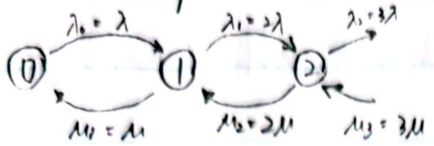
EE503 HW9 Chih-Cheng Hsieh

1.



$$E(T_0) = \frac{1}{\lambda}$$

$$E(T_1) = \frac{1}{2\lambda} + \frac{\mu}{2\lambda}E(T_0) = \frac{1}{2\lambda} + \frac{\mu}{2\lambda}\frac{1}{\lambda} = \frac{1}{2\lambda}\left(1 + \frac{\mu}{\lambda}\right)$$

$$E(T_2) = \frac{1}{3\lambda} + \frac{2\mu}{3\lambda}E(T_1) = \frac{1}{3\lambda} + \frac{\mu}{3\lambda^2} + \frac{\mu^2}{3\lambda^3} = \frac{1}{3\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2\right)$$

$$E(T_3) = \frac{1}{4\lambda} + \frac{3\mu}{4\lambda}E(T_2) = \frac{1}{4\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3\right)$$

$$E(T_4) = \frac{1}{5\lambda} + \frac{4\mu}{5\lambda}E(T_3) = \frac{1}{5\lambda}\left(1 + \frac{\mu}{\lambda} + \left(\frac{\mu}{\lambda}\right)^2 + \left(\frac{\mu}{\lambda}\right)^3 + \left(\frac{\mu}{\lambda}\right)^4\right)$$

a) $E[0 \text{ to } 4] = E(T_0) + E(T_1) + E(T_2) + E(T_3)$ ※

b) $E[2 \text{ to } 5] = E(T_2) + E(T_3) + E(T_4)$ ※

2.

a) $\begin{cases} \mu_n = n\mu, & n \geq 1 \\ \lambda_n = n\lambda + \theta, \ n < N; & \lambda_n = n\lambda, \ n \geq N \end{cases}$

Let $X(t)$: population at time t

Let $M(t) = E[X(t)]$

$$M(t+h) = E[X(t+h)] = E[X(t+h) \mid X(t)]$$

$$X(t+h) = \begin{cases} X(t)+1, & \text{w.p. } (X(t)\lambda + \theta)h + o(h) \\ X(t)-1, & \text{w.p. } (X(t)\mu)h + o(h) \\ X(t), & \text{w.p. } 1 - [X(t)\lambda + \theta + X(t)\mu]h + o(h) \end{cases}$$

$$E[X(t+h) \mid X(t)] = (X(t)+1)((X(t)\lambda+\theta)h) + (X(t)-1)(X(t)\mu h) + X(t)(1 - [X(t)\lambda + \theta + X(t)\mu]h)$$

$$= (X(t)\lambda + \theta)h - X(t)\mu h + X(t) + o(h)$$

$$= X(t) + (X(t)\lambda + \theta - X(t)\mu)h + o(h)$$

b) Find $\sum_{i=3}^{\infty} P_i$, $N=3$, $\lambda=\theta=1$, $\mu=2$

$\lambda_k P_k = \mu_{k+1} P_{k+1}$

$P_1 = \frac{\theta}{\mu} P_0$

$P_2 = \frac{\lambda+\theta}{2\mu} P_1 = \frac{\theta(\lambda+\theta)}{2\mu^2} P_0$

$P_3 = \frac{2\lambda+\theta}{2\mu} P_2 = \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{4\mu^3} P_0$

For $k \geq 4$, $P_k = \frac{(k-1)\lambda}{k\mu} P_{k-1} = \frac{(k-1)(k-2)\cdots 4\cdot 3}{k(k-1)\cdots 5\cdot 4}\left(\frac{\lambda}{\mu}\right)^{k-3} = \frac{3}{k}\left(\frac{\lambda}{\mu}\right)^{k-3} P_3$

$\Rightarrow \sum_{k=3}^{\infty} P_k = \sum_{k=3}^{\infty} \frac{3}{k}\left(\frac{\lambda}{\mu}\right)^{k-3} P_3 = \left(\frac{\mu}{\lambda}\right)^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left(\frac{\lambda}{\mu}\right)^k$

$\because \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{\lambda}{\mu}\right)^k = \log\left(\frac{1}{1-\frac{\lambda}{\mu}}\right) = \log\left(\frac{\mu}{\mu-\lambda}\right)$, if $\frac{\lambda}{\mu} < 1$

