A Minimal Control Multiagent for Collision Avoidance and Velocity Alignment

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***Abstract*—This paper investigates a group of multiagents moving on a 2-D plane with a constant speed but maneuverable headings. It is called a minimal control multiagent model (MCMA) when each agent employs a static decentralized control law that relies on its neighbors' relative positions with respect to its local reference frame. In other words, the control law does not involve any complicated velocity measurement or estimation mechanism. Various minimal multiagent models have been investigated and extensively simulated in terms of their collaborative behaviors. This paper, for the first time, gives rigorous theoretical proofs for the functionalities of an MCMA model in both collision avoidance and velocity alignment.**

***Index Terms*—Collective motions, collision avoidance, cooperative control, flocking, multiagent systems (MASs), synchronization.**

I. Introduction

It IS appealing that highly coordinated collective behaviors in natural systems, such as congregation, synchronization, migration, and cooperative hunting, emerge by simple individual intelligence and communication. For control engineering systems, it is also an interesting topic to design a simple control algorithm for multiple agents of simple dynamics to accomplish complicated collective missions. A simple control algorithm has obvious advantages in saving costs. Also, it is scientifically interesting to reveal minimal agent intelligence required for specified collective behaviors.

To describe simple individual intelligence and communication, a class of "minimal models" has been studied by statistical physicists and mathematicians. It was summarized in a recent survey paper [1] that "the simplest (most minimal) model...looks like this: The particles are trying to maintain a given absolute velocity and the only interaction between them is a repulsive linear force within a short distance (i.e., they do not 'calculate' the average of the velocity of their neighbors, and the only interaction is through a pair-wise central force)." Analogous results on simple models were discussed in [2]–[5] and the references therein.

Motivated by the aforementioned minimal model, we define a minimal control multiagent (MCMA) model in this paper as it satisfies the following three properties.

1) Each agent moving on a 2-D plane has a simple kinematic model with a constant speed but maneuverable heading.
2) Each agent employs a static control law for heading.
3) The control law is decentralized and only relies on the measurement of its neighbors' relative positions with respect to its local reference frame.

The main objective of this paper is to investigate an MCMA model and provide theoretical analysis in terms of its functionalities for achieving two remarkable collective behaviors, collision avoidance and velocity alignment.

It is well accepted that a three layer model with the rules of separation, alignment, and cohesion, plays a typical mechanism for collective behaviors (see [6]). In particular, the separation rule aims at collision avoidance for too close agents, which is caused by interagent repulsive force. In most existing works, a potential field can be used for characterizing such repulsive force. The idea can be traced back to 1980s in [7] where the artificial potential field approach was introduced for collision avoidance. Koditschek and Rimon [8], [9] studied the local minima problem of the potential field approach and a navigation function was developed to navigate a robot through a field with spherical obstacles. Later, various theoretic frameworks for obstacle/collision avoidance were studied in [10]–[14], among others. Collision avoidance and flocking behaviors were also studied in a unified framework. Recently, it was explicitly discussed in [15] that how much control is enough for collision avoidance when agents move along the gradient of a potential energy function.

The essential mechanism for the aforementioned references is as follows. Along the gradient of a potential energy function, or called a navigation function, an agent has the intelligence to modify its velocity to avoid collision when it is too close to other agents and/or obstacles. Obviously, none of the existing works based on a navigation function is applicable for an MCMA model that is equipped with a constant speed due to the aforementioned property 1). In this paper, we prove a novel mechanism that collision can be avoided for agents by simply turning their headings while maintaining a constant speed. The alignment rule of the three layer model (see [6]) attempts

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to match velocity with nearby group mates. Essentially, each agent measures and calculates the average velocity (in a certain sense) of its neighbors and use the information to adjust its own behavior. As a result, a multiagent system achieves velocity alignment, usually, called flocking. In the aforementioned works for collision avoidance, a flocking phenomenon using the alignment rule has also well studied (see [16]–[20]). However, this alignment rule is not applicable for an MCMA model as a neighbor's velocity is not measurable due to the property 3). It is also the reason that a minimal model is classified as "a model without explicit alignment rule" in [1].

It is worth mentioning that direct measurement of neighbor's velocities can be relaxed by a dynamic velocity observation technique in [21]–[23]. However, this technique is not applicable for an MCMA model either due to the property 2). In this paper, we prove another novel mechanism that velocity alignment can be achieved for agents by the guidance of their motion space boundary without relying on neighbors' velocity information.

The remainder of this paper is organized as follows. Section II gives the formulation of an MCMA model. Collision avoidance and velocity alignment are investigated in Sections III and IV, respectively. A simulation example is shown in Section V. Finally, some concluding remarks are drawn in Section VI.

II. Minimal Control Multiagent Model

Consider a group of $N \geq 2$ agents, marked by the integers $1, \dots, N$. Assume the agents move between to parallel walls, marked by the two letters L and R. Denote $\mathbb{N} := \{1, \dots, N\}$ and $\mathbb{N}^+ := \{1, \dots, N, L, R\}$. Also, denote the global 2-D reference frame as Σ with x–y axes. The position of the agent i in Σ is denoted as $p_i = [x_i, y_i]^\top \in \mathbb{R}^2$ and the two walls are along the y-axis, i.e., $x = x_L$ and $x = x_R$, respectively. Without loss of generality, $x_R > x_L$.

Also, it is assumed that all the agents move with a constant speed $v_o > 0 \in \mathbb{R}$. Denote the velocity of the agent i in the global reference frame Σ as $v_i \in \mathbb{R}^2$. Let $\theta_i \in \mathbb{R}$ be the moving heading (i.e., the angle of velocity) with respect to the global reference frame Σ. Then, the agent i has the kinematic equation

$$\dot{p}_i = v_i = \begin{bmatrix} \mu_i \\ w_i \end{bmatrix} = v_o \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix}, \quad i \in \mathbb{N}. \tag{1}$$

The main target of this paper is on analyzing a decentralized strategy for controlling each agent's moving heading θ_i to achieve desired collaborative behaviors.

The relative position between each two agents i and j is denoted as

$$p_{ij} := p_i - p_j, \quad i, j \in \mathbb{N} \tag{2}$$

and the relative position between each agent i and the wall L or R is

$$p_{ij} := \begin{bmatrix} x_i - x_j \\ 0 \end{bmatrix}, \quad i \in \mathbb{N}, \quad j = L, R. \tag{3}$$

Assume each agent has a sensing range r, based on which, one can define a neighborhood for each agent i as follows:

$$\mathcal{N}_i := \{j \in \mathbb{N}^+ \mid \|p_{ij}\| \leq r, \ j \neq i\}, \ i \in \mathbb{N}. \tag{4}$$

Any element of $\mathcal{N}_i$ is called a neighbor of the agent i.

