Report for exercise 3 from group K

Tasks addressed:	5
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Last compiled:	2021–05–26
Source code:	https://github.com/Linnore/MLCMS-EX3-GroupK

The work on tasks was divided in the following way:

Oliver Beck (03685783)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%
Junle Li (03748878)	Task 1	33%
	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%
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	Task 2	33%
	Task 3	33%
	Task 4	33%
	Task 5	33%

Abstract

In this exercise, we studied qualitative changes of dynamical systems over changes of their parameters. These changes in the qualitative behavior of the system are called bifurcations.

The goals for this exercise are

to familiarize yourself with the mathematical notation of bifurcation theory,

to understand topological equivalence between systems,

to know several basic bifurcations present in almost all dynamical systems in the world,

to be able to visualize qualitative changes of a dynamical system in a bifurcation diagram,

to apply these ideas to crowd dynamics (the SIR model).

Even though it is possible to analyze a given mathematical model formally regarding its bifurcations, there are many systems where such an analysis is not possible. The most difficult examples are systems in the real world, where you cannot study the behavior on pen and paper, only through observations. This is where Machine Learning can assist you, to produce data-driven bifurcation diagrams for real systems (or large, complex simulated systems such as the ones you studied with Vadere) from data.

Python	3.8.8
Jupyter Notebook	6.3.0

Table 1: Software versions

Report on task 1, Vector fields, orbits, and visualization

Task description Consider the following linear dynamical system, with state space $X = \mathbb{R}^2$, $I = \mathbb{R}$, parameter $\alpha \in \mathbb{R}$, and flow ϕ_α defined by

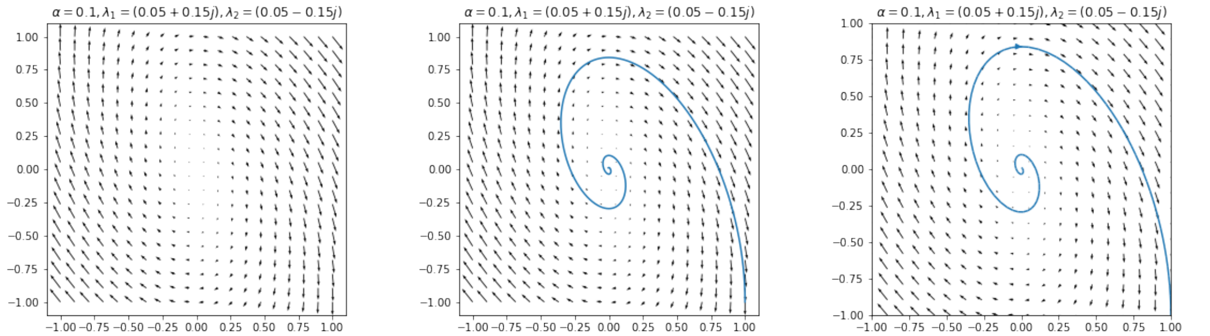
$$\frac{d\phi_\alpha(t,x)}{dt}|_{t=0} = A_\alpha x,$$

where $A_\alpha \in \mathbb{R}^{2\times 2}$ is a parametrized matrix. With the linear system defined above, construct a figure that summarizes topological classification of hyperbolic equilibria on the plane. Specify the value of the parameter for each of the phase portraits. Are the systems in your figure topologically equivalent? Why, or why not?

Solutions and Results

Preparation To start with visualizing dynamical systems, we first tried to reproduce a similar figure as Figure 2 in the exercise sheet. After reading Matplotlib documentation, we found out two methods of matplotlib.pyplot can help us visualize the vector field and phase portrait. The first method is "matplotlib.pyplot.quiver()", which can visualize the vector field using multiple small arrows. The second method is "matplotlib.pyplot.streamplot()", which can visualize phase portraits by plotting representative orbits in the vector field.

