Lecture 8

Coding & Interleaving

Outline

- Review: Effect of fading
- Introduction to error control coding
  - FEC code, channel code
- Block Code
  - Hamming distance
  - Linear block code
  - Hamming code
- Interleaving
- Required reading:
  - Textbook, Chapter 8.1, 8.2.1, 8.2.2, 8.2.3, 8.8.1

△Flat Fading Channel.

$$\overline{P_s} = \int_0^{+\infty} P_s(\gamma)\, P_{\gamma_s}(\gamma)\, d\gamma$$

SNR per symbol. ↓概率密度函数

Rayleigh: $\vec{r} = h \cdot \vec{S_i} + \vec{n}$

$$|h| \sim p(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

Suppose average transmission power is α:

$$\alpha = \frac{1}{T_s}\int_0^{T_s} S_i^2(t)\,dt$$

Receiving signal Power $= \alpha |h|^2$

Noise power $= N_0 B = \frac{N_0}{T_s}$

Receiving SNR per symbol. $= \frac{T_s \alpha |h|^2}{N_0}$

$N_0 = 2\sigma_n^2$ ↓ $\frac{T_s \alpha |h|^2}{2\sigma_n^2} \triangleq \gamma_s$

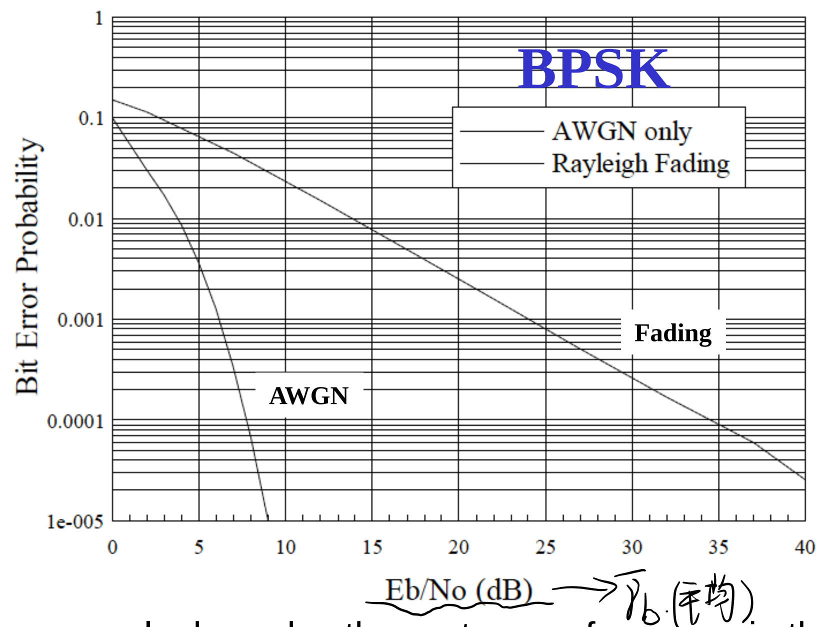
$$P_{\gamma_s}(\gamma) = \frac{1}{\overline{\gamma_s}} e^{-\gamma/\overline{\gamma_s}}, \quad \overline{\gamma_s} = \frac{\sigma^2 T_s \alpha}{\sigma_n^2}$$

Average Error Prob for BPSK and Rayleigh fading

$$\overline{P_s} = \int_0^{+\infty} Q(\sqrt{2\gamma_s})\, \frac{1}{\overline{\gamma_s}} e^{-\frac{\gamma}{\overline{\gamma_s}}}\, d\gamma$$

QPSK: $\overline{P_s} = \int_0^{+\infty} \{1 - [1 - Q(\sqrt{\gamma_s})]^2\} \frac{1}{\overline{\gamma_s}} e^{-\frac{\gamma}{\overline{\gamma_s}}}\, d\gamma$