$\therefore \sum_{k=3}^{\infty} \frac{1}{k}\left(\frac{\lambda}{\mu}\right)^k = \log\left(\frac{\mu}{\mu-\lambda}\right) - \frac{\lambda}{\mu} - \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 = \log 2 - \frac{1}{2} - \frac{1}{8} = 0.0681$

$\frac{1\cdot 2\cdot 3}{32}$

$\because \sum_{k=0}^{\infty} P_i = 1$

$\Rightarrow P_0 + \frac{\theta}{\mu} P_0 + \frac{\theta(\lambda+\theta)}{2\mu^2} P_0 + \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{4\mu^3} P_0 + \left(\frac{\mu}{\lambda}\right)^3 \frac{\theta(\lambda+\theta)(2\lambda+\theta)}{4\lambda^3} P_0 \left(\log\left(\frac{\mu}{\mu-\lambda}\right) - \frac{\lambda}{\mu} - \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2\right) = 1$

$\Rightarrow P_0 = \left(1 + \frac{1}{2} + \frac{3}{8} + \frac{3}{16} + \frac{3}{2}\left(\log 2 - \frac{1}{2} - \frac{1}{8}\right)\right)^{-1} = 2.18$

$\sum_{k=3}^{\infty} P_k = \left(\frac{\mu}{\lambda}\right)^3 P_3 \sum_{k=3}^{\infty} \frac{1}{k}\left(\frac{\lambda}{\mu}\right)^k = 8 \cdot \frac{3}{16} \cdot 2.18 \cdot 0.0681 = 0.2227$

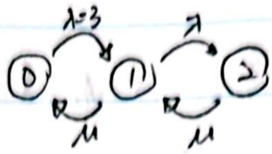
#

$3P_0 = 4P_1 \Rightarrow P_1 = \frac{3}{4}P_0$

$4P_2 = 3P_1 \Rightarrow P_2 = \frac{3}{4}P_1$

$\frac{12}{16}$ $\quad \frac{21}{16} / 37$

3. $\lambda = 3, \quad \mu = 4$



$\because \sum_{i=0}^{2} P_i = 1$

$\Rightarrow P_0 + \frac{3}{4}P_0 + \frac{9}{16}P_0 = 1 \Rightarrow P_0 = \frac{16}{37}$

$P_1 = \frac{3}{4} \cdot \frac{16}{37} = \frac{12}{37}$

$P_2 = \frac{9}{16} \cdot \frac{16}{37} = \frac{9}{37}$

a) $E(\text{customers}) = \sum_{n=0}^{2} n P_n = 0 \cdot \frac{16}{37} + 1 \cdot \frac{12}{37} + 2 \cdot \frac{9}{37} = \frac{30}{37}$ #

b) $P_0 + P_1 = \frac{28}{37}$ ✗

c) $\lambda = 3, \quad \mu = 8$ $\qquad \frac{24}{64} \quad \frac{64}{33} / 97$

$P_0 = \frac{1}{1 + \frac{3}{8} + \frac{9}{64}} = \frac{64}{97}$

$P_1 = \frac{3}{8} \cdot \frac{64}{97} = \frac{24}{97}$

$P_2 = \frac{9}{64} \cdot \frac{64}{97} = \frac{9}{97}$

$P_0 + P_1 = \frac{88}{97}$

rate of customers add $= \lambda \left(\overset{0.907}{\frac{88}{97}} - \overset{0.757}{\frac{28}{37}} \right) = 3 \times 0.15 = 0.45$ #

4. $\lambda = 3$, $\mu = 2 \times 2 = 4$

a) $P(0) = \left(\frac{\lambda}{\mu}\right)^0 \left(1 - \frac{\lambda}{\mu}\right) = 1 - \frac{3}{4} = \frac{1}{4}$

