Assignments for Chapter 1
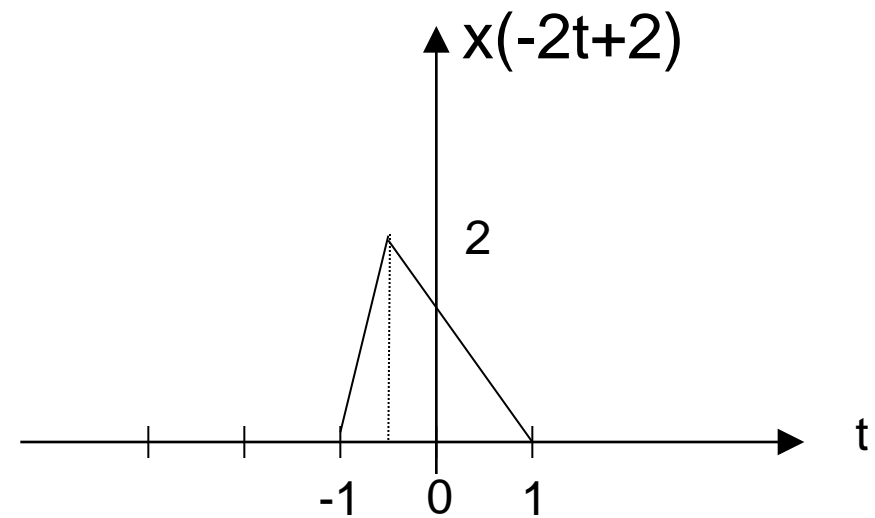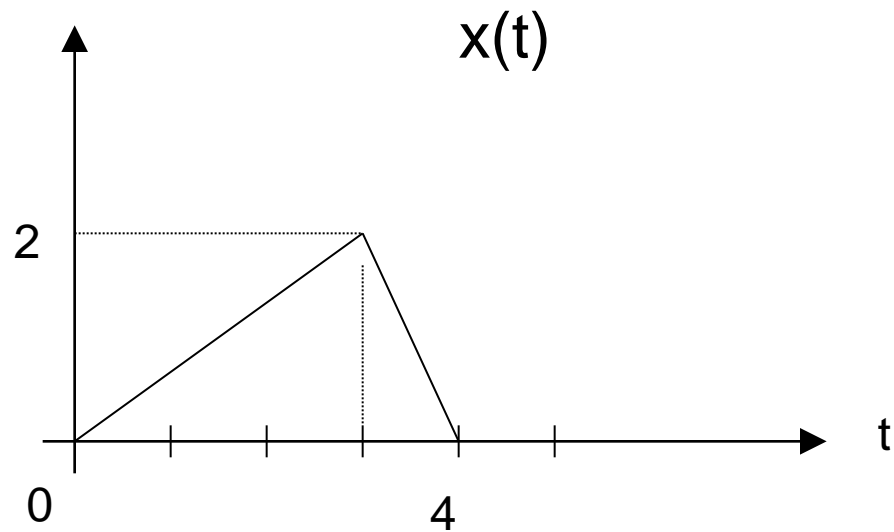
- **1.20**
- **1.21 (c) (f)**
- **1.24 (a)**
- **1.26**
- **1.27 (a) (f)**
- **1.41**

Tutorial Questions (Week 3)

- **Basic Problems with Answers 1.15, 1.18**
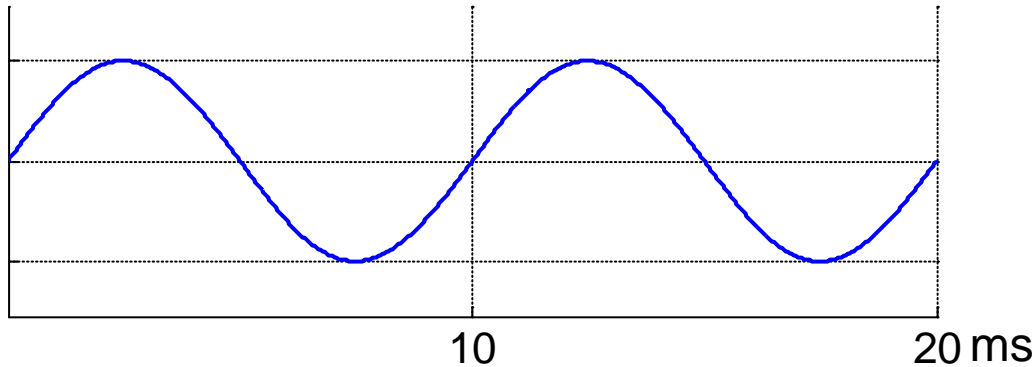- **Basic Problems 1.29, 1.31**
- **Advanced Problems 1.33, 1.42**
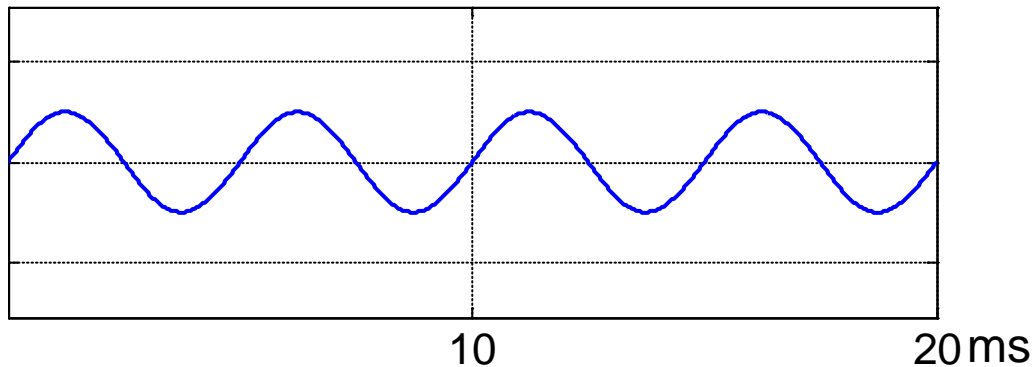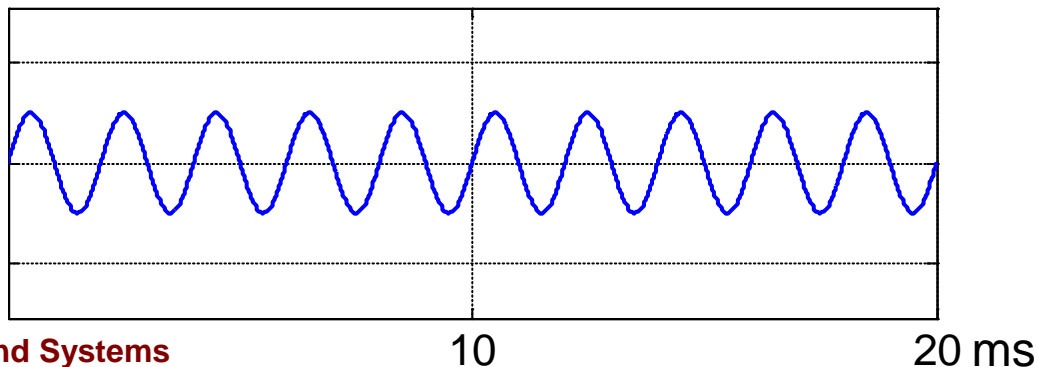
- **1教111**
- **周1、2、3、4晚上9:00-10:00**

Class problem

x(t)

x(-2t+2)

2

0   4

2

-1 0 1   t

x(-2t+2)   ?

