Outline 幅度

$m(t)$ $f_c(t)$ $A_c \cos(w_c t)$

① Introduction

② Amplitude Modulation

③ Angle Modulation { FM 调频, PM 调相

思路：$\theta(t)$ of $f_c(t)$

$\theta(t) \propto m(t)$

Angle Modulation

Features:

- The angle of the carrier signal is varied according to the baseband modulating signal (message signal). → is being modulated
- The amplitude of the carrier signal is maintained constant. 幅度不变

Advantage:

- Provides better discrimination against noise and interference than Amplitude Modulation (AM).

Tradeoff:

- Performance improvement is achieved at the expense of increased transmission bandwidth.

只有宽带FM才有性能提高

代码宽宽为代价

SNR output ↑

Angle Modulation - Topics

I. Basics of FM & PM
  - Definitions
  - Single-tone 單音 modulation $m(t) = A_m \cos(2\pi f_m t)$

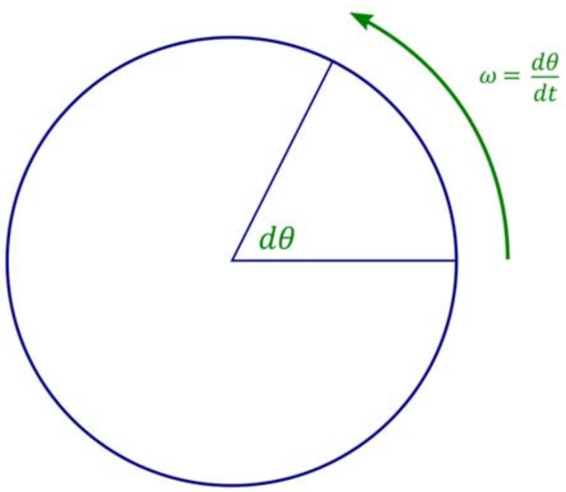
II. Narrow-Band and Wide-Band FM

III. Bandwidth of FM Signals

IV. Generation of FM Signals

V. Demodulation and Performance of FM Signals

Angle, Angular Frequency, Frequency



Angular frequency (or angular speed), in radian/second, is a measure of how fast an object is rotating around its axis.

- Angular frequency, ω, is also referred to as angular speed.
- $\omega = 2\pi f$, where f denotes frequency (in Hz or 1/second).
- For a constant ω (not changing with time), $\omega = \frac{2\pi\ (radians)}{T\ (seconds)}$.
- In general,

$$\boxed{\begin{aligned} \omega(t) &= \frac{d\theta(t)}{dt} \\ \Longleftrightarrow \theta(t) &= \int_{-\infty}^{t} \omega(\tau)d\tau \\ &= 2\pi \int_{-\infty}^{t} f(\tau)d\tau \end{aligned}}$$

Basic Definitions: FM & PM

Recall the general expression for a carrier waveform:

$$f(t) = A_c \cos[2\pi f_c t + \phi(t)] = A_c \cos\theta(t) \Rightarrow f_i(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + \frac{1}{2\pi}\frac{d\phi(t)}{dt}$$

A_c : carrier amplitude

f_c : carrier frequency (i.e., frequency of unmodulated carrier)

$\phi(t)$: instantaneous phase of carrier

$\theta(t)$: instantaneous angle of carrier

$\theta(t)$ can be varied in some manner with the message signal.

There are 2 commonly used angle modulation schemes:

– Phase modulation (PM)

– Frequency modulation (FM)

Phase modulation

- in which the instantaneous phase $\phi(t)$ is varied linearly with the message signal $m(t)$, i.e.,

$$\boxed{\phi(t) = \phi_0 + k_p m(t)}$$

ϕ_0 : initial phase; assumed to be zero unless otherwise specified.

k_p : phase sensitivity, a constant (*radian/volt*)

($m(t)$ is assumed to be a voltage signal.)

$k_p m(t)$: instantaneous phase deviation, $\Delta\phi(t)$ 定义

The phase modulated (PM) signal $f_{PM}(t)$ is then given by

$$\boxed{f_{PM}(t) = A_c \cos[2\pi f_c t + \underbrace{k_p m(t)}_{\phi(t)}].}$$

Frequency Modulation

- in which the instantaneous carrier frequency, $f_i(t)$, is <u>varied</u> <u>linearly</u> with the message signal $m(t)$, i.e.,

$$f_i(t) = f_c + \boxed{k_f m(t)}$$

Instantaneous carrier frequency deviation, $\Delta f_i(t)$

f_c : carrier frequency, a constant

k_f : frequency sensitivity, a constant (*Hz/volt*).

Using the relationship between angle and freq., it follows that

$$\theta_i(t) = 2\pi\int_{-\infty}^{t} f_i(\tau)d\tau = 2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$$

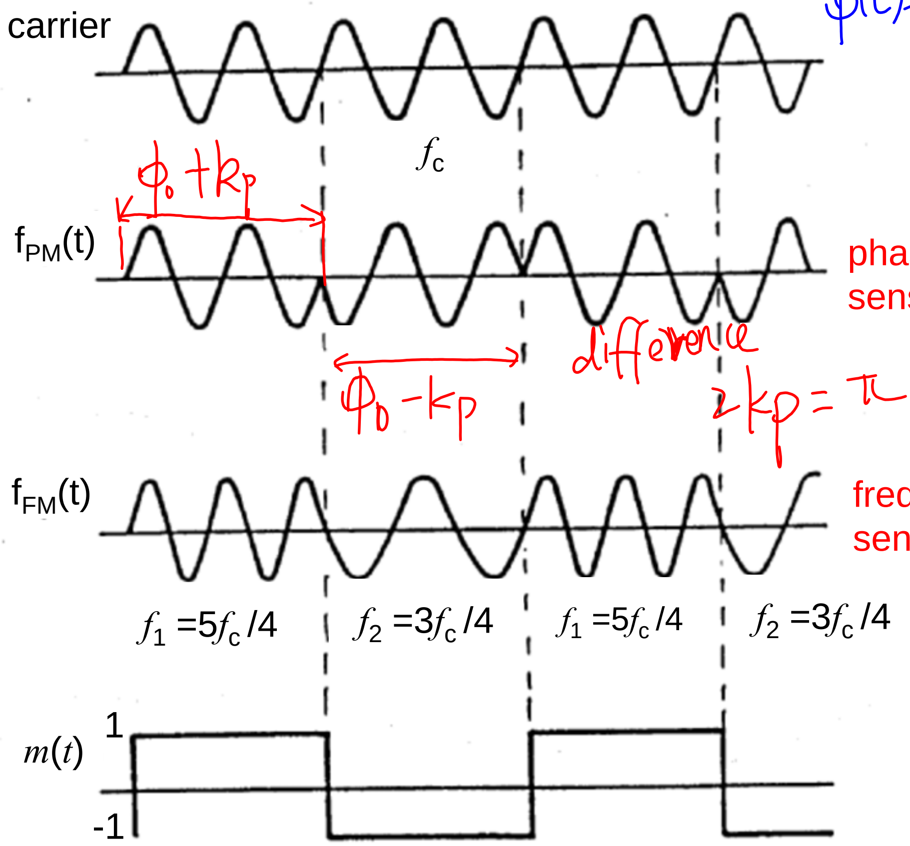
The frequency modulated (FM) signal is given by

$$\boxed{f_{FM}(t) = A_c \cos[2\pi f_c t + \underbrace{2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau}_{\phi(t)}]}$$

$$\boxed{m(t) = \frac{1}{2\pi k_f}\frac{d\phi(t)}{dt}}$$

Exercise:

$\phi(t) = \varphi_0 + k_p m(t)$

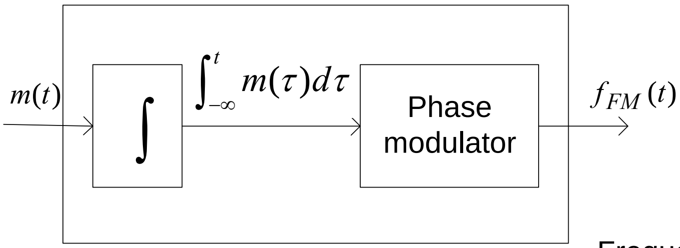


phase sensitivity, k_p= ? $\frac{\pi}{2}$

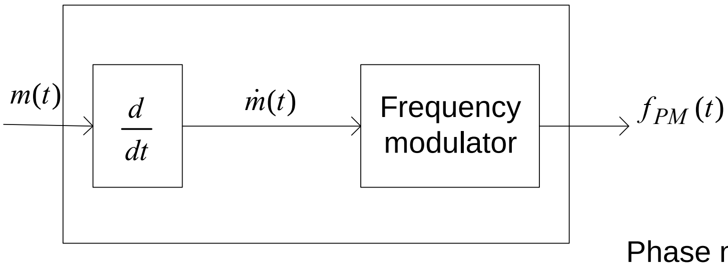
frequency sensitivity, k_f= ? $\frac{f_c}{4}$

Relationship between FM & PM

FM may be regarded as PM, after replacing $\int_{-\infty}^{t} m(\tau)d\tau$ with $m(t)$.

