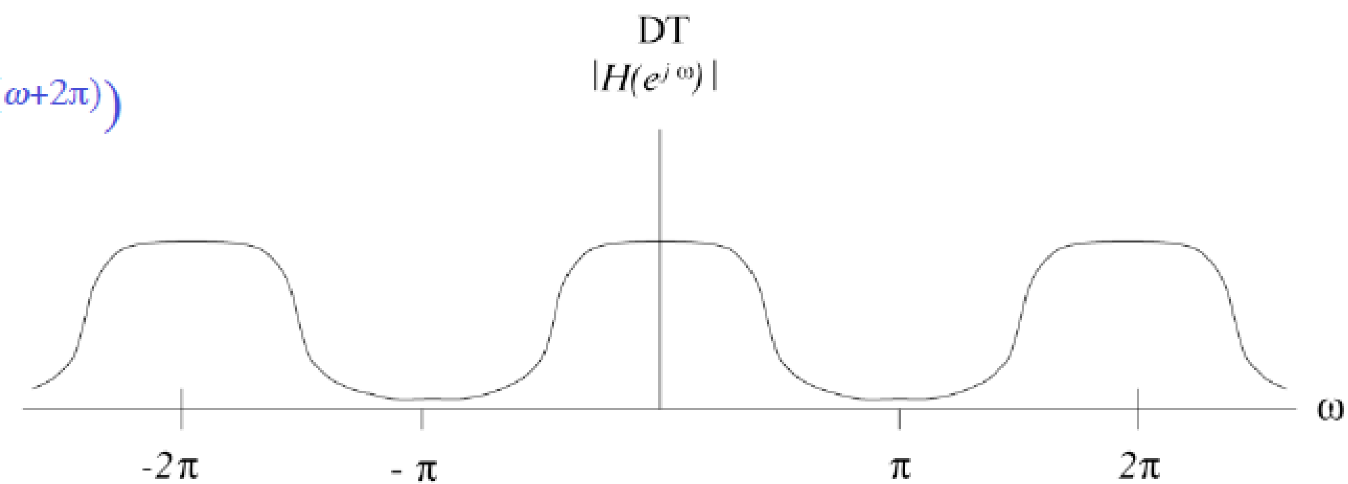
For audio signals, the amplitude is much more important than the phase.

Example #2: Frequency Selective Filters

Lowpass Filters:
Only show
amplitude here.