Harmonically Related Signal Sets

T_0=10 ms
f_0 =100 Hz

T_1=5 ms
f_1 =200 Hz

T_2=2 ms
f_2 =500 Hz

Harmonically Related Signal Sets (cont.)

- A set of periodic exponentials which have a <u>common period</u> T_0.

$$\{\phi_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots\}$$

must be integer multiple

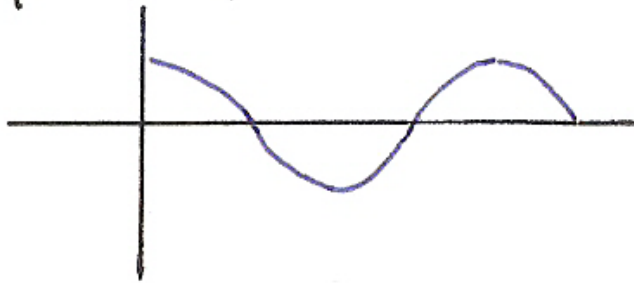fundamental frequency $|k\omega_0|$

fundamental period $\quad T_k = \dfrac{2\pi}{|k\omega_0|} = \dfrac{T_0}{|k|}, \quad T_0 = \dfrac{2\pi}{\omega_0}$

- The *k*th harmonic $\phi_k(t)$ is periodic with period T_0, as it goes through $|k|$ of its fundamental periods T_k in duration of length T_0.
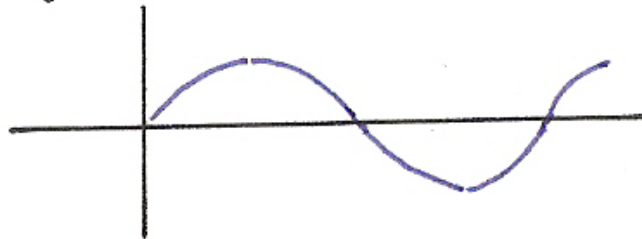
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Periodic Complex Exponential/Sinusoidal Signals

$$x(t) = e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

$\mathrm{Re}\left\{e^{j\omega_0 t}\right\} = \cos \omega_0 t$

$\mathrm{Im}\left\{e^{j\omega_0 t}\right\} = \sin \omega_0 t$

-Fundamental frequency: ω_0

-Fundamental period: $T_0 = \dfrac{2\pi}{\omega_0}$

-In CT, $e^{j\omega_0 t}$ **always periodic**

-Distinct signals for distinct values of ω_0.

-Rapid variation with large ω_0

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Periodicity Properties of DT Complex Exponentials

Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals

- For discrete-time, signals with frequencies ω_0 and $\omega_0 + m \cdot 2\pi$ are identical. This is Not true for continuous-time.

Proof:

$$e^{j(\omega_0 + m \cdot 2\pi)n} = e^{j\omega_0 n} \cdot e^{jm \cdot 2\pi n} = e^{j\omega_0 n}$$

$$e^{j(\omega_0 + X)t} = e^{j(\omega_0 + X)t}$$

or $e^{j(\omega_0 + m \cdot 2\pi)t} \neq e^{j\omega_0 t}$

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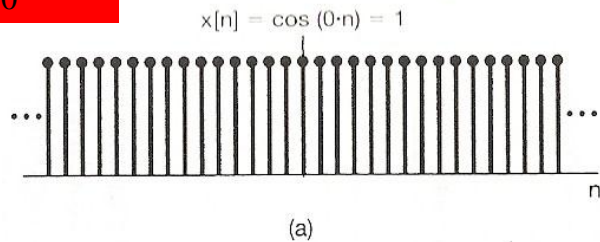# Periodicity Properties of DT Complex Exponentials (cont.)

Understanding:

- We need only consider a frequency interval of length 2π, and on most cases, we use the interval: $0 \le \omega_0 < 2\pi$, or $-\pi \le \omega_0 < \pi$

- $e^{j\omega_0 n}$ does **not** have a continually increasing rate of oscillation as ω_0 is increased in magnitude.

low-frequency (slowly varying): ω_0 near 0, 2π, …, or $2k \cdot \pi$

high-frequency (rapid variation): ω_0 near $\pm \pi$, …, or $(2k+1) \cdot \pi$
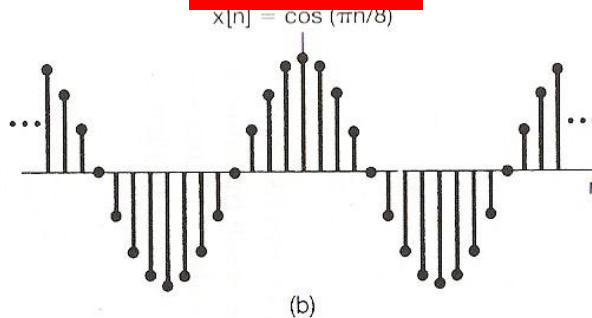
$$e^{j(2k+1)\pi n} = e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

$$e^{j2\pi n} = (e^{j2\pi})^n = (1)^n = 1$$

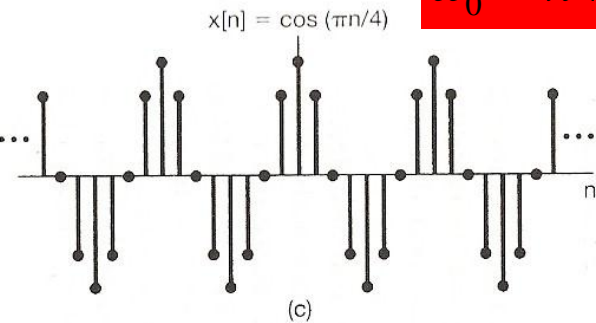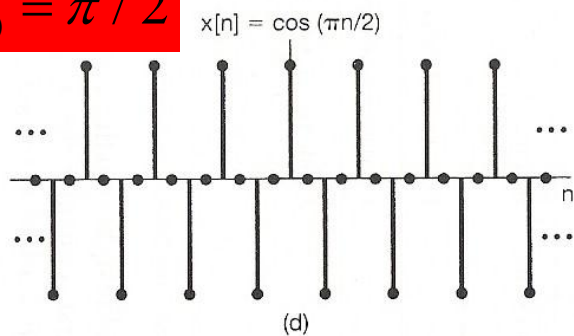南方科技大学 SUSTC

$$\omega_0 = 0$$

$$\omega_0 = \pi/8$$

$$\omega_0 = \pi/4$$

x[n] = cos (0·n) = 1

(a)

x[n] = cos (πn/8)

(b)

x[n] = cos (πn/4)

(c)

$$\omega_0 = \pi/2$$

$$\omega_0 = \pi$$

$$\omega_0 = 3\pi/2$$

x[n] = cos (πn/2)

(d)

x[n] = cos πn

(e)

x[n] = cos (3πn/2)

(f)
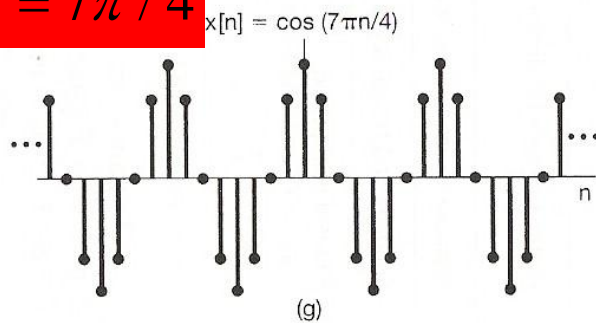
$$\omega_0 = 7\pi/4$$

$$\omega_0 = 15\pi/8$$

$$\omega_0 = 2\pi$$

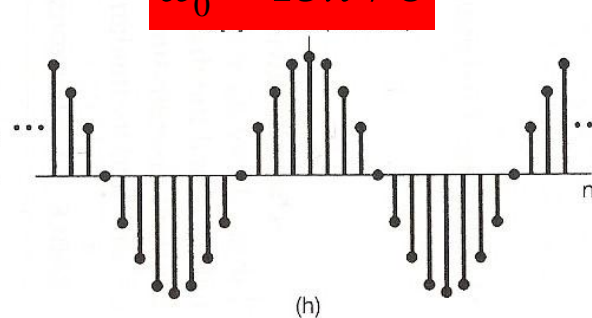x[n] = cos (7πn/4)