In a general scenario, the global reference frame Σ is not available for any agent. Therefore, p_{ij} is not measurable for any agent. For each agent i, only the distances between the agent i and its neighbors and its neighbors' position angle relative to its velocity direction θ_i are measurable. More specifically, the distance and the relative position angle are

$$d_{ij} := \|p_{ij}\|, \ \rho_{ij} := \angle p_{ij} - \theta_i, \ j \in \mathcal{N}_i$$

respectively, for every agent $i \in \mathbb{N}$. In other words, the relative position between the agent i and its neighbors with respect to its moving direction is

$$\begin{aligned} \bar{p}_{ij} &= d_{ij} \begin{bmatrix} \cos\rho_{ij} \\ \sin\rho_{ij} \end{bmatrix} = d_{ij} R(-\theta_i) \begin{bmatrix} \cos\angle p_{ij} \\ \sin\angle p_{ij} \end{bmatrix} \\ &= R(-\theta_i) p_{ij}, \ j \in \mathcal{N}_i \end{aligned} \tag{5}$$

which is measurable. Here the matrix

$$R(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{6}$$

is a rotation matrix.

To derive a control protocol for θ_i, we define the repulsive force that is applied on the agent i but generated from its neighbor $j \in \mathcal{N}_i$, as follows:

$$F_{ij} = \frac{\bar{p}_{ij}\big(r - \|\bar{p}_{ij}\|\big)}{\|\bar{p}_{ij}\|} \mathrm{sat}_\gamma \left(\frac{1}{\|\bar{p}_{ij}\| - d} \right) \in \mathbb{R}^2 \tag{7}$$

which is measurable as $\bar{p}_{ij}$. The saturation function is defined as

$$\mathrm{sat}_\gamma(x) := \begin{cases} x, & \gamma \geq x \geq 0 \\ \gamma, & x > \gamma \end{cases}$$

for $\gamma > 0$. The constant $d > 0$ is regarded as the collision radius for each agent, which must satisfy $d < r$ and $2d < x_R - x_L$ for a well posed scenario. It can be verified that $\|F_{ij}\| \leq r\gamma$ for $j \in \mathcal{N}_i$.

The structure of (7) is further explained as follows. The factor $\bar{p}_{ij}/\|\bar{p}_{ij}\|$ determines the force direction applied on an agent with respect to its moving heading. Two agents affect each other only when they are within the specified sensing range r and the effect is inversely proportional to their distance according to $r - \|\bar{p}_{ij}\|$. When two agents get closer, more significant repulsive force is required for collision avoidance according to $\mathrm{sat}_\gamma(1/(\|\bar{p}_{ij}\| - d))$.

The rotational acceleration caused by F_{ij} is calculated as

$$\omega_{ij} = \begin{cases} -\alpha_i \|F_{ij}\| c(\phi_{ij}) s(\phi_{ij}), & \phi_{ij} \in (\pi/2, 3\pi/2) \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

where $\alpha_i \in \mathbb{R}$ is any constant positive number representing the updating rate and $\phi_{ij} \in [0, 2\pi)$ is angle of F_{ij} with respect to the moving direction θ_i. The function c is arbitrarily selected to satisfy

$$-1 \leq c(\phi_{ij}) < 0, \ \forall \phi_{ij} \in (\pi/2, 3\pi/2). \tag{9}$$

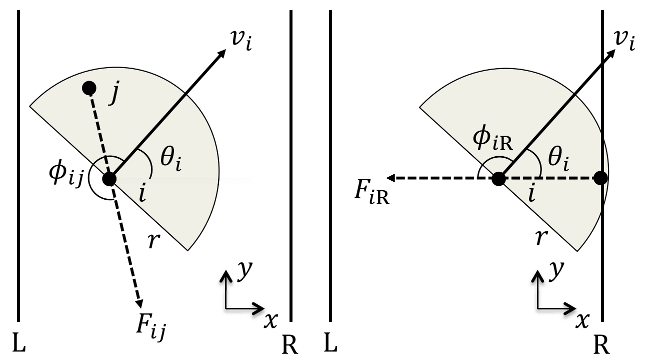

Fig. 1. Illustration of a minimal controller mechanism. Agent (labeled i) is able to detect any neighbored agent (the agent j in the left graph and the right wall R in the right graph) within its front semicircle of radius r. The neighbored agent j is located from the left front (left graph) and causes the agent i to turn right or clockwise. The neighbored wall R is located from the right front (right graph) and causes the agent i to turn left or counterclockwise.

It can be a cosine function throughout the paper. In (8), $s(\phi_i) := \operatorname{sgn}\sin(\phi_i)$ with the sign function defined as

$$\operatorname{sgn}(x) := \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}.$$

Now, it is ready to give the so-called minimal controller

$$\dot{\theta}_i = \sum_{j \in \mathcal{N}_i} \omega_{ij} + \sigma_i \tag{10}$$

where σ_i represents time varying disturbances, typically, measurement errors. These disturbances are assumed to be bounded for analysis of collision avoidance but vanish for velocity alignment.

The controller (10) is called a minimal controller as it satisfied the properties 2) and 3) specified in Section I. Therefore, the closed-loop system composed of (1) that satisfies the property 1), and (10) is called an MCMA model.

The physical mechanism underlying the controller (10) is illustrated in Fig. 1 and explained below. When $\phi_{ij} \in [0, \pi/2] \cup [3\pi/2, 2\pi)$, the repulsive force F_{ij} (in terms of its projection on v_i) and v_i has the same direction. It means that the agent i detects the effective force source from the back (an agent defines its front and back facing its moving direction) and ignore it with $\omega_{ij} = 0$. When $\phi_{ij} \in (\pi/2, \pi]$, the repulsive force F_{ij} and v_i has opposite directions. It means that the agent i detects the effective force source from the front, in particular, from the right front (an agent also defines its left and right facing its moving direction). As a result, the agent turns counter-clockwise with $\omega_{ij} = -\alpha_i \|F_{ij}\| \cos(\phi_{ij}) s(\phi_{ij}) = -\alpha_i \|F_{ij}\| \cos(\phi_{ij}) > 0$. When $\phi_{ij} \in (\pi, 3\pi/2)$, the agent i detects the effective force source from the front, in particular, from the left front. As a result, the agent turns right or clockwise with $\omega_{ij} = -\alpha_i \|F_{ij}\| \cos(\phi_{ij}) s(\phi_{ij}) = \alpha_i \|F_{ij}\| \cos(\phi_{ij}) < 0$.

In this paper, we will show that the MCMA model formulated in this section can achieve collision avoidance and velocity alignment.

III. Collision Avoidance

For any pair of agents (or an agent and a wall) $i, j \in \mathbb{N}^+$, we say they marginally collide at t if $\|p_{ij}(t)\| = d$; and collide at t if $\|p_{ij}(t)\| < d$. In this section, we aim to show that the minimal controller (10) can guarantee collision not occur, i.e., $\|p_{ij}(t)\| \geq d, \ \forall t \geq 0$. Let $t_1, t_2 \in [0, t_\infty]$ be two discrete time instants when two pairs of agents start to marginally collide, respectively. For any distribution over $[0, t_\infty]$ with a finite probability density function, the probability $P(t_1 = t_2)$ is always zero. In other words, almost no more than one pairs of agents marginally collide simultaneously.