Review: Effect of Flat Fading

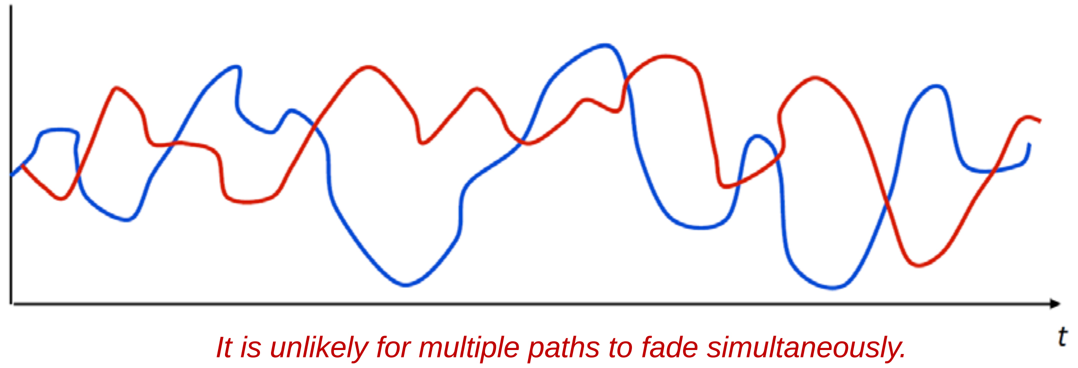


- Fading severely degrades the system performance in the presence of AWGN.

Remedies

- Remedies that combat the effects of fading:
  - Diversity
  - Channel coding, interleaving
  - Adaptive modulation
  - Power control
  - Spread spectrum
  - etc.

Channel gains of two independent paths:



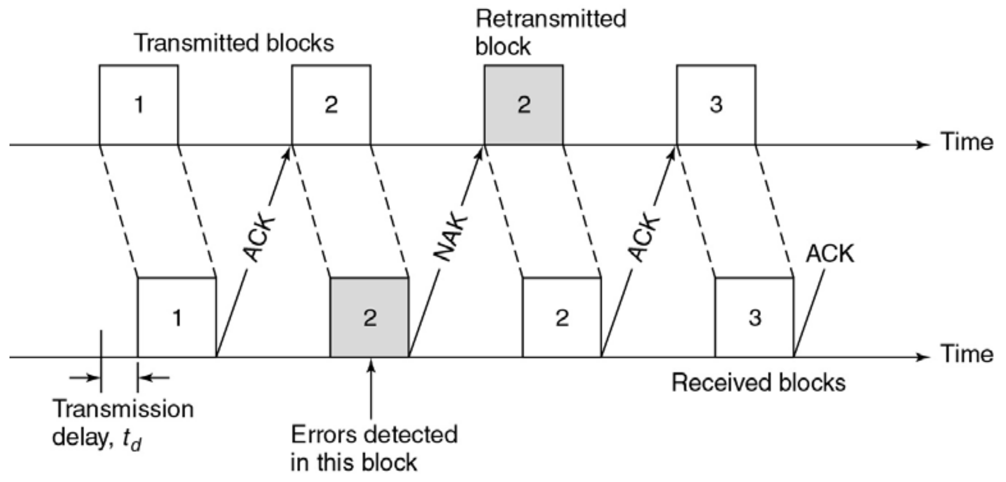
It is unlikely for multiple paths to fade simultaneously.

Error Control Coding

冗余

- Claude Shannon: Controlled redundancy in digital communications allows transmission at arbitrarily low bit error rates
- Error control coding (ECC) uses this controlled redundancy to detect and correct errors
  - When ECC is used to correct errors, this is called forward error correction code （FEC） or **channel code**
- The key in error control coding research is to
  - Find a way to add redundancy to the channel such that the receiver can utilize the redundancy to detect and correct the errors and to improve coding gain --- the effective lowering of the power required or improvement in throughput

Error Detection and ARQ



- ARQ (automatic repeat request) could lead to a large delay in data transmission. ∴Not suitable for real-time data communications.
  - Can be used in applications which may tolerate large delay (packet switched data networks).
  - Can add some FEC coding to reduce the # of repeat requests.

