EE503 HW8 Chih-Cheng Hsieh

1.

a) MC is irreductable means every states communicate and chain is finite means j can be reach in a finite number of steps

$\therefore P_{ij} = 1$

b) $x_i = P(\text{visit } N \text{ before } 0 \mid \text{start in } i)$

$= P_{iN} \cdot 1 + P_{i(N-1)} \cdot x_{N-1} + \cdots + P_{i1} \cdot x_1 + P_{i0} \cdot 0$

$= P_{iN} + P_{i(N-1)} \cdot x_{n-1} + \cdots + P_{i1} x_1$ #

c) $x_i = P_{iN} + P_{i(N-1)} \cdot x_{n-1} + \cdots + P_{i1} x_1$

$= P_{iN} \cdot \frac{N}{N} + P_{i(N-1)} \cdot \frac{N-1}{N} + \cdots + P_{i1} \cdot \frac{1}{N}$

$= \frac{1}{N} \cdot \left(\sum_j j P_{ij} \right) = \frac{1}{N} \cdot i = \frac{i}{N}$

2.

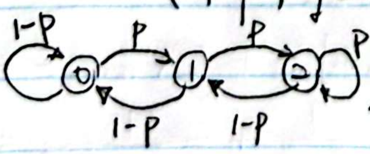
a) Yes, $\{X_n, n \geq 0\}$ is a Markov chain. The number of white balls in the urn next stage (future) depends only on the current state and is independent if the past states.

b) The only class of $\{X_n, n \geq 0\}$ is communication class.

The period is 2.

The chain is recurrent

c) $P_{ij} = \begin{cases} P, & j = i+1 \\ 1-P, & j = i-1 \end{cases}$

d) 

$\begin{cases} \Pi_0 = (1-P)\Pi_0 + (1-P)\Pi_1 \\ \Pi_1 = P\Pi_0 + (1-p)\Pi_2 \\ \Pi_2 = P\Pi_1 + p\Pi_2 \end{cases}$

e) $\Pi_0 = \frac{1}{3}$

$\Pi_1 = \frac{1}{3}$

$\Pi_2 = \frac{1}{3}$

f) $\sum_{j=0}^{2} \Pi_j = \Pi_0 + \Pi_1 + \Pi_2 = (1-p)\Pi_0 + (1-p)\Pi_1 + p\Pi_0 + (1-p)\Pi_2 + p\Pi_1 + p\Pi_2$

$= (1-p+1-p+p+1-p+p+p)\cdot\frac{1}{3} = 3\cdot\frac{1}{3} = 1$

#

g) $0 \xrightarrow{p} \cdots \xrightarrow{p} i \xrightarrow{p} \cdots \xrightarrow{p} N$

$S_{iN} = (N-i)p = N-i$ #

3. $P(A_1) = P(X > Y_i) = e^{-\lambda Y_1}$

$P(A_2 | A_1) = e^{-\lambda Y_2}$

$\vdots$

$P(A_n | A_1 \cdots A_{n-1}) = e^{-\lambda Y_n}$

$P = P(A_1)P(A_2|A_1)\cdots = e^{-\lambda Y_1} e^{-\lambda Y_2} \cdots e^{-\lambda Y_n}$

$= e^{-\lambda(Y_1 + \cdots + Y_n)}$ #

4.

a) $E[S_4] = \frac{1}{\lambda}\cdot 4 = \frac{4}{\lambda}$

b) $E(S_4 | N(1)=2) = E(S_2 | N(1)=2) + E(S_1)\times 2 = \int_0^1 \frac{2!}{1} y\,dy + \frac{2}{\lambda} = 1 + \frac{2}{\lambda}$

(c) $E[N(4) - N(2) | N(1) = 3] = (4-2)\lambda = 2\lambda$

5. $p_i = p\,P_{i+1} + q\,P_{i-1} = \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^k}$

$p = P(X = i+1) = \dfrac{e^{-\lambda_1}\lambda_1^{i+1}}{(i+1)!}$

$q = P(X = i-1) = \dfrac{e^{-\lambda_2}\lambda_2^{i-1}}{(i-1)!}$ #

$P(T_1 = a+k \mid T_2 = a)$

$= \dfrac{P(T_1 = a+k \cap T_2 = a)}{P(T_2 = a)}$

or $= \dfrac{P(T_1 = a+k)\,P(T_2 = a)}{P(T_2 = a)}$

$= \dfrac{e^{-\lambda_1}\lambda_1^{a+k}}{(a+k)!}$ #

6.

a) The dist. is sum of exp. r.v. with rate λ

$\therefore P(T \mid N=n) \sim \exp(p\lambda)$

b) $P_{N \mid T=t}(k) = \dfrac{e^{-\lambda(1-p)t}(\lambda(1-p)t)^k}{k!} \sim \text{Poisson}(\lambda(1-p)t)$

7.

a) $P(X_1 < X_2 < X_3) = P(X_1 = \min(X_1, X_2, X_3)) \times P(X_2 < X_3 \mid X_1 = \min(X_1, X_2, X_3))$

