EE503 HW4 Chih-Cheng Hsieh

1. Let $U = \max(X_1, X_2)$

$F_U(u) = P(U \le u) = P(\max(X_1, X_2) \le u)$

$= P(X_1 \le u \cap X_2 \le u) = P(X_1 \le u)\, P(X_2 \le u)$

$= F_{X_1}(u)\, F_{X_2}(u)$

if X_1, X_2 are discrete r.v.

$E[\max(X_1, X_2)] = \sum_{X_1}\sum_{X_2} \max(X_1, X_2) F_{X_1}(u) F_{X_2}(u)$

Let $V = \min(X_1, X_2)$

$F_V(v) = P(V \le v) = 1 - P(V > v) = 1 - P(X_1 > v, X_2 > v)$

$= 1 - P(X_1 > v) P(X_2 > v) = 1 - (1 - F_{X_1}(v))(1 - F_{X_2}(v))$

$E[\min(X_1, X_2)] = \sum_{X_1}\sum_{X_2} \min(X_1, X_2)\left(1 - (1 - F_{X_1}(v))(1 - F_{X_2}(v))\right)$

$G = \sum_{X_1}\sum_{X_2} \max(X_1, X_2) F_{X_1}(u) F_{X_2}(u) + \sum_{X_1}\sum_{X_2} \min(X_1, X_2)\left(1 - (1 - F_{X_1}(v))(1 - F_{X_2}(v))\right)$

or

$\mathrm{Min}(X_1 X_2) = \frac{1}{2}(X_1 + X_2) - |X_1 - X_2|$

$\mathrm{Max}(X_1 X_2) = \frac{1}{2}(X_1 + X_2) + |X_1 - X_2|$

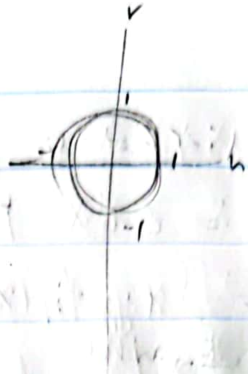
$E[\mathrm{Max}(X_1, X_2)] + E[\mathrm{Min}(X_1, X_2)]$

$= \frac{1}{2} E(X_1 + X_2) - \cancel{E|X_1 - X_2|} + \frac{1}{2} E(X_1 + X_2) + \cancel{E|X_1 - X_2|}$

$= E(X_1 + X_2) = E(X_1) + E(X_2)$

#

2. $f_{UV}(u,v) = C$



$-1 \le u \le 1$

$-1 \le v \le 1$

$\iint f_{UV}(u,v)\, du\,dv = 1$

$\Rightarrow \pi \cdot 1^2 \cdot c = 1$

$\Rightarrow c = \frac{1}{\pi}$

$$f_{UV}(u,v) = \begin{cases} \frac{1}{\pi} & ;\ 0 \le u^2+v^2 < 1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$$P(R \le r) = \iint_{0 \le u^2+v^2 \le 1} f(u,v)\, dA = \int_0^{2\pi}\int_0^{r} \frac{1}{\pi} \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{1}{2\pi} r^2 \Big|_0^r\right] d\theta = \int_0^{2\pi} \frac{1}{2\pi} r^2\, d\theta = \frac{r^2}{2\pi}\left[\theta \Big|_0^{2\pi}\right]$$

$$= \frac{r^2}{\cancel{2\pi}} \cdot \cancel{2\pi} = r^2$$

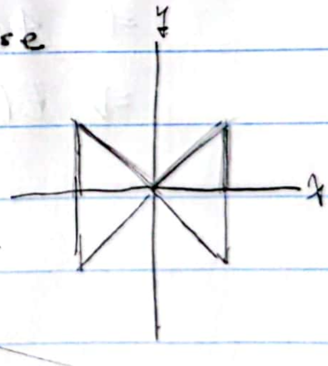
\#

3.

$$f_{XY}(x,y) = \begin{cases} k & ,\ 0 \le |x| \le |y| \ ,\ 0 \le |y| \le 1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$0 \le y \le 1$, $0 \le x \le y$

