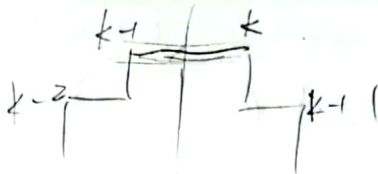


$$C_{\infty}^{\infty} \frac{3!}{2!1!} (n+1)p =$$



EE510 HW2

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1. if $(n+1)p$ is integer, $P(X)$ is discrete.

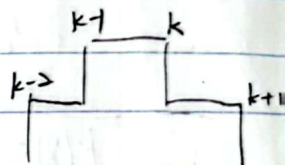
$$(a) \frac{P(X=k)}{P(X=k-1)} = 1 = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k-1} p^{k-1} (1-p)^{n-k+1}} = \frac{\frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}}{\frac{n!}{(k-1)!(n-k+1)!} p^{k-1} (1-p)^{n-k+1}}$$

$$= \frac{\frac{1}{k}}{\frac{1}{n-k+1} \cdot \frac{1}{p} \cdot (1-p)} = \frac{\frac{1}{k}}{\frac{1-p}{(n-k+1)p}} = \frac{(n-k+1)p}{k(1-p)} = 1$$

$$\Rightarrow np - kp + p = k - kp \Rightarrow k = p(n+1) \Rightarrow k-1 = p(n+1) - 1$$

$\therefore (n+1)p$ is a integer

$\therefore k = p(n+1)$ or $p(n+1) - 1$ reaches its largest value



(b) if $(n+1)p$ is not integer, $P(X)$ is count.

$$\text{Let } \frac{P(X=k)}{P(X=k-1)} > 1 \Rightarrow k = p(n+1), k-1 = p(n+1) - 1$$

$\therefore (n+1)p$ isn't a integer

$\therefore P(X)$ reaches largest value when $p(n+1) - 1 < k < p(n+1)$

2. from Binomial r.v. $\Rightarrow P = (\text{number of distinct partition}) \times (p_1^{x_1} p_2^{x_2} \dots)$

$$\# \text{ of distinct partition} = \binom{n}{x_1} \binom{n-x_1}{x_2} \dots \binom{n-x_1-\dots-x_{r-1}}{x_r}$$

$$= \frac{n!}{x_1!(n-x_1)!} \cdot \frac{(n-x_1)!}{x_2!(n-x_1-x_2)!} \cdot \dots \cdot \frac{(n-x_1-\dots-x_{r-1})!}{x_r!(n-x_1-\dots-x_{r-1}-x_r)!}$$

$$\rightarrow n-x_1-\dots-x_{r-1}=x_r$$

$$= \frac{n!}{x_1! x_2! \dots x_{r-1}! x_r!}$$

$$\Rightarrow P = \frac{n!}{x_1! x_2! \dots x_r!} p_1^{x_1} p_2^{x_2} \dots p_r^{x_r}$$

#



$$3. \quad P(\eta) = C_3^6 p^4 (1-p)^3 + C_3^6 p^3 (1-p)^4 \quad \times$$

$$= 20 p^4 (1-p)^3 + 20 p^3 (1-p)^4$$

$$= 20 p^3 (1-p)^3$$

$$20 \cdot \frac{1}{8} \cdot \frac{1}{8} = \frac{20}{64} = \frac{5}{16}$$

if $P'(\eta) = 0 \Rightarrow$ The roots have maximum or minimum value

$$\Rightarrow P'(\eta) = \frac{d}{dx} 20 p^3 (1-p)^3 dp = 60 p^2 (1-p)^3 - 60 p^3 (1-p)^2$$

$$= 60 p^2 (1-p)^2 [(1-p) - p] = 60 p^2 (1-p)^2 (1-2p) = 0$$

$$p = 0, 1, \frac{1}{2}$$

$$\Rightarrow P(\eta) = \begin{cases} 0, & \text{when } p=0 \\ 0, & \text{when } p=1 \\ \frac{5}{16}, & \text{when } p=\frac{1}{2} \end{cases}$$

