EE 503 : Homework 7

Due : 11/07/2023, Tuesday before class.

1. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov Chain. How many states are needed?

   (Hint: Define a state as what happened in the last three days)

2. Consider a process $\{X_n, n = 0, 1, \dots\}$, which takes on the values $0, 1$, or 2. Suppose

$$P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i + 0\} = \begin{cases} P_{ij}^{I} & \text{when } n \text{ is even} \\ P_{ij}^{II} & \text{when } n \text{ is odd} \end{cases}$$

   where $\sum_{j=0}^{2} P_{ij}^{I} = \sum_{j=0}^{2} P_{ij}^{II} = 1, i = 0, 1, 2$. Is $\{X_n, n = 0, 1, \dots\}$ a time-homogeneous Markov chain? If not, then show how, by enlarging the state space, we may transform it into a time-homogeneous Markov chain.

3. Let the transition probability matrix of a two-state Markov Chain be given by

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

   Show by mathematical induction that

$$P^{(n)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{bmatrix}$$

4. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1? Suppose that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

5. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

$$P_1 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad P_4 = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. Coin 1 comes up heads with probability 0.6 and coin 2 with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

   a) What proportion of flips use coin 1?

   b) If we start the process with coin 1 what is the probability that coin 2 is used on the fifth flip?

7. A transition probability matrix **P** is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \text{f or all } j$$

   If such a chain is irreducible and aperiodic and consists of $M+1$ states $0, 1, \ldots, M$, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \ldots, M$$

8. Let P be the $n * n$ transition matrix of a Markov chain with a finite state space $S = \{1, 2, ..., n\}$. Show that π is the stationary distribution of the Markov chain, i.e., $\pi P = \pi$, $\sum_{i=1}^{n} \pi_i = 1$ if and only if $\pi(I - P + 11^T) = 1^T$ where I is the $n * n$ identity matrix and $1^T = [11...1]$ is a $1 * n$ row vector with all components being 1.

9. **Simulation problem:** An individual possesses r umbrellas that he employs in going from his home to office, and vice versa. If he is at home (the office) in the morning (in the evening) and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains in the morning (in the evening) with probability p. Let X_n be the number of umbrellas at his home at the end of the n^{th} day. Then, X_n defines a Markov chain. Let $r = 3$.

   a) Complete the following transition probability matrix P for this Markov chain. (For example, if $X_n = 0$, then the probability that $X_{n+1} = 0$ is: P_{00} = prob(rain in the morning, no rain in the evening) + prob(no rain in the morning and evening) $= p(1-p) + (1-p)^2$).

$$\mathrm{P} = \begin{pmatrix} p(1-p) + (1-p)^2 & ? & 0 & 0 \\ ? & ? & p(1-p) & 0 \\ 0 & ? & ? & ? \\ 0 & 0 & ? & p^2 + 1 - p \end{pmatrix}$$

b) For $r = 3$, use the formula given in class to confirm that the limiting probabilities are given by:

$$\pi_i = \begin{cases} \frac{q}{r+q}, & \text{if } i = 0 \\ \frac{1}{r+q}, & \text{if } i = 1, \cdots, r \end{cases}$$

where $q = 1 - p$.

c) Let $p = 0.5$. Use Python to plot P_{i2}^n, $i = 0, 1, 2, 3$ for different values of n. Do P_{i2}^n's converge as n grows? If yes, what is the relationship between $\lim_{n\to\infty} P_{i2}^n$, $i = 0, 1, 2, 3$ and π_2?

d) Repeat part c with $p = 0.1$. If P_{i2}^n's converge in both cases, which one converge faster?