EE206: Communications Principles Tutorial

Assignment 6

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1. An FM modulator is followed by an ideal band-pass filter with centre frequency of 500 Hz and bandwidth of 72 Hz. The gain of the filter is 1 in the pass-band. The message signal $m(t) = 10\cos(20\pi t)$ and the carrier signal is $f(t)=10\cos(1000\pi t)$. The modulation frequency sensitivity $k_f = 7$ Hz/volt.

Single tone message signal

   a. Draw the amplitude spectra of the FM signal at the input and output of the band-pass filter, respectively.

   b. Determine the signal power at the input and output of the band-pass filter.

$$f_m = 10\text{Hz}, A_m = 10\text{V}, k_f = 7\text{Hz/V}.$$

$$\beta = \frac{k_f A_m}{f_m} = \frac{7\times 10}{10} = 7 > 0.2$$

=> Wideband FM

a. Draw the amplitude spectra of the FM signal at the input and output of the band-pass filter, respectively.

	7
$n=0$	0.3001
	−0.004683
	−0.3014
	−0.1676
	0.1578
	0.3479
	0.3392
	0.2336
	0.1280
	0.05892
$n=10$	0.02354
	0.008335
	0.002656

$$f_{FM}(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$

$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

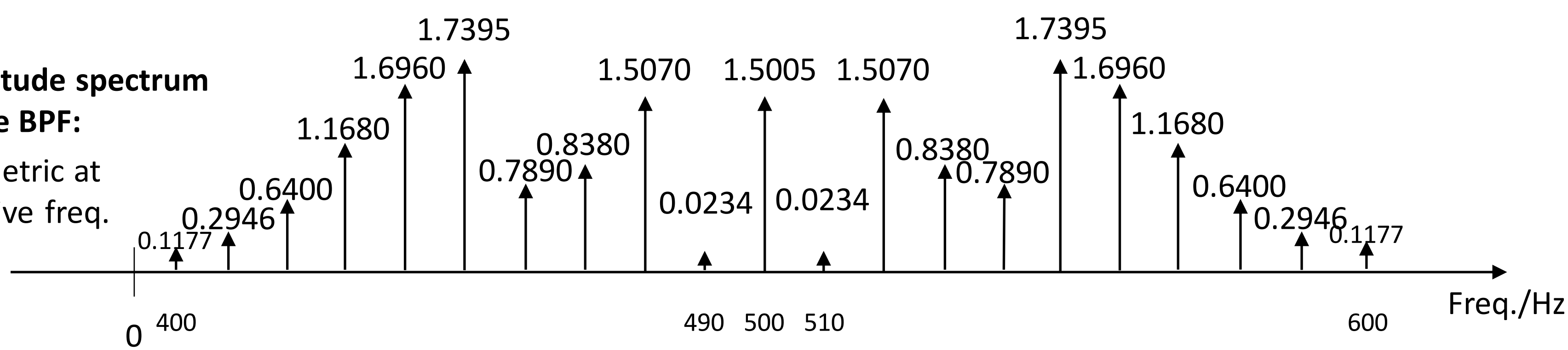
$$F_{FM}(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

$$|F_{FM}(f)| = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} |J_n(\beta)|[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

$$\beta = 7,\ f_m = 10\text{Hz},\ A_c = 10V,\ f_c = 500\text{Hz}.$$

Amplitude spectrum before BPF:

Symmetric at negative freq.

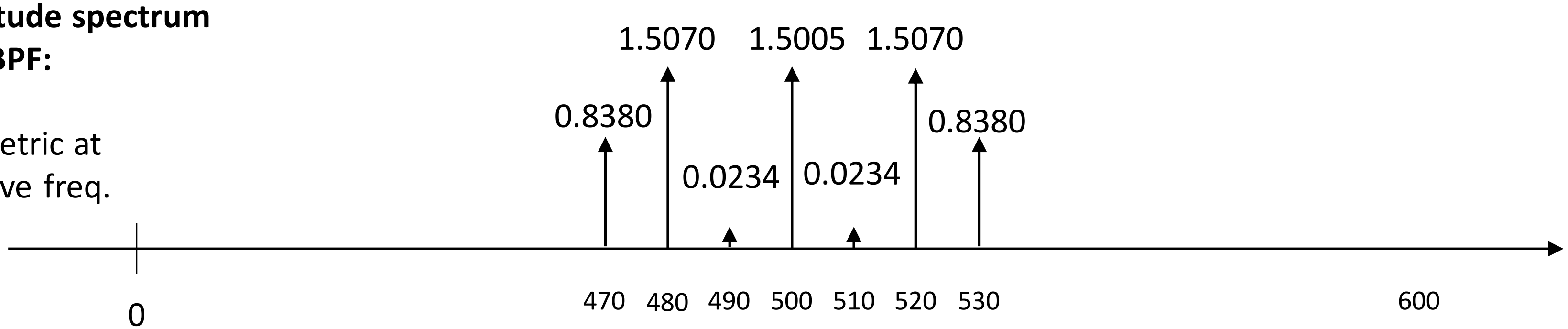


a. Draw the amplitude spectra of the FM signal at the input and output of the band-pass filter, respectively.

