EE 503 : Homework 8

Due : 11/14/2023, Tuesday before class.

1. Consider an irreducible finite Markov chain with states $0, 1, \ldots, N$.

   a) Starting in state i, what is the probability the process will ever visit state j? Explain.

   b) Let $x_i = P\{$ visit state N before state $0|$ start in $i\}$. Compute a set of liner equations that the x_i satisfy, $i = 0, 1, \ldots, N$.

   c) If $\sum_j jP_{ij} = i$ for $i = 0, \ldots, N-1$, show that $x_i = i/N$ is a solution to the equation in part b).

2. At all times, an urn contains N balls - some white balls and some black balls. At each stage, a coin having probability $p, 0 < p < 1$, of landing heads is flipped. If heads appears, then a ball is chosen at random from the urn and is replaced by a white ball; if tails appears, then a ball is chosen from the urn and is replaced by a black ball. Let X_n denote the number of white balls in the urn after the nth stage.

   a) Is $\{X_n, n \geq 0\}$ a Markov chain? Explain.

   b) What are its classes? What are their periods? Are they transient or recurrent?

   c) Compute the transition probabilities.

   d) Let $N = 2$. Find the proportion of time at each state.

   e) Based on your answer in part d) and your intuition guess the answer for the limiting probabilities in the general case.

   f) Prove your guess in e) by showing that the equations for the limiting probabilities are satisfied.

   g) If $p = 1$, what is the expected time until there are only white balls in the urn if initially there are i white and $N - ii$ black?

3. Let $X, Y_1, \ldots, Y_n$ be independent exponential random variables; X having rate λ, and Y_i having rate μ. Let A_j be the event that the j^{th} smallest of these $n+1$ random variables is one of the Y_i. Find $p = P\{X > max_i Yi\}$, using the identity

$$p = P(A_1 A_2 \ldots A_n) = P(A_1)P(A_2|A_1) \ldots P(A_n|A_1 A_2 \ldots A_{n-1})$$

4. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the nth event. Find:

   a) $E[S_4]$

b) $E[S_4|N(1) = 2]$

c) $E[N(4) - N(2)|N(1) = 3]$

5. Teams 1 and 2 are playing a match. The teams score points according to independent Poisson processes with respective rates λ_1 and λ_2. If the match ends when one of the teams has scored k more points than the other, find the probability that team 1 wins.
Hint: Relate this to the gambler's ruin problem.

6. Shocks occur according to a Poisson process with rate λ and each shock independently causes a certain system to fail with probability p. Let T denote the time at which the system fails and let N denote the number of shocks that it takes.

   a) Find the conditional distribution of T given that $N = n$.

   b) Calculate the conditional distribution of N, given that $T = t$, and notice that it is distributed as 1 plus a Poisson random variable with mean $\lambda(1 - p)t$.

Problem 7 is an optional question, those who solve it will get extra credits.

7. If X_i, $i = 1, 2, 3$, are independent exponential random variables with rates λ_i, $i = 1, 2, 3$, find

   a) $P[X_1 < X_2 < X_3]$,

   b) $P[X_1 < X_2| \max(X_1, X_2, X_3) = X_3]$

   c) $E[\max X_i|X_1 < X_2 < X_3]$

   d) $E[\max X_i]$

8. **Simulation problem:** Suppose that electrical shocks occur according to a Poisson process with rate $\lambda = 0.25$ per hour. Each shock may cause a certain system to fail. We know that the system can tolerate an average of 100 electrical shocks. We start observing a newly replaced system at $t = 0$.

   a) What is the expected time until the 100^{th} shock occurs?

   b) Use Python to create 100 i.i.d exponential random variables with rate λ.

   c) Add them together. This value (call it t_1) represents one instance of the Poisson process until the 100^{th} shock occurs.

   d) Repeat part c n times for different values of n to create other instances of the Poisson process. Plot the sample mean $\frac{\sum_{i=1}^{n} t_i}{n}$ versus n.

   e) What is minimum value of n for which the sample mean in part d differs from the expected time in part a by at most 2% for all $n \in \{n, n + 1, \ldots, n + 49\}$, i.e. for 50 consecutive values?