(g)

(h)

x[n] = cos 2πn

(i)
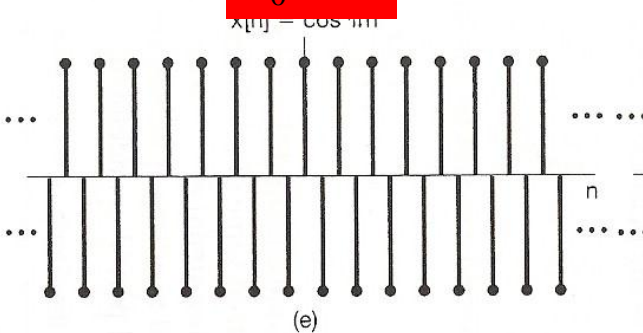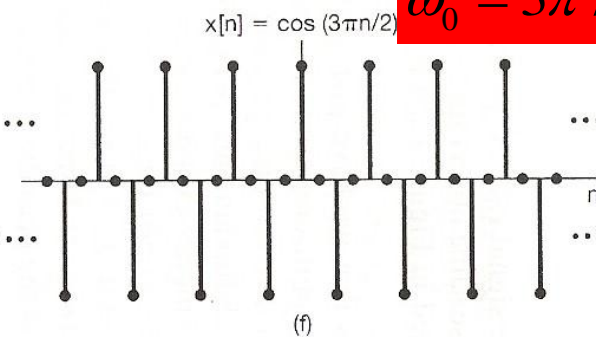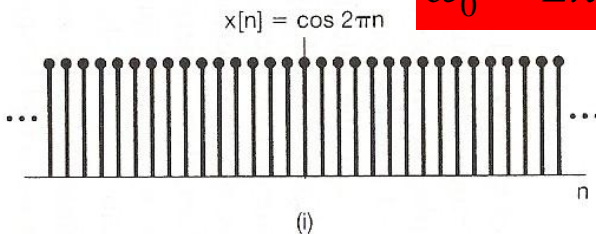
Figure 1.27   Discrete-time sinusoidal sequences for several different frequencies.

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Periodicity Properties of DT Complex Exponentials (cont.)

Important Differences Between Continuous-time and Discrete-time Exponential/Sinusoidal Signals

– For discrete-time, ω_0 is usually defined only for $[-\pi, \pi]$ or $[0, 2\pi]$. For continuous-time, ω_0 is defined for $(-\infty, \infty)$

– For discrete-time, the signal is periodic only when $\omega_0 N = 2\pi m$

Why?

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \quad \rightarrow \quad e^{j\omega_0 N} = 1 \quad \rightarrow \quad \omega_0 N = 2\pi m$$

$$\text{or} \quad \frac{\omega_0}{2\pi} = \frac{m}{N}$$
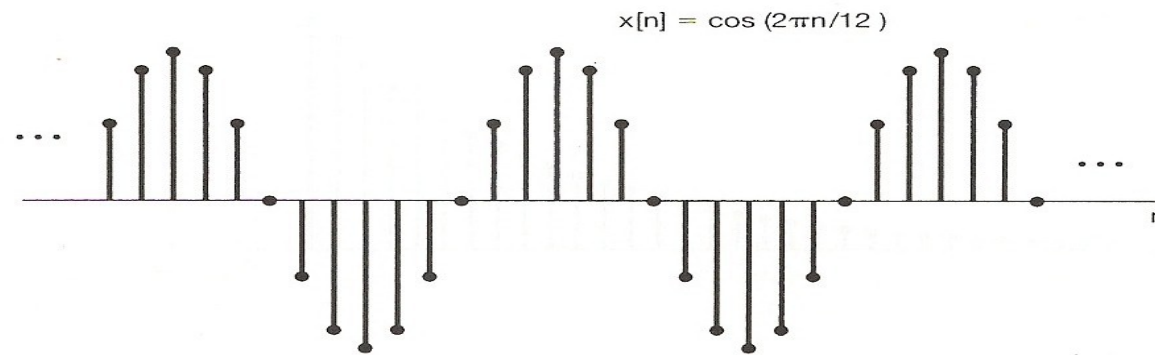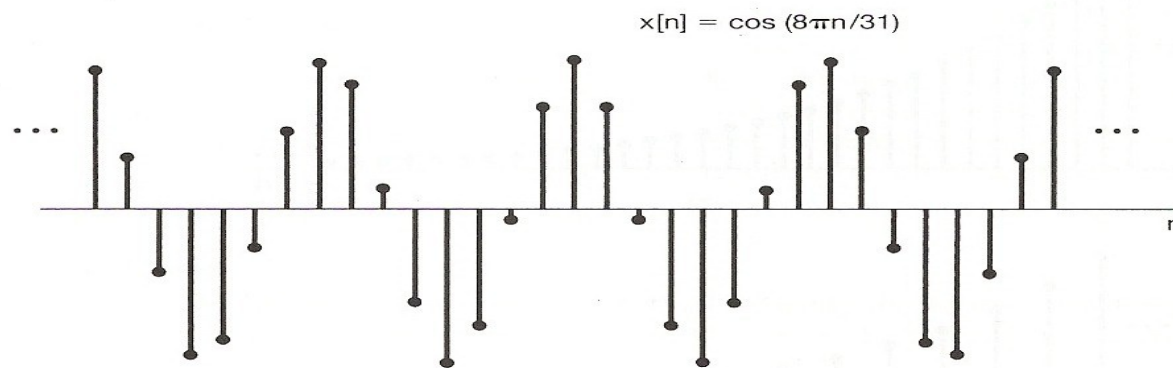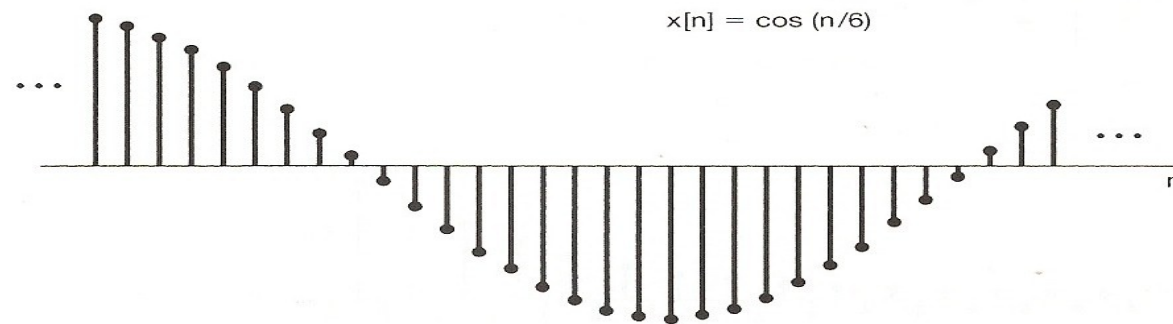
Figure 1.25   Discrete-time sinusoidal signals.

$$\frac{\omega_0}{2\pi} = \frac{2\pi/12}{2\pi} = \frac{1}{12}$$

$$\frac{\omega_0}{2\pi} = \frac{8\pi/31}{2\pi} = \frac{4}{31}$$

$$\frac{\omega_0}{2\pi} = \frac{1/6}{2\pi} = \frac{1}{12\pi}$$

Signals and Systems

Periodicity Properties of DT Complex Exponentials (cont.)

