EE503 HW5 Chih-Cheng Hsieh

1. $P(X=k \mid Y=3)$ $\quad 3W \quad X$

$= \dfrac{P(X=x, Y=3)}{P(Y=3)}$ $\quad C_6^{14}$ $\quad 6R$

$5B \quad Y$

$P(Y=3) = C_3^5 C_3^9 \,/\, C_6^{14}$

$P(X=k, Y=3) = C_k^3 C_{3-k}^6 C_3^5 \,/\, C_6^{14}$

$\Rightarrow P(X=k \mid Y=3) = \dfrac{C_k^3 C_{3-k}^6 \cancel{C_3^5}}{\cancel{C_3^5} C_3^9} = \dfrac{C_k^3 C_{3-k}^6}{C_3^9}$ #

$E[X=k \mid Y=1] = \sum_{k=0} k\, P(X=k \mid Y=1)$

$= \dfrac{0 + 1\cdot C_1^3 C_4^6 + 2\cdot C_2^3 C_3^6 + 3\cdot C_3^3 C_2^6}{C_5^9}$

$= \dfrac{0+45+120+45}{126} = \dfrac{210}{126} = \dfrac{5}{3}$ #

2. $\begin{cases} E[E[N_k \mid N_{k-1}]] = E[N_k] \\ E[N_k \mid N_{k-1}=n_{k-1}] = \sum N_k P(N_k \mid N_{k-1}=n_{k-1}) \end{cases}$

$= (n_{k-1}+1)\cdot \frac{1}{m} + (n_{k-1} + E[N_k])(1-\frac{1}{m})$

$= n_{k-1} + \frac{1}{m} + (1-\frac{1}{m}) E[N_k]$

$\Rightarrow E[N_k] = E[n_{k-1} + \frac{1}{m} + (1-\frac{1}{m}) E[N_k]]$

$= E[N_{k-1}] + \frac{1}{m} + (1-\frac{1}{m}) E[N_k]$

$\Rightarrow \frac{1}{m} E[N_k] = E[N_{k-1}] + \frac{1}{m}$

$\Rightarrow E[N_k] = m E[N_{k-1}] + 1$

$= m(m E[N_{k-2}] + 1) + 1$

$= m^2 E[N_{k-2}] + m + 1$

$\vdots$

$= m^{k-1} + m^{k-2} + \dots + m + 1 = \dfrac{m^k - 1}{m-1}$ #

$\frac{\overset{15}{\cancel{30}}}{2}$ $\quad \frac{\cancel{6}\,5\,4}{\cancel{2}\,\cancel{3}}$ $\quad \frac{6\,5}{2}$ $\quad \frac{9\,8\,7\,6}{\cancel{2}\,3\,\cancel{4}}$ $\quad \frac{14}{9}$ $\quad 126$

$p + pE[X] + 2p - 2p^2 + pE[X] - p^2E[X] + (2+\frac{1}{p})(1-p)^2$

$1-2p+p^2$

$(p-1)^2$

$p(2-p^2)E[X]$

$p(3-2p)$

3. $E[X] = E[E[X|Y]]$

$E[X|Y] = \Sigma x P[X|Y=y] = E[X|H]p + E[X|T](1-p)$

$= (1+E[X])p + E[X|T](1-p)$

$E[X|T] = E[X|TH]p + E[X|TT](1-p)$

$= (2+E[X])p + (2+\frac{1}{p})(1-p)$

$\Rightarrow E[X|Y] = p(1+E[X]) + (2+E[X])p(1-p) + (2+\frac{1}{p})(1-p)^2$

$= p(2-p)E[X] + \frac{1}{p}$

$\Rightarrow E[X] = E[p(2-p)E[X] + \frac{1}{p}] = p(2-p)E[X] + \frac{1}{p}$

$\Rightarrow (1-p(2-p))E[X] = \frac{1}{p} \Rightarrow (p-1)^2 E[X] = \frac{1}{p}$

$\Rightarrow E[X] = \frac{1}{p(p-1)^2}$

#

4.

a) $E(X|Y) = \int x f_{X|Y}(x|y)\,dx$

$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \Rightarrow f_Y(y) = \int f_{XY}(x,y)\,dx = \int_0^1 (x+y)\,dx$

