Lecture 4 Capacity of Wireless Channels

EE313/EE5028 Wireless Communications

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AWGN Channel

real channel.

$X \rightarrow \boxed{AWGN} \rightarrow Y$

$\uparrow$ Noise

$Y = X + N$.

real channel

$N \sim (0, \underbrace{\frac{N_0 B}{2}}_{})$ (方差，平均功率) B is bandwidth.

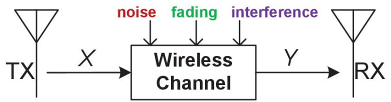
X一直在变 ⇒ X is random variable ⇒ Y is r.v.

$p(x, y) = \underbrace{p(x)}_{Tx} \cdot \underbrace{p(y|x)}_{\text{Channel}}$

Overview

- Capacity in AWGN
- Capacity of Flat-Fading Channels
  - Capacity with Channel Distribution Information (CDI)
  - Capacity with Receiver Channel State Information (CSI)
  - Capacity with both TX and RX CSI
- Capacity of Frequency-Selective Fading Channels

What Is Channel Capacity?



Point-to-point transmission.

TX sent X with rate R to RX, output alphabet is Y, what is the maximal R achieving an arbitrarily small error probability?

For a memoryless time-invariant channel, the mutual information is

$$I(X;Y) = \sum_{x\in X, y\in Y} p(x,y)\log\left(\frac{p(x,y)}{p(x)p(y)}\right) = H(Y) - H(Y|X),$$

where $H(X) = -\sum_{x\in X} p(x)\log p(x)$, **(information entropy)**

$$H(Y|X) = -\sum_{x\in X, y\in Y} p(x,y)\log p(y|x), \text{ (noise entropy)}$$

→ 除去Y中从X传递到来的信息量，Y中还剩多少信息量.

记

Channel capacity: $C = \max_{p(x)} I(X;Y) = \max_{p(x)} (H(Y) - H(Y|X)) = \frac{1}{2}\log_2(1+\gamma)$ → γ = 信噪比 = SNR = $\frac{E[X^2]}{E[N^2]} = \frac{P/2}{N_0B/2} = \frac{P}{N_0B}$

real channel

(when channel are real channel) capacity per symbol.

$I(X;Y) = H(Y) - H(Y|X)$

$p(x,y) = p(x)\,p(y|x)$

Tx channel.

$C = \max I(X;Y)$ Capacity.

当传输的数据率低于Capacity时，存在一种传输方式无差错

AWGN: $C = B\log_2(1+\frac{P}{N_0 B})$ → Tx/Rx signal power.

Consider an AWGN $y[i] = x[i] + n[i]$, with bandwidth B and TX power P. SNR is $\gamma = \frac{P}{N_0 B}$, where $N_0/2$ is the noise PSD of $n[i]$, then

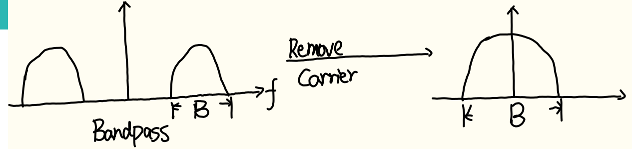
$$C = B\log_2(1+\gamma) \text{ bits/second (bps)}$$

Shannon proved that a code exists for achieving R arbitrarily close to C. It is *impossible* to achieve error-free transmissions with rate $R > C$. For the AWGN channel, the maximizing input distribution is Gaussian.

EXAMPLE 1.

Consider an AWGN channel where power falloff with distance follows $P_r(d) = P_t(d_0/d)^3$ for $d_0 = 10$m. $B = 30$KHz and $N_0 = 10^{-9}$W/Hz, $P_t = 1$W, find C for a transmit-receive distance of 100m and 1Km.

Solution: $\gamma_{100} = \frac{P_r(100)}{N_0 B} = 33 = 15\text{dB}$, $\gamma_{1000} = \frac{P_r(1000)}{N_0 B} = 0.033 = -15\text{dB}$,
$C_{100} = B\log_2(1+\gamma_{100}) = 30000\log_2(1+33) = 152.6\text{Kbps}$,
$C_{1000} = B\log_2(1+\gamma_{1000}) = 30000\log_2(1+0.033) = 1.4\text{Kbps}$



baseband $\frac{B}{2}$ 采样频率：$2\cdot\frac{B}{2}=B$.

In wireless communication, the sampling frequency for a signal with bandwidth B is also B.

There are B samples per second. Samples are complex. } 2B real AWGN channel.

Capacity $= 2B\cdot\frac{1}{2}\log_2(1+\gamma) = B\log_2(1+\gamma)$

capacity per second.

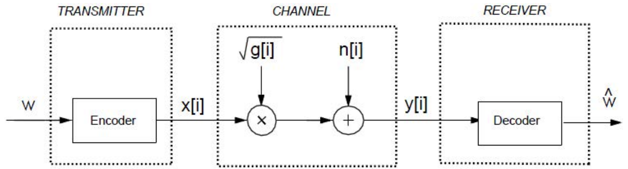
Capacity for complex AWGN channel, $Y = X + N$.

(Y, X, N are complex).

