Signals and Systems Tutorial
Week 13

7.41 7.44 7.47 7.49

Problem 1

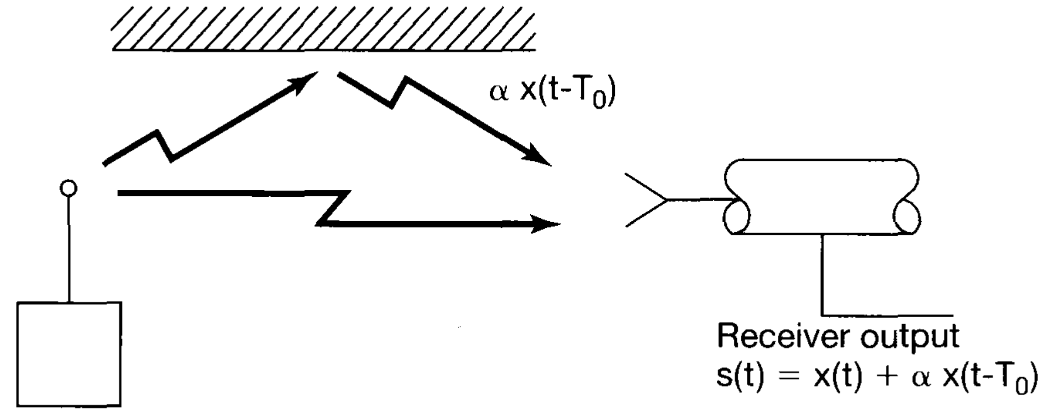
7.41. In many practical situations a signal is recorded in the presence of an echo, which we would like to remove by appropriate processing. For example, in Figure P7.41(a), we illustrate a system in which a receiver simultaneously receives a signal $x(t)$ and an echo represented by an attenuated delayed replication of $x(t)$. Thus, the receiver output is $s(t) = x(t) + \alpha x(t - T_0)$, where $|\alpha| < 1$. This output is to be processed to recover $x(t)$ by first converting to a sequence and then using an appropriate digital filter $h[n]$, as indicated in Figure P7.41(b).

Assume that $x(t)$ is band limited [i.e., $X(j\omega) = 0$ for $|\omega| > \omega_M$] and that $|\alpha| < 1$.

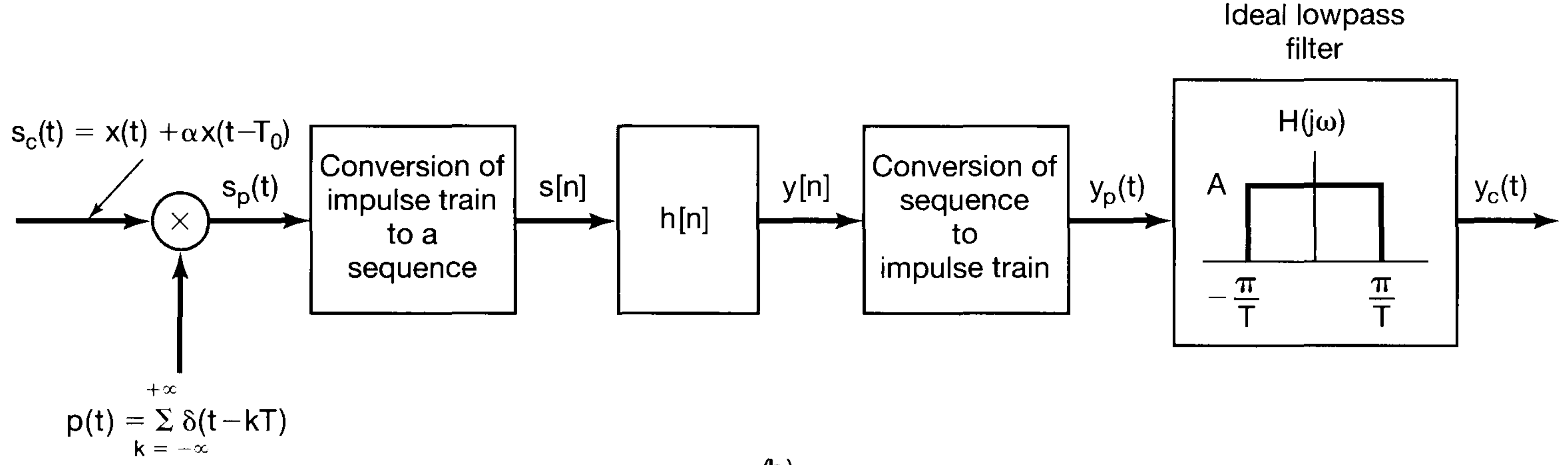
(a) If $T_0 < \pi/\omega_M$, and the sampling period is taken to be equal to T_0 (i.e., $T = T_0$), determine the difference equation for the digital filter $h[n]$ such that $y_c(t)$ is proportional to $x(t)$.

