Notes for Lecture 2

September 13, 2022

1 Signal Model

Bandpass communication signal:

$$s(t) = s_I(t)cos(2\pi f_c t) - s_Q(t)cos(2\pi f_c t).$$

Equivalent baseband/lowpass signal:

$$u(t) = s_I(t) + js_Q(t).$$

The carrier frequency f_c is hidden in this expression.

Their relation:

$$\begin{aligned} s(t) &= Re\{u(t)e^{j2\pi f_c t}\} \\ &= Re\{s_I(t)e^{j2\pi f_c t} + js_Q(t)e^{j2\pi f_c t}\} \\ &= s_I(t)cos(2\pi f_c t) - s_Q(t)sin(2\pi f_c t). \end{aligned}$$

Suppose that the bandpass signal travels a distance of d with path power gain a^2, then the received signal can be written as

$$r(t) = as_I(t)cos(2\pi f_c t - 2\pi d/\lambda) - as_Q(t)sin(2\pi f_c t - 2\pi d/\lambda).$$

Its equivalent baseband signal can be written as

$$u_r(t) = au(t)e^{-j2\pi d/\lambda}.$$

This is because

$$\begin{aligned} & Re\{u_r(t)e^{j2\pi f_c t}\} \\ =\; & Re\{au(t)e^{j(2\pi f_c t - 2\pi d/\lambda)}\} \\ =\; & Re\{as_I(t)e^{j(2\pi f_c t - 2\pi d/\lambda)} + jas_Q(t)e^{j(2\pi f_c t - 2\pi d/\lambda)}\} \\ =\; & as_I(t)cos(2\pi f_c t - 2\pi d/\lambda) - as_Q(t)sin(2\pi f_c t - 2\pi d/\lambda) \\ =\; & r(t). \end{aligned}$$

Remember

$$\text{Bandpass signal} = Re\{\text{Baseband expression} \times e^{2\pi f_c t}\}$$

2 Path Loss

The path loss is defined as

$$P_L = \frac{P_t}{P_r},$$

$$P_L \text{ dB} = 10 \log_{10} \frac{P_t}{P_r} \text{ dB}.$$

The path gain is defined as

$$P_G = \frac{P_r}{P_t},$$

$$P_G \text{ dB} = 10 \log_{10} \frac{P_r}{P_t} \text{ dB} = -P_L \text{ dB}.$$

Hence, $P_r = P_t/P_L$ and $P_r = P_t \times P_G$.

Furthermore, $P_r \text{ dBm} = 10 \log_{10} P_r \text{ mW} = 10 \log_{10} \frac{P_t \text{ mW}}{P_L} = P_t \text{ dBm} - P_L \text{ dB}$.

Notice that the path loss is about the power relation between the transmission and receiving signals, instead of the phase.

2.1 Free Space Model

In free space (Line-of-Sight, LoS), the path gain is given by

$$P_G = \left[\frac{\sqrt{G_l}\lambda}{4\pi d} \right]^2.$$

Let $u(t)$ be the transmission signal (in baseband), the received signal can be expressed as

$$u_r(t) = \sqrt{P_G} u(t) e^{-j2\pi d/\lambda} = \frac{\sqrt{G_l}\lambda e^{-j2\pi d/\lambda}}{4\pi d} u(t).$$

The received signal (bandpass) is

$$r(t) = Re\{u_r(t) e^{j2\pi f_c t}\} = Re\{\frac{\sqrt{G_l}\lambda e^{-j2\pi d/\lambda}}{4\pi d} u(t) e^{j2\pi f_c t}\}.$$

Observations:

- The power of received signal decays with the order of $\frac{1}{d^2}$;
- High carrier frequency suffers from large path loss.

2.2 Two Path Model

Let $u(t)$ be the transmission signal, the received signal of the first path (Line-of-Sight, LoS) is

$$u_1(t) = \frac{\sqrt{G_l}\lambda e^{-j2\pi l/\lambda}}{4\pi l} u(t).$$

The received signal of the reflected path (None-Line-of-Sight, NLoS) is

$$u_2(t) = \frac{R\sqrt{G_r}\lambda e^{-j2\pi(x+x')/\lambda}}{4\pi(x+x')} u(t).$$

Hence, the total received signal is

$$u_r(t) = \frac{\lambda}{4\pi}\left[\frac{\sqrt{G_l}}{l} + \frac{R\sqrt{G_r}}{x+x'}e^{-j2\pi(x+x'-l)/\lambda}\right]e^{-j2\pi l/\lambda}u(t).$$

The time difference of two paths is $(x+x'-l)/c$, which is named as *delay spread.*