- How to determine the fundamental frequency and fundamental period of a periodic signal $e^{j\omega_0 n}$?

   ◆ This is different from CT periodic signal $e^{j\omega_0 t}$

   ◆ We have $\omega_0 N = 2\pi m$

   Fundamental frequency: $\dfrac{2\pi}{N} = \dfrac{\omega_0}{m}$

   Fundamental period: $N = m(\dfrac{2\pi}{\omega_0})$

See comparison between CT and DT signals in Table 1.1

Unit Impulse (or Unit Sample) Function

Discrete-time

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$
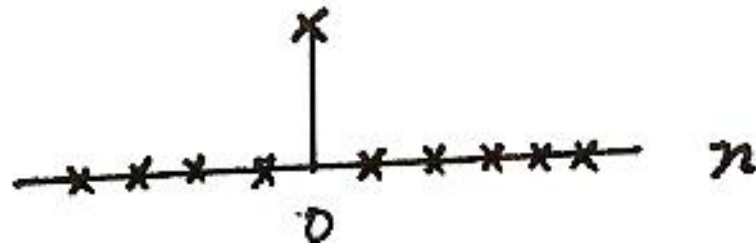
- As a basic building function, we can use unit impulse function to represent other different signals.

e.g.1

e.g.2

$$x[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

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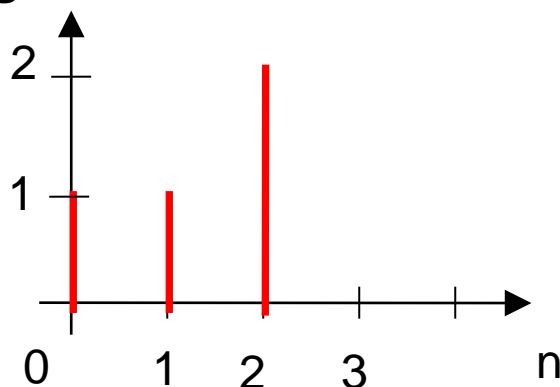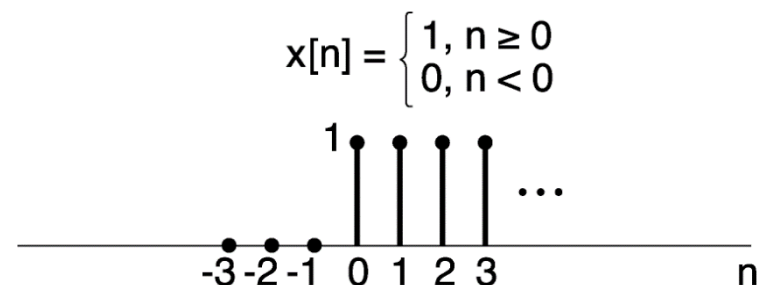
Unit Impulse Function (cont.)

● Sampling property

$x[n]\ \delta[n]\quad = x[0]\ \delta[n]$

$x[n]\ \delta[n-n_0]\ = x[n_0]\ \delta[n-n_0]$

x[n]   **δ[n-2]**   **x[2] δ[n-2]**

$\times$ $=$

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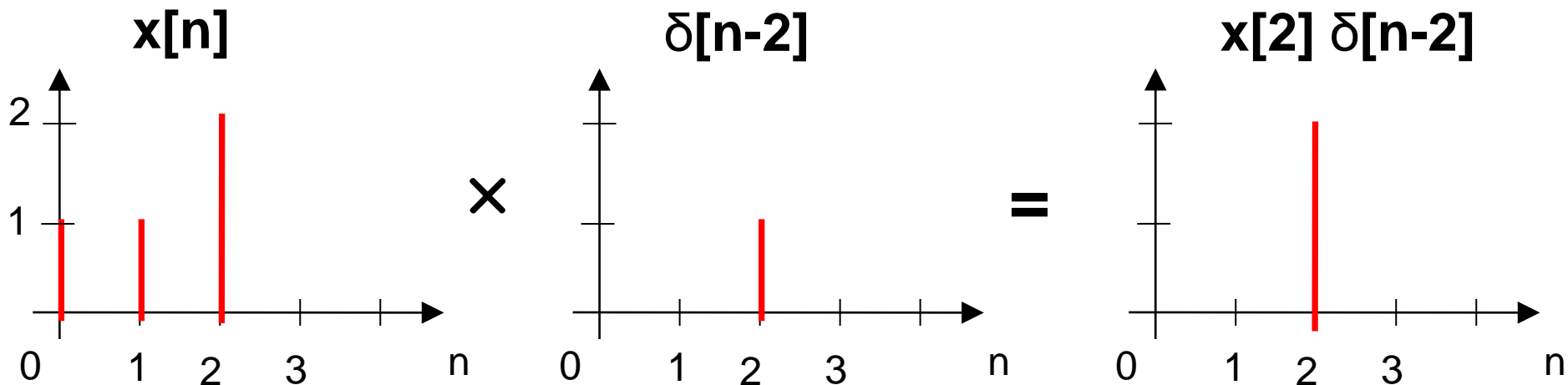
Unit Step Function

Discrete-time

$$x[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$
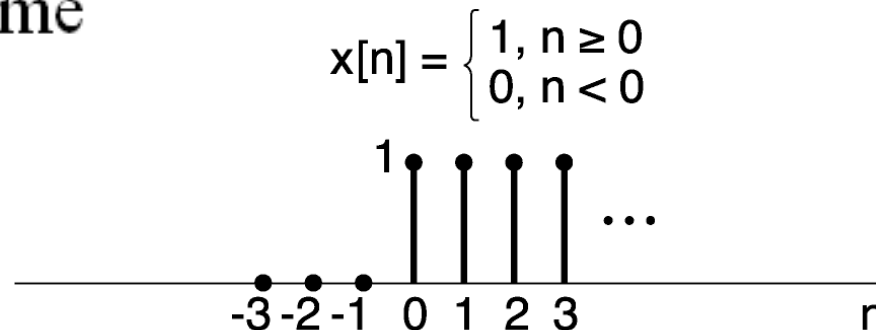
Relation between unit impulse and unit step functions

– First difference

$$\delta [n] = u[n] - u[n-1]$$

– Running Sum

$$u[n] = \sum_{m=-\infty}^{n} \delta [m] \quad \begin{cases} =0,\ n<0 \\ =1,\ n\geq 0 \end{cases}$$

$$u[n] = \sum_{k=0}^{\infty} \delta [n-k]$$

Unit Step Function: First Difference

Discrete-time

$\delta[n]$
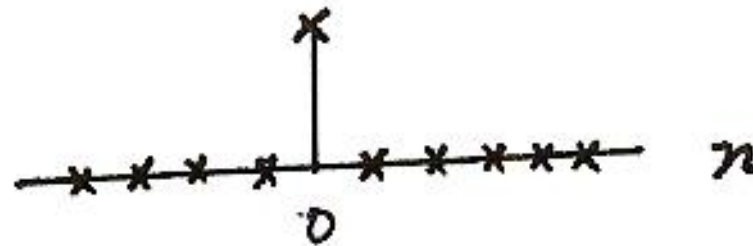
$u[n]$
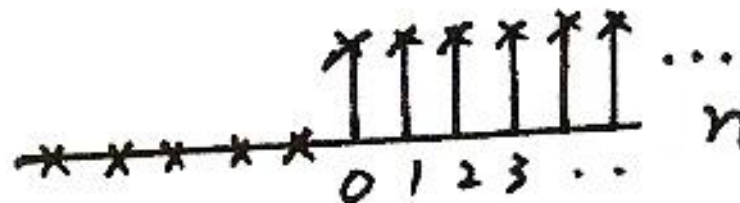
$u[n-1]$
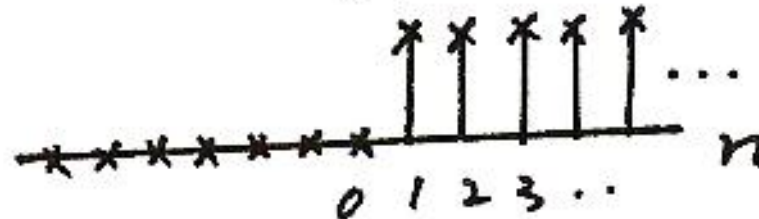
$$\delta[n] = u[n] - u[n-1]$$

