EE 503 : Homework 2

Due : 09/12/2023, Tuesday before class.

1. Let X be binomially distributed with parameters n and p. Show that as k goes from 0 to n, $P(X = k)$ increases monotonically, then decreases monotonically reaching its largest value

   a) in the case that $(n+1)p$ is an integer, when k equals either $(n+1)p - 1$ or $(n+1)p$,

   b) in the case that $(n+1)p$ is not an integer, when k satisfies $(n+1)p - 1 < k < (n+1)p$.

   Hint: Consider $P(X = k)/P(X = k-1)$ and see for what values of k it is greater or less than 1.

2. Suppose that an experiment can result in one of r possible outcomes, the ith outcome having probability P_i, $i = 1, \cdots, r, \sum_{i=1}^{r} p_i = 1$. If n of these experiments are performed, and if the outcome of any one of the n does not affect the outcome of the other $n-1$ experiments, then compute the probability that the first outcome appears x_1 times, the second x_2 times and the rth x_r times, where $x_1 + x_2 + \cdots + x_r = n$.

3. Suppose that two teams A and B are playing a series of games, each of which is independently won by team A with probability p and by team B with probability $1-p$. The winner of the series is the first team to win i games. If $i = 4$, find the probability that a total of 7 games are played. Also show that this probability is maximized when $p = 1/2$.

4. Consider n independent flips of a coin having probability p of landing heads. Say a changeover occurs whenever an outcome differs from the one preceding it. For instance, if the results of the flips are H, H, T, H, T, H, H, T, then there are a total of five changeovers. If $p = 1/2$, what is the probability there are k changeovers?

5. The number of orders waiting to be processed is given by a Poisson random variable with parameter $\alpha = \frac{\lambda}{n\mu}$, where λ is the average number of orders that arrive in a day, μ is the number of orders that can be processed by an employee per day, and n is the number of employees. Let $\lambda = 3$ and $\mu = 1$. Find the number of employees required so the probability that more than 4 orders are waiting is less than 0.9. What is the probability that there are no orders waiting?

6. The rth percentile, $\pi(r)$, of a random variable X is defined by $P(X \leq \pi(r)) = r/100$.

   a) Find the 90, 95 and 99 percentiles of the exponential random variable with parameter λ.

   b) Repeat part a) for the Gaussian random variable with parameters m and σ.

7. The entropy of a discrete random variable X, taking values in a finite set $\mathcal{X}$, is denoted by $H(X)$ and is given by the following expression

$$H(X) = -\sum_{x \in \mathcal{X}} P(X = x) \log_2 P(X = x)$$

   a) Show that $H(X) \geq 0$.
   b) Evaluate $H(X)$ when X is uniformly distributed.

8. **Coin flip experiment:** Use random.uniform() to generate output [Heads, Tail] of a coin flip experiment.

   a) Estimate the probability of heads in a fair coin flip by generating a lot of instances of experiment.
   b) Estimate the probability of heads in a biased coin flip with a bias of 0.8, and find the number of heads and tails in a random experiment.
   c) Plot/Display estimate as a function of number of experiments (n) and argue how do you decide how many times you need to repeat the experiment to get a good estimate.