From geometry,

$$\begin{aligned} x+x'-l &= \sqrt{(h_t+h_r)^2+d^2} - \sqrt{(h_t-h_r)^2+d^2} \\ &\approx d(1+\frac{1}{2}(\frac{h_t+h_r}{d})^2) - d(1+\frac{1}{2}(\frac{h_t-h_r}{d})^2) \\ &= \frac{2h_th_r}{d}, \end{aligned}$$

where the approximation is made for $d >> h_t + h_r$. Hence,

$$u_r(t) = \frac{\lambda}{4\pi}\left[\frac{\sqrt{G_l}}{l} + \frac{R\sqrt{G_r}}{x+x'}e^{-j\frac{4\pi h_t h_r}{\lambda d}}\right]e^{-j2\pi l/\lambda}u(t).$$

For sufficiently large d, $x + x' \approx l \approx d$. Let $G_l \approx G_r$ and $R \approx -1$, the path gain can be written as

$$\begin{aligned} P_G &= \left|\frac{\lambda}{4\pi}\left[\frac{\sqrt{G_l}}{l} + \frac{R\sqrt{G_r}}{x+x'}e^{-j\frac{4\pi h_t h_r}{\lambda d}}\right]e^{-j2\pi l/\lambda}\right|^2 \\ &= \left[\frac{\lambda\sqrt{G_l}}{4\pi d}\right]^2\left[1-e^{-j\frac{4\pi h_t h_r}{\lambda d}}\right]^2 \\ &\approx \left[\frac{\lambda\sqrt{G_l}}{4\pi d}\right]^2\left[\frac{4\pi h_t h_r}{\lambda d}\right]^2 \\ &= \left[\frac{\sqrt{G_l}h_t h_r}{d^2}\right]^2. \end{aligned} \tag{1}$$

Hence,

$$P_G \text{ dB} = 20\log_{10}(\sqrt{G_l}h_t h_r) - 40\log_{10} d,$$

and

$$P_r \text{ dBm} = P_t \text{ dBm} + 20\log_{10}(\sqrt{G_l}h_t h_r) - 40\log_{10} d.$$

2.3 Simplified Path Loss Model

Let K be the free space path loss at a reference distance d_0 with $G_l = 1$. Thus,

$$K = \left[\frac{\lambda}{4\pi d_0}\right]^2.$$

The path gain of the simplified path loss model with a distance d is

$$P_G = \frac{P_r}{P_t} = K\left[\frac{d_0}{d}\right]^\gamma, 2 \leq \gamma \leq 8.$$

The path loss exponent γ depends on the environment. When $\gamma = 2$, the model reduces to the free space model.

3 Shadowing

When both path loss and shadowing are considered,

$$\frac{P_t}{P_r} = P_L \Psi \text{ or } \frac{P_r}{P_t} = P_G/\Psi.$$

Let $\Psi_{dB} = 10\log_{10}\Psi$, we have

$$\begin{aligned} P_r \text{ dB} - P_t \text{ dB} &= P_G \text{ dB} - \Psi_{dB} \\ &= 10\log_{10} K - 10\gamma\log_{10}\left(\frac{d}{d_0}\right) - \Psi_{dB} \text{ for simplified path loss model.} \end{aligned}$$

or

$$P_r(d) \text{ dB} = P_t \text{ dB} + 10\log_{10} K - 10\gamma\log_{10}\left(\frac{d}{d_0}\right) - \Psi_{dB} \text{ for simplified path loss model.}$$

We usually assume Ψ_{dB} follows normal distribution $N(0, \sigma^2_{\Psi_{dB}})$. Thus,

$$p(\psi_{dB}) = \frac{1}{\sqrt{2\pi}\sigma_{\Psi_{dB}}} \exp\left[-\frac{\psi^2_{dB}}{2\sigma^2_{\Psi_{dB}}}\right].$$

In other words, Ψ follows log-normal distribution.

4 Outage Probability

$$\begin{aligned} & \Pr[P_r(d) \text{ dB} \leq P_{min}] \\ = \ & \Pr[P_t \text{ dB} + P_G \text{ dB} - \Psi_{dB} \leq P_{min}] \\ = \ & \Pr[\Psi_{dB} \geq P_t \text{ dB} + P_G \text{ dB} - P_{min}] \\ = \ & \Pr[-\frac{\Psi_{dB}}{\sigma_{\Psi_{dB}}} \leq \frac{P_{min} - P_t \text{ dB} - P_G \text{ dB}}{\sigma_{\Psi_{dB}}}] \end{aligned}$$

$$
\begin{aligned}
&= 1 - Q\left(\frac{P_{min} - P_t \text{ dB} - P_G \text{ dB}}{\sigma_{\Psi_{dB}}}\right) \\
&= 1 - Q\left(\frac{P_{min} - P_t \text{ dB} - 10\log_{10} K + 10\gamma\log_{10}\left(\frac{d}{d_0}\right)}{\sigma_{\Psi_{dB}}}\right)
\end{aligned}
$$