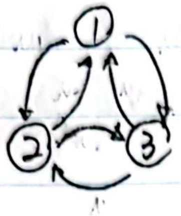
EE503 HW10 Chih-Cheng Hsieh

1.



$$(\mu_2+\mu_3)P_1 = \mu_1(P_2+P_3)$$
$$(\mu_1+\mu_3)P_2 = \mu_2(P_1+P_3)$$
$$(\mu_1+\mu_2)P_3 = \mu_3(P_1+P_2)$$
$$P_1+P_2+P_3 = 1$$

$$P_1 = \frac{\mu_1^2+\mu_1\cdot\mu_2+\mu_1\cdot\mu_3}{(\mu_1+\mu_2+\mu_3)^2}$$

$$P_2 = \frac{\mu_2^2+\mu_1\cdot\mu_2+\mu_2\cdot\mu_3}{(\mu_1+\mu_2+\mu_3)^2}$$

$$\frac{\lambda(1+k\rho^{k+1}-(k+1)\rho^k)}{\lambda(\mu-\lambda)(1-\rho^{k+1})\left(1-\rho^k\frac{1-\rho}{1-\rho^{k+1}}\right)}$$

$$P_3 = \frac{\mu_3^2+\mu_3\cdot\mu_1+\mu_3\cdot\mu_2}{(\mu_1+\mu_2+\mu_3)^2} \quad \#$$

2.

a) $$P_0 = \frac{1-\rho}{1-\rho^{k+1}} = \frac{1-\frac{\lambda}{\mu}}{1-(\frac{\lambda}{\mu})^{k+1}} \quad \#$$

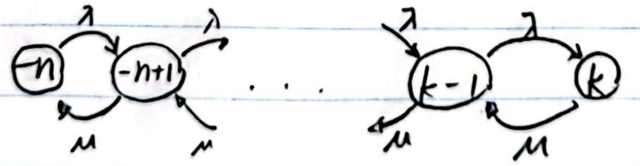
b) $$w = \frac{L}{\lambda_a} = \frac{L}{\lambda(1-P_k)} = \frac{\lambda\left(1+k(\frac{\lambda}{\mu})^{k+1}-(k+1)(\frac{\lambda}{\mu})^k\right)}{\lambda\left(1-\frac{(\frac{\lambda}{\mu})^k(1-\frac{\lambda}{\mu})}{1-(\frac{\lambda}{\mu})^{k+1}}\right)(\mu-\lambda)(1-(\frac{\lambda}{\mu})^{k+1})} = \frac{1+k(\frac{\lambda}{\mu})^{k+1}-(k+1)(\frac{\lambda}{\mu})^k}{(\mu-\lambda)(1-(\frac{\lambda}{\mu})^k)} \quad \#$$

$$L = \sum_{n=0}^{k} nP_n = \sum_{n=0}^{k} n\rho^n\frac{1-\rho}{1-\rho^{k+1}} = \frac{1-\rho}{1-\rho^{k+1}}\sum_{n=0}^{k} n\rho^n$$
$$= \frac{1-\rho}{1-\rho^{k+1}}\cdot\rho\cdot\frac{k\rho^{k+1}-(k+1)\rho^k+1}{(\rho-1)^2} = \frac{\rho(k\rho^{k+1}-(k+1)\rho^k+1)}{(1-\rho^{k+1})(1-\rho)}$$

c) $$= \frac{\frac{\lambda}{\mu}\left(k(\frac{\lambda}{\mu})^{k+1}-(k+1)(\frac{\lambda}{\mu})^k+1\right)}{(1-(\frac{\lambda}{\mu})^{k+1})(1-\frac{\lambda}{\mu})} = \frac{\lambda\left(1+k(\frac{\lambda}{\mu})^{k+1}-(k+1)(\frac{\lambda}{\mu})^k\right)}{(\mu-\lambda)(1-(\frac{\lambda}{\mu})^{k+1})} \quad \#$$

