Schwarz's Inequality

If two complex functions $\phi_1(t)$ and $\phi_2(t)$ satisfying

$$\int_{-\infty}^{\infty} |\phi_1(t)|^2 dt < \infty$$

and

$$\int_{-\infty}^{\infty} |\phi_2(t)|^2 dt < \infty ,$$

then we have

$$\left| \int_{-\infty}^{\infty} \phi_1(t)\phi_2^*(t)\, dt \right|^2 \leq \int_{-\infty}^{\infty} \left| \phi_1(t) \right|^2 dt \int_{-\infty}^{\infty} \left| \phi_2(t) \right|^2 dt ,$$

The equality holds if, and only if

$$\phi_1(t) = k\phi_2(t).$$

Solution of Optimal h(t)

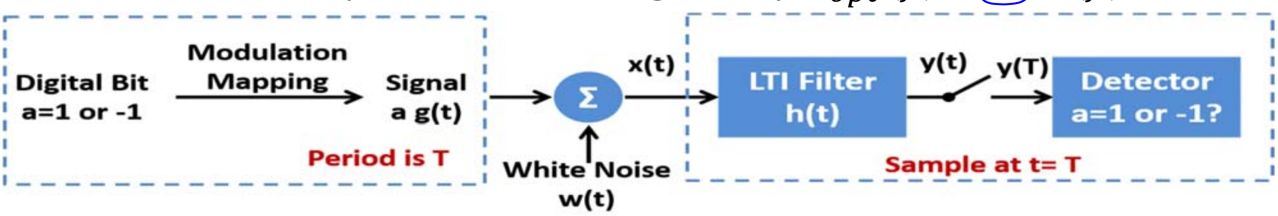
1. Receiver's SNR $\eta = \dfrac{\left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT} df \right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df}$

2. Let $\phi_1(f) = H(f)$ and $\phi_2^*(f) = G(f)e^{j2\pi fT}$. Use Schwarz's inequality, we have $\eta = \dfrac{\left| \int_{-\infty}^{\infty} H(f)G(f)e^{j2\pi fT} df \right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df} \leq \dfrac{\int_{-\infty}^{\infty} \cancel{|H(f)|^2} df \int_{-\infty}^{\infty} \left|G(f)e^{j2\pi fT}\right|^2 df}{\frac{N_0}{2}\int_{-\infty}^{\infty} |\cancel{H(f)}|^2 df}$.

3. Hence $\eta \leq \frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2 df$, and the equality holds if and only if $H(f) = kG^*(f)e^{-j2\pi fT}$, k is a an arbitrary non-zero constant

任意非零常数

4. As a result, the optimal receiver is given by $H_{opt}(f) = kG^*(f)e^{-j2\pi fT}$



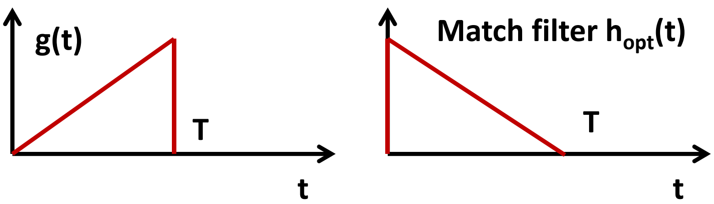
Match Filter

- The optimal filter used at the receiver is
  - $H_{opt}(f) = kG^*(f)e^{-j2\pi fT}$
- In time domain

