Tutorial Problems (Week 10)

Basic Problems with Answers 5.1, 5.3, 5.4
Advanced Problems 5.41

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5.1. Use the Fourier transform analysis equation (5.9) to calculate the Fourier transforms of:

(a) $(\frac{1}{2})^{n-1}u[n-1]$ **(b)** $(\frac{1}{2})^{|n-1|}$

Sketch and label one period of the magnitude of each Fourier transform.

5.1 (a) let x[n]= $(1/2)^{n-1}$u[n-1].Using the Fourier transform analysis equation (5.9).the Fourier transform $X(e^{jw})$ of this signal is

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn}$$

$$= \sum_{n=1}^{\infty} (1/2)^{n-1} e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (1/2)^{n} e^{-jw(n+1)}$$

$$= e^{-jw} \frac{1}{(1-(1/2)e^{-jw})}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \tag{5.8}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \tag{5.9}$$

discrete-time Fourier transform pair

Example 5.1

Consider the signal

$$x[n] = a^n u[n], \qquad |a| < 1.$$

In this case,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n}$$

$$\boxed{= \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1-ae^{-j\omega}}.}$$

(b) Let $x[n]=(1/2)^{|n-1|}$.Using the Fourier transform analysis equation (5.9).The Fourier transform $X(e^{jw})$ of signal is

$$X(e^{jw})=\sum_{n=-\infty}^{\infty} x[n]e^{-jwn} = \sum_{n=-\infty}^{0} (1/2)^{-(n-1)} e^{-jwn} + \boxed{\sum_{n=1}^{\infty} (1/2)^{n-1} e^{-jwn}}$$

The second summation in the right—hand side of the above equation is exactly the same as result of part (a).Now ,

$$\sum_{n=-\infty}^{0} (1/2)^{-(n-1)} e^{-jwn} = \sum_{n=0}^{\infty} (1/2)^{n+1} e^{jwn} =(1/2) \frac{1}{(1-(1/2)e^{jw})}$$

Therefore

$$X(e^{jw}) =(1/2) \frac{1}{(1-(1/2)e^{jw})} + e^{-jw} \frac{1}{(1-(1/2)e^{-jw})} = \frac{0.75e^{-jw}}{(1.25-\cos w)}$$

$$\cos x = \frac{e^{ix}+e^{-ix}}{2}$$

5.3. Determine the Fourier transform for $-\pi \le \omega < \pi$ in the case of each of the following periodic signals:

(a) $\sin(\frac{\pi}{3}n + \frac{\pi}{4})$ **(b)** $2 + \cos(\frac{\pi}{6}n + \frac{\pi}{8})$

5.3 We note from section 5.2 that a periodic signal with Fourier series representation

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

has a Fourier transform

$$X(e^{jw}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(w - \frac{2\pi k}{N})$$

(a) Consider the signal $x_1[n] = \sin(\frac{\pi}{3}n + \frac{\pi}{4})$ .We note that the fundamental period of the signal $x_1[n]$ is N=6.

The signal may be written as

$$x_1[n] = \frac{1}{2j}e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - \frac{1}{2j}e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})} = \frac{1}{2j}e^{j\frac{\pi}{4}}\boxed{e^{j\frac{2\pi}{6}n}} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}$$

K=1 K= -1

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$

$$x_1[n]= \frac{1}{2j}e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} - \frac{1}{2j}e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})}= \frac{1}{2j}e^{j\frac{\pi}{4}}e^{j\frac{2\pi}{6}n} - \frac{1}{2j}e^{-j\frac{\pi}{4}}e^{-j\frac{2\pi}{6}n}$$

Form this , we obtain the non-zero Fourier series coefficients a_k of $x_1[n]$ the range

$-2 \le k \le 3$ as

$$a_1 = (1/2j)e^{j\frac{\pi}{4}} \qquad a_{-1} = -(1/2j)e^{-j\frac{\pi}{4}}$$

Therefore , in the range $-\pi \le w \le \pi$,we obtain

$$X(e^{jw}) = 2\pi a_1\delta(w-\frac{2\pi}{6}) + 2\pi a_{-1}\delta(w+\frac{2\pi}{6})$$

$$=(\pi/j)\{e^{j\pi/4}\delta(w-2\pi/6) - e^{-j\pi/4}\delta(w+2\pi/6)\}$$

(b) consider the signal $x_2[n] = 2+\cos(\frac{\pi}{6}n+\frac{\pi}{8})$.we note that the fundamental period of the signal $x_1[n]$ is N=12.the signal maybe written as

$$x_1[n] = 2+(1/2)e^{j(\frac{\pi}{6}n+\frac{\pi}{8})} + (1/2)e^{-j(\frac{\pi}{6}n+\frac{\pi}{8})}$$

$$= 2+(1/2)\boxed{e^{j\frac{2\pi}{12}n}}e^{j\frac{\pi}{8}} + (1/2)e^{-j\frac{2\pi}{12}n}e^{-j\frac{\pi}{8}}$$

Form this ,we obtain the non-zero Fourier series coefficients a_k of $x_2[n]$ in the range $-5 \le k \le 6$ as

$$a_0 = 2 \qquad a_1 = (1/2)e^{j\frac{\pi}{8}} \qquad a_{-1} = (1/2)e^{-j\frac{\pi}{8}}$$

$$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\,\delta\left(\omega - \frac{2\pi k}{N}\right)}$$