$\frac{\mu-\lambda}{\mu}$

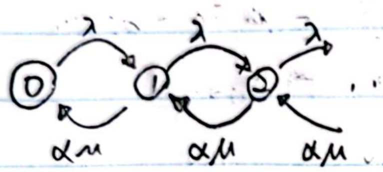
d)



state
$$\begin{cases} -n: & \lambda P_{-n} = \mu P_{-n+1} \\ -n<j<k: & (\lambda+\mu)P_j = \lambda P_{j-1}+\mu P_{j+1} \\ k: & \lambda P_{k-1} = \mu P_k \end{cases}$$

$$\sum_{j=-n}^{k} P_j = 1 \quad \#$$

e) $\alpha = \sum_{j=-n}^{-1} |j| P_j$ #

3. 

a) state $\begin{cases} 0 : & \lambda P_0 = \alpha\mu P_1 \\ n \geq 1 : & (\lambda + \alpha\mu) P_n = \lambda P_{n-1} + \alpha\mu P_{n+1} \end{cases}$ #

b) $E(W) = \sum_n \frac{n}{\mu} P_n = \frac{1}{\mu} \sum_n n P_n = \frac{L}{\mu} = \frac{\lambda}{(\alpha\mu - \lambda)\mu}$ #

c) $P(\text{enter } n \text{ times}) = (1-\alpha)^{n-1} \alpha$ #

d) $E[n] = \sum_n n P(\text{enter } n \text{ times}) = \sum_n n(1-\alpha)^{n-1} \alpha$

$= \frac{\alpha}{1-\alpha} \sum_n n(1-\alpha)^n = \frac{\alpha}{\cancel{1-\alpha}} \frac{\cancel{1-\alpha}}{(1-1+\alpha)^2} = \frac{1}{\alpha}$

$E[\text{amount of time}] = E[nT] = E[n]E[T] = \frac{1}{\alpha} \cdot \frac{1}{\mu} = \frac{1}{\alpha\mu}$ #

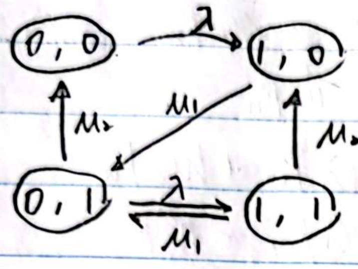
e) customers can join the queue after service

$\Rightarrow$ total time being served is affected by history

$\Rightarrow$ not memoryless #

4. $\lambda = 2$, $\mu_1 = 4$, $\mu_2 = 2$

a)



$$\Rightarrow \begin{cases} \lambda P_{00} = \mu_2 P_{01} \\ \mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11} \\ (\mu_2 + \lambda) P_{01} = \mu_1 P_{10} + \mu_1 P_{11} \\ (\mu_1 + \mu_2) P_{11} = \lambda P_{01} \end{cases}$$

$$\Rightarrow \begin{cases} P_{01} = P_{00} \\ 4P_{10} = 2P_{00} + 2P_{11} = 2P_{00} + \frac{2}{3}P_{00} = \frac{8}{3}P_{00} \Rightarrow P_{10} = \frac{2}{3}P_{00} \\ 4P_{01} = 4P_{10} + 4P_{11} \\ 6P_{11} = 2P_{01} \Rightarrow P_{11} = \frac{1}{3}P_{00} \\ \Sigma P = 1 \end{cases}$$

$$\Rightarrow \Sigma P = P_{00} + \frac{2}{3}P_{00} + P_{00} + \frac{1}{3}P_{00} = 3P_{00} = 1 \Rightarrow P_{00} = \frac{1}{3}$$

$$\Rightarrow \begin{cases} P_{00} = \frac{1}{3} \\ P_{01} = \frac{1}{3} \\ P_{10} = \frac{2}{9} \\ P_{11} = \frac{1}{9} \end{cases}$$