(b) With the assumptions of part (a), specify the gain A of the ideal lowpass filter such that $y_c(t) = x(t)$.

(c) Now suppose that $\pi/\omega_M < T_0 < 2\pi/\omega_M$. Determine a choice for the sampling period T, the lowpass filter gain A, and the frequency response for the digital filter $h[n]$ such that $y_c(t)$ is proportional to $x(t)$.


$\alpha\, x(t-T_0)$
Receiver output
$s(t) = x(t) + \alpha\, x(t-T_0)$

(a)


$s_c(t) = x(t) + \alpha x(t-T_0)$
$s_p(t)$
Conversion of impulse train to a sequence
$s[n]$
$h[n]$
$y[n]$
Conversion of sequence to impulse train
$y_p(t)$
Ideal lowpass filter
$H(j\omega)$
A
$-\frac{\pi}{T}$
$\frac{\pi}{T}$
$y_c(t)$
$p(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)$

(b)

Solution 1

(a) The Nyquist rate for the signal $x(t)$ is $2\omega_M$ (i.e. $T < \pi / \omega_M$).

$y_c(t)$ will be proportional to $x(t)$ as long as $y[n] = \beta x[n]$. Now

$$\begin{aligned} s[n] &= x(nT_0) + \alpha x(nT_0 - T_0) \\ &= x[n] + \alpha x[n-1] \\ &= \frac{1}{\beta}\left(y[n] + \alpha y[n-1]\right) \end{aligned}$$

Without loss of generality, we set $\beta = 1$. Therefore, the difference equation for the filter $h[n]$ is

$$y[n] + \alpha y[n-1] = s[n]$$

(b) From (a) we have $y[n] + \alpha y[n-1] = s[n]$. Therefore, we have

$$H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{S\left(e^{j\Omega}\right)} = \frac{X\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right) + \alpha e^{-j\Omega} X\left(e^{j\Omega}\right)} = \frac{1}{1 + \alpha e^{-j\Omega}}$$

From Figure P7.41(b), we have

$$H_{eq}(j\omega) = \frac{A}{T_0} H\left(e^{j\omega T_0}\right) = \frac{A}{T_0} \frac{1}{1 + \alpha e^{-j\omega T_0}}, \qquad (1.1)$$

where $H_{eq}(j\omega)$ is the system response of the overall continuous-time system.

Also, since we require $y_c(t) = x(t)$, we have

$$H_{eq}(j\omega) = \frac{Y_c(j\omega)}{S_c(j\omega)} = \frac{X(j\omega)}{S_c(j\omega)} = \frac{X(j\omega)}{X(j\omega) + \alpha e^{-j\omega T_0} X(j\omega)} = \frac{1}{1 + \alpha e^{-j\omega T_0}} \quad (1.2)$$

Comparing eq. (1.1) and (1.2), we get $A = T_0$

(c) We require a T to avoid aliasing. Therefore, $T < \pi / \omega_M$.

With the analysis in (b), we know $H_{eq}(j\omega) = \frac{A}{T} H\left(e^{j\omega T}\right)$ and $H_{eq}(j\omega) = \frac{1}{1 + \alpha e^{-j\omega T_0}}$.

For these to be consistent, we have $A = T$ and $H\left(e^{j\omega T}\right) = \frac{1}{1 + \alpha e^{-j\omega T_0}}$. Let $\Omega = \omega T$, we have

$$H\left(e^{j\Omega}\right) = \frac{1}{1 + \alpha e^{-j\Omega T_0 / T}}$$

Problem 2

7.44. Suppose we wish to design a continuous-time generator that is capable of producing sinusoidal signals at any frequency satisfying

$$\omega_1 \le \omega \le \omega_2,$$

where ω_1 and ω_2 are given positive numbers.