BPF:

Amplitude spectrum after BPF:

Symmetric at negative freq.



b. Determine the signal power at the input and output of the band-pass filter.

Before BPF:

$$P_{in} = 2 \times (\frac{A_c}{2})^2 \sum_{n=-\infty}^{\infty} J_n{}^2(\beta) \\ = \frac{A_c{}^2}{2} \sum_{n=-\infty}^{\infty} J_n{}^2(\beta) \\ = 50\text{W}$$

Some useful properties of $J_n(\beta)$

1. $J_n(\beta)$ are real valued.
2. $J_n(\beta) = J_{-n}(\beta)$, for even n
3. $J_n(\beta) = -J_{-n}(\beta)$, for odd n
4. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

$$f_{FM}(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t] \\ = A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

After BPF:

$$P_{out} = \frac{A_c{}^2}{2} \sum_{n=-3}^{3} J_n{}^2(\beta) \\ = \frac{A_c{}^2}{2} [J_0{}^2(\beta) + 2[J_1{}^2(\beta) + J_2{}^2(\beta) + J_3{}^2(\beta)]] \\ = 50 \times [0.3001^2 + 2 \times (0.004683^2 + 0.3014^2 + 0.1676^2)] \\ = 16.4\text{W}$$

2. Show that unlike AM, the mean power of an FM signal in the form of $A_c \cos[\omega_c t + \beta \sin \omega_m t]$ is independent of modulation index, β (Hint: make use of the property $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$).

- The message contains a single tone/frequency component, i.e., $m(t) = A_m \cos 2\pi f_m t$.
- The single-tone AM is given by
  $x(t) = A_c[1 + \mu \cos 2\pi f_m t] \cos 2\pi f_c t$, where $\mu = k_a A_m$.
- To avoid overmodulation, μ must be kept below 1.

Modulation index of AM

$$\overline{f_{FM}{}^2(t)} = \overline{\{A_c \cos[\omega_c t + \beta \sin(\omega_m t)]\}^2}$$

$$= \overline{[A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c t + n\omega_m t)]^2}$$

$$= \frac{A_c{}^2}{2} \sum_{n=-\infty}^{\infty} J_n^2(\beta) \quad (\because \sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1)$$

$$= \frac{A_c{}^2}{2}$$

3. For an FM signal $f_{FM}(t) = 6\cos(2\pi 10^9 t + 4\sin 2\pi 10^3 t)$, calculate the total mean power of the significant sideband components and the carrier component within the bandwidth, where

   a. the bandwidth is determined using 1% rule, and

   b. the bandwidth is determined using Carson's rule.

	4
$n = 0$	−0.3971
	−0.06604
	0.3641
	0.4302
	0.2811
	0.1321
	0.04909
$n = 7$	0.01518
	0.004029

Q3. **(a)** $f_{FM}(t) = 6\cos(2\pi 10^9 t + 4\sin 4\pi 10^3 t) \;\Rightarrow\; \beta = 4.$

$|J_n(4)|>0.01$

From the Bessel function table, no. of significant side-band pairs n'=7.

$$P = \frac{A_c^2}{2}\sum_{n=-7}^{7} J_n^{\,2}(\beta) = \frac{6^2}{2}\{(0.3971)^2 + 2[(0.06604)^2 + (0.3641)^2 + (0.4302)^2$$

$$+ (0.2811)^2 + (0.1321)^2 + (0.04909)^2 + (0.01518)^2]\} = \frac{6^2}{2}\times \boxed{0.99991} = 17.99838$$

$\boxed{99.991\%}$ of the total signal power is included in the bandwidth.

3. For an FM signal $f_{FM}(t) = 6\cos(2\pi 10^9 t + 4\sin 2\pi 10^3 t)$, calculate the total mean power of the significant sideband components and the carrier component within the bandwidth, where

a. the bandwidth is determined using 1% rule, and

b. the bandwidth is determined using Carson's rule.

(b) Using Carson's rule

$$BW = 2(\beta+1)f_m = 2(4+1)f_m = 10 f_m$$

$\Rightarrow$ The first 5 side-band pairs are included in the bandwidth.

$$P = \frac{A_c^2}{2}\sum_{n=-5}^{5} J_n^{\,2}(\beta) = \frac{6^2}{2}\ \{(0.3971)^2 + 2[(0.06604)^2 + (0.3641)^2 + (0.4302)^2$$

$$+ (0.2811)^2 + (0.1321)^2]\} = \frac{6^2}{2} \times \boxed{0.99464} = 17.90352$$

$\boxed{99.464\%}$ of the total signal power is included in the bandwidth.

4. A message signal $m(t) = 5\sin(2000\pi t)$ phase modulates a cosine wave of 100 MHz. The PM signal has peak-phase deviation of $\pi/2$ and amplitude $A_c = 100$ volts.

   a. Determine the amplitude spectrum of the PM signal.

   b. Determine the approximate bandwidth which contains 99% of total power of the PM signal.

   c. Determine the approximate bandwidth using Carson's rule and compare the results with the analytical result obtained in part (b).