$\vec{m}=(m_1,m_2,\cdots m_k) \longrightarrow \boxed{\text{Encoder}} \longrightarrow \vec{c}=(c_1,c_2,\cdots c_n) \longrightarrow \boxed{\text{mod}} \xrightarrow{S_i(t)}$

$S_i(t)+n(t) \longrightarrow \boxed{\text{Dem}} \xrightarrow{\vec{r}=(\hat{c}_1,\cdots \hat{c}_n)} \boxed{\text{Decoder}} \longrightarrow (\hat{m}_1,\hat{m}_2,\cdots \hat{m}_n)$

$\Uparrow$ $\min \Delta(\vec{r},\vec{c}_i)$, $i\in\{1,\cdots 2^k\}$

k/n code rate

hamming distance $\Delta(\vec{c}_1,\vec{c}_2)$

$2^k \Rightarrow 2^k \{\vec{c}_1,\vec{c}_2,\cdots \vec{c}_{2^k}\}$

minimum hamming dist $d=\min\limits_{(i\neq j)}\Delta(\vec{c}_i,\vec{c}_j) \longrightarrow$ codeword 的差异性的大小和检错能力. c_i与c_j易混淆.

if detect $d-1$ bit errors, 则可以认为传输过程可现错误.

corret $\lfloor\frac{d-1}{2}\rfloor$ bit errors. 最近的code 唯一.

Channel Coding

- Classification
  - Block codes or convolutional codes. 无记忆 有记忆
  - The classification depends on the absence or presence of memory
- Block code
  - No memory
  - Divides the information sequence into message blocks, each containing k information symbols over an alphabet set Σ, i.e., a message can be represented as a k-tuple

$$\vec{m}=(m_1,\cdots,m_k)\in\Sigma^k$$

  - The encoder transforms the message into a n-tuple codeword $\vec{c}=(c_1,\cdots,c_n)\in\Sigma^n \quad n\geq k$

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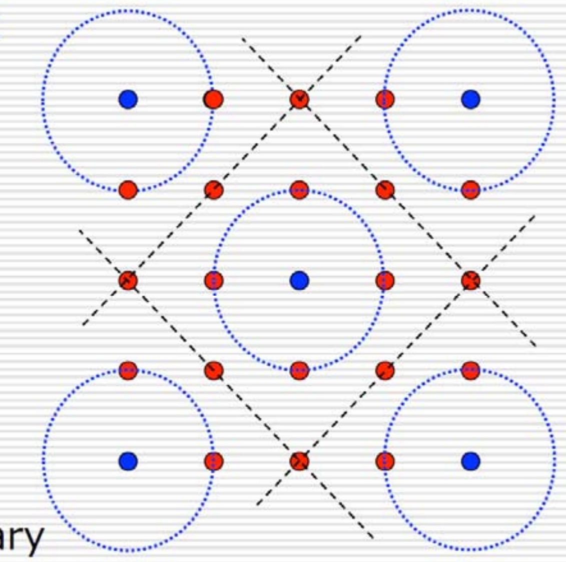
Block Code

模2运算

Tx:

- The encoder adds n-k redundant symbols
- Code rate: k/n
- There are in total $|\Sigma^k|$different possible codewords. However, there are totally $|\Sigma^n|$possible received sequences.
- The redundancy helps to detect and correct errors.
- Most codes are defined over binary sets, i.e., m and c are 0, 1 bits



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△Channel code.

$(m_1,m_2,\cdots m_k)$ $\vec{m} \longrightarrow \boxed{\text{Encoder}} \longrightarrow \boxed{\text{Mod}} \xrightarrow{S(t)}$

$\vec{c}=(c_1,c_2,\cdots,c_n)$ codeword, $n>k$ → 每个codeword 有n个bit.

Code Rate $=\frac{k}{n}\leq 1$, codeword中每个bit携带的信息.

2^k messages $\Rightarrow$ $\underbrace{2^k \text{ code words}}_{\text{codebook}}$.

? Received codewords. 2^n

Decoding.

