EE 503 : Homework 10

Due : 12/05/2023, Tuesday online.

1. Two customers move about among three servers. Upon completion of service at server i, the customer leaves that server and enters service at whichever of the other two servers is free. (Therefore, there are always two busy servers). If the service times at server i are exponential with rate μ_i, $i = 1, 2, 3$, what proportion of time is server i idle?

2. A facility produces items according to a Poisson process with rate λ. However, it has shelf space for only k items and so it shuts down production whenever k items are present. Customers arrive at the facility according to a Poisson process with rate μ. Each customer wants one item and will immediately depart either with the item or empty handed if there is no item available.

   a) Find the proportion of customers that go away empty handed.

   b) Find the average time that the item is on the shelf.

   c) Find the average number of items on the shelf.

   Suppose now that when a customer does not find any available items it joins the "customers' queue" as long as as there are no more than $n-1$ other customers waiting at that time. If there are n waiting customers then the new arrival departs without an item.

   d) Set up the balance equations.

   e) In terms of the solution of the balance equations, what is the average number of customers in the system?

3. Consider a single-server queue with Poisson arrivals and exponential service times having the following variation: Whenever a service is completed a departure occurs only with probability α. With probability $1-\alpha$ the customer, instead of leaving, joins the end of the queue. Note that a customer may be serviced more than once.

   a) Set up the balance equations and solve for the steady-state probabilities, stating conditions for it to exist.

   b) Find the expected waiting time of a customer from the time he arrives until he enters service for the first time.

   c) What is the probability that a customer enters service exactly n times, $n = 1, 2, ...$?

   d) What is the expected amount of time that a customer spends in service (which does not include the time he spends waiting in line)? **Hint:** Use part c)

   e) What is the distribution of the total length of time a customer spends being served? **Hint:** Is it memoryless?

4. Consider a sequential-service system consisting of two servers, A and B. Arriving customers will enter this system only if server A is free. If a customer does enter, then he is immediately served by server A. When his service by A is completed, he then goes to B if B is free, or leaves the system if B is busy. Upon completion of service at server B, the customer departs. Assuming that the (Poisson) arrival rate is two customers an hour, and that A and B serve at respective (exponential) rates of four and two customers an hour,

   a) what proportion of customers enter the system?

   b) what proportion of entering customers receive service from B?

   c) what is the average number of customers in the system?

   d) what is the average amount of time that an entering customer spends in the system?

5. Customers arrive at a two-server system according to a Poisson process having rate $\lambda = 5$. An arrival finding server 1 free will begin service with that server. An arrival finding server 1 busy and server 2 free will enter service with server 2. An arrival finding both servers busy goes away. Once a customer is served by either server, he departs the system. The service times at server t are exponential with rates μ_i, where $\mu_1 = 4$, $\mu_2 = 2$.

   a) What is the average time an entering customer spends in the system?

   b) What proportion of time is server 2 busy?

6. Consider a network of three stations. Customers arrive at stations 1, 2, 3 in accordance with Poisson processes having respective rates 5, 10, 15. The service times at the three stations are exponential with respective rates 10, 50, 100. A customer completing service at station 1 is equally likely to (i) go to station 2, (ii) go to station 3, or (iii) leave the system. A customer departing service at station 2 always goes to station 3. A departure from service at station 3 is equally likely to either go to station 2 or leave the system.

   a) What is the average number of customers in the system (consisting of all three stations)?

   b) What is the average time a customer spends in the system?

7. Consider a model of a computer CPU connected to m I/O devices as shown in Figure 1. Jobs enter the system according to a Poisson process with rate λ, use the CPU, and with probability $p_i, i = 1, \ldots, m$ are routed to the i^{th} I/O device, while with probability p_0 they exit the system. The service time of a job at the CPU (or the i^{th} I/O device) is exponentially distributed with mean $\frac{1}{\mu_0}$ (or $\frac{1}{\mu_i}$, respectively). We assume that all job service times at all queues are independent (including the times of successive visits to the CPU and I/O devices of the same job). Find the occupancy distribution of the system and construct an equivalent system with $m + 1$ queues in tandem that has the same occupancy distribution.

8. Customers arrive at a single-server station in accordance with a Poisson process having rate λ. Each customer has a value. The successive values of customers are independent and come from a uniform distribution on $(0, 1)$. The service time of a customer having value x is a random variable with mean $3 + 4x$ and variance 5.

   a) What is the average time a customer spends in the system?

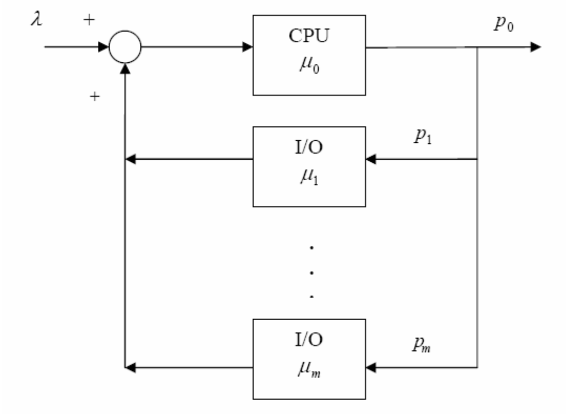

Figure 1: Figure for Problem 7.

b) What is the average time a customer having value x spends in the system?

9. For the $M/G/1$ queue, let X_n denote the number in the system left behind by the n^{th} departure.

a) If

$$X_{n+1} = \begin{cases} X_n - 1 + Y_n, & \text{if } X_n \geq 1 \\ Y_n, & \text{if } X_n = 0 \end{cases}$$

What does Y_n represent?

b) Rewrite the preceding as

$$X_{n+1} = X_n - 1 + Y_n + \delta_n \tag{1}$$

where

$$\delta_n = \begin{cases} 1, & \text{if } X_n = 0 \\ 0, & \text{if } X_n \geq 1 \end{cases}$$

Take expectations and let $n \to \infty$ in (1) to obtain

$$E[\delta_\infty] = 1 - \lambda E[S]$$

c) Square both sides of (1), take expectations and then let $n \to \infty$ to obtain

$$E[X_\infty] = \frac{\lambda E[S^2]}{2(1 - \lambda E[S])} + \lambda E[S]$$

d) Argue that $E[X_\infty]$, the average number as seen by a departure, is equal to L.