$-1 \le y < 0$, $y \le x \le 0$



(a) $\int_0^1 \int_{-|y|}^{|y|} k\, dx\,dy$

$= k\int_0^1 2|y|\, dy$

$= 2k\left(\int_0^1 y\,dy + \int_0^1 (-y)\,dy\right)$

$= 2k\left(\left[\frac{1}{2}y^2\Big|_0^1\right] + \left[-\frac{1}{2}y^2\Big|_1^0\right]\right)$

$= 2k \cdot \left(\frac{1}{2} + \frac{1}{2}\right) = 2k = 1 \Rightarrow k = \frac{1}{2}$

\#

(b) $f_X(x) = \int_0^1 f_{XY}(x,y)\,dy + \int_{-1}^0 f_{XY}(x,y)\,dy = \int_{-1}^{1} \frac{1}{2}dy = \frac{1}{2}\left[y\Big|_{-1}^{1}\right] = 1$

$$f_X(x) = \begin{cases} 1 & ,\ -1 \le x \le 1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$f_Y(y) = \int_0^y f_{XY}(x,y)\,dx + \int_y^0 f_{XY}(x,y)\,dx = \frac{1}{2}\left[x\Big|_{-y}^{y}\right] = y$

$$f_Y(y) = \begin{cases} y & ,\ -1 \le y \le 1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$1 \cdot y \ne \frac{1}{2} \Rightarrow$ not independent

\#

$\frac{2}{3}$ $\frac{y^3}{2} + \frac{y^3}{2}$

(c.) if $Cov(X, Y) = 0 \Rightarrow X \,\&\, Y$ uncorrelated

$Cov(X, Y) = E(X, Y) - EX\,EY$

$E(X, Y) = \iint xy\, f_{XY}(x, y)\, dx\, dy = \int_{-1}^{1}\int_{-|y|}^{|y|} x\, y \cdot \frac{1}{2}\, dx\, dy$

$= \frac{1}{2}\int_{-1}^{1} y \left[\frac{1}{2}x^2 \Big|_{-|y|}^{|y|}\right] dy = \frac{1}{2}\int_{-1}^{1} y^3\, dy = \frac{1}{2}\left[\frac{1}{4}y^4 \Big|_{-1}^{1}\right] = \frac{1}{4}$

$EX = \int_{-|y|}^{|y|} x \cdot 2\, dx = \left[x^2 \Big|_{-|y|}^{|y|}\right] = 2y^2$

$EY = \int_{-1}^{1} y \cdot 2y\, dy = \left[\frac{2}{3}y^3 \Big|_{-1}^{1}\right] = \frac{4}{3}$

$Cov(X, Y) = \frac{1}{2} - \frac{4}{3} \cdot 2y^2 \neq 0 \Rightarrow$ not uncorrelated (correlated) $\times$

4. $E((X+aY)^2) \geq 0$

$\Rightarrow E((X+aY)^2) = E(X^2 + 2aXY + a^2Y^2)$

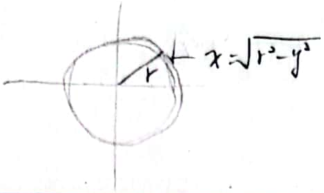
$= E(X^2) + E(2aXY) + a^2E(Y^2) \geq 0$

$\Rightarrow E(2aXY)^2 - 4a^2E(X^2)E(Y^2) \leq 0$

$\Rightarrow \cancel{4a^2}E(X^2Y^2) \leq \cancel{4a^2}E(X^2)E(Y^2)$

$\Rightarrow E(XY)^2 \leq E(X^2)E(Y^2)$

\#



5. $X, Y \sim G(0, \sigma^2)$, $f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$

(a) $R = \sqrt{X^2+Y^2}$

$F_R(r) = P(R \le r) = P(x^2+y^2 \le r^2) = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{XY}(x,y)\,dx\,dy$

$f_R(r) = \frac{dF_R(r)}{dr} = \int_{-r}^{r} \frac{d}{dr}\left(\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{XY}(x,y)\,dx\right) dy = \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$ #

or

$|J| = \begin{vmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{vmatrix} = R\cos^2\theta + R\sin^2\theta = R$

$f(R,\theta) = \frac{1}{2\pi\sigma^2} e^{-\frac{R^2}{2\sigma^2}} \cdot R$

$f_R(R) = \int_0^{2\pi} \frac{R}{2\pi\sigma^2} e^{-\frac{R^2}{2\sigma^2}} d\theta = \frac{R}{2\pi\sigma^2} e^{-\frac{R^2}{2\sigma^2}} \cdot 2\pi = \frac{R}{\sigma^2} e^{-\frac{R^2}{2\sigma^2}}$ #