$\frac{1}{6} + \frac{1}{2} = \frac{4}{6} = \frac{2}{3}$

$\frac{3}{6}$

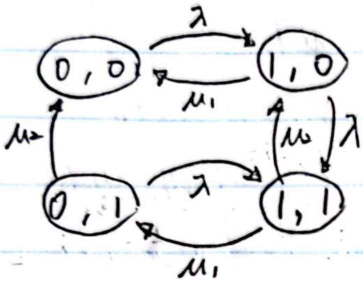
$$P(\text{enter the system}) = P_{00} + P_{01} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \quad \#$$

b) $$P(\text{enter } B) = \frac{P_{00}}{P_{00}+P_{01}} \cdot P(\text{enter } B \mid P_{00}) + \frac{P_{01}}{P_{00}+P_{01}} \cdot \overbrace{P(\text{enter } B \mid P_{01})}^{\frac{\mu_2}{\mu_2+\mu_1}}$$
$$= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{2}{6} = \frac{2}{3} \quad \#$$

c) $$L = 1 \cdot P_{01} + 1 \cdot P_{10} + 2 \cdot P_{11} = \frac{1}{3} + \frac{2}{9} + \frac{2}{9} = \frac{7}{9} \quad \#$$

d) $$W = \frac{L}{\lambda_a} = \frac{\frac{7}{9}}{2 \cdot \frac{2}{3}} = \frac{7}{9} \cdot \frac{3}{4} = \frac{7}{12} \quad \#$$

5. $\lambda = 5, \quad \mu_1 = 4, \quad \mu_2 = 2$



$$\begin{cases} \lambda P_{00} = \mu_2 P_{01} + \mu_1 P_{10} \\ (\lambda+\mu_2) P_{01} = \mu_1 P_{11} \\ (\lambda+\mu_1) P_{10} = \lambda P_{00} + \mu_2 P_{11} \\ (\mu_1+\mu_2) P_{11} = \lambda P_{01} + \lambda P_{10} \end{cases}$$

$\frac{42}{7}$ 8

$\frac{42}{7}$ $\frac{62}{7}$

$$\Rightarrow \begin{cases} 5P_{00} = 2P_{01} + 4P_{10} \Rightarrow \\ 7P_{01} = 4P_{11} \Rightarrow P_{01} = \frac{4}{7}P_{11} \\ 9P_{10} = 5P_{00} + 2P_{11} \Rightarrow 5P_{00} = \frac{198}{35}P_{11} - \frac{70}{35}P_{11} \Rightarrow P_{00} = \frac{128}{175}P_{11} \\ 6P_{11} = 5P_{01} + 5P_{10} \Rightarrow P_{10} = \frac{(6-\frac{20}{7})P_{11}}{5} = \frac{22}{35}P_{11} \end{cases}$$

$$\Rightarrow \sum_{\forall i,j} P_{ij} = 1 \Rightarrow P_{11} + \frac{4}{7}P_{11} + \frac{128}{175}P_{11} + \frac{22}{35}P_{11} = 1 \Rightarrow P_{11} = \frac{175}{513}$$

$$\Rightarrow \begin{cases} P_{00} = \frac{128}{513} \\ P_{01} = \frac{100}{513} \\ P_{10} = \frac{110}{513} \\ P_{11} = \frac{175}{513} \end{cases}$$

$57 + 55 = \frac{112}{338} =$

$\frac{46}{560} = \frac{112}{56} \frac{338}{169}$

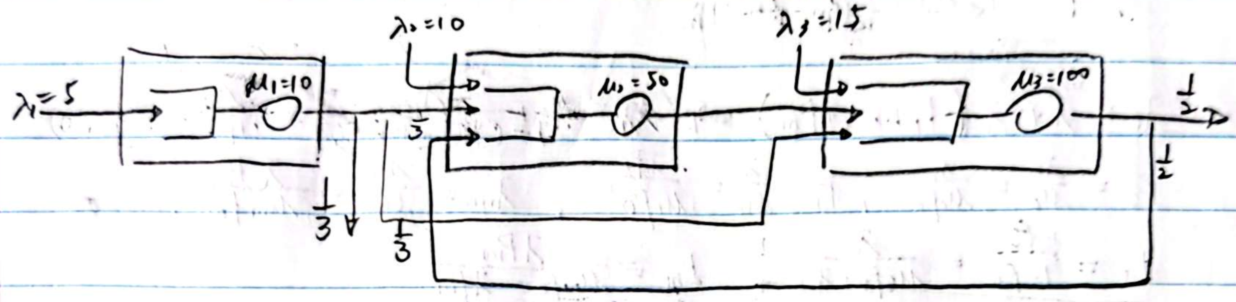
a) $W = \frac{L}{\lambda_a} = \frac{P_{01} + P_{10} + 2 \cdot P_{11}}{\lambda(1 - P_{11})} = \frac{\frac{100}{513} + \frac{110}{513} + \frac{350}{513}}{5 \cdot (1 - \frac{175}{513})} = \frac{56}{169}$ #