Taking the parametrized matrix in the exercise sheet $A_\alpha = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix}$ and $\alpha = 0.1$ as an example, details of the visualization steps will be reported in the following. The first step for using quiver() and streamplot() is always constructing a grid, which will define the resolution of the vector field or phase portrait. Following this, velocity should be calculated based on the grid and passed into both methods for plotting. Figure 1(a) shows the vector field plotted by quiver() method using a grid of size 20x20 and Figure shows an additional representative orbit plotted by streamplot() method on the vector field. In particular, streamplot() method plots many representative orbits to show a phase portrait by defaults. We specified only one starting point (1,-1) with backward integration configuration in streamplot() to get figure . Another way to plot representative orbits is by using Euler's Algorithm described in the exercise sheet. We also implemented a little framework for Euler's Algorithm (check the jupyter notebook at the root of our repository for details), which will solve a trajectory forward or backward based on the given velocity function, i.e. $\frac{d\phi_\alpha(t,x)}{dt}|_{t=0}$, with user-specified maximum number of steps and step size. Figure 1(b) shows the same representative orbit starts from (1, -1) solved by Euler's Algorithm. In this simple example, the orbit solved by Euler's Algorithm is basically same as the one solved by streamplot().



(a) The vector field plotted by quiver(). (b) An additional representative orbit on the vector field plotted by streamplot()) (c) An additional representative orbit on the vector field solved by Euler's Algorithm.

Figure 1: Getting start to visualize dynamical systems using vector fields and phase portraits.

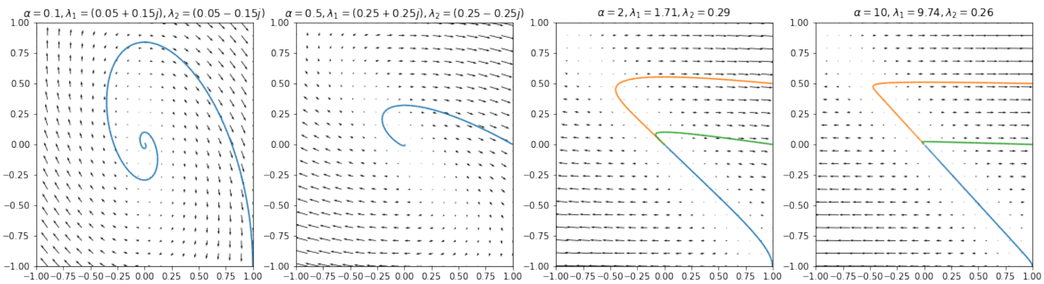


Figure 2: Reproduced Figure 2 in the exercise sheet (with minor differences).

Reproduce Figure 2 in the Exercise Sheet Now that we have explored basic visualization on 2D dynamical system, we reproduced figure 2 in the exercise sheet as Figure 2. Note that for $\alpha = 2$ and $\alpha = 10$, we plotted 3 representative orbits in different colors. Next, we move on to construct the table for topological classification of hyperbolic equilibria on the plane.

Topological Classification of Hyperbolic Equilibria on the Plane To cover all the possible classes, we construct 2 forms of parametrized matrix:

$$A_\alpha^{(1)} = \begin{bmatrix} \alpha & \alpha \\ -1 & \alpha \end{bmatrix}, A_\alpha^{(2)} = \begin{bmatrix} \alpha & \alpha \\ 1 & \alpha \end{bmatrix}.$$

Both of them can produce 3 classes of hyperbolic equilibria respectively and in total 5 classes in which they shared saddle-point class. Figure 4 summarizes our exploration of these two parametrized matrices with different values of α.

In figure 4, the phase portraits in the same (n_+, n_-) category are topological equivalent. That is, the systems with parametrized matrix $A_{-1.5}^{(1)}$ and $A_{-0.5}^{(2)}$ are equivalent; the systems with parametrized matrix $A_{-1.5}^{(2)}$ and $A_{0.5}^{(1)}$ are equivalent. This is called Node-Focus equivalence in [Kuznetsov, p43]. The main ideas to show the node-focus equivalence is to focus at the neighbourhood of the fixed points. Figure 3 shows such neighbourhoods of the category $(n_+, n_-) = (0, 2)$, where the dynamical system is called stable. Based on figure 3 We can make the following observations:

Observation 1 For neighbourhood of the fixed point in the stable Node cases (figure 3a), if the radius of neighbourhood is sufficiently small, then all trajectories can be regarded as pointing to the fixed point directly.