(b) $f(\theta) = \int_0^\infty \frac{R}{2\pi\sigma^2} e^{-\frac{R^2}{2\sigma^2}} dR = \frac{1}{2\pi\sigma^2}\int_0^\infty R e^{-\frac{R^2}{2\sigma^2}} dR = \frac{1}{2\pi\sigma^2}\left[-\sigma^2 e^{-\frac{R^2}{2\sigma^2}} \Big|_0^\infty\right]$

$= \frac{-1}{2\pi}\left[e^{-\infty} - e^0\right] = -\frac{1}{2\pi}[0-1] = \frac{1}{2\pi}$ #

6.

(a) $F(X_i)$ is a uniform r.v.

$\Rightarrow p = P(F(X_1) < F(X_2) > F(X_3) < F(X_4))$

$= P(U_1 < U_2 > U_3 < U_4)$

$\Rightarrow U_i$ are independent uniform r.v. #

(b) $p = \int_0^1\int_{x_1}^1\int_0^{x_2}\int_{x_3}^1 dx_4\,dx_3\,dx_2\,dx_1 = \int_0^1\int_{x_1}^1\int_0^{x_2} (1-x_3)\,dx_3\,dx_2\,dx_1$

$= \int_0^1\int_{x_1}^1 (x_2 - \frac{1}{2}x_2^2)\,dx_2\,dx_1 = \int_0^1 \left[\frac{1}{2}x_2^2 - \frac{1}{6}x_2^3\Big|_{x_1}^1\right] dx_1$

$= \int_0^1 \left(\frac{1}{3} - (\frac{1}{2}x_1^2 - \frac{1}{6}x_1^3)\right) dx_1 = \left[\frac{1}{3}x_1 - \frac{1}{6}x_1^3 + \frac{1}{24}x_1^4 \Big|_0^1\right]$

$= \frac{1}{3} - \frac{1}{6} + \frac{1}{24} = \frac{8}{24} - \frac{4}{24} + \frac{1}{24} = \frac{5}{24}$ #

(C) $X_4 > X_2 > X_3 > X_1$

$X_4 > X_2 > X_1 > X_3$

$X_2 > X_4 > X_3 > X_1$

$X_2 > X_4 > X_1 > X_3$

$X_2 > X_1 > X_4 > X_3$

$\Rightarrow$ 5 situations can satisfy $X_1 < X_2 > X_3 < X_4$

$p = \frac{5}{4!} = \frac{5}{24}$ ※

7.(a)

```python
import numpy as np

alpha = 2
n = 1000
m = 1
pareto_sp = np.random.pareto(alpha, n) + m

pareto_mean = sum(pareto_sp)/n
print('Pareto expected value')
print(pareto_mean)

print('Pareto n instances')
print(pareto_sp)
```

```
Pareto expected value
1.983697639960465
Pareto n instances
[ 1.08053106  1.04766102  1.24903473  4.09372869  1.24299727  1.82025849
  1.08970901  1.18981287  1.03726724  4.50177948  1.18822937  1.50526825
  1.35315961  1.63364008  1.04035862  2.19196992  1.655347    8.43922521
  1.11709233  1.17710964  1.18948096  2.85624135  1.12627754  1.01305689
  1.68429766  1.3115396   1.0131616   1.5483312   2.75108476  1.83597015
  2.18069962  1.18775485  5.82901134  1.14334492  1.55205764  1.14536735
  1.08393815  1.0259811   4.01026005  1.24069944  1.76557036  1.01597399
  2.19975727  1.44015744  1.14499363  1.18637897  1.66933652  1.13790246
  1.52779394  1.86056364  1.11563953  1.10437874  1.61769524  3.2967876
  1.33810723  1.18030384  1.20524541  1.08853155  1.74226485  1.88731153
  3.67407949  1.38636712 42.96317696  1.29260218  1.62513842  1.40166752
  2.12003822  7.23508002  1.59888586  2.06907459  2.44713535  2.12423305
  1.06277808  5.55825533  1.35592956  2.07411313  2.06802459  5.41587872
  3.65663529  1.31030709  1.53674506  1.75485483  4.13001461  1.08581991
  1.34596005  1.02616042  1.3924529   1.35009954  1.85321101  4.29942394
  4.34154867  1.10368827  1.19961055  1.60641551  1.27695211  1.6939387
  1.47037358  1.41572883  1.06377987  2.07388736  1.01668593  1.10157332
  4.00579581  2.94194528  1.58808729  1.93620081  1.30507274  1.16402569
  2.75950324  1.97243112  1.41302462  1.32005427  2.24497601  1.56965836
  1.88899268  1.21730704  2.64759912  4.97608042  2.43833553  1.52211617
  1.1626627   1.41709459  1.599238    2.91270691  5.26669203  2.98569213
  1.03172587  1.24242746  1.56897697  4.60174987 13.43474453  1.06566333
  1.39373987  1.08533633  4.8052882   1.19039869  5.28778557  1.65803719
  1.61147497  1.32899975  1.55729221  1.59692848  1.08523256  5.12093085
  1.63376254  1.17263608  1.12184605  1.4406705   3.04672829  4.12601783
  1.61317833  2.31466928  1.55837351  1.52680085  1.3212437   1.83815424
```

