Chapter 9
Band-Pass Transmission of Digital Signals

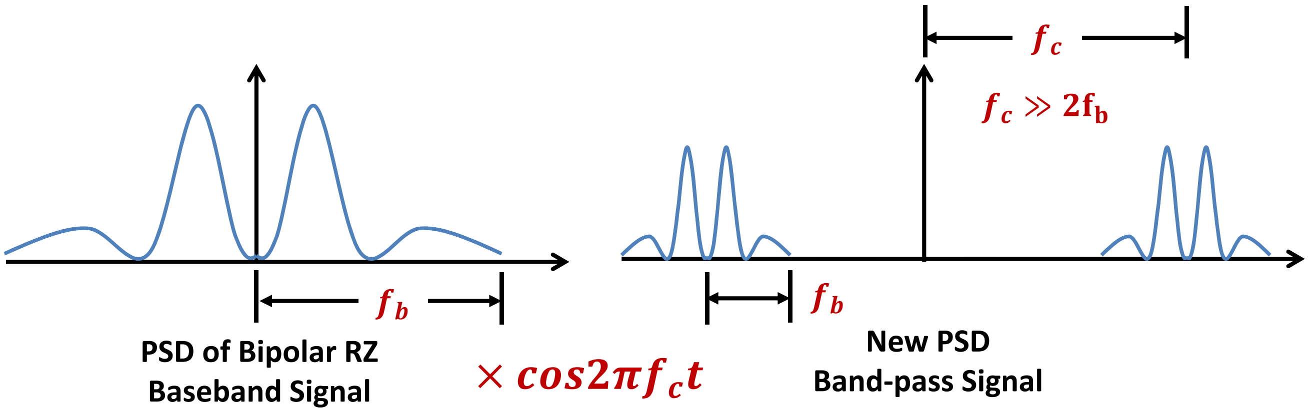
psk -> phase shift keying

Outline

- Introduction of band-pass signals
  - Generation
  - Complex envelope
  - Frequency division multiplexing
- 2-ary band-pass modulation
  - Binary FSK
- M-ary band-pass modulation
  - Quadrature Phase-Shift Keying (QPSK)
  - Quadrature Amplitude Modulation (QAM)

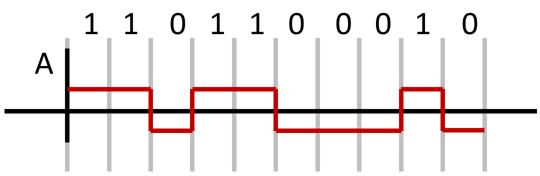
Band-Pass Signals

- Band-pass signals: center of spectrum >> signal bandwidth
- Why band-pass signals? Wireless