Observation 2 Similarly, for neighbourhood of the fixed point in the stable Focus cases (figure 3b), if the radius of neighbourhood is sufficiently small, then all trajectories can be regarded as pointing to the fixed point in some spiral manner.

To show they are topological equivalent, the key is to find a one-to-one mapping between the sets of trajectories in these two cases, and this is where mathematical derivation is needed. However, without math work of specific derivation for the mapping, we could still argue that such mapping exists. Since trajectories only have one shared point-the fixed point-in both the node and focus system, the mapping we need to derive would be some spiral transformation from straight trajectories to spiral trajectories and it exists. In this way, the node-focus equivalence is shown for the $(n_+, n_-) = (0, 2)$ category.

As for the $(n_+, n_-) = (2, 0)$ category, the ideas to show the node-focus equivalence should be the same as above, except that all trajectories have the opposite directions (fixed point becomes source point instead of end point of the trajectories).

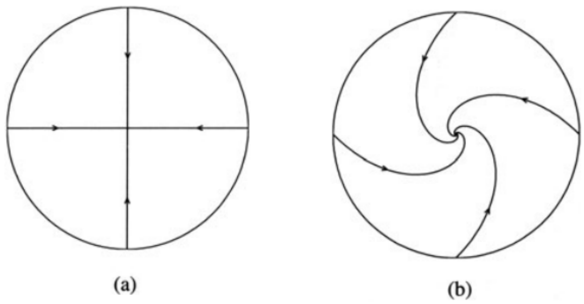


Figure 3: Node-Focus Equivalence for stable dynamical system, Figure 2.2 in [Kuznetsov, p44]

Other than the equivalence discussed above, phase portraits that are in different (n_+, n_-) categories are NOT considered as topological equivalent. Systems in category $(n_+, n_-) = (0, 2)$ and the ones in category $(n_+, n_-) = (2, 0)$ are not topological equivalent, as their trajectories have opposite directions and they have different stability. Systems in category $(n_+, n_-) = (1, 1)$ are not topological equivalent to systems in the other two categories. If checking the neighbourhood of the saddle point, we can see not all trajectories are pointing to and ending up at the saddle point, since some trajectories will never reach the saddle point. This is much different to the situation in the other two categories, and thus no one-to-one mapping of trajectories exists between systems in category $(n_+, n_-) = (1, 1)$ and systems in the other two categories.

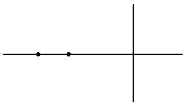
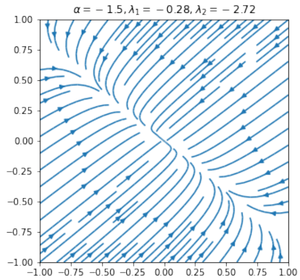
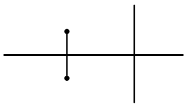
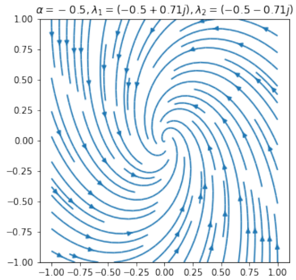
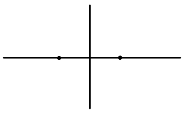
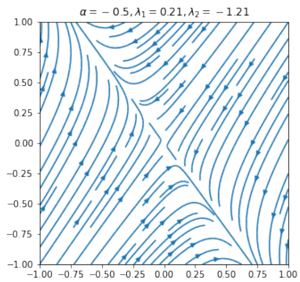
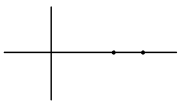
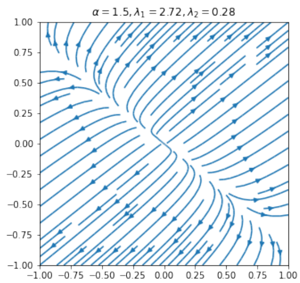
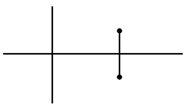
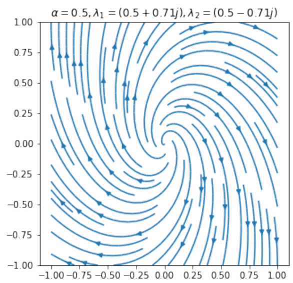
(n_+, n_-)	Eigenvalues	Parametrized Matrix	Phase Portrait		Stability
$(0,2)$		$A_\alpha^{(1)} = \begin{bmatrix} \alpha & \alpha \\ -1 & \alpha \end{bmatrix}$		Node	Stable
		$A_\alpha^{(2)} = \begin{bmatrix} \alpha & \alpha \\ 1 & \alpha \end{bmatrix}$		Focus	
$(1,1)$		$A_\alpha^{(1)} = \begin{bmatrix} \alpha & \alpha \\ -1 & \alpha \end{bmatrix}$		Saddle	Unstable
$(2,0)$		$A_\alpha^{(2)} = \begin{bmatrix} \alpha & \alpha \\ 1 & \alpha \end{bmatrix}$		Node	Unstable
		$A_\alpha^{(1)} = \begin{bmatrix} \alpha & \alpha \\ -1 & \alpha \end{bmatrix}$		Focus	