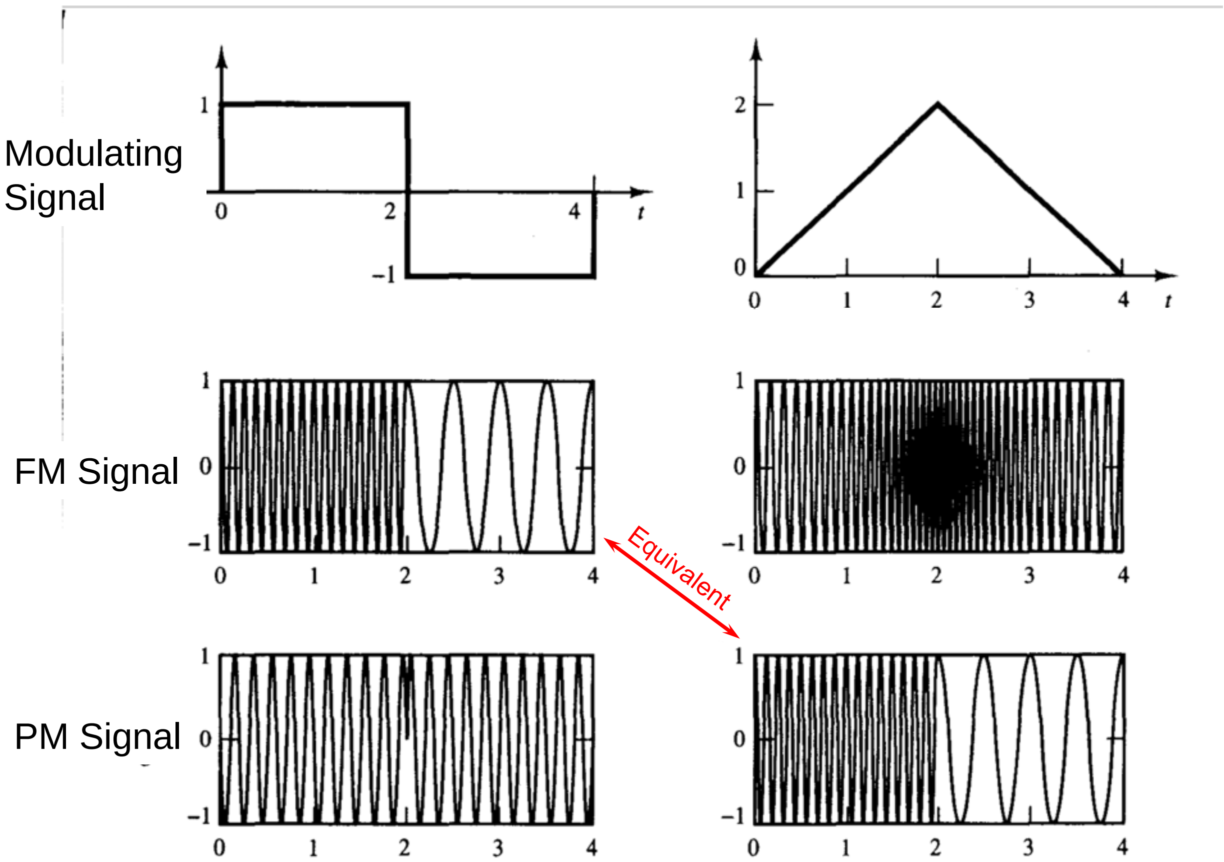


Frequency modulator



Phase modulator

Phase and frequency modulations are inseparable. All the properties of PM signals can be deduced from those of FM signals.



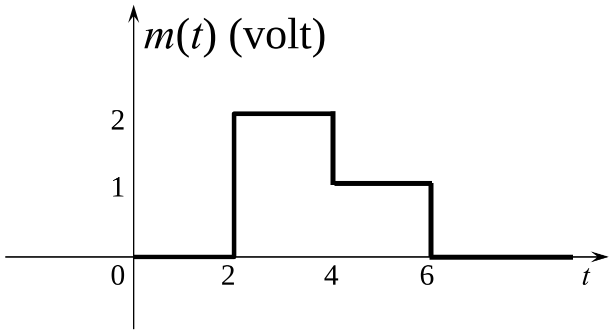
- PM - instantaneous phase deviation $\Delta\phi(t) = k_p m(t) = \phi(t)$
  - peak phase deviation $\Delta\phi = k_p \left| m(t) \right|_{\max}$
  - instantaneous frequency $f_i(t) = f_c + \dfrac{k_p}{2\pi}\dfrac{dm(t)}{dt}$ ✓
  - instantaneous frequency deviation $\Delta f_i(t) = \dfrac{k_p}{2\pi}\dfrac{dm(t)}{dt}$
  - peak frequency deviation $\Delta f = \dfrac{k_p}{2\pi}\left|\dfrac{dm(t)}{dt}\right|_{\max}$

$f_i(t) = f_c + k_f m(t) = d$

$$\boxed{m(t) = \frac{1}{2\pi k_f}\frac{d\phi(t)}{dt}}$$

- FM – instantaneous frequency $f_i(t) = f_c + k_f m(t)$
  - instantaneous frequency deviation $\Delta f_i(t) = k_f m(t)$
  - peak frequency deviation $\Delta f = k_f \left| m(t) \right|_{\max}$
  - peak-to-peak frequency deviation = $k_f [m(t)_{\max} - m(t)_{\min}]$
  - instantaneous phase deviation $\Delta\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$
  - peak phase deviation $\Delta\phi = 2\pi k_f \left| \int_{-\infty}^{t} m(\tau)d\tau \right|_{\max}$

Example:
The following message signal is frequency modulated to generate an FM signal with frequency sensitivity k_f = 10 Hz/volt. Sketch a) the instantaneous frequency deviation; and b) the instantaneous phase deviation of the obtained FM signal.



从这开始

Single Tone Modulation

Now we consider a special message signal, $m(t) = A_m \cos 2\pi f_m t$ (referred to as a <u>single tone</u> message signal).

<u>PM</u>:

$$\theta_i(t) = 2\pi f_c t + k_p A_m \cos 2\pi f_m t = 2\pi f_c t + \beta_p \cos 2\pi f_m t$$

where $\beta_p = k_p A_m$ is the "peak phase deviation", i.e., the <u>maximum</u> amount of phase difference between $f_{PM}(t)$ and the unmodulated carrier $A_c \cos 2\pi f_c t$.

The single-tone PM signal is given by

$$\boxed{f_{PM}(t) = A_c \cos[2\pi f_c t + \beta_p \cos 2\pi f_m t]}$$

β_p *is also called <u>modulation index</u>* (for PM).

FM: **(Single tone modulation)**

$$f_i(t) = f_c + k_f A_m \cos 2\pi f_m t = f_c + \Delta f \cos 2\pi f_m t$$

where $\Delta f = k_f A_m$ is the "peak frequency deviation", i.e., the maximum departure of $f_i(t)$, the instantaneous freq. of FM signal, from f_c .

→积分得来

Thus, $\theta_i(t) = 2\pi \int_0^t f_i(\tau) d\tau = 2\pi f_c t + \frac{\Delta f}{f_m} \sin 2\pi f_m t = 2\pi f_c t + \beta \sin 2\pi f_m t$

where $\boxed{\beta \triangleq \frac{\Delta f}{f_m}}$ is referred to as **modulation index** of FM.

β also denotes the peak phase deviation of the FM signal.

The single tone FM Signal is given by

$$\boxed{f_{FM}(t) = A_c \cos[2\pi f_c t + \beta \sin 2\pi f_m t]}$$

Remarks

- ✓ Small β $\Rightarrow$ narrow-band FM.
- ✓ Large β $\Rightarrow$ wide-band FM.

- From now onward, we assume single-tone modulation, unless otherwise specified. For notational simplicity, we also use ω and $2\pi f$ interchangeably.

Angle Modulation - Topics

I. Basics of FM & PM

II. Narrow-Band and Wide-Band FM

- Narrow-band FM and phasor diagram
- Wide-band FM and amplitude spectrum

III. Bandwidth of FM Signals

IV. Generation of FM Signals

V. Demodulation and Performance of FM Signals

Narrow-Band FM

$$f_{FM}(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$ ← Trigo Identity

$$= A_c[\cos \omega_c t \cos(\beta \sin \omega_m t) - \sin \omega_c t \sin(\beta \sin \omega_m t)]$$

For β <<1,

$$\beta \sin \omega_m t << 1, \quad \Rightarrow \cos(\beta \sin \omega_m t) \approx 1, \quad \sin(\beta \sin \omega_m t) \approx \beta \sin \omega_m t,$$

$$f_{NBFM}(t) \approx A_c[\cos \omega_c t - \beta \boxed{\sin \omega_m t \sin \omega_c t}]$$ ← Trigo Identity

$$= A_c \cos \omega_c t + \frac{\beta A_c}{2} \cos(\omega_c + \omega_m)t - \frac{\beta A_c}{2} \cos(\omega_c - \omega_m)t$$

Recall conventional AM modulation $f_{AM}(t) = A_c(1 + \mu \cos \omega_m t) \cos \omega_c t$ (where μ = modulation index for AM). We have

$$f_{AM}(t) = A_c \cos \omega_c t + \frac{\mu A_c}{2} \cos(\omega_c + \omega_m)t + \frac{\mu A_c}{2} \cos(\omega_c - \omega_m)t$$

造成两者时域波形不同

Observations

- Narrow-band FM (NBFM) has same spectral components as an AM signal, except for phase relationship between the sidebands and the carrier.

- Both NBFM & AM have a carrier term and sidebands centered at $\pm f_c$.

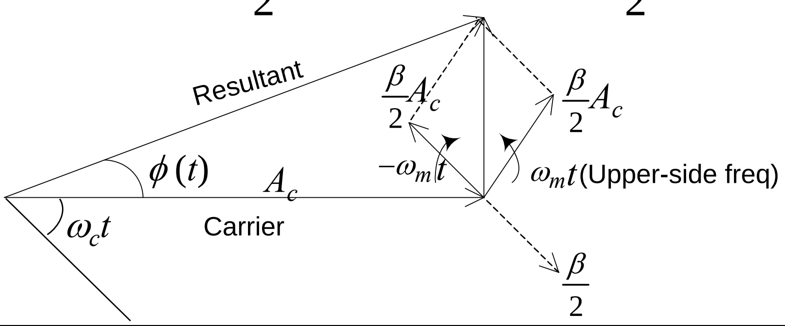
- Both have identical transmission BW, which is $2 f_m$.

- FM & AM waveforms are very dissimilar. For AM signal, freq. is constant and amplitude varies with time, whereas for FM signal, amplitude is constant and freq. varies with time [to be verified by ***phasor diagram***].

Phasor Representation

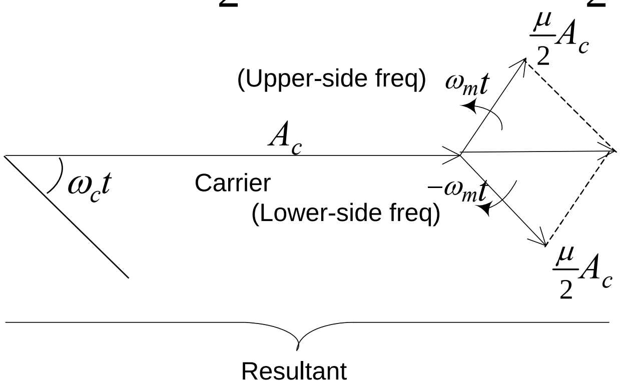
$$f_{NBFM}(t) = A_c \cos\omega_c t + \frac{\beta A_c}{2}\cos(\omega_c + \omega_m)t - \frac{\beta A_c}{2}\cos(\omega_c - \omega_m)t$$

NBFM:



$$f_{AM}(t) = A_c \cos\omega_c t + \frac{\mu A_c}{2}\cos(\omega_c + \omega_m)t + \frac{\mu A_c}{2}\cos(\omega_c - \omega_m)t$$

AM:



Remarks

- Resultant phasor of NBFM has almost the same amplitude as the carrier phasor, but the two phasors have different phases.
- Resultant phasor of AM has an amplitude different from that of the carrier phasor, but the two phasors always have the same phase.
- There is slight amplitude modulation in NBFM waveform.
- To be a valid NBFM signal, a commonly-used criterion for β is $\boxed{\beta \le 0.2}$.