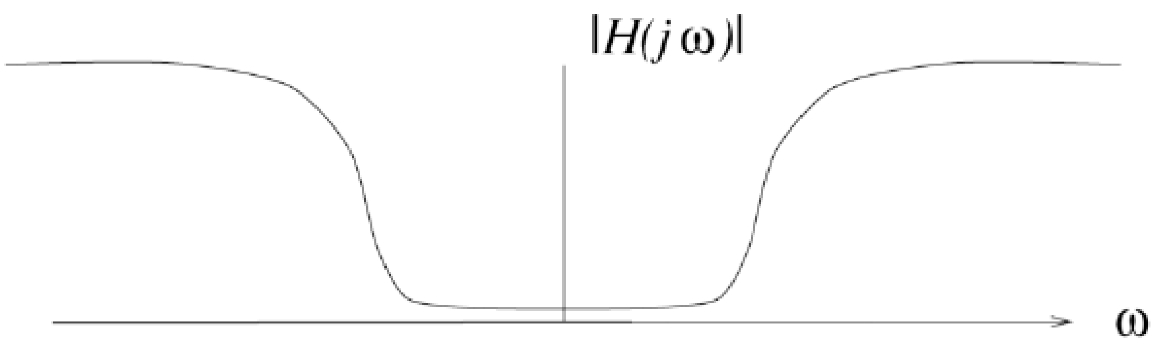


Note for DT:
$H(e^{j\omega}) = H(e^{j(\omega+2\pi)})$



Highpass Filters

CT

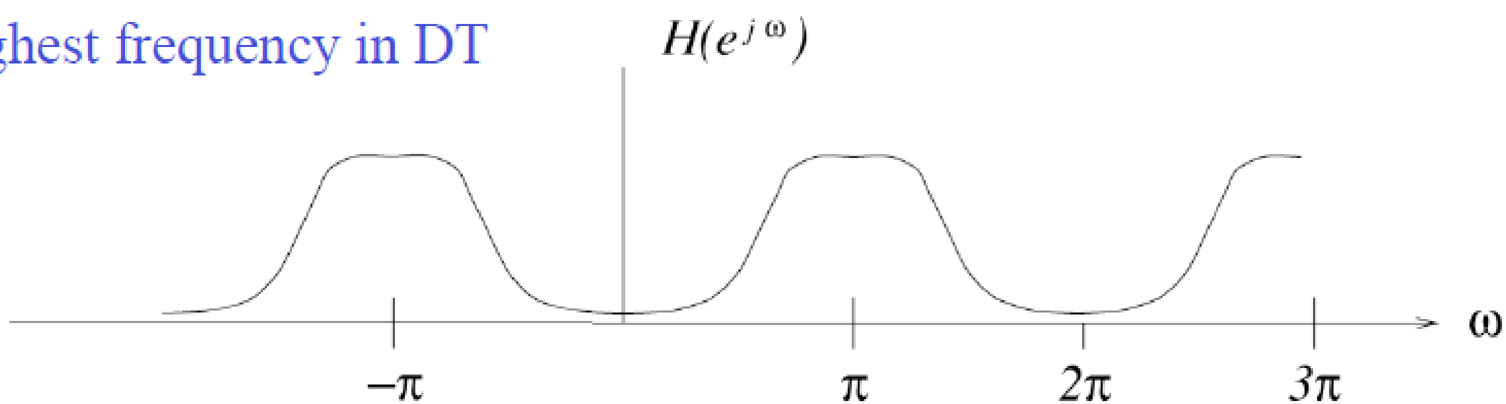


Remember:

