chapter-4 习题课（hw6&7）

2022.11.14

4-1. Capacity in AWGN is given by $C = B\log_2(1 + P/N_0B)$, where P is the received signal power, B is the signal bandwidth, and $N_0/2$ is the noise PSD. Find capacity in the limit of infinite bandwidth $B \to \infty$ as a function of P.

$$C = B\log_2\left(1 + \frac{S}{N_0 B}\right)$$

$$C = \frac{\log_2 \;\; 1+\frac{S}{N_0 B}}{\frac{1}{B}}$$

As $B \to \infty$ by L'Hospital's rule

$$C = \frac{S}{N_0}\frac{1}{\ln 2}$$

4-2. Consider an AWGN channel with bandwidth 50 MHz, received signal power 10 mW, and noise PSD $N_0/2$ where $N_0 = 2 \cdot 10^{-9}$ W/Hz. How much does capacity increase by doubling the received power? How much does capacity increase by doubling the channel bandwidth?

$B = 50$ MHz
$P = 10$ mW
$N_0 = 2 \times 10^{-9}$ W/Hz
$N = N_0 B$
$C = 6.87$ Mbps.

$P_{\text{new}} = 20$ mW, $C = 13.15$ Mbps (for $x \ll 1$, $\log(1+x) \approx x$)
$B = 100$ MHz, Notice that both the bandwidth and noise power will increase. So $C = 7$ Mbps.

4-4. Consider a flat fading channel of bandwidth 20 MHz and where, for a fixed transmit power $\bar{P}$, the received SNR is one of six values: $\gamma_1 = 20$ dB, $\gamma_2 = 15$ dB, $\gamma_3 = 10$ dB, $\gamma_4 = 5$ dB, $\gamma_5 = 0$ dB, and $\gamma_6 = -5$ dB. The probabilities associated with each state are $p_1 = p_6 = .1$, $p_2 = p_4 = .15$, and $p_3 = p_5 = .25$. Assume that only the receiver has CSI.

(a) Find the Shannon capacity of this channel.

(b) Plot the capacity versus outage for $0 \le P_{\text{out}} < 1$ and find the maximum average rate that can be correctly received (maximum C_{out}).

(a) Ergodic Capacity (with Rcvr CSI only)$= B[\sum_{i=1}^{6} \log_2(1+\gamma_i)p(\gamma_i)] = 2.8831 \times B = 57.66$ Mbps.

(b) $p_{\text{out}} = Pr(\gamma < \gamma_{\text{min}})$

$C_o = (1\text{-}p_{\text{out}})B\log_2(1+\gamma_{min})$

For

$\gamma_{\text{min}} > 20\text{dB}$, $p_{\text{out}} = 1$, $C_o = 0$

$15\text{dB} < \gamma_{\text{min}} < 20\text{dB}$, $p_{\text{out}} = .9$, $C_o = 0.1B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx 20$dB.

$10\text{dB} < \gamma_{\text{min}} < 15\text{dB}$, $p_{\text{out}} = .75$, $C_o = 0.25B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx 15$dB.

$5\text{dB} < \gamma_{\text{min}} < 10\text{dB}$, $p_{\text{out}} = .5$, $C_o = 0.5B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx 10$dB.

$0\text{dB} < \gamma_{\text{min}} < 5\text{dB}$, $p_{\text{out}} = .35$, $C_o = 0.65B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx 5$dB.

$-5\text{dB} < \gamma_{\text{min}} < 0\text{dB}$, $p_{\text{out}} = .1$, $C_o = 0.9B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx 0$dB.

$\gamma_{\text{min}} < -5\text{dB}$, $p_{\text{out}} = 0$, $C_o = B\log_2(1+\gamma_{\text{min}})$, max C_o at $\gamma_{\text{min}} \approx$ -5dB.

4-4. Consider a flat fading channel of bandwidth 20 MHz and where, for a fixed transmit power $\bar{P}$, the received SNR is one of six values: $\gamma_1 = 20$ dB, $\gamma_2 = 15$ dB, $\gamma_3 = 10$ dB, $\gamma_4 = 5$ dB, $\gamma_5 = 0$ dB, and $\gamma_6 = -5$ dB. The probabilities associated with each state are $p_1 = p_6 = .1$, $p_2 = p_4 = .15$, and $p_3 = p_5 = .25$. Assume that only the receiver has CSI.