Band-Pass Signal Generation

- Suppose $g_I(t)$ is the signal after baseband digital modulation

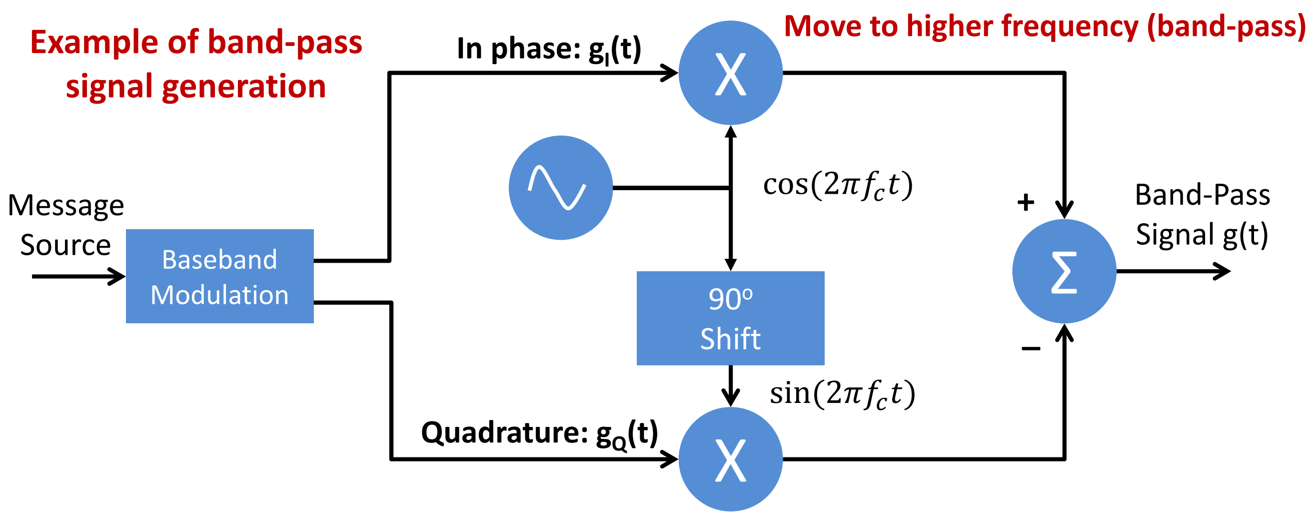


- Move the signal to higher frequency: $g_I(t)\cos(2\pi f_c t)$
- If $g_Q(t)$ is another modulated baseband signal, can $g_I(t)\cos(2\pi f_c t)$ and $g_Q(t)\sin(2\pi f_c t)$ be transmitted simultaneously?

$$\left[g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)\right] \times \cos(2\pi f_c t)$$
$$= \frac{1}{2} g_I(t)\cos(4\pi f_c t) + \frac{1}{2} g_I(t) - \frac{1}{2} g_Q(t)\sin(4\pi f_c t)$$
$$\left[g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)\right] \times \sin(2\pi f_c t)$$
$$= \frac{1}{2} g_I(t)\sin(4\pi f_c t) - \frac{1}{2} g_Q(t) + \frac{1}{2} g_Q(t)\cos(4\pi f_c t)$$

Band-Pass Signal Generation

- Two different baseband signals are transmitted in the same spectrum
- $g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$
- In phase component $g_I(t)$ and quadrature component $g_Q(t)$
- Distinguishability: sine and cosine are separable at the receiver



Complex Envelop

- Definition: if $g(t) = Re\{\tilde{g}(t)e^{j2\pi f_c t}\}$, $\tilde{g}(t)$ is called complex envelop of $g(t)$ given the carrier frequency f_c.

- $g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$
  $= Re\{[g_I(t) + jg_Q(t)]e^{j2\pi f_c t}\}$
- Therefore, $\tilde{g}(t) = g_I(t) + jg_Q(t)$

- Observations
  - In phase component of $g(t)$ = Real part of $\tilde{g}(t)$
  - Quadrature component of $g(t)$ = Imaginary part of $\tilde{g}(t)$
  - Use a complex baseband signal to represent a band-pass signal

- **We usually use complex envelop to represent a band-pass signal**

Property

- If $g(t) \leftrightarrow G(f)$ and $\tilde{g}(t) \leftrightarrow \tilde{G}(f)$, then

$$G(f) = \frac{1}{2}[\tilde{G}(f - f_c) + \tilde{G}^*(-f - f_c)]$$

- Proof:

$$g(t) = Re\{\tilde{g}(t)e^{j2\pi f_c t}\} = \frac{1}{2}[\tilde{g}(t)e^{j2\pi f_c t} + \tilde{g}^*(t)e^{-j2\pi f_c t}]$$

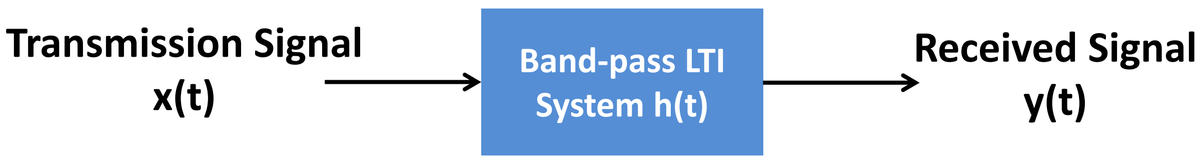
Therefore,

$$F\{g(t)\} = F\left\{\frac{1}{2}\left[\tilde{g}(t)e^{j2\pi f_c t} + \tilde{g}^*(t)e^{-j2\pi f_c t}\right]\right\}$$

