$C_2^3 \; \frac{3!}{2!\,1!} \quad (n+1)P =$

EEJ10 HW2   6024588048   ~~[illegible]~~ Chih-Cheng Hsieh

1. if $(n+1)p$ is integer, $P(X)$ is decrete.

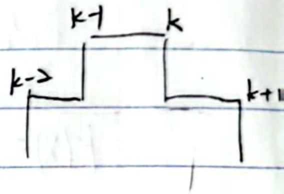
(a)
$$\frac{P(X=k)}{P(X=k-1)}=1=\frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}}=\frac{\cancel{\frac{n!}{k!(n-k)!}}\,\cancel{p^k}\,\cancel{(1-p)^{n-k}}}{\cancel{\frac{n!}{(k-1)!(n-k+1)!}}\,\cancel{p^k}\cdot p^{-1}\,\cancel{(1-p)^{n-k}}(1-p)}$$

$$=\frac{\frac{1}{k}}{\frac{1}{n-k+1}\cdot\frac{1}{p}\cdot(1-p)}=\frac{\frac{1}{k}}{\frac{1-p}{(n-k+1)p}}=\frac{(n-k+1)p}{k(1-p)}=1$$

$$\Rightarrow np-\cancel{kp}+p=k-\cancel{kp}\Rightarrow k=p(n+1)\Rightarrow k-1=p(n+1)-1$$

$\because$ $(n+1)p$ is a integer

$\therefore$ $k=p(n+1)$ or $p(n+1)-1$ reachs its largest value



(b) if $(n+1)p$ is not integer, $P(X)$ is count.

Let $\frac{P(X=k)}{P(X=k-1)}=1 \Rightarrow k=p(n+1)$, $k-1=p(n+1)-1$

$\because$ $(n+1)p$ isn't a integer

$\therefore$ $P(X)$ reaches largest value when $p(n+1)-1<k<p(n+1)$

2. from Binomial r.v. $\Rightarrow P=(\text{number of distinct partition})\times(p_1^{x_1}p_2^{x_2}\cdots\cdots)$

$\#$ of distinct partition $=\binom{n}{x_1}\binom{n-x_1}{x_2}\cdots\cdots\binom{n-x_1-\cdots-x_{r-1}}{x_r!}$

$$=\frac{n!}{x_1!\cancel{(n-x_1)!}}\cdot\frac{\cancel{(n-x_1)!}}{x_2!(n-x_1-x_2)!}\cdot\cdots\cdots\cdot\frac{\cancel{(n-x_1-\cdots-x_{r-2})!}}{x_{r-1}!\,\underbrace{(n-x_1-\cdots-x_{r-2}-x_{r-1})}!}$$

$\longrightarrow n-x_1-\cdots-x_{r-1}=x_r$

$$=\frac{n!}{x_1!\,x_2!\cdots x_{r-1}!\,x_r!}$$

$$\Rightarrow P=\frac{n!}{x_1!\,x_2!\cdots x_r!}\,p_1^{x_1}p_2^{x_2}\cdots p_r^{x_r}$$

$\#$

$8\ 5\ 4$

$\rightarrow 3$

3. $P(7) = C_3^6 p^4(1-p)^3 + C_3^6 p^3(1-p)^4$ ※

$= 20p^4(1-p)^3 + 20p^3(1-p)^4$ $\quad 20 \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{20}{64} = \frac{10}{32} = \frac{5}{16}$

$= 20p^3(1-p)^3$

if $P'(7) = 0 \Rightarrow$ The roots have maximum or minimum value

$\Rightarrow P'(7) = \frac{d}{dx} 20p^3(1-p)^3 dp = 60p^2(1-p)^3 - 60p^3(1-p)^2$

$= 60p^2(1-p)^2[(1-p)-p] = 60p^2(1-p)^2(1-2p) = 0$

$p = 0, 1, \frac{1}{2}$

$\Rightarrow P(7) = \begin{cases} 0, & \text{when } p=0 \\ 0, & \text{when } p=0 \\ \frac{5}{16}, & \text{when } p=\frac{1}{2} \end{cases}$