$C = \log(1+\gamma)$

Channel and System Model

Consider a discrete-time channel with stationary and ergodic time-varying gain $\sqrt{g[i]}$, which follows a given $p(g)$. In a block fading channel, $\sqrt{g[i]}$ is constant over a blocklength T and then changes to a new independent one. $\bar{P}$ is the average transmit power, then $\gamma[i] = \frac{\bar{P}g[i]}{N_0 B}$, $E[\gamma] = \frac{\bar{P}\bar{g}}{N_0 B}$.

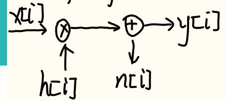


Flat-Fading Channel and System Model.

The channel gain $g[i]$ is the **channel side information (CSI)**.

$p(g)$ is the **channel distribution information (CDI)**.

Flat fading Channel.



令 $|h[i]|^2 = g[i] \rightarrow$ channel power gain

$y[i] = h[i]\, x[i] + n[i]$

Channel state info (CSI)

$\rightarrow \mathcal{CN}(0, N_0B)$

complex channel.

复高斯分布.

△ If CSI is available at the Rx.

等价于 $\rightarrow$ CSIR is available.

$y'[i] = y'[i]/h[i] = x[i] + \frac{n[i]}{h[i]}$

$\mathcal{CN}\left(0, \frac{N_0B}{g[i]}\right)$

$\rightarrow$ AWGN.

when CSI is available, flat fading channel changes into AWGN.

signal power $= E[|x[i]|^2] = P$

Noise power $= \frac{N_0B}{g[i]}$

$SNR = \frac{P\,g[i]}{N_0B}$

Capacity $= B\log_2\left(1 + \frac{g[i]P}{N_0B}\right) = C$.

△ with CSIR, but without CSIT (Scheme 1)

Tx fixes a data rate R.

$R = B\log(1+\gamma_{min})$

令第一种情况中 $\frac{g[i]P}{N_0B} = \gamma$

$\begin{cases} \gamma \geq \gamma_{min}. & C \geq R \rightarrow \text{error-free} \\ \gamma < \gamma_{min} & R > C \rightarrow \text{packet outage.} \end{cases}$

Outage Probability.

$P_{out} = P_r[R > C] = P_r[\gamma_{min} > \gamma]$

意义：发射机以固定速率R传输，包丢失的概率.

$\Rightarrow R(1-P_{out})$ goodput, average successful datarate.

Channel Distribution Information (CDI) Known

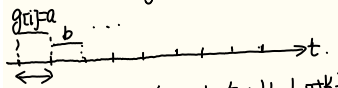
When $p(g)$, or equivalently, the distribution of SNR $p(\gamma)$ is known to both the TX and RX. The channel capacity is defined as

$$C = \max_{p(x)} I(X;Y) = \max_{p(x)} \sum_{x,y} p(x,y) \log\left(\frac{p(x,y)}{p(x)p(y)}\right)$$

Finding the capacity-achieving $p(x)$ and corresponding C under CDI known to both TX and RX, remains an open problem for almost all wireless channel distributions (*except for Rayleigh fading and finite-state Markov chains*).

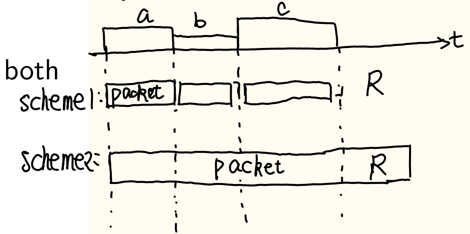
△ with CSIR, without CSIT (scheme 2)

block fading.



将时间段分为许多block,每个block时长为coherent time.

在每个block时间段中,power gain认为恒定.



Average capacity (Ergodic capacity)

$$\bar{C} = \int_0^{+\infty} B\log_2(1+\gamma)\, p(\gamma)\, d\gamma$$

$\begin{cases} R \le \bar{C} \Rightarrow \text{error-free} \\ R > \bar{C} \Rightarrow \text{outage} \end{cases}$

$\dfrac{R=\bar{C}}{\text{CDI}}$ (Channel distribution Info)

advantage: 只需要知道CDI,就可以使$R=\bar{C}$,不丢包.

disadvantage: transmission delay is large

△ With CSIT and CSIR.

$C_a = B\log\left(1+\frac{aP}{N_0B}\right)$

$R=C_a$, $R=C_b$, ……

$$C = \int_0^{+\infty} B\cdot\log\left(1+\frac{gP}{N_0B}\right)\cdot p(g)\cdot dg.$$

$$= \int_0^{\infty} B\cdot\log(1+\gamma)\, p(\gamma)\, d\gamma$$

advantage: ① Ergodic capacity with short delay.

Channel Side Information at Receiver

$g[i]$ is known at RX, and $p(g)$ is known to both TX and RX.

Shannon or ergodic capacity: the maximum data rate that can be sent over the channel with asymptotically small error probability.

Capacity with outage: the maximum rate that can be sent over a channel with some outage probability corresponding to the probability that the transmission cannot be decoded with negligible error probability.

Shannon capacity: $C = \int_0^\infty B \log_2(1+\gamma)p(\gamma)d\gamma$.

Since TX has no CSI, the transmission rate is constant, and TX cannot adapt its power relative to the CSI.

By Jensen's inequality,

$$C = E[B \log_2(1+\gamma)] \leq B \log_2(1+E[\gamma]) = B \log_2(1+\bar{\gamma})$$

Shannon capacity of a fading channel with RX CSI only is less than the Shannon capacity of an AWGN channel with the same average SNR.