$$= \frac{1}{2}F\{\tilde{g}(t)e^{j2\pi f_c t}\} + \frac{1}{2}F\{\tilde{g}^*(t)e^{-j2\pi f_c t}\}$$

$$= \frac{1}{2}\tilde{G}(f - f_c) + \frac{1}{2}\tilde{G}^*(-f - f_c)$$

Complex Envelop in LTI System



- We already know
  - y(t) = h(t) * x(t)
  - Y(f) = H(f) X(f)
- Suppose the complex envelop of x(t), y(t) and h(t) are $\tilde{x}(t), \tilde{y}(t)$ and $\tilde{h}(t)$. If x(t) is a band-pass signal and h(t) is a band-pass system, then

$$\boldsymbol{\tilde{y}(t) = \frac{1}{2}\tilde{h}(t) * \tilde{x}(t)}$$

- The convolution property of LTI systems also applies on the complex envelop

Proof

Since

$$H(f) = \frac{1}{2}[\tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c)]$$

and

$$X(f) = \frac{1}{2}[\tilde{X}(f - f_c) + \tilde{X}^*(-f - f_c)],$$

we have

$$Y(f) = \frac{1}{4}[\tilde{H}(f - f_c) + \tilde{H}^*(-f - f_c)][\tilde{X}(f - f_c) + \tilde{X}^*(-f - f_c)]$$

$$= \frac{1}{4}\Big[\tilde{H}(f - f_c)\tilde{X}(f - f_c) + \underbrace{\tilde{H}^*(-f - f_c)\tilde{X}(f - f_c)}_{=0} + \underbrace{\tilde{H}(f - f_c)\tilde{X}^*(-f - f_c)}_{=0}$$

$$+ \tilde{H}^*(-f - f_c)\tilde{X}^*(-f - f_c)\Big]$$

$$= \frac{1}{2}[\frac{1}{2}\tilde{H}(f - f_c)\tilde{X}(f - f_c) + \frac{1}{2}\tilde{H}^*(-f - f_c)\tilde{X}^*(-f - f_c)]$$

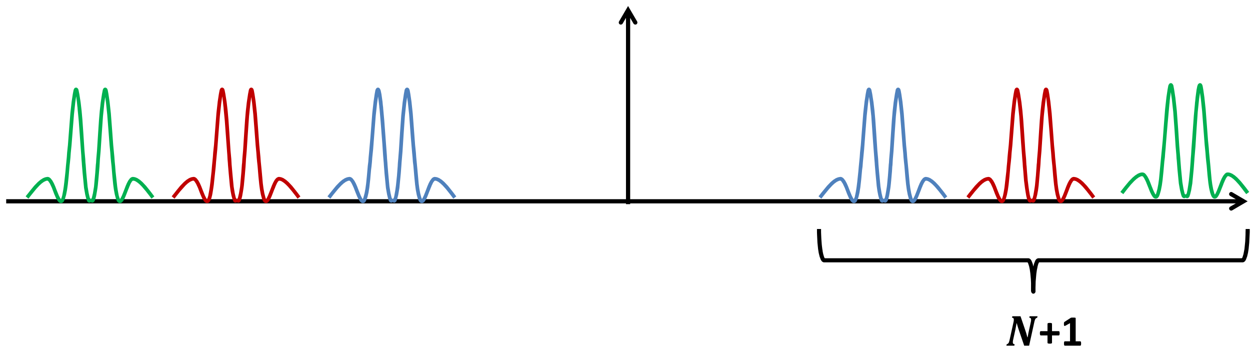
Because $Y(f) = \frac{1}{2}[\tilde{Y}(f - f_c) + \tilde{Y}^*(-f - f_c)]$, we have

$$\tilde{Y}(f - f_c) = \frac{1}{2}\tilde{H}(f - f_c)\tilde{X}(f - f_c) \;\; or \;\; \tilde{Y}(f) = \frac{1}{2}\tilde{H}(f)\tilde{X}(f)$$

