Lecture 11

Multicarrier Modulation

ISI Countermeasures

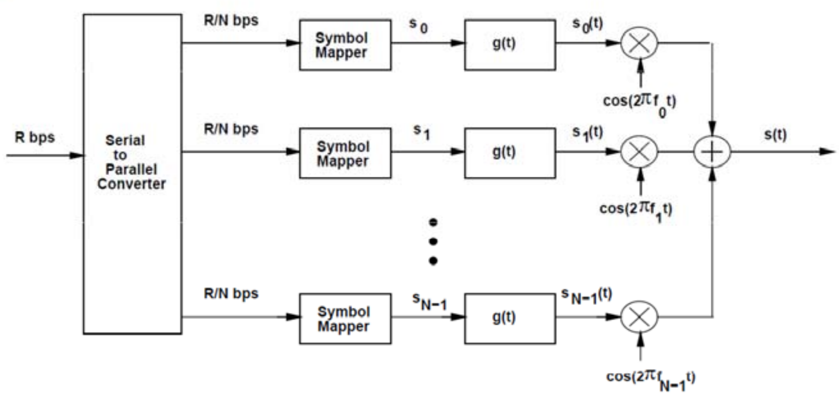
- Frequency-selective fading channel with ISI: $y(n) = \sum_{l=0}^{L-1} h_l x(n-l) + z(n)$
- Equalization
  - Signal processing at receiver to eliminate ISI (similar to MIMO detection)
  - Can be very complex at high data rates, and performs poorly in fast-changing channels
  - Not that common in state-of-the-art wireless systems
- Spread spectrum
  - Superimpose a fast (wideband) spreading sequence on top of data sequence, allowing resolution for multipath combining (Rake Receiver)
- Multicarrier Modulation
  - Break data stream into lower-rate substreams modulated onto narrowband flat-fading subchannels, each with less severe ISI

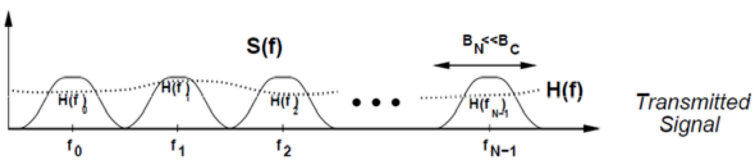
Outline

- Multicarrier Modulation Basics
- Mitigation of Subchannel Fading
- Digital Implementation of Multicarrier Modulation
- OFDM
  - OFDM transmitter & receiver
  - Matrix representation
  - MIMO decomposition
- Vector Coding
- Required reading:
  - Textbook, Chapter 12.1–12.4, 12.6