(a) Find the Shannon capacity of this channel.

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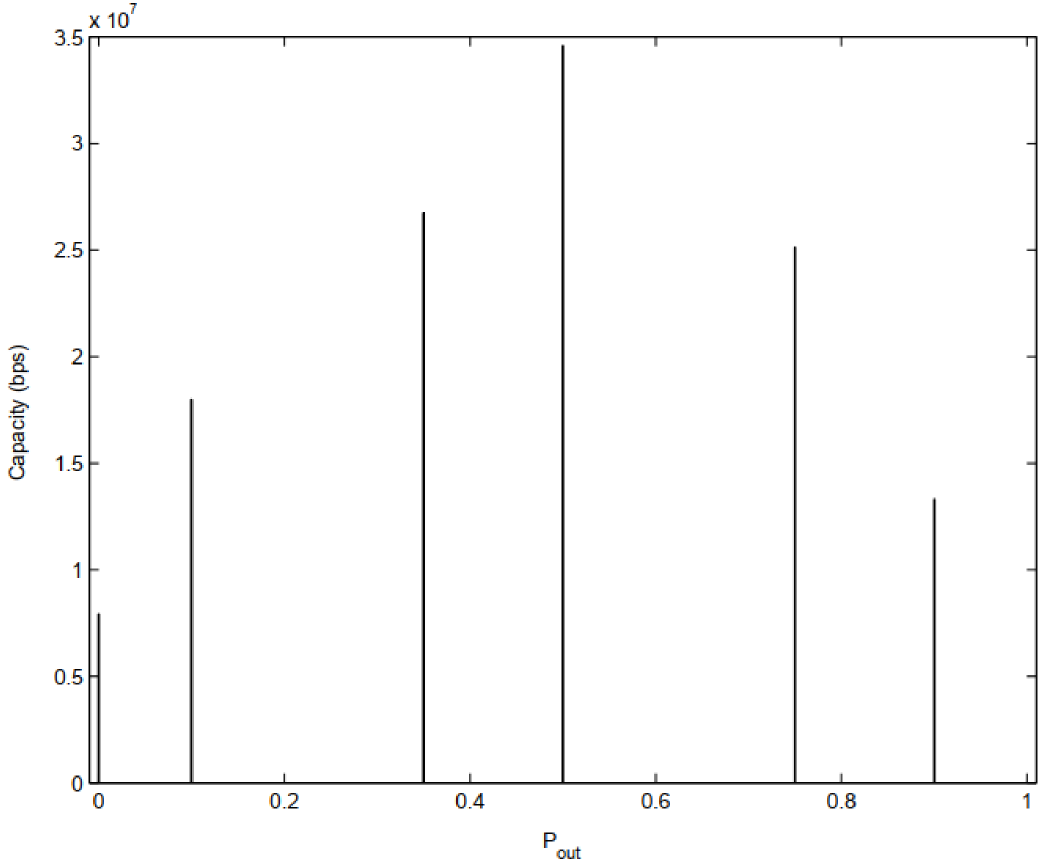
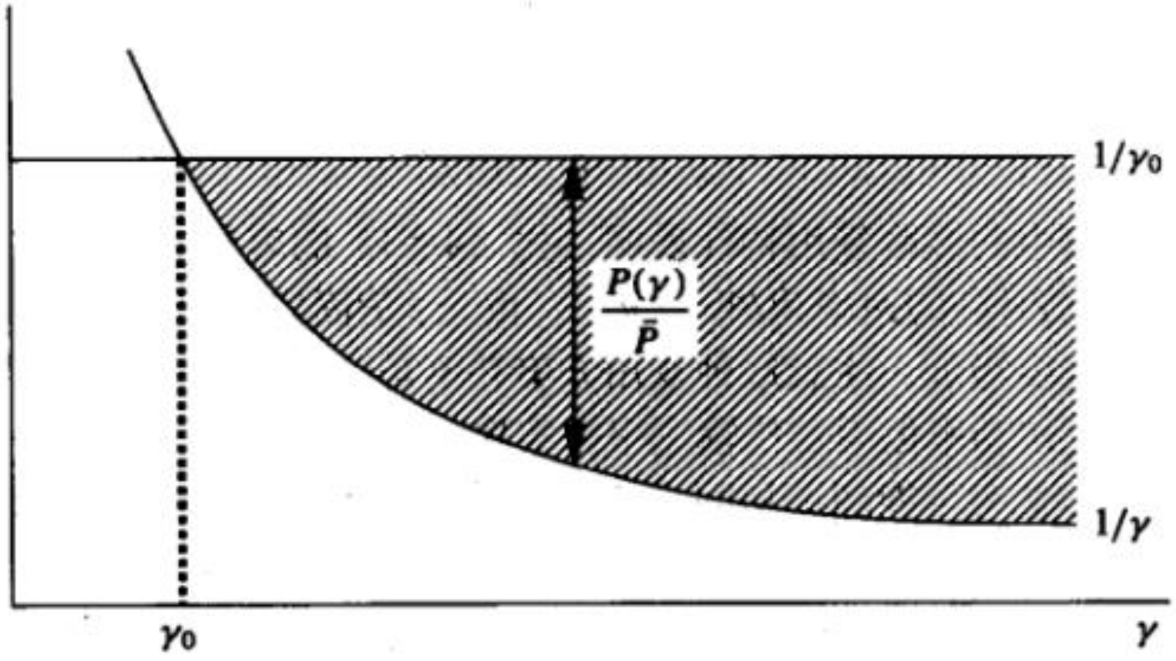