Reference: Chapter 2.9 & 2.10

Frequency Division Multiplexing

- $g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$
- FDM: multiple band-pass signal can be delivered simultaneously via different carrier frequency



- $g(t) = \sum_{k=0}^{N}[g_{I,k}(t)\cos(2\pi f_k t) - g_{Q,k}(t)\sin(2\pi f_k t)]$

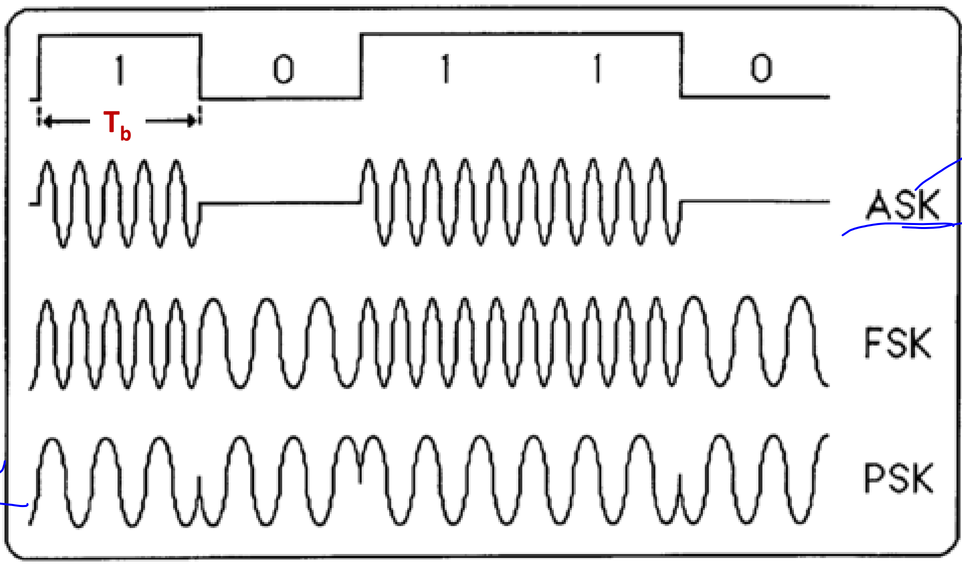
Outline

- Introduction of band-pass signals
  - Generation
  - Complex envelope
  - Frequency division multiplexing
- 2-ary band-pass modulation
  - Binary FSK
- M-ary band-pass modulation
  - Quadrature Phase-Shift Keying (QPSK)
  - Quadrature Amplitude Modulation (QAM)

2-ary Band-Pass Modulation

- Amplitude-shift keying: $g_k = A_k \cos(2\pi f_c t)$, $A_k = 0,1$
- Frequency-shift keying: $g_k = \cos(2\pi f_k t)$, $f_k = f_0, f_1$
- Phase-shift keying: $g_k = \cos(2\pi f_c t + \phi_k)$, $\phi_k = \pi, 0$

