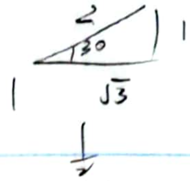


EE503 HW3 Chuh-Cheng Hsieh

1. $X = X_1 + X_2 + \cdots + X_n = \sum_{i=1}^{n} X_i$ #

2. $E(X) = \sum_{i=1}^{n} X_i P(X_i) = \sum_{i=1}^{n} 1 \cdot P(X_i)$

$$= \frac{1}{m+n} + \frac{1}{m+(n-1)} + \cdots + \frac{1}{m+n-(n-1)}$$

$$= \frac{1}{m+n} + \frac{1}{m+n-1} + \cdots + \frac{1}{m+1}$$

#

2. $X \sim U[-\pi, \pi] \Rightarrow f_X(x) = \begin{cases} \frac{1}{\pi-(-\pi)} = \frac{1}{2\pi} & ; -\pi < x < \pi \\ 0 & ; \text{otherwise} \end{cases}$

$F_Y(y) = P(Y \le y) = P(\tan X \le y) = P(X \le \tan^{-1} y) = F_X(\tan^{-1} y)$

$$F_X(\tan^{-1} y) = \int f_X(x)\,dx = \int_{-\pi}^{\tan^{-1} y} \frac{1}{2\pi}\,dx = \frac{1}{2\pi} x \Big|_{-\pi}^{\tan^{-1} y}$$

$$= \frac{1}{2\pi}\left(\tan^{-1} y - (-\pi)\right) = \frac{1}{2\pi}\tan^{-1} y + \frac{1}{2}$$

#

3. $X \sim N(\mu, \sigma) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$,

$F_Y(y) = P(Y \le y) = P\left(\frac{(X-\mu)^2}{\sigma^2} \le y\right) = P(X \le \mu + \sqrt{\sigma^2 y}) = F_X(\mu + \sqrt{\sigma^2 y})$

$\frac{(x-\mu)^2}{\sigma^2} \le y \Rightarrow x - \mu \le \sqrt{\sigma^2 y} \quad x \le \mu +$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\mu + \sqrt{\sigma^2 y}) = \frac{1}{2\sqrt{\sigma^2 y}} f_X(\mu + \sqrt{\sigma^2 y})$$

$$= \frac{1}{2\sqrt{\sigma^2 y}} \times \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{\frac{-(\mu + \sqrt{\sigma^2 y} - \mu)^2}{2\sigma^2}}\right) = \frac{1}{2\sigma^2\sqrt{2\pi y}} e^{\frac{-y^2}{2}}$$

#

4. $X = \begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ 2 & \text{w.p. } \frac{1}{8} \\ 3 & \text{w.p. } \frac{3}{8} \end{cases}$ Type 3 coutomers are impossible to reach 15 sec. since they require const. 2 sec.

$F_{X_1}(x) = \begin{cases} 1-e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, $F_{X_2}(x) = \begin{cases} 1-e^{-\frac{x}{10}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$P(X_1 > 15) = 1 - P(X_1 \leq 15) = 1-(1-e^{-15}) = e^{-15}$

$P(X_2 > 15) = 1 - P(X_2 \leq 15) = 1-(1-e^{-\frac{15}{10}}) = e^{-\frac{3}{2}}$

$P(X) = \frac{1}{2}e^{-15} + \frac{1}{8}e^{-\frac{3}{2}} + \frac{3}{8} \cdot 0 = 0.0279$ #

5. $E(X) = \mu = 65$, $\sigma = 10$ $10k+65 = 95$

a) $P(X-\mu \geq k\sigma) \leq \frac{1}{k^2}$

$\Rightarrow$ Let $X \geq \mu + k\sigma = 95 \Rightarrow 65 + k \cdot 10 = 95 \Rightarrow k = \frac{95-65}{10} = 3$

$\Rightarrow P(X > 95) = \frac{1}{3^2} = \frac{1}{9} = 0.1111$ #

b) $X \sim N(\mu, \sigma)$

$P(X > 95) = P(\frac{X-\mu}{\sigma}) > \frac{95-65}{10} = P(Z > 3)$

$= 1 - P(Z < 3) = 1 - 0.99865 = 0.00135$ #

The prob. of normal distribution is less than prob. of upper bond. #

6.

