chapter-3 习题课
（hw2&3）

2022.10.17

3-1. Consider a two-ray channel consisting of a direct ray plus a ground-reflected ray, where the transmitter is a fixed base station at height h and the receiver is mounted on a truck (also at height h). The truck starts next to the base station and moves away at velocity v. Assume that signal attenuation on each path follows a free-space path-loss model. Find the time-varying channel impulse at the receiver for transmitter–receiver separation $d = vt$ sufficiently large for the length of the reflected ray to be approximated by $r + r' \approx d + 2h^2/d$.

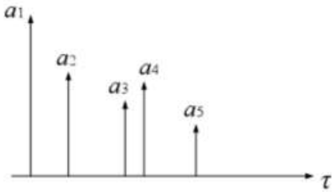
$$c(\tau,t) = \sum_{n=1}^{N(t)} \alpha_n(t) e^{-j\phi_n(t)} \delta(\tau - \tau_n(t))$$

$c(\tau,t)$: **impulse response of the channel at time t to the impulse input at time t-τ**

$\alpha_n(t)$: **random amplitude**

$\varphi_n(t)$: **random phase**

$\tau_n(t)$: **random delay**



$d = vt$

$r + r' = d + \frac{2h^2}{d}$

Equivalent low-pass channel impulse response is given by

$$c(\tau,t) = \alpha_0(t)e^{-j\phi_0(t)}\delta(\tau - \tau_0(t)) + \alpha_1(t)e^{-j\phi_1(t)}\delta(\tau - \tau_1(t))$$

3-1. Consider a two-ray channel consisting of a direct ray plus a ground-reflected ray, where the transmitter is a fixed base station at height h and the receiver is mounted on a truck (also at height h). The truck starts next to the base station and moves away at velocity v. Assume that signal attenuation on each path follows a free-space path-loss model. Find the time-varying channel impulse at the receiver for transmitter–receiver separation $d = vt$ sufficiently large for the length of the reflected ray to be approximated by $r + r' \approx d + 2h^2/d$.

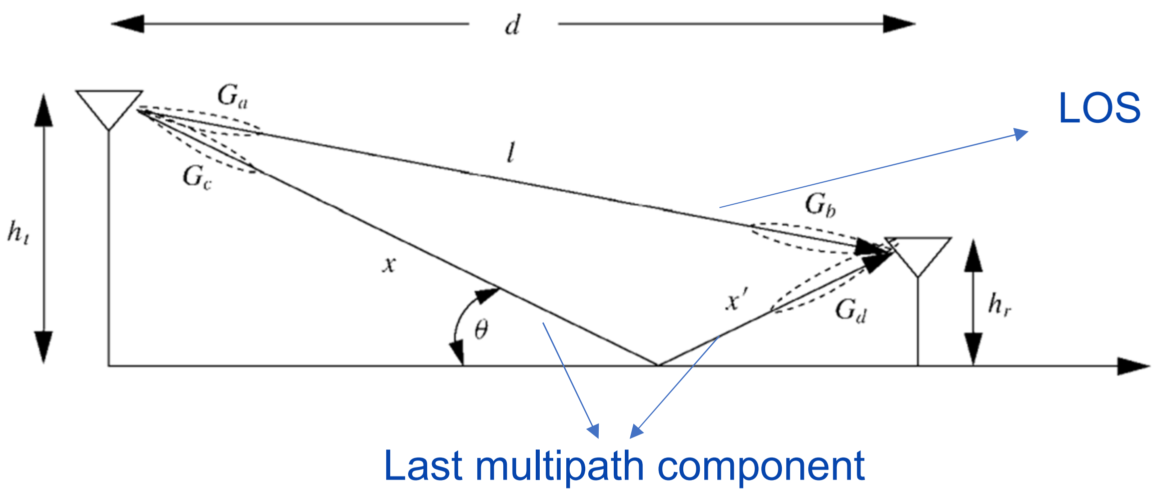
$$\alpha_0(t) = \frac{\lambda\sqrt{G_l}}{4\pi d} \text{ with } d = vt$$
$$\phi_0(t) = 2\pi f_c \tau_0(t) - \phi_{D_0}$$
$$\tau_0(t) = d/c$$
$$\phi_{D_0} = \int_t 2\pi f_{D_0}(t)dt$$
$$f_{D_0}(t) = \frac{v}{\lambda}\cos\theta_0(t)$$
$$\theta_0(t) = 0\ \forall t$$

