EE 503 Midterm Exam, Fall 2020

Professor: Konstantinos Psounis

Student name:________________________

Instructions:

1. Please be fair with your fellow students: Do not turn to the next page before everybody in the room has a copy of the exam. Also, when the Professor/TA declares that time is up, stop writing. Don't keep on writing while the Professor/TA collects the exams from other students.

2. You can ask the Professor/TA for clarifications if you feel that there is an ambiguity in the problem statement. But do not ask them for hints.

3. Exam is open "everything", but you cannot bring in devices with wireless capabilities.

4. You can directly use any formula derived in the class, i.e. that can be found in the lecture notes. Also, you can use any "well-known" formula that we didn't derive in class, but please cite your source, e.g. a book, in this case.

5. Use the space provided for your answers. If needed, you can use extra sheets of paper.

Problem 1 (25 points): Investing

A model for the movement of a financial index (a quantity whose value changes over time) assumes that, given the present value V_n of the index at time slot n, $n = 0, 1, 2, \ldots$, after one time slot it will be $V_{n+1} = U \cdot V_n$ with probability p or $V_{n+1} = D \cdot V_n$ with probability $1 - p$. Assume that successive movements of the value are independent, $U = 1.01$, $D = 0.99$, $p = 0.51$, and V_0 is known. (Note: Even though I am giving you specific values for quantitates, you do not need to make numerical computations once you express things as a function of those numerical values.)

(i) (4 points) Let X_n be a random variable dictating whether the index will go up or down after time slot n, that is, $V_{n+1} = X_n \cdot V_n$. Express X_n as a function of U and V and compute its mean and variance.

(ii) (5 points) Write the value of the index at time n, V_n, as a function of X_i's, $i = 0, 1, 2, \ldots, n$ and the starting value V_0.

(iii) (10 points) Approximate the probability that the index's value will be up at least 10% after the first $1,000$ time slots. Hint: Recall that $\log(\prod_i X_i) = \sum_i \log(X_i)$. Think whether the random variables $\log(X_i)$, $i = 0, 1, \ldots$ are i.i.d. and work from there. (Recall it is not required to work out the final numerical solution.)

(iv) (6 points) Let the above probability be P_{up} and let P_{down} be the probability that the index's value will be down at least 10% after the first $1,000$ time slots. Which one is larger, P_{up} or P_{down}?

Solution (You can use the back of the page too.)

Problem 2 (25 points): Random Variables

Let X and N be independent random variables. X is defined as follows:

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{3} \\ 0 & \text{with probability } \frac{1}{3} \\ -1 & \text{with probability } \frac{1}{3} \end{cases}$$

and N is Gaussian $\mathcal{N}(0, \sigma^2)$, with CDF $F_N(n)$ and PDF $f_N(n) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{-\frac{n^2}{2\sigma^2}}$.
Define $Y = X + N$.
Note: You can give your answers below as a function of $F_N(\cdot)$ and $f_N(\cdot)$ and if integrals are involved you don't have to compute them.

(i) (6 points) Find the CDF $F_Y(y)$ and PDF $f_Y(y)$.

(ii) (5 points) Find the conditional CDF $F_{Y|X=1}(y)$ and the conditional PDF $f_{Y|X=1}(y)$.

(iii) (5 points) Find $P(X = 1|Y \geq 1)$.

(iv) (6 points) Find $P\left((X + Y)^2 \leq \frac{1}{2}\right)$.

(v) (3 points) Find $E(Y)$ and $Var(Y)$.

Solution (You can use the back of the page too.)

Problem 3 (25 points): Inequalities

Label each of the following statements with $=, \leq, \geq$, or NONE. Label a statement with $=$ if equality always holds, with $\leq$ or $\geq$ if this inequality or strict inequality holds, and with NONE if none of the above holds. Provide justification for your answers in order to get credit.

(i) (5 points) $P(X^4 < 2\mathbb{E}[X^4])$ vs $\frac{1}{3}$.

(ii) (5 points) $\rho_{X_1X_2}$ vs $\rho_{X_1X_3}$ if $\underline{X} = [X_1, X_2, X_3]^T$ is a Gaussian random vector with covariance matrix $\Sigma_{\underline{X}} = \begin{bmatrix} 1 & 0.1 & 0.2 \\ 0.1 & 2 & 0.3 \\ 0.2 & 0.3 & 4 \end{bmatrix}$.

(iii) (5 points) $f_{XY|Z}(x, y|z)$ vs. $f_{X|Z}(x|z)f_{Y|Z}(y|z)$ if X and Y are independent.

(iv) (5 points) $P(Y \leq \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i-1)}{n})$ vs $1/4$ if $X_i, i = 1, 2, \ldots$ are i.i.d with distribution $U[0, 4]$ and $Y \sim \mathcal{N}(1, 2)$.

(v) (5 points) $P(\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i}{\sqrt{n}} \leq 2)$ vs $P(Y \leq 1)$, if $X_i, i = 1, 2, \ldots$ are i.i.d with $E[X_i] = 2$, $Var[X_i] = 4$, and $Y \sim \mathcal{N}(2, 4)$.

Solution (You can use the back of the page too.)

Problem 4 (25 points): Urns with balls

Consider an urn containing 6 red balls and 4 blue balls. Two players A and B play a game as follows: in round 1, player A picks 3 balls at random from the urn, notes the colors, and return them to the urn. In round 2, player B picks 3 balls at random, notes the colors, and then return them back to the urn, and so on. A's objective is to obtain 3 red balls, and B's objective is to obtain 2 red balls and 1 blue ball. If a player reaches his or her objective at one round, the game ends (therefore, the game can end in the first, second, third, etc. round).

(i) (6 points) Find the probability that A picks a winning set of balls at a round. Find the probability that B picks a winning set of balls at a round.

(ii) (7 points) Find the probability that A wins the game, that is, he is the first to win a round. Hint: for this and the next question you may want to derive a recursive formula.

(iii) (6 points) Find the expected number of rounds played until one player wins.

(iv) (6 points) Assume that the game changes as follows: at each round, player A picks 1 ball and notes the color, then, without returning the ball back to the urn, player B picks a ball and notes the color, then without returning the ball, player A picks another ball, and so on until both players have picked 3 balls. Assume the players have the same objective as before. Similar to (i) above, find the probability that A and the probability that B pick a winning set of balls at a round. Hint: We are only concerned about a single round. Don't be shy to dive into combinatorics for this one.

Solution (You can use the back of the page too.)