幅度调制的数字版本

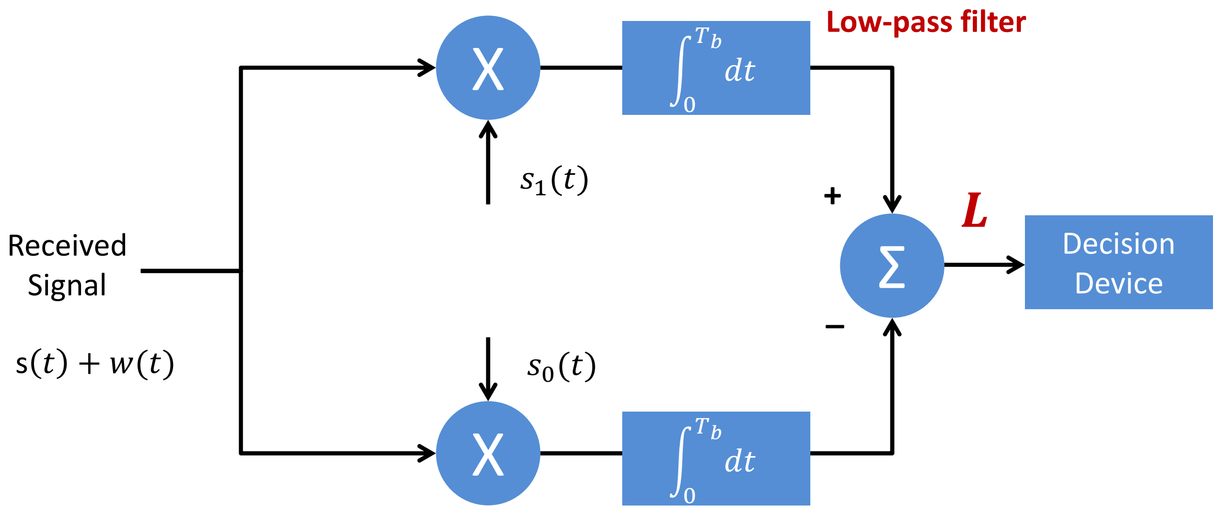


最好的方式

Coherent Detection of Binary FSK

- Received signal of AWGN channel: $x(t) = s(t) + w(t)$

$$Bit\ 1:\ \ s(t) = s_1(t) = A_c\cos(2\pi f_1 t) \quad 0 \le t \le T_b$$
$$Bit\ 0:\ \ s(t) = s_0(t) = A_c\cos(2\pi f_0 t) \quad 0 \le t \le T_b$$



Coherent Detection of Binary FSK --- Cont.

- Two hypotheses

$$H_0: x(t) = s_0(t) + w(t)$$
$$H_1: x(t) = s_1(t) + w(t)$$

- Receiver:

$$L = \int_0^{T_b} x(t)[s_1(t) - s_0(t)]dt$$

- If bit 1 is transmitted,

$$H_1:\ \ L = \int_0^{T_b} s_1(t)[s_1(t) - s_0(t)]dt + \int_0^{T_b} w(t)[s_1(t) - s_0(t)]dt$$

分布一样，
(分布参数不一样)

- If bit 0 is transmitted,

$$H_0:\ \ L = \int_0^{T_b} s_0(t)[s_1(t) - s_0(t)]dt + \int_0^{T_b} w(t)[s_1(t) - s_0(t)]dt$$

- $\int_0^{T_b} s_1^2(t)dt = \mathrm{A_c^2} \int_0^{T_b} \cos^2(2\pi f_1 t)\, dt = \frac{A_c^2 T_b}{2}$
- $\int_0^{T_b} s_0^2(t)dt = \mathrm{A_c^2} \int_0^{T_b} \cos^2(2\pi f_0 t)\, dt = \frac{A_c^2 T_b}{2}$
- Therefore we define

$$Signal\ Energy:\ E_b = \int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} s_0^2(t)dt$$

- Let ρ be the correlation coefficient between s_0 and s_1

$$Correlation\ Coefficient:\ \rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{E_b} \in [-1,1]$$

- Hence, $\int_0^{T_b} s_1(t)[s_1(t) - s_0(t)]dt = \mathrm{E_b} - \mathrm{E_b}\rho = E_b(1-\rho)$
  $\int_0^{T_b} s_0(t)[s_1(t) - s_0(t)]dt = -E_b(1-\rho)$