$= [\frac{1}{2}x^2 + yx|_0^1] = \frac{1}{2} + y$

$= \frac{x+y}{\frac{1}{2}+y}$

$\Rightarrow E(X|Y) = \int x \cdot \frac{x+y}{\frac{1}{2}+y}\,dx = \frac{2}{1+2y}\int_0^1 x^2 + xy\,dx = \frac{2}{1+2y}[\frac{1}{3}x^3 + \frac{y}{2}x^2|_0^1]$

$= \frac{2}{1+2y}(\frac{1}{3}+\frac{y}{2}) = \frac{2}{1+2y}\,\frac{2+3y}{6} = \frac{2+3y}{3(1+2y)}$

#

b) let $z = E(X|Y) = \frac{2+3y}{3(1+2y)}$

$\Rightarrow 3z + 6yz = 2+3y \Rightarrow (6z-3)y = 2-3z \Rightarrow y = \frac{2-3z}{6z-3}$ $\frac{2-3z}{2z-1}$

$f_Z(z) = \Sigma_i \frac{f_Y(y_i)}{|g'(y_i)|} = \frac{\frac{1}{2}+y}{-\frac{1}{3(2y+1)^2}} = -\frac{1}{2}(1+2y)\cdot 3(1+2y)^2 = -\frac{3}{2}(1+2y)^3$

$= -\frac{3}{2}\cdot\frac{1}{3^3}\left(\frac{(2z-1)+(2-3z)}{2z-1}\right)^3 = \begin{cases} -\frac{1}{18}\left(\frac{-z+1}{2z-1}\right)^3, & \frac{5}{9} \leq z \leq \frac{2}{3} \\ 0, & \text{otherwise} \end{cases}$

#

$\int fg' = fg - \int f'g \quad f = x \quad g' = e^{-\lambda x}$

$f' = 1 \quad g =$

$f_{X|\Lambda}(x|\lambda) = \frac{f_{X\Lambda}(x,\lambda)}{f_\Lambda(\lambda)}$

$-\frac{8}{3} z^{-\frac{8}{3}} \quad \lambda^{2} \cdot \frac{1}{3}x^{2} - \frac{8}{3} - 1 = \frac{-11}{3} \quad (\lambda^{-1})' = -\lambda^{-2} \quad \frac{2}{3} + \frac{6}{3}$

5. $E(X|\Lambda = \lambda) = \int x f_{X|\Lambda}(x|\lambda)\,dx$ $\quad f(\lambda) = \begin{cases} \frac{5}{3}\lambda^{\frac{2}{3}}, & 0 \le \lambda \le 1 \\ 0, & \text{otherwise} \end{cases}$

$= \int_0^\infty x \cdot \lambda e^{-\lambda x}\,dx = \frac{1}{\lambda}$

let $z = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{z}$

$\Rightarrow f_Z(z) = \sum \frac{f(\lambda)}{|g'(\lambda)|} = \frac{\frac{5}{3}\lambda^{\frac{2}{3}}}{+\lambda^{-2}} = +\frac{5}{3}\lambda^{\frac{8}{3}} = \begin{cases} +\frac{5}{3} z^{-\frac{8}{3}}, & z \ge 1 \\ 0, & \text{otherwise} \end{cases}$

$E[X] = E[E[X|\Lambda = \lambda]] = E[z]$

$= \int_1^\infty z(+\frac{5}{3}z^{-\frac{8}{3}})\,dz = \int_1^\infty +\frac{5}{3}z^{-\frac{5}{3}}\,dz = [-\frac{5}{2}z^{-\frac{2}{3}}\Big|_1^\infty]$

$= 0 + \frac{5}{2} = \frac{5}{2}$ #

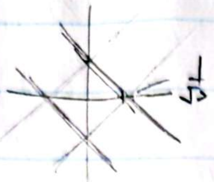
6. $E[X] = E[E[X|N]] = E[np] = p \cdot E[n] = \lambda p$ #

$Var(X) = Var(\sum_{i=1}^{n} Z_i) = \lambda \cdot p(1-p)$ #

$x+y=-1$

1.

a)



$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \cdot 4 \cdot c = 1$$

$$\Rightarrow \quad c = 1$$

\#

b) $f_X(x) = \int f_{XY}(x,y)\,dy = \int_{-(\frac{1}{\sqrt{2}}-|x|)}^{\frac{1}{\sqrt{2}}-|x|} 1\,dy = \left[y \Big|_{-\frac{1}{\sqrt{2}}+|x|}^{\frac{1}{\sqrt{2}}-|x|} \right]$