\Rightarrow when $p = \frac{1}{2}$, $P(\eta)$ is maximized



$$P(N=k) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}$$

$$n \quad p = \frac{1}{2} \quad k \quad \rightarrow$$

HT	HHH	THT
HHT	HHHT	TTH
TH	HTH	THT
TT	HTT	TTH

4. $H T H H T T \Rightarrow n = n-1, p = \frac{1}{2}$
 $1 \quad 1 \quad 0 \quad 1 \quad 0$

$$\rightarrow P = \binom{n}{k} p^k (1-p)^{n-k} = \binom{n-1}{k} \frac{1}{2}^k \frac{1}{2}^{n-1-k} = \binom{n-1}{k} \left(\frac{1}{2}\right)^{n-1}$$

5. From Poisson r.v. $\rightarrow P(N=k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$$\lambda = \frac{\lambda}{n \mu} \left(\frac{\text{avg \# a day}}{\# \text{ an employee can processed}} \times \frac{\# \text{ of employees}}{\# \text{ of employees}} \right) = \frac{3}{n} (\text{probability of event occur}) \quad \frac{\lambda - n \mu}{n \mu}$$

$$\Rightarrow \lambda = x = \frac{3}{n}$$

$$P(X > 4) < 0.9 \Rightarrow P(X \leq 4) + P(X > 4) = 1 \Rightarrow P(X \leq 4) = 1 - P(X > 4)$$

$$\Rightarrow P(X \leq 4) > 0.1 \Rightarrow P(0) + P(1) + P(2) + P(3) + P(4) > 0.1$$

$$\Rightarrow \frac{(\frac{3}{n})^0}{0!} e^{-\frac{3}{n}} + \frac{(\frac{3}{n})^1}{1!} e^{-\frac{3}{n}} + \frac{(\frac{3}{n})^2}{2!} e^{-\frac{3}{n}} + \frac{(\frac{3}{n})^3}{3!} e^{-\frac{3}{n}} + \frac{(\frac{3}{n})^4}{4!} e^{-\frac{3}{n}} > 0.1 \quad \frac{27}{8} = \frac{9}{2}$$

$$\Rightarrow e^{-\frac{3}{n}} \left(1 + \frac{3}{n} + \frac{9}{2n^2} + \frac{9}{2n^3} + \frac{27}{8n^4} \right) > 0.1 \quad \frac{81}{24} = \frac{27}{8}$$

$$\Rightarrow n > 0.3753 \Rightarrow 1 \text{ or more employees are required}$$

$$\text{Let } n=1$$

$$\Rightarrow P(1) = \frac{(\frac{3}{1})^1}{1!} e^{-\frac{3}{1}} = 0.0498$$

$$F(x) = 1 - e^{-\lambda x}$$

$$\frac{x^2 - 2x\mu + \mu^2}{2\sigma^2}$$

6. From exponential random variable: $F(x) = 1 - e^{-\lambda x}$

$$(a) F(\pi(90)) = 1 - e^{-\lambda(\pi(90))} = 0.9 \Rightarrow e^{-\lambda(\pi(90))} = 0.1$$

$$\Rightarrow -\lambda(\pi(90)) = \ln(0.1)$$

$$\Rightarrow \pi(90) = -\frac{\ln(0.1)}{\lambda} \quad \#$$

$$F(\pi(95)) = 1 - e^{-\lambda(\pi(95))} = 0.95 \Rightarrow \pi(95) = -\frac{\ln(0.05)}{\lambda} \quad \#$$

$$F(\pi(99)) = 1 - e^{-\lambda(\pi(99))} = 0.99 \Rightarrow \pi(99) = -\frac{\ln(0.01)}{\lambda} \quad \#$$

(b) From Gaussian r.v. $\Rightarrow Z = \frac{X - \mu}{\sigma}$

$$Z(\pi(90)) = \frac{\pi(90) - \mu}{\sigma} = 1.29 \quad (\text{from table})$$

$$\Rightarrow \pi(90) = 1.29\sigma + \mu \quad \#$$

$$Z(\pi(95)) = \frac{\pi(95) - \mu}{\sigma} = 1.65$$

$$\Rightarrow \pi(95) = 1.65\sigma + \mu \quad \#$$

$$Z(\pi(99)) = \frac{\pi(99) - \mu}{\sigma} = 2.33$$

$$\Rightarrow \pi(99) = 2.33\sigma + \mu \quad \#$$

7.