Unit Impulse and Unit Step Functions

Continuous-time

$$u(t)=\begin{cases} 0,\ t<0 \\ 1,\ t>0 \end{cases}$$

u(t)

1

0   t

δ(t)

1

0   t
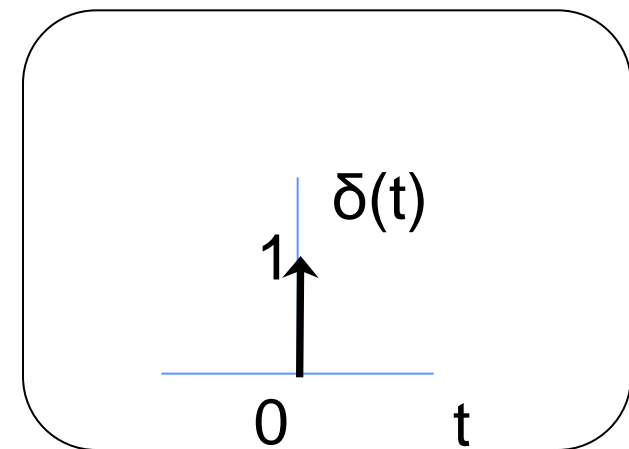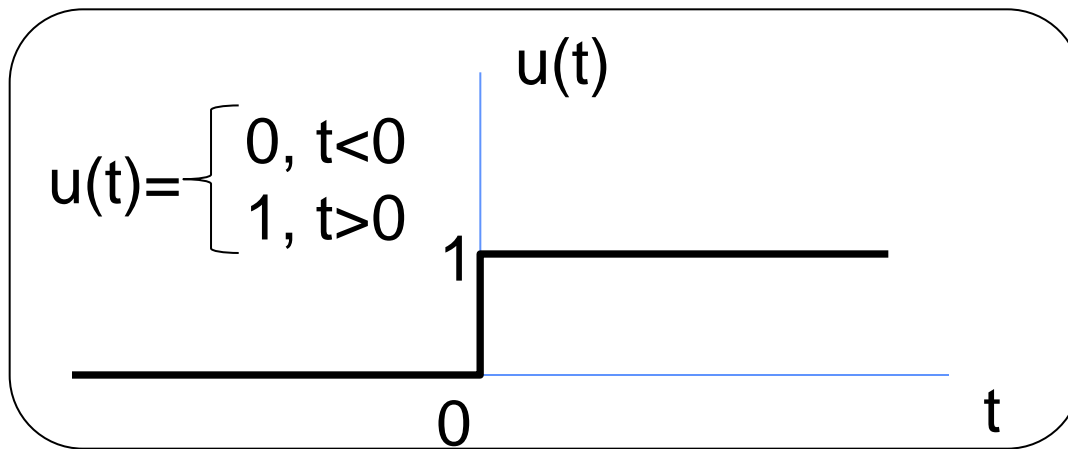
Relation between unit impulse and unit step functions

– First Derivative

$$\delta(t) = \frac{du(t)}{dt}$$

– Running Integral

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau$$

Signals and Systems

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Unit Impulse and Unit Step Functions (Cont.)
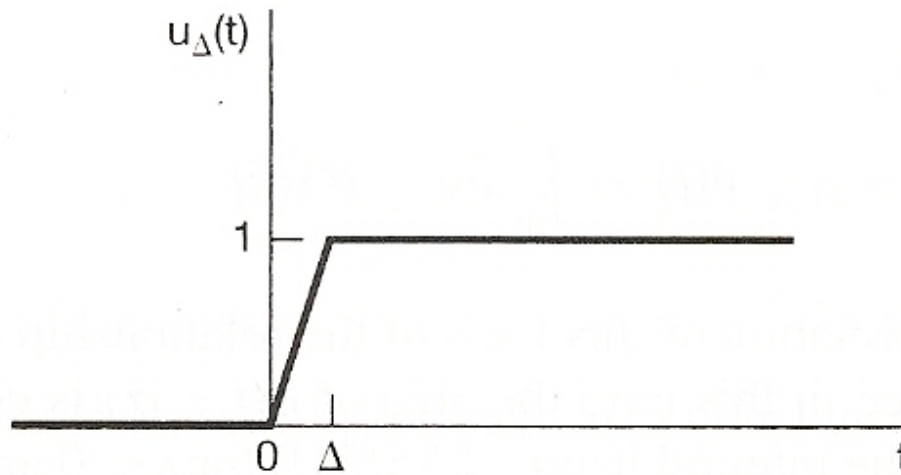
Figure 1.33 Continuous approximation to the unit step, $u_\Delta(t)$.

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$
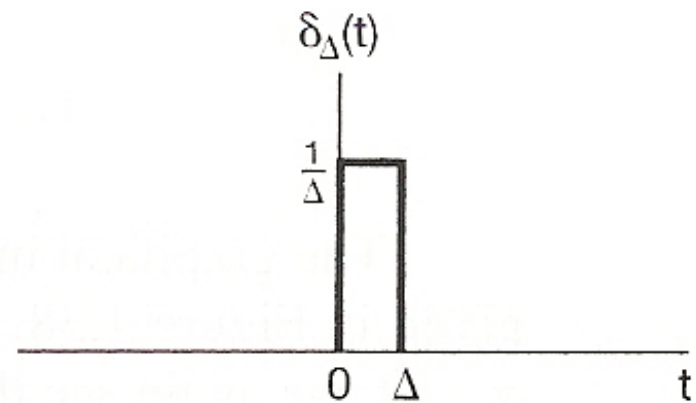
Figure 1.34 Derivative of $u_\Delta(t)$.

