Tutorial Problems

- 7.25, 7.37, 7.40

7.25. In Figure P7.25 is a sampler, followed by an ideal lowpass filter, for reconstruction of $x(t)$ from its samples $x_p(t)$. From the sampling theorem, we know that if $\omega_s = 2\pi/T$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then the reconstructed signal $x_r(t)$ will exactly equal $x(t)$. If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will *not* equal $x(t)$. We seek to show in this problem that if $\omega_c = \omega_s/2$, then for any choice of T, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants; that is,

$$x_r(kT) = x(kT),\ k = 0, \pm 1, \pm 2, \ldots.$$

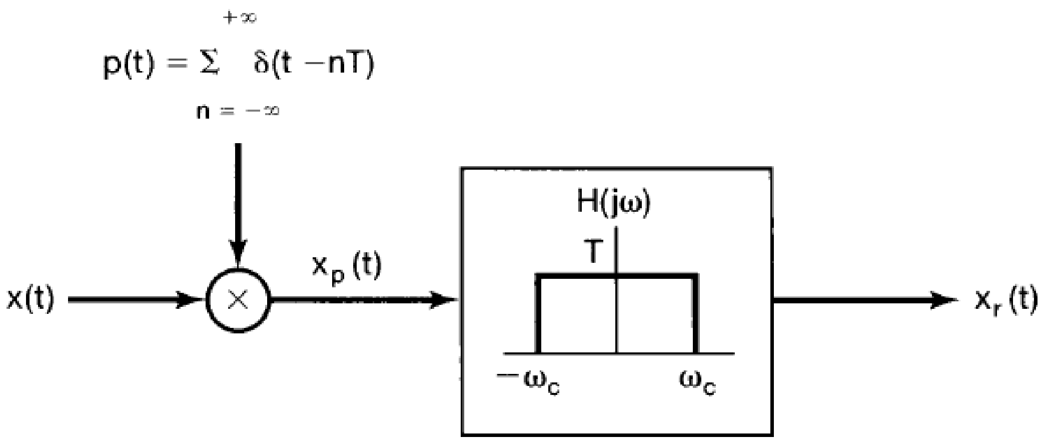


Figure P7.25

To obtain this result, consider eq. (7.11), which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)T\frac{\omega_c}{\pi}\frac{\sin[\omega_c(t-nT)]}{\omega_c(t-nT)}.$$

With $\omega_c = \omega_s/2$, this becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\frac{\pi}{T}(t-nT)\right]}{\frac{\pi}{T}(t-nT)}. \qquad \text{(P7.25–1)}$$

By considering the values of α for which $[\sin(\alpha)]/\alpha = 0$, show from eq. (P7.25–1) that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of k.