Therefore ,in the range ,we obtain

$$X(e^{jw}) = 2\pi a_0\delta(w) + 2\pi a_1\delta(w-\frac{2\pi}{12}) + 2\pi a_{-1}\delta(w+\frac{2\pi}{12})$$

$$= 4\pi\delta(w) + \pi\{e^{j\pi/8}\delta(w-\frac{\pi}{6}) + e^{-j\pi/8}\delta(w+\frac{\pi}{6})\}$$

5.4. Use the Fourier transform synthesis equation (5.8) to determine the inverse Fourier transforms of:

(a) $X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\{2\pi\delta(\omega - 2\pi k) + \pi\delta(\omega - \frac{\pi}{2} - 2\pi k) + \pi\delta(\omega + \frac{\pi}{2} - 2\pi k)\}$

(b) $X_2(e^{j\omega}) = \begin{cases} 2j, & 0 < \omega \le \pi \\ -2j, & -\pi < \omega \le 0 \end{cases}$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega, \qquad (5.8)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}. \qquad (5.9)$$

discrete-time Fourier transform pair

5.4 (a)Using the Fourier transform synthesis equation (5.8)

$$x_1[n] = (1/2\pi)\int_{-\pi}^{\pi} X_1(e^{jw})e^{jwn}dw$$

Only one period

$$= (1/2\pi)\int_{-\pi}^{\pi}[2\pi\delta(w) + \pi\delta(w-\pi/2) + \pi\delta(w+\pi/2)]e^{jwn}dw$$

$$= e^{j0} + (1/2)e^{j(\pi/2)n} + (1/2)e^{-j(\pi/2)n}$$

$$= 1 + \cos(\pi n/2)$$

(b)Using the transform synthesis equation (5.8)

$$x_2[n] = (1/2\pi)\int_{-\pi}^{\pi} X_2(e^{jw})e^{jwn}dw$$

$$= -(1/2\pi)\int_{-\pi}^{0} 2je^{jwn}dw + (1/2\pi)\int_{0}^{\pi} 2je^{jwn}dw$$

$$= (j/\pi)[-\frac{1-e^{-j\pi n}}{jn} + \frac{e^{j\pi n}-1}{jn}]$$

$$= -(4/(n\pi))\sin^2(n\pi/2)$$

5.41 Let $\tilde{x}[n]$ be a periodic signal with period N. A finite-duration signal $x[n]$ is related to $\tilde{x}[n]$ through

$$x[n] = \begin{cases} \tilde{x}[n], & n_0 \le n \le n_0 + N - 1 \\ 0, & \text{otherwise} \end{cases},$$

for some integer n_0. That is, $x[n]$ is equal to $\tilde{x}[n]$ over one period and zero elsewhere.

(a) If $\tilde{x}[n]$ has Fourier series coefficients a_k and $x[n]$ has Fourier transform $X(e^{j\omega})$, show that

$$a_k = \frac{1}{N} X(e^{j2\pi k/N})$$

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n},$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}.$$

regardless of the value of n_0.

(b) Consider the following two signals:

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}.$$

$$x[n] = u[n] - u[n-5]$$

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

where N is a positive integer. Let a_k denote the Fourier coefficients of $\tilde{x}[n]$ and let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$.

(i) Determine a closed-form expression for $X(e^{j\omega})$.

(ii) Using the result of part (i), determine an expression for the Fourier coefficients a_k.

5.41. (a)The Fourier transform $X(e^{jw})$ of the signal x[n] is

$$X(e^{jw})=\sum_{n=-\infty}^{\infty}x[n]e^{-jwn}=\sum_{n_0}^{n_0+N-1}x[n]e^{-jwn}$$

Therefore,

$$X(e^{j2\pi k/N})=\sum_{n=n_0}^{n_0+N-1}x[n]e^{-j(2\pi/N)kn}$$

S5.41-1

Now, we may write the expression for the FS coefficients of $\tilde{x}[n]$ as

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n},$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

$$a_k=\frac{1}{N}\sum_{<N>}\tilde{x}[n]e^{-j(2\pi/N)kn}=\frac{1}{N}\sum_{n=n_0}^{n_0+N-1}x[n]e^{-j(2\pi/N)kn}$$

(Because x[n]=$\tilde{x}[n]$in the range $n_0\le n\le n_0+N-1$). Comparing the above equation with eq. (S5.41-1), we get

$$a_k=\frac{1}{N}X(e^{j2\pi k/N})$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.$$

(b) (i) From the given information,

$$X(e^{jw})=1+e^{-jw}+e^{-2jw}+e^{-3jw}$$

$$=e^{-j(3/2)w}\{e^{j(3/2)w}+e^{-j(3/2)w}\}+e^{-j(3/2)w}\{e^{j(1/2)w}+e^{-j(1/2)w}\}$$

$$=2e^{-j(3/2)w}\{\cos(3w/2)+\cos(w/2)\}$$

(ii)From part (a),

$$a_k=\frac{1}{N}X(e^{j2\pi k/N})=\frac{1}{N}2e^{-j(3/2)2\pi k/N}\{\cos(6\pi k/(2N))+\cos(\pi k/N)\}$$