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More on CT unit impulse function:

● **δ(t) has in effect no duration, but unit area.**

area = k

area = 1

CT unit impulse

Scaled impulse
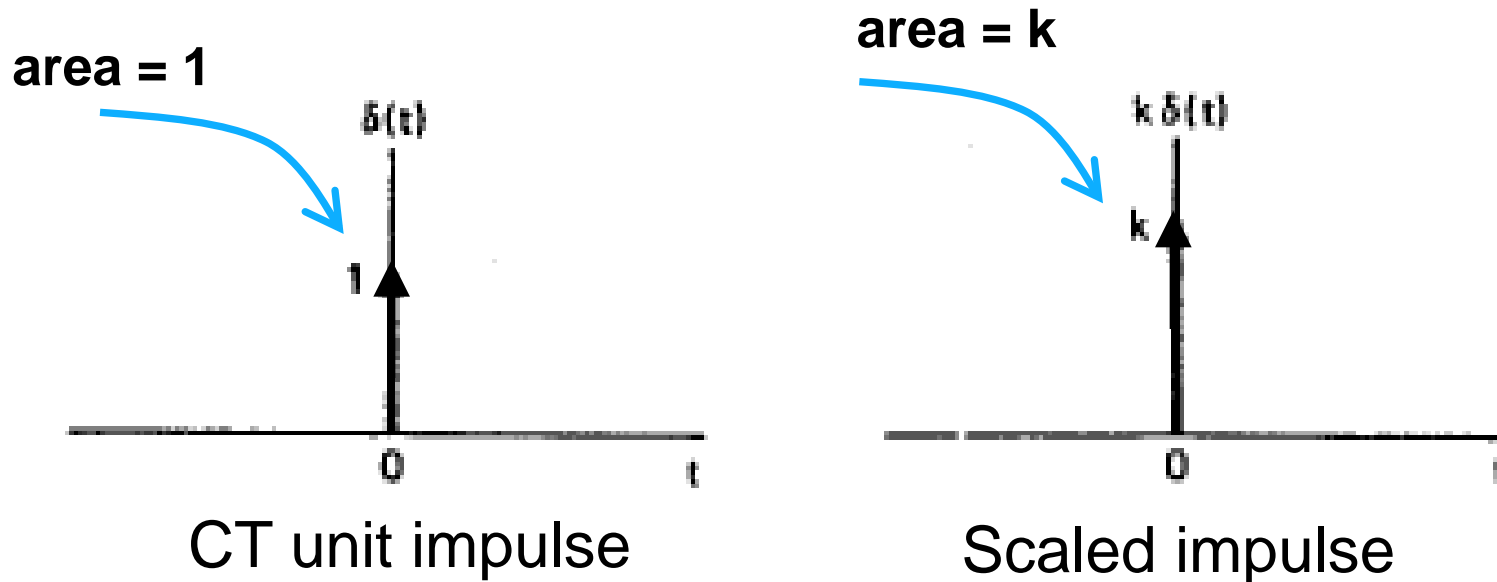
● **Or the integration of CT unit impulse function is unit.** $\int_{-\infty}^{\infty} \delta(t)dt = 1$

Sampling Property

– Sampling property

$$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$$

System Examples

- *Systems* are described from input/output perspective, that is, input to the system *x* causes the output *y*

$$x(t) \longrightarrow \boxed{\text{CT System}} \longrightarrow y(t) \qquad x[n] \longrightarrow \boxed{\text{DT System}} \longrightarrow y[n]$$
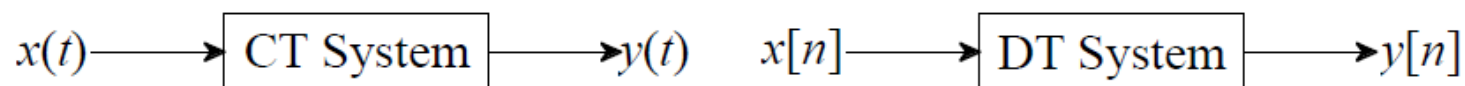
Ex. #1   RLC circuit — an electrical system
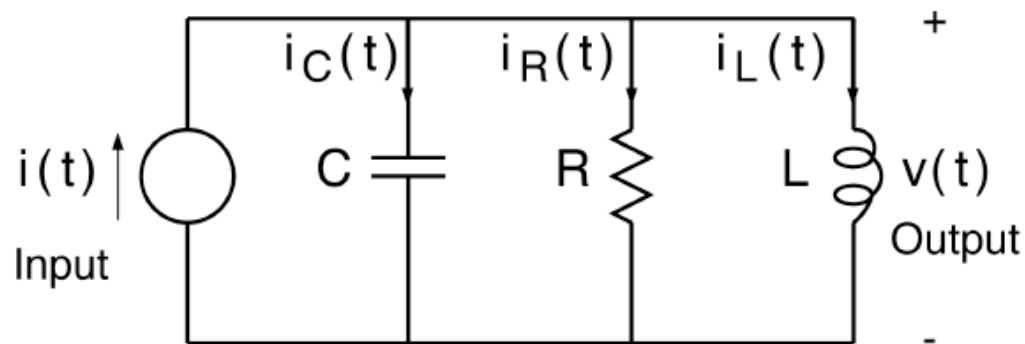
$$i(t) = \underbrace{C\frac{dv(t)}{dt}}_{\text{capacitance}} + \underbrace{\frac{v(t)}{R}}_{\text{resistance}} + \underbrace{\frac{1}{L}\int_{-\infty}^{t}v(\tau)d\tau}_{\text{inductance}}.$$

Ex. #2 A shock absorber – a mechanical system
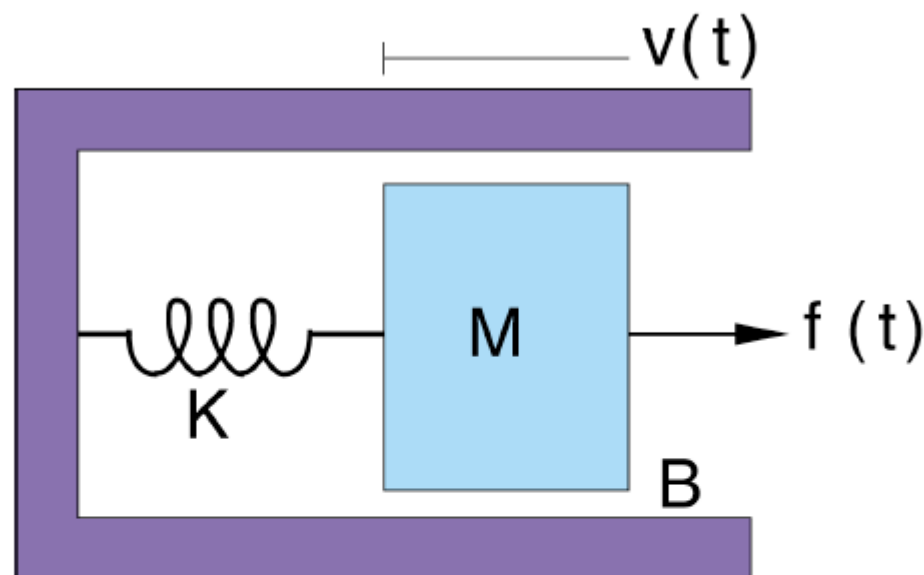
Force Balance:

$$f(t) = \underbrace{M\frac{dv(t)}{dt}}_{inertial\ force} + \underbrace{Bv(t)}_{friction} + \underbrace{K\int_{-\infty}^{t} v(\tau)d\tau}_{spring\ force}.$$

This equation looks quite familiar, we just saw it earlier.