b) $P_{01} + P_{11} = \frac{275}{513}$ #

b_1 $\lambda_1 = 5$, $\mu_1 = 10$

$\lambda_2 = 10$ $\quad \mu_2 = 50$

$\lambda_3 = 15$ $\quad \mu_3 = 100$



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a) $\begin{cases} \lambda_1' = \lambda_1 = 5 \\ \lambda_2' = 10 + \frac{1}{3}\lambda_1 + \frac{1}{2}\lambda_3 \Rightarrow \lambda_2' = 10 + \frac{5}{3} + \frac{1}{2}\lambda_3 \Rightarrow \lambda_2' - \frac{1}{2}\lambda_3' = \frac{35}{3} \\ \lambda_3' = 15 + \frac{1}{3}\lambda_1 + \lambda_2 \Rightarrow \lambda_3' = 15 + \frac{5}{3} + \lambda_2 \Rightarrow \lambda_2' - \lambda_3' = -\frac{50}{3} \end{cases}$

$\frac{1}{2}\lambda_3 = \frac{85}{3} \Rightarrow \lambda_3 = \frac{170}{3}$

$\lambda_2 = 40$

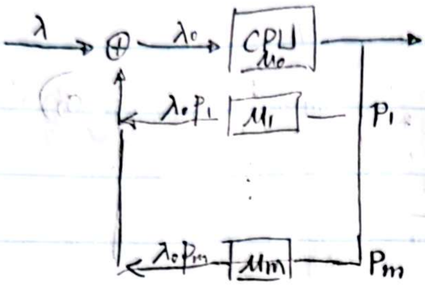
$\Rightarrow \begin{cases} \lambda_1' = 5 \\ \lambda_2' = 40 \\ \lambda_3' = \frac{170}{3} \end{cases} \Rightarrow \begin{cases} \rho_1 = \frac{5}{10} = \frac{1}{2} \\ \rho_2 = \frac{40}{50} = \frac{4}{5} \\ \rho_3 = \frac{170}{300} = \frac{17}{30} \end{cases}$

$\Rightarrow \begin{cases} L_1 = \frac{\rho_1}{1-\rho_1} = 1 \\ L_2 = 4 \\ L_3 = \frac{17}{13} \end{cases}$

$L = L_1 + L_2 + L_3 = \frac{82}{13}$ #

b) $W = \frac{L}{\lambda} = \frac{\frac{82}{13}}{5+10+15} \approx 0.21$ #

7.



$P_{10}=1$, $P_{20}=1$, ...

$P_{01}=P_1$, $P_{02}=P_2$, ...

$\lambda_0 = \lambda + \lambda_1 \cdot 1 + \dots + \lambda_m \cdot 1$

$\lambda_1 = \lambda_0 \cdot P_1$

$\lambda_2 = \lambda_0 \cdot P_2$

$\vdots$

$\lambda_0 = \lambda + \lambda_0 (P_1 + \dots + P_m) \Rightarrow \lambda_0 = \frac{\lambda}{1-(P_1+\dots+P_m)} = \frac{\lambda}{P_0}$

$\rho_0 = \frac{\lambda_0}{\mu_0} = \frac{\lambda}{\mu_0 P_0}$, $\rho_1 = \frac{\lambda_1}{\mu_1} = \frac{\lambda P_1}{\mu_1 P_0}$, $\rho_m = \frac{\lambda_m}{\mu_m} = \frac{\lambda P_m}{\mu_m P_0}$