EXAMPLE 2.

Consider a flat-fading channel with i.i.d. $\sqrt{g[i]}$ taking on 3 possible values: $p_1 = p(\sqrt{g[i]} = 0.05) = 0.1$, $p_2 = p(\sqrt{g[i]} = 0.5) = 0.5$, and $p_2 = p(\sqrt{g[i]} = 1) = 0.4$. $P_t = 10\text{mW}$, $N_0 = 10^{-9}\text{W/Hz}$, $B = 30\text{KHz}$. Find the Shannon capacity for RX CSI only, and the capacity of AWGN with the same average SNR.

Solution: Since $\gamma_1 = \frac{P_t g_1}{N_0 B} = 0.8333 = -0.79\text{dB}$, $\gamma_2 = 83.333 = 19.2\text{dB}$, $\gamma_3 = 333.33 = 25\text{dB}$, then $C_{CSI} = \sum_i B\log_2(1+\gamma_i)p(\gamma_i) = 199.26\text{Kbps}$.

For AWGN with the same $\bar{\gamma}$, $\bar{\gamma} = \sum_i \gamma_i p(\gamma_i) = 175.08 = 22.43\text{dB}$, then $C_{AWGN} = B\log_2(1+\bar{\gamma}) = 223.8\text{Kbps}$.

Capacity with Outage: allows bits sent over a given transmission burst with some error probability. TX fixes a minimum received SNR γ_{min} and sets a data rate $C = B\log_2(1+\gamma_{min})$. The probability of outage is thus $p_{out} = p(\gamma < \gamma_{min})$, and $C_o = (1 - p_{out})C$.

EXAMPLE 3.

Assume the same channel as in EX. 2, find the capacity versus outage, and the average rate correctly received for outage probabilities $p_{out} < 0.1$, $p_{out} = 0.1$ and $p_{out} = 0.6$.

Solution: For $p_{out} < 0.1$, $\gamma_{min} = \gamma_1$, $C = B\log_2(1+\gamma_{min}) = 26.23$Kbps, $C_o = C = 26.23$Kbps, since no outage will occur.

For $0.1 \leq p_{out} < 0.6$, we can decode incorrectly when the channel is in the weakest state only, $\gamma_{min} = \gamma_2$, $C = 191.94$Kbps, $C_o = 0.9C = 172.75$Kbps.

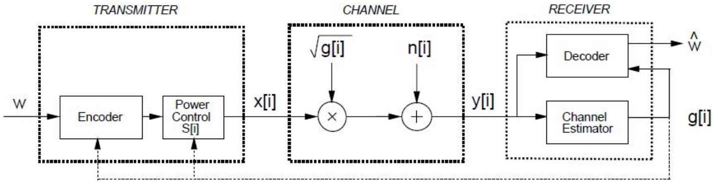
For $0.6 \leq p_{out} < 1$, we can decode incorrectly if the channel has received SNR γ_1 or γ_2, $\gamma_{min} = \gamma_3$, and $C = 251.55$Kbps, $C_o = 0.4C = 125.78$Kbps.

191.94Kbps needs 10% retransmission probability (complicated).

26.23Kbps data rate does not need retransmissions.

Channel Side Information at TX and RX

When both TX and RX have CSI, TX can adapt its transmission strategy relative to this CSI.



System Model with TX and RX CSI.

In this case there is no notion of capacity versus outage where TX sends bits that cannot be decoded, since TX knows the channel and thus will not send bits unless they can be decoded correctly.

We only have Shannon Capacity.

Theorem: For a stationary and ergodic process taking values on a finite set S of discrete memoryless channels, C_s is the capacity of a particular $s \in S$ with probability $p(s)$, then the capacity of time-varying channel is

$$C = \sum_{s \in S} C_s p(s)$$

Then, Shannon capacity of the fading channel with TX and RX CSI is thus

$$\checkmark\; C = \int_0^\infty C_\gamma p(\gamma) d\gamma = \int_0^\infty B \log_2(1+\gamma) p(\gamma) d\gamma$$

the same with the capacity of RX CSI only, showing that TX CSI does not increase capacity unless power is also adapted, i.e., allowing transmit power $P(\gamma)$ to vary with γ, subject to an average power constraint $\bar{P}$,

$$\int_0^\infty P(\gamma) p(\gamma) d\gamma \leq \bar{P}$$

We thus have

$$\checkmark\; C = \max_{P(\gamma): \int_0^\infty P(\gamma)p(\gamma)d\gamma = \bar{P}} \int_0^\infty B \log_2\left(1 + \frac{P(\gamma)\gamma}{\bar{P}}\right) p(\gamma) d\gamma$$

How to achieve this C?