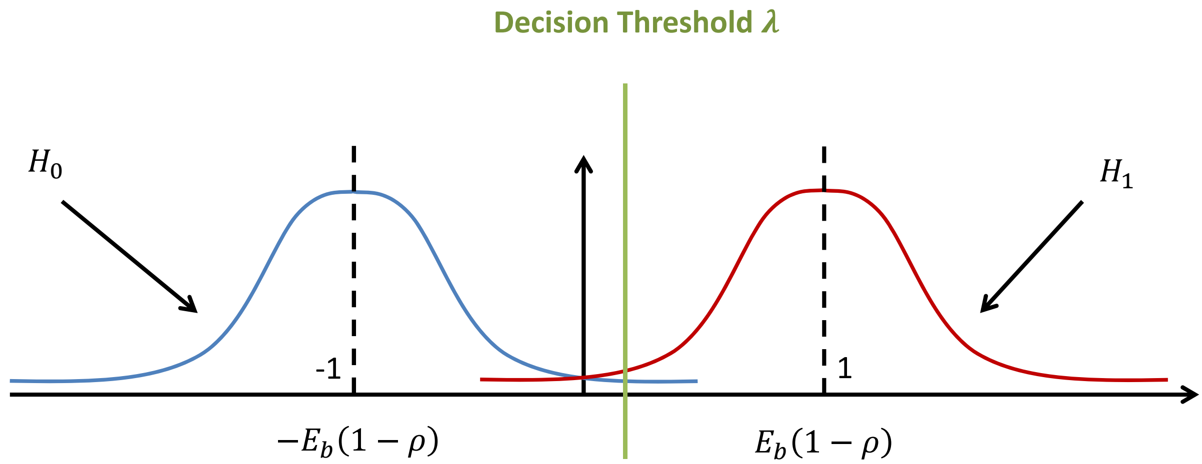
- Define

$$Equivalent\ Noise: N = \int_0^{T_b} w(t)[s_1(t) - s_0(t)]dt$$

$$E(N) = \int_0^{T_b} E[w(t)][s_1(t) - s_0(t)]dt = 0$$

$$Var(N) = E(N^2) = \int_0^{T_b} \frac{N_0}{2}[s_1(t) - s_0(t)]^2 dt = N_0 E_b(1-\rho)$$

- N is Gaussian RV with $E(N) = 0$ and $Var(N) = N_0 E_b(1-\rho)$
- $\mathrm{H_1}$: $L = E_b(1-\rho) + N$
- $\mathrm{H_0}$: $L = -E_b(1-\rho) + N$
- Similar to the baseband case, we formulate the optimal threshold design as a optimization problem



Formulation

- Probability of detection error (BER)

$$
\begin{aligned}
P_e &= \Pr(H_1)\Pr(Detection\ Error|H_1) \\
&\quad + \Pr(H_0)\Pr(Detection\ Error|H_0) \\
&= \Pr(H_1)\Pr(L < \lambda|H_1) + \Pr(H_0)\Pr(L > \lambda|H_0)
\end{aligned}
$$

- Problem formulation: find the optimal threshold such that the BER is minimized

$$\min_{\lambda} \Pr(H_1)\Pr(L < \lambda|H_1) + \Pr(H_0)\Pr(L > \lambda|H_0)$$

Solution

- With $\Pr(H_0) = \Pr(H_1) = 0.5$, it can be solved that $\lambda = 0$
- Hence,

$$P_e = (L < 0|H_1) = \Pr\big(N < -E_b(1-\rho)\big)$$

概统知识、

$$= \Pr\left(\frac{N}{\sqrt{N_0 E_b(1-\rho)}} < -\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

$$= \Pr\left(\frac{N}{\sqrt{N_0 E_b(1-\rho)}} > \sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

ρ越小越好

$$= Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$$

Discussion

- $P_e = Q\left(\sqrt{\frac{E_b(1-\rho)}{N_0}}\right)$
- The BER performance of binary FSK depends on the correlation coefficient ρ
- When ρ decreases, BER becomes better
- Problem 9.8, $\rho \approx sinc[2(f_1 - f_0)T_b]$
- When $\rho = 0$, $P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$
- How to compare with 2-ary baseband modulation?

Outline

- Introduction of band-pass signals
  - Generation
  - Complex envelope
  - Frequency division multiplexing
- 2-ary band-pass modulation
  - Binary FSK
- M-ary band-pass modulation
  - Quadrature Phase-Shift Keying (QPSK)
  - Quadrature Amplitude Modulation (QAM)

M-Ary Modulation

- Binary ASK, PSK and FSK
  - Transmitter has two band-pass signals: $s_1(t)$ and $s_0(t)$
  - Each signal can represent 1 information bit
- M-ary Modulation
  - Transmitter has M different signals
  - Each signal can represent $\log_2 M$ information bits
- Example:
  - QPSK
  - M-QAM