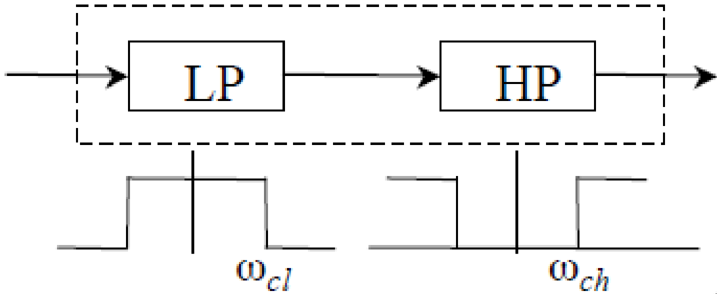
$(-1)^n = e^{j\pi n}$

— π = highest frequency in DT

DT

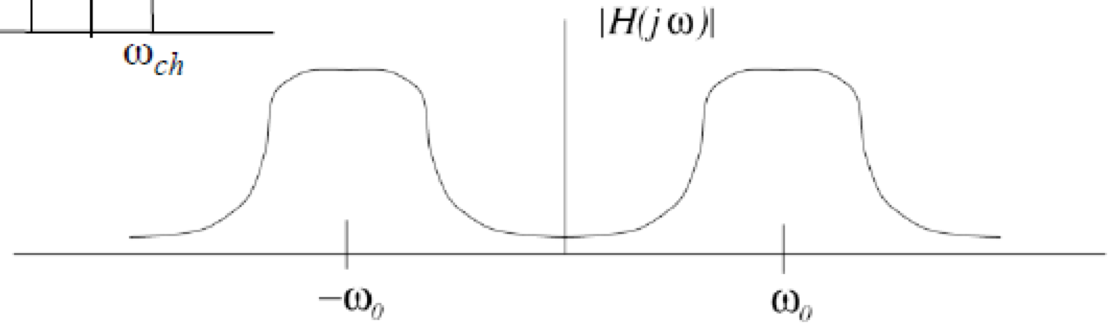


Bandpass Filters

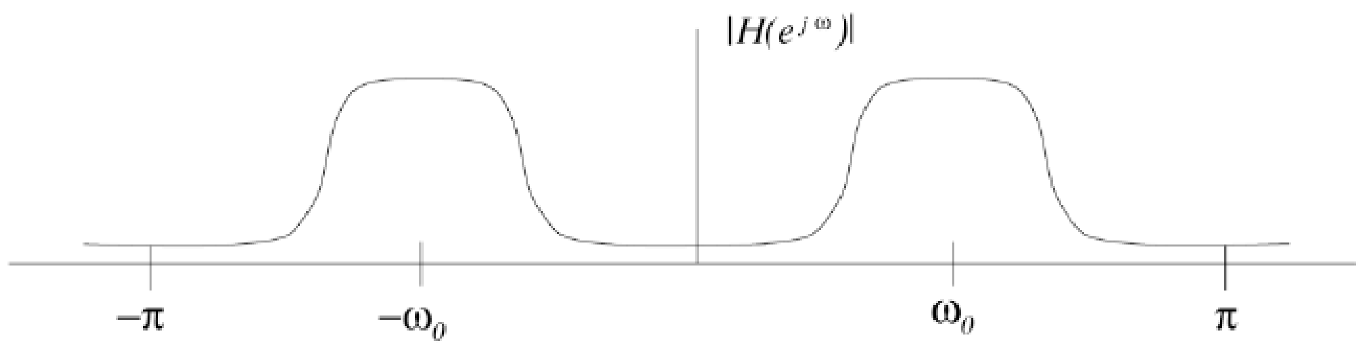


BPF if $\omega_{cl} > \omega_{ch}$

CT

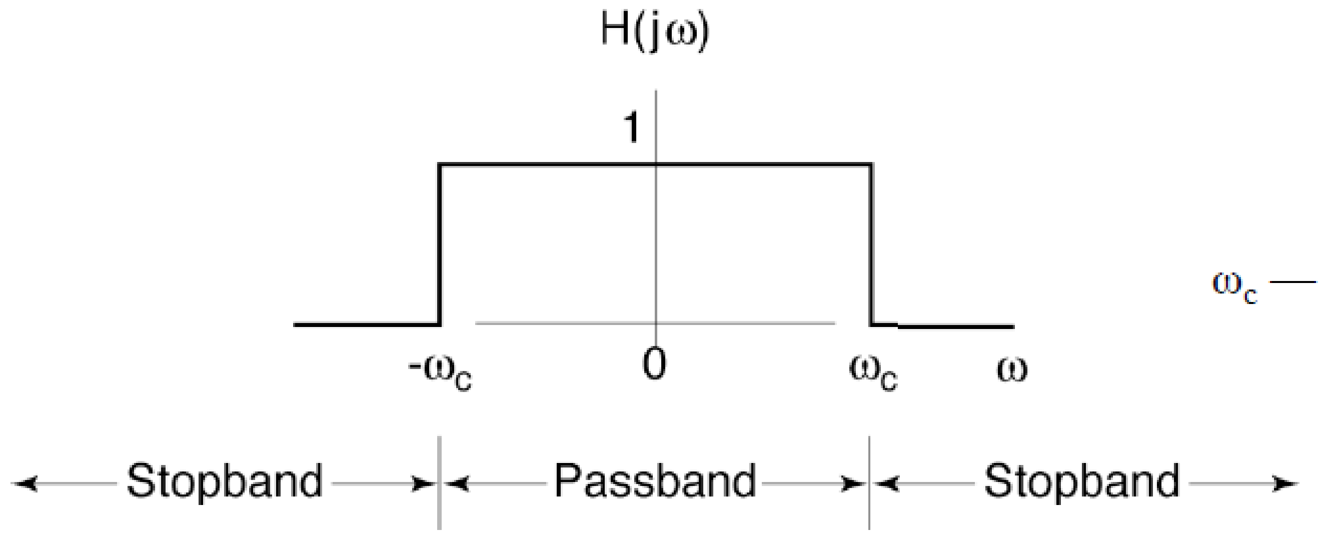