$\Rightarrow$ when $p = \frac{1}{2}$, $P(7)$ is maximized

$$P(N=k)=\frac{(\lambda T)^k}{k!}e^{-\lambda T}$$

n $\quad p=\frac{1}{2}$ $\quad k$ $\quad 3$

n	p	k
2	$\frac{1}{2}$	

HT HHH TTT
HH HHT TTH
TH HTH THT
TT HTT THH

4. H T H H T T $\Rightarrow n=n-1,\ p=\frac{1}{2}$

1 1 0 1 0

$$\Rightarrow P=\binom{n}{k}p^k(1-p)^{n-k}=\binom{n-1}{k}\frac{1}{2}^k\frac{1}{2}^{n-1-k}=\binom{n-1}{k}\left(\frac{1}{2}\right)^{n-1}$$ #

5. From Poisson r.v. $\Rightarrow P(N=k)=\frac{\lambda^k}{k!}e^{-\lambda}$

$$\alpha=\frac{\lambda}{n\mu}\frac{(\text{avg \# a day})}{(\text{\# an employee can processed})\times(\text{\# of employee})}=\frac{3}{n}(\text{probability of event occur})$$ $\quad \frac{\lambda-n\mu}{n\mu}$

$$\Rightarrow \lambda=\alpha=\frac{3}{n}$$

$$P(X>4)<0.9 \Rightarrow P(X\le4)+P(X>4)=1\Rightarrow P(X\le4)=1-P(X>4)$$

$$\Rightarrow P(X\le4)>0.1\Rightarrow P(0)+P(1)+P(2)+P(3)+P(4)>0.1$$

$$\Rightarrow \frac{(\frac{3}{n})^0}{0!}e^{-\frac{3}{n}}+\frac{(\frac{3}{n})^1}{1!}e^{-\frac{3}{n}}+\frac{(\frac{3}{n})^2}{2!}e^{-\frac{3}{n}}+\frac{(\frac{3}{n})^3}{3!}e^{-\frac{3}{n}}+\frac{(\frac{3}{n})^4}{4!}e^{-\frac{3}{n}}>0.1$$ $\quad \frac{27}{6}=\frac{9}{2}$

$$\Rightarrow e^{-\frac{3}{n}}\left(1+\frac{3}{n}+\frac{9}{2n^2}+\frac{9}{2n^3}+\frac{27}{8n^4}\right)>0.1$$ $\quad \frac{81}{24}=\frac{27}{8}$

$\Rightarrow n>0.3753 \Rightarrow$ 1 or more employees are required #

Let $n=1$

$$\Rightarrow P(0)=\frac{(\frac{3}{1})^0}{0!}e^{-\frac{3}{1}}=0.0498$$ #

$F(x) = 1 - e^{-\lambda x}$ $\qquad \dfrac{x^2 - 2x\mu + \mu^2}{2\sigma^2}$

6. From exponential random variable : $F(x) = 1 - e^{-\lambda x}$

(a) $F(\pi(90)) = 1 - e^{-\lambda(\pi(90))} = 0.9 \Rightarrow e^{-\lambda(\pi(90))} = 0.1$

$\Rightarrow -\lambda(\pi(90)) = \ln(0.1)$

$\Rightarrow \pi(90) = -\dfrac{\ln(0.1)}{\lambda}$ #

$F(\pi(95)) = 1 - e^{-\lambda(\pi(95))} = 0.95 \Rightarrow \pi(95) = -\dfrac{\ln(0.05)}{\lambda}$ #

$F(\pi(99)) = 1 - e^{-\lambda(\pi(99))} = 0.99 \Rightarrow \pi(99) = -\dfrac{\ln(0.01)}{\lambda}$ #

(b) From Gaussian r.v. $\Rightarrow Z = \dfrac{X - m}{\sigma}$

$Z(\pi(90)) = \dfrac{\pi(90) - m}{\sigma} = 1.29$ (from table)

$\Rightarrow \pi(90) = 1.29\sigma + m$ #

$Z(\pi(95)) = \dfrac{\pi(95) - m}{\sigma} = 1.65$

$\Rightarrow \pi(95) = 1.65\sigma + m$ #

$Z(\pi(99)) = \dfrac{\pi(99) - m}{\sigma} = 2.33$

$\Rightarrow \pi(99) = 2.33\sigma + m$

#

7.