Quadrature Phase-Shift Keying (QPSK)

- BPSK

$$Bit\ 1: s_1(t) = \mathrm{A_c}\cos(2\pi f_c t)$$

能量不发生改变

$$Bit\ 0: s_0(t) = A_c \cos(2\pi f_c t + \pi)$$

- QPSK: 4 different phases to deliver 2 bits per transmission

$$Bits\ 11:\ s_0(t) = \sqrt{2}A_c \cos(2\pi f_c t + \frac{\pi}{4}) = A_c \cos 2\pi f_c t - A_c \sin 2\pi f_c t \quad (+\,+)$$

$$Bits\ 01:\ s_1(t) = \sqrt{2}A_c \cos(2\pi f_c t + \frac{3\pi}{4}) = -A_c \cos 2\pi f_c t - A_c \sin 2\pi f_c t \quad (-\,+)$$

$$Bits\ 00:\ s_2(t) = \sqrt{2}A_c \cos(2\pi f_c t + \frac{5\pi}{4}) = -A_c \cos 2\pi f_c t + A_c \sin 2\pi f_c t \quad (-\,-)$$

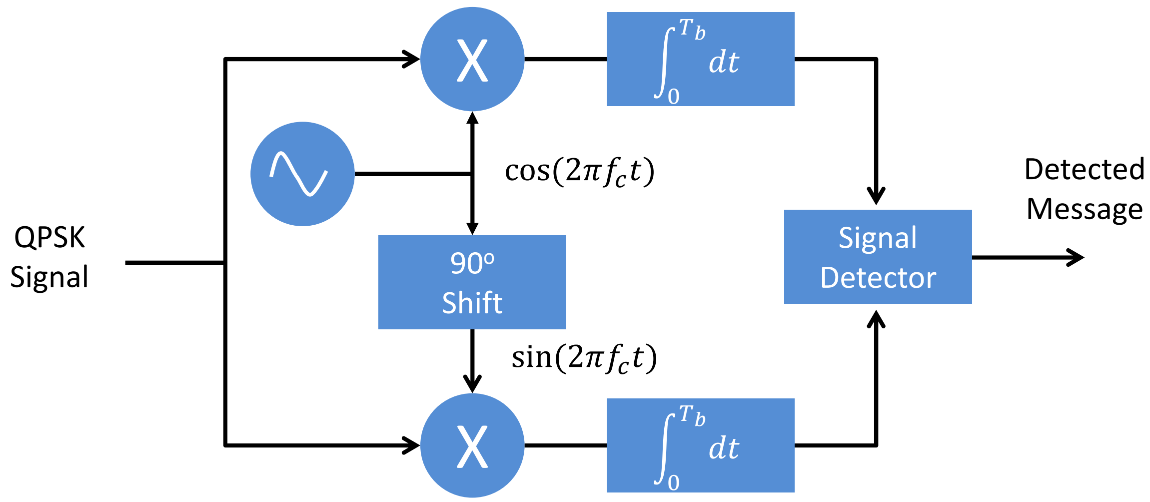
$$Bits\ 10:\ s_3(t) = \sqrt{2}A_c \cos(2\pi f_c t + \frac{7\pi}{4}) = A_c \cos 2\pi f_c t + A_c \sin 2\pi f_c t \quad (+\,-)$$