$P(1) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$

$P(2) = \frac{3}{4} \cdot \frac{3}{16} = \frac{9}{64}$

16
12
9

$P(0) + P(1) + P(2) = \frac{1}{4} + \frac{3}{16} + \frac{9}{64} = \frac{37}{64}$ #

b) $\lambda = 3$, $\mu = 4$

$\Rightarrow P(0) + P(1) + P(2) = \frac{37}{64}$ #

5. $\mu_1 P_1 = p\lambda P_0 \Rightarrow P_1 = \frac{p\lambda}{\mu_1} P_0$

$\mu_2 P_2 = (1-p)\lambda P_0 \Rightarrow P_2 = \frac{(1-p)\lambda}{\mu_2} P_0$

$\because \sum_i P_i = 1$

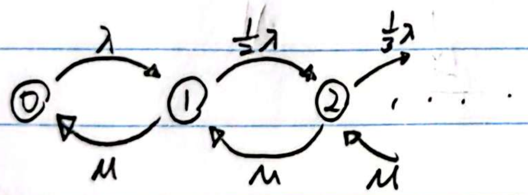
$\Rightarrow P_0 + P_1 + P_2 = P_0 \left(1 + \frac{p\lambda}{\mu_1} + \frac{(1-p)\lambda}{\mu_2}\right) = 1$

$P_0 = \frac{\mu_1 \mu_2}{\mu_1 \mu_2 + \mu_2 p\lambda + \mu_1 (1-p)\lambda}$ #

$P_1 = \frac{p\lambda \mu_2}{\mu_1 \mu_2 + \mu_2 p\lambda + \mu_1 (1-p)\lambda}$ #

$P_2 = \frac{\mu_1 (1-p)\lambda}{\mu_1 \mu_2 + \mu_2 p\lambda + \mu_1 (1-p)\lambda}$ #

6.



$\lambda P_0 = \mu P_1 \Rightarrow P_1 = \frac{\lambda}{\mu} P_0$

$\frac{1}{2}\lambda P_1 = \mu P_2 \Rightarrow P_2 = \frac{\lambda}{2\mu} P_1 = \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 P_0$

$\frac{1}{3}\lambda P_2 = \mu P_3 \Rightarrow P_3 = \frac{\lambda}{3\mu} P_2 = \frac{1}{6}\left(\frac{\lambda}{\mu}\right)^3 P_0$

$\vdots$

$\Rightarrow P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$

$\sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$

$= P_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n$

$= P_0 e^{\frac{\lambda}{\mu}} = 1$

$P_0 = e^{-\frac{\lambda}{\mu}}$

$\Rightarrow P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\lambda}{\mu}}$

1.

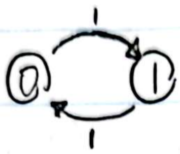
i) $P_0 = 10P_1 \Rightarrow P_1 = \frac{1}{10}P_0$

$P_0 + P_1 = 1 \Rightarrow P_0 + \frac{1}{10}P_0 = 1 \Rightarrow \begin{cases} P_0 = \frac{10}{11} \\ P_1 = \frac{1}{11} \end{cases}$ #

ii) $Q = \begin{bmatrix} -1 & 1 \\ 10 & -10 \end{bmatrix}$ $\quad P = \begin{bmatrix} \frac{10}{11} & \frac{1}{11} \end{bmatrix}$

$PQ = -\frac{10}{11} + \frac{10}{11} + \frac{10}{11} - \frac{10}{11} = 0$ #

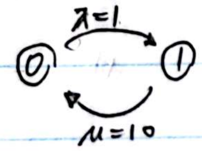
iii) $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



iv) $\begin{cases} \Pi = \Pi P \\ \sum \Pi = 1 \end{cases} \Rightarrow \begin{cases} \Pi_0 = \Pi_1 \\ \Pi_0 + \Pi_1 = 1 \end{cases} \Rightarrow \Pi_0 = \Pi_1 = \frac{1}{2}$ #

V) Yes, we can reconstruct CTMC because we know stationary dist. and relationship between state 0 and state I #

vi) from q_{ij} I know $\lambda = 1$, $\mu = 10$



#

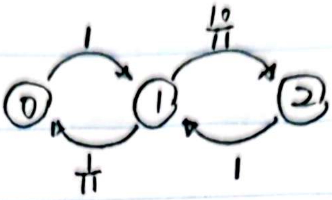
vii) $P_0 = 10P_1 \Rightarrow P_1 = \frac{1}{10}P_0$