$L_0 = \frac{\rho_0}{1-\rho_0} = \frac{\lambda}{\mu_0 P_0 - \lambda}$, $L_m = \frac{\lambda P_m}{\mu_m P_0 - \lambda P_m}$

$L = \sum_{i=0}^{m} L_i = \dots$

$W = \frac{L}{\lambda} = \frac{\sum_{i=0}^{m} L_i}{\lambda}$ ※

8. $E[S|U] = 3+4U$, $Var[S|U] = 5$

$E[S] = E[E[S|U]] = E[3+4U] = 3 + 4E[U] = 3 + 4 \cdot \frac{1+0}{2} = 5$

$Var[S] = E[Var(S|U)] + Var(E[S|U])$

$= 5 + Var(3+4U) = 5 + 16\,Var(U) = 5 + \cancel{16}^{4} \frac{1}{\cancel{12}_{3}} = \frac{19}{3}$

$E[S^2] = Var(S) + E[S]^2 = \frac{19}{3} + 25 = \frac{94}{3}$

a) $W = W_Q + E(S) = \frac{\lambda E(S^2)}{2(1-\lambda E(S))} + E(S)$

$= \frac{\lambda \cdot \frac{94}{3}}{2(1-5\lambda)} + 5$ ※

b) $W_Q + E[S|U=x] = \frac{\frac{94\lambda}{3}}{2(1-5\lambda)} + (3+4x)$ ※

9.

a) When $X_n = 0 \Rightarrow X_{n+1} = Y_n$

$\therefore Y_n$ is the number of arrivals between Nth departure and $(n+1)$th departure ※

b) $X_{n+1} = X_n - 1 + Y_n + \delta_n$

$$\Rightarrow E[X_{n+1}] = E[X_n - 1 + Y_n + \delta_n]$$
$$= E[X_n] - 1 + E[Y_n] + E[\delta_n]$$
$$\Rightarrow E[\delta_n] = E[X_{n+1}] - E[X_n] + 1 - E[Y_n]$$
$$\Rightarrow E[\delta_\infty] = \cancel{E[X_\infty] - E[X_\infty]} + 1 - E[Y_\infty]$$
$$= 1 - \lambda E[S]$$ ※

service time

c) $X_{n+1}^2 = (X_n - 1 + Y_n + \delta_n)^2$

$$= X_n^2 + 1 + Y_n^2 + \delta_n^2 - 2X_n + 2X_n Y_n + 2X_n\delta_n - 2Y_n - 2\delta_n + 2Y_n\delta_n$$
$$\Rightarrow E[X_{n+1}^2] = E[X_n^2] + 1 + E[Y_n^2] + E[\delta_n^2] - 2E[X_n] + 2E[X_n]E[Y_n]$$
$$+ 2E[X_n]E[\delta_n] - 2E[Y_n] - 2E[\delta_n] + 2E[Y_n]E[\delta_n]$$
$$\Rightarrow \cancel{E[X_\infty^2]} = \cancel{E[X_\infty^2]} + 1 + E[Y_\infty^2] + E[\delta_\infty^2] - 2E[X_\infty] + 2E[X_\infty]E[Y_\infty]$$
$$+ 2E[X_\infty]E[\delta_\infty] - 2E[Y_\infty] - 2E[\delta_\infty] + 2E[Y_\infty]E[\delta_\infty]$$
$$\Rightarrow 0 = 1 + \lambda^2 E[S^2] + \cancel{1} - 2E[X_\infty] + 2\lambda E[S]E[X_\infty]$$
$$- 2\lambda E[S] - 2(1 - \lambda E[S]) + 2\lambda E[S](1 - \lambda E[S])$$
$$\Rightarrow E[X_\infty] = \frac{\lambda^2 E[S^2]}{2(1 - \lambda E[S])} + \lambda E[S]$$ ※

d) $L = \sum_{n=0}^{\infty} n P(X = n) = E[X_\infty]$ ※