Figure 4: Topological classification of hyperbolic equilibria on the plane.

Report on task 2, Common bifurcations in nonlinear systems

Task Description Consider a dynamic system on the real line $X = \mathbb{R}$, time $I = \mathbb{R}$, with the evolution described by

$$\dot{x} = \alpha - x^2 \tag{1}$$

For $\alpha > 0$, this system has two steady states at $x_0 = \pm\sqrt{\alpha}$, and for $\alpha < 0$, there are no steady states. What type of bifurcation happens at $\alpha = 0$? Plot the bifurcation diagram of the system for values of α in (-1, 1), visually indicating the stability of the steady states. Then, do the same for the following system:

$$\dot{x} = \alpha - 2x^2 - 3 \tag{2}$$

Are the systems (1) and (2) at $\alpha = 1$ topologically equivalent? Why, or why not? What about the systems at $\alpha = -1$? Argue why the systems (1) and (2) have the same normal form.

Solutions and Results

In this task, we are dealing with the dynamical systems defined on 1D space, which is hard to visualize directly. To better show the features of such systems, we made use of some augmented 2D system based on equation (1) and (2) , where we treated the original x component as x_1 and added an x_2 component which is independent of x_1.

For the system (1) The augmented system is:

$$\begin{aligned} \dot{x_1} &= \alpha - x_1^2, \\ \dot{x_2} &= -x_2. \end{aligned} \tag{3}$$

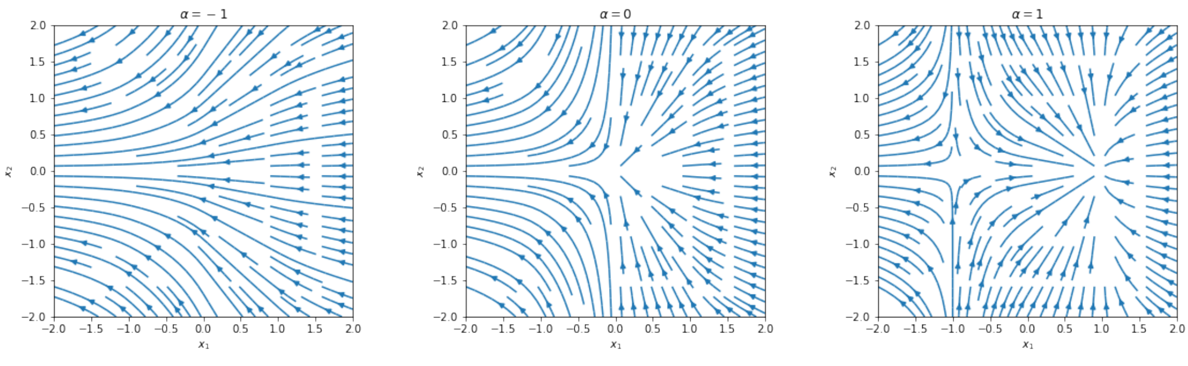
Figure 5 shows the phase portraits of the above augmented system at $\alpha = -1, 0,$ and 1. By observing the "fixed value" of x_1, we can learn information about the fixed point of system (1).