Hamming distance $\Delta(\vec{c}_1,\vec{c}_2)=$ number of different bits.

let $\underbrace{J=2^k}_{\text{num of message}}$, $c=\{\vec{c}_1,\vec{c}_2,\cdots \vec{c}_J\}$ 总可能组合.

let $\vec{r}$ be the Rx codeword.

Estimate message $=\arg\min\limits_{i}\Delta(\vec{r},\vec{c}_i)$

$\underset{\substack{1\times k\\(k\text{ bits})}}{\vec{m}}\ \underset{\substack{k\times n\\\text{generecter}\\\text{matrix}}}{\vec{G}}=\underset{1\times n}{\vec{c}}$ $n>k$.

$G=[\underset{k\times k}{I}\ \ \underset{k\times(n-k)}{P}] \longrightarrow \vec{c}=[\underbrace{\vec{m}}_{k}\ \underbrace{\vec{m}P}_{n-k}]$

Rx: parity check matrix $H=[\underset{(n-k)\times k}{P^T}\ \ \underset{(n-k)\times(n-k)}{I}]$

when no error: $\vec{c}H^T=[\vec{m}\ \vec{m}P]\begin{bmatrix}P\\I\end{bmatrix}=\vec{m}P+\vec{m}P=0$ 模2运算

correct codeword $\vec{c}=$"1101"

receive codeword: "1111" $=\vec{c}+$"0010", "0101" $=\vec{c}+$"1000"

$\vec{r}=\vec{c}+\vec{e}$ (error code) → 哪些位置错了该位bit就为1.

$\vec{r}H^T=\vec{c}H^T+\vec{e}H^T=\vec{e}H^T$

Hamming Distance

- The hamming distance $\Delta(\vec{c}_1, \vec{c}_2)$ between two codewords is defined as the number of different positions between the codewords $\vec{c}_1$ and $\vec{c}_2$
- Example: the hamming distance from
  - [0101] to [0110] is 2 bits
  - [1011101] to [1001001] is 2 bits
- A code with minimum hamming distance *d* can
  - Detect *d*-1 errors
  - Correct $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors

Linear Block Code

- Linear code: the sum of any two codeword is another codeword
  - The all-zero sequence is a codeword in every linear block code
- Generation of linear block code

$$\vec{c} = \vec{m}\mathbf{G}$$

  - Here, **G** is the generator matrix
- Special case: systematic code

k Information bits	*n-k* Check bits

  - **G** takes the form

$$\mathbf{G} = \begin{bmatrix} \mathbf{I} & \mathbf{P} \end{bmatrix}$$

Hamming Code

- A binary linear block code
- A perfect code: a code that maximizes the minimum hamming distance with a given code rate
- Can correct single-bit errors
- Key elements
  - Generator matrix $\mathbf{G} = \begin{bmatrix} \mathbf{I} & \mathbf{P} \end{bmatrix}$
  - Parity check matrix $\mathbf{H} = \begin{bmatrix} \mathbf{P}^{\mathrm{T}} & \mathbf{I} \end{bmatrix}$

$$\boxed{\mathbf{G}\mathbf{H}^T = \mathbf{0}}$$

Encoding and Decoding

- Encoding: $\vec{c} = \vec{m}\,\mathbf{G}$
- Receive: $\vec{r} = \vec{c} + \vec{e}$
- Decoding: calculate $\vec{s} = \vec{r}\mathbf{H}^T$
  - $\vec{s} = \mathbf{0}$ if there is no error, i.e., $\vec{e} = \mathbf{0}$
  - Otherwise, there is an error
  - The pattern of $\vec{s}$ tells which bit is in error