$= \frac{1}{\sqrt{2}} - |x| + \frac{1}{\sqrt{2}} - |x| = \sqrt{2} - 2|x| \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$ \#

$f_Y(y) = \int f_{XY}(x,y)\,dx = \int_{-(\frac{1}{\sqrt{2}}-|y|)}^{\frac{1}{\sqrt{2}}-|y|} 1\,dx = \left[x \Big|_{-(\frac{1}{\sqrt{2}}-|y|)}^{\frac{1}{\sqrt{2}}-|y|} \right]$

$= \sqrt{2} - 2|y|, \quad -\frac{1}{\sqrt{2}} \le y \le \frac{1}{\sqrt{2}}$

$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{2}-2|y|}$

\#

c) $f_{XY}(x,y) = f_X(x) f_Y(y) \Rightarrow$ indep.

$\Rightarrow 1 \neq (\sqrt{2}-2|x|)(\sqrt{2}-2|y|) \Rightarrow$ X & Y are not independent.

(d) Define $Z = |X| + |Y|$

$f_{XY}(x,y) = \begin{cases} 1, & |x|+|y| \le \frac{1}{\sqrt{2}} \\ 0, & \text{otherwise} \end{cases}$

$w = |x|$

$x = \pm w$

$f_X(x) = \sqrt{2} - 2|x|, \quad -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$|y| = z - (\pm w)$

$f_Y(y) = \sqrt{2} - 2|y|, \quad -\frac{1}{\sqrt{2}} \le y \le \frac{1}{\sqrt{2}}$

$y = \pm(z + (\pm w)) = z+w,\ z-w$

$-z-w,\ -z+w$

$w = |X|, \quad z = |X| + |Y|$

$J = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$

$f_{ZW}(z,w) = f_{XY}(w, z-w) = 1$

$f_Z(z) = \int f_{XY}(w, z-w)\,dw = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 1\,dx = \left[w \Big|_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \right] = \sqrt{2}$

\#

$X = k \quad k = 1, 2, 3 \dots \quad \text{wp} \quad p(1-p)^{k-1}$

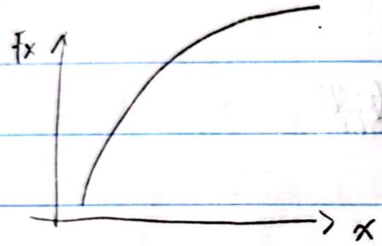
$P(X=k) = p(1-p)^{k-1}$

8. $X \sim G(p)$

a) $F_X(x|A) = P(X > k) = 1 - P(X \le k)$

$= 1 - \sum_{i=1}^{k} p(1-p)^{i}$

b) $F_X(x|A) = P(X \le k) = \sum_{i=1}^{k} p(1-p)^{i}$



9. $x_1 \sim n(1,1)$, $x_2 \sim n(5,4)$, $x_3 \sim n(2,9)$

a) i)

$$f_{x_1}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_1-1)^2}{2}}$$

ii)

$$E(x_2+x_3) = E(x_2) + E(x_3) = 7 = \mu$$

$$Var(x_2+x_3) = Var(x_2) + Var(x_3) + 2Cov(x_2, x_3)$$

$$= 4+9+2\cdot 0 = 13$$

$$z = x_2+x_3 \sim n(7, 13)$$

$$f_z(z) = \frac{1}{\sqrt{2\pi\cdot 13}} e^{\frac{-(z-7)^2}{2\cdot 13}} = \frac{1}{\sqrt{26\pi}} e^{\frac{-(z-7)^2}{26}}$$

iii)

$$\mu = E(2x_1+x_2+x_3) = 2+5+2 = 9$$

$$\sigma = Var(2x_1+x_2+x_3) = 4Var(x_1) + Var(x_2) + Var(x_3) + 2\cdot 2\,Cov(x_1 x_2)$$

$$+ 2\cdot 2\,Cov(x_1, x_3) + 2Cov(x_2, x_3)$$

$$= 4+4+9+4\cdot 1 + 4\cdot 0 + 0 = 21$$

$$m = 2x_1+x_2+x_3$$

$$\Rightarrow f_m(m) = \frac{1}{\sqrt{2\pi\cdot 21}} e^{-\frac{(m-9)^2}{2\cdot 21}} = \frac{1}{\sqrt{42\pi}} e^{-\frac{(m-9)^2}{42}}$$