7.25 Here, $x_T(kT)$ can be written as

$$X_T(kT)= \sum_{k=-\infty}^{\infty} \frac{\sin[\pi(k-n)]}{\pi(k-n)} x(nT)$$

Note that when $n \neq k$,

$$\frac{\sin[\pi(k-n)]}{\pi(k-n)} = 0$$

And when n=k,

$$\frac{\sin[\pi(k-n)]}{\pi(k-n)} = 1$$

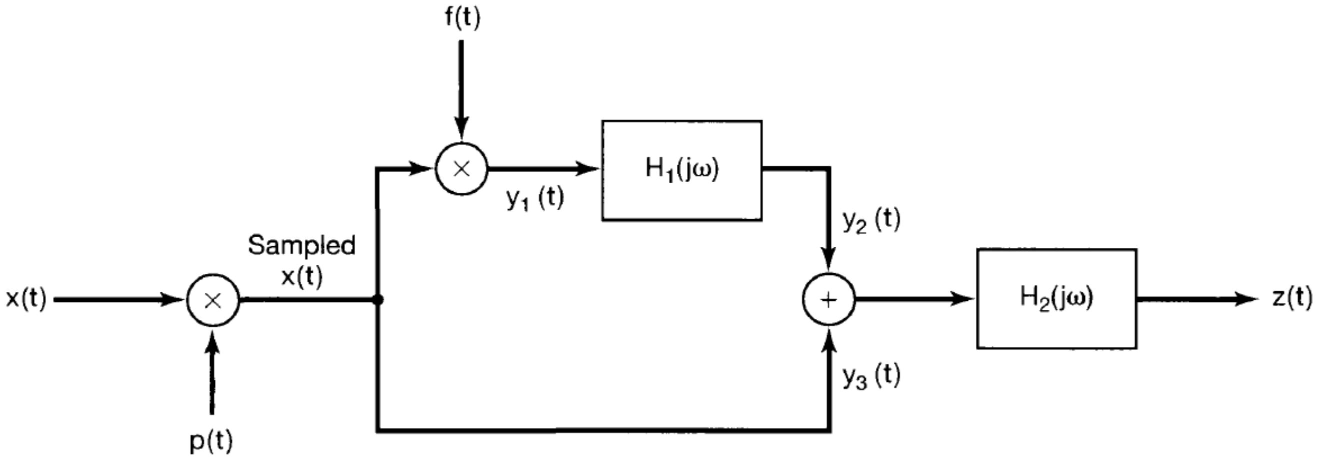
Therefore,

$$x_T(kT)=x(kT)$$

7.37. A signal limited in bandwidth to $|\omega| < W$ can be recovered from nonuniformly spaced samples as long as the average sample density is $2(W/2\pi)$ samples per second. This problem illustrates a particular example of nonuniform sampling. Assume that in Figure P7.37(a):

1. $x(t)$ is band limited; $X(j\omega) = 0, |\omega| > W$.
2. $p(t)$ is a nonuniformly spaced periodic pulse train, as shown in Figure P7.37(b).
3. $f(t)$ is a periodic waveform with period $T = 2\pi/W$. Since $f(t)$ multiplies an impulse train, only its values $f(0) = a$ and $f(\Delta) = b$ at $t = 0$ and $t = \Delta$, respectively, are significant.
4. $H_1(j\omega)$ is a 90° phase shifter; that is,

$$H_1(j\omega) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}.$$



(a)

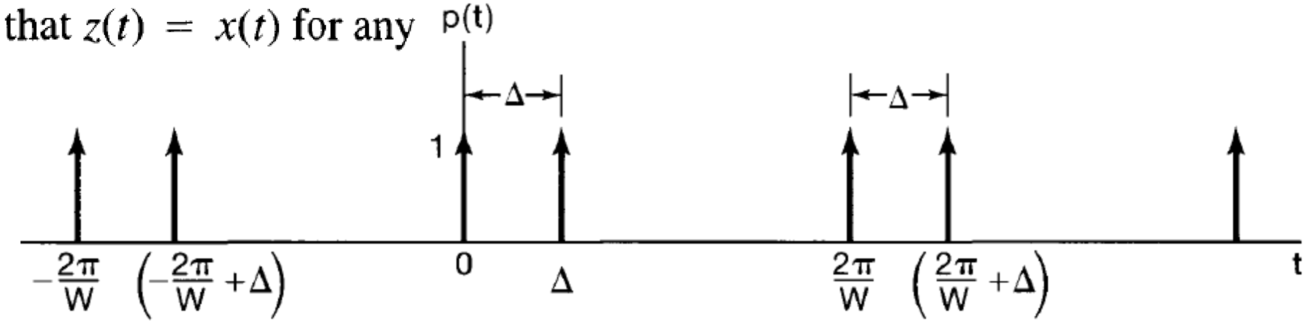
5. $H_2(j\omega)$ is an ideal lowpass filter; that is,

$$H_2(j\omega) = \begin{cases} K, & 0 < \omega < W \\ K^*, & -W < \omega < 0, \\ 0, & |\omega| > W \end{cases}$$

where K is a (possibly complex) constant.