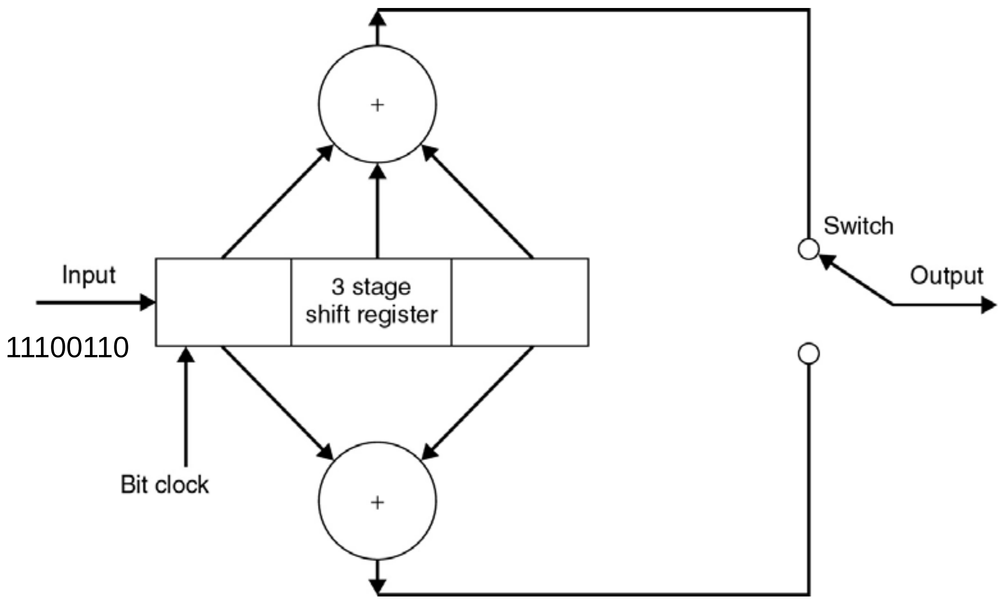
Example: (7,4) Hamming Code

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

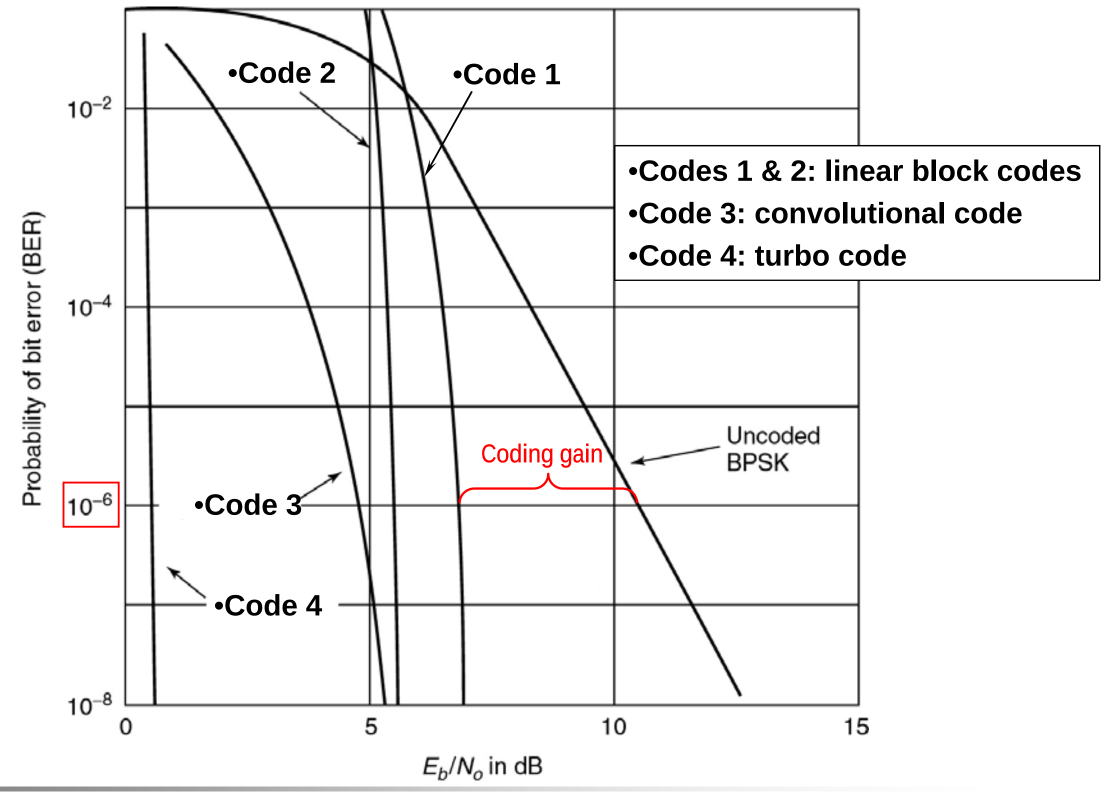
- Suppose the message is m=[0 0 1 1]
- Then, the codeword c=[0 0 1 1 0 1 0]
- Assume the 6th bit is received in error, i.e., r=[0 0 1 1 0 0 0]
- For decoding, calculate $\vec{s} = \vec{r}\,\mathbf{H}^T = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