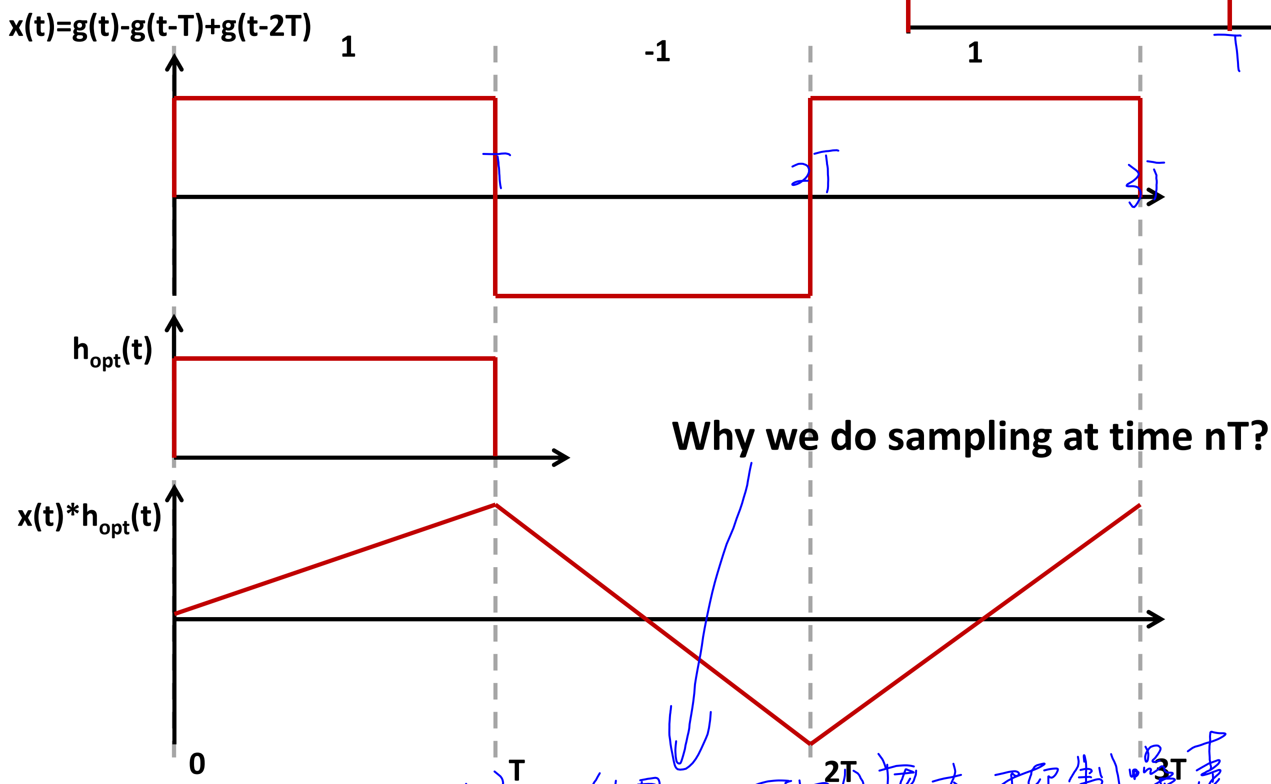
$$h_{opt}(t) = \int_{-\infty}^{\infty} H_{opt}(f)e^{j2\pi ft}df \; = k\int_{-\infty}^{\infty} G^*(f)e^{-j2\pi f(T-t)}df$$

$$= k\int_{-\infty}^{\infty} G(-f)e^{-j2\pi f(T-t)}df = k\int_{-\infty}^{\infty} G(f)e^{j2\pi f(T-t)}df = kg(T-t)$$

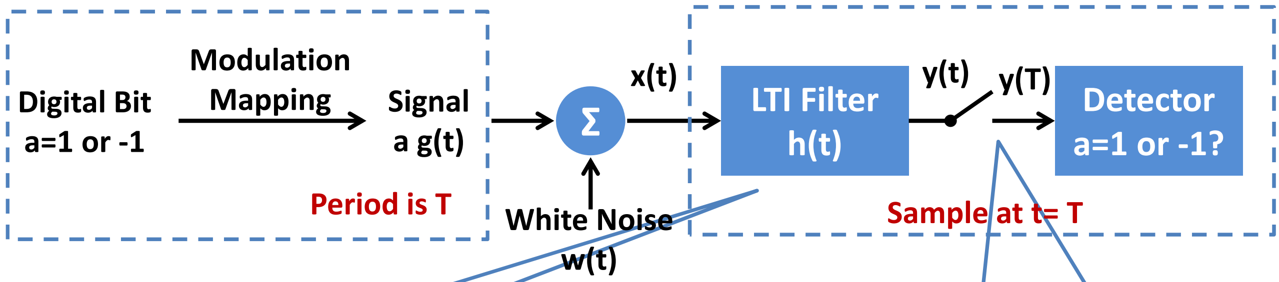
- **Match Filter**: receiver's filter matches the shape of transmitted pulse