同理，

$$\alpha_1(t) = \frac{\lambda R\sqrt{G_l}}{4\pi(r+r')} = \frac{\lambda R\sqrt{G_l}}{4\pi(d+\frac{2h^2}{d})} \text{ with } d = vt$$
$$\phi_1(t) = 2\pi f_c \tau_1(t) - \phi_{D_1}$$
$$\tau_1(t) = (r + r')/c = (d + \frac{2h^2}{d})/c$$
$$\phi_{D_1} = \int_t 2\pi f_{D_1}(t)dt$$
$$f_{D_1}(t) = \frac{v}{\lambda}\cos\theta_1(t)$$
$$\theta_1(t) = \pi - \arctan\frac{h}{d/2}\ \forall t$$

3-2. Find a formula for the multipath delay spread T_m for a two-ray channel model. Find a simplified formula when the transmitter–receiver separation is relatively large. Compute T_m for $h_t = 10$ m, $h_r = 4$ m, and $d = 100$ m.

- Delay spread:
  - Def: difference between the time of arrival of the **earliest** significant multipath component (typically the LOS) and the time of arrival of the **last** multipath components. ---- (from Wiki)



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For the 2 ray model:

$$\tau_0 = \frac{l}{c}$$

$$\tau_1 = \frac{x + x'}{c}$$

$$\therefore \text{delay spread}(T_m) = \frac{x + x' - l}{c} = \frac{\sqrt{(h_t + h_r)^2 + d^2} - \sqrt{(h_t - h_r)^2 + d^2}}{c}$$

when $d \gg (h_t + h_r)$

$$T_m = \frac{1}{c}\frac{2h_t h_r}{d}$$

$$h_t = 10m,\ h_r = 4m,\ d = 100m$$

$$\therefore T_m = 2.67 \times 10^{-9} s$$

3-3. Consider a time-invariant indoor wireless channel with LOS component at delay 23 ns, a multipath component at delay 48 ns, and another multipath component at delay 67 ns. Find the delay spread assuming that the demodulator synchronizes to the LOS component. Repeat assuming that the demodulator synchronizes to the first multipath component.

Delay for LOS component $= \tau_0 = 23$ ns
Delay for First Multipath component $= \tau_1 = 48$ ns
Delay for Second Multipath component $= \tau_2 = 67$ ns

$\tau_c =$ Delay for the multipath component to which the demodulator synchronizes.

$$T_m = \max_m \tau_m - \tau_c$$

So, when $\tau_c = \tau_0$, $T_m = 44$ ns. When $\tau_c = \tau_1$, $T_m = 19$ ns.

3-4. Show that the minimum value of $f_c\tau_n$ for a system at $f_c = 1$ GHz with a fixed transmitter and a receiver separated by more than 10 m from the transmitter is much greater than 1.

$$f_c = 10^9 Hz$$
$$\tau_{n,min} = \frac{10}{3\times10^8} s$$
$$\therefore \min f_c\tau_n = \frac{10^{10}}{3\times10^8} = 33 \gg 1$$

注意，对于典型的载波频率来说，第n径会满足$f_c\tau_n(t)>>1$。例如室内系统的典型时延是$\tau_n=50$ns，若$f_c=1$GHz，则$f_c\tau_n=50>>1$。室外系统的多径时延要比50ns大得多，因而也同样满足这一点。当$f_c\tau_n(t)>>1$时，第n条径上路径时延的微小变化将导致相位$\phi_n(\tau)=2\pi f_c\tau_n(t)-\phi_{D_n}-\phi_0$极大的变化。各径上相位的快速变化将造成剧烈的干涉现象，从而使接收信号强度发生快速的变化。这种现象叫作*衰落*（fading），下面几节将对此进行详细的讨论。

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3-5. Prove, for X and Y independent zero-mean Gaussian random variables with variance σ^2, that the distribution of $Z = \sqrt{X^2 + Y^2}$ is Rayleigh distributed and that the distribution of Z^2 is exponentially distributed.