(b)

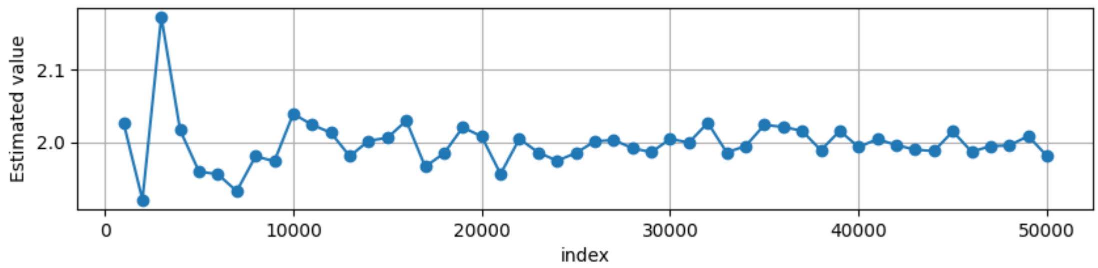
```python
import numpy as np

alpha = 2
m = 1
n0 = 1000
s = 1000
n1 = n0 + 50 * s

pareto_means = []
index = []

for n in range(n0, n1, s):
    pareto_sp = np.random.pareto(alpha, n) + m
    pareto_mean = sum(pareto_sp)/n
    pareto_means.append(pareto_mean)
    index.append(n)

plt.figure(figsize = (10, 2))
plt.plot(index, pareto_means, marker='o', linestyle='-')
plt.xlabel('index')
plt.ylabel('Estimated value')
plt.grid(True)
plt.show()
```



(c)

```python
import numpy as np
pareto_ov_means = sum(pareto_means)/50

alpha = 2
m = 1
n0 = 1000
s = 1000
N = n0

count = 0
while count < 50:
    pareto_sp1 = np.random.pareto(alpha, N) + m
    pareto_mean1 = np.mean(pareto_sp1)

    if (abs(pareto_mean1 - pareto_ov_means) / pareto_ov_means) <= 0.02:
        count += 1
    else:
        count = 0
    N += s
N
```

```
114000
```