Example



该处值最大，可区分度大，抑制噪声



$$\boldsymbol{h(t) = kg(T-t)}$$

$$\boldsymbol{y(t) = h(t) * x(t) = \underbrace{h(t) * ag(t)}_{g_0(t)} + \underbrace{h(t) * w(t)}_{n(t)}}$$

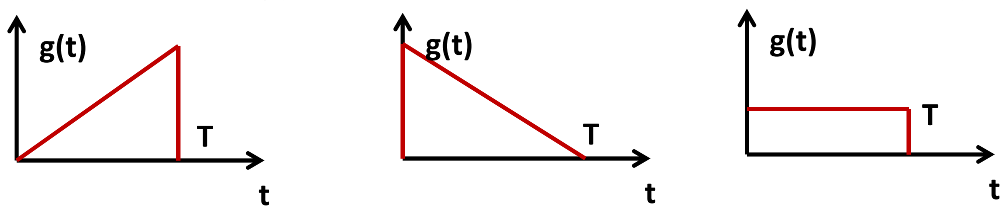
SNR of y(T) = $\frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df$

How to detect a from y(T)?

最优化

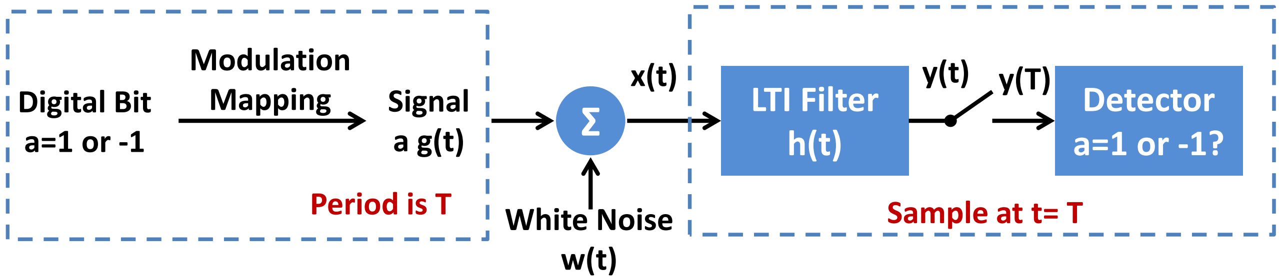
Optimized SNR

- Optimized SNR of y(T) $=\frac{2}{N_0}\int_{-\infty}^{\infty}|G(f)|^2 df = \frac{2E}{N_0}$, where E=$\int_0^T |g(t)|^2 dt$ is the energy of g(t)
- Observations
  - SNR is determined by the PSD of noise + energy of pulse
  - Given pulse energy, SNR is independent of the pulse shape



As long as their energies are the same, the SNRs are the same

Detector Design Issue



- **Detector design: given y(T), how to guess a?**
- $y(T) = g_0(T) + n(T) = \int_{-\infty}^{\infty} aG(f)kG^*(f)e^{-j2\pi fT}e^{j2\pi fT}df + n(T) = akE + n(T)$
- $\frac{y(T)}{kE} = a + \frac{n(T)}{kE}$: Signal + Noise
- What is the noise $Z = \frac{n(T)}{kE}$ like?

$\frac{y(T)}{kE} \sim N(a, \frac{N_0}{2E})$ $SNR = \frac{k^2E^2}{\frac{EN_0k^2}{2}} = \frac{2E}{N_0}$

Feature of Noise

随机过程

- $n(T) = w(t) * h(t)|_{t=T} = k\int_{-\infty}^{\infty} w(\tau)g(\tau)d\tau$ is a Gaussian R.V.
- Mean: $E[n(T)] = k\int_{-\infty}^{\infty} E[w(\tau)]g(\tau)d\tau = 0$
- Variance:

$$E[n^2(T)] = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} \left|kG^*(f)e^{-j2\pi fT}\right|^2 df$$