DT



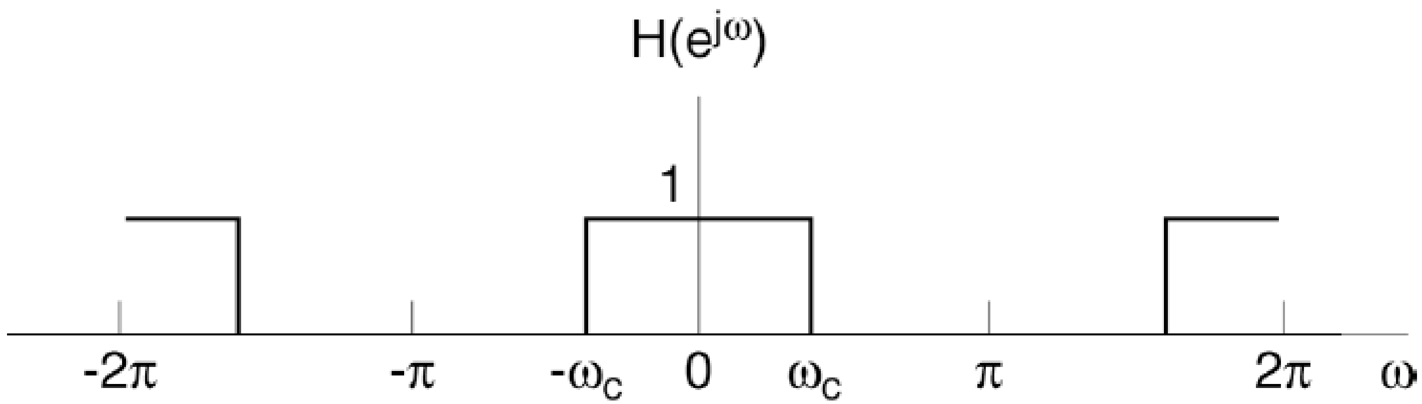
Idealized Filters

CT



ω_c — cutoff frequency

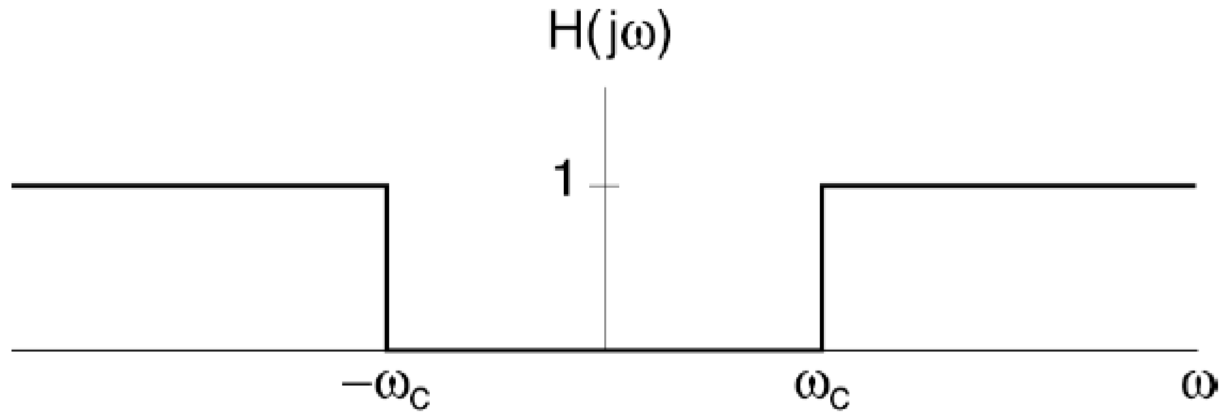
DT



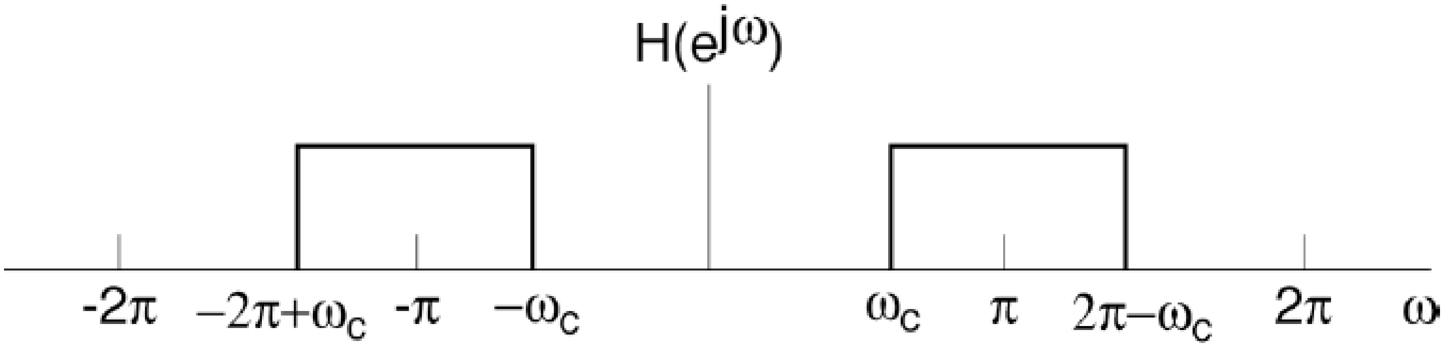
Note: $|H| = 1$ and $\angle H = 0$ for the ideal filters in the passbands, no need for the phase plot.