△ CSIR + CSIT + Power adaptation (信道的功率)

$P = \bar{P} \rightarrow P(\gamma)$, st $E[P(\gamma)] = \bar{P}$

$\int_0^{+\infty} P(\gamma) p(\gamma) d\gamma = \bar{P}$, $\gamma \triangleq \frac{g \cdot \bar{P}}{N_0 B}$

$E_\gamma\left[B \log_2\left[1 + P(\gamma)\frac{\gamma}{\bar{P}}\right]\right]$

$= \int_0^{+\infty} B \log_2\left(1 + P(\gamma)\frac{\gamma}{\bar{P}}\right) p(\gamma) d\gamma$

$C = \max_{P(\gamma)} \int_0^{+\infty} B \log_2\left(1 + P(\gamma)\frac{\gamma}{\bar{P}}\right) p(\gamma) d\gamma$ ← object.

s.t. $\int_0^{+\infty} P(\gamma) p(\gamma) d\gamma = \bar{P}$ ← constraint

Convex Optimization 凸优化

Lagrangian

$J(\lambda) = \int_0^{+\infty} B \log_2\left(1 + P(\gamma)\frac{\gamma}{\bar{P}}\right) p(\gamma) d\gamma$

$- \lambda\left(\int_0^{+\infty} P(\gamma) p(\gamma) d(\gamma) - \bar{P}\right)$

$C = \min_{\lambda}\left(\max_{P(\gamma)} J(\lambda)\right)$

$\max_{P(\gamma)} \int_0^{+\infty} \left[B \log_2\left(1 + \frac{P(\gamma)\gamma}{\bar{P}}\right) - \lambda P(\gamma)\right] p(\gamma) d\gamma + \lambda \bar{P}$

$\rightarrow \max_{P(\gamma)} B \log_2\left(1 + \frac{P(\gamma)\gamma}{\bar{P}}\right) - \lambda P(\gamma) \xrightarrow{求导} \frac{P(\gamma)}{\bar{P}} = \frac{B}{\ln 2 \cdot \lambda \cdot \bar{P}} - \frac{1}{\gamma}$

let $\gamma_0 = \frac{\ln 2 \cdot \lambda \cdot \bar{P}}{B}$

$\rightarrow \frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}, & \gamma \geq \gamma_0 \\ 0, & \gamma < \gamma_0 \end{cases} = \max\left\{0, \frac{1}{\gamma_0 - \gamma}\right\}$

$\int_0^{+\infty} \frac{P(\gamma)}{\bar{P}} p(\gamma) d\gamma = 1$

$\rightarrow \int_0^{+\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right)^+ p(\gamma) d\gamma = 1$

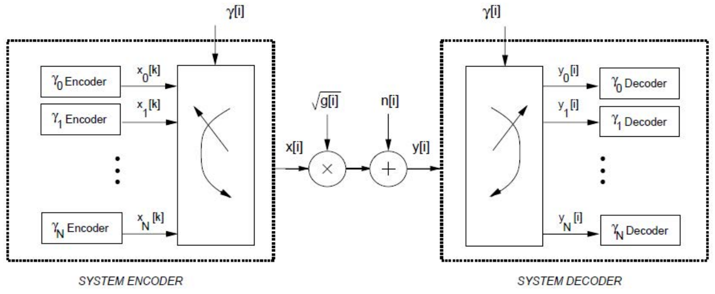
$\rightarrow \underbrace{\int_{\gamma_0}^{+\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p(\gamma) d\gamma}_{\phi(\gamma_0)} = 1$ → 对γ_0的函数.

$\phi(\gamma_0) = 1$

$\Rightarrow \gamma_0 = ? \Rightarrow \lambda = ?$

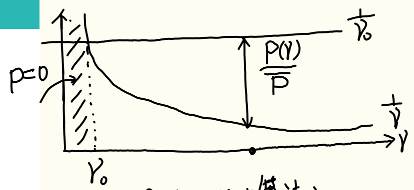
Optimal $\frac{P(\gamma)}{\bar{P}} = \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right)^+$, $\gamma \triangleq \frac{g \cdot \bar{P}}{N_0 B}$

The main idea behind the proof is a “time diversity” system with multiplexed input and demultiplexed output.



Multiplexed Coding and Decoding.

We first quantize channel gain to a finite set $\{\gamma_j : 1 \leq j \leq N\}$. For each γ_j, we design an encoder/decoder pair for an AWGN channel with SNR γ_j. This effectively reduces the time-varying channel to a set of time-invariant channels in parallel, where the j_{th} channel only operates when $\gamma[i] = \gamma_j$.



water filling(注水算法)

$$C=\int_0^{\infty} B\log_2\left(1+\frac{P(\gamma)}{\bar{P}}\gamma\right)\cdot p(\gamma)d\gamma$$

$$=\int_{\gamma_0}^{\infty} B\log_2\left[1+\left(\frac{1}{\gamma_0}-\frac{1}{\gamma}\right)\cdot\gamma\right]\cdot p(\gamma)d\gamma$$

$$=\int_{\gamma_0}^{\infty} B\log_2\left(\frac{\gamma}{\gamma_0}\right)p(\gamma)d\gamma$$

= maxmized ergodic capacity

→信道好的时候 传输功率大.

信道差的时候,传输功率小.

△CSIR+CSIT+ channel version(信道的稳定性)

Rx $SNR=\frac{P(\gamma)\cdot g}{N_0 B}=\frac{\gamma P(\gamma)}{\bar{P}}$ to σ. $\rightarrow P(\gamma)=\frac{\sigma\bar{P}}{\gamma}$

Capacity= $C=B\log(1+\sigma)$, 与平均发射功率相关

$\int_0^{+\infty}P(\gamma)p(\gamma)d\gamma=\bar{P} \Rightarrow \int_0^{+\infty}\frac{\sigma}{\gamma}p(\gamma)d\gamma=1 \rightarrow \sigma=\frac{1}{\int_0^{+\infty}\frac{1}{\gamma}p(\gamma)d\gamma}=\frac{1}{E[\frac{1}{\gamma}]}$

$C=B\log\left(1+\frac{1}{E[\frac{1}{\gamma}]}\right)$, $\gamma\triangleq\frac{g\bar{P}}{N_0B}$

→ zero-outage capacity.

disadvantage:

Rayleigh Fading, $\forall\bar{P}<\infty$, $E[\frac{1}{\gamma}]\to\infty$

$C=0$.