It should be noted that the sign of ω_{ij} defined in (8) impulsively changes at $\phi_{ij} = \pi$. In particular, the force F_{ij} with the disturbance σ_{ij} may cause the sign of ω_{ij} frequently change. In an extreme situation, we call the change frequency is infinitely high at a time t^*, if, $\phi_{ij}(t^*) = \pi$ and for any small $\tau > 0$, there exist $\tau_1, \tau_2 < \tau$ such that $\phi_{ij}(t^* - \tau_1) > \pi$ and $\phi_{ij}(t^* - \tau_2) \leq \pi$. In other words, in any short time $(t^* - \tau, t^*)$, with τ approaching zero, the sign of $\omega_{ij}(t)$ changes.

In this paper, we consider a *generic* setup (in terms of generic initial conditions and disturbances for the fixed closed-loop system) that neither the zero probability scenario with more than one pairs marginally colliding simultaneously nor the aforementioned infinitely high frequency change happens during the entire evolution. The main theorem is stated below.

Theorem 1: Consider the MCMA model composed of (1) and (10) in a generic setup. If the initial conditions satisfy $\|p_{ij}(0)\| > d, \ \forall i \in \mathbb{N}, \ j \in \mathbb{N}^+$ and the disturbances are bounded in the sense of $\|\sigma_i(t)\| \leq \sigma, \ \forall i \in \mathbb{N}, \ \forall t \geq 0$, for a constant σ, then there exists finite repulsive force such that collision does not occur during the entire evolution, i.e., $\|p_{ij}(t)\| \geq d, \ \forall i \in \mathbb{N}, j \in \mathbb{N}^+, \forall t \geq 0$.

Proof: Let $T > 0$ be the time instant when marginal collision starts to occur for the pair $\ell \in \mathbb{N}$ and $\hbar \in \mathbb{N}^+$ with $\|p_{\ell\hbar}(T)\| = d$ and $\|p_{\ell\hbar}(t)\| > d$ for $t \in [T - \tau_o, T)$ for some $\tau_0 > 0$. In particular, there exists $\tau_0 > 0$ such that

$$0 < \frac{d(-\|p_{\ell\hbar}(t)\|)}{dt} \leq 2v_o, \forall t \in [T - \tau_0, T).$$

Define a temporary reference frame Σ^* with $p_\ell(T)$ as the origin and $p_{\ell\hbar}(T)$ as the negative x-axis. In the reference frame Σ^*, the position coordinate of the agent ℓ is $p^*_\ell(T) = [0, 0]^\intercal$ and that of the agent $\hbar$ (or the projection of ℓ on the wall $\hbar$) is $p^*_\hbar(T) = [d, 0]^\intercal$. So, the relative position between ℓ and $\hbar$ is $p^*_{\ell\hbar}(T) = [-d, 0]^\intercal$. Also, in the reference frame Σ^*, denote the moving direction of the agent ℓ as $\theta^*_\ell \in [0, 2\pi)$.

We will prove the claim that $\theta^*_\ell(T) \in [\pi/2, 3\pi/2]$. If the claim is not true, one has $\theta^*_\ell(T) \in [0, \pi/2) \cup (3\pi/2, 2\pi)$ and will derive contradiction below.

First, we define a constant δ_1 according to

$$2\delta_1 = \begin{cases} \pi/2 - \theta^*_\ell(T), & \theta^*_\ell(T) \in [0, \pi/2) \\ \theta^*_\ell(T) - 3\pi/2, & \theta^*_\ell(T) \in (3\pi/2, 2\pi). \end{cases}$$

Obviously, one has $\pi/4 \geq \delta_1 > 0$ and $\theta^*_\ell(T) = \pi/2 - 2\delta_1$ or $3\pi/2 + 2\delta_1$. It implies that

$$\theta^*(T) \in [0, \pi/2 - 2\delta_1] \cup [3\pi/2 + 2\delta_1, 2\pi).$$

As $\theta^*_\ell(t)$ is continuous in t. There exists $\tau_1 > 0$ such that

$$\theta^*(t) \in [0, \pi/2 - \delta_1) \cup (3\pi/2 + \delta_1, 2\pi), \forall t \in [T - \tau_1, T]. \tag{11}$$

Next, as it is assumed that no more than one pairs marginally collide simultaneously, the distance between the agent ℓ and any agent/wall other than $\hbar$ is larger than d, at T. In particular, we denote

$$\bar{d} = \min_{i\in\mathbb{N}^+, i\neq\hbar} \|p_{\ell i}(T)\| > d.$$

Let $\tilde{d} = (\bar{d} - d)/2$. As $p_{\ell i}(t)$ is a continuous function in t, there exists $\tau_2 > 0$ such that

$$\|p_{\ell i}(t)\| \geq \bar{d} - \tilde{d} = d + \tilde{d},\ \forall t \in [T - \tau_2, T].$$

From (10), one has $\dot{\theta}_\ell = \omega_{\ell\hbar} + \tilde{\omega}_\ell$ for $\tilde{\omega}_\ell = \sum_{i\in\mathcal{N}_\ell, i\neq\hbar} \omega_{\ell i} + \sigma_\ell$. On one hand, by noting $\|\bar{p}_{\ell i}\| = \|p_{\ell i}\|$, one has

$$\|\tilde{F}_{\ell i}\| = \left\| \frac{\bar{p}_{\ell i}}{\|\bar{p}_{\ell i}\|} \frac{r - \|\bar{p}_{\ell i}\|}{\|\bar{p}_{\ell i}\| - d} \right\| \leq \frac{r-d}{(d+\tilde{d}) - d} = \frac{r-d}{\tilde{d}}$$

for $\gamma > 1/\tilde{d}$. Therefore

$$\|\tilde{\omega}_\ell(t)\| < \frac{\alpha_\ell N(r-d)}{\tilde{d}} + \sigma := \Delta,\ \forall t \in [T - \tau_2, T] \tag{12}$$

for a finite constant $\Delta > 0$. On the other hand

$$\lim_{t\to T} \|F_{\ell\hbar}(t)\| = \lim_{t\to T} \frac{r - \|\bar{p}_{\ell\hbar}(t)\|}{\|\bar{p}_{\ell\hbar}(t)\| - d} = \infty.$$

Since $\angle F_{\ell\hbar}(T) = \angle \bar{p}_{\ell\hbar}(T)$, one has

$$\begin{aligned} \phi_{\ell\hbar}(T) &= \angle F_{\ell\hbar}(T) = \angle \bar{p}_{\ell\hbar}(T) = \mathrm{mod}\{\angle p_{\ell\hbar}(T) - \theta_\ell(T)\} \\ &= \mathrm{mod}\big\{\angle p^*_{\ell\hbar}(T) - \theta^*_\ell(T), 2\pi\big\} = \mathrm{mod}\big\{\pi - \theta^*_\ell(T)\big\}. \end{aligned}$$

Therefore, $\phi_{\ell\hbar}(T) \in (\pi/2, 3\pi/2)$ as $\theta^*_\ell(T) \in [0, \pi/2) \cup (3\pi/2, 2\pi)$.