At $\alpha = -1$, we can see from figure 7(a) that the flow won't converge to a fixed value of x_1; therefore, the system (1) is unstable at $\alpha = -1$.

At $\alpha = 0$, we can observe from figure 7(b) that trajectories on the right hand side are pointing to (0, 0), while trajectories on the left hand side will never converge to a fixed value of x_1. Correspondingly, there is where a fixed point $x_1^0 = 0$ appears in the system (1). Such fixed point is called saddle-node, and the bifurcation happens here is called Saddle-node Bifurcation.

At $\alpha = 1$, we can observe from figure 7(c) that trajectories start from $x_1 < -1$ will never converge to a fixed x_1 value, and trajectories start from $x_1 > -1$ are all pointing to (1,0). I.e., (-1, 0) is a saddle and (1, 0) is a node. Correspondingly, two fixed points $x_1^0 = \pm 1$ appear in the system (1), while $x_1^0 = -1$ is unstable and $x_1^0 = 1$ is stable.

Figure 6 is the bifurcation diagram of system (3). The stability of steady states is encoded using different colors. The blue steady states are stable, while the red ones are unstable.



(a) (b) (c)

Figure 5: Phase portraits of system defined by equation (3)

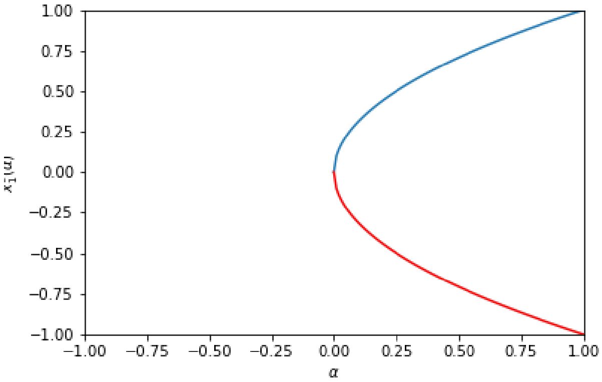


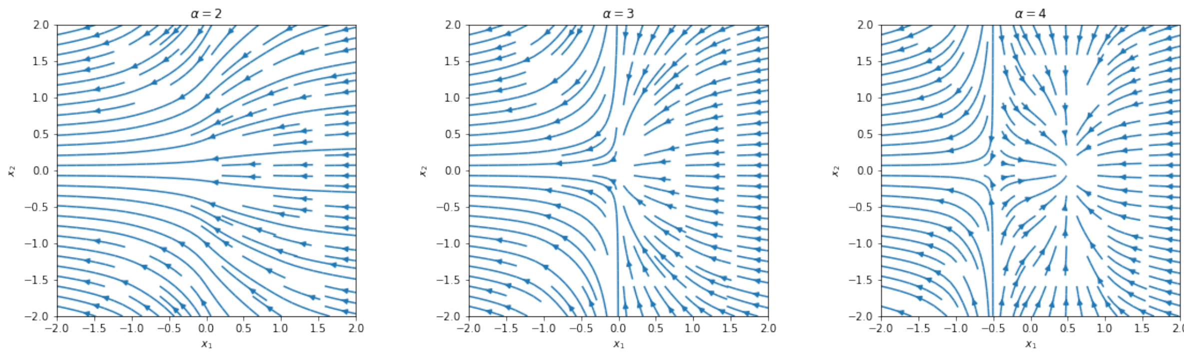
Figure 6: Bifurcation Diagram of system defined by equation (1). Blue: Stable, Red: Unstable.

For the system (2) Solving $0 = \alpha - 2x_1^2 r - 3$ gives information of steady states of the system. For $\alpha > 3$, the system has two steady states at $x_0 = \pm\sqrt{\frac{\alpha-3}{2}}$; for $\alpha = 3$, again the steady state $x_0 = 0$ is a saddle-node; and for $\alpha < 3$, there are no steady states.