$P_1 = 10P_2 \Rightarrow P_2 = \frac{1}{10}P_1 = \frac{1}{100}P_0$

$P_0 + \frac{1}{10}P_0 + \frac{1}{100}P_0 = 1 \Rightarrow \begin{cases} P_0 = \frac{100}{111} \\ P_1 = \frac{1}{10}\frac{100}{111} = \frac{10}{111} \\ P_2 = \frac{1}{100}\frac{100}{111} = \frac{1}{111} \end{cases}$ #

$$Q=\begin{bmatrix} -1 & 1 & 0 \\ 1 & -11 & 10 \\ 0 & 10 & -10 \end{bmatrix}, \quad P=\begin{bmatrix} \frac{100}{111} & \frac{10}{111} & \frac{1}{111} \end{bmatrix}$$ #

$$R=\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{11} & 0 & \frac{10}{11} \\ 0 & 1 & 0 \end{bmatrix}$$



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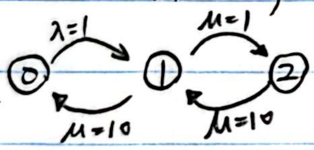
$$\begin{cases} \pi_0 = \frac{1}{11}\pi_1 \\ \pi_1 = \pi_0 + \pi_2 \\ \pi_2 = \frac{10}{11}\pi_1 \end{cases} \Rightarrow \frac{1}{11}\pi_1 + \pi_1 + \frac{10}{11}\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{2}$$

$$\begin{cases} \pi_0 = \frac{1}{22} \\ \pi_1 = \frac{1}{2} \\ \pi_2 = \frac{5}{11} \end{cases}$$ #

We can't reconstruct CTMC only by P_{ij}, we don't know the relationship between state 0 and state 2 directly #

from $q_{ij} \Rightarrow \lambda = 1, \ \mu = 10$



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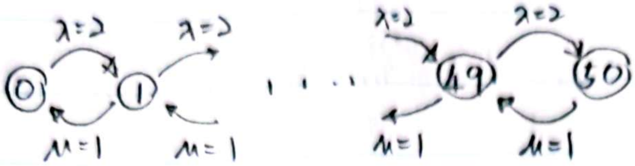
8.

a)

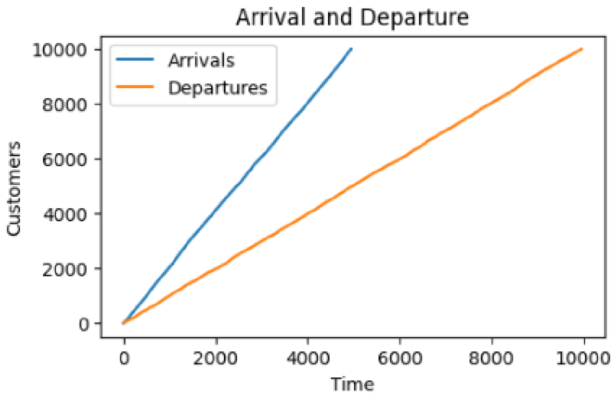


b)

(b)