Wide-Band FM

Consider single-tone FM wave for an arbitrary value of β **.**

$$f_{FM}(t) = A_c \cos[\omega_c t + \beta \sin \omega_m t]$$

In complex notation, $f_{FM}(t) = R_e\left\{A_c e^{j\omega_c t} e^{j\beta \sin \omega_m t}\right\}$

$e^{j\beta \sin \omega_m t}$ - A periodic function with fundamental angular freq. ω_m

Therefore, $e^{j\beta \sin \omega_m t} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}$ (Fourier series),

where $C_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{j\beta \sin \omega_m t} e^{-jn\omega_m t} dt$ and $T = \frac{2\pi}{\omega_m}$.

Letting $x = \omega_m t$, we have $C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$.

It is actually the *n*-th order Bessel function of the 1st kind, i.e.,

$$C_n = J_n(\beta) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx$$

Using $e^{j\beta\sin\omega_m t}=\sum_{n=-\infty}^{\infty}J_n(\beta)e^{jn\omega_m t}$, we have

重點

$$f_{FM}(t)=R_e\left\{A_c e^{j\omega_c t}\sum_{n=-\infty}^{\infty}J_n(\beta)e^{jn\omega_m t}\right\}\boxed{=A_c\cos[\omega_c t+\beta\sin\omega_m t]}$$

$$=A_c\sum_{n=-\infty}^{\infty}J_n(\beta)R_e\left\{e^{j(\omega_c+n\omega_m)t}\right\}\boxed{=A_c\sum_{n=-\infty}^{\infty}J_n(\beta)\cos(\omega_c+n\omega_m)t}$$

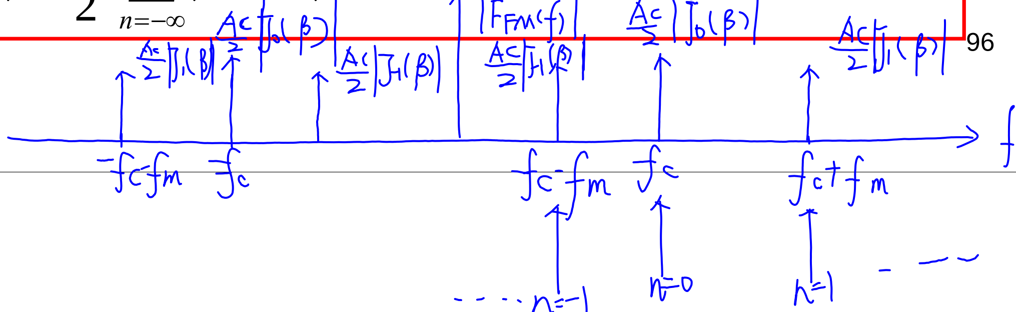
The **spectrum** (including amplitude & phase) of $f_{FM}(t)$ is

→ 对特定的 β，带宽永远是无限宽 (详情见 Bessel function)

$$\boxed{F_{FM}(f)=\frac{A_c}{2}\sum_{n=-\infty}^{\infty}J_n(\beta)[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)]}$$

So, the **amplitude spectrum** is

$$\boxed{|F_{FM}(f)|=\frac{A_c}{2}\sum_{n=-\infty}^{\infty}|J_n(\beta)|[\delta(f-f_c-nf_m)+\delta(f+f_c+nf_m)]}$$



Some useful properties of $J_n(\beta)$

lower BW upper

1. $J_n(\beta)$ are real valued.
2. $J_n(\beta)=J_{-n}(\beta)$, for even n
3. $J_n(\beta)=-J_{-n}(\beta)$, for odd n
4. $\sum_{n=-\infty}^{\infty}J_n^2(\beta)=1$

In the case of $\beta<<1 \Rightarrow J_n(\beta)\approx 0$ for $|n|\geq 2$

$$f_{FM}(t)=A_c\sum_n J_n(\beta)\cos(W_c+nW_m)t$$

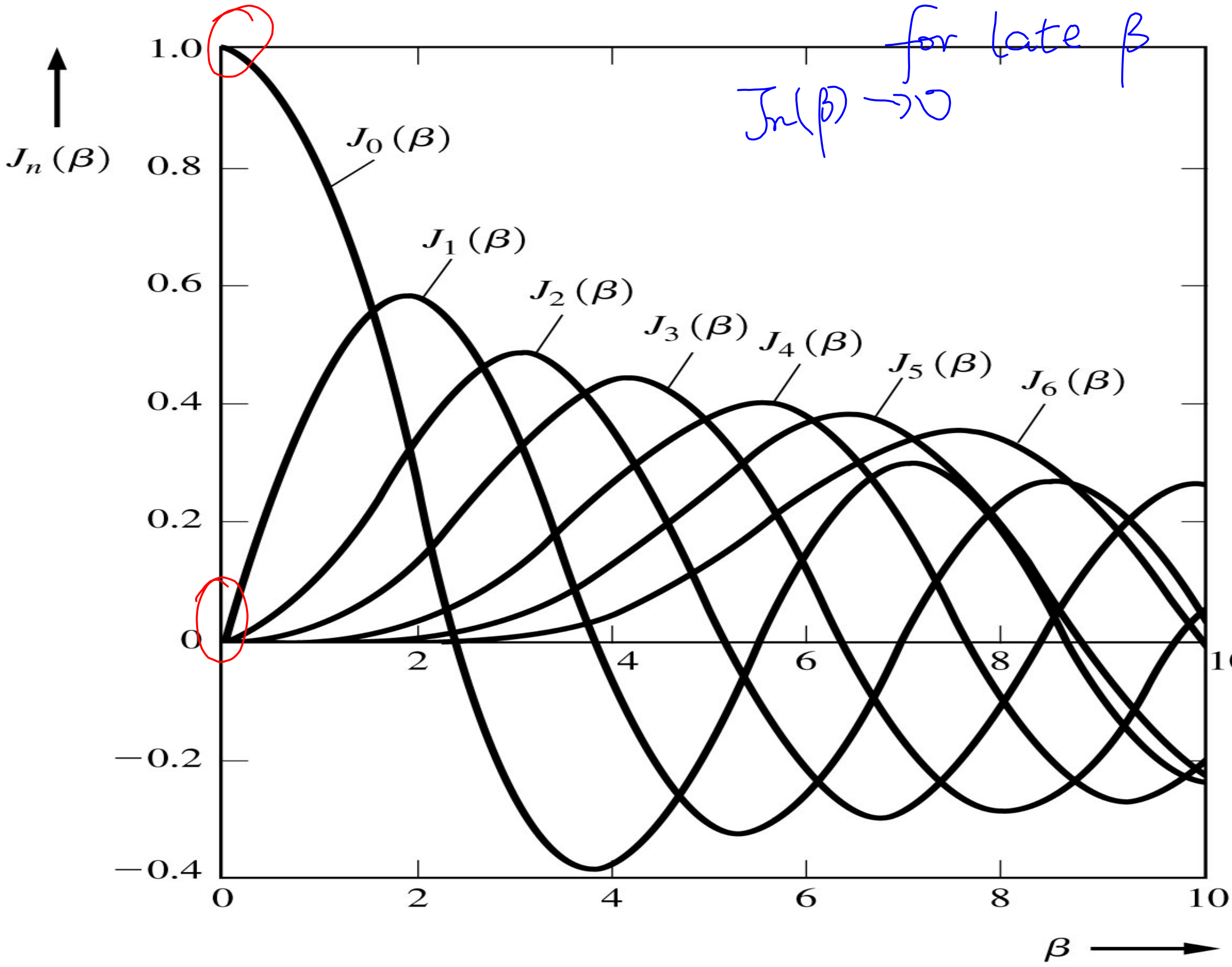
W_c+W_m

$$=A_c J_0(\beta)\cos W_c t+A_c J_1(\beta)\cos[(W_c+W_m)t]-A_c J_1(\beta)\cos[(W_c-W_m)t]$$

Compare with FNBFM

$$F_{NBFM}=A_c\cos(W_c t)+\frac{A_c\beta}{2}\cos[(W_c+W_m)t]-\frac{A_c\beta}{2}\cos[(W_c-W_m)t]$$

两者一致



Values of the Bessel Functions $J_n(\beta)$

n \ β	0.5	1	2	3	4	5	6	7	8	9	10
0	0.9385	0.7652	0.2239	−0.2601	−0.3971	−0.1776	0.1506	0.3001	0.1717	+0.09033	−0.2459
1	0.2423	0.4401	0.5767	0.3391	−0.06604	−0.3276	−0.2767	−0.004683	0.2346	0.2453	0.04347
2	0.03060	0.1149	0.3528	0.4861	0.3641	0.04657	−0.2429	−0.3014	−0.1130	0.1448	0.2546
3	0.002564	0.01956	0.1289	0.3091	0.4302	0.3648	0.1148	−0.1676	−0.2911	−0.1809	0.05838
4		0.002477	0.03400	0.1320	0.2811	0.3912	0.3576	0.1578	−0.1054	−0.2655	−0.2196
5			0.007040	0.04303	0.1321	0.2611	0.3621	0.3479	0.1858	−0.05504	−0.2341
6			0.001202	0.01139	0.04909	0.1310	0.2458	0.3392	0.3376	0.2043	−0.01446
7				0.002547	0.01518	0.05338	0.1296	0.2336	0.3206	0.3275	0.2167
8					0.004029	0.01841	0.05653	0.1280	0.2235	0.3051	0.3179
9						0.005520	0.02117	0.05892	0.1263	0.2149	0.2919
10						0.001468	0.006964	0.02354	0.06077	0.1247	0.2075
11							0.002048	0.008335	0.02560	0.06222	0.1231
12								0.002656	0.009624	0.02739	0.06337
13									0.003275	0.01083	0.02897
14									0.001019	0.003895	0.01196
15										0.001286	0.004508
16											0.001567

0

Discussions

1. At $\beta = 0$, $J_0(0) = 1$ while all other $J_n's$ are zero. There is no modulation ($\beta = 0$). Only carrier is present while all sidebands have zero amplitude.

2. When β departs slightly from zero, $J_1(\beta)$ becomes a significant value while all higher order J_n terms are comparatively negligible.

