EE 503 : Homework 3

Due : 09/19/2023, Tuesday before class.

1. An urn contains $n+m$ balls, of which n are red and m are black. They are drawn from the urn one at a time, without replacement. Let X be the number of red balls removed before the first black ball is chosen. We are interested in determining $E[X]$. To obtain this quantity, number the red balls from 1 to n. Now define the random variables X_i, $i = 1, \cdots n$, by

$$X_i = \begin{cases} 1, & \text{if red ball } i \text{ is taken before any black ball is chosen} \\ 0, & \text{otherwise} \end{cases}$$

   a) Express X in terms of the X_i.
   b) Find $E[X]$.

2. Let X be a uniformly distributed random variable in the interval $[-\pi, \pi]$. What is the cdf of $Y = \tan X$?

3. If $X \sim \mathcal{N}(\mu, \sigma)$, what is the pdf of $Y = (X - \mu)^2/\sigma^2$?

4. Three types of customers arrive at a service station. The times required to service type 1 and type 2 customers are exponential random variables with respective means 1 and 10 seconds. Type 3 customers require a constant service time of 2 seconds. Suppose that the proportion of type 1, 2 and 3 customers is 1/2, 1/8 and 3/8, respectively. Find the probability that an arbitrary customer requires more than 15 seconds of service time.

5. The average score in the final exam of a course is 65 and the standard deviation is 10.

   a) Give an upper bound on the probability of a student scoring more than 95?
   b) Suppose the scores follow a normal distribution. Compute the probability of a student scoring more than 95 and compare it to the bound obtained in a).

6. The number X of electrons counted by a receiver in an optical communication system is a Poisson random variable with rate λ_1 when a signal is present and with rate $\lambda_0 < \lambda_1$ when a signal is absent. Suppose that a signal is present with probability p.

   a) Find $P[\text{signal present}|X = k]$ and $P[\text{signal absent}|X = k]$.
   b) The receiver uses the following decision rule:
      If $P[\text{signal present}|X = k] > P[\text{signal absent}|X = k]$, decide signal present; otherwise decide signal absent
      Show that this decision rule leads to the following threshold rule: If $X > T$, decide signal present; otherwise, decide signal absent.

c) What is the probability of error for the above decision rule?

7. **Exponential Random Variable:**

    a) Generate instances of exponential random distribution from a uniform distributed random variable, random.uniform(0,1).

    b) Use the built-in function random.exponential() to generate the same number of instances.

    c) Compare the histograms of a and b.