  which is the 6th column of H. Therefore, we conclude that the 6th bit is in error

A Typical Rate ½ Convolutional Code

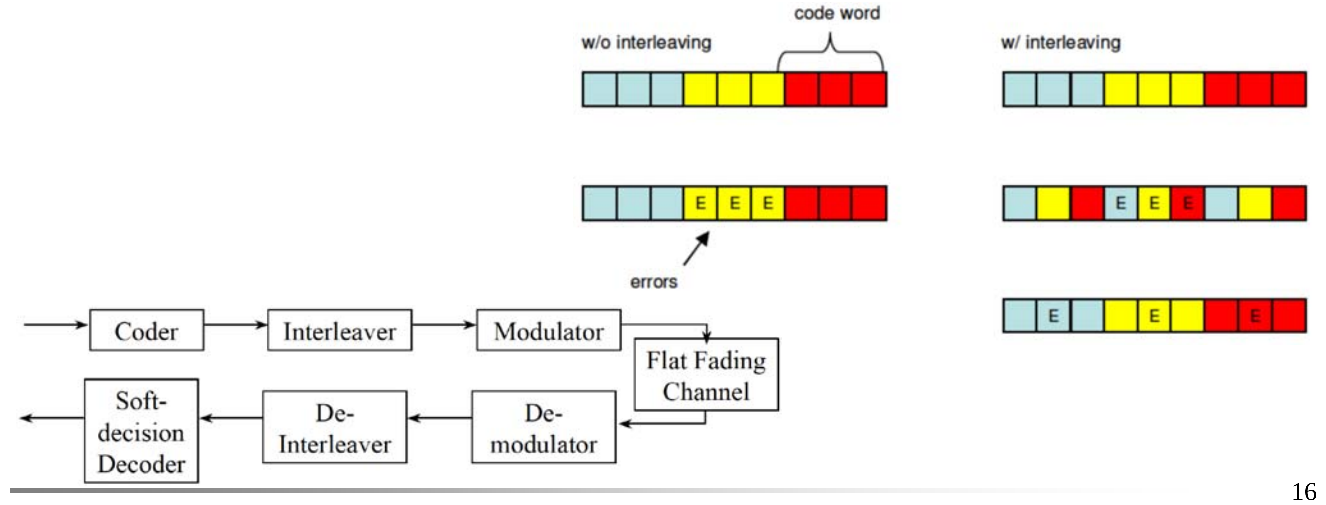


Performance of Typical Channel Codes



Interleaving

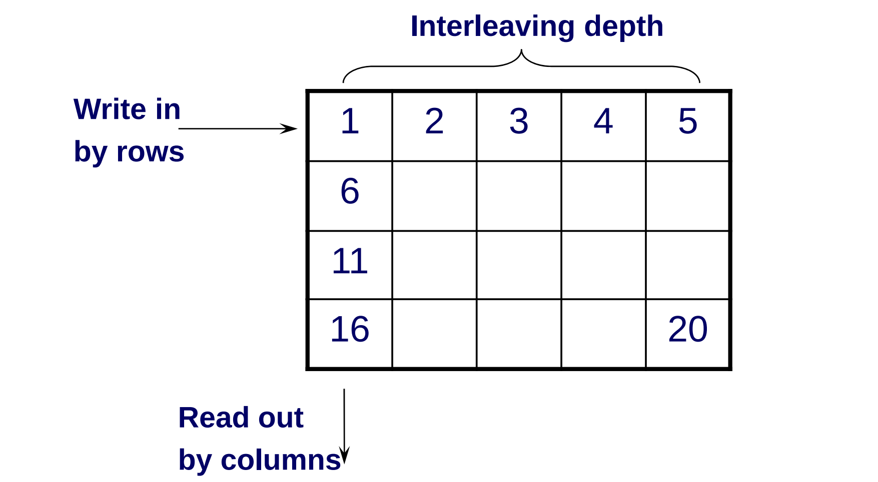
- Wireless channels are not memoryless. Errors typically occur in bursts rather than independently.
- Most FEC fails to correct bursty errors
- Interleaving is to rearrange the data so that 连续的 consecutive bits are transmitted over independent channels



Interleaving:
一般离散的错误比特可用 channel code纠错.

Block Interleaving

- Write row-by-row
- Read column-by-column (or another way round)



Summary

- In a fading channel, channel coding with interleaving provides diversity gain and higher SNR (i.e., coding gain), and the former is much more important than the latter.
- Interleaving depth has to be larger than the coherence time; otherwise, no diversity due to correlated fades. Interleaving depth is limited in real-time applications.
- Channel coding can only be achieved with bandwidth expansion or rate reduction.

AWGN $\vec{y} = \vec{x} + \vec{n}$

2-D: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

Baseband $(y_1 + j y_2) = (x_1 + j x_2) + (n_1 + j n_2)$

$y = x + n$