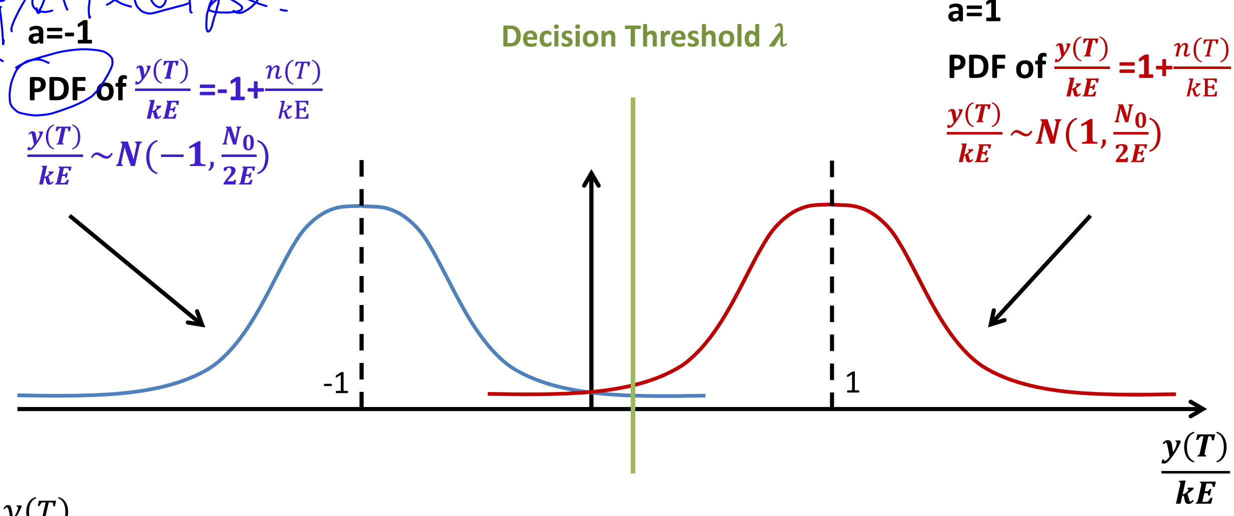
$$= \frac{EN_0k^2}{2}$$

- $n(T) \sim N(0, \frac{EN_0k^2}{2})$
- $\boldsymbol{Z = \frac{n(T)}{kE} \sim N(0, \frac{N_0}{2E})}$

Remember, We just proved
SNR = $\frac{2E}{N_0} = \frac{1}{N_0/2E}$

Observation

概率密度函数.



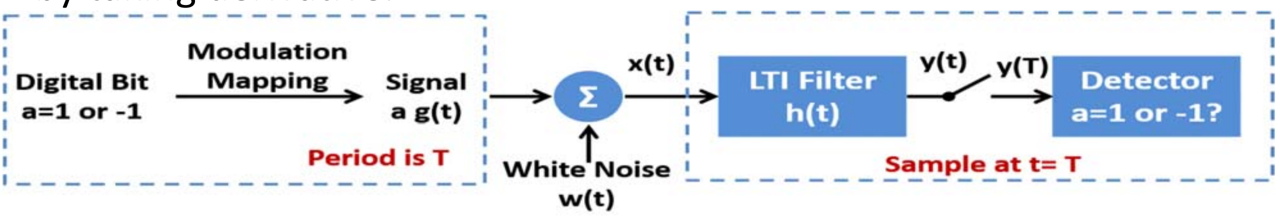
- $\frac{y(T)}{kE}$ is stochastically larger for a=1
- Threshold-based detector
  - If $\frac{y(T)}{kE} > \lambda$, we believe a=1; If $\frac{y(T)}{kE} < \lambda$ we believe a=-1

Optimal Detector Design

- Generally speaking, we want to

$$\min_{\lambda} Pr(make\ a\ wrong\ guess\ on\ a)$$

- Wrong guess
  - Error of the first kind: a=-1, $\frac{y(T)}{kE} > \lambda$
  - Error of the second kind: a=1, $\frac{y(T)}{kE} < \lambda$
- Methodology: find the math expression of the probability, optimize λ by taking derivative.