Our design is to take the following form: We have stored a discrete-time cosine wave of period N; that is, we have stored $x[0], \ldots, x[N-1]$, where

$$x[k] = \cos\left(\frac{2\pi k}{N}\right).$$

Every T seconds we output an impulse weighted by a value of $x[k]$, where we proceed through the values of $k = 0, 1, \ldots, N-1$ in a cyclic fashion. That is,

$$y_p(kT) = x(k \text{ modulo } N),$$

or equivalently,

$$y_p(kT) = \cos\left(\frac{2\pi k}{N}\right),$$

and

$$y_p(t) = \sum_{k=-\infty}^{+\infty} \cos\left(\frac{2\pi k}{N}\right) \delta(t - kT).$$

(a) Show that by adjusting T, we can adjust the frequency of the cosine signal being sampled. That is, show that

$$y_p(t) = (\cos \omega_0 t) \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

where $\omega_0 = 2\pi/NT$. Determine a range of values for T such that $y_p(t)$ can represent samples of a cosine signal with a frequency that is variable over the full range

$$\omega_1 \leq \omega \leq \omega_2.$$

(b) Sketch $Y_p(j\omega)$.

The overall system for generating a continuous-time sinusoid is depicted in Figure P7.44(a). $H(j\omega)$ is an ideal lowpass filter with unity gain in its passband; that is,

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}.$$

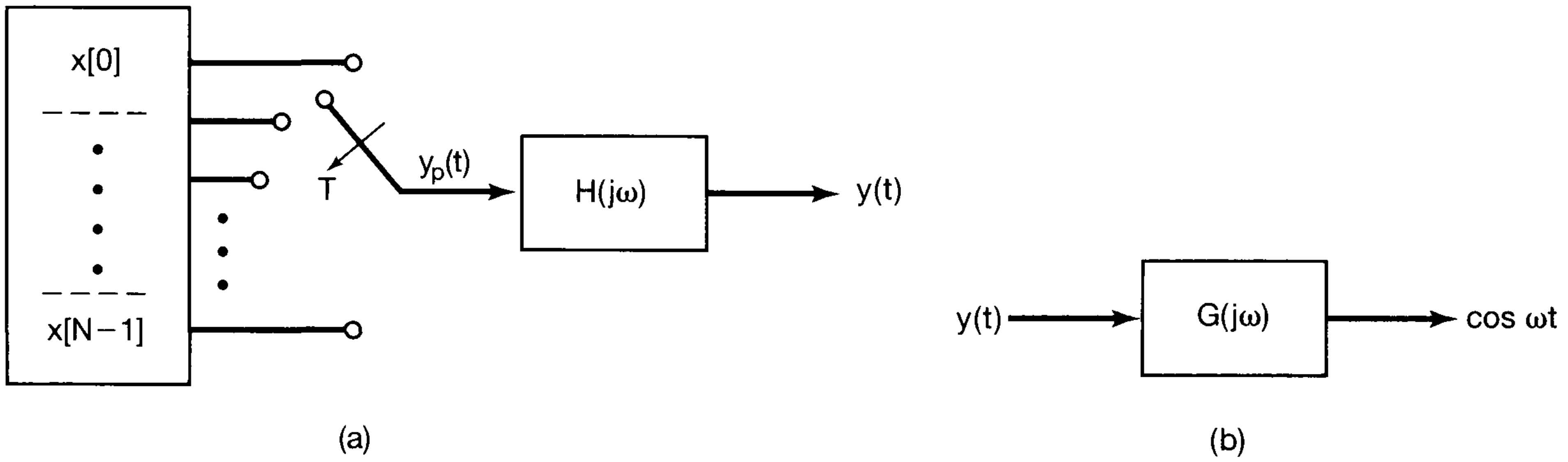


Figure P7.44

The parameter ω_c is to be determined so that $y(t)$ is a continuous-time cosine signal in the desired frequency band.

(c) Consider any value of T in the range determined in part (a). Determine the minimum value of N and some value for ω_c such that $y(t)$ is a cosine signal in the range $\omega_1 \leq \omega \leq \omega_2$.

(d) The amplitude of $y(t)$ will vary, depending upon the value of ω chosen between ω_1 and ω_2. Thus, we must design a system $G(j\omega)$ that normalizes the signal as shown in Figure P7.44(b). Find such a $G(j\omega)$.