Ideal Highpass Filter

CT

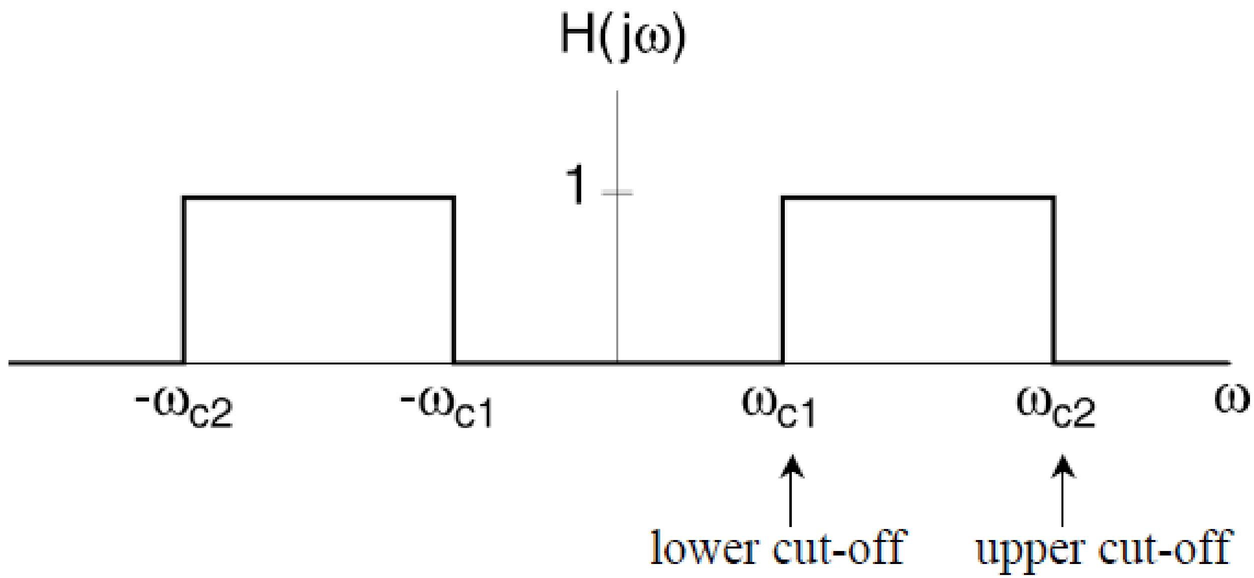


DT

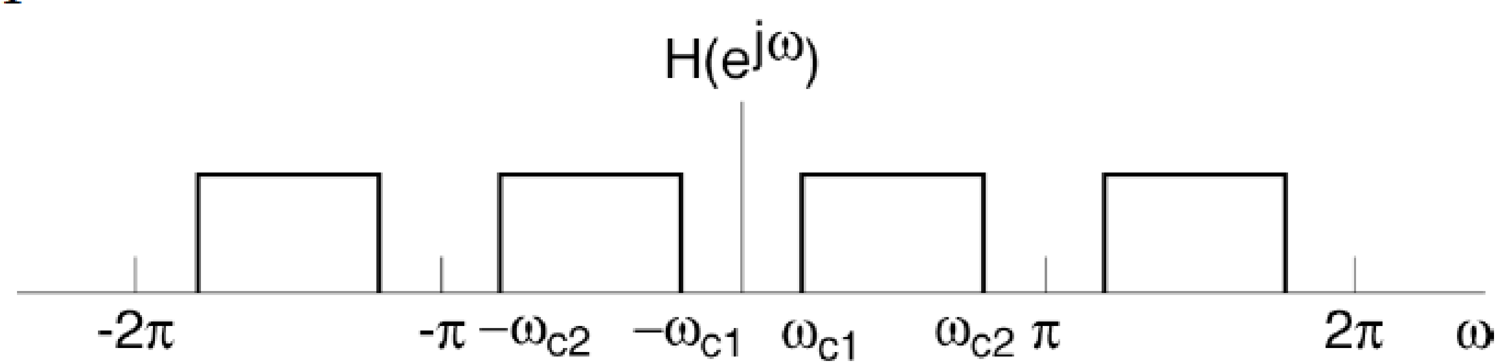


Ideal Bandpass Filter

CT



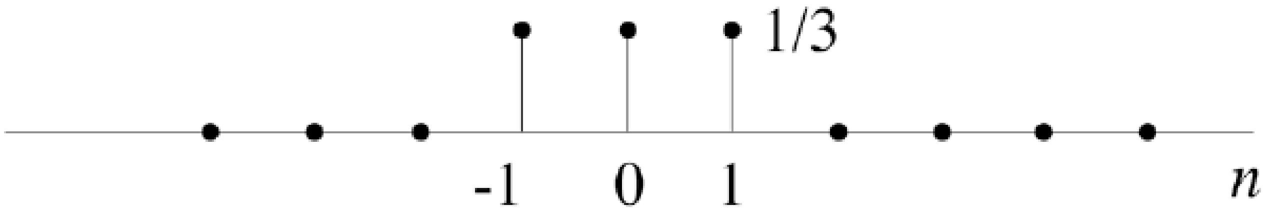
DT



Example #3: DT Averager/Smoother

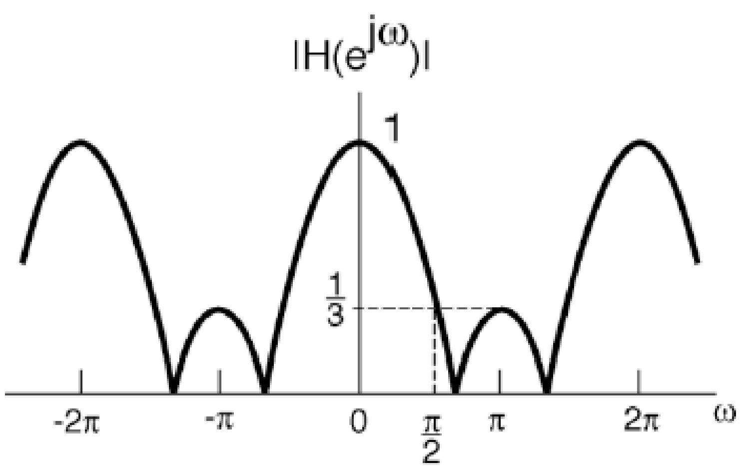
$$y[n] = \frac{1}{3}\{x[n-1] + x[n] + x[n+1]\}$$

$$h[n] = \frac{1}{3}\{\delta[n-1] + \delta[n] + \delta[n+1]\}$$