Given: $J_0(\pi/2)=0.4720$, $J_1(\pi/2)=0.5668$, $J_2(\pi/2)=0.2497$, $J_3(\pi/2)=0.0690$, $J_4(\pi/2)=0.0140$.

Single Tone Modulation

Now we consider a special message signal, $m(t) = A_m \cos 2\pi f_m t$ (referred to as a single tone message signal).

PM:

$$\theta_i(t) = 2\pi f_c t + k_p A_m \cos 2\pi f_m t = 2\pi f_c t + \beta_p \cos 2\pi f_m t$$

where $\beta_p = k_p A_m$ is the "peak phase deviation", i.e., the maximum amount of phase difference between $f_{PM}(t)$ and the unmodulated carrier $A_c \cos 2\pi f_c t$.

The single-tone PM signal is given by

$$\boxed{f_{PM}(t) = A_c \cos[2\pi f_c t + \beta_p \cos 2\pi f_m t]}$$

β_p *is also called modulation index* (for PM).

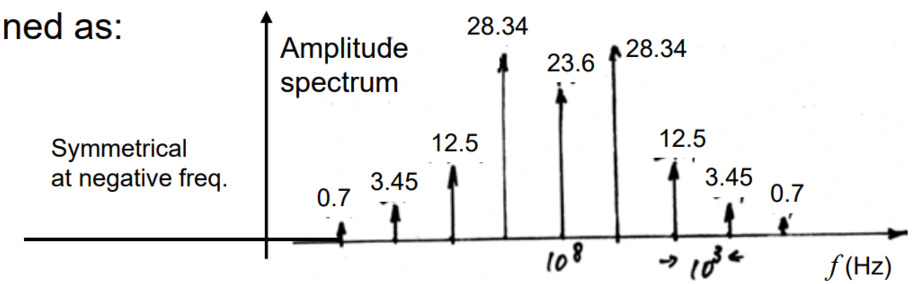
$$f_{FM}(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$
$$= A_c \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m)t$$

Q4. (a) $f_c = 10^8 Hz, \quad f_m = 1000 Hz, \quad \beta_p = \frac{\pi}{2}, \quad A_c = 100V$

$$f_{PM}(t) = 100 \cos[2\pi 10^8 t + \frac{\pi}{2} \sin 2000\pi t] = 100 \sum_{n=-\infty}^{\infty} J_n(\frac{\pi}{2}) \cos[2\pi(10^8 + n10^3)t]$$

Using $J_0(\pi/2)=0.4720$, $J_1(\pi/2)=0.5668$, $J_2(\pi/2)=0.2497$, $J_3(\pi/2)=0.0690$, $J_4(\pi/2)=0.0140$,

The amplitude spectrum is obtained as:



(b) The total signal power is

$$\frac{A_c^2}{2} = \frac{100^2}{2} = 5000 \qquad (W)$$

To find the BW (containing 99% of total power), we need to find the minimum integer *K* with

$$\frac{100^2}{2} \sum_{n=-K}^{K} J_n^2(\frac{\pi}{2}) \geq 0.99 \times 5000$$

From the given Bessel function values, we can find *K = 2.*

$$BW_{effective} = 2Kf_m = 4000 \quad (Hz)$$

(C) Using Carson's rule, $BW = 2(\frac{\pi}{2} + 1) f_m \approx 5140 \quad (Hz)$

Bessel Function Table

Values of the Bessel Functions $J_n(\beta)$

n \ β	0.5	1	2	3	4	5	6	7	8	9	10
0	0.9385	0.7652	0.2239	−0.2601	−0.3971	−0.1776	0.1506	0.3001	0.1717	+0.09033	−0.2459
1	0.2423	0.4401	0.5767	0.3391	−0.06604	−0.3276	−0.2767	−0.004683	0.2346	0.2453	0.04347
2	0.03060	0.1149	0.3528	0.4861	0.3641	0.04657	−0.2429	−0.3014	−0.1130	0.1448	0.2546
3	0.002564	0.01956	0.1289	0.3091	0.4302	0.3648	0.1148	−0.1676	−0.2911	−0.1809	0.05838
4		0.002477	0.03400	0.1320	0.2811	0.3912	0.3576	0.1578	−0.1054	−0.2655	−0.2196
5			0.007040	0.04303	0.1321	0.2611	0.3621	0.3479	0.1858	−0.05504	−0.2341
6			0.001202	0.01139	0.04909	0.1310	0.2458	0.3392	0.3376	0.2043	−0.01446
7				0.002547	0.01518	0.05338	0.1296	0.2336	0.3206	0.3275	0.2167
8					0.004029	0.01841	0.05653	0.1280	0.2235	0.3051	0.3179
9						0.005520	0.02117	0.05892	0.1263	0.2149	0.2919
10						0.001468	0.006964	0.02354	0.06077	0.1247	0.2075
11							0.002048	0.008335	0.02560	0.06222	0.1231
12								0.002656	0.009624	0.02739	0.06337
13									0.003275	0.01083	0.02897
14									0.001019	0.003895	0.01196
15										0.001286	0.004508
16											0.001567