To find the optimal power allocation $P(\gamma)$, we form the Lagrangian

$$J(P(\gamma)) = \int_0^\infty B\log_2(1+\frac{P(\gamma)\gamma}{\bar{P}})p(\gamma)d\gamma - \lambda(\int_0^\infty P(\gamma)p(\gamma)d\gamma - \bar{P})$$

We differentiate the Lagrangian and set the derivative equal to zero

$$\frac{\partial J(P(\gamma))}{\partial P(\gamma)} = [\frac{B/\ln 2}{1+\gamma P(\gamma)/\bar{P}} - \lambda]p(\gamma) = 0$$

We have

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma}, \text{if } & \gamma \geq \gamma_0 \\ 0, \quad \text{if } & \gamma < \gamma_0 \end{cases}$$

for some "cutoff" value γ_0. If $\gamma[i]$ is below γ_0 then no data is transmitted over the i_{th} time interval, so the channel is only used if $\gamma_0 \leq \gamma[i] < \infty$. Then the optimal capacity can be obtained as

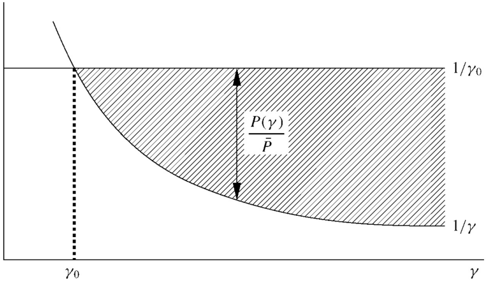
$$C = \int_{\gamma_0}^\infty B\log_2(\frac{\gamma}{\gamma_0})p(\gamma)d\gamma$$

Since $\int_0^\infty \frac{P(\gamma)}{\bar{P}} p(\gamma) d\gamma = 1$, showing γ_0 must satisfy

$$\int_{\gamma_0}^{\infty} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right) p(\gamma) d\gamma = 1$$ → 怎么确定 γ_0.

γ_0 cannot be solved in closed form for typical continuous PDFs $p(\gamma)$ and thus must be found numerically.

Since γ is time-varying, the maximizing power adaptation policy is a "water-filling" formula in time, the power is poured into the bowl to a constant level $\frac{1}{\gamma_0}$. The amount of power allocated for a given γ is $\frac{1}{\gamma_0} - \frac{1}{\gamma}$.



Optimal Power Allocation: Water-Filling.

EXAMPLE 4.

Assume the same channel as in EX. 3. Find the ergodic capacity assuming both TX and RX have instantaneous CSI.

Solution: Since $\sum_{\gamma_i \geq \gamma_0}(\frac{1}{\gamma_0} - \frac{1}{\gamma})p(\gamma_i) = 1$. We first assume that all channel states are used to obtain γ_0, i.e., assume $\gamma_0 \leq \gamma_1$, and see if the resulting γ_0 is below that of the weakest channel,

$$\sum_{i=1}^{3} \frac{p(\gamma_i)}{\gamma_0} - \sum_{i=1}^{3} \frac{p(\gamma_i)}{\gamma_i} = 1 \Longrightarrow \frac{1}{\gamma_0} = 1.13 \Longrightarrow \gamma_0 = 0.88 > \gamma_1$$

Then redo the calculation for $\gamma_1 < \gamma_0 \leq \gamma_2$, which yields

$$\sum_{i=2}^{3} \frac{p(\gamma_i)}{\gamma_0} - \sum_{i=2}^{3} \frac{p(\gamma_i)}{\gamma_i} = 1 \Longrightarrow \frac{1}{\gamma_0} = 1.0072 \Longrightarrow \gamma_0 = 0.99 < \gamma_2$$

The weakest channel with SNR γ_1 is not used. Then the capacity is

$$C = \sum_{i=2}^{3} B \log_2(\frac{\gamma_i}{\gamma_0})p(\gamma_i) = 200.82\text{Kbps}$$

Zero-Outage Capacity and Channel Inversion

Consider a suboptimal TX adaptation scheme where TX uses the CSI to maintain a constant received power, i.e., it inverts the channel fading. The channel then appears to the encoder and decoder as a time-invariant AWGN channel. This power adaptation, called channel inversion, is given by $\frac{P(\gamma)}{\bar{P}} = \frac{\sigma}{\gamma}$, where σ is the constant received SNR can be maintained with the transmit power constraint. Thus, $\int \frac{\sigma}{\gamma} p(\gamma) d\gamma = 1$, $\sigma = 1/E[1/\gamma]$. Fading channel capacity with channel inversion is

$$C = B\log_2(1+\sigma) = B\log_2[1+\frac{1}{E[1/\gamma]}]$$

This has the advantage of maintaining a fixed data rate over the channel regardless of channel conditions. For this reason the channel capacity is called zero-outage capacity, since the data rate is fixed under all channel conditions and there is no channel outage.