Probability Analysis

$$\boldsymbol{Pr(make\ a\ wrong\ guess\ on\ a)}$$

$$= \mathbf{Pr}(\boldsymbol{guess\ a\ wrongly}|\boldsymbol{a} = -\mathbf{1})\,\mathbf{Pr}(\boldsymbol{a} = -\mathbf{1}) + \mathbf{Pr}(\boldsymbol{guess\ a\ wrongly}|\boldsymbol{a} = \mathbf{1})\,\mathbf{Pr}(\boldsymbol{a} = \mathbf{1})$$

$$= \mathbf{Pr}\left(\frac{n(T)}{k\mathrm{E}} - \mathbf{1} > \boldsymbol{\lambda} \middle| \boldsymbol{a} = -\mathbf{1}\right)\mathbf{Pr}(\boldsymbol{a} = -\mathbf{1}) + \boldsymbol{Pr}\left(\frac{n(T)}{k\mathrm{E}} + \mathbf{1} < \boldsymbol{\lambda} \middle| \boldsymbol{a} = \mathbf{1}\right)\boldsymbol{Pr}(\boldsymbol{a} = \mathbf{1})$$

$$= \mathbf{Pr}\left(\frac{n(T)}{k\mathrm{E}} > \boldsymbol{\lambda} + \mathbf{1}\right)\mathbf{Pr}(\boldsymbol{a} = -\mathbf{1}) + \mathbf{Pr}\left(\frac{n(T)}{k\mathrm{E}} < \boldsymbol{\lambda} - \mathbf{1}\right)\boldsymbol{Pr}(\boldsymbol{a} = \mathbf{1})$$

Probability Analysis --- Cont'd

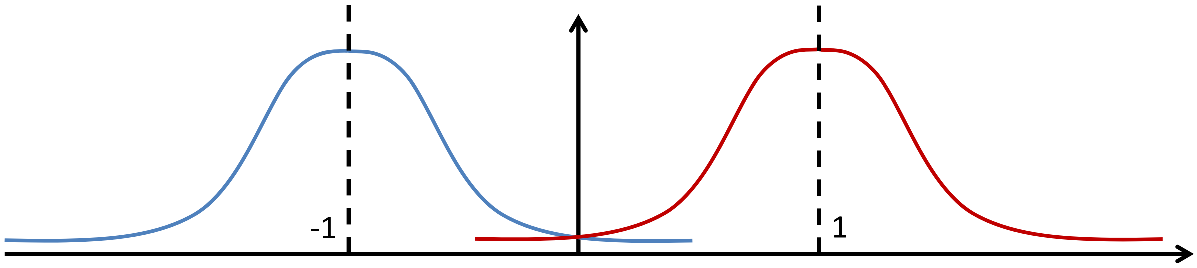
- **PDF of Z=$\frac{n(T)}{kE}$ ~N(0, $\frac{N_0}{2E}$) :** $f_Z(z) = \frac{1}{\sqrt{\pi N_0/E}} e^{-\frac{z^2}{N_0/E}}$
- $\mathbf{Pr}(\boldsymbol{Z} > \boldsymbol{\lambda} + \mathbf{1}) = \int_{\lambda+1}^{\infty} f_Z(z)dz = \int_{\lambda+1}^{\infty} \frac{1}{\sqrt{\pi N_0/E}} e^{-\frac{z^2}{N_0/E}} dz = \int_{\frac{\lambda+1}{\sqrt{N_0/2E}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$
- $\mathbf{Pr}(\boldsymbol{Z} < \boldsymbol{\lambda} - \mathbf{1}) = \int_{-\infty}^{\lambda-1} f_Z(z)dz = \int_{-\infty}^{\lambda-1} \frac{1}{\sqrt{\pi N_0/E}} e^{-\frac{z^2}{N_0/E}} dz = \int_{-\infty}^{\frac{\lambda-1}{\sqrt{N_0/2E}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$
- $\mathbf{Pr}(\boldsymbol{make\ a\ wrong\ guess\ on\ a}) =$
  $p_{-1} \int_{\frac{\lambda+1}{\sqrt{N_0/2E}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \mathrm{p}_1 \int_{-\infty}^{\frac{\lambda-1}{\sqrt{N_0/2E}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$
  - Priori probability: p_{-1} = Pr(a=-1); p_1=Pr(a=1)

Formulation & Solution

The determination of optimal threshold can be formulated as the following problem:

$$\min_{\lambda} p_{-1} \int_{\frac{\lambda+1}{\sqrt{N_0/2E}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz + \mathrm{p}_1 \int_{-\infty}^{\frac{\lambda-1}{\sqrt{N_0/2E}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