$\frac{2\pi}{4} = \frac{\pi}{2}$ $\qquad \frac{2\pi}{8} = \frac{\pi}{4}$

$\frac{6}{4}\pi$ $\qquad \frac{2\pi}{4} \times 3$

$\underline{\frac{\pi}{2} + \pi.}$

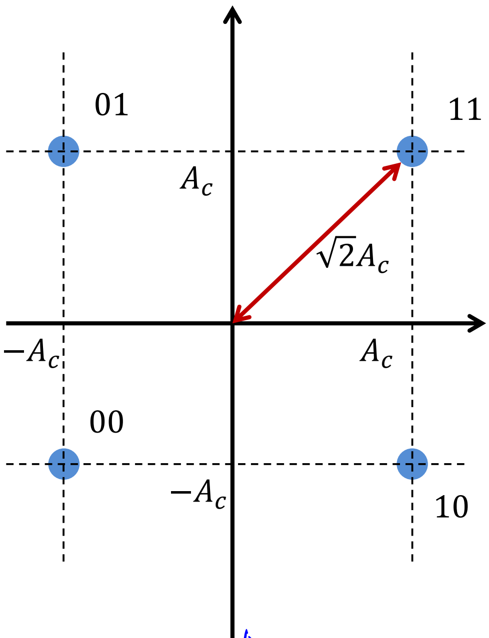
QPSK Receiver



AWGN Channel BER $P_e = Q\left(\sqrt{\frac{E}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$

E: Energy per symbol
E_b: Energy per bit

QPSK Constellation



8PSK. → $\sqrt{3}A_c$.

- Complex envelope at receiver

$$Bits\ 11:\ \tilde{s}_0 = A_c + A_c j$$
$$Bits\ 01:\ \tilde{s}_1 = -A_c + A_c j$$
$$Bits\ 00:\ \tilde{s}_2 = -A_c - A_c j$$
$$Bits\ 10:\ \tilde{s}_3 = A_c - A_c j$$

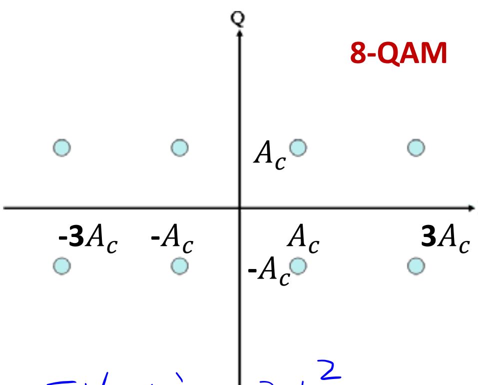
- Constellation:
  - X-axis: in-phase component
  - Y-axis: quadrature component
  - A signal can be represented by one point on the constellation.
- Distance^2 is proportional to signal power
  - Signal Power= ½ (In-Phase^2 + Quadrature^2) =1/2 Distance^2
- Constellation mapping (demapping)
  - Mapping between constellation point and binary bits

M-QAM

- M-Array Quadrature Amplitude Modulation

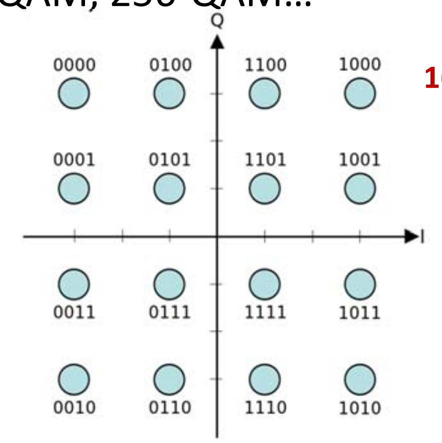
$$s(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t)$$
$$g_I = \pm A_c, \pm 3A_c, \dots, \pm(2N_I+1)A_c$$
$$g_Q = \pm A_c, \pm 3A_c, \dots, \pm(2N_Q+1)A_c$$

- Complex envelope $\tilde{s} = g_I + jg_Q = \pm(2n_I+1)A_c \pm (2n_Q+1)A_c j$
- QPSK = 4-QAM, 8-QAM, 16-QAM, 64-QAM, 256-QAM…



8-QAM

平均功率: $3A_c^2$



16-QAM

Homework #D7

- D7.1

Please design a receiver of the following band-pass modulation for AWGN channel. What is the BER?

$$Bit\ 1\text{:}\ \ s(t) = s_1(t) = A_c \cos(2\pi f_c t) \qquad 0 \le t \le T_b$$
$$Bit\ 0\text{:}\ \ s(t) = s_0(t) = A_c \cos(2\pi f_c t + \phi) \quad 0 \le t \le T_b$$

$\phi=0$, BER $= 50\%$

- D7.2

What is the average transmission power of the following 8-QAM modulation scheme? Suppose each symbol is transmitted with equal probability, and the symbol duration is T.