Similarly, we define a constant δ_2 according to

$$2\delta_2 = \begin{cases} \phi_{\ell\hbar}(T) - \pi/2, & \phi_{\ell\hbar}(T) \in (\pi/2, \pi] \\ 3\pi/2 - \phi_{\ell\hbar}(T), & \phi_{\ell\hbar}(T) \in (\pi, 3\pi/2) \end{cases}.$$

Obviously, one has $\pi/4 \geq \delta_2 > 0$ and $\phi_{\ell\hbar}(T) = \pi/2 + 2\delta_2$ or $3\pi/2 - 2\delta_2$. It implies that

$$\phi_{\ell\hbar}(T) \in [\pi/2 + 2\delta_2, 3\pi/2 - 2\delta_2].$$

As $\phi_{\ell\hbar}(t)$ is continuous in t, there exists $\tau_3 > 0$ such that

$$\phi_{\ell\hbar}(t) \in (\pi/2 + \delta_2, 3\pi/2 - \delta_2),\ \forall t \in [T - \tau_3, T].$$

With the generic setup, no change of $s(\phi_{\ell\hbar}(t))$ happens with infinitely high frequency. Therefore, there exists $0 < \tau_4 \leq \tau_3$ such that

$$\phi_{\ell\hbar}(t) \in (\pi/2 + \delta_2, \pi],\ \forall t \in [T - \tau_4, T), \text{ or} \tag{13}$$

$$\phi_{\ell\hbar}(t) \in (\pi, 3\pi/2 - \delta_2],\ \forall t \in [T - \tau_4, T). \tag{14}$$

In what follows, it is assumed (13) holds. The proof is similar for the case with (14).

We pick a constant $0 < \tau \leq \min\{\tau_0, \tau_1, \tau_2, \tau_3, \tau_4\}$. Some calculation follows. Noting (13), one has

$$\begin{aligned} \int_{T-\tau}^T \omega_{\ell\hbar}(t)dt &= \int_{T-\tau_1}^T -\alpha_\ell \|F_{\ell\hbar}\| \cos(\phi_{\ell\hbar}) s(\phi_{\ell\hbar}) dt \\ &= \int_{T-\tau}^T -\alpha_\ell \|F_{\ell\hbar}\| \cos(\phi_{\ell\hbar}) dt \\ &\geq \int_{T-\tau}^T -\alpha_\ell \|F_{\ell\hbar}\| \cos(\pi/2 + \delta_2) dt \\ &= \int_{T-\tau}^T \alpha_\ell \|F_{\ell\hbar}\| \sin(\delta_2) dt. \end{aligned}$$

It, together with (12), implies

$$\begin{aligned} \int_{T-\tau}^T \dot{\theta}(t)dt &= \int_{T-\tau}^T \omega_{\ell\hbar}(t)dt + \int_{T-\tau}^T \tilde{\omega}_\ell(t)dt \\ &\geq \int_{T-\tau}^T \omega_{\ell\hbar}(t)dt - \int_{T-\tau}^T \|\tilde{\omega}_\ell(t)\|dt \\ &\geq \int_{T-\tau}^T \alpha_\ell \|F_{\ell\hbar}\| \sin(\delta_2)dt - \tau\Delta \\ &\geq \alpha_\ell(r-d)\sin(\delta_2) \int_{T-\tau}^T \mathrm{sat}_\gamma\left(\frac{1}{\|\bar{p}_{\ell\hbar}\| - d}\right)dt - \tau\Delta. \end{aligned}$$

For

$$\begin{aligned} &\int_{T-\tau}^T \frac{1}{\|\bar{p}_{\ell\hbar}\| - d}dt \\ &= \int_d^{\|p_{\ell\hbar}(T-\tau)\|} \frac{1}{\|\bar{p}_{\ell\hbar}\| - d} \frac{1}{d(-\|p_{\ell\hbar}\|)/dt} d(-\|\bar{p}_{\ell\hbar}\|) \\ &\geq \frac{1}{2v_o} \int_d^{\|p_{\ell\hbar}(T-\tau)\|} \frac{1}{\|\bar{p}_{\ell\hbar}\| - d} d\|\bar{p}_{\ell\hbar}\| \\ &= \frac{1}{2v_o}\big[\ln(\|p_{\ell\hbar}(T-\tau)\| - d) - \ln(0)\big] = \infty \end{aligned}$$

there exists a constant $\tau > \epsilon > 0$ such that

$$\int_{T-\tau}^{T-\epsilon} \frac{1}{\|\bar{p}_{\ell\hbar}\| - d}dt \geq \frac{\pi + \tau\Delta}{\alpha_\ell(r-d)\sin(\delta_2)}.$$

The following finite γ (implying finite repulsive force):

$$\gamma = \max_{t\in[T-\tau, T-\epsilon]} \left\{ \frac{1}{\|\bar{p}_{\ell\hbar}(t)\| - d} \right\} \tag{15}$$

gives

$$\int_{T-\tau}^T \mathrm{sat}_\gamma\left(\frac{1}{\|\bar{p}_{\ell\hbar}\| - d}\right)dt \geq \frac{\pi + \tau\Delta}{\alpha_\ell(r-d)\sin(\delta_2)}$$

and hence

$$\int_{T-\tau}^T \dot{\theta}(t)dt \geq \alpha_\ell(r-d)\sin(\delta_2)A - \tau\Delta \geq \pi.$$

It contradicts (11). The claim that $\theta^*_\ell(T) \in [\pi/2, 3\pi/2]$ is thus proved.

Denote the velocity of ℓ along the x-axis of the reference frame Σ^* is $v^*_{x\ell}$. From the definition, one has, at T

$$v^*_{x\ell}(T) = v_o \cos\theta^*_\ell(T) \leq 0.$$

Define the coordinate of the agent ℓ in the x-axis of the reference frame Σ^* as x^*_ℓ. By by noting $x^*_\ell(T) = 0$, one has $x^*_\ell(t) \leq 0,\ \forall t \in [T, T + \tilde{T})$ for some $\tilde{T} > 0$.

If $\hbar$ is a wall, then the x-coordinate of the wall in the reference frame Σ^* is a constant $x^*_\hbar(t) = d,\ \forall t \geq 0$. Therefore, one has $\|p_{\ell\hbar}(t)\| = \|x^*_\ell(t) - x^*_\hbar(t)\| \geq d,\ \forall t \in [T, T + \tilde{T})$.

If $\hbar$ is an agent, the velocity of $\hbar$ along the x-axis of the reference frame Σ^* is $v^*_{x\hbar}$. For the same argument, one has $v^*_{x\hbar}(T) \geq 0$. As a result

$$v^*_{x\ell}(T) - v^*_{x\hbar}(T) \leq 0.$$

By noting $x^*_\ell(T) - x^*_\hbar(T) = -d$, one has $x^*_\ell(t) - x^*_\hbar(t) \leq -d,\ \forall t \in [T, T + \tilde{T})$ for some $\tilde{T} > 0$. As a result, one has $\|p_{\ell\hbar}(t)\| \geq \|x^*_\ell(t) - x^*_\hbar(t)\| \geq d, \forall t \in [T, T + \tilde{T})$.