Solution 2

(a) We have

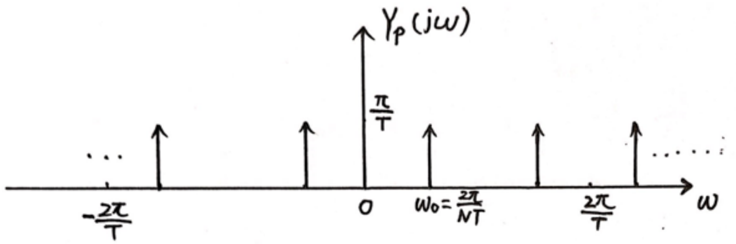
$$y_p(t) = \sum_{k=-\infty}^{\infty} \cos\left(\frac{2\pi k}{N}\right)\delta(t-kT).$$

Since $\omega_0 = 2\pi / NT$, we have

$$\begin{aligned} y_p(t) &= \sum_{k=-\infty}^{\infty} \cos(\omega_0 kT)\delta(t-kT) \\ &= \sum_{k=-\infty}^{\infty} \cos(\omega_0 t)\delta(t-kT) \\ &= \cos(\omega_0 t)\sum_{k=-\infty}^{\infty} \delta(t-kT) \end{aligned}$$

Since $\omega_1 \le \omega \le \omega_2$, we have $\omega_1 \le \dfrac{2\pi}{NT} \le \omega_2$, Thus $\dfrac{2\pi}{N\omega_2} \le T \le \dfrac{2\pi}{N\omega_1}$.

(b) Let $c(t)=\cos(\omega_0 t)$. We have $Y_p(j\omega)=\frac{1}{T}\sum_{k=-\infty}^{\infty} C\left(j\left(\omega-k\frac{2\pi}{T}\right)\right)$. This is shown below.



(c) To avoid aliasing, we require that $\frac{2\pi}{T}>2\omega_0=\frac{4\pi}{NT}$. This implies that $N>2$. Therefore, the minimum value of N is 3. By inspection of $Y_p(j\omega)$, we have $\frac{2\pi}{NT}<\omega_c<\frac{2\pi}{T}-\frac{2\pi}{NT}$. For $N=3$, we have $\frac{2\pi}{3T}<\omega_c<\frac{4\pi}{3T}$.

(d) We have

$$G(j\omega)=\begin{cases} T, & -\omega_c\le\omega\le\omega_c \\ \text{arbitrary}, & \text{otherwise} \end{cases}$$

Problem 3

7.47. Suppose $x[n]$ has a Fourier transform that is zero for $\pi/3 \leq |\omega| \leq \pi$. Show that

$$x[n] = \sum_{k=-\infty}^{\infty} x[3k]\left(\frac{\sin(\frac{\pi}{3}(n-3k))}{\frac{\pi}{3}(n-3k)}\right).$$

Solution 3

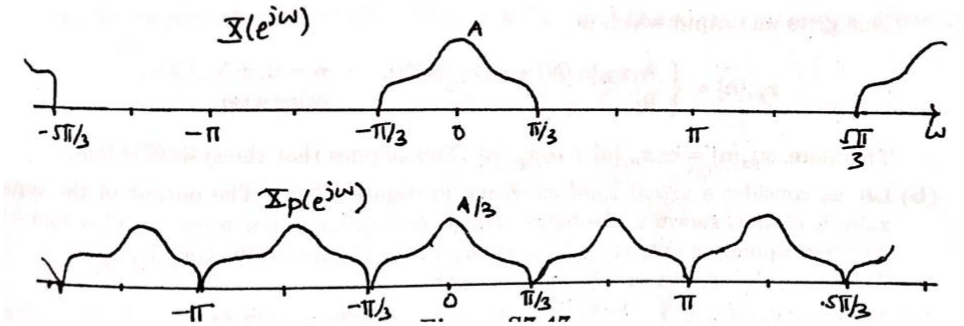
Let us define a signal

$$x_p[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-3k] = \sum_{k=-\infty}^{\infty}x[3k]\delta[n-3k].$$

From Section 7.5.1, we know that the Fourier transform of $x_p[n]$ is

$$X_p\left(e^{j\omega}\right) = \frac{1}{3}\sum_{k=0}^{2}X\left(e^{j(\omega-2\pi k/3)}\right).$$

Since $X_p\left(e^{j\omega}\right)=0$ for $\pi/3 \le |\omega| \le \pi$, there is no aliasing among the replicas of $X\left(e^{j\omega}\right)$ in $X_p\left(e^{j\omega}\right)$. This is shown below.