4.5826

b) $$P(m<0) = P\left(\frac{m-1\cdot 9}{\sqrt{21}\sqrt{1}} < \frac{0-9}{\sqrt{21}\sqrt{1}} = -1.9640\right)$$

$$= 0.025$$

$3 \,\lfloor 216$ $\quad 3\,\lfloor 72$ $\quad 4\,\lfloor 24$ $\quad 16$ $\qquad 6\sqrt{6}$

c) $AX \sim n(A\mu_x,\ A\Sigma A^T)$

$$A\mu_x = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \end{bmatrix}$$

$$A\Sigma A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & -3 & 9 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 6 \\ 6 & 12 \end{bmatrix} \qquad |\Sigma| = 21\cdot 12 - 36 = 216$$

$$\Rightarrow \Sigma^{-1} = \frac{1}{21\cdot 12 - 6\cdot 6}\begin{bmatrix} 12 & -6 \\ -6 & 21 \end{bmatrix} = \begin{bmatrix} \frac{1}{18} & -\frac{1}{36} \\ -\frac{1}{36} & \frac{7}{72} \end{bmatrix}$$

$$f_Y(y) = \frac{1}{2\sqrt{216}\pi} e^{-\frac{1}{2}(y-\mu_y)^T \Sigma^{-1}(y-\mu_y)}$$

$$(y-\mu_y)^T \Sigma^{-1}(y-\mu_y) = \begin{bmatrix} y_1-9 & y_2+2 \end{bmatrix}\begin{bmatrix} \frac{1}{18} & -\frac{1}{36} \\ -\frac{1}{36} & \frac{7}{72} \end{bmatrix}\begin{bmatrix} y_1-9 \\ y_2+2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{y_1-9}{18} - \frac{y_2+2}{36} & -\frac{y_1-9}{36} + \frac{7y_2+14}{72} \end{bmatrix}\begin{bmatrix} y_1-9 \\ y_2+2 \end{bmatrix}$$

$\frac{2y_1-18}{36} - y_2 - 2$

$$= \frac{2y_1-y_2-20}{36}(y_1-9) + \frac{-2y_1+7y_2+32}{72}(y_2+2)$$

$(2y_1-y_2-20)(y_1-9)$

$-2y_1+18+7y_2+14$

$-2y_1+7y_2+32$

\#

$$\rho_{xy} = \frac{Cov(x,y)}{\sqrt{Var(x)\,Var(y)}}$$

$\mu_1 \quad \mu_2 \qquad 9, -2$

$\sigma_{y_1}^2 \quad \sigma_{y_2}^2 \qquad 21 \quad 12$

$\rho_{xy} = \frac{1}{\sqrt{7}}$

or

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sqrt{21}\sqrt{2\pi}\sqrt{12}\sqrt{\frac{6}{7}}} e^{[\cdots]} = \frac{1}{12\sqrt{6}\pi} e^{[\cdots]}$$

$$[\cdots] = \frac{-7}{6}\left(\frac{(y_1-9)^2}{42} + \frac{(y_2+2)^2}{24} - \frac{1}{\sqrt{7}} \frac{(y_1-9)(y_2+2)}{\sqrt{21}\sqrt{12}} \right)$$

$7 \cdot 7 \cdot 3 \cdot 3 \cdot 4$

$\overset{3}{\cancel{21}} \cdot 12 \cdot \frac{6}{\cancel{7}} = 3 \cdot 3 \cdot 4 \cdot 3 \cdot 2 = 6\sqrt{6}$

$2\pi \cdot 2\sqrt{2}\ \sqrt{0.75}\quad \frac{\sqrt{3}}{\sqrt{2}}\quad 4\quad \sqrt{\frac{3}{4}}$

$4\pi \frac{\sqrt{3}}{\sqrt{2}} = 2\sqrt{6}\pi\quad \frac{3}{2} = \quad \frac{15}{10}\quad \frac{0.75}{1.50}$