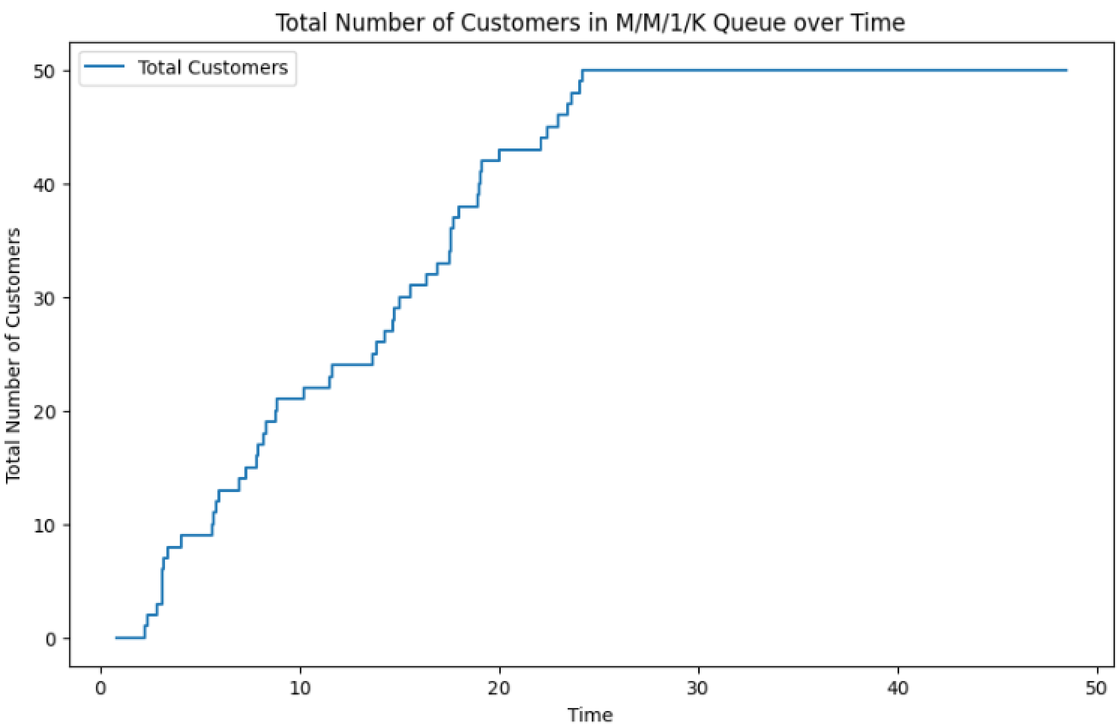
```
[87] def simulate_MM1K_queue(num_transitions, arrival_rate, service_rate, max_queue_length):
        arrivals = np.random.exponential(1/arrival_rate, num_transitions)
        departures = np.random.exponential(1/service_rate, num_transitions)

        return arrivals, departures
```

```
import numpy as np
import matplotlib.pyplot as plt

num_transitions = 10000
arrival_rate = 2
service_rate = 1
max_queue_length = 50

arrivals, departures = simulate_MM1K_queue(num_transitions, arrival_rate, service_rate, max_queue_length)
arrival_times = np.cumsum(arrivals)
departure_times = np.cumsum(departures)

plt.figure(figsize=(5, 3))

plt.step(arrival_times, range(1, num_transitions + 1), label='Arrivals', where='post')
plt.step(departure_times, range(1, num_transitions + 1), label='Departures', where='post')

plt.xlabel('Time')
plt.ylabel('Customers')
plt.title('Arrival and Departure')
plt.legend()
plt.show()
```



(c)

```python
import numpy as np
import matplotlib.pyplot as plt

num_transitions = 100
arrival_rate = 2
service_rate = 1
max_queue_length = 50

arrivals, departures = simulate_MM1K_queue(num_transitions, arrival_rate, service_rate, max_queue_length)

arrival_times = np.cumsum(arrivals)
departure_times = np.cumsum(departures)

total_customers = np.zeros(num_transitions)

state = 0
for i in range(num_transitions):
    total_customers[i] = state

    if arrival_times[i] < departure_times[i]:
        if state < max_queue_length:
            state += 1
    else:
        if state > 0:
            state -= 1

plt.figure(figsize=(10, 6))

plt.plot(arrival_times, total_customers, label='Total Customers', drawstyle='steps-post')

plt.xlabel('Time')
plt.ylabel('Total Number of Customers')
plt.title('Total Number of Customers in M/M/1/K Queue over Time')
plt.legend()
plt.show()
```



(d)

```python
import numpy as np

num_transitions_list = [100, 1000, 10000]
arrival_rate = 2
service_rate = 1
max_queue_length = 50

for num in num_transitions_list:
    arrivals, departures = simulate_MM1K_queue(num, arrival_rate, service_rate, max_queue_length)

    state = 0
    for i in range(num):

        if arrivals[i] < departures[i]:
            if state < max_queue_length:
                state += 1
        else:
            if state > 0:
                state -= 1
    print(f'Afer {num} times of transitions: {state}')
```

```
Afer 100 times of transitions: 26
Afer 1000 times of transitions: 49
Afer 10000 times of transitions: 49
```

(e) Result of 1000 and 10000 times are closest of the theoretical results.

```python
import numpy as np

num_transitions_list = 10000
arrival_rate = 2
service_rate = 1
max_queue_length = 50

arrivals, departures = simulate_MM1K_queue(num, arrival_rate, service_rate, max_queue_length)

state = 0
queue_lengths = [state]

for i in range(num):
    if arrivals[i] < departures[i]:
        if state < max_queue_length:
            state += 1
    else:
        if state > 0:
            state -= 1
    queue_lengths.append(state)

plt.figure(figsize=(5, 3))
plt.hist(queue_lengths, bins=np.arange(0, max(queue_lengths)+2)-0.5, density=True, align='mid')
plt.xlabel('Customers in System')
plt.ylabel('Probability')
plt.show()
```