EE 503 : Homework 9

Due : 11/21/2023, Tuesday before class.

1. Consider a birth and dean process with birth rates $\lambda_i = (i+1)\lambda, i \geq 0$, and death rates $\mu_i = i\mu, i \geq 0$.

   a) Determine the expected time to go from state 0 to state 4.

   b) Determine the expected time to go from state 2 to state 5.

2. Each individual in a biological population is assumed to give birth at an exponential rate λ, and to die at an exponential rate μ. In addition, there is an exponential rate of increase θ due to immigration. However, immigration is not allowed when the population size is N or larger.

   a) Set this up as a birth and death model.

   b) If $N = 3, 1 = \theta = \lambda, \mu = 2$, determine the proportion of time that immigration is restricted.

3. A small barbershop, operated by a single barber, has room for at most two customers. Potential customers arrive at a Poisson rate of three per hour, and the successive service times are independent exponential random variables with mean 1/4 hour.

   a) What is the average number of customers in the shop?

   b) What is the proportion of potential customers that enter the shop?

   c) If the barber could work twice as fast, how much more business would he do?

4. A service center consists of two servers, each working at an exponential rate of two services per hour. If customers arrive at a Poisson rate of three per hour, then, assuming a system capacity of at most three customers,

   a) what fraction of potential customers enter the system?

   b) what would the value of part (a) be if there was only a single server, and his rate was twice as fast (that is, $\mu = 4$)?

5. Each time a machine is repaired it remains up for an exponentially distributed time with rate λ. It then fails, and its failure is either of two types. If it is a type 1 failure, then the time to repair the machine is exponential with rate μ_1; if it is a type 2 failure, then the repair time is exponential with rate μ_2. Each failure is, independently of the time it took the machine to fail, a type 1 failure with probability p and a type 2 failure with probability $1-p$. What proportion of time is the machine down due to a type 1 failure? What proportion of time is it down due to a type 2 failure? What proportion of time is it up?

6. Customers arrive at a single-server queue in accordance with a Poisson process having rate λ. However, an arrival that finds n customers already in the system will only join the system with probability $1/(n+1)$. That is, with probability $n/(n+1)$ such an arrival will not join the system. Show that the limiting distribution of the number of customers in the system is Poisson with mean λ/μ.

7. Consider the continuous time Markov Chain with the state transition diagram as shown in Figure 1.

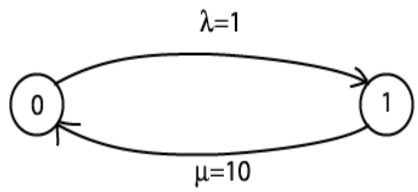

Figure 1: The Markov Chain

   (i) Find the stationary distributions P_0 and P_1.
   (ii) Find $Q = [q_{ij}]$ and verify that the set of equations $PQ = 0$ and $\sum_i P_i = 1$, where $P = [P_0 \;\; P_1]$ give the same values for P_i as the ones you found in part (i).
   (iii) Draw the transition diagram of the corresponding embedded discrete time Markov Chain. Find the transition probabilities using the corresponding formula **and** by inspection.
   (iv) Find π_0 and π_1 using the corresponding formula **and** by inspection.
   (v) Based only on the P_{ij}'s, can we reconstruct the original continuous time Markov Chain? Why/why not?
   (vi) If you know q_{ij}, how would you reconstruct the original continuous time Markov Chain from the discrete time Markov Chain?
   (vii) Repeat all of the above for the continuous time Markov Chain shown in Figure 2.

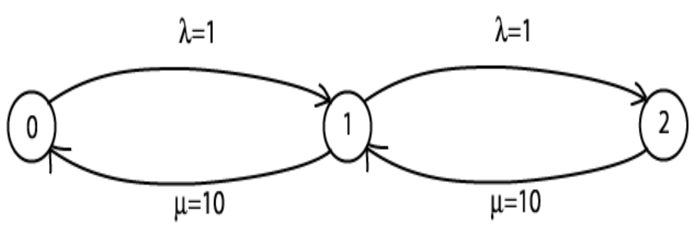

Figure 2: The Markov Chain for part (vii)

8. Consider a M/M/1/K queuing system, where we have Poisson arrival with rate 2 customers /second and Poisson departure with rate 1 customer /second, 1 server to service the customers, and maximum length of queue = 50. To simulate this model, we can create two exponential

variable realizations corresponding to their respective service and arrival time at time instant t = 1. The lowest among the two would determine the next state of our model. Likewise, at next state, again simulate two exponential variable realizations and determine the next state. Continue this process for 100, 1000 and 10000 transitions.

a). Draw the state diagram of the system and write the generator matrix.

b). Plot the number of customers versus time instants for both the departure process and arrival processes for 10000 transition case. Remember both are Poisson processes therefore, both plots will be an increasing function with respect to time.

c). Plot the total number of customers present in the system versus time.

d). Compute stationary distributions of all the states from the simulation results and plot them (probability versus state name/number) for each case 100, 1000 and 10000 transitions.

e). Now, compute stationary distributions from the theoretical formulas and compare it to the three cases 100, 1000 and 10000 transitions. Which one of the simulation results are closest of the theoretical results.