(d)

```python
import numpy as np

alpha = 2
n = 1000
m = 1
exp_sp = np.random.default_rng().exponential(scale = 2, size = n
exp_sp_dec = np.vectorize(lambda x: '{:.8f}'.format(x))(exp_sp)

exp_mean = sum(exp_sp)/n
print('Pareto expected value')
print(exp_mean)

print('Pareto n instances')
print(exp_sp_dec)
```

```
Pareto expected value
2.0049024459682303
Pareto n instances
['0.41772096' '0.53777227' '0.21788360' '2.86284139' '0.18086242'
 '6.20351019' '0.91941810' '0.89671100' '1.22495257' '1.83536115'
 '10.52622547' '1.06236664' '7.25597900' '3.96365787' '4.18028153'
 '0.48922893' '0.88888353' '0.28241952' '1.20431018' '0.78942299'
 '5.02080953' '2.30879061' '1.79468531' '3.81639275' '1.28174049'
 '0.81024434' '1.53942710' '1.47990245' '1.32719158' '0.99196533'
 '0.86973927' '0.50861860' '0.19339552' '1.63723840' '4.24646056'
 '1.09578311' '1.37376977' '0.24190463' '4.09196377' '1.60930060'
 '2.21985173' '0.19388860' '0.70885753' '6.68615599' '2.40234256'
 '4.19385089' '0.25532336' '1.56486095' '7.42683472' '3.49801079'
 '0.48452312' '1.28995084' '2.61873373' '0.07201491' '0.22331962'
 '0.37250094' '1.45879904' '0.78171181' '5.10641977' '2.20607054'
 '1.03480734' '7.51055430' '1.30800650' '0.77479881' '0.87993638'
 '4.00792593' '3.87162065' '1.11988315' '0.62708037' '4.26017801'
 '3.72730046' '4.01459959' '3.72779827' '2.44775487' '0.80813462'
 '0.17408890' '0.48256920' '0.60524278' '0.14038111' '7.36383856'
 '0.88025100' '1.95656273' '3.67354787' '0.89715675' '1.77980236'
 '0.12191509' '0.19766046' '0.04376215' '0.99273147' '2.59763373'
 '4.34231949' '2.45934316' '1.06766422' '0.83659872' '2.25228668'
 '3.00867010' '2.83001088' '4.04312105' '2.53277850' '0.32685491'
 '4.86003053' '5.52873038' '1.00633480' '0.18568488' '4.37418311'
 '0.25831971' '1.59955804' '0.95383779' '2.45539358' '2.53504782'
 '2.29358767' '9.55681529' '5.94194866' '3.11999540' '0.66050173'
 '2.02161422' '0.68655346' '0.79358820' '2.51252814' '0.24730033'
 '3.30193445' '2.82496789' '4.06118818' '1.37765831' '2.45844923'
```

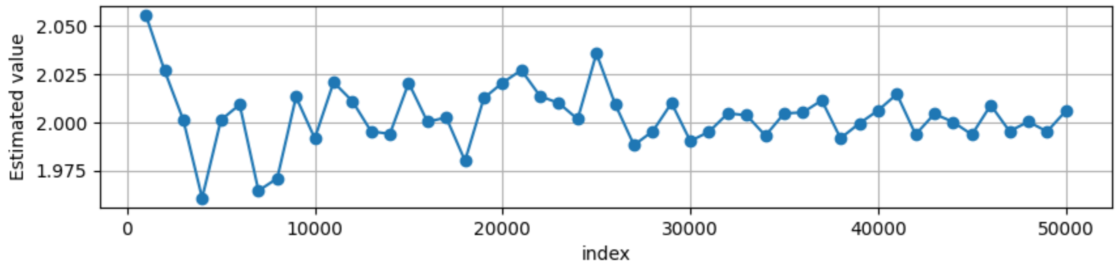
```python
import numpy as np

alpha = 2
m = 1
n0 = 1000
s = 1000
n1 = n0 + 50 * s

exp_means = []
index = []

for n in range(n0, n1, s):
    exp_sp = np.random.default_rng().exponential(scale = 2, size = n)
    exp_mean = sum(exp_sp)/n
    exp_means.append(exp_mean)
    index.append(n)

plt.figure(figsize = (10, 2))
plt.plot(index, exp_means, marker='o', linestyle='-')
plt.xlabel('index')
plt.ylabel('Estimated value')
plt.grid(True)
plt.show()
```




```python
import numpy as np
exp_ov_means = sum(exp_means)/50

alpha = 2
m = 1
n0 = 1000
s = 1000
N = n0

count = 0
while count < 50:
    exp_spl = np.random.default_rng().exponential(scale = 2, size = n)
    exp_meanl = np.mean(exp_spl)

    if (abs(exp_meanl - exp_ov_means) / exp_ov_means) <= 0.02:
        count += 1
    else:
        count = 0
    N += s
N
```

51000

(e) The N1 value for the Pareto distribution is 114000, while for the exponential distribution, it is 51000. Based on these results, it appears that the N1 value for the exponential distribution is significantly smaller than that of the Pareto distribution. This suggests that the results from the exponential distribution are more stable, while the Pareto distribution may more frequently experience sudden divergences leading to the resetting of the counter and necessitating a larger N1 value.