Homework

TA

I. Question 10-1

(a) From $(AB)^H = B^H A^H$, we have:

$$(AA^H)^H = AA^H$$

For eigendecomposition in complex form, we have: $X = V\Sigma V^H$. Assume the decomposition of AA^H is $V\Sigma V^H$, actually Σ must be a matrix with real number, that is, the eigenvalue of AA^H is real. Besides, different eigenvalue will correspond to orthogonal eigenvector.

$$\lambda v_2^H v_1 = v_2^H AA^H v_1 = v_1^H AA^H v_2 = \mu v_1^H v_2 = \mu v_2^H v_1.$$

(b) From definition of positive semidefinite: If $x^H Ax \geq 0, \forall x$,we say matrix A is positive semidefinite. For AA^H, we have $x^H AA^H x = (A^H x)^H (A^H x) = \|A^H x\|_2^2 \geq 0$.

(c) Similar to positive semidefinite, definition of positive definite is: If $x^H Ax > 0, \forall x \neq 0$,we say matrix A is positive definite. Therefore, $x^H (I_M + AA^H)x = \|x\|_2^2 + \|A^H x\|_2^2 > 0$.

(d) Do SVD decomposition to matrix A, we have $A = U\Lambda V$. Therefore, the matrix $I_N + A^H A = I_N + V^H \Lambda^H U^H U \Lambda V = I_N + V^H \Lambda^H \Lambda V$. Similarly, we have $I_M + AA^H = I_M + U\Lambda\Lambda^H U^H$. Apparently:

$$\det[I_N + V^H \Lambda^H \Lambda V] = \det[V^H] \det[I_N + \Lambda^H \Lambda] \det[V] = \det[I_N + \Lambda^H \Lambda]$$

By the property of SVD decomposition, we know singular matrix only contains value in *diagonal*. That is the nonzero component of matrix $\Lambda^H \Lambda$ is the same as $\Lambda\Lambda^H$. Therefore, $\det[I_N + \Lambda^H \Lambda] = \det[I_M + \Lambda\Lambda^H] \rightarrow \det[I_N + A^H A] = \det[I_M + AA^H]$.

II. Question 10-2

$$HH^H - \lambda I = \begin{bmatrix} 1.05 - \lambda & 0.63 & 0.79 \\ 0.63 & 1.11 - \lambda & 1.16 \\ 0.79 & 1.16 & 1.27 - \lambda \end{bmatrix}$$

Let $\det[HH^H - \lambda I] = 0$, we have:

$$\begin{aligned} &(1.05 - \lambda)\left((1.11 - \lambda)(1.27 - \lambda) - 1.16^2\right) \\ &\qquad - 0.63\,(0.63\,(1.27 - \lambda) - 1.16 \cdot 0.79) + 0.79\,(0.63 \cdot 1.16 - 0.79\,(1.11 - \lambda)) \\ &\qquad = -x^3 + (3.43x^2) - (1.5421x) + 0.0252 \end{aligned}$$

Solve this equation results in eigenvalue: $\lambda_1 = 2.9015, \lambda_2 = 0.5115, \lambda_3 = 0.0169$

With these eigenvalues, we can compute the eigenvector of HH^T as:

$$u_1 = \begin{bmatrix} 0.4783 \\ 0.5896 \\ 0.6508 \end{bmatrix}, u_2 = \begin{bmatrix} 0.8685 \\ -0.4272 \\ -0.2513 \end{bmatrix}, u_3 = \begin{bmatrix} 0.1298 \\ 0.6855 \\ -0.7164 \end{bmatrix}$$

With singular value as $\sigma_1 = 1.7034, \sigma_2 = 0.7152, \sigma_3 = 0.13$

The computation of vector v_i follows $\sigma^{-1}H^T u_i$:

$$v_1 = \begin{bmatrix} 0.3458 \\ 0.5708 \\ 0.7116 \\ 0.2198 \end{bmatrix}, v_2 = \begin{bmatrix} 0.6849 \\ 0.2192 \\ -0.6109 \\ 0.3312 \end{bmatrix}, v_3 = \begin{bmatrix} -0.4269 \\ -0.0709 \\ -0.0145 \\ 0.9030 \end{bmatrix}$$

The SVD matrix is:

$$U = [u_1, u_2, u_3], \Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 \end{bmatrix}, V = [v_1, v_2, v_3, v_4]$$

where v_4 is the vector perpendicular to v_1, v_2, v_3.

III. Question 10-5

This problem can be interpreted as:

$$\max_{\sigma_i} \sum_{i=1}^{R_{\mathbf{H}}} B \log_2(1 + \sigma_i^2 \rho / M_t)$$
$$\text{s.t} \quad \sum_{i=1}^{R_{\mathbf{H}}} \sigma_i = \sigma$$

Apparently, the bound must be obtained to get max throughput.

Since the constraint and object function is both have continuous first derivative, we can obtain a local maxima by Lagrange multiplier theorem. Besides, the object function is a concave one, that is, the local maxima point is exactly the global maxima.

$$\mathcal{L}(\sigma_0, \cdots, \sigma_{R_{\mathbf{H}}}, \lambda) = \sum_{i=1}^{R_{\mathbf{H}}} B \log_2(1 + \sigma_i^2 \rho / M_t) + \lambda(\sum_{i=1}^{R_{\mathbf{H}}} \sigma_i - \sigma)$$
$$\frac{\partial \mathcal{L}(\sigma_0, \cdots, \sigma_{R_{\mathbf{H}}}, \lambda)}{\partial \sigma_i} = B \frac{1}{\ln 2} \frac{1}{(1 + \sigma_i^2 \rho / M_t)} 2\sigma_i \rho / M_t + \lambda = 0$$

Apparently, the equation $B \frac{1}{\ln 2} \frac{1}{(1+\sigma_i^2 \rho/M_t)} 2\sigma_i \rho / M_t$ is monotone for $\sigma \geq 0$ (by second order derivative). Therefore, the maximum throughput is attained for equal σ.