- **Observation: different systems could be described by the same input/output relations**
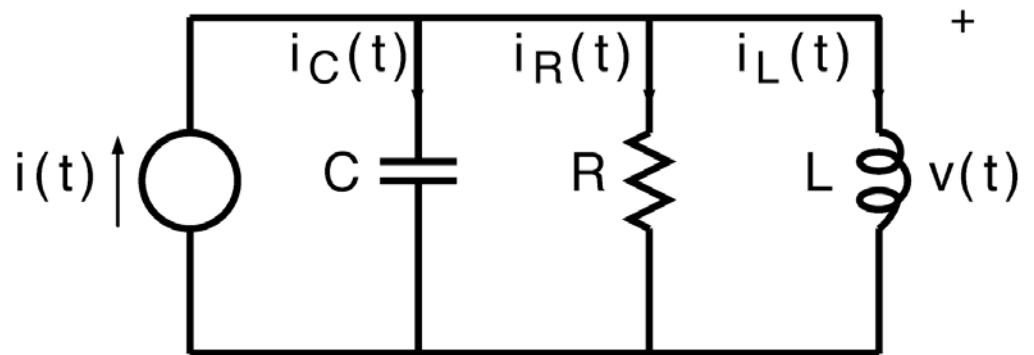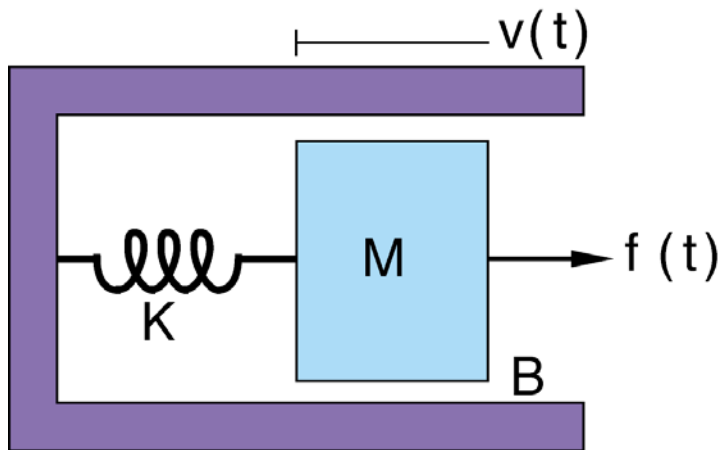
Observations

- A very rich class of systems are described by differential/difference equations.

- Such an equation, by itself, does not completely describe the input-output behaviour of a system: we need auxiliary conditions (initial conditions, boundary conditions).

- Very different physical systems may have very similar or same mathematical descriptions.

Interconnection of Systems

- **Series (cascade)**

- **Parallel**

addition
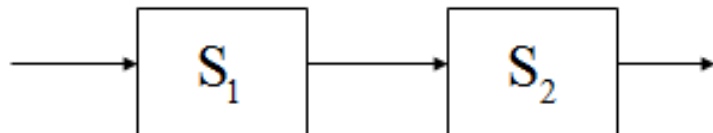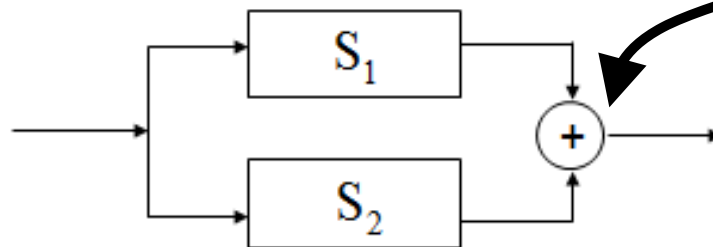
- **Feedback**
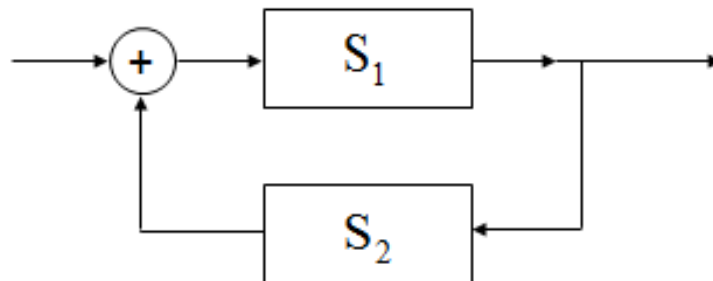
System Properties
(Causality, Linearity, Time-invariance, etc.)

- Why bother with such general properties?
  - Important practical/physical implications. We can make many important predictions of the system behaviours without having to do any mathematical derivations.
  - They allow us to develop powerful tools (*transformations*, more on this later) for analysis and design.

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1) Memoryless or With Memory

Memoryless : output at a given time depends only on the input at the same time

eg.
$$y[n] = (ax[n] - x^2[n])^2$$

With Memory

eg.
$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

summer or accumulator

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Memoryless or With Memory (cont.)

- In physical system, memory is associated with the storage of energy, e.g., capacitor in electric circuit.

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2) Invertability

invertible : distinct inputs lead to distinct outputs, i.e.
an inverse system exits

x $\rightarrow$ [S] $\rightarrow$ y   y $\rightarrow$ [**Inverse system**] $\rightarrow$ x

eg.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$z[n] = y[n] - y[n-1] = x[n]$$

3) Causality

- **<u>Causality</u>**: A system is causal if the output does **not** anticipate future values of the input, i.e., if the output at any time depends only on values of the input up to that time.

- All real-time physical systems are **causal**, because time only moves forward, and effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)

  - ◆ Do **not** apply to spatially varying signals. (We can move both left and right, up and down.)

  - ◆ Do **not** apply to systems processing *recorded (or non-realtime)* signals, e.g. taped sports games vs. live broadcast.

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Causal or Non-causal?

- $y(t) = x^2(t-1)$

- $y(t) = x(t+1)$

- $y(t) = x(t)\cos(t+1)$

- $y[n] = x[-n]$

- $y[n] = (1/2)^{n+1} x^3[n-1]$

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4) Stability

- Small input leads to response that does not diverge.
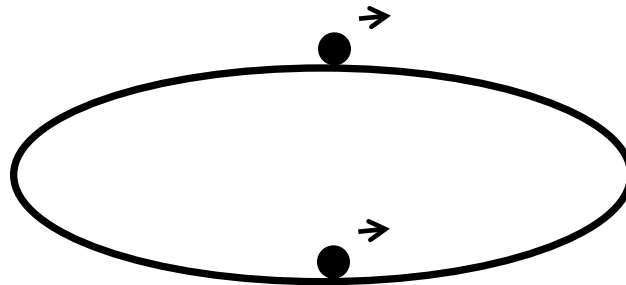
- ◆ If the input to a stable system is bounded, the output must also be bounded.

- ◆ e.g.: S_1: y(t) = t x(t)

        S_2: y(t) = $e^{x(t)}$

Stability (cont.)

- Stability of physical systems are due to the presence of mechanisms that dissipate energy, e.g., resistor, friction.

- Stable or not (Example 1.13 in textbook)?

  - Find specific example, e.g, for S: $y(t)=t\,x(t)$

  - To prove that all bounded inputs lead to bounded output.

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5) Time Invariance (TI)

- Informally, a system is time-invariant (TI) if its behavior does not depend on the choice of $t = 0$. Then two identical experiments will yield the same results, regardless the starting time.

  - Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for *any* input $x[n]$ and *any* time shift n_0

$$\text{If} \quad x[n] \rightarrow y[n]$$
$$\text{then} \quad x[n - n_0] \rightarrow y[n - n_0] .$$

  - Similarly for CT time-invariant system

$$\text{If} \quad x(t) \rightarrow y(t)$$
$$\text{then} \quad x(t - t_o) \rightarrow y(t - t_o) .$$

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Time-invariant or Time-varying?

- **Steps:**
1) **Calculate $y_1(t) \leftarrow x_1(t)$**
2) **Calculate $y_2(t) \leftarrow x_2(t)=x_1(t-t_0)$**
3) **Does $y_1(t-t_0)$ equal $y_2(t)$?**

e.g.: $y[n] = \left(\dfrac{1}{2}\right)^{n+1} x^3[n-1]$

① $y_1[n] = \left(\dfrac{1}{2}\right)^{n+1} x_1^3[n-1]$

② $x_2[n] = x_1[n-n_0]$

$y_2[n] = \left(\dfrac{1}{2}\right)^{n+1} x_2^3[n-1]$

$= \left(\dfrac{1}{2}\right)^{n+1} x_1^3[n-n_0-1]$

③ $y_1[n-n_0] = \left(\dfrac{1}{2}\right)^{n-n_0+1} \cdot x_1^3[n-n_0-1]$

$\therefore y_1[n-n_0] \neq y_2[n]$

$\therefore$ Time-varying

Now we can deduce something:

- If the input to a TI system is periodic, then the output is also periodic with the same period (Problem 1.43 (a)).