3. Consistent with case when $\beta \leq 0.2$, an NBFM signal is composed of a carrier and one pair of sidebands at freq. $f_c \pm f_m$.

Discussions (cont'd)

4. As $\beta \uparrow$, amplitude J_1 of the 1st sideband pair and amplitude J_2 of the 2nd sideband pair become significant. As β continues $\uparrow$, J_3, J_4 , etc. begin to acquire significant value, giving rise to sideband pairs at freq. $f_c \pm 3f_m$, $f_c \pm 4f_m$, etc.

5. Unlike AM wave, the amplitude of the carrier component of FM is dependent on the modulation index β .

$f_i(t) = f_c + k_f m(t)$ $\quad \theta(t) = \int 2\pi f_i(t)\,dt = 2\pi f_c t + 2\pi k_f \int 5\cos(\omega_m t)\,dt$

$= 2\pi f_c t + 5 \cdot 2\pi \frac{k_f}{\omega_m} \sin(\omega_m t)$, $\quad \omega_m = 2\pi f_m$

Example

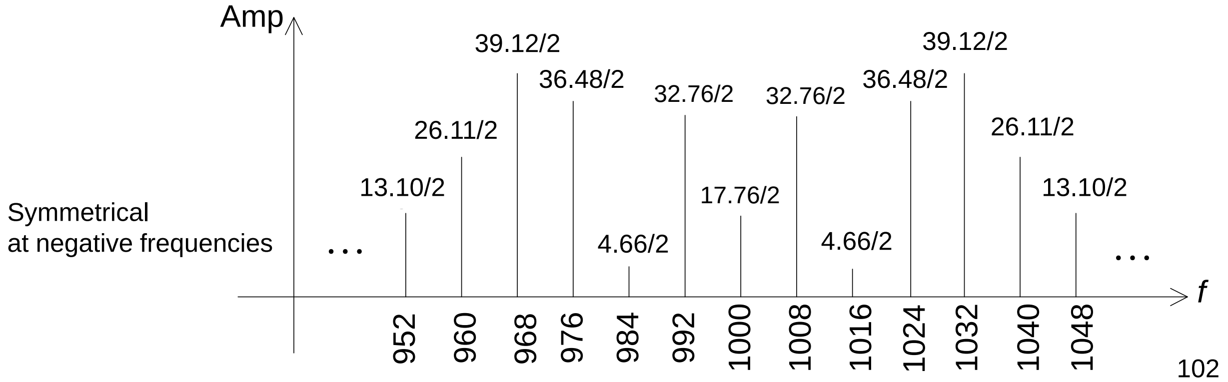
$m(t)$=5cos[2π(8)t] → **FM modulator** ($A_c = 100$, $f_c = 10^3\ Hz$, $k_f = 8\ (Hz/volt)$) → $f_{FM}(t)$ → **BP filter** with center freq = $10^3 Hz$, BW = 56 Hz → Output Power?

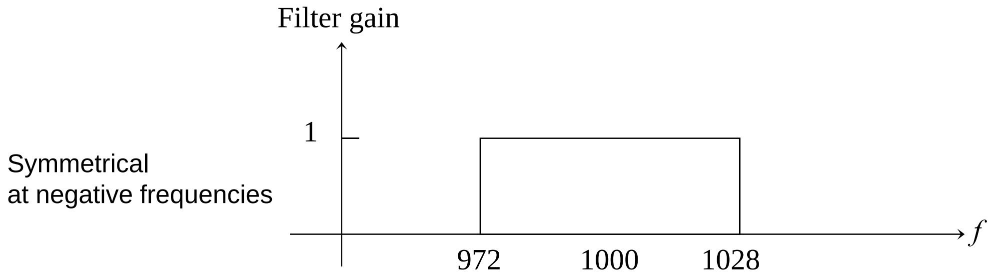
$$\frac{5\times 8}{8} = 5 = \beta$$

Solution:

$$\beta = \frac{k_f A_m}{f_m} = \frac{8\times 5}{8} = 5.$$

The amplitude spectrum is given by





The filter passes 3 components on each side of f_c.

So the power at the filter output

$$= \left\{ \left(\frac{17.76}{2}\right)^2 + 2\left[\left(\frac{32.76}{2}\right)^2 + \left(\frac{4.66}{2}\right)^2 + \left(\frac{36.48}{2}\right)^2 \right] \right\} \times 2 = 2583.43$$

symmetrical double-sideband $\quad$ symmetrical spectrum at negative freq.

Alternatively, based on the spectrum formula on Slide 96, the power at the filter output can be calculated as

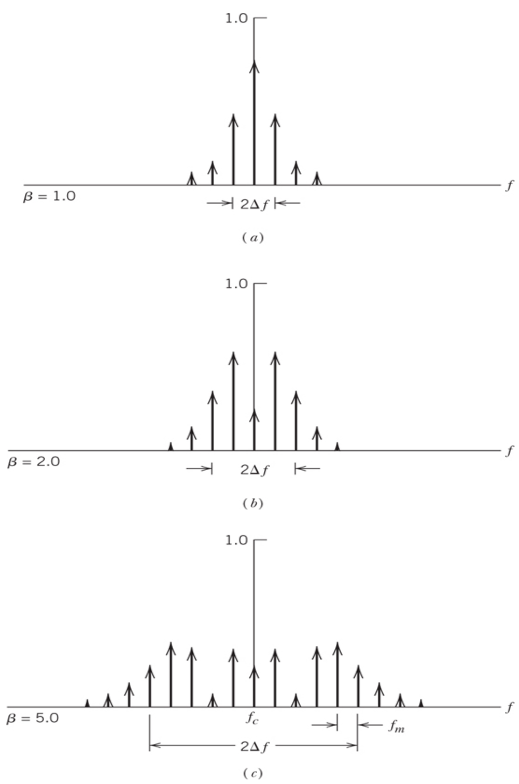
$$= \left(\frac{A_c}{2}\right)^2 \left[J_0^2(5) + 2[J_1^2(5) + J_2^2(5) + J_3^2(5)] \right] \times 2 = 2500 \times 1.033373 = 2583.43$$

Angle Modulation - Topics

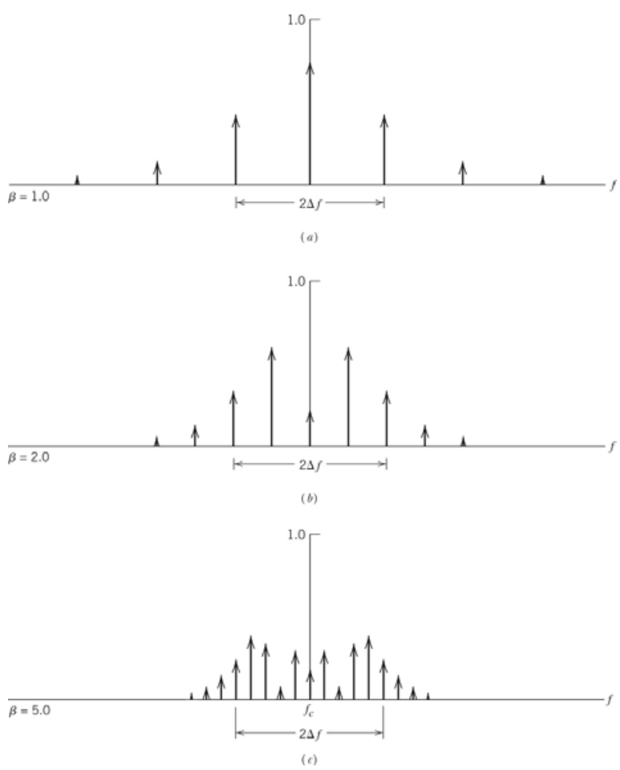
I. Basics of FM & PM
II. Narrow-Band and Wide-Band FM
III. Bandwidth of FM Signals
   - Two rules
   - Bandwidth of arbitrary message signals
IV. Generation of FM Signals
V. Demodulation and Performance of FM Signals

Spectra of FM Signals

f_m is fixed:



Δf is fixed:



Bandwidth (BW) of FM Signals

- In principle, FM signal has an infinite number of sidebands and therefore the BW (in Hz) required to encompass the signal is infinite.
- In practice, for any given β, a large portion of the power is in the sidebands which lie within some "finite" BW. No serious distortion of the signal results if the sidebands outside this BW are lost.
- Two commonly used approximation rules:
  - 1% rule – Bessel table
  - Carson's rule – Convenient

single tone

1% rule

- An accurate assessment of FM wave BW.
- Based on retaining the max. no. of significant sideband freq. pairs while all the higher order amplitudes are all less than 1% of the carrier amplitude obtained when the modulation is removed.
- Consider the single tone case. If the FM wave has n' significant sideband frequency pairs, with $|J_{n'}(\beta)| \geq 0.01$, and for all $n > n'$, $|J_n(\beta)| < 0.01$, then, $\boxed{BW = 2n' f_m}$.

Carson's rule

- Very convenient approximation, as 'a rule of thumb', by John R. Carson.

$$\boxed{BW = 2(\beta+1)f_m \quad = 2(\Delta f + f_m)}$$

Recall $\beta \triangleq \frac{\Delta f}{f_m}$.

- For $\beta << 1$, $BW \approx 2f_m$
- For $\beta >> 1$, $BW \approx 2\beta f_m = 2\Delta f$

BW for arbitrary *m*(*t*)

→ $m(t) \neq A_m\cos(2\pi f_m t)$ 不为 single tone

Consider a general modulating signal *m*(*t*) with its highest freq. component denoted by *W*. Suppose Δf corresponds to max. freq. deviation caused by *m*(*t*).

Define a deviation ratio $D = \frac{\Delta f}{W}$. *D* plays the same role for non-single-tone modulation as β plays for the case of single-tone modulation. Both 1% and Carson's rules apply, after replacing β by *D* and f_m by *W*.

Note:
- Carson's rule tends to <u>underestimate</u> BW.

↓ 低估

Example

For commercial radio FM broadcasting,

$\Delta f = 75KHz$ is the fixed max. value of freq. dev.

$W = 15KHz$ is the typical max. audio freq. of interest in FM

Then, $D = \dfrac{\Delta f}{W} = 5$.

Using Carson's rule, $BW = 2(D+1)W = 180KHz$.

Using 1% rule with $\beta = 5, n' = 8$ and $BW = 2n'W = 240KHz$.

In this case, Carson's rule underestimates BW by 25% compared with 1% rule.