The augmented system for system (2) is:

$$\begin{aligned} \dot{x_1} &= \alpha - 2x_1^2 - 3, \\ \dot{x_2} &= -x_2. \end{aligned} \tag{4}$$

Checking the phase portraits of the augmented system (4) at $\alpha = 2, 3$ and 4, we observed basically the same facts as the phase portraits of system (3) (comparing figure 5 and figure 7). Figure 8 shows the bifurcation diagram for the augmented system (4), which has the same shape as the one of system (3).



(a) (b) (c)

Figure 7: Phase portraits of system defined by equation (4)

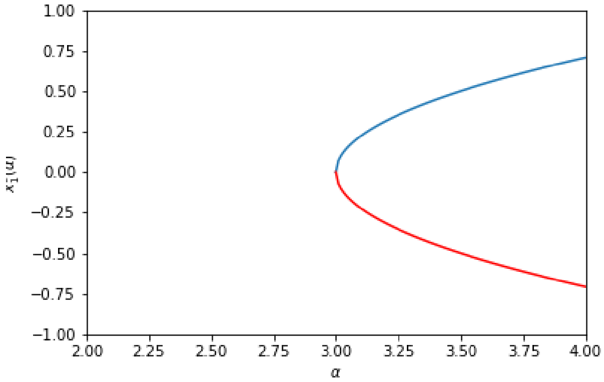


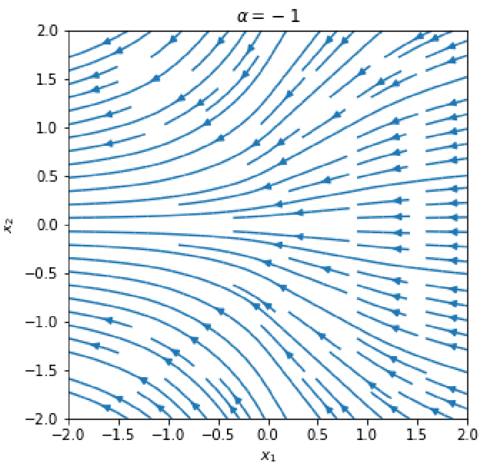
Figure 8: Bifurcation Diagram of system defined by equation (2). Blue: Stable, Red: Unstable.

Comparison of system (1) and (2) Based on the analysis above, especially the bifurcation diagrams (figure 6 and 8), we can now compare the two systems with specific values of α.

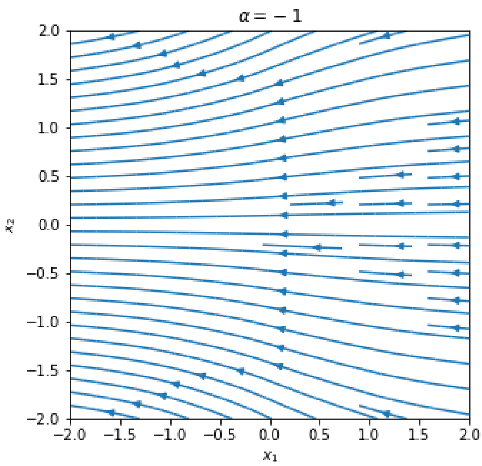
At $\alpha = 1$, system (1) has two steady states–one stable and the other unstable, while system (2) has no steady states. Therefore, they are not topological equivalent.

At $\alpha = -1$, however, both systems have no steady states. We can first have a look at the phase portraits of the augmented systems. Figure 9 shows the phase portraits of the corresponding augmented systems at $\alpha = -1$. In both phase portraits, we can observe that the flows are all flowing to negative x_1 direction. Correspondingly, we can conclude that systems (1) and (2) are both unstable and tends to the same direction, and thus they are topological equivalent.

Based on the bifurcation diagrams of both systems, we can now argue that they have the same normal form. The bifurcation diagrams these two systems share the same shape. From $\alpha = -\infty$ to $\alpha = \infty$, both system experience the same changes in terms of steady states: (1) no steady states (2) one saddle-node steady state (3) one saddle steady state and one node steady state. With each of the above three stages, both systems share exactly the same topological features, including the relative position of the saddle and node steady points in stage (3). Therefore, they have the same normal form.