(a) Let $0 \le P(X=x) \le 1$

$\Rightarrow \log_2 P(X=x) \le \log_2 1 \Rightarrow \log_2 P(X=x) \le 0$

$\Rightarrow P(X=x)\log_2 P(X=x) \le 0 \cdot P(X=x) = 0$

$\Rightarrow \sum_{x\in X} P(X=x)\log_2 P(X=x) \le 0$

$\Rightarrow -\sum_{x\in X} P(X=x)\log_2 P(X=x) \ge 0 \Rightarrow H(x) \ge 0$ #

(b) $P(X=x) = \frac{1}{n}$

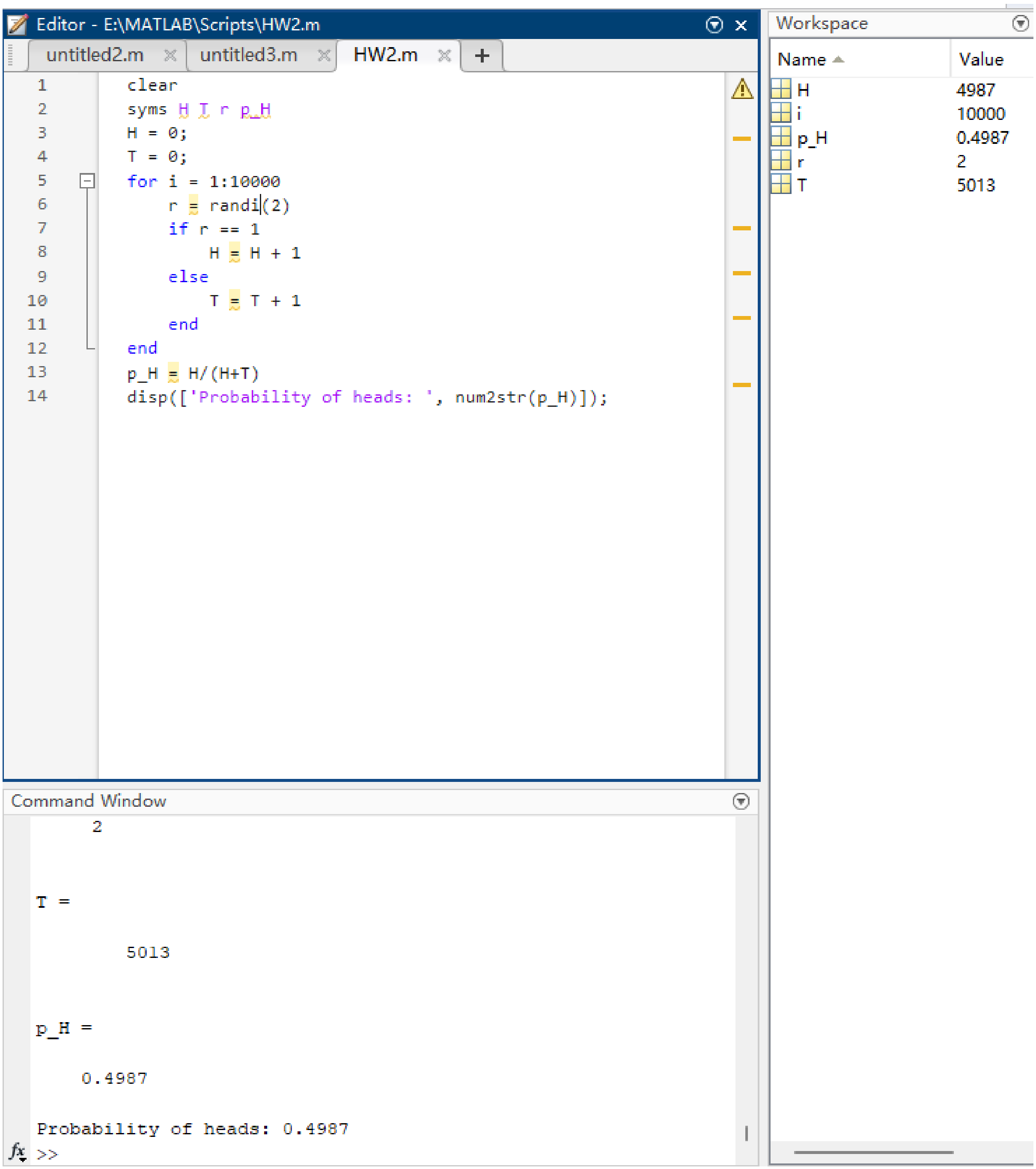
$H(x) = -\sum_{k=0}^{n} \frac{1}{n}\log_2 \frac{1}{n} = -\frac{1}{n}\log_2\left(\frac{1}{n}\right) \sum_{k=0}^{n} 1 = -\frac{1}{n}\log_2\left(\frac{1}{n}\right)\cdot n = \log_2(n)$ #

8.