Angle Modulation - Topics

I. Basics of FM & PM

II. Narrow-Band and Wide-Band FM

III. Bandwidth of FM Signals

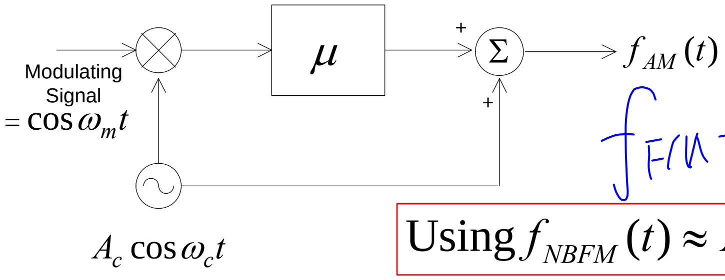
IV. Generation of FM Signals
- Generation of narrow-band FM signals
- Generation of wide-band FM signals

V. Demodulation and Performance of FM Signals

Generation of NBFM signals

Recall for conventional AM signal, $f_{AM}(t) = A_c\cos\omega_c t + A_c\mu\cos\omega_m t\cos\omega_c t$

$f_i = f_c + k_f m(t)$

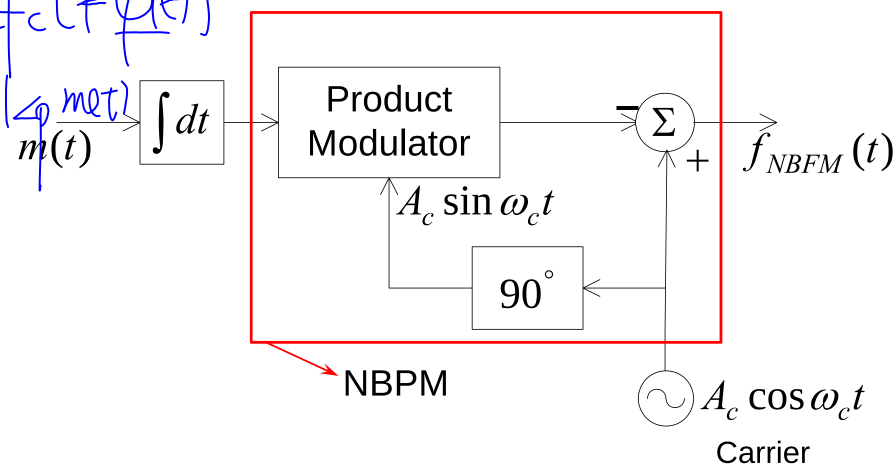


$f_{FM} = A_c\cos\theta(t)$

$\boxed{\text{Using } f_{NBFM}(t) \approx A_c[\cos\omega_c t - \beta\sin\omega_m t\sin\omega_c t]}$

This method can be used to generate NBFM & NBPM signals.

$A_c\cos[2\pi f_c t + \phi(t)]$



Generation of WBFM signals

- Indirect Method (multi-stage)
  - The message signal is first used to generate an NBFM signal, and frequency multiplication is used next to increase frequency deviation (hence modulation index β) to produce a WBFM signal.
- Direct Method (one stage)
  - The instantaneous freq. of the carrier signal (oscillator's output frequency) is varied directly in accordance with the message signal. That is,

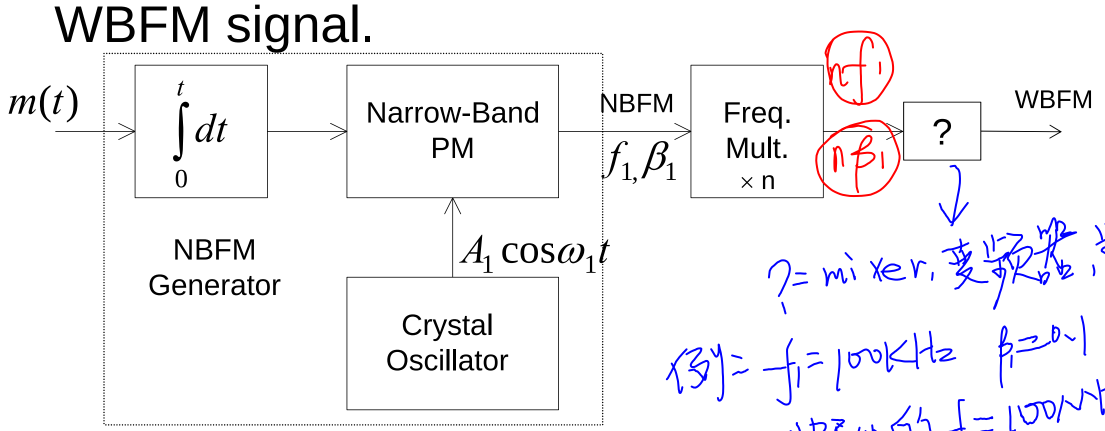
$$f_i(t) = f_c + k_f m(t). \rightarrow \text{input}$$

Indirect Method (Armstrong Method)

- Message signal $m(t)$ is first integrated and then used to phase-modulate a crystal-controlled oscillator.
- β is kept small to minimize distortion. ($\beta \leq 0.2$)

NBFM

- Frequency multiplier is used next to produce the WBFM signal.



?= mixer, 变频器, 频移

例: $f_1 = 100kHz$ $\beta_1 = 0.1$

WBFM的 $f = 100MHz$ $\beta = 10$

则?为上变频、

Block Diagram

Observations

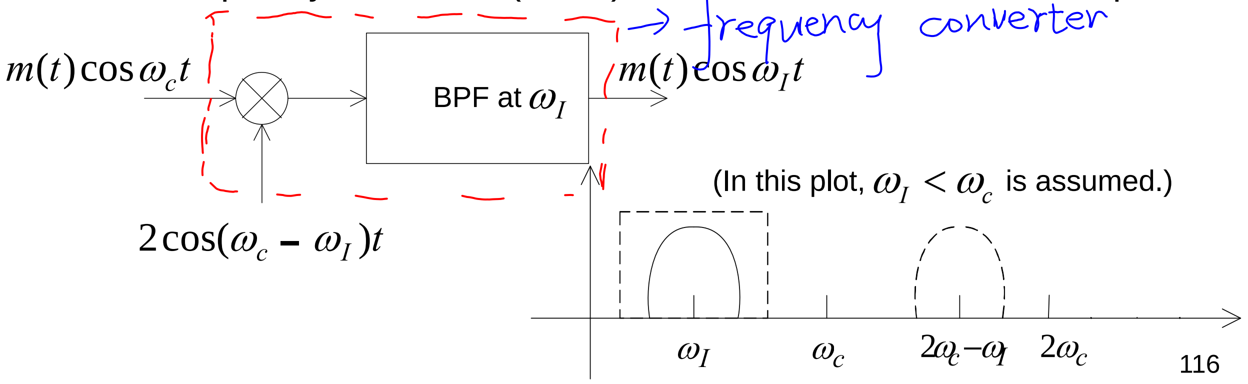
1. How to obtain a freq. deviation multiplied by *n*?
   - Use a device with *n* degrees of non-linearity, and choose the appropriate filter to get the *n*-th multiplier value.
2. May end up with an unwanted carrier freq. when trying to achieve a desired modulation index!
   - Use a frequency converter after the stage of frequency multiplication to control the value of carrier frequency.
   - The frequency converter translates the spectrum of the signal by a given amount but does not alter its spectral content ($\Rightarrow$ β *value is not affected*).

Remarks

1. Frequency multiplier (*n* times):
   - Including a non-linear device with order *n*:

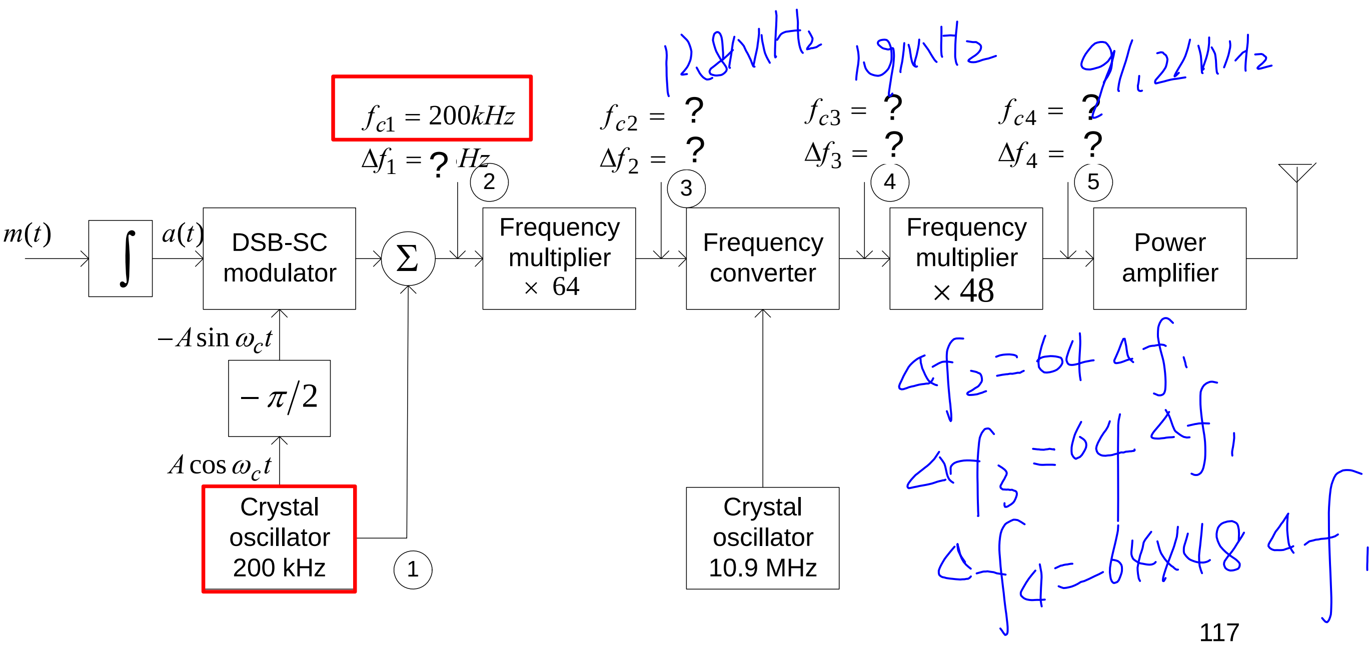
     $s_{out}(t) = k_0 + k_1 s_1(t) + \ldots + k_n s_1^n(t)$, where $s_1(t)$ is an NBFM signal with f_1 and Δf_1.