Proof: Suppose $x(t + T) = x(t)$
and $x(t)$ $\rightarrow$ $y(t)$

Then by TI

$$x(t + T) \quad \rightarrow \quad y(t + T)$$

$\uparrow$ $\uparrow$

But these are   So these must be
the same input!   the same output,

$i.e., \ y(t) = y(t+T)$

6) Linear and Nonlinear Systems

- Many systems are nonlinear.
  - ◆ **e.g.:** economic system with
    input: fiscal and monetary policies, labor, resources, etc.
  - → output: GDP, inflation, etc.
  - ◆ System behaviour is very unpredictable, because it is highly nonlinear.
- We will deal with only linear systems, which are good approximations of nonlinear systems in certain ranges.
  - ◆ **e.g.:** small-signal conductance of a nonlinear diode.
- Linear systems can be analysed accurately.

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Linearity

We have $x_1(t) \rightarrow y_1(t)$, and $x_2(t) \rightarrow y_2(t)$.

The system is linear, if:

1) **<u>Additivity</u> property:** $x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$

2) **<u>Scaling (or homogeneity)</u> property:**

$$a\ x_1(t) \rightarrow a\ y_1(t)$$

where a is a complex number

e.g.: $y(t) = 2\ x(t)$ $\qquad y(t) = x^2(t)$

Linearity (cont.)

A (CT) system is linear if it has the superposition property:

If $\qquad x_1(t) \rightarrow y_1(t) \quad$ and $\qquad x_2(t) \rightarrow y_2(t)$

then $\qquad ax_1(t) + bx_2(t) \quad \rightarrow \quad ay_1(t) + by_2(t)$

- For linear systems, zero input $\rightarrow$ zero output

"Proof" $\qquad 0 = 0 \cdot x[n] \rightarrow 0 \cdot y[n] = 0$

Question: $\qquad$ Is the system $y = A + x$ linear?

● *Incrementally linear system!*

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Linear system or not?

● **Steps**

1) Have $y_1(t)$ and $y_2(t)$ as output signals to $x_1(t)$ and $x_2(t)$

2) Have $y_3(t)$ as output signal to $x_3(t) = a\,x_1(t) + b\,x_2(t)$

3) Does $y_3(t)$ equal "$a\,y_1(t) + b\,y_2(t)$"?

More examples on textbook

Read Example 1.17 ~ 1.20

Linearity (cont.)

- Superposition

If
$$x_k[n] \;\rightarrow\; y_k[n]$$

Then
$$\sum_k a_k x_k[n] \;\rightarrow\; \sum_k a_k y_k[n]$$

- This property seems to be almost trivial now, but it is one of the most important ones

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Linear Time-invariant (LTI) Systems

- Focus of most of this course
  - Practical importance (**e.g.** #1-2 earlier this lecture are both LTI systems.)
  - The powerful analysis tools (*transformation*) associated with LTI systems
- A basic fact: If we know the response of an LTI system to some inputs, we actually know the response to many inputs.

Example: DT LTI System

$x_1 \rightarrow y_1$

$x_2 = f(x_1)$
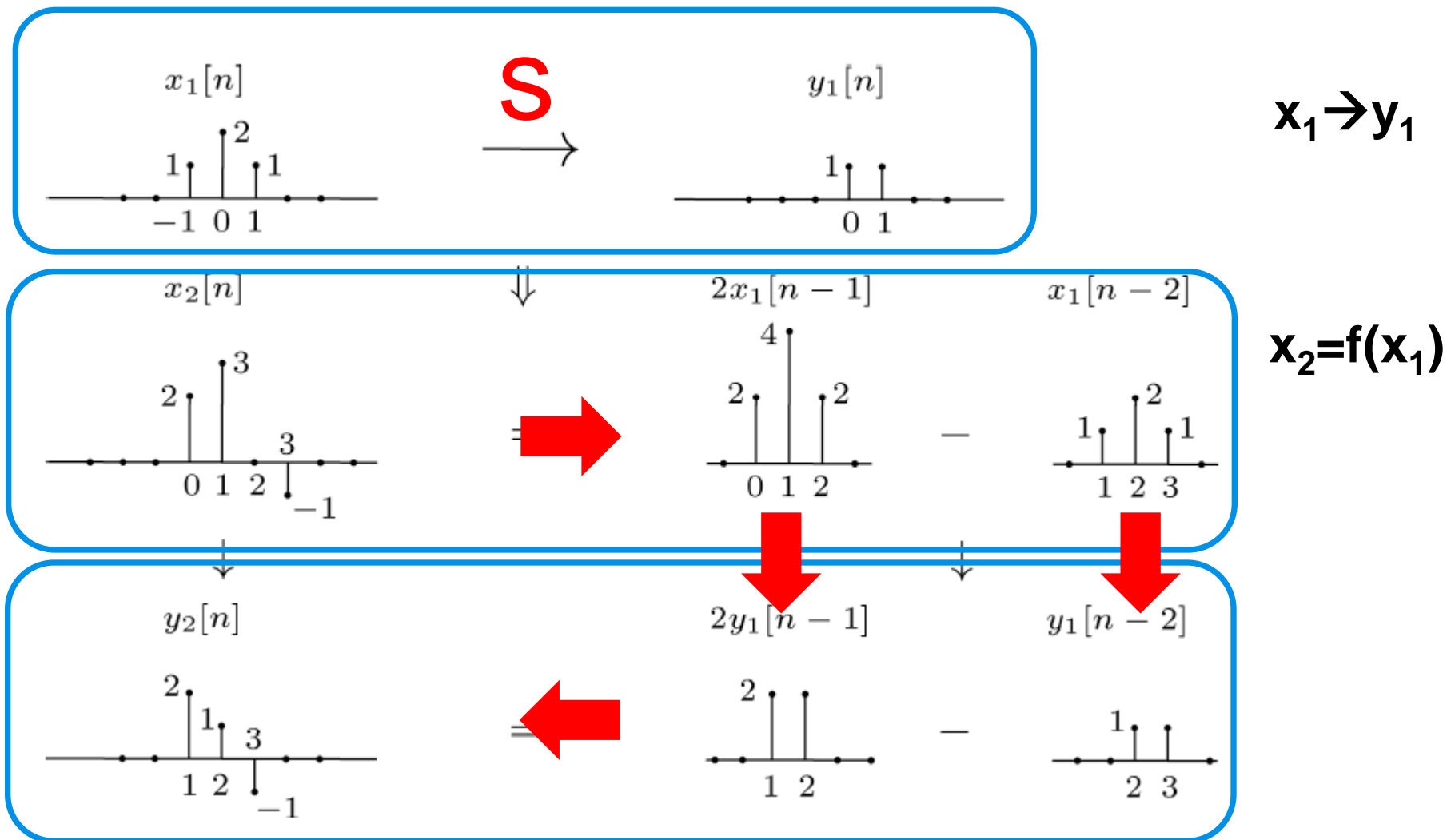
Summary

- **Signals**
  - ◆ **Exponential signal (Difference between CT and DT)**
  - ◆ **Unit impulse function**
  - ◆ **Unit step function**
- **Systems**
  - ◆ **Properties**
  - ◆ **Linear, & time-invariant**