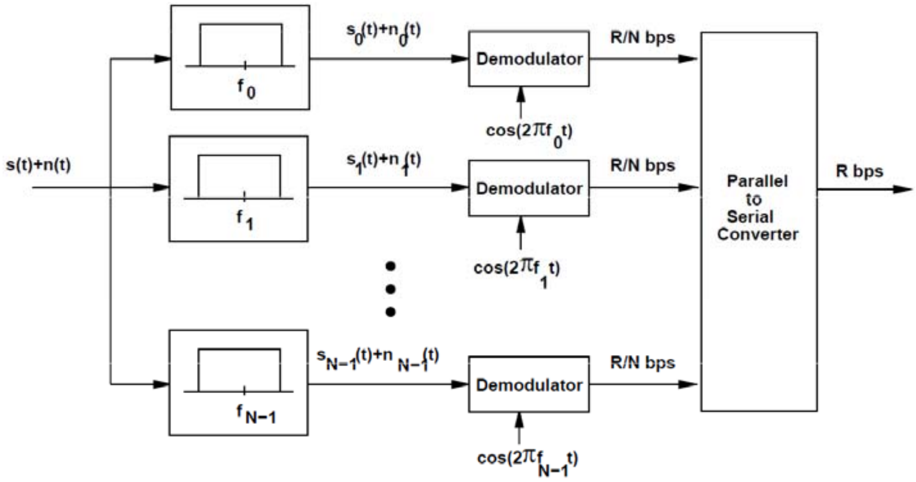
Multicarrier Modulation

- Non-overlapping Subchannels
  - Transmitter

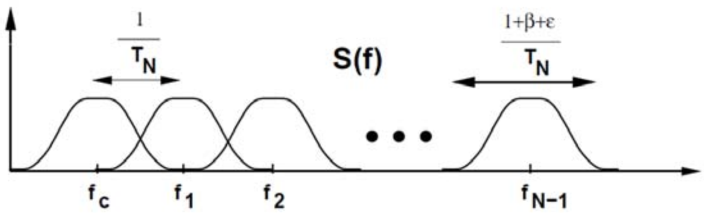




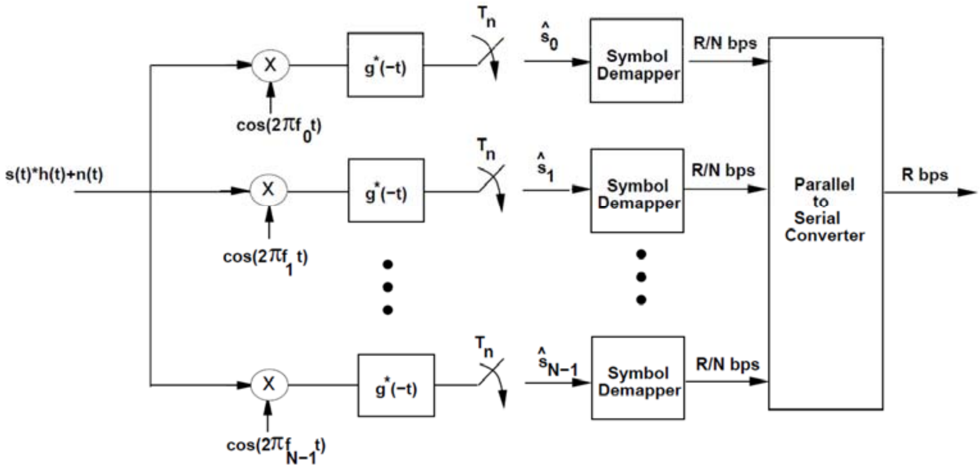
– Receiver



- Overlapping Subchannels
  - Transmitter (similar to that of non-overlapping case)



– Receiver



Mitigation of Subchannel Fading

- Frequency equalization (SNR per subchannel unchanged)
- Precoding (similar to channel inversion power control)
- Coding with interleaving over time and frequency
- Adaptive power and rate loading over time and frequency (similar to adaptive modulation)

Digital Implementation of Multicarrier Modulation

- Orthogonal Frequency Division Multiplexing (OFDM)
  - Adopt cyclic prefix to mitigate ISI
  - Channel decomposition based on eigenvalue decomposition
  - Channel not needed to be known at Tx
  - Power loss due to cyclic prefix
- Vector Coding
  - Adopt guard interval to eliminate ISI
  - Channel decomposition based on singular value decomposition
  - Channel needed to be known at Tx
  - No power loss in guard interval