     - with spectra at $f_1, 2f_1, \ldots, nf_1$
     - peak freq. dev. at $\Delta f_1, 2\Delta f_1, \ldots, n\Delta f_1$
   - And an appropriate filter to obtain nf_1 and $n\Delta f_1$ only.
2. Frequency Converter (Mixer) is then used to translate the spectrum.



Example:

Given a single-tone message $m(t)$ with f_m = 200Hz, and f_{c1} = 200KHz, find suitable values for all the unknown frequencies that lead to $80\text{KHz} \geq \Delta f_4 \geq 75\text{KHz}$ and $100\text{ MHz} \geq f_{c4} \geq 90\text{MHz}$.



Steps:

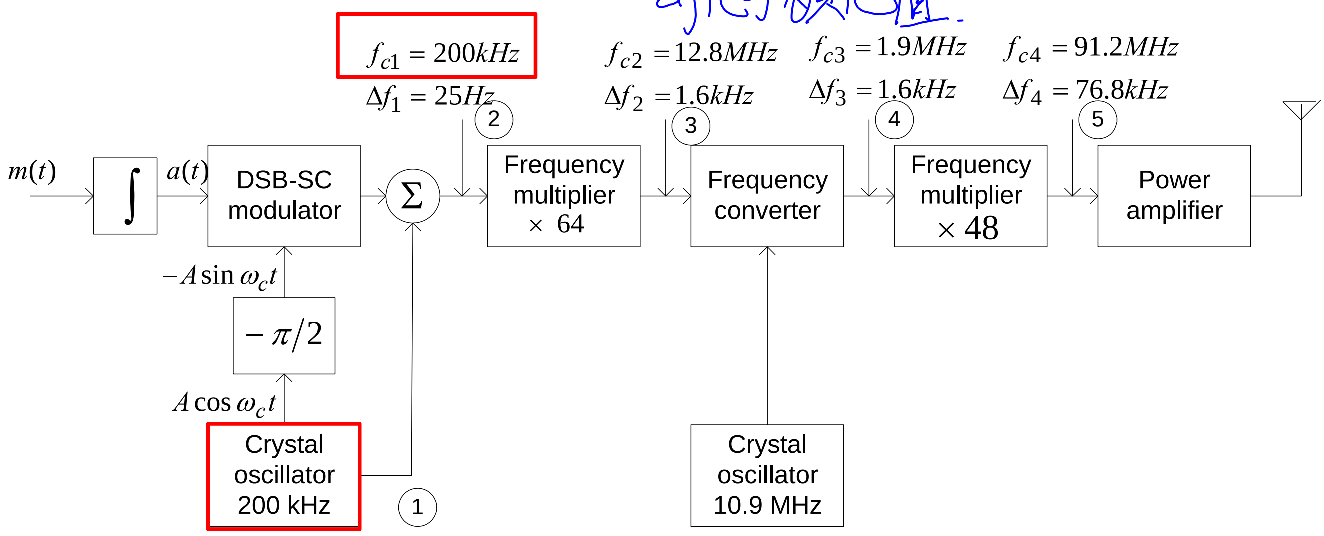
1. Choose Δf_1 to be 25Hz.
2. $\beta = \dfrac{\Delta f_1}{f_m} = \dfrac{25Hz}{f_m}$ = 0.125 < 0.2 → NBFM
3. $f_{c2} = f_{c1} \times 64, \quad \Delta f_2 = 64 \times \Delta f_1$
4. $f_{c3} = f_{c2} - 10.9MHz \ , \quad \Delta f_3 = \Delta f_2$
5. $f_{c4} = f_{c3} \times 48 \ , \quad \Delta f_4 = 48 \times \Delta f_3$

✓ $48 \times 64 = 2^4 \times 3 \times 2^6, \qquad 2^x - doubler, \qquad 3^y - tripler$

In this example, 10 doublers and 1 tripler are used.

✓ Stable freq. but suffers from noise caused by frequency multipliers.

答案不唯一，以下答案是基于$\Delta f_1=25Hz$推出来的.
Δf_1也可为其他值.



Direct Method

A common method used for generating FM directly is to vary the inductance or capacitance of a tuned electronic oscillator (or voltage – controlled oscillator - VCO). For a simple tuned circuit, the freq. of oscillation is

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \propto f_c + k_f m(t)$$

– if L or C is varied, the output freq. will also vary.

Angle Modulation - Topics

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- Frequency discriminator & FM receiver model
- Capture effect 捕获效应
- Input / Output Signal-to-Noise Ratios

Demodulation of FM Signals

There are a number of ways to recover the modulating signal $m(t)$ from the FM wave, and the overall objective is to provide an output signal whose amplitude is linearly proportional to $f_i(t)$ of the input waveform. Basically, there are 2 approaches.

1. Direct approach – <u>Frequency Discriminator</u> (i.e., system that has a linear freq-to-voltage transfer function).
   – Time Differentiator

2. Indirect approach – <u>Phase-Lock Loop (PLL)</u> (i.e., to place a frequency modulator in the return branch of a feedback system).

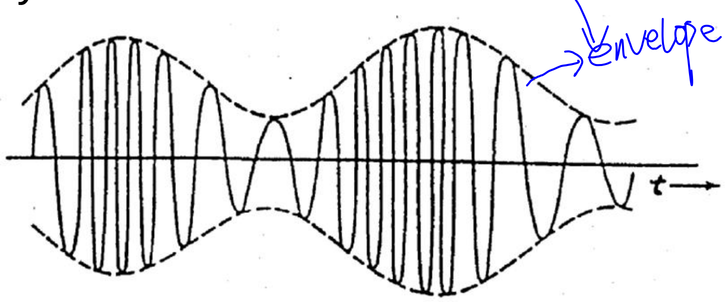
Read Section 4.4 on pages 127-133

Time Differentiator

If applying on $f_{FM}(t) = A_c \cos[\omega_c t + 2\pi k_f \int_0^t m(\tau)d\tau]$ time differentiation, the output is

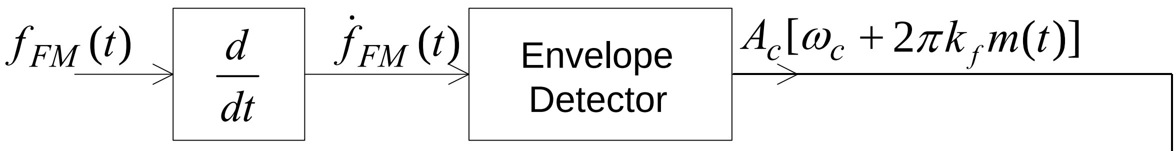
$$\dot{f}_{FM}(t) = \frac{d}{dt} f_{FM}(t)$$
$$= -\boxed{A_c[\omega_c + 2\pi k_f m(t)]} \sin[\omega_c t + 2\pi k_f \int_0^t m(\tau)d\tau]$$

which may look like:

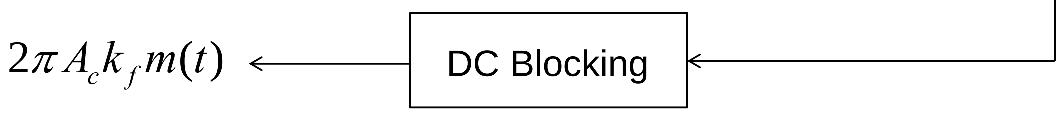


How is $m(t)$ recovered?

$\dot{f}_{FM}(t)$ is both amplitude and frequency modulated, with its envelope (i.e., the amplitude variations) being linearly proportional to $m(t)$. $m(t)$ can be obtained with $\dot{f}_{FM}(t)$ passing through an "envelope detector".

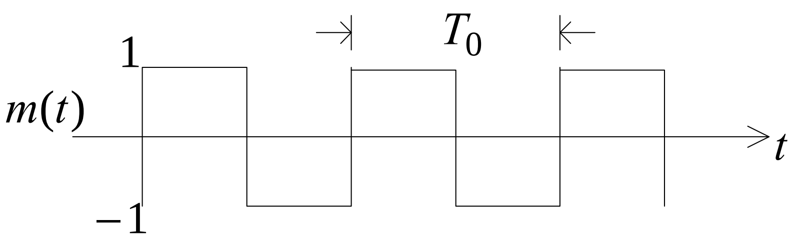


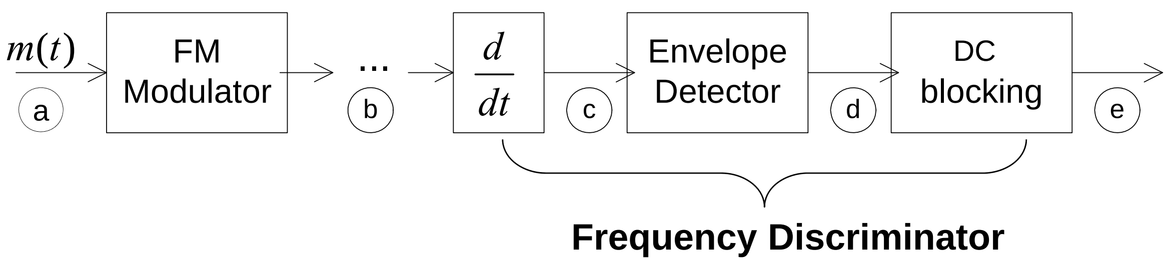
Then, a DC-blocking circuit is applied to remove the DC component such that $m(t)$ can be recovered.