10. $X \sim n(0,2)\qquad Y \sim n(0,4)$

a) $f(X|Y=1) = \frac{f(X,y=1)}{f_Y(1)}\qquad \sqrt{0.75}\qquad 4\pi\sqrt{2}\cdot\sqrt{0.75}$

$$f(x,Y=1) = \frac{1}{\sqrt{2\pi}\sqrt{2}\sqrt{2\pi}\ 2\cdot\sqrt{1-0.5^2}}\ e^{[\cdot\cdot]} = \frac{1}{2\sqrt{6}\pi}\ e^{[\cdot\cdot]}$$

$$[\cdots] = \frac{-1}{1-0.5^2}\left(\frac{(x-0)^2}{2\cdot 2} + \frac{(y-0)^2}{2\cdot 4} - 0.5\ \frac{xy}{2\sqrt{2}}\right)$$

$$= \frac{-4}{3}\left(\frac{x^2}{4} + \frac{y^2}{8} - \frac{xy}{4\sqrt{2}}\right)$$

$$= -\frac{4}{3}\left(\frac{x^2}{4} - \frac{x}{4\sqrt{2}} + \frac{1}{8}\right)\qquad -\frac{1}{8}$$

$$f_Y(1) = \frac{1}{\sqrt{2\pi}\cdot 2}\ e^{-\frac{1}{8}}$$

$-\frac{1}{6} - \frac{1}{8}$

$= -\frac{8}{48} - \frac{6}{48}$

$= \frac{-14}{48}\quad \frac{1}{2}\sqrt{8}$

$$\Rightarrow \frac{f(x,y=1)}{f_Y(1)} = \frac{1}{\sqrt{3\pi}}\ e^{-\frac{x^2}{3} + \frac{x}{3\sqrt{2}} - \frac{7}{24}}$$

$0.5\sqrt{8}$

$= \frac{7}{24}\quad \sqrt{\frac{8}{4}}$

#

b) $Z = 2X + Y + 2$

$E(Z) = 2E(X) + E(Y) + 2 = 2\qquad \rho_{XY} = \frac{Cov(XY)}{\sqrt{Var X\ Var Y}}$

$Var(Z) = 4Var(X) + Var(Y) + 4Cov(X,Y)\qquad Cov(XY) =$

$= 8 + 4 + 2 = 14$

$Z \sim n(2, 14)\qquad Cov(Z,Y) = Cov(2X+Y, Y)$

$= Cov(2X,Y) + Cov(Y,Y)$

$E[Z|Y=1] = \int_{-\infty}^{+\infty} z\ f_{Z|Y=1}(z|Y=1)\,dz$

$= 1 + 4 = 5$

$$f(z, Y=1) = \frac{1}{\sqrt{2\pi}\sqrt{14}\sqrt{2\pi}\ 2\cdot\sqrt{\left(1-\frac{25}{56}\right)}}\ e^{[\cdot\cdot]}\qquad \rho_{zy} = \frac{5}{\sqrt{14\cdot 4}} = \frac{5}{2\sqrt{14}}\quad \frac{25}{4\cdot 14}$$

$\frac{56}{25}\ \ 31\quad \frac{25}{56}$

$$[\cdots] = \frac{-1}{1-\left(\frac{5}{2\sqrt{14}}\right)^2}\left(\frac{(z-2)^2}{2\cdot 14} + \frac{1}{2\cdot 4} - \frac{5}{2\sqrt{14}}\ \frac{(z-2)(1)}{\sqrt{14\cdot 4}}\right)$$

#

$\begin{matrix} 5 \| 125 \\ 5 \lfloor 25 \\ 5 \end{matrix}$ 750 11.18

11. $n = \left(\frac{z}{E}\right)^2 \cdot p(1-p) = \frac{(1.96)^2}{(0.02)^2} \cdot 0.1 \cdot 0.9 = 864.36 \approx 865$ #

12. $P(X > 6) = 1 - P(X \leq 6)$

$EX_i = 0 \quad , \quad Var(X_i) = \frac{0.5-(-0.5)}{12} = \frac{1}{12}$

2.0868

$P(X \leq 6) = P\left(\frac{X}{\sqrt{\frac{1}{12}}\sqrt{100}} \leq \frac{6}{\sqrt{\frac{100}{12}}} = 2.078\right) = 0.9812$

$\Rightarrow P(S > 100y + 6) + P(S < 100y - 6)$

$= 2(1 - 0.9812) = 0.0376$ #

13.

a) $X_i \overset{iid}{\sim} U[-1, 1]$

b) $E[Z_n] = E\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} E[X_i] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1+(-1)}{2} = 0$ #