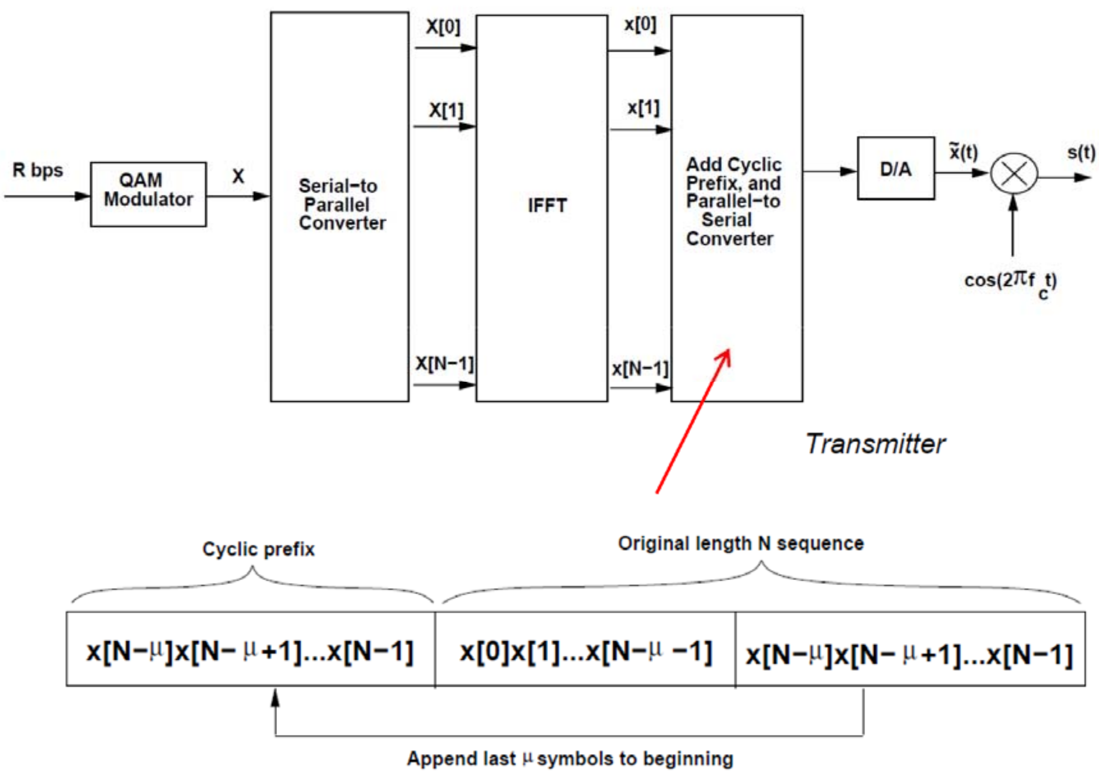
Block-Based Transmission

- Signals are transmitted in consecutive N-symbol blocks over time:

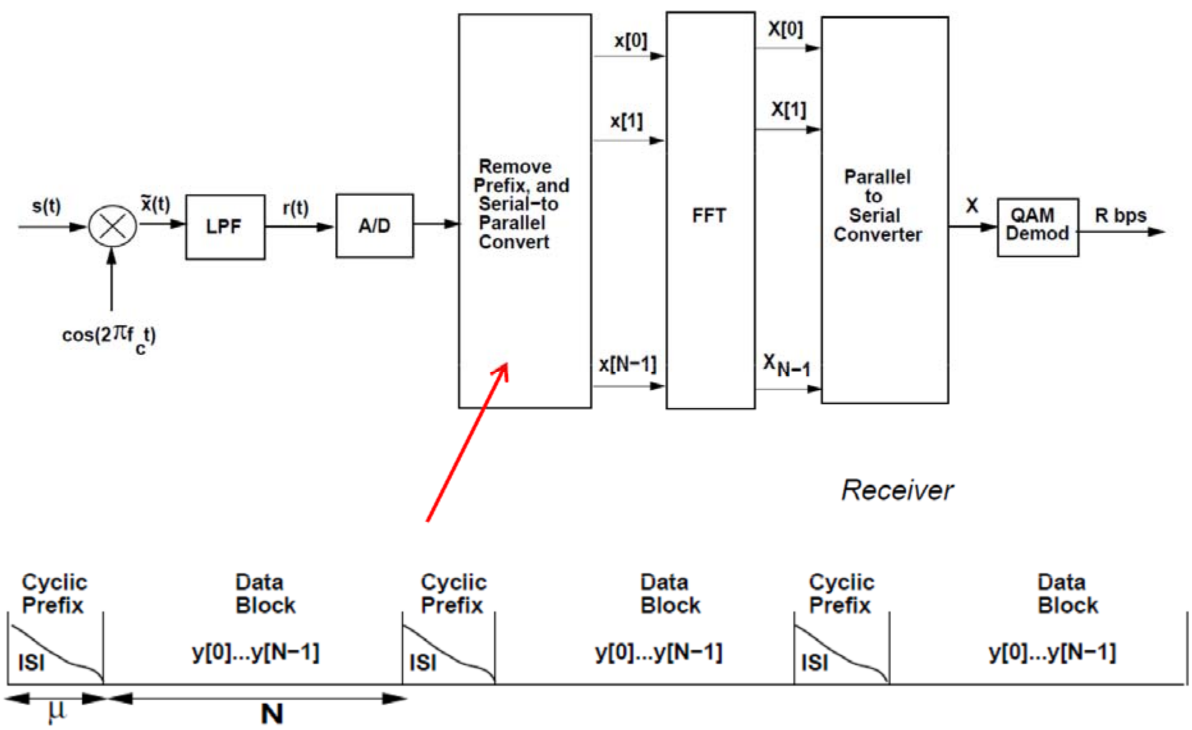
$$y(n) = \sum_{l=0}^{L-1} h_l x(n-l) + z(n), \;\; n = 0, \ldots, N-1 \tag{1}$$