Frequency response:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \frac{1}{3}[e^{-j\omega} + 1 + e^{j\omega}] = \frac{1}{3} + \frac{2}{3}\cos\omega$$



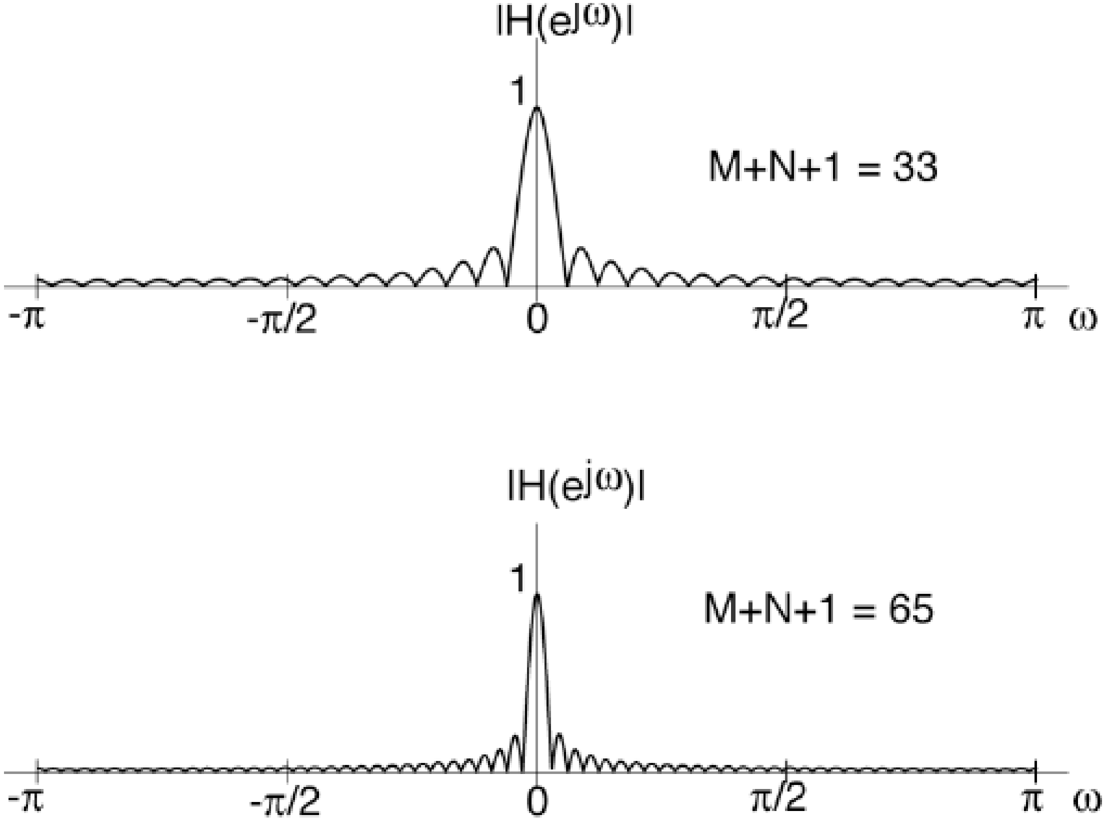
A LPF

Example #4: Nonrecursive DT (FIR) filters

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k] \longrightarrow h[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} \delta[n-k]$$

