EE 503 Sample Midterm Exam

October 2021

Problem 1 (15 points)

You have a fair coin and 2 dice. The first die is fair and the second die is biased as follows:the probability of getting 6 equals $\frac{1}{3}$ and the rest outcomes are equally probable. We toss the coin first. If the result is tail, we choose the first die for our experiment. Otherwise, we use the second die. And then we roll the die once.

(i) (2 points) For the second die, find the probability of getting $1, 2, \ldots 6$.

(ii) (3 points) What is the probability that the rolled die showed a 6.

(iii) (5 points) If the rolled die showed a 3, what is the probability that the coin showed "Tail"?

(iv) (5 points) We throw the chosen die until we get two consecutive 6. What is the expected number of throws needed?

Solution (You can use the back of the page too)

Problem 2 (25 points)

Label each of the following statements with $=, \leq, \geq$, or NONE. Label a statement with $=$ if equality always holds, with $\leq$ or $\geq$ if such inequality holds in general and strict inequality holds sometimes, and with NONE if none of the above holds. Provide brief justification for your answers in order to get full credit.

(i) (5 points) $P(\bigcup_{i=1}^{n} A_i)$ vs $\sum_{i=1}^{n} P(A_i)$ if A_i are independent.

(ii) (5 points) $f_{XYZ}(x,y,z)$ vs. $f_X(x) f_Y(y) f_{Z|XY}(z|x,y)$ if X and Y are independent.

(iii) (5 points) $P(X > 2E[X])$ vs. $\frac{E[X^2]}{(E[X])^2} - 1$ if $X > 0$

(iv) (5 points) $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} 3(X_i^2 - Var[X_i])}{n}$ vs. $2(E[X_i])^2$, if X_i, $i = 1, 2, \ldots$ are i.i.d.

(v) (5 points) $P(lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - 2n}{5\sqrt{n}} \leq 5)$ vs $P(Y \leq 10)$ if $E(X_i) = 2$, $Var(X_i) = 25$, and $Y \sim n(0,1)$.

Solution (You can use the back of the page too.)

Problem 3 (25 points)

Two random variables X and Y have the following joint probability density function:

$$f_{XY}(x,y) = \begin{cases} c(x+y) & 0 \leq y \leq x \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

(i) (5 points) Find c.

(ii) (5 points) Find the pdf of Y, $f_Y(y)$.

(iii) (5 points) Find $E(X|Y)$.

(iv) (5 points) Are X and Y independent? Justify your answer.

(v) (5 points) Let $Z = (2y+1)^2$. Find the pdf of Z, $f_Z(z)$.

Solution (You can use the back of the page too)

Problem 4 (25 points): Investing

A model for the movement of a financial index (a quantity whose value changes over time) assumes that, given the present value V_n of the index at time slot n, $n = 0, 1, 2, \ldots$, after one time slot it will be $V_{n+1} = U \cdot V_n$ with probability p or $V_{n+1} = D \cdot V_n$ with probability $1 - p$. Assume that successive movements of the value are independent, $U = 1.01$, $D = 0.99$, $p = 0.51$, and V_0 is known. (Note: Even though I am giving you specific values for quantitates, you do not need to make numerical computations once you express things as a function of those numerical values.)

(i) (4 points) Let X_n be a random variable dictating whether the index will go up or down after time slot n, that is, $V_{n+1} = X_n \cdot V_n$. Express X_n as a function of U and V and compute its mean and variance.

(ii) (5 points) Write the value of the index at time n, V_n, as a function of X_i's, $i = 0, 1, 2, \ldots, n$ and the starting value V_0.

(iii) (10 points) Approximate the probability that the index's value will be up at least 10% after the first $1,000$ time slots. Hint: Recall that $\log(\prod_i X_i) = \sum_i \log(X_i)$. Think whether the random variables $\log(X_i)$, $i = 0, 1, \ldots$ are i.i.d. and work from there. (Recall it is not required to work out the final numerical solution.)

(iv) (6 points) Let the above probability be P_{up} and let P_{down} be the probability that the index's value will be down at least 10% after the first $1,000$ time slots. Which one is larger, P_{up} or P_{down}?

Solution (You can use the back of the page too.)