Zero-outage capacity can exhibit a large data rate reduction relative to Shannon capacity in extreme fading environments.

EXAMPLE 5.

Assume the same channel as in EX. 3. Find the the zero-outage capacity of this channel assuming both TX and RX have instantaneous CSI.

Solution: The zero-outage capacity is $B\log_2[1+\frac{1}{E[1/\gamma]}]$. Since

$$E[1/\gamma] = \frac{0.1}{0.8333} + \frac{0.5}{83.33} + \frac{0.4}{333.33} = 0.1272$$

Then $C = 30000\log_2(1+\frac{1}{0.1272}) = 94.43\text{Kbps}$.

Outage Capacity and Truncated Channel Inversion

By suspending transmission in outage states, we can maintain a higher constant rate in other states. Outage capacity is achieved with a truncated channel inversion policy that only compensates for fading above γ_0,

$$\frac{P(\gamma)}{\bar{P}} = \begin{cases} \frac{\sigma}{\gamma}, \text{if } \ \gamma \geq \gamma_0 \\ 0, \text{if } \ \gamma < \gamma_0 \end{cases}$$

Maximum outage capacity: $C = \max_{\gamma_0} B \log_2(1 + 1/E[\frac{1}{\gamma}])p(\gamma \geq \gamma_0)$

EXAMPLE 6.

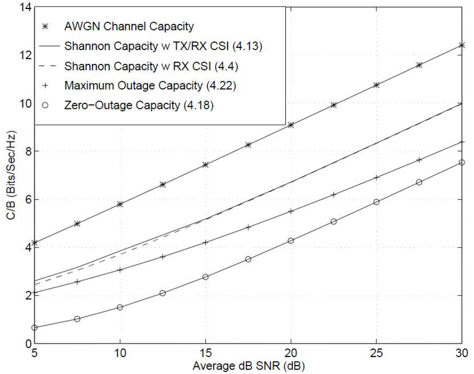
Assume the same channel as in EX. 3. Find the outage capacity and associated outage probabilities for $\gamma_0 = 0.84$ and $\gamma_0 = 83.4$.

Solution: For $\gamma_0 = 0.84$, we use the channel when the SNR is γ_2 or γ_3, then $\sigma = E[\frac{1}{\gamma}] = \sum_{i=2}^{3} \frac{p(\gamma_i)}{\gamma_i} = 0.0072$. The outage capacity can be obtained as $C = B \log_2(1 + 1/E[\frac{1}{\gamma}])p(\gamma \geq \gamma_0) = 192.457$Kbps.

For $\gamma_0 = 83.34$, we can only use the channel when SNR is γ_3, hence $\sigma = E[\frac{1}{\gamma}] = \frac{p(\gamma_3)}{\gamma_3} = 0.0012$, and $C = B \log_2(1 + \sigma)p(\gamma_3) = 116.45$Kbps.

Capacity Comparisons

$\mu_{dB} = 10\log_{10}\mu - \sigma_{dB}^2 \frac{\ln 10}{20}$, $\sigma_{dB} = 8$.



Capacity in Log-Normal Shadowing.

Shannon Capacity with TX/RX CSI and with RX CSI only are almost the same for high SNR, indicating that TX adaptation yields a negligible capacity gain relative to using only RX CSI.

发射功率：$\int_0^{+\infty} P(\gamma) f(\gamma) d\gamma = \bar{P}$

$\int_0^{+\infty} \frac{\sigma}{\gamma} f(\gamma) d\gamma = 1$

$\sigma = \frac{1}{\underbrace{\int_0^{\infty} \frac{1}{\gamma} f(\gamma) d\gamma}_{\text{期望}}} = \frac{1}{E(\frac{1}{\gamma})}$

$C = B\log_2(1 + \frac{1}{E[\frac{1}{\gamma}]})$ zero-outage capacity.

under Rayleigh Channel $\forall \bar{P} < \infty$ $E[\frac{1}{\gamma}] = \infty \Rightarrow C = 0$.

[大量功率被用于拯救γ很低的信道].

⇓ 增加一个门限.

- CSIR + CSIT Truncated Channel Inv.

$P(\gamma) = \begin{cases} 0, & \gamma \le \gamma_0 \\ \frac{\sigma \bar{P}}{\gamma}, & \gamma > \gamma_0 \end{cases} \Rightarrow B\log_2(1+\sigma)$.

$\Rightarrow$ Total Capacity [outage capacity]: $C = B\log_2(1+\sigma) P_r[\gamma \ge \gamma_0]$

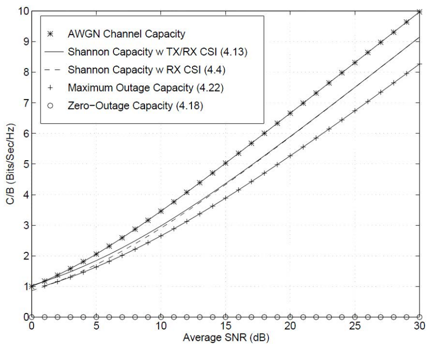
Insight: σ应该与$\bar{P}$, γ_0有关 ⇒ 约束 $E[P(\gamma)] = \bar{P}$