Frequency response:

$$H(e^{j\omega}) = \frac{1}{N+M+1} \sum_{k=-N}^{M} e^{-jk\omega} = \frac{1}{N+M+1} e^{j\omega(N-M)/2} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}$$



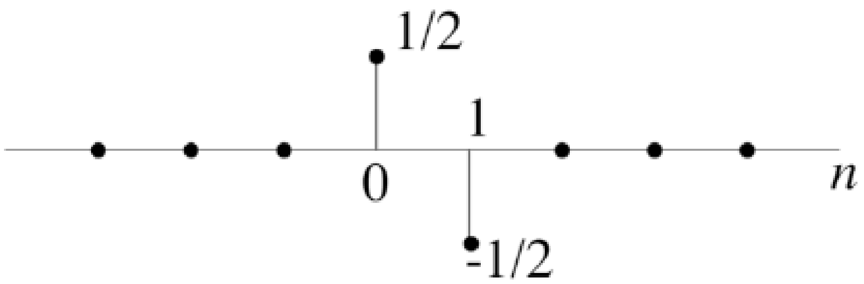
Rolls off at lower ω as M+N+1 increases

Example #5: Simple DT "Edge" Detector

— DT 2-points "differentiator"

$$y[n] = \frac{1}{2}[x[n] - x[n-1]]$$

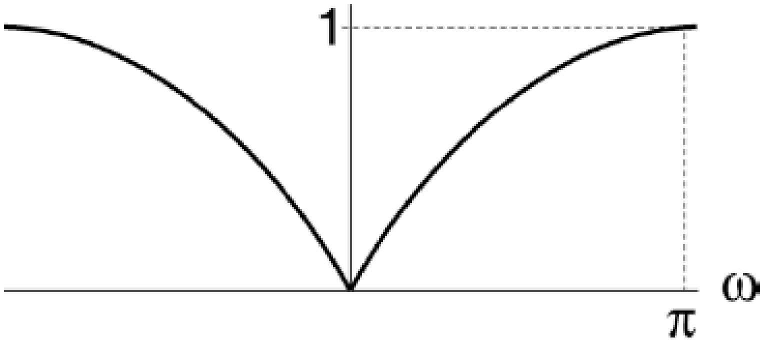
$$h[n] = \frac{1}{2}[\delta[n] - \delta[n-1]]$$



Frequency response:

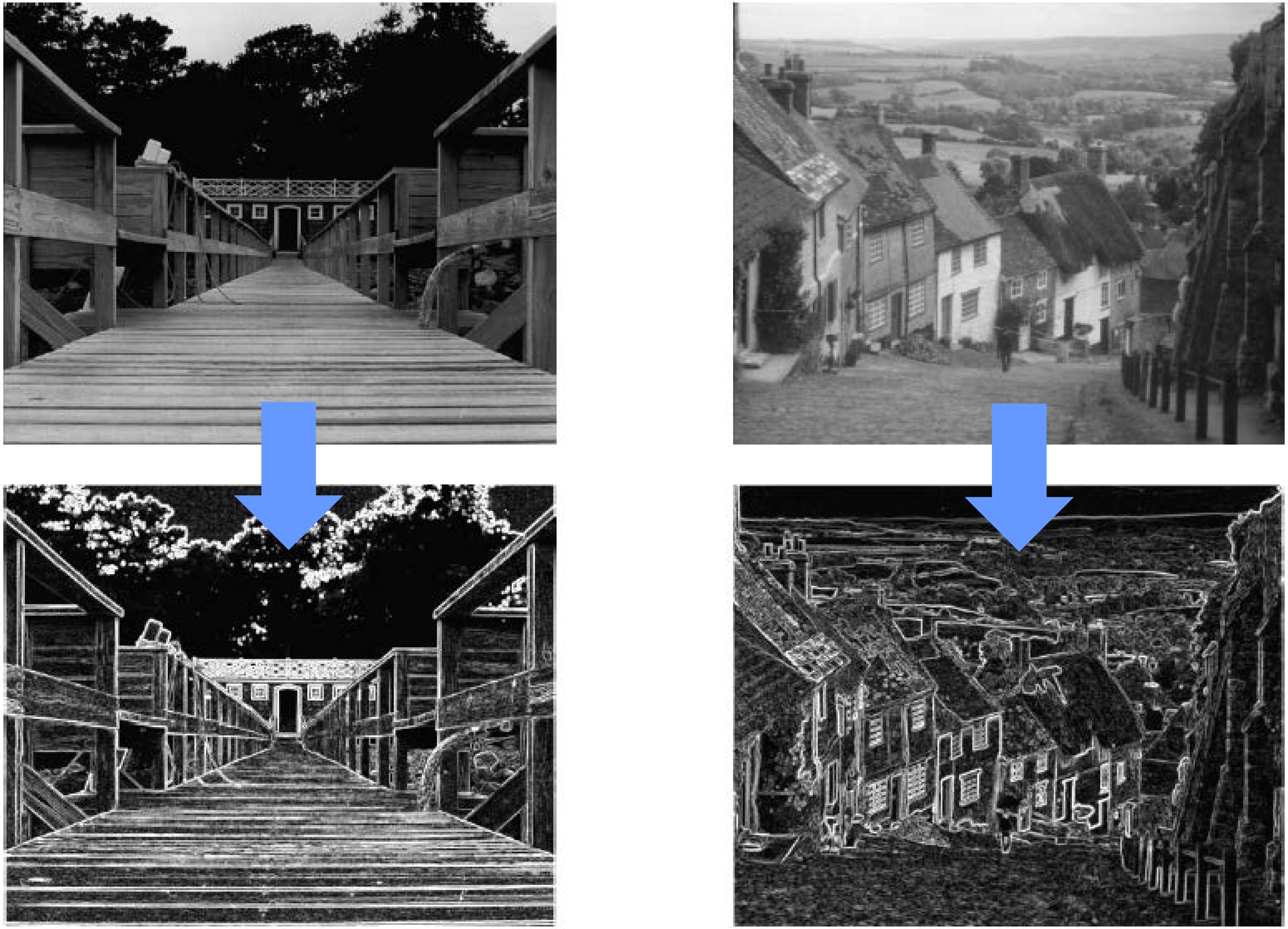
$$H(e^{j\omega}) = \frac{1}{2}(1 - e^{-j\omega}) = je^{j\omega/2}\sin(\omega/2)$$

