EE 503 : Homework 5

Due : 10/10/2023, Tuesday before class.

1. An urn contains three white, six red, and five black balls. Six of these balls are randomly selected from the urn. Let X and Y denote respectively the number of white and black balls selected. Compute the conditional probability mass function of X given that $Y = 3$. Also compute $E[X|Y = 1]$.

2. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed until the same outcome occurs k consecutive times. If N denoted the number of trips, show that:

$$E[N] = \frac{m^k - 1}{m - 1}$$

(Hint: Before computing $E[N]$ try to compute $E[N_i]$, where N_i is the time until the same outcome occurs i times. Also note that $1 + m + \ldots + m^{(k-1)} = \frac{m^k-1}{m-1}$)

3. A coin that comes up heads with probability p is continually flipped until the pattern T, T, H appears. (That is, you stop flipping when the most recent flip lands heads, and the two immediately preceding it lands tails.) Let X denote the number of flips made, and find $E[X]$.

4. Let X and Y be two random variables with the joint pdf

$$f_{XY}(x,y) = \begin{cases} x + y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   a) Find $E(X \mid Y)$.
   b) Find the pdf $f_Z(z)$ of $Z = E(X \mid Y)$

Hint: The pdf of a function of a random variable Y, for example $Z = g(Y)$ is given to be

$$f_Z(z) = \sum_i \frac{f_Y(y_i)}{\mid g'(y_i) \mid}$$

where y_i are real roots of the equation $z = g(y)$, i.e. $y_i = g^{-1}(z)$.

5. Let Λ and X be two random variables with

$$\Lambda \sim f(\lambda) = \begin{cases} \frac{5}{3}\lambda^{\frac{2}{3}}, & 0 \le \lambda \le 1 \\ 0, & \text{otherwise} \end{cases}$$

and $f(X \mid \Lambda = \lambda) \sim exp(\lambda)$. Find $E(X)$.

6. Let $N \sim P(\lambda)$, i.e. Poisson with parameter λ be the number of photons arriving at a photodetector per unit time. Each photon is detected with probability p and undetected with probability

$(1-p)$ independent of N and other photons. Let X be the number of detected photons per unit time. Thus $X = \sum_{i=1}^{N} Z_i$, where

$$Z_i = \begin{cases} 1, & \text{photon } i \text{ is detected} \\ 0, & \text{photon } i \text{ is not detected} \end{cases}$$

and for each $N = n, \{Z_1, Z_2, \ldots, Z_n\}$ are independent. Find the mean and variance of X.

7. Two continuous random variables X and Y are described by the pdf

$$f_{XY}(x,y) = \begin{cases} c, & \text{if } |\, x \,| + |\, y \,| \leq \frac{1}{\sqrt{2}} \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant.

a) Find c.

b) Find $f_X(x)$ and $f_{X|Y}(x \mid y)$.

c) Are X and Y independent random variables? Justify your answer.

Problem d is an optional question, those who solve it will get extra credits.

d) Define the random variable $Z = (|\, X \,| + |\, Y \,|)$. Find the pdf $f_Z(z)$.

8. Let X be a geometric random variable. Find and plot $F_X(x|A)$ if:

a) $A = \{X > k\}$ where k is a positive integer

b) $A = \{X < k\}$ where k is positive integer

9. Let $\mathbf{X} = (x_1\ x_2\ x_3)^T$ be a Gaussian random vector and $\mathbf{X} \sim \eta(\mu_{\mathbf{X}}, \sigma_{\mathbf{X}})$ where $\mu_{\mathbf{X}} = (1\ 5\ 2)^T$ and

$$\sigma_{\mathbf{X}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

a) What are the pdfs of

i) x_1

ii) $x_2 + x_3$

iii) $2x_1 + x_2 + x_3$

b) What is $P(2x_1 + x_2 + x_3 < 0)$?

c) What is the pdf of $\mathbf{Y} = \mathbf{AX}$ where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

10. Let $\mathbf{X}$ and $\mathbf{Y}$ be jointly gaussian random variables with zero mean, variance $\sigma_{\mathbf{X}}^2 = 2$ and $\sigma_{\mathbf{Y}}^2 = 4$, and normalized correlation coefficient $\rho_{\mathbf{XY}} = 0.5$.

    a) Find the pdf of $X|Y = 1$.

    b) Let $Z = 2X + Y + 2$. Find $E[Z|Y = 1]$.

11. Suppose that 10% of voters are in favour of certain legislation. A large number n of voters are polled and a relative frequency estimate $f_A(n)$ for the above proportion is obtained. How many voters should be polled in order that the probability is at least 0.95 that $f_A(n)$ differs from 0.10 by less than 0.02?

12. The sum of a list of 100 real numbers is to be computed. To make the addition easier, suppose the numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Estimate the probability that the total error in the sum of the 100 numbers exceeds 6.

13. Consider the random sequence generated by $Z_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$

    a) You will be generating sum of iid uniform random variables on $[-1, 1]$. Thus, $X_i \sim U[-1, 1]$.

    b) Analytically derive $\mathrm{E}\{Z_n\}$ and $\mathrm{Var}\{Z_n\}$.

    c) Do X_i meet the conditions of Central Limit Theorem?

    d) Generate at least 1000 realizations of Z_n for n = 1,2,3,10,20. Plot the pdf for each n of these realizations.

    e) Compare these plots with a plot of the Gaussian densities $\mathcal{N}(E\{Z_n\}, Var\{Z_n\})$.