$\Rightarrow \int_{\gamma_0}^{+\infty} P(\gamma) f(\gamma) d\gamma = \bar{P} \Rightarrow \int_{\gamma_0}^{+\infty} \frac{\sigma \bar{P}}{\gamma} f(\gamma) d\gamma = \bar{P} \Rightarrow \sigma = \frac{1}{\int_{\gamma_0}^{+\infty} \frac{1}{\gamma_0} f(\gamma) d\gamma} = \frac{1}{E_{\gamma_0}[\frac{1}{\gamma}]}$

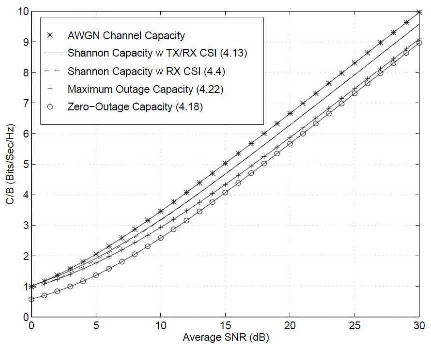
只是一个记号.

maximized outage capacity

$\max_{\gamma_0} C(\gamma_0) = B\log_2(1 + \frac{1}{E_{\gamma_0}[\frac{1}{\gamma_0}]}) P[\gamma \ge \gamma_0]$.



Capacity in Rayleigh Fading.

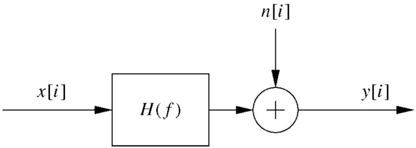


Capacity in Nakagami Fading ($m = 2$).

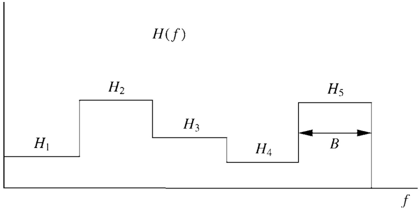
1. At low SNRs, AWGN and fading channel with TX/RX CSI have almost the same capacity (larger than the corresponding AWGN capacity).
2. Capacity difference ↓ as fading severity ↓, and their respective capacities approach that of the AWGN channel.
3. A tradeoff between capacity and complexity: CSI feedback and RX complexity.

Time-Invariant Frequency-Selective (FS) Fading Channels

Assume a total transmit power constraint P, $H(f)$ is known at TX/RX. $H(f)$ is block-fading, frequency is divided into subchannels of bandwidth B, and $H(f) = H_j$ is constant over each block, thus consisting of AWGN in parallel with SNR $\gamma_j = \frac{|H_j|^2 P_j}{N_0 B}$ on the j_{th} channel, and $\sum_j P_j \leq P$.



Time-Invariant FS Fading Channel.



Block FS Fading.

The capacity of FS fading channel is

$$C = \max_{P_j : \sum_j P_j \leq P} \sum B \log_2\left(1 + \frac{|H_j|^2 P_j}{N_0 B}\right)$$

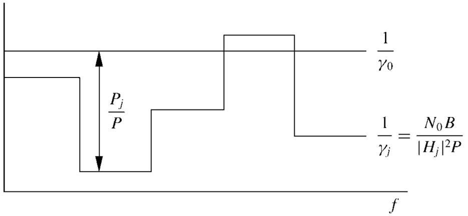
This is similar to the capacity and optimal power allocation for flat-fading, with power and rate changing over frequency in a deterministic way, i.e., water-filling in the frequency domain.

$$\checkmark \quad \frac{P_j}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma_j}, \text{if } & \gamma_j \geq \gamma_0 \\ 0, \quad \text{if } & \gamma_j < \gamma_0 \end{cases}$$

→ 怎么确定的

for some cutoff value γ_0, which satisfies $\sum_j \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_j}\right) = 1$. ✓ Then

$$\checkmark \quad C = \sum_{j:\gamma_j \geq \gamma_0} B \log_2\left(\frac{\gamma_j}{\gamma_0}\right)$$



Water-Filling in Block Frequency-Selective Fading.

When $H(f)$ is continuous, the capacity under power constraint P is

$$C = \max_{P(f):\int P(f)df \leq P} \int \log_2\left(1 + \frac{|H(f)|^2 P(f)}{N_0}\right)df$$

The power allocation over frequency, $P(f)$ is also water-filling

$$\frac{P(f)}{P} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma(f)}, & \text{if } \gamma(f) \geq \gamma_0 \\ 0, & \text{if } \gamma(f) < \gamma_0 \end{cases}$$

which results in channel capacity $C = \int_{f:\gamma(f)\geq\gamma_0} \log_2\left(\frac{\gamma(f)}{\gamma_0}\right)df$

EXAMPLE 7.

A time-invariant FS block fading channel consists of 3 subchannels of $B = 1\text{MHz}$, $H_1 = 1$, $H_2 = 2$, $H_3 = 3$, $P = 10\text{mW}$, $N_0 = 10^{-9}\text{W/Hz}$. Find the Shannon capacity and the optimal power allocation.

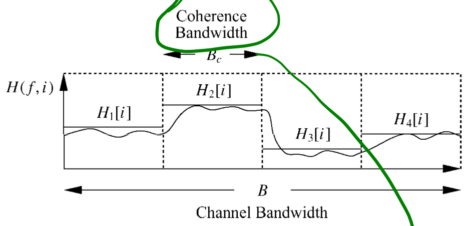
Solution: $\gamma_1 = 10$, $\gamma_2 = 40$, $\gamma_3 = 90$. First assume all subchannels are allocated power, $\frac{3}{\gamma_0} = 1 + \sum_j \frac{1}{\gamma_j}$, then $\gamma_0 = 2.64 < \gamma_1$, consistent!