  - h_l denotes the complex channel gain for the lth delayed path, $l = 0, \ldots, L-1$, where L is the number of resolvable paths
  - $\mathbb{E}[|x(n)|^2] \leq P$
  - $z(n) \sim \mathcal{CN}(0, \sigma_z^2)$
  - Assume time-invariant multipath channel (no fading)

OFDM Transmitter



OFDM Receiver



Matrix Representation

- Transmitted signal block of length $N+\mu$ with Cyclic Prefix (CP):

$$x[N-\mu], \ldots, x[N-1], x[0], x[1], \ldots, x[N-1] \qquad (2)$$

- CP length satisfies $\mu \geq L-1$ to eliminate the inter-block interference
- Let $\boldsymbol{x} = [x[0], x[1], \ldots, x[N-1]]^T$ represent the transmitted signal vector without CP
- $\boldsymbol{x}$ is generated by OFDM modulation as

$$\boldsymbol{x} = \boldsymbol{W}^H \boldsymbol{X} \qquad (3)$$

- $\boldsymbol{X} = [X[0], X[1], \ldots, X[N-1]]^T$ is the information signal vector

- $\boldsymbol{W} \in \mathbb{C}^{N\times N}$ is the Discrete Fourier Transform (DFT) matrix

$$\boldsymbol{W} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{\frac{-j2\pi}{N}} & \cdots & e^{\frac{-j2\pi(N-1)}{N}} \\ \vdots & & & \\ 1 & e^{\frac{-j2\pi(N-1)}{N}} & \cdots & e^{\frac{-j2\pi(N-1)(N-1)}{N}} \end{bmatrix} \tag{4}$$

where $[\boldsymbol{W}]_{k,m} = e^{\frac{-j2\pi km}{N}}, m = 0, \ldots, N-1, k = 0, \ldots, N-1$

- Check power constraint: $\mathbb{E}[\|\boldsymbol{x}\|^2] = \mathbb{E}[\|\boldsymbol{X}\|^2] \leq NP$, since $\boldsymbol{W}\boldsymbol{W}^H = \boldsymbol{W}^H\boldsymbol{W} = \boldsymbol{I}_N$
- In practice, IDFT/DFT is efficiently implemented as IFFT/FFT
- Received signal block of length $N + \mu$ is

$$y[-\mu], \ldots, y[-1], y[0], y[1], \ldots, y[N-1] \tag{5}$$

- After CP removal, the resulted vector is $\boldsymbol{y} = [y[0], y[1], \ldots, y[N-1]]^T$
- The equivalent MIMO channel is

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} h_0 & 0 & \cdots & 0 & h_{L-1} & \cdots & h_1 \\ h_1 & h_0 & & 0 & 0 & & h_2 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ h_{L-1} & h_{L-2} & & 0 & 0 & & 0 \\ 0 & h_{L-1} & & 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & h_{L-1} & h_{L-2} & \cdots & h_0 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} + \boldsymbol{z} \tag{6}$$

where $\boldsymbol{z} = [z[0], \ldots, z[N-1]]^T$

- Alternatively, the MIMO channel is expressed as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z} \tag{7}$$

Eigenvalue Decomposition of Circulant Matrix

- For circulant matrix $\boldsymbol{H}$, the following EVD exists:

$$\boldsymbol{H} = \boldsymbol{W}^{-1}\boldsymbol{\Delta}\boldsymbol{W} \tag{8}$$

where $\boldsymbol{\Delta} \in \mathbb{C}^{N\times N}$ is a diagonal matrix, with diagonal elements given by

$$H[k] = \sum_{l=0}^{L-1} h_l e^{\frac{-j2\pi kl}{N}}, k = 0, \ldots, N-1 \tag{9}$$

  - Since $\boldsymbol{W}\boldsymbol{W}^H = \boldsymbol{W}^H\boldsymbol{W} = \boldsymbol{I}_N$, $\boldsymbol{W}^{-1} = \boldsymbol{W}^H$
  - $[H[0], \ldots, H[N-1]]$ is the N-point DFT of $[h_0, \ldots, h_{L-1}, 0, \ldots, 0]$ (adding $N-L$ zeros), without the normalization by $1/\sqrt{N}$