Maximum at $\gamma_{\min} = 10$dB, p_{out}=0.5 and $C_o = 34.59$ Mbps.



$$\int_{\gamma_0}^{\infty}\left(\frac{1}{\gamma_0}-\frac{1}{\gamma}\right)p(\gamma)\,\mathrm{d}\gamma = 1$$

$$\sum_{\gamma_i > \gamma_0}\left(\frac{1}{\gamma_0}-\frac{1}{\gamma_i}\right)p(\gamma_i) = 1$$

4-5. Consider a flat fading channel in which, for a fixed transmit power $\bar{P}$, the received SNR is one of four values: $\gamma_1 = 30$ dB, $\gamma_2 = 20$ dB, $\gamma_3 = 10$ dB, and $\gamma_4 = 0$ dB. The probabilities associated with each state are $p_1 = .2$, $p_2 = .3$, $p_3 = .3$, and $p_4 = .2$. Assume that both transmitter and receiver have CSI.

(a) Find the optimal power adaptation policy $P[i]/\bar{P}$ for this channel and its corresponding Shannon capacity per unit hertz (C/B).

We suppose that all channel states are used

$$\frac{1}{\gamma_0} = 1 + \sum_{i=1}^{4} \frac{1}{\gamma_i} p_i \Rightarrow \ \gamma_0 = 0.8109$$

$$\frac{1}{\gamma_0} - \frac{1}{\gamma_4} > 0 \therefore true$$

$$\frac{S(\gamma_i)}{\overline{S}} = \frac{1}{\gamma_0} - \frac{1}{\gamma_i}$$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 1.2322 & \gamma = \gamma_1 \\ 1.2232 & \gamma = \gamma_2 \\ 1.1332 & \gamma = \gamma_3 \\ 0.2332 & \gamma = \gamma_4 \end{cases}$$

$$\frac{C}{B} = \sum_{i=1}^{4} \log_2 \left(\frac{\gamma_i}{\gamma_0} \right) p(\gamma_i) = 5.2853 bps/Hz$$

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(b) Find the channel inversion power adaptation policy for this channel and associated zero-outage capacity per unit bandwidth.

$$\sigma = \frac{1}{E[1/\gamma]} = 4.2882$$

$$\frac{S(\gamma_i)}{\overline{S}} = \frac{\sigma}{\gamma_i}$$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 0.0043 & \gamma = \gamma_1 \\ 0.0029 & \gamma = \gamma_2 \\ 0.4288 & \gamma = \gamma_3 \\ 4.2882 & \gamma = \gamma_4 \end{cases}$$

$$\frac{C}{B} = \log_2(1+\sigma) = 2.4028 bps/Hz$$

4-6. Consider a cellular system where the power falloff with distance follows the formula $P_r(d) = P_t(d_0/d)^\alpha$, where $d_0 = 100$ m and α is a random variable. The distribution for α is $p(\alpha = 2) = .4$, $p(\alpha = 2.5) = .3$, $p(\alpha = 3) = .2$, and $p(\alpha = 4) = .1$. Assume a receiver at a distance $d = 1000$ m from the transmitter, with an average transmit power constraint of $P_t = 100$ mW and a receiver noise power of .1 mW. Assume that both transmitter and receiver have CSI.

