EE 503 : Homework 4

Due : 09/28/2023, Thursday before class

1. Let X_1 and X_2 be two random variables. Find

$$G = E\left[\max\left(X_1, X_2\right)\right] + E\left[\min\left(X_1, X_2\right)\right]$$

2. A point is uniformly distributed within the disk of radius 1, i.e., its density is

$$f(u, v) = C, \quad 0 \leq u^2 + v^2 \leq 1$$

   Find the probability that its distance from the origin is less than x, $0 \leq x \leq 1$.

3. Two continuous random variables X and Y are described by the pdf

$$f_{XY}(x, y) = \begin{cases} k, & \text{if } 0 \leq\mid x \mid\leq\mid y \mid,\ 0 \leq\mid y \mid\leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   where k is a constant.

   a) Find k.
   b) Are X and Y independent?
   c) Are X and Y uncorrelated?

   (Reminder: Two random variables X and Y are uncorrelated if $E(XY) = E(X)E(Y)$)

4. *Cauchy-Schwartz Inequality*: Prove the following inequality:

$$[E(XY)]^2 \leq E(X^2)E(Y^2)$$

   (Hint: Use the fact that for any real a, $E((X + aY)^2) \geq 0$.)

5. X and Y are two i.i.d Gaussian random variables with mean 0 and variance σ^2. Find the pdf of

   a) $R = \sqrt{X^2 + Y^2}$
   b) $\theta = \arctan(Y/X)$

   Hint: $X = R\cos\theta$, $Y = R\sin\theta$

6. Let X_1, X_2, X_3 and X_4 be independent continuous random variables with a common density function f and let $p = P(X_1 < X_2 > X_3 < X_4)$

   a) Argue that the value of p is the same for all continuous density functions f. (Hint: Remember the inverse transformation method of generating random variables)

b) Find p by integrating the joint density function over the appropriate region.

c) Find p by using the fact that all 4! possible orderings of $X_1, \cdots, X_4$ are equally likely.

7. **Simulation Problem**: Let $x_1, x_2, x_3, \cdots, x_n$ be independent random variables having a common distribution X. We can use $\frac{\sum_{i=1}^{n} x_i}{n}$ as an estimation for the expected value of X. Let $N_0 = 1000$, and $S = 1000$.

   a) Use *np.random.pareto(α, n)* from the Numpy library to create n instances of a Pareto distribution with parameters $\alpha = 2$ and $m = 1$.

   b) Choose a value $N_1 > N_0 + 50S$. Vary n from N_0 to N_1 by steps of size S, and estimate the expected value of the Pareto distribution for each n. Plot your estimation for each value of n.

   c) By observing the plot of part b, find (by trial and error) the value N_1 for which your estimation lies within 2% of the expected value of the Pareto distribution for 50 consecutive steps before N_1, i.e., your estimation must lie within 2% of the expected value for $n = N_1 - 50S, N_1 - 49S, N_1 - 48S, \cdots, N_1$.

   d) Repeat parts a - c for an exponential distribution with parameter $\lambda = 2$. You can use *np.random.exponential(λ, n)* from the Numpy library.

   e) Compare N_1 derived in part c for the Pareto and exponential distributions. What is your observation?