a) $P(\text{present}) = P(p) = p$

$P(\text{absent}) = P(a) = 1-p$

$P(X=k \mid p) = \frac{\lambda_1^k}{k!} e^{-\lambda_1}$, $P(X=k \mid a) = \frac{\lambda_0^k}{k!} e^{-\lambda_0}$

$$P(X=k) = P(X=k \mid p)P(p) + P(X=k \mid a)P(a)$$
$$= \frac{\lambda_1^k}{k!} e^{-\lambda_1} p + \frac{\lambda_0^k}{k!} e^{-\lambda_0}(1-p)$$

$$\Rightarrow P(p \mid X=k) = \frac{P(X=k \mid p)P(p)}{P(X=k)} = \frac{\frac{\lambda_1^k}{k!} e^{-\lambda_1} p}{\frac{\lambda_1^k}{k!} e^{-\lambda_1} p + \frac{\lambda_0^k}{k!} e^{-\lambda_0}(1-p)}$$

$$P(a \mid X=k) = \frac{P(X=k \mid a)P(a)}{P(X=k)} = \frac{\frac{\lambda_0^k}{k!} e^{-\lambda_0}(1-p)}{\frac{\lambda_1^k}{k!} e^{-\lambda_1} p + \frac{\lambda_0^k}{k!} e^{-\lambda_0}(1-p)}$$

#

b) $P(p \mid X=k) = P(a \mid X=k)$

$$\Rightarrow \frac{\lambda_1^k}{\cancel{k!}} e^{-\lambda_1} p = \frac{\lambda_0^k}{\cancel{k!}} e^{-\lambda_0}(1-p)$$

$$\Rightarrow k \ln\lambda_1 - \lambda_1 + \ln p = k \ln\lambda_0 - \lambda_0 + \ln(1-p)$$

$$\Rightarrow k(\ln\lambda_1 - \ln\lambda_0) = \lambda_1 - \lambda_0 + \ln(1-p) - \ln(p)$$

$$\Rightarrow k = \frac{\lambda_1 - \lambda_0 + \ln\frac{1-p}{p}}{\ln\frac{\lambda_1}{\lambda_0}} = T$$

#

c) $P_{error} = P(X<T \mid p) + P(X \geq T \mid a)$

$$= \sum_{k=0}^{T} \frac{\lambda_1^k}{k!} e^{-\lambda_1} p + \sum_{k=T+1}^{\infty} \frac{\lambda_0^k}{k!} e^{-\lambda_0}(1-p)$$

#

7.

(a)(b)

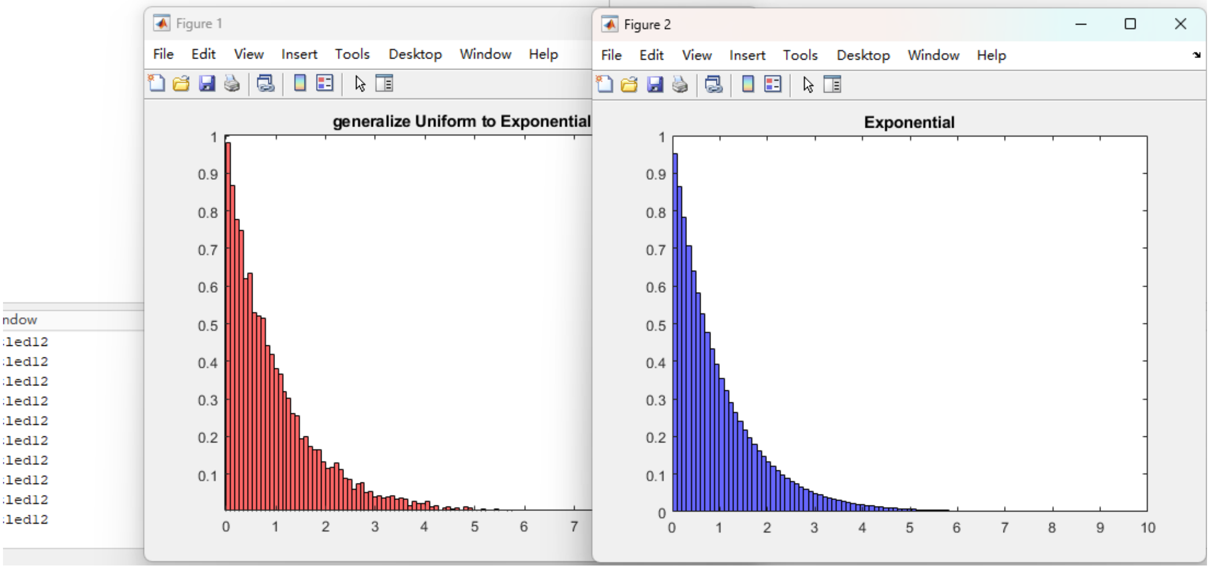
```
clear
n_size = 10000;
uni = rand(1, n_size);
l = 1;
exp_uni = -log(1 - uni) / l;

exp_rand = exprnd(l, n_size);

figure;
histogram1 = histogram(exp_uni, 100, 'Normalization', 'pdf', 'FaceColor', 'r');
title('generalize Uniform to Exponential');

figure;
histogram2 = histogram(exp_rand, 200, 'Normalization', 'pdf', 'FaceColor', 'b');
title('Exponential');

xlim([0, 10]);
```



(c) The distribution generated by uniform distribution doesn't match the ideal result very well, nevertheless, the overall distribution is similar to exponential.