As $T > 0$ can be an arbitrary time instant when marginal collision starts to occur, one has $\|p_{ij}(t)\| \geq d$, $\forall i \in \mathbb{N}, j \in \mathbb{N}^+$, $\forall t \geq 0$. ■

Remark 1: The proof starts from the time instant $T > 0$ where marginal collision starts to occur for the pair $\ell \in \mathbb{N}$ and $\hbar \in \mathbb{N}^+$ with $\|p_{\ell\hbar}(T)\| = d$ and reaches collision avoidance by analyzing the agents' behavior in the time neighborhood $[T - \tau, T)$. The underling assumption is that all agent trajectories are continuous in time due to a constant speed. More specifically, for a larger speed v_o, the agent distribution takes quicker changes, that is, τ and hence ϵ take smaller values throughout the proof. As a result, $\|p_{\ell\hbar}(t)\|$ is closer to d for $t \in [T - \tau, T - \epsilon]$ which leads to a larger γ due to (15). It is noted that explicit calculation of the parameter γ is difficult. Nevertheless, a practical controller can be applied without explicitly involving the saturation function upon the theoretical guarantee that the repulsive force never approaches infinity.

IV. Velocity Alignment

In this section, we investigate velocity alignment for the group of agents. As the MCMA model composed of (1) and (10) does not involve velocity measurement or other alignment mechanism, velocity alignment by the collaboration of the agents is unpractical. In fact, the main idea is to form velocity alignment with the guidance of the two walls.

For rigorous investigation, we define the concepts of *top boundary* and *bottom boundary* of the group. Let

$$y_T(t) = \max\{y_i(t),\ i \in \mathbb{N}\}$$
$$\mathbb{N}_T(t) = \{i \in \mathbb{N} \mid y_i(t) = y_T(t)\}$$
$$w_T(t) = \max\{w_i(t),\ i \in \mathbb{N}_T(t)\}$$

be the top boundary position, top boundary agents, and top boundary position velocity of the group of agent $\mathbb{N}$ at time t, respectively. Similarly, we can define the corresponding bottom boundary concepts as follows:

$$y_B(t) = \min\{y_i(t),\ i \in \mathbb{N}\}$$
$$\mathbb{N}_B(t) = \{i \in \mathbb{N} \mid y_i(t) = y_B(t)\}$$
$$w_B(t) = \min\{w_i(t),\ i \in \mathbb{N}_B(t)\}.$$

The main result of this paper is based on the following lemma.

Lemma 1: Consider the MCMA model composed of (1) and (10) without disturbances, i.e., $\|\sigma_i(t)\| = 0$. If there exists $t_o \geq 0$ such that $w_T(t_o) > 0$, then $\lim_{t\to\infty} w_T(t) = v_0$.

Proof: There are two steps in this proof. In the first step, we show that $w_T(t)$ is a monotonically increasing function of t based on the following statements.

Statement 1: If there exist $t_2 > t_1 \geq t_o$ such that $w_T(t_1) > 0$ and $w_T(t) = w_\ell(t)$, $\forall t \in [t_1, t_2)$ for the agent ℓ, then $w_T(t)$ is a monotonically increasing function of t during the interval $[t_1, t_2)$.

Obviously, for $t \in [t_1, t_2)$, ℓ is one of the top boundary agents, i.e., $\ell \in w_T(t)$. Since $w_T(t_1) > 0$, one has $w_T(t) = w_\ell(t) > 0$, $\forall t \in [t_1, \bar{t}_2)$ for some $t_1 < \bar{t}_2 \leq t_2$. As a result, one has

$$\mathrm{mod}\{\theta_\ell(t), 2\pi\} \in (0, \pi),\ \forall t \in [t_1, \bar{t}_2).$$

Next, we will show $w_\ell(t)$ is a monotonically increasing function of t during the interval $[t_1, \bar{t}_2)$ by considering three cases.

1) *mod$\{\theta_\ell(t), 2\pi\} \in (0, \pi/2)$:* We aim to show that $\omega_{\ell j} \geq 0$, $\forall j \in \mathcal{N}_\ell$. By (10) with $\|\sigma_i(t)\| = 0$, it implies that $\dot{\theta}_\ell(t) \geq 0$ and hence $\dot{w}_\ell(t) = d(v_o \sin\theta_\ell(t))/dt = v_o \cos\theta_\ell(t)\dot{\theta}_\ell(t) \geq 0$. In fact, for $j = L$, one has $\phi_{\ell L} = \mathrm{mod}\{-\theta_\ell, 2\pi\} \in (3\pi/2, 2\pi)$ and hence $\omega_{\ell L} = 0$. For $j = R$, one has $\phi_{\ell R} = \mathrm{mod}\{\pi - \theta_\ell, 2\pi\} \in (\pi/2, \pi)$ and hence $\omega_{\ell R} > 0$. If j is a normal agent (not a wall), one has $\angle F_{\ell j} = \angle p_{\ell j} = \angle(p_\ell - p_j) \in [0, \pi]$. It implies $\phi_{\ell j} = \mathrm{mod}\{\angle F_{\ell j} - \theta_\ell, 2\pi\} \in (3\pi/2, 2\pi) \cup [0, \pi)$ and hence $\omega_{\ell j} \geq 0$.
2) *mod$\{\theta_\ell(t), 2\pi\} = \pi/2$:* We aim to show that $\omega_{\ell j} = 0$, $\forall j \in \mathcal{N}_\ell$. By (10), it implies that $\dot{\theta}_\ell(t) = 0$ and $\dot{w}_\ell(t) = d(v_o \sin\theta_\ell(t))/dt = v_o \cos\theta_\ell(t)\dot{\theta}_\ell(t) = 0$. In fact, for $j = L$, one has $\phi_{\ell L} = 3\pi/2$ and hence $\omega_{\ell L} = 0$. For $j = R$, one has $\phi_{\ell R} = \pi/2$ and hence $\omega_{\ell R} = 0$. If j is a normal agent (not a wall), one has $\angle F_{\ell j} = \angle p_{\ell j} = \angle(p_\ell - p_j) \in [0, \pi]$. It implies $\phi_{\ell j} = \mathrm{mod}\{\angle F_{\ell j} - \theta_\ell, 2\pi\} \in [3\pi/2, 2\pi) \cup [0, \pi/2]$ and hence $\omega_{\ell j} = 0$.
3) *mod$\{\theta_\ell(t), 2\pi\} \in (\pi/2, \pi)$:* We aim to show that $\omega_{\ell j} \leq 0$, $\forall j \in \mathcal{N}_\ell$. By (10), it implies that $\dot{\theta}_\ell(t) \leq 0$ and hence $\dot{w}_\ell(t) = d(v_o \sin\theta_\ell(t))/dt = v_o \cos\theta_\ell(t)\dot{\theta}_\ell(t) \geq 0$. In fact, for $j = L$, one has $\phi_{\ell L} = \mathrm{mod}\{-\theta_\ell, 2\pi\} \in (\pi, 3\pi/2)$ and hence $\omega_{\ell L} < 0$. For $j = R$, one has $\phi_{\ell R} = \mathrm{mod}\{\pi - \theta_\ell, 2\pi\} \in (0, \pi/2)$ and hence $\omega_{\ell R} = 0$. If j is a normal agent (not a wall), one has $\angle F_{\ell j} = \angle p_{\ell j} = \angle(p_\ell - p_j) \in [0, \pi]$. It implies $\phi_{\ell j} = \mathrm{mod}\{\angle F_{\ell j} - \theta_\ell, 2\pi\} \in (\pi, 2\pi) \cup [0, \pi/2)$ and hence $\omega_{\ell j} \leq 0$.