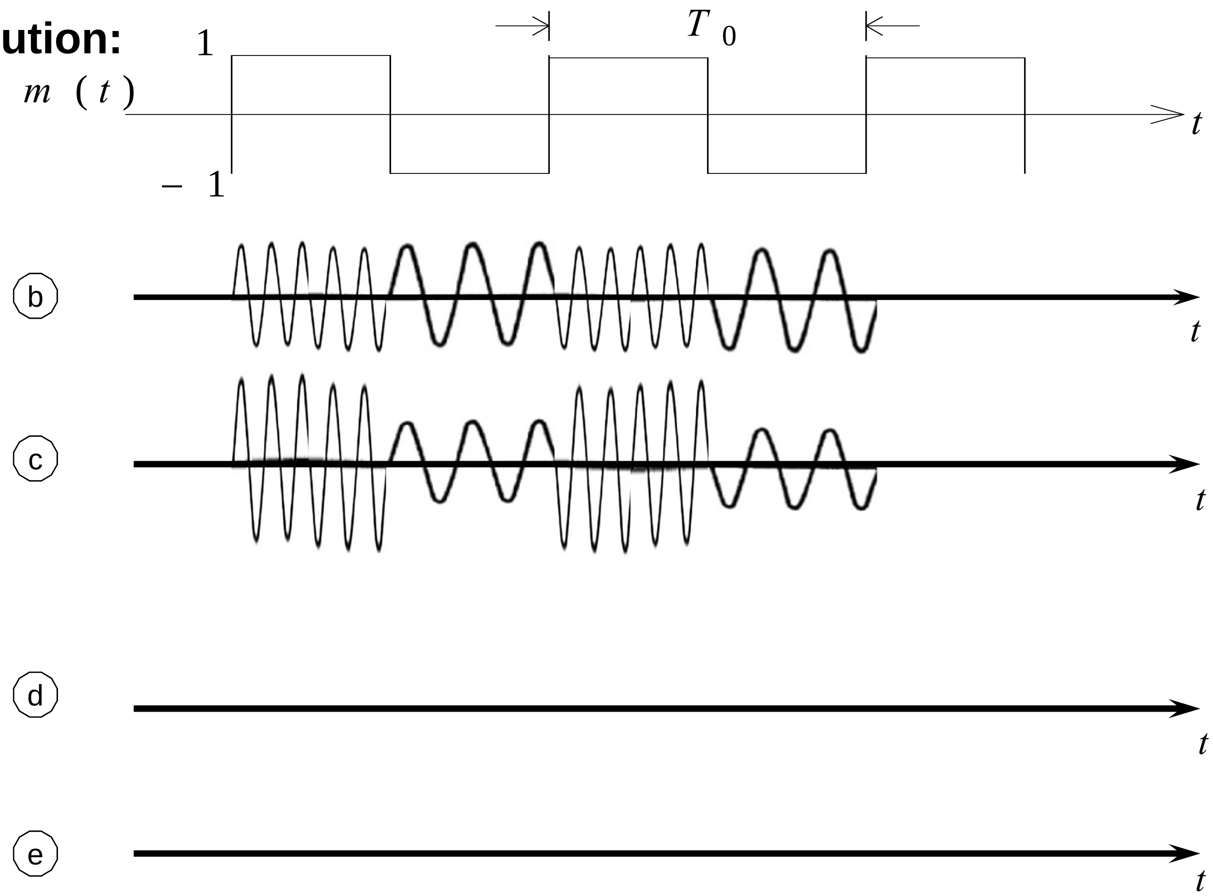
Example

A periodic square wave $m(t)$ frequency modulates a carrier of frequency f_c =10 KHz with Δf =1KHz. The carrier amplitude is *A*. This obtained FM signal is demodulated as shown. Sketch the waveforms at points b, c, d & e.





Solution:



"Limiter" - To remove carrier amplitude variation

- We assumed constant amplitude A_c for the received FM signal.
- Due to channel noise and fading, it is often that A_c is a function of time. If this is the case,

$$\frac{d}{dt} f_{FM}(t) = -A_c(t)[\omega_c + 2\pi k_f m(t)]\sin[\omega_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau]$$

$$+ \text{ terms containing } \frac{d}{dt} A_c(t)$$

random unknown

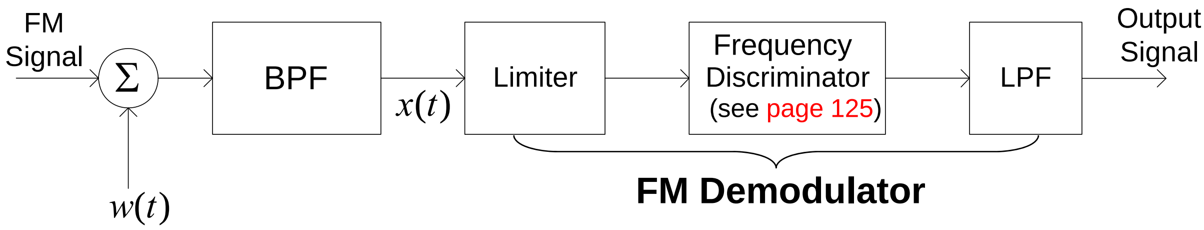
✓ Even if the term containing $\frac{d}{dt}A_c(t)$ is neglected, the output of envelope detector will be $A_c(t)[\omega_c + 2\pi k_f m(t)]$, from which it is impossible to separate $A_c(t)$ and $m(t)$.

✓ Therefore, "**limiter**" is incorporated in FM demodulation before frequency discriminator stage, in a way that the carrier amplitude variation is removed.

✓ The signal amplitude at the "limiter" output is a constant.

FM Receiver Model

- The following model is used to evaluate the performance of an FM receiver in the presence of <u>additive white Gaussian noise</u> $w(t)$ of zero mean and double-sided power spectral density (PSD) $\eta/2$.



✓ The BPF filter is assumed to be an ideal bandpass filter with centre frequency f_c and bandwidth equal to the FM signal's bandwidth.

✓ The filtered noise $n(t)$ can be represented in terms of its in-phase and quadrature components.

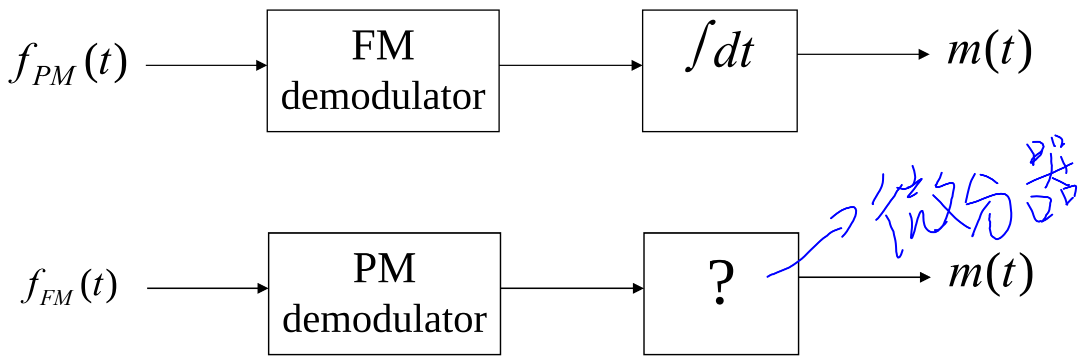
✓ The LPF is used to further suppress the noise, by choosing a bandwidth equal to the message's bandwidth.

PM Signal Demodulation

Given an angle-modulated wave $f(t) = A_c \cos(\omega_c t + \phi(t))$,

for PM, $\phi(t) \propto m(t)$; for FM, $\frac{d}{dt}\phi(t) \propto m(t)$.

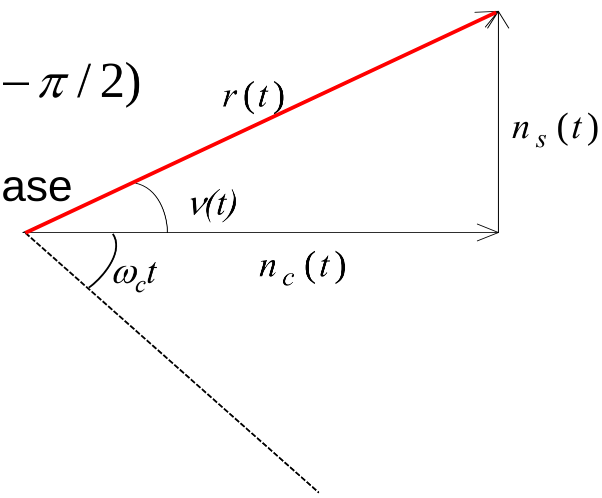
Thus, FM demodulator can be used to demodulate PM signals if an integrator is appended at the output, i.e.,



Band-Pass Noise Representation

The band-pass noise $n(t)$ at the BPF output may be written as

$$n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$$
$$= n_c(t)\cos\omega_c t - n_s(t)\cos(\omega_c t - \pi/2)$$



or in terms of its amplitude and phase

$$n(t) = r(t)\cos[\omega_c t + v(t)]$$

where

$$r(t) = \sqrt{n_c^2(t) + n_s^2(t)}$$

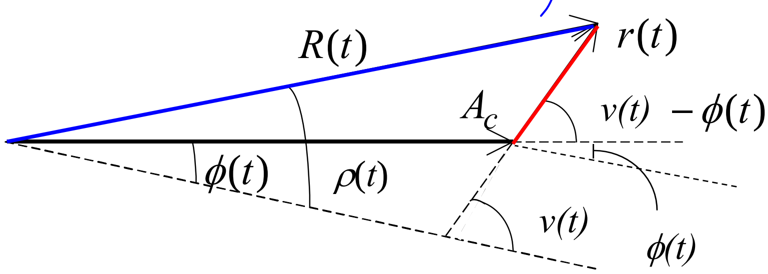
$$v(t) = \tan^{-1}\frac{n_s(t)}{n_c(t)}$$

Capture Effect

The total signal (signal + noise) at the BPF output is

$$x(t) = f_{FM}(t) + n(t)$$
$$= A_c\cos[\omega_c t + \phi(t)] + r(t)\cos[\omega_c t + v(t)]$$
$$= R(t)\cos[\omega_c t + \rho(t)]$$

该图已经假设
$A_c > r(t)$



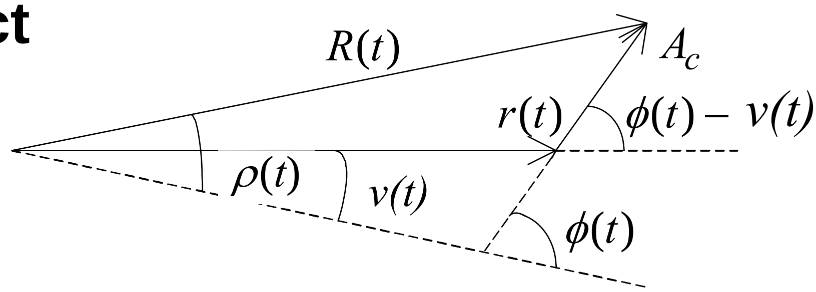
For $A_c >> r(t)$

$$\rho(t) = \phi(t) + \tan^{-1}\frac{r(t)\sin[v(t)-\phi(t)]}{A_c + r(t)\cos[v(t)-\phi(t)]}$$

$$\approx \phi(t) + \frac{r(t)}{A_c}\sin[v(t)-\phi(t)]$$

Capture Effect

For $r(t) >> A_c$



$$\rho(t) = v(t) + \tan^{-1}\frac{A_c \sin[\phi(t) - v(t)]}{r(t) + A_c \cos[\phi(t) - v(t)]}$$

$$\approx v(t) + \frac{A_c}{r(t)}\sin[\phi(t) - v(t)]$$

<u>Remarks</u>

When $A_c >> r(t)$, the phase to be demodulated (i.e., $\rho(t)$) includes an extra term $\frac{r(t)}{A_c}\sin[v(t) - \phi(t)]$ due to noise. This term ↓ as A_c ↑. Thus, $m(t)$ can still be recovered from $\phi(t) = 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau$ at the frequency discriminator output.