(a) Compute the distribution of the received SNR.

(b) Derive the optimal power adaptation policy for this channel and its corresponding Shannon capacity per unit hertz (C/B).

$$SNR_{recvd} = \frac{P_\gamma(d)}{P_{noise}} = \begin{cases} 10dB & w.p.\ 0.4 \\ 5dB & w.p.\ 0.3 \\ 0dB & w.p.\ 0.2 \\ -10dB & w.p.\ 0.1 \end{cases}$$

Assume all channel states are used

$$\frac{1}{\gamma_0} = 1 + \sum_{i=1}^{4} \frac{1}{\gamma_i} p_i \Rightarrow \gamma_0 = 0.4283 > 0.1 \quad \therefore \text{not possible}$$

4-6. Consider a cellular system where the power falloff with distance follows the formula $P_r(d) = P_t(d_0/d)^\alpha$, where $d_0 = 100$ m and α is a random variable. The distribution for α is $p(\alpha = 2) = .4$, $p(\alpha = 2.5) = .3$, $p(\alpha = 3) = .2$, and $p(\alpha = 4) = .1$. Assume a receiver at a distance $d = 1000$ m from the transmitter, with an average transmit power constraint of $P_t = 100$ mW and a receiver noise power of .1 mW. Assume that both transmitter and receiver have CSI.

(c) Determine the zero-outage capacity per unit bandwidth of this channel.

Now assume only the best 3 channel states are used

$$\frac{0.9}{\gamma_0} = 1 + \sum_{i=1}^{3} \frac{1}{\gamma_i} p_i \Rightarrow \gamma_0 = 0.6742 < 1 \quad \therefore \text{ok!}$$

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} 1.3832 & \gamma = \gamma_1 = 10 \\ 1.1670 & \gamma = \gamma_2 = 3.1623 \\ 0.4832 & \gamma = \gamma_3 = 1 \\ 0 & \gamma = \gamma_4 = 0.1 \end{cases}$$

$$C/B = 2.3389 bps/Hz$$

(c).

$$\sigma = 0.7491$$
$$C/B = \log_2(1+\sigma) = 0.8066 bps/Hz$$

4-7. Assume a Rayleigh fading channel, where the transmitter and receiver have CSI and the distribution of the fading SNR $p(\gamma)$ is exponential with mean $\bar{\gamma} = 10$ dB. Assume a channel bandwidth of 10 MHz.

(a) Find the cutoff value γ_0 and the corresponding power adaptation that achieves Shannon capacity on this channel.

Maximize capacity given by

$$C = \max_{S(\gamma):\int S(\gamma)p(\gamma)d\gamma=\overline{S}} \int_\gamma B\log\left(1+\frac{S(\gamma)\gamma}{\overline{S}}\right)p(\gamma)d\gamma.$$

Construct the Lagrangian function

$$\mathcal{L} = \int_\gamma B\log\left(1+\frac{S(\gamma)\gamma}{\overline{S}}\right)p(\gamma)d\gamma - \lambda\int\frac{S(\gamma)}{\overline{S}}p(\gamma)d\gamma$$

Taking derivative with respect to $S(\gamma)$, (refer to discussion section notes) and setting it to zero, we obtain,

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{\gamma_0} - \frac{1}{\gamma} & \gamma \geq \gamma_0 \\ 0 & \gamma < \gamma_0 \end{cases}$$

Now, the threshold value must satisfy

$$\int_{\gamma_0}^{\infty}\left(\frac{1}{\gamma_0} - \frac{1}{\gamma}\right)p(\gamma)d\gamma = 1$$

4-7. Assume a Rayleigh fading channel, where the transmitter and receiver have CSI and the distribution of the fading SNR $p(\gamma)$ is exponential with mean $\bar{\gamma} = 10$ dB. Assume a channel bandwidth of 10 MHz.

(a) Find the cutoff value γ_0 and the corresponding power adaptation that achieves Shannon capacity on this channel.