(a) Find the Fourier transforms of $p(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$.

(b) Specify the values of a, b, and K as functions of Δ such that $z(t) = x(t)$ for any band-limited $x(t)$ and any Δ such that $0 < \Delta < \pi/W$.



7.37. We may write p(t) as

$$p(t)=p_1(t)+p_1(t-\Delta),$$

where

$$p_1(t)=\sum_{k=-\infty}^{\infty}\delta(t-2\pi k/W)$$

Therefore,

$$p(j\omega)=(1+e^{-j\Delta\omega})p_1(j\omega)$$

where

$$p_1(j\omega)=w\sum_{k=-\infty}^{\infty}\delta(\omega-kW)$$

Let us denote the product p(t)f(t) by g(t). Then,

$$g(t)=p(t)f(t)=p_1(t)f(t)+p_1(t-\Delta)f(t)$$

This may be written as

$$g(t)=ap_1(t)+bp_1(t-\Delta)$$

Therefore,

$$G(j\omega)=(a+be^{-j\omega\Delta})p_1(j\omega)$$

with $p_1(j\omega)$ is specified in eq.(s7.37-1). Therefore

$$G(j\omega)=w\sum_{k=-\infty}^{\infty}\left[a+be^{-jk\Delta w}\right]\delta(\omega-kw)$$

We now have

$$y_1(t)=x(t)p(t)f(t)$$

Therefore,

$$Y_1(j\omega)=\frac{1}{2\pi}[G(j\omega)*x(j\omega)]$$

This give us

$$Y_1(j\omega)=\frac{W}{2\pi}\sum\left[a+be^{-jk\Delta w}\right]x(j(\omega-kW))$$

In the range 0< ω <W, we may specify $Y_1(j\omega)$ as

$$Y_1(j\omega)=\frac{w}{2\pi}\left[(a+b)x(j\omega)+(a+be^{-jk\Delta w})x(j(\omega-W))\right]$$

since $Y_2(j\omega)=Y_1(j\omega)H_1(j\omega)$, in the range 0< ω <W we may specify $Y_2(j\omega)$ as

$$Y_2(j\omega)=\frac{jW}{2\pi}\left[(a+b)x(j\omega)+(a+be^{-j\Delta W})x(j(\omega-W))\right]$$

Since $y_3(t)=x(t)p(t)$, in the range 0< ω <W we may specify $Y_3(j\omega)$ as

$$Y_3(j\omega)=\frac{W}{2\pi}\left[2x(j\omega)+(1+e^{-j\Delta W})x(j(\omega-W))\right]$$

$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$
$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$

$$Y(j\omega)=\delta(\omega-\omega_c)*X(j\omega)=X(j(\omega-\omega_c))$$

Give that 0<W△< π, we require that $Y_2(j\omega)+Y_3(j\omega)=\frac{1}{K}X(j\omega)$ for 0< ω <W.

That is

$$\frac{W}{2\pi}\left[(a+ja+jb)x(j\omega)\right]+\frac{W}{2\pi}\left[(1+e^{-j\Delta w})x(j(\omega-W))\right]=\frac{1}{K}X(j\omega)$$

This implies that

$$1+e^{-j\Delta W}+ja+jbe^{-j\Delta W}=0$$

$$a=\sin(W\Delta)+\frac{(1+\cos(W\Delta))}{\tan(W\Delta)} \quad \text{and} \quad b=-\frac{1+\cos(W\Delta)}{\sin(W\Delta)}$$

Finally, we also get $k=\frac{2\pi}{W}\left[1/(2+ja+jb)\right]$

7.40. Consider a disc on which four cycles of a sinusoid are painted. The disc is rotated at approximately 15 revolutions per second, so that the sinusoid, when viewed through a narrow slit, has a frequency of 60 Hz.