零均值高斯分布:
$$X \sim \mathcal{N}(0,\ \sigma^2) \Rightarrow Y = \frac{X}{\sigma} \sim \mathcal{N}(0,\ 1)$$

瑞利分布:
$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, x > 0$$

指数分布:
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$F(x;\lambda) = \begin{cases} 1 - e^{-\lambda x} & , x \ge 0, \\ 0 & , x < 0. \end{cases}$$

3-5. Prove, for X and Y independent zero-mean Gaussian random variables with variance σ^2, that the distribution of $Z = \sqrt{X^2 + Y^2}$ is Rayleigh distributed and that the distribution of Z^2 is exponentially distributed.

Use CDF strategy.

$$F_z(z) = P[x^2 + y^2 \le z^2] = \int\int_{x^2+y^2\le z^2} \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y)^2}{2\sigma^2}} dxdy = \int_0^{2\pi}\int_0^{z} \frac{1}{2\pi\sigma^2} e \frac{-r^2}{2\sigma^2} r dr d\theta = 1 - e^{\frac{-z^2}{2\sigma^2}} (z \ge 0)$$

$$\frac{df_z(z)}{dz} = \frac{z}{\sigma^2} e^{-\frac{z^2}{2\sigma^2}} \rightarrow Rayleigh$$

For Power:

$$F_{z^2}(z) = P[Z \le \sqrt{z}] = 1 - e \frac{-z^2}{2\sigma^2}$$

$$f_z(z) = \frac{1}{2\sigma^2} e \frac{-z}{2\sigma^2} \rightarrow Exponential$$

3-6. Assume a Rayleigh fading channel with average signal power $2\sigma^2 = -80$ dBm. What is the power outage probability of this channel relative to the threshold $P_0 = -95$ dBm? How about $P_0 = -90$ dBm?

Recall Rayleigh envelop distribution

$$p_z(z) = \frac{1}{\bar{P}_r} e^{-\frac{z^2}{2\sigma^2}}$$

Power distribution can be obtained from the above distribution function

$$P_{z^2}(x) = \frac{1}{2\sigma^2} e^{-\frac{x}{2\sigma^2}}$$

where $2\sigma^2 = -80dBm = 10^{-8}\,mW$

Power outage probability is

$$Pr(P_0) = Prob(x < P_0) = 1 - e^{-P_0/2\sigma^2}$$

So for the threshold is $\boldsymbol{P_0}$ =-95dBm=$10^{-9.5}mW$, the outage probability is
***Pr*($10^{-9.5}$)=0.031**
and for $\boldsymbol{P_0}$ =-90dBm=$10^{-9}mW$
***Pr*(10^{-9})=0.095**

3-7. Suppose we have an application that requires a power outage probability of .01 for the threshold $P_0 = -80$ dBm. For Rayleigh fading, what value of the average signal power is required?

$f_{\alpha_I,\alpha_Q}(\alpha_I,\alpha_Q)$: Independent Gaussian with variance σ^2

$$f_\Phi(\varphi)=\frac{1}{2\pi} \quad \text{if } \varphi\in[0,2\pi) \quad : \text{Uniform Phase}$$

$$f_R(r)=\frac{r}{\sigma^2}\exp(-\frac{r^2}{2\sigma^2}) \quad \text{if } r>0 \quad : \text{Rayleigh Amplitude}$$

$$f_P(p)=\frac{1}{P_o}\exp(-\frac{p}{P_o}) \quad \text{if } p>0 \quad : \text{Exponential Channel Power Gain}$$

where $p=r^2$ and P_o=$2\sigma^2$ is mean channel power gain

For Rayleigh fading channel

$$P_{outage} = 1 - e^{-P_0/2\sigma^2}$$

$$0.01 = 1 - e^{-P_0/P_r}$$

$$\therefore P_r = -60dBm$$

for $\ln 0.99 = 0.01$