Evaluating this with $p(\gamma) = \frac{1}{10}e^{-\gamma/10}$, we have

$$\begin{aligned} 1 &= \frac{1}{10\gamma_0}\int_{\gamma_0}^{\infty} e^{-\gamma/10} d\gamma - \frac{1}{10}\int_{\gamma_0}^{\infty} \frac{e^{-\gamma/10}}{\gamma} d\gamma && (1) \\ &= \frac{1}{\gamma_0} e^{-\gamma_0/10} - \frac{1}{10}\int_{\frac{\gamma_0}{10}}^{\infty} \frac{e^{-\gamma}}{\gamma} d\gamma && (2) \\ &= \frac{1}{\gamma_0} e^{-\gamma_0/10} - \frac{1}{10}\mathrm{EXPINT}(\gamma_0/10) && (3) \end{aligned}$$

where EXPINT is as defined in matlab. This gives $\gamma_0 = 0.7676$. The power adaptation becomes

$$\frac{S(\gamma)}{\overline{S}} = \begin{cases} \frac{1}{0.7676} - \frac{1}{\gamma} & \gamma \geq 0.7676 \\ 0 & \gamma < 0.7676 \end{cases}$$

(b) Compute the Shannon capacity of this channel.

(c) Compare your answer in part (b) with the channel capacity in AWGN with the same average SNR.

(d) Compare your answer in part (b) with the Shannon capacity when only the receiver knows $\gamma[i]$.

(e) Compare your answer in part (b) with the zero-outage capacity and outage capacity when the outage probability is .05.

(b) Capacity can be computed as

$$C/B = \frac{1}{10}\int_{0.7676}^{\infty} \log\left(\gamma/0.7676\right) e^{-\gamma/10} d\gamma = 2.0649 \quad \text{nats/sec/Hz.}$$

Note that I computed all capacites in nats/sec/Hz. This is because I took the natural log. In order to get the capacity values in bits/sec/Hz, the capacity numbers simply need to be divided by natural log of 2.

(c) AWGN capacity $C/B = \log(1+10) = 2.3979$ nats/sec/Hz.

(d) Capacity when only receiver knows γ

$$C/B = \frac{1}{10}\int_{0}^{\infty} \log\left(1+\gamma\right) e^{-\gamma/10} d\gamma = 2.0150 \quad \text{nats/sec/Hz.}$$

(b) Compute the Shannon capacity of this channel.

(c) Compare your answer in part (b) with the channel capacity in AWGN with the same average SNR.

(d) Compare your answer in part (b) with the Shannon capacity when only the receiver knows $\gamma[i]$.

(e) Compare your answer in part (b) with the zero-outage capacity and outage capacity when the outage probability is .05.

(e) Capacity using channel inversion is ZERO because the channel can not be inverted with finite average power. Threshold for outage probability 0.05 is computed as

$$\frac{1}{10}\int_{\gamma 0}^{\infty} e^{-\gamma/10} d\gamma = 0.95$$

which gives $\gamma_0 = 0.5129$. This gives us the capacity with truncated channel inversion as

$$\begin{aligned} C/B &= \log\left[1+\frac{1}{\frac{1}{10}\int_{\gamma_0}^{\infty}\frac{1}{\gamma}e^{-\gamma/10}d\gamma}\right] * 0.95 && (4)\\ &= \log\left[1+\frac{1}{\frac{1}{10}\mathrm{EXPINT}(\gamma_0/10)}\right] * 0.95 && (5)\\ &= 1.5463 \quad \text{nats/s/Hz.} && (6) \end{aligned}$$

(f) Repeat parts (b), (c), and (d) – that is, obtain the Shannon capacity with perfect transmitter and receiver side information, in AWGN for the same average power, and with just receiver side information – for the same fading distribution but with mean $\bar{\gamma} = -5$ dB. Describe the circumstances under which a fading channel has higher capacity than an AWGN channel with the same average SNR and explain why this behavior occurs.

Channel Mean=-5 dB = 0.3162. So for perfect channel knowledge at transmitter and receiver we compute $\gamma_0 = 0.22765$ which gives capacity $C/B = 0.36$ nats/sec/Hz.
With AWGN, $C/B = \log(1 + 0.3162) = 0.2748$ nats/sec/Hz.
With channel known only to the receiver $C/B = 0.2510$ nats/sec/Hz.

Capacity with AWGN is always greater than or equal to the capacity when only the reciever knows the channel. This can be shown using Jensen's inequality. However the capacity when the transmitter knows the channel as well and can adapt its power, can be higher than AWGN capacity specially at low SNR. At low SNR, the knowledge of fading helps to use the low SNR more efficiently.