The arrangement is indicated in Figure P7.40. Let $v(t)$ denote the position of the line seen through the slit. Then

$$v(t) = A\cos(\omega_0 t + \phi), \omega_0 = 120\pi.$$

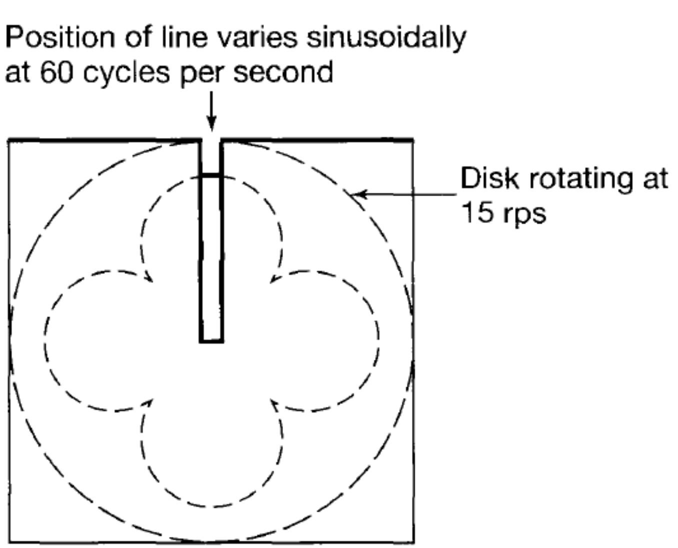


Figure P7.40

For notational convenience, we will normalize $v(t)$ so that $A = 1$. At 60 Hz, the eye is not able to follow $v(t)$, and we will assume that this effect can be explained by modeling the eye as an ideal lowpass filter with cutoff frequency 20 Hz.

Sampling of the sinusoid can be accomplished by illuminating the disc with a strobe light. Thus, the illumination can be represented by an impulse train; that is,

$$i(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

where $1/T$ is the strobe frequency in hertz. The resulting sampled signal is the product $r(t) = v(t)i(t)$. Let $R(j\omega)$, $V(j\omega)$, and $I(j\omega)$ denote the Fourier transforms of $r(t)$, $v(t)$, and $i(t)$, respectively.

(a) Sketch $V(j\omega)$, indicating clearly the effect of the parameters ϕ and ω_0.

(b) Sketch $I(j\omega)$, indicating the effect of T.

(c) According to the sampling theorem, there is a maximum value for T in terms of ω_0 such that $v(t)$ can be recovered from $r(t)$ using a lowpass filter. Determine this value of T and the cutoff frequency of the lowpass filter. Sketch $R(j\omega)$ when T is slightly less than the maximum value.

If the sampling period T is made greater than the value determined in part (c), aliasing of the spectrum occurs. As a result of this aliasing, we perceive a lower frequency sinusoid.

(d) Suppose that $2\pi/T = \omega_0 + 20\pi$. Sketch $R(j\omega)$ for $|\omega| < 40\pi$. Denote by $v_a(t)$ the apparent position of the line as we perceive it. Assuming that the eye behaves as an ideal lowpass filter with 20-Hz cutoff and unity gain, express $v_a(t)$ in the form

$$v_a(t) = A_a \cos(\omega_a + \phi_a),$$

where A_a is the apparent amplitude, ω_a the apparent frequency, and ϕ_a the apparent phase of $v_a(t)$.

(e) Repeat part (d) for $2\pi/T = \omega_0 - 20\pi$.

7.40. (a) The Fourier transform $V(j\omega)$ is as shown in Figure S7.40.