As $w_\ell(t)$ is a monotonically increasing function of t during the interval $[t_1, \bar{t}_2)$, one has $w_\ell(\bar{t}_2) \geq w_\ell(t_1)$. As a result, we can extend the interval $[t_1, \bar{t}_2)$ to $[t_1, t_2)$. Statement 1 is thus proved.

Statement 2: If there exist $t_2 > t_o$ such that $w_T(t_2-) = w_\ell(t_2-)$ but $w_T(t_2) = w_\hbar(t_2)$ for two agents $\hbar \neq \ell$, then $w_T(t_2) \geq w_T(t_2-)$.

This statement is clearly true based on the definition of w_T. In particular, it is noted that $w_T(t_2) = w_\hbar(t_2) \geq w_\ell(t_2) = w_\ell(t_2-) = w_T(t_2-)$.

By repeatedly using the above two statements, we conclude that $w_T(t)$ is a monotonically increasing function of t for $t \geq t_o$.

In the second step, we aim to show that $\lim_{r\to\infty} w_T(t) = v_0$. It is ready to see that, the conclusion that $w_T(t)$ is a monotonically increasing function of t for $t \geq t_o$, together with $w_T(t) \leq v_o$, implies a limit $\lim_{r\to\infty} w_T(t) = \bar{v}_o$ exists. What is left is to show $\bar{v}_o = v_o$. Specifically, we aim to derive a contradiction if $0 < \bar{v}_o < v_o$.

As $\lim_{r\to\infty} w_T(t) = \bar{v}_o$, for any $\bar{v}_o/2 > \epsilon > 0$, there exists $t_1 > t_o$ such that $\bar{v}_o - w_T(t_1) \leq \epsilon$. Define $a = \sin^{-1}(\bar{v}_o/v_o)$ that satisfies $0 < a < \pi/2$ and $\varepsilon = a - \sin^{-1}((\bar{v}_o - \epsilon)/v_o)$ that satisfies $0 < \varepsilon < a - \sin^{-1}(\sin a/2)$. Then

$$\sin a = \frac{\bar{v}_o}{v_o},\quad \sin(a - \varepsilon) = \frac{\bar{v}_o - \epsilon}{v_o}$$

and ε approaches zero as ϵ does. Next, pick σ such that $\pi/2-\sin^{-1}(\sin a/2) < a+\sigma < \pi/2$ and pick δ such that

$$\sin(a+\sigma) = \frac{\bar{v}_o + \delta}{v_o}.$$

Clearly, one has $\bar{v}_o + \delta = v_o \sin(a+\sigma) < v_o$ and $v_o \sin(\pi/2 - a - \sigma) < v_o \sin(\sin^{-1}(\sin a/2)) = \bar{v}_o/2$.

Assume $w_T(t_1) = w_\ell(t_1)$ for a top boundary agent ℓ. From the above definition, one has

$$\bar{v}_o \geq w_T(t_1) = w_\ell(t_1) = v_o \sin\theta_\ell(t_1) \geq \bar{v}_o - \epsilon.$$

Since $y_\ell(t_1) = y_T(t_1)$, there exists $T > 0$ such that $y_T(t) - y_\ell(t) \leq d\cos(a+\sigma)$, $\forall t \in [t_1, t_1+T)$. Next, we will consider the behavior of the agent ℓ during $[t_1, t_1+T)$. In fact, we have the following statement.

Statement 3: For $t \in [t_1, t_1+T]$, $\dot{w}_\ell(t) \geq 0$ if $w_\ell(t) \in (\bar{v}_o/2, \bar{v}_o + \delta]$.

The proof is similar to that for Statement 1. We note that $w_\ell(t) = v_o \sin\theta_\ell(t) \in (\bar{v}_o/2, \bar{v}_o+\delta]$ implies $\mathrm{mod}\{\theta_\ell(t), 2\pi\} \in (\pi/2 - a - \sigma, a+\sigma] \cup [\pi - (a+\sigma), \pi - (\pi/2 - a - \sigma))$. Due to the symmetry, we only study the case with $\mathrm{mod}\{\theta_\ell(t), 2\pi\} \in (\pi/2 - a - \sigma, a + \sigma]$. The case with $\mathrm{mod}\{\theta_\ell(t), 2\pi\} \in [\pi - (a+\sigma), \pi - (\pi/2 - a - \sigma))$ follows the similar procedure.

For $\mathrm{mod}\{\theta_\ell(t), 2\pi\} \in (\pi/2 - a - \sigma, a+\sigma] \subset (0, \pi/2)$, we aim to show that $\omega_{\ell j} \geq 0, \forall j \in \mathcal{N}_\ell$. By (10), it implies that $\dot{\theta}_\ell(t) \geq 0$ and hence $\dot{w}_\ell(t) = d(v_o \sin\theta_\ell(t))/dt = v_o \cos\theta_\ell(t)\dot{\theta}_\ell(t) \geq 0$. In fact, for $j = L$, one has $\phi_{\ell L} = \mathrm{mod}\{-\theta_\ell, 2\pi\} \in (3\pi/2, 2\pi)$ and hence $\omega_{\ell L} = 0$. For $j = R$, one has $\phi_{\ell R} = \mathrm{mod}\{\pi - \theta_\ell, 2\pi\} \in (\pi/2, \pi)$ and hence $\omega_{\ell R} > 0$. If j is a normal agent (not a wall), one has $\angle F_{\ell j} = \angle p_{\ell j} = \angle(p_\ell - p_j)$. As $y_j(t) - y_\ell(t) \leq d\cos(a+\sigma)$ and $\|p_\ell(t) - p_j(t)\| \geq d$, one has $\angle(p_\ell - p_j) \in [a+\sigma-\pi/2, 3\pi/2 - a - \sigma]$. It implies that $\phi_{\ell j} = \mathrm{mod}\{\angle F_{\ell j} - \theta_\ell, 2\pi\} \in [3\pi/2, 2\pi) \cup [0, \pi]$ and hence $\omega_{\ell j} \geq 0$. Statement 3 is thus proved.

From Statement 3, on one hand, we can conclude that $w_\ell(t) < \bar{v}_o + \delta < v_o$, $\forall t \in [t_1, t_1+T)$. Otherwise, there exists $0 \leq \bar{T} \leq T$ such that $w_\ell(t_1 + \bar{T}) \geq \bar{v}_o + \delta$. Statement 3 implies that $w_\ell(t) \geq \bar{v}_o + \delta > w_T(t)$, $\forall t \in [t_1+\bar{T}, t_1+T)$. As a result, $y_T(t) - y_\ell(t) \leq y_T(t_1+\bar{T}) - y_\ell(t_1+\bar{T}) < d\cos(a+\sigma)$, $\forall t \in [t_1+\bar{T}, t_1+T)$. From the definition of T, T can be selected arbitrarily large, that is, $w_\ell(t) \geq \bar{v}_o + \delta > w_T(t)$, $\forall t \in [t_1+\bar{T}, \infty)$, which makes a contradiction to the definition of w_T.