Then, $C = \sum_{j=1}^{3} B \log_2\left(\frac{\gamma_j}{\gamma_0}\right) = 10.93\text{Mbps}$.

Time-Varying Channels

$H(f) = H(f, i)$, the channel varies over both frequency and time. It is difficult to determine the capacity of time-varying FS fading channels, even when $H(f, i)$ is known perfectly at TX/RX, due to the random effects of self-interference (ISI).

The capacity of time-varying FS fading is in general unknown, however upper and lower bounds and limiting formulas exist.



Channel Division in Frequency-Selective Fading.

Each subchannel is independent, time-varying flat-fading with $H(f, i) = H_j[i]$ on the j_{th} subchannel, $C = \sum_j \int_{\gamma_0}^{\infty} B_c \log_2(\frac{\gamma_j}{\gamma_0}) p(\gamma_j) d\gamma_j$.

$\vec{S_i} = (S_{i1}, S_{i2}, \cdots S_{iN})$

length of $\|\vec{S_i}\| = \sqrt{\sum_{j=1}^{N} S_{ij}^2}$

distance between $\vec{S_i}$ and $\vec{S_R}$

$$\|\vec{S_i} - \vec{S_R}\| = \sqrt{\sum_{j=1}^{N} (S_{ij} - S_{kj})^2}$$

$$= \sqrt{\int_0^T [S_i(t) - S_k(t)]^2 dt}$$

inner product

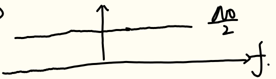
$$\langle \vec{S_i}, \vec{S_k} \rangle = \sum_{j=1}^{N} S_{ij} S_{kj} = \langle S_i(t), S_k(t) \rangle$$

Orthogonal. $\langle \vec{S_i}, \vec{S_k} \rangle = 0$, $\vec{S_i} \perp \vec{S_k}$

$\triangle$ AWGN. $r(t) = S(t) + \underline{n(t)} \leftarrow$ Gaussian Process.

$E[n(t)] = 0$, $E[n(t)n(\tau)] = \frac{N_0}{2} \delta(t-\tau)$

PSD



$r(t) = h s(t) + n(t)$

$\Downarrow$

$\frac{r(t)}{h} = s(t) + \frac{n(t)}{h}$.

Main Points

- Capacity of AWGN channel
- Fundamental capacity of flat-fading channels depends on what is known at TX and RX.
  - Capacity with TX/RX CSI: variable-rate variable-power transmission (water-filling) optimal
  - Almost same capacity as with RX CSI only
  - Channel inversion practical, but should truncate
- Capacity of wideband channel obtained by breaking up channel into subbands
  - Similar to multicarrier modulation

Digital Modulation.

Info bits $\rightarrow$ analog signals.

messages | symbols

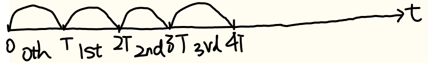
$\{m_1, m_2, \cdots M_M\}$

$k = \log_2 M$

$\{b_1, b_2, \cdots b_k\}$

$m_i \rightarrow \boxed{Tx} \rightarrow$

$m_i \rightarrow S_i(t), \forall i = 1 \cdots M$.



$S_1(t)$ $S_2(t)$ $S_3(t)$ $S_1(t)$

$S(t) = S_1(t) + S_2(t-T) + S_3(t-2T) + S_1(t-3T)$

Suppose message $m(k)$ is transmitted in the k-th period:

$$S(t) = \sum_{k=0}^{+\infty} S_{m(k)}(t - kT)$$

Energy of $S_i(t)$: $E_{S_i} = \int_0^T S_i^2(t) dt$

$\triangle$ Geometric Representation of Signal.

Orthonormal Basis functions

$\Phi = \{\phi_1(t); \phi_2(t), \cdots, \phi_m(t)\}$, $t \in [0, T)$

$\langle f_1(t), f_2(t) \rangle = \int_0^T f_1(t) f_2(t) dt$

$$\begin{cases} \langle \phi_i(t), \phi_i(t) \rangle = \int_0^T \phi_i^2(t) dt = 1 \\ \langle \phi_i(t), \phi_k(t) \rangle = \int_0^T \phi_i(t) \phi_k(t) dt = 0 \end{cases}$$

$i \neq k$ $\phi_i(t) \perp \phi_k(t)$

$\uparrow$

Eg: $N=2$ $\begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \\ \phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \end{cases}$

$\forall i$ $S_i(t) = \sum_{j=1}^{N} S_{ij} \phi_j(t)$ $0 \le t < T$.

$S_{ij} = \langle S_i(t), \phi_j(t) \rangle$

$= \int_0^T \sum_{k=1}^{N} S_{ik} \phi_k(t) \cdot \phi_j(t) dt$

$= S_{ij} \int_0^T \phi_j(t) \cdot \phi_j(t) dt$

$= S_{ij}$

$S_i(t) \Rightarrow (S_{i1}, S_{i2}, \cdots S_{im}) \triangleq \vec{S_i}$

$\forall i = 1, \cdots, M$.