(a) Let $0 \leq P(X=x) \leq 1$

$$\Rightarrow \log_2 P(X=x) \leq \log_2 1 \Rightarrow \log_2 P(X=x) \leq 0$$

$$\Rightarrow P(X=x) \log_2 P(X=x) \leq 0 \cdot P(X=x) = 0$$

$$\Rightarrow \sum_{x \in X} P(X=x) \log_2 P(X=x) \leq 0$$

$$\Rightarrow - \sum_{x \in X} P(X=x) \log_2 P(X=x) \geq 0 \Rightarrow H(X) \geq 0 \quad \#$$

$$(b) P(X=x) = \frac{1}{n}$$

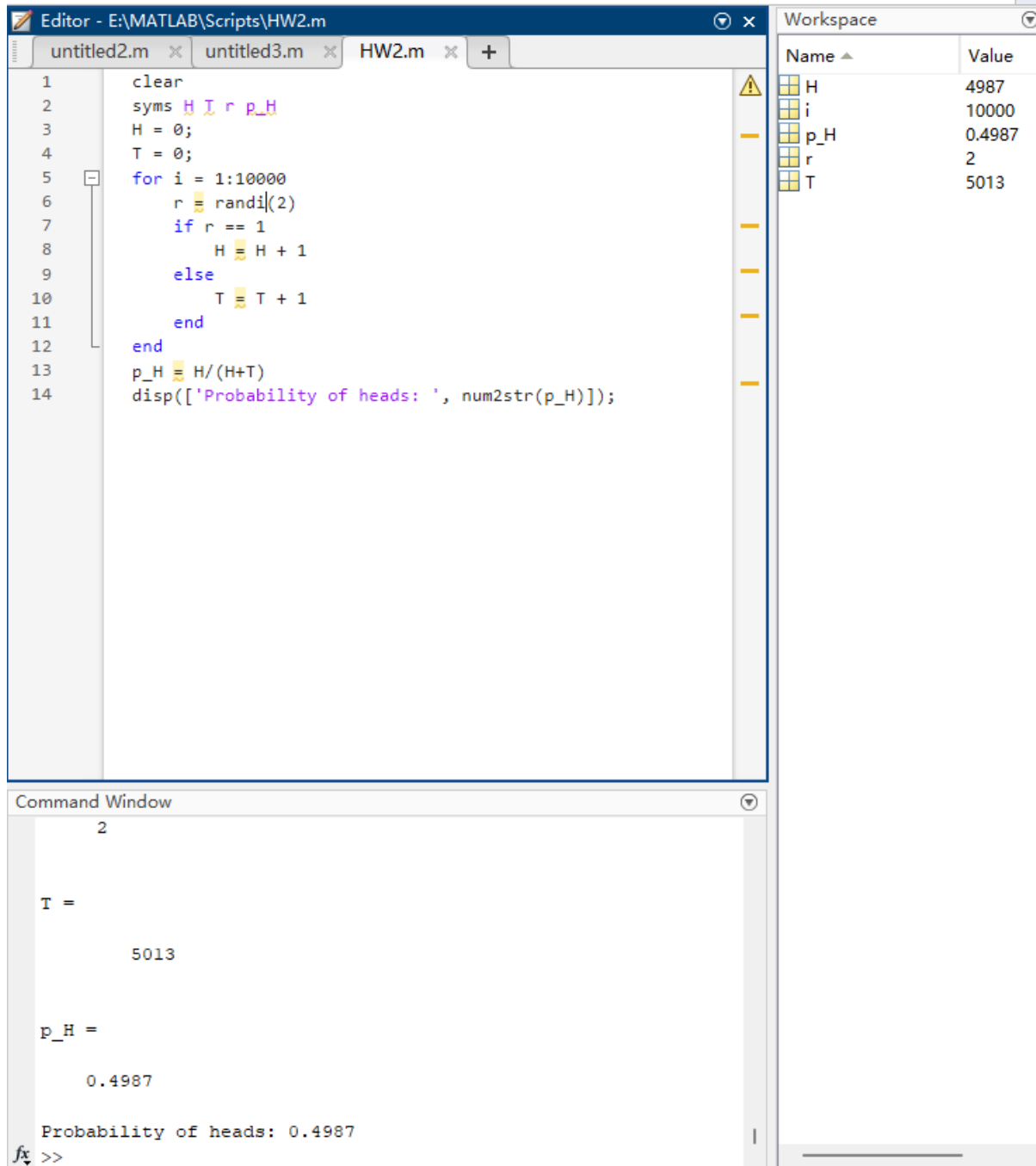
$$H(X) = - \sum_{k=0}^n \frac{1}{n} \log_2 \frac{1}{n} = - \frac{1}{n} \log_2 \left(\frac{1}{n} \right) \sum_{k=0}^n 1 = - \frac{1}{n} \log_2 \left(\frac{1}{n} \right) \cdot n = \log_2(n) \quad \#$$