On the other hand, we have $w_\ell(t) \geq \bar{v}_o - \epsilon > \bar{v}_o/2$, $\forall t \in [t_1, t_1+T)$ as $w_\ell(t_1) \geq \bar{v}_o - \epsilon > \bar{v}_o/2$. It implies that T can be selected as

$$T = \frac{d\cos(a+\sigma)}{\epsilon} = \frac{d\cos(a+\sigma)}{\bar{v}_o - (\bar{v}_o - \epsilon)} \leq \frac{d\cos(a+\sigma)}{w_T(t) - w_\ell(t)}.$$

As ϵ can be arbitrarily small, T can be arbitrarily large.

As $\bar{v}_o/2 < w_\ell(t) < \bar{v}_o + \delta$, $\forall t \in [t_1, t_1+T)$, one has $\mathrm{mod}\{\theta_\ell(t), 2\pi\} \in (\pi/2 - a - \sigma, a+\sigma]$ due to the symmetry as explained in the proof of Statement 3. As a result

$$\begin{aligned}\mu_\ell(t) &= \sqrt{v_o^2 - w_\ell^2(t)} > \sqrt{v_o^2 - (\bar{v}_o + \delta)^2}\\ &= \sqrt{v_o^2 - (v_o\sin(a+\sigma))^2}\\ &= v_o\cos(a+\sigma),\ \forall t \in [t_1, t_1+T).\end{aligned}$$

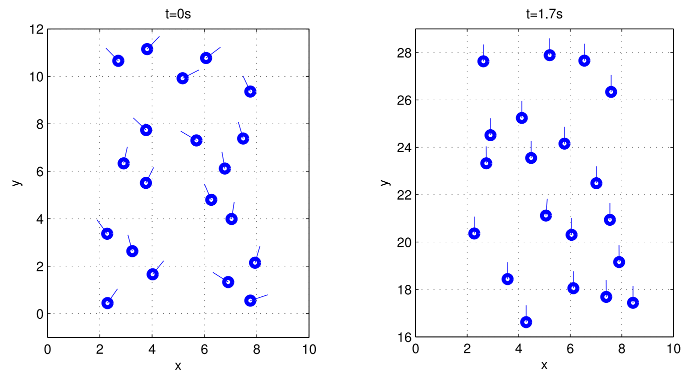

Fig. 2. Snapshots of agent distribution with positions represented by the circles and the headings by the short lines. Left: initial distribution at $t = 0$ s; right: velocity alignment at $t = 1.7$ s.

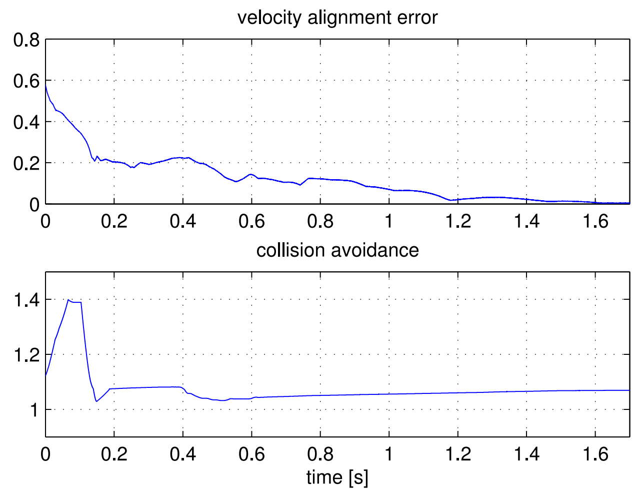

Fig. 3. Profiles of velocity alignment and collision avoidance with initial headings in $[0, \pi)$. Top: profile of the index $J_a(t)$; bottom: profile of the index $J_c(t)$.

So, the agent ℓ displacement in the positive x-axis is

$$\int_{t_1}^{t_1+T} \mu_\ell(t)dt > v_o\cos(a+\sigma)T$$

during the interval $[t_1, t_1+T)$, which contradicts with $v_o\cos(a+\sigma)T > x_R - x_L$ for an arbitrarily large T. The proof is thus complete.

More specifically, we can show that $\mu_\ell(t)$ approaches zero in finite time during $[t_1, t_1+T)$ and hence the contradiction if the function c is selected such that $|c(x)/\cos^\varsigma(x)| > c_o > 0$, $\forall x \in \mathbb{R}$ for some constants $c_o > 0$ and $0 < \varsigma < 1$. From the proof of Statement 3, one has $\omega_{\ell j} \geq 0$. Let $V(t) = \mu_\ell(t)$, $t \in [t_1, t_1+T)$ and

$$\begin{aligned}\dot{V}(t) &= -v_o\sin\theta_\ell\dot{\theta}_\ell = -v_o\sin\theta_\ell\sum_{j\in\mathcal{N}_\ell}\omega_{\ell j}\\ &\leq -v_o\sin\theta_\ell\omega_{\ell R} = v_o\sin\theta_\ell\alpha_\ell\|F_{\ell L}\|c(\phi_{\ell R})s(\phi_{\ell R})\\ &= v_o^\varsigma v_o^{1-\varsigma}\sin\theta_\ell\alpha_\ell\|F_{\ell R}\|\frac{c(\phi_{iR})}{\cos^\varsigma(\phi_{\ell R})}\cos^\varsigma(\phi_{\ell R})\\ &\leq -v_o^\varsigma v_o^{1-\varsigma}\sin\theta_\ell\alpha_\ell\|F_{\ell R}\|c_o\cos^\varsigma(\theta_\ell)\\ &= -\mu_\ell^\varsigma\left[v_o^{1-\varsigma}\sin\theta_\ell\alpha_\ell\|F_{\ell R}\|c_o\right]\end{aligned}$$

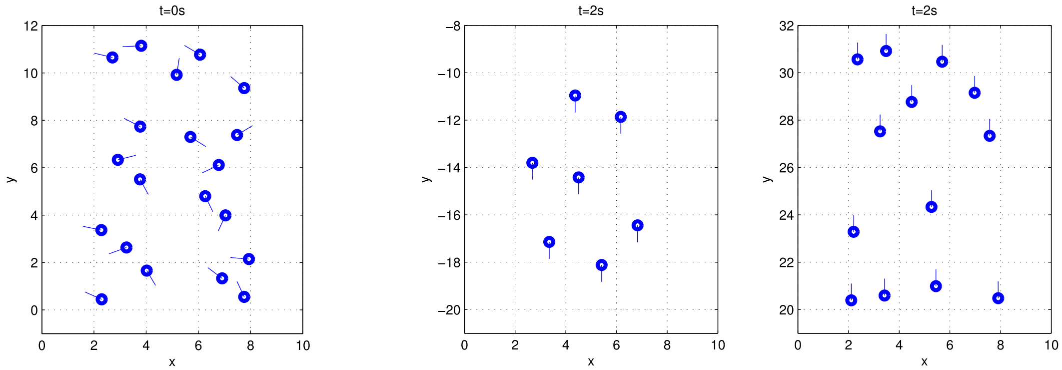

Fig. 4. Snapshots of agent distribution with positions represented by the circles and the headings by the short lines. Left: initial distribution at $t = 0$ s; right: velocity alignment for two clusters moving in opposite directions at $t = 2$ s.

where we note $\phi_{\ell R} = \text{mod}\{\pi - \theta_\ell, 2\pi\} \in (\pi/2, \pi)$. After a finite time, one has $[v_o^{1-\varsigma} \sin\theta_\ell \alpha_\ell \|F_{\ell R}\| c_o] > c_1$ for some constant $c_1 > 0$. As a result, $\dot{V}(t) \le -c_1 V^\varsigma(t)$, which implies that $\mu_\ell(t)$ approaches zero in finite time. ∎

A similar statement is as follows.