$$|H(e^{j\omega})| = |\sin(\omega/2)|$$



Amplifies high-frequency components

Example #6: Edge enhancement using DT differentiator



Summary

- DT Fourier Series pair
  - Understand the difference between CT and DT
- Frequency response
  - How to determine frequency response?
- Filtering

Problem 3.44

Suppose we are given the following facts about a signal x(t)

1. x(t) is a real signal
2. x(t) is periodic with period T=6, and has Fourier series coefficients a_k
3. a_k =0 for k=0 and k>2
4. x(t) = -x(t-3)
5. $$\frac{1}{6}\int_{-3}^{3}|x(t)|^2\, dt = 1/2$$
6. a_1 is a positive real number

Show that x(t)=A cos(Bt+C), and determine the values of A, B, C

1. x(t) is a real signal

2. x(t) is periodic with period T=6, and has Fourier series coefficients a_k

3. a_k =0 for k=0 and |k|>2

4. x(t) = -x(t-3)

5. $$\frac{1}{6}\int_{-3}^{3}|x(t)|^2\,dt = 1/2$$

6. a_1 is a positive real number

From 2), $\omega_0=2\pi/T=\pi/3$

From 3), $$x(t)=\sum_{k=-2,-1}^{1,2} a_k e^{jk\pi t/3}$$

From 4), $$-x(t-3)=-\sum_{k=-2,-1}^{1,2} a_k e^{jk\pi(t-3)/3}$$

$$=-\sum_{k=-2,-1}^{1,2} a_k e^{jk\pi t/3}e^{-jk\pi}=-\sum_{k=-2,-1}^{1,2} a_k e^{jk\pi t/3}(-1)^k=\sum_{k=-2,-1}^{1,2} a_k e^{jk\pi t/3}(-1)^{k+1}$$

$\therefore a_k = (-1)^{k+1} a_k$

$a_2 = -a_2$

$a_{-2} = -a_{-2}$

$\therefore a_2 = a_{-2} = 0$

$\Rightarrow$ $$x(t)=a_1 e^{j\pi t/3}+a_{-1}e^{-j\pi t/3}$$

From 1), $a_{-k}=a_k^*$ $\mathrm{Re}\{a_k\}$ is even , $\mathrm{Im}\{a_k\}$ is odd

From 6), $a_{-1}=a_1=\mathrm{Re}\{a_1\}$, From 5), $2|a_1|^2=1/2$, $\therefore a_1=1/2$ $\Rightarrow$ $$x(t)=\frac{1}{2}e^{j\pi t/3}+\frac{1}{2}e^{-j\pi t/3}=\cos(\pi t/3)$$