- Solution: take derivative with respect to λ
- If $p_{-1} = p_1, \lambda = 0$

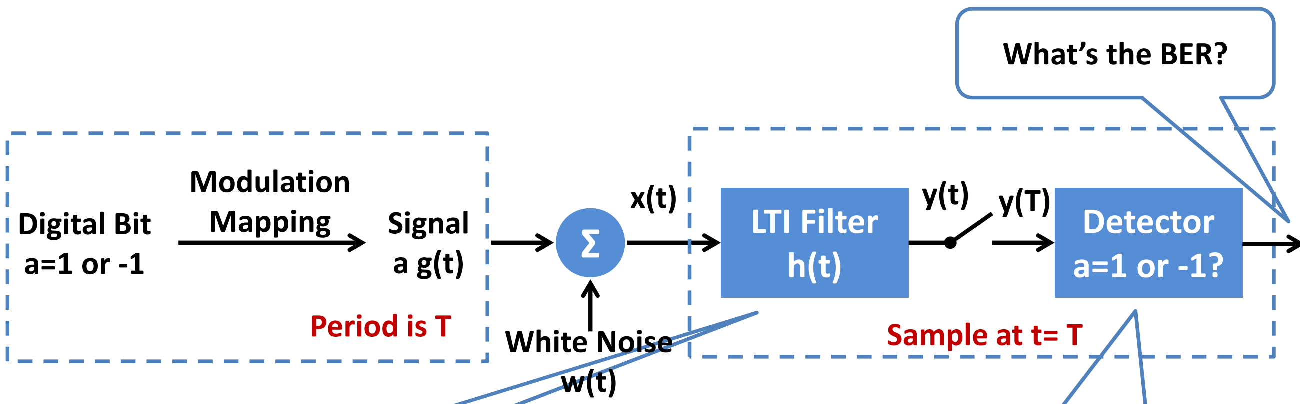


- If $p_{-1} \neq p_1, \lambda = 0$?

$\lambda \neq 0$

if $p_1 > p_{-1}$, $\lambda < 0$

if $p_1 < p_{-1}$, $\lambda > 0$



$$\boldsymbol{h(t) = kg(T-t)}$$

How to detect a from y(T)?
If $p_{-1} = p_1$,
y(T)>0 => a=1 and y(T)<0 => a=-1

Bit Error Rate

- Bit Error Rate (BER): the probability one bit is guessed wrongly
- Suppose $p_{-1} = p_1 = 1/2$, $\rightarrow \lambda = 0$

$$\textit{BER} \;\; P_e = p_{-1}\int_{\frac{\lambda+1}{\sqrt{N_0/2E}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \mathrm{p}_1\int_{-\infty}^{\frac{\lambda-1}{\sqrt{N_0/2E}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$= \frac{1}{2}\int_{\frac{1}{\sqrt{N_0/2E}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz + \frac{1}{2}\int_{-\infty}^{\frac{-1}{\sqrt{N_0/2E}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

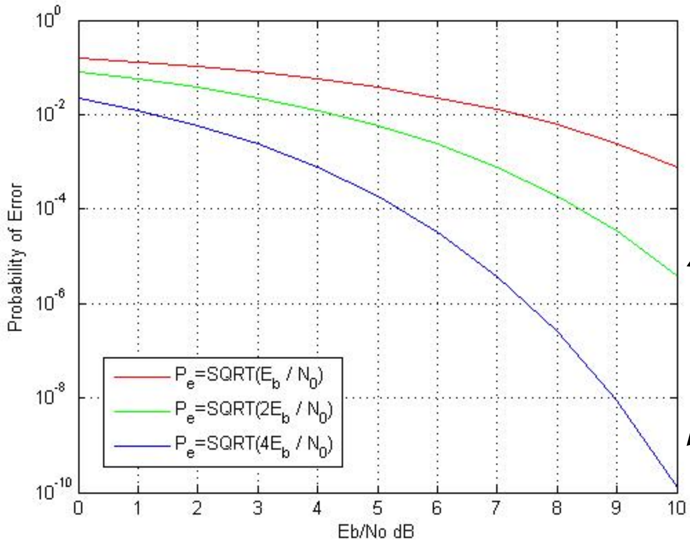
$$= \int_{\frac{1}{\sqrt{N_0/2E}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