Lemma 2: Consider the MCMA model composed of (1) and (10) without disturbances, i.e., $\|\sigma_i(t)\| = 0$. If there exists $t_o \ge 0$ such that $w_B(t_o) < 0$, then $\lim_{t\to\infty} w_B(t) = -v_0$.

Now, we assume the group of agent has the following *generic* setup. Consider any subgroup, there exists a time t_o, such that: 1) $w_T(t_o) > 0$ or $w_B(t_o) < 0$ and 2) for any agent i, $y_T(t) - y_i(t) \ge 0$ and $y_i(t) - y_B(t) \ge 0$ are either bounded or approach infinity.

Now, the main theorem is given below.

Theorem 2: Consider the MCMA model composed of (1) and (10) without disturbances, i.e., $\|\sigma_i(t)\| = 0$, in a generic setup. If the initial conditions satisfy $\|p_{ij}(0)\| > d$, $\forall i \in \mathbb{N}$, $j \in \mathbb{N}^+$. Then, there exists finite repulsive force such that collision does not occur during the entire evolution and velocity alignment is achieved in the sense of

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t w_i(t)dt = v_0 \text{ or } -v_o, \ i \in \mathbb{N}. \tag{16}$$

Proof: Without loss of generality, there exists a time t_o, such that $w_T(t_o) > 0$. By Lemma 1, one has $\lim_{t\to\infty} w_T(t) = v_0$. It implies that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t w_T(t)dt = v_0. \tag{17}$$

Let $\mathbb{N}_1$ be the set of agents which satisfy

$$0 \le y_T(t) - y_i(t) < C, \ i \in \mathbb{N}_1$$

for a constant C. Then

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t}\int_0^t w_i(t)dt &= \lim_{t\to\infty} \frac{1}{t}[y_i(t) - y_i(0)]dt \\ &= \lim_{t\to\infty} \frac{1}{t}[y_i(t) - y_T(t) \\ &\quad + y_T(0) - y(0) + y_T(t) - y_T(0)]dt \\ &= \lim_{t\to\infty} \frac{1}{t}[y_i(t) - y_T(t) + y_T(0) - y(0)]dt \\ &\quad + \lim_{t\to\infty} \frac{1}{t}\int_0^t w_T(t)dt = v_0, \ i \in \mathbb{N}_1.\end{aligned}$$

In a generic setup, all the agents in $\mathbb{N} - \mathbb{N}_1$ satisfy

$$\lim_{t\to\infty} y_T(t) - y_i(t) = \infty, \ i \in \mathbb{N} - \mathbb{N}_1.$$

That is, after a certain finite, the agents in $\mathbb{N}_1$ have no effect on those in $\mathbb{N} - \mathbb{N}_1$. We can repeat the above analysis for the subgroup $\mathbb{N} - \mathbb{N}_1$ until (16) is achieved for all $i \in \mathbb{N}$. ∎

V. Numerical Simulation

In this section, we consider an MCMA model composed of (1) and (10) with $N = 20$. The agents move in a plane between two walls $x = x_L = 0$ and $x = x_R = 10$. The collision radius is $d = 1$ and the sensing range $r = 5$.

The initial headings of agents are arbitrarily selected from $[0, \pi]$. It is observed that collision does not occur during the entire evolution and velocity alignment is achieved in Fig. 2. To quantitatively illustrate the performance, we define two indexes

$$\begin{aligned}J_a(t) &= 1/N\sum_{i=1}^N |\text{mod}\{\theta_i(t), 2\pi\} - \pi/2| \\ J_c(t) &= \min_{i\in\mathbb{N}, j\in\mathbb{N}^+, i\ne j}\big\{\|p_{ij}(t)\|\big\}.\end{aligned}$$

In particular, J_a denotes the deviation of the agents' headings away from the positive y-axis (i.e., $\text{mod}\{\theta_i, 2\pi\} = \pi/2$). When velocity alignment is achieved, one has $\lim_{t\to\infty} J_a(t) = 0$ as illustrated in the top graph of Fig. 3. The index J_c denotes the shortest distance between two agents (or one agent and one wall). The profile that $J_c(t) > d = 1$ in the bottom graph of Fig. 3 illustrates collision avoidance during the entire evolution.

If the initial headings of agents are arbitrarily selected from $[0, 2\pi)$, some of agents achieve their velocity alignment along the positive y-axis and the others along the negative y-axis as expected in Theorem 2. It is observed again that collision does not occur during the entire evolution and velocity alignment is achieved in Fig. 4. To quantitatively illustrate this profile, we use the modified index

$$\hat{J}_a(t) = 1/N\sum_{i=1}^N |\text{mod}\{\theta_i(t), \pi\} - \pi/2|$$

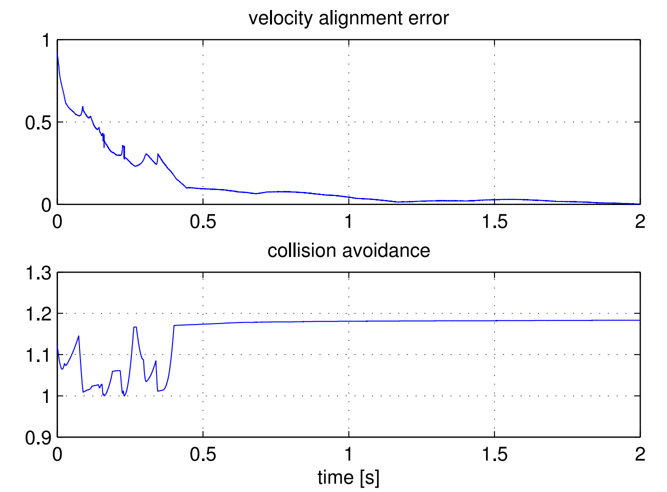

Fig. 5. Profiles of velocity alignment and collision avoidance with initial headings in $[0, 2\pi)$. Top: profile of the index $\hat{J}_a(t)$; bottom: profile of the index $J_c(t)$.

that achieves $\lim_{t\to\infty} \hat{J}_a(t) = 0$ when the bipartisan velocity alignment is achieved. In this case, collision avoidance is still guaranteed as $J_c(t) > d = 1$. The profiles are shown in Fig. 5.

VI. Conclusion

This paper has introduced the definition of an MCMA model. For the first time, it has given rigorous theoretical proofs for the functionalities of a MCMA model in both collision avoidance and velocity alignment. The proposed mechanisms used for both collision avoidance and velocity alignment are brand new in control society. Their simplicity would have great potentials in engineering applications.

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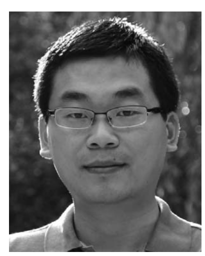

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