Remarks

- When $r(t) >> A_c$, the phase to be demodulated is equal to noise phase $v(t)$ plus the term $\frac{A_c}{r(t)}\sin[\phi(t) - v(t)]$, with signal phase $\phi(t)$ buried inside, which cannot be extracted by demodulation process. Thus, noise predominates at the demodulator output, and the noise "<u>captures</u>" the signal.
- Similarly, when the interference produced by another frequency modulated signal with approximately same carrier frequency is the stronger one of the two, the receiver locks on to the stronger one and thereby suppresses (captures) the desired FM signal.

Signal-to-Noise Ratios (SNRs) in FM

Define “**output SNR**” at the output of an FM demodulator as

$$\frac{S_0}{N_0} = \frac{\text{mean power of message signal (after demodulation)}}{\text{mean power of noise}}$$

Define “**input SNR**” at the input of an FM demodulator (output of BPF filter) as

$$\frac{S_i}{N_i} = \frac{\text{mean power of FM signal}}{\text{mean power of noise within FM signal's BW}}$$

✓ It can be easily shown that $\frac{S_i}{N_i} = \frac{A_c^2/2}{\eta B_{FM}}$.

S_0 Calculation

假设前提

- We assume an ideal limiter in the FM receiver such that all amplitude variations are removed, and the signal amplitude at the limiter output is 1. So, the demodulator output signal is $2\pi k_f m(t)$.

- The mean-square value of the output signal is

$$S_0 = \overline{(2\pi k_f m(t))^2}$$

$$\Rightarrow \quad \boxed{S_0 = 4\pi^2 k_f^2 \overline{m^2(t)}}$$

N_0 Calculation

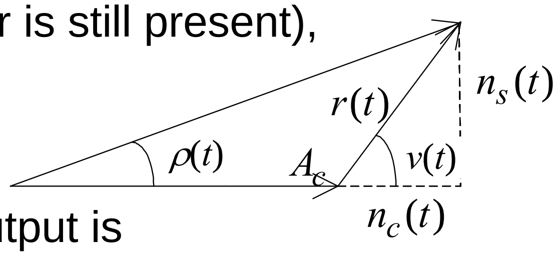
Recall earlier that in the FM receiver model,

$$x(t) = f_{FM}(t) + n(t) \quad \text{and} \quad n(t) = r(t)\cos[\omega_c t + v(t)]$$

For $A_c >> r(t)$, $\quad \rho(t) \approx \phi(t) + \frac{r(t)}{A_c}\sin[v(t) - \phi(t)]$

where $\rho(t)$ is the phase to be demodulated. In the absence of signal $m(t)$ (the carrier is still present),

$$\rho(t) \approx \frac{r(t)}{A_c}\sin v(t) = \frac{n_s(t)}{A_c}$$



And noise at FM demodulator output is

$$n_0(t) = \frac{d\rho(t)}{dt} = \frac{1}{A_c}\frac{d}{dt}n_s(t)$$

Recall the transfer function of $h(t) = d/dt$ is $H(\omega) = j\omega$.

↓ 假设 limiter 输出为 1V

The PSD of $n_0(t)$ is $S_{n_0}(\omega) = \frac{1}{A_c^2}S_{n_s}(\omega)|H(\omega)|^2$. Thus,

$$S_{n_0}(\omega) = \frac{1}{A_c^2}S_{n_s}(\omega)|j\omega|^2 \quad = \frac{1}{A_c^2}\omega^2 S_{n_s}(\omega)$$

Note that the differentiation operation in FM discriminator emphasized the spectral components of noise at high frequency.

To determine $S_{n_s}(\omega)$, recall $n(t) = n_c(t)\cos\omega_c t - n_s(t)\sin\omega_c t$.

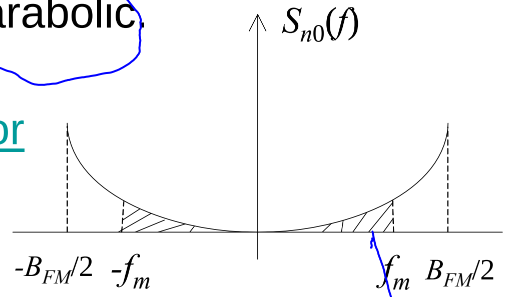
It can be shown that $S_{n_s}(\omega) = \eta, \text{ for } |f| \le B_{FM}/2.$

Thus,

$$\boxed{S_{n_0}(\omega) = \frac{1}{A_c^2}\omega^2\eta, \text{ for } |f| \le B_{FM}/2.}$$

抛物线

The equation $S_{n_0}(\omega) = \frac{1}{A_c^2}\omega^2\eta = \frac{(2\pi)^2}{A_c^2} f^2\eta$ shows that if a white noise with a two-sided PSD $\frac{\eta}{2}$ is assumed for the demodulator input, the output noise PSD will be parabolic.



Since the BW of the freq. discriminator output is limited by an LPF with bandwidth f_m, the power of output noise is

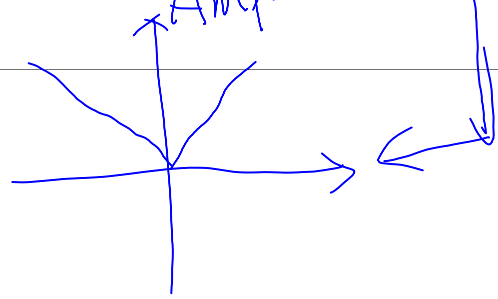
$$N_0 = \overline{n_0^2(t)} = 2\int_0^{f_m} S_{n_0}(f)df = \frac{2\eta\times(2\pi)^2}{A_c^2}\int_0^{f_m} f^2 df$$

$$\Rightarrow \boxed{N_0 = \frac{8\pi^2\eta f_m^3}{3A_c^2}}$$

→假设 limiter 输出为1

Ac为limiter切之前的Ac

Amplitude



Noise Quieting

The mean power of carrier, $S_c = \frac{A_c^2}{2}$ (where A_c is the carrier amplitude)

(Note: the mean power of an FM signal is $S_i = \overline{f_{FM}^2(t)} = \frac{A_c^2}{2}$)

Recall

$$N_0 = \frac{8\pi^2\eta f_m^3}{3A_c^2} = \frac{4\pi^2\eta f_m^3}{3}\frac{1}{S_c}$$

where the output noise power decreases as the mean FM carrier power increases. This is referred to as "noise quieting".

前提是系统以信号换取噪声，S_i 需 >10dB

Output Signal-to-Noise Ratio

From $S_0 = 4\pi^2 k_f^2 \overline{m^2(t)}$ & $N_0 = \dfrac{8\pi^2 \eta f_m^3}{3A_c^2}$, $\Rightarrow$

$$\boxed{\frac{S_0}{N_0} = \frac{3A_c^2 k_f^2 \overline{m^2(t)}}{2\eta f_m^3}}$$

For single-tone modulation,

$m(t) = A_m \cos \omega_m t$, $\Delta f = A_m k_f$, $\overline{m^2(t)} = \dfrac{A_m^2}{2}$, $\beta = \Delta f / f_m$.

It can thus be shown that $\boxed{\dfrac{S_0}{N_0} = \dfrac{3A_c^2 \beta^2}{4\eta f_m}}$. → 可以通过 A_c, β 改变 S/N output

✓ For WBFM, BW increases linearly with β. $\therefore \dfrac{S_0}{N_0} \uparrow$ with (BW)[2]

With $\boxed{\dfrac{S_i}{N_i} = \dfrac{A_c^2/2}{\eta B_{FM}} = \dfrac{A_c^2/2}{\eta 2(\beta+1) f_m}}$ (using Carson's rule), it can be easily shown that $\dfrac{S_0}{N_0} = 3\beta^2(\beta+1)\dfrac{S_i}{N_i}$.

不能无限制增加

Performance Comparison of FM & AM

For DSB-SC AM and conventional AM receiver for sinusoidal modulation, it can be shown that (both AM and FM signals have the same A_c)

$$\left[\frac{S_0}{N_0}\right]_{AM} = \frac{A_c^2/2}{\eta B_{AM}} = \frac{A_c^2}{4\eta f_m}$$

For AM, assume $s(t) = A_c \cos \omega_m t$.

Recall that

$$\left[\frac{S_0}{N_0}\right]_{FM} = \frac{3A_c^2 \beta^2}{4\eta f_m}$$

Hence,

$$\boxed{\left[\frac{S_0}{N_0}\right]_{FM} = 3\beta^2 \left[\frac{S_0}{N_0}\right]_{AM}}$$

Example

Typical FM broadcast uses a $\Delta f = 75 \text{ KHz}$ & the highest modulating frequency = 15 KHz.

$$\therefore \beta = \frac{75 \text{ KHz}}{15 \text{ KHz}} = 5$$

$\Rightarrow$ FM improvement over AM is

$$= 3\beta^2 = 75$$

or in dB

$$= 10 \log_{10} 75 = 18.8 \text{ dB}$$

Further improvement is possible by further suppressing the noise.

Remarks

- Improvement of FM over AM can be done by increasing the modulation index β, but at the expense of an increase in BW.

- Exchange of BW for $\left[{S_0}/{N_0}\right]_{FM}$ cannot be increased indefinitely as the noise power increases with the BW of the FM signal. If the signal power is fixed, the noise power may eventually become comparable with the signal power (i.e., the signal no longer captures the noise).

Threshold Phenomenon 门限效应

- The output SNR formula was derived based on the small noise assumption. For large noise, "threshold phenomenon" occurs in the FM receiver.
- As discussed in the capture effect, when noise is small, the desired signal captures the noise.
- When noise is strong, it can effectively capture the signal.
- The "threshold" is defined as the minimum S_i / N_i that validates the output SNR formula and makes the FM receiver work properly.

- This threshold occurs when $\frac{S_i}{N_i} \approx 10$ dB.
- The system experiences rapid quieting of noise at the output as $\frac{S_i}{N_i}$ is increased above 10 dB.
- At low $\frac{S_i}{N_i}$ (< 10 dB), FM actually performs poorer than AM in terms of $\frac{S_0}{N_0}$.
- For a fixed $\frac{S_i}{N_i}$ (> 10 dB), larger β results in higher $\frac{S_0}{N_0}$, but it also needs wider BW.