- Notice the eigenvalues $H[k]$'s are in general complex numbers, and thus not equal to the singular values of $\boldsymbol{H}$, which are always non-negative real numbers

- Let the SVD of $\boldsymbol{H}$ be by $\boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^H$, with singular values $\lambda_0, \ldots, \lambda_{N-1}$
- We thus have

$$\boldsymbol{H}\boldsymbol{H}^H = \boldsymbol{U}\boldsymbol{\Lambda}^2\boldsymbol{U}^H \tag{10}$$

- Since $\boldsymbol{H} = \boldsymbol{W}^H\boldsymbol{\Delta}\boldsymbol{W}$, we also have

$$\boldsymbol{H}\boldsymbol{H}^H = \boldsymbol{W}^H\boldsymbol{\Delta}\boldsymbol{\Delta}^H\boldsymbol{W} \tag{11}$$

- Since the eigenvalues of $\boldsymbol{H}\boldsymbol{H}^H$ must be unique (but eigenvectors may not be unique), we obtain that

$$\boldsymbol{\Lambda}^2 = \boldsymbol{\Delta}\boldsymbol{\Delta}^H \tag{12}$$

- Thus we have the following relationship between singular values and eigenvalues of circulant matrix $\boldsymbol{H}$:

$$\lambda_k = |H[k]|, \quad k = 0, \ldots, N-1 \tag{13}$$

MIMO Decomposition

- From (3) and (7), we have

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z} = \boldsymbol{H}\boldsymbol{W}^H\boldsymbol{X} + \boldsymbol{z} = \boldsymbol{W}^H\boldsymbol{\Delta}\boldsymbol{W}\boldsymbol{W}^H\boldsymbol{X} + \boldsymbol{z} = \boldsymbol{W}^H\boldsymbol{\Delta}\boldsymbol{X} + \boldsymbol{z} \tag{14}$$

- OFDM demodulation by applying DFT matrix to $\boldsymbol{y}$, yielding

$$\boldsymbol{Y} = \boldsymbol{W}\boldsymbol{y} = \boldsymbol{W}\boldsymbol{W}^H\boldsymbol{\Delta}\boldsymbol{X} + \boldsymbol{W}\boldsymbol{z} = \boldsymbol{\Delta}\boldsymbol{X} + \boldsymbol{Z} \tag{15}$$

where $\boldsymbol{Z} = \boldsymbol{W}\boldsymbol{z} \sim \mathcal{CN}(0, \sigma_z^2 \boldsymbol{I}_N)$

- From (15), we see that the equivalent MIMO channel is decomposed into N parallel SISO channels given by

$$Y[k] = H[k]X[k] + Z[k], \; k = 0, \ldots, N-1 \tag{16}$$

where $\boldsymbol{Y} = [Y[0], \ldots, Y[N-1]]^T$, and $\boldsymbol{Z} = [Z[0], \ldots, Z[N-1]]^T$

- Let $\mathbb{E}[|X[k]|^2] = p_k, k = 0, \ldots, N-1$, with $\frac{1}{N}\sum_{k=0}^{N-1} p_k \leq P$
- The receiver SNR for the kth subchannel/subcarrier is

$$\gamma_k = \frac{|H[k]|^2 p_k}{\sigma_z^2}, \; k = 0, \ldots, N-1 \tag{17}$$

- The maximum achievable rate (in bps/Hz) for the OFDM system is

$$R = \frac{N}{\mu + N}\frac{1}{N}\sum_{k=0}^{N-1} \log_2(1 + \gamma_k) \tag{18}$$

where the factor $\frac{N}{\mu+N}$ accounts for the rate loss due to the CP insertion

- In the case of known CSIT, frequency-domain WF is optimal:

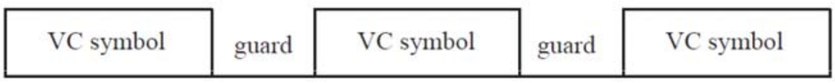
$$p_k = \left(\nu - \frac{\sigma_z^2}{|H[k]|^2}\right)^+, \; k = 0, \ldots, N-1 \tag{19}$$

where ν is the water-level with which $\sum_{k=0}^{N-1} p_k = NP$

Case Study: IEEE 802.11a Wireless LAN

- Total bandwidth: 300MHz, divided into 15 20-MHz OFDM channels
- $N = 64$ subcarriers (SCs), in which only 48 SCs are used for data transmission
- OFDM symbol period without CP: $64/(20 \times 10^6)$=3.2us (why?)
- CP length: $\mu = 16$
- OFDM symbol period with CP: 3.2us$\times$(1+16/64)=4us
- QAM constellation size for each SC: $M \in \{2, 4, 16, 64\}$
- Code rate: $r \in \{1/2, 2/3, 3/4\}$
- Maximum throughput for each 20-MHz channel ($M = 64$, $r = 3/4$): $48\times(3/4)\times6/(4 \cdot 10^{-6})$=54Mbps

Vector Coding



- OFDM has power loss due to CP insertion
- Vector coding (VC) replaces CP by Guard Interval (GI), thus avoiding power loss
- Consider VC based block transmission with N data symbols per block and GI length equivalent to μ data symbols (similar to OFDM)

- The equivalent MIMO channel is

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N+\mu-1] \end{bmatrix} = \begin{bmatrix} h_0 & 0 & \cdots \\ h_1 & h_0 & \cdots \\ \vdots & \vdots & \cdots \\ h_{L-1} & h_{L-2} & \cdots \\ 0 & h_{L-1} & \cdots \\ \vdots & \vdots & \cdots \\ 0 & 0 & \cdots \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} + \begin{bmatrix} z[0] \\ z[1] \\ \vdots \\ z[N+\mu-1] \end{bmatrix} \quad (23)$$

- Alternatively, the MIMO channel is expressed as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{z} \quad (24)$$

where $\boldsymbol{H} \in \mathbb{C}^{(N+\mu)\times N}$

- Notice $\boldsymbol{H}$ is not circulant matrix for VC

- Let the *truncated* SVD of $\boldsymbol{H}$ be denoted by

$$\boldsymbol{H} = \tilde{\boldsymbol{U}}\tilde{\boldsymbol{\Lambda}}\tilde{\boldsymbol{V}}^H \quad (25)$$

where $\tilde{\boldsymbol{\Lambda}}$ is an $N \times N$ diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_N$

- Apply eigenmode transmission, with $\boldsymbol{x} = \tilde{\boldsymbol{V}}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X}$, $\boldsymbol{Y} = \tilde{\boldsymbol{U}}^H\boldsymbol{y}$
- Decomposed MIMO channel: $\boldsymbol{Y} = \boldsymbol{\Lambda}\boldsymbol{\Sigma}^{\frac{1}{2}}\boldsymbol{X} + \boldsymbol{Z}$, with $\boldsymbol{Z} = \tilde{\boldsymbol{U}}^H\boldsymbol{z}$
- VC achievable rate: $\frac{1}{N+\mu}\sum_{i=1}^{N}\log_2(1+\lambda_i^2 p_i/\sigma_z^2)$
- Optimize power allocation $\boldsymbol{\Sigma}$ by WF under $\sum_{i=1}^{N} p_i \le (N+\mu)P$ (why?)
- Notice for VC, $\boldsymbol{H}$ needs to be known at Tx to obtain precoding matrix $\tilde{\boldsymbol{V}}$
- In contrast, for OFDM, the precoding (IDFT) matrix $\boldsymbol{W}^H$ is independent of $\boldsymbol{H}$, thus $\boldsymbol{H}$ needs not to be known at Tx: (DFT/IDFT matrix is universal eigenvectors for circulant matrix)

Summary

- Multicarrier vs. Single-Carrier Modulation (an ongoing debate)
- Multicarrier digital implementation
  – OFDM vs. Vector Coding (OFDM wins for wireless)
- With cyclic prefix, the precoding matrix becomes an IDFT matrix, which is independent of the channel matrix. (advantage of OFDM over Vector Coding)
- Challenges of OFDM
  – PAPR, frequency and timing offset
- SVD/EVD-based channel decomposition is a universal approach for optimizing MIMO transmission (over time and/or space)
- Water-Filling is a universal approach to optimizing power allocation over parallel AWGN channels (in time, frequency, and/or space)

The End

- **Thank you !**
- **Good luck for the final exam !!**
- **Have a great winter holiday !!!**
- **Merry Christmas and Happy 2020 !!!!**