In order to be able to recover $x[n]$ from $x_p[n]$, it is clear that we need to pass $x_p[n]$ to a lowpass filter with cutoff frequency $\pi/3$ and passband gain 3. Therefore,

$$\begin{aligned} x[n] &= x_p[n] * \frac{3\sin(\pi n/3)}{\pi n} \\ &= \left\{\sum_{k=-\infty}^{\infty}x[3k]\delta[n-3k]\right\} * \frac{3\sin(\pi n/3)}{\pi n} \\ &= \sum_{k=-\infty}^{\infty}x[3k]\frac{\sin(\pi(n-3k)/3)}{\pi(n-3k)/3} \end{aligned}$$

Problem 4

7.49. As discussed in Section 7.5 and illustrated in Figure 7.37, the procedure for interpolation or upsampling by an integer factor N can be thought of as the cascade of two operations. The first operation, involving system A, corresponds to inserting $N - 1$ zero-sequence values between each sequence value of $x[n]$, so that

$$x_p[n] = \begin{cases} x_d\left[\frac{n}{N}\right], & n = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

For exact band-limited interpolation, $H(e^{j\omega})$ is an ideal lowpass filter.

(a) Determine whether or not system A is linear.

(b) Determine whether or not system A is time invariant.

(c) For $X_d(e^{j\omega})$ as sketched in Figure P7.49 and with $N = 3$, sketch $X_p(e^{j\omega})$.

(d) For $N = 3$, $X_d(e^{j\omega})$ as in Figure P7.49, and $H(e^{j\omega})$ appropriately chosen for exact band-limited interpolation, sketch $X(e^{j\omega})$.

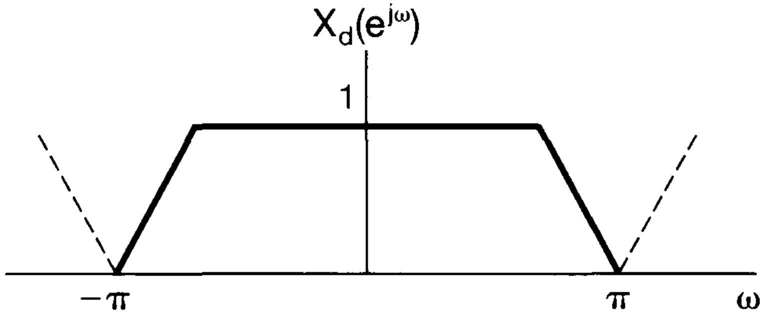


Figure P7.49

Solution 4

(a) Let the signals $x_{d_1}[n]$ and $x_{d_2}[n]$ be inputs to system A. Let the corresponding outputs be $x_{p_1}[n]$ and $x_{p_2}[n]$. Now consider an input of the form $x_{d_3}[n]=\alpha_1 x_{d_1}[n]+\alpha_2 x_{d_2}[n]$. This gives the output which is

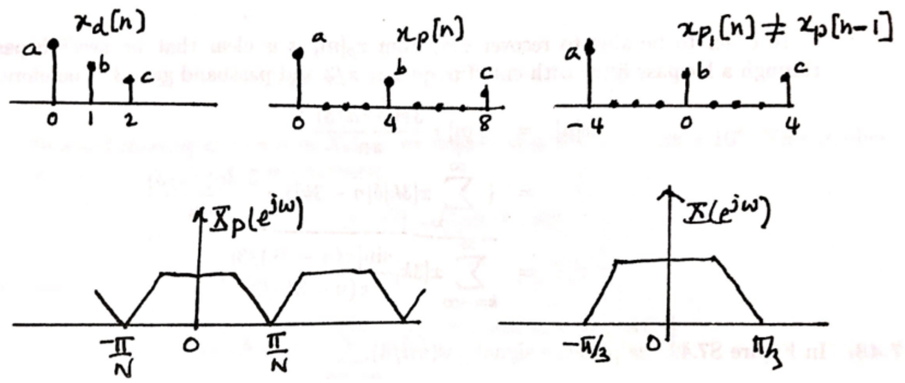
$$x_{p_3}[n]=\begin{cases}\alpha_1 x_{d_1}[n/N]+\alpha_2 x_{d_2}[n/N], & n=0,\pm N,\pm 2N,\ldots\\ 0, & \text{otherwise}\end{cases}$$

Therefore, $x_{p_3}[n]=\alpha_1 x_{p_1}[n]+\alpha_2 x_{p_2}[n]$. Hence, the system is linear.

(b) Let us consider a signal $x_d[n]$ as shown in the figure below. The output of the system $x_p[n]$ is then as shown in the figure. Let us define a new input $x_{d_1}[n]=x_d[n-1]$. The corresponding $x_{p_1}[n]$ is shown in the figure. Clearly, $x_{p_1}[n]\neq x_p[n]$. Hence, the system is not time invariant.

(c) We have $X_p\left(e^{j\omega}\right)=X_d\left(e^{j\omega N}\right)$. This is shown below.

(d) $X\left(e^{j\omega}\right)$ is shown below.



Q & A