$\xrightarrow[x]{x(t)} \boxed{\text{AWGN}} \xrightarrow[x+n]{x(t)+n(t)}$

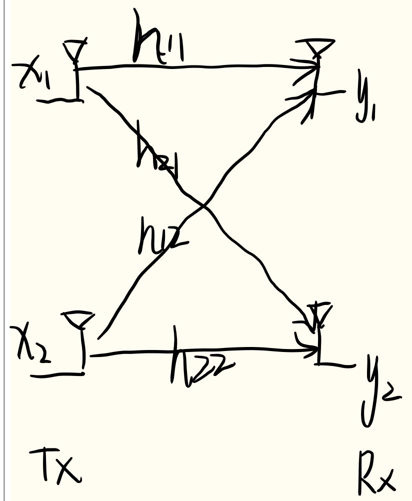
Narrowband / flat fading channel

$y = hx + n$

eg: rayleigh → 实部 虚部高斯分布
模为 rayleigh分布
相位为$(0, 2\pi)$的均匀分布

singal input singal output

△ Narrowband MIMO model.



$y_1 = h_{11} x_1 + h_{12} x_2 + n_1 = (h_{11} \ h_{12}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + n_1$

$y_2 = h_{21} x_1 + h_{22} x_2 + n_2 = (h_{21} \ h_{22}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + n_2$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$$

Multiple-input-Multiple-output (MIMO)

(SIMO) (MISO)

M_t: Tx antennas M_r: Rx antennas

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{M_r} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1M_t} \\ h_{21} & h_{22} & \cdots & h_{2M_t} \\ \vdots & \vdots & \ddots & \vdots \\ h_{M_r 1} & h_{M_r 2} & \cdots & h_{M_r M_t} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{M_t} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_{M_r} \end{pmatrix}$$

h_{ji}: from i-th Tx antenna to j-th Rx antenna.

$\vec{y} = H\vec{x} + \vec{n}$ 剩下的看左边

$n_c \sim CN(0, \sigma^2)$ $R_n = E[\vec{n}\vec{n}^H] = \sigma^2 I$. $\sigma^2 = E[n_i n_i^H]$

$|x_i|^2 = x_{i1}^2 + x_{i2}^2$ $E|x_i|^2 = P_i$.