$Var[Z_n] = Var\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} Var(X_i) = \frac{1}{n}\sum_{i=1}^{n}\frac{(1-(-1))^2}{12}$

$= \frac{1}{\cancel{n}} \cdot \cancel{n} \cdot \frac{1}{3} = \frac{1}{3}$ #

c) No, only when n is very large that meet the condition of CLT

d) code

(d)

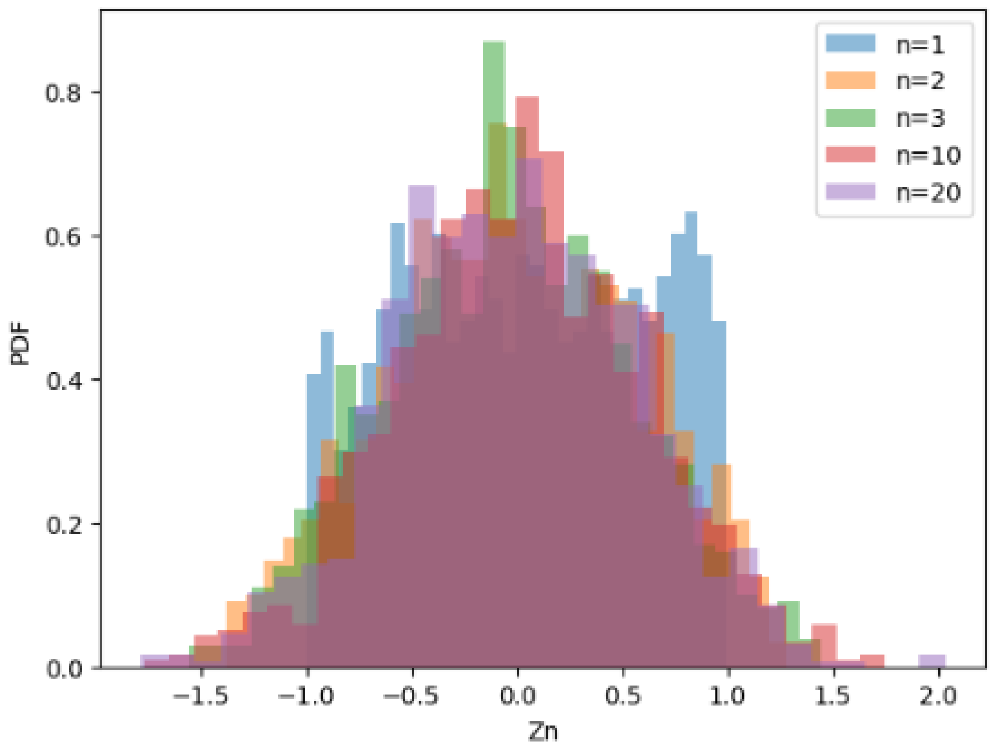
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
from sklearn.neighbors import KernelDensity

num_realizations = 1000
n_values = [1, 2, 3, 10, 20]
realizations = {}

sum = 0
for i in n_values:
    Zn_values = []
    for m in range(num_realizations):
        Xi = np.random.uniform(-1, 1, i)
        Zn = np.sum(Xi) / np.sqrt(i)
        Zn_values.append(Zn)
    realizations[i] = Zn_values

    plt.hist(Zn_values, bins=30, density=True, alpha=0.5, label=f'n={i}')
plt.xlabel('Zn')
plt.ylabel('PDF')
plt.legend()

# Show the plot
plt.show()
```



(e)

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
from sklearn.neighbors import KernelDensity

plt.style.use('default')
num_realizations = 1000
n_values = [1, 2, 3, 10, 20]
realizations = {}

sum = 0
for i in n_values:
    Zn_values = []
    for m in range(num_realizations):
        Xi = np.random.uniform(-1, 1, i)
        Zn = np.sum(Xi) / np.sqrt(i)
        Zn_values.append(Zn)
    realizations[i] = Zn_values

    plt.hist(Zn_values, bins=30, density=True, alpha=0.5, label=f'n={i}')


mu, sigma = 0, np.sqrt(1/3)
s = np.random.normal(mu, sigma, 1000)

# Plot a line chart for the normal distribution
plt.plot(np.sort(s), norm.pdf(np.sort(s), mu, sigma), color='lime', label='Gaussian')

plt.xlabel('Zn')
plt.ylabel('PDF')
plt.legend()

# Show the plot
plt.show()
```