Q-Function

- Q-function is usually utilized in communication system analysis
- Definition
  - $Q(u) = \frac{1}{\sqrt{2\pi}}\int_u^{\infty} e^{-z^2/2}dz$
- BER $P_e = \int_{\frac{1}{\sqrt{N_0/2E}}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$ $\rightarrow$ bit

BER Curve

- BER $P_e = Q(\sqrt{2E_b/N_0})$
- Relation between P_e and E_b/N_0 shows the robustness against noise



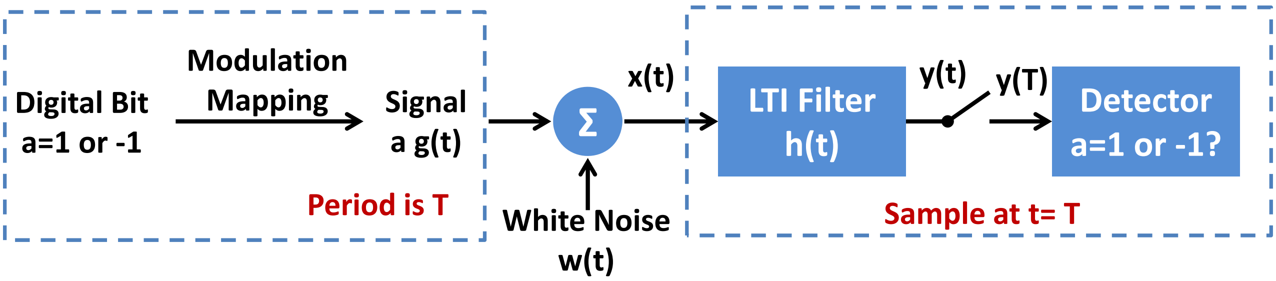
$P_e = Q(\sqrt{E_b/N_0})$

$P_e = Q(\sqrt{2E_b/N_0})$

$P_e = Q(\sqrt{4E_b/N_0})$

Which curve is the best?

AWGN Channel Summary



- AWGN channel
  - Optimal h(t): maximize the SNR of y(T)
  - Optimal decision threshold λ: minimize the error probability
- AWGN channel assumes impulse response is $\delta(t)$
- It is an approximation when signal bandwidth is relatively narrow
- Most of the channels are not AWGN

实际上

$ag(t) \longrightarrow$ Channel $\longrightarrow \oplus \longrightarrow$ Receiver

$\delta(t)$ $w(t)$

- What we know so far
  - Baseband digital modulation
  - Receiver design for AWGN channel
  - BER analysis for AWGN channel

- What's going on
  - Receiver deign for dispersive channel 分散的
  - BER analysis for dispersive channel

Advanced Knowledge – Parameter Estimation

另外一种比较方法

- $\frac{y(T)}{kE} = a + \frac{n(T)}{k\mathrm{E}}$
- In the detector design, we have a noisy observation $\mathrm{Y} = \frac{y(T)}{kE}$ and want to guess a.
- Likelihood function: $\mathrm{p}(Y|a)$ a=1 or -1

最大似然估计

$$\mathrm{p}(Y|a=-1) = \frac{1}{\sqrt{2\pi\frac{N_0}{2E}}} e^{-\frac{(Y-(-1))^2}{\frac{N_0}{E}}} \quad and \quad \mathrm{p}(Y|a=1) = \frac{1}{\sqrt{2\pi\frac{N_0}{2E}}} e^{-\frac{(Y-1)^2}{\frac{N_0}{E}}}$$

- Maximum likelihood (ML) detection: $\max_a \mathrm{p}(Y|a)$
- Which is smaller, $|Y-(-1)|$ or $|Y-1|$?
- ML detector is the same as our previous design for equal probability of 0 and 1.

Homework #D5

- D5.1

(a) If the Unipolar NRZ code as follows is used at the transmitter, how to design the receiver? What is the BER?

(b) If the Split-Phase Manchester Code as follows is used at the transmitter, how to design the receiver? What is the BER?