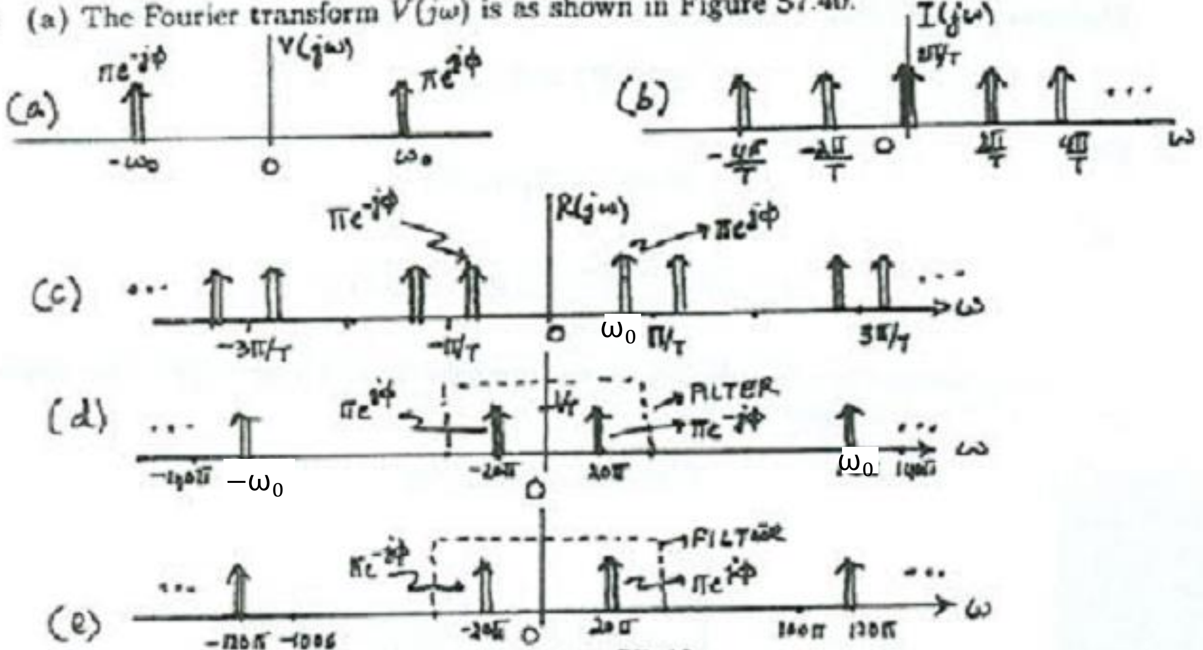


Figure S7.40

(b) The Fourier transform $I(j\omega)$ is

$$I(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - 2\pi k/T).$$

This is as shown in Figure S7.40.

(c) The Nyquist rate for $v(t)$ is $2\omega_0$. Therefore,

$$\frac{2\pi}{T_{max}} = 2\omega_0 \quad \Rightarrow \quad T_{max} = \frac{\pi}{\omega_0}.$$

The cutoff frequency of the lowpass filter has to be ω_0.

(d) Now,

$$R(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} V(j(\omega - 2\pi k/T)).$$

(a) $V(t) = \cos(\omega_0 t + \phi)$

$= \cos[\omega_0(t + \frac{\phi}{\omega_0})]$

由 $x(t - t_0) \longrightarrow e^{-j\omega t_0} X(j\omega)$

$\cos\omega_0 t \longrightarrow \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

得 $V(j\omega) = \pi \cdot e^{j\frac{\omega\phi}{\omega_0}} \cdot [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

$$\sum_{n=-\infty}^{+\infty}\delta(t - nT) \qquad \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$$

Therefore, $v_a(t)$ obtained by passing $r(t)$ through a lowpass filter with cutoff frequency $2\pi(20)$ rad/sec is

$$v_a(t) = \frac{1}{T}\cos(20\pi t - \phi).$$

Therefore,

$$\omega_a = 20\pi, \quad \phi_a = -\phi, \quad \text{and} \quad A_a = \frac{1}{T}.$$

(e) Here, $2\pi/T = 120\pi - 20\pi = 100\pi$. Therefore, $R(j\omega)$ is as shown in Figure S7.40. It follows that

$$v_a(t) = \frac{1}{T}\cos(20\pi t + \phi).$$

and

$$\omega_a = 20\pi, \quad \phi_a = \phi, \quad \text{and} \quad A_a = \frac{1}{T}.$$