Programming language: MATLAB

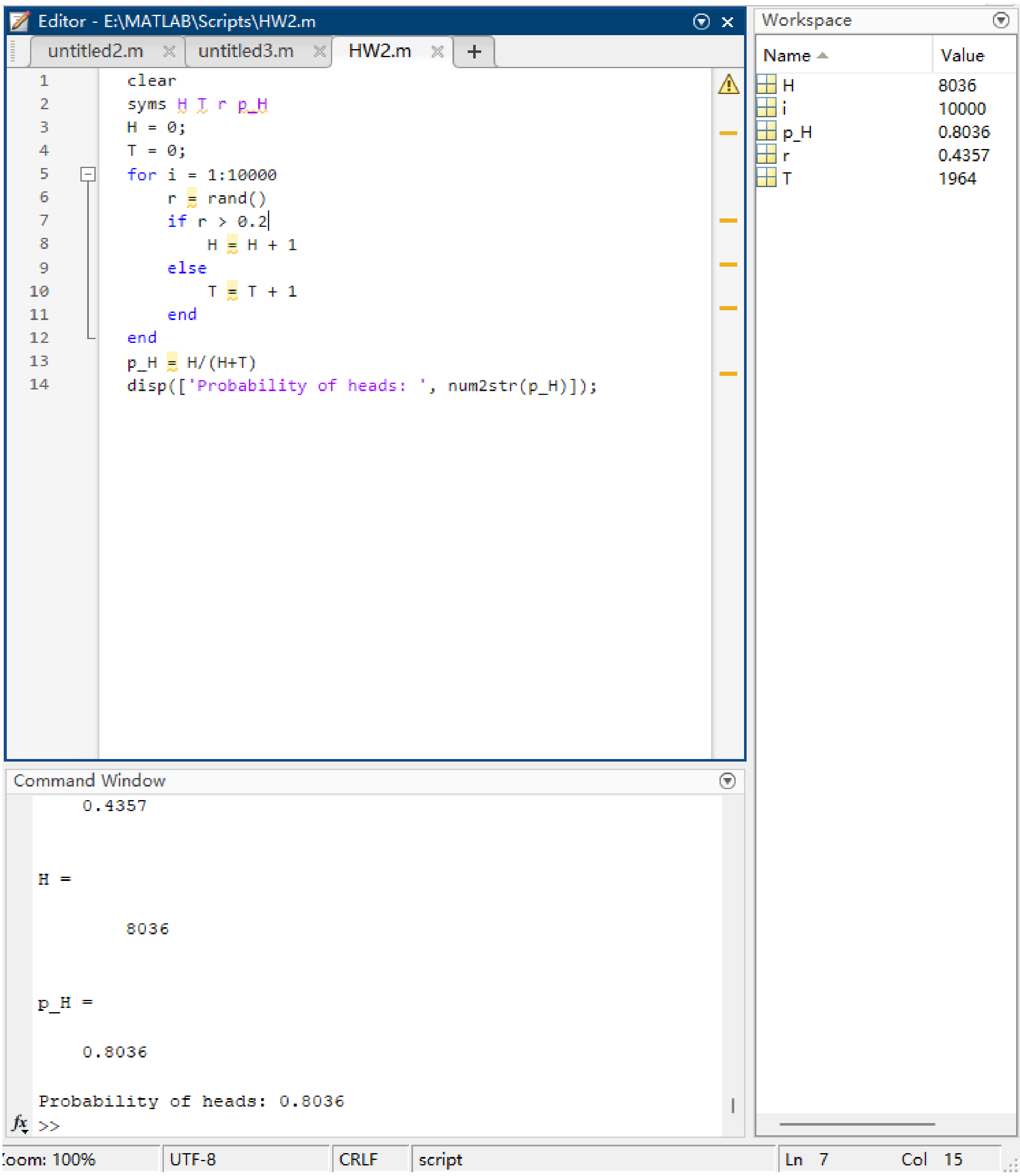
(a) Fair coin flip for 10k times

P(Head) = 0.4987



(b) Coin flip with a bias of 0.8 for 10k times

P(Head) = 0.8036



(b) Plot of fair coin flip for 10k times

The probability can converge to the ideal value swiftly, as shown in the plot below. Both plots reach an approximate estimate after around the 50th toss. Nevertheless, it is possible that the test results diverge and require more tests (2000~4000 times in plot) to stabilize. Therefore, if we have limited computational power, we can obtain good performance results for 50 tosses. However, if we have sufficient computational power, we should conduct as many tests as possible.