$= P(X_1 = \min(X_1, X_2, X_3)) \times P(X_2 < X_3) = \dfrac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \dfrac{\lambda_2}{\lambda_2+\lambda_3}$ #

b) $P(X_1 < X_2 \mid \max(X_1, X_2, X_3) = X_3) = \dfrac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)}$

$= \dfrac{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3}}{\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_2}{\lambda_2+\lambda_3} + \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \cdot \frac{\lambda_1}{\lambda_1+\lambda_3}} = \dfrac{\lambda_1+\lambda_3}{\lambda_1+\lambda_2+2\lambda_3}$ #

c) $E[\max X_i \mid X_1 < X_2 < X_3]$

$= E[X_1 + (X_2 - X_1) + (X_3 - X_2) \mid X_1 < X_2 < X_3]$

$= E[X_1 \mid X_1 < X_2 < X_3] + E[X_2 - X_1 \mid X_1 < X_2 < X_3] + E[X_3 - X_2 \mid X_1 < X_2 < X_3]$

$= E[X_i \mid X_1 < X_2 < X_3] + E[X_2 \mid X_2 < X_3] + E[X_3]$

$= E[\min X_i \mid X_1 < X_2 < X_3] + E[\min\{X_2, X_3\} \mid X_2 < X_3] + E[X_3]$

$= E[\min X_i] + E[\min\{X_2, X_3\}] + E[X_3]$

$= \dfrac{1}{\lambda_1+\lambda_2+\lambda_3} + \dfrac{1}{\lambda_2+\lambda_3} + \dfrac{1}{\lambda_3}$

#

d) $E(\max X_i) = \sum_{i \neq j \neq k} E(\max X_i \mid X_i < X_j < X_k)\, P(X_i < X_j < X_k)$

$$= \sum_{i \neq j \neq k} \left[\left(\frac{1}{\lambda_i + \lambda_j + \lambda_k} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k} \right) \frac{1}{\lambda_i + \lambda_j + \lambda_k} \quad \frac{\lambda_j}{\lambda_j + \lambda_k} \right]$$

#

8.

(a)(b)(c)

```python
import numpy as np

lambda_rate = 0.25
num_shocks = 100

expected_time = num_shocks / lambda_rate
print(f"a) The expected time is: {expected_time} hours")

t1 = 0
shocks = 0

while shocks < num_shocks:
    t1 += np.random.exponential(1/lambda_rate)
    shocks += 1

print(f"c) The value t1 representing one instance of the Poisson process until the 100th shock occurs is: {t1:.2f} hours")
```

```
a) The expected time is: 400.0 hours
c) The value t1 representing one instance of the Poisson process until the 100th shock occurs is: 354.15 hours
```

(d)

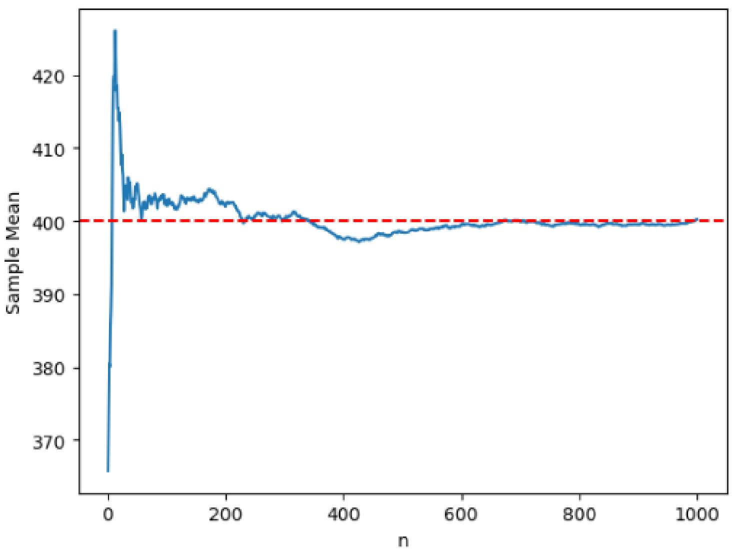
```python
import matplotlib.pyplot as plt

lambda_rate = 0.25
num_shocks = 100
expected_time = num_shocks / lambda_rate

n_values = range(1, 1001)
sample_means = []
t1 = 0
shocks = 0

for n in n_values:
        while shocks < num_shocks * n:
            t1 += np.random.exponential(1/lambda_rate)
            shocks += 1
        sample_means.append(t1/n)

# 繪製圖表
plt.plot(n_values, sample_means)
plt.axhline(expected_time, color='r', linestyle='--', label='Expected Time')
plt.xlabel('n')
plt.ylabel('Sample Mean')
plt.show()
```



(e)

```
count = 0
i = 0
while count < 50:
    if(np.abs(sample_means[i] - expected_time) < expected_time* 0.02):
        count = count + 1
    else: count = 0
    i = i + 1

print(f"e)  minimum value of n is: {i - 50}")
```

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e) minimum value of n is: 24
```