8.

Programming language: MATLAB

(a) Fair coin flip for 10k times

$P(\text{Head}) = 0.4987$



The image shows a MATLAB environment with the Editor window displaying a script for simulating 10,000 coin flips. The script initializes variables H and T to 0, then enters a loop where it generates random numbers r and increments H or T based on the value of r. After the loop, it calculates the probability p_H as H/(H+T) and displays the result. The Workspace window on the right shows the current values of the variables: H=4987, i=10000, p_H=0.4987, r=2, and T=5013. The Command Window at the bottom shows the execution of the script, including the final display of the probability of heads as 0.4987.

```
1 clear
2 syms H T r p_H
3 H = 0;
4 T = 0;
5 for i = 1:10000
6     r = randi(2)
7     if r == 1
8         H = H + 1
9     else
10        T = T + 1
11    end
12 end
13 p_H = H/(H+T)
14 disp(['Probability of heads: ', num2str(p_H)]);
```

Name	Value
H	4987
i	10000
p_H	0.4987
r	2
T	5013

```
2

T =

    5013

p_H =

    0.4987

Probability of heads: 0.4987
fx >>
```

(b) Coin flip with a bias of 0.8 for 10k times
 $P(\text{Head}) = 0.8036$

The screenshot displays the MATLAB environment with the following components:

- Editor:** Contains a script for simulating 10,000 coin flips with a bias of 0.2 (probability of heads = 0.8). The script uses a for loop and conditional statements to count heads (H) and tails (T).
- Workspace:** A table showing the values of variables defined in the script.
- Command Window:** Shows the output of the script, including the calculated probability of heads.

Name	Value
H	8036
i	10000
p_H	0.8036
r	0.4357
T	1964

```
1 clear
2 syms H T r p_H
3 H = 0;
4 T = 0;
5 for i = 1:10000
6     r = rand()
7     if r > 0.2
8         H = H + 1
9     else
10        T = T + 1
11    end
12 end
13 p_H = H/(H+T)
14 disp(['Probability of heads: ', num2str(p_H)]);
```

Command Window Output:

```
0.4357

H =

    8036

p_H =

    0.8036

Probability of heads: 0.8036
fx >>
```

Status Bar: Zoom: 100% | UTF-8 | CRLF | script | Ln 7 | Col 15

(b) Plot of fair coin flip for 10k times

The probability can converge to the ideal value swiftly, as shown in the plot below. Both plots reach an approximate estimate after around the 50th toss. Nevertheless, it is possible that the test results diverge and require more tests (2000~4000 times in plot) to stabilize. Therefore, if we have limited computational power, we can obtain good performance results for 50 tosses. However, if we have sufficient computational power, we should conduct as many tests as possible.