(a) System 3



(b) System 4

Figure 9: Comparison of the two augmented systems at $\alpha = -1$.

Report on task 3, Bifurcations in higher dimensions

Task Description Bifurcations can happen for dynamical systems with state spaces of arbitrary dimension, and also in more than one parameter. Some bifurcations do not occur if the state space is one-dimensional (and the system is continuous). Examine the systems with following normal form:

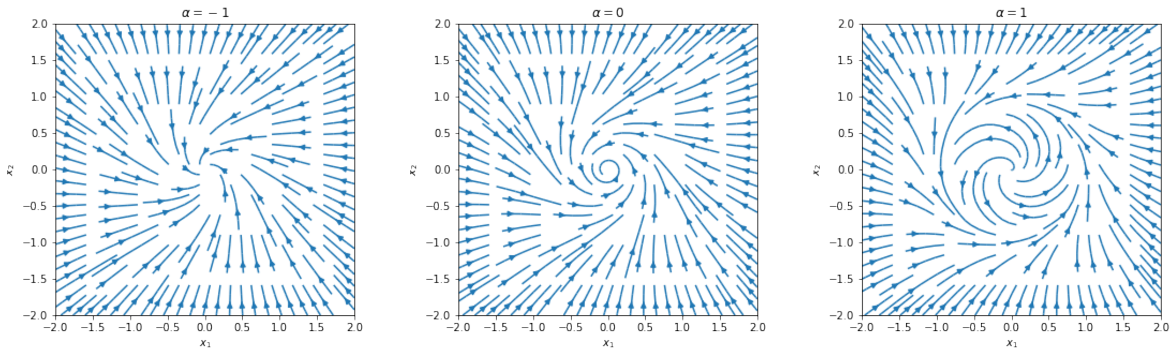
$$\begin{aligned} \dot{x} &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x} &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2). \end{aligned} \tag{5}$$

and

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3. \tag{6}$$

Solutions and Results

For the system (5) Figure 10 shows three representative phase portraits of system (5) at $\alpha = -1, 0$ and 1. At $\alpha = -1$, the system has one focus fixed point at (0,0). At $\alpha = 0$, we can see that the trajectories are spiraling even more. This is where the so-called Andronov-Hopf bifurcation happens. At $\alpha = 1$, a limit cycle is observed together with an unstable focus at the center.

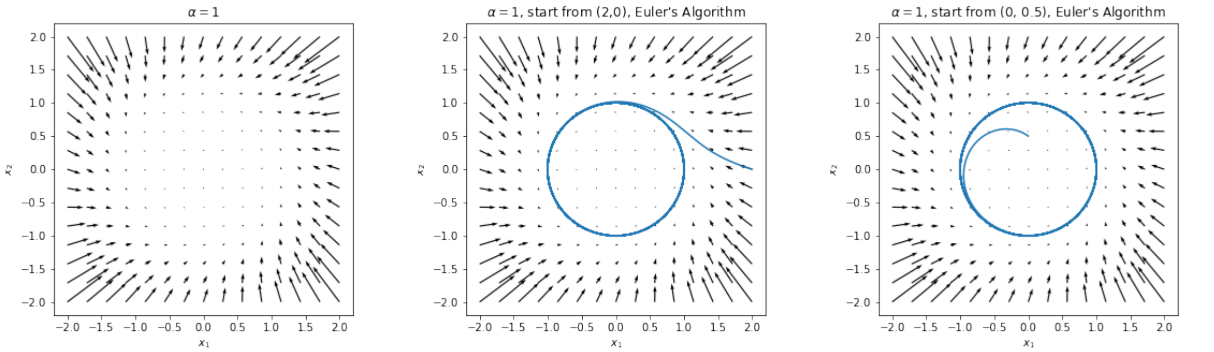


(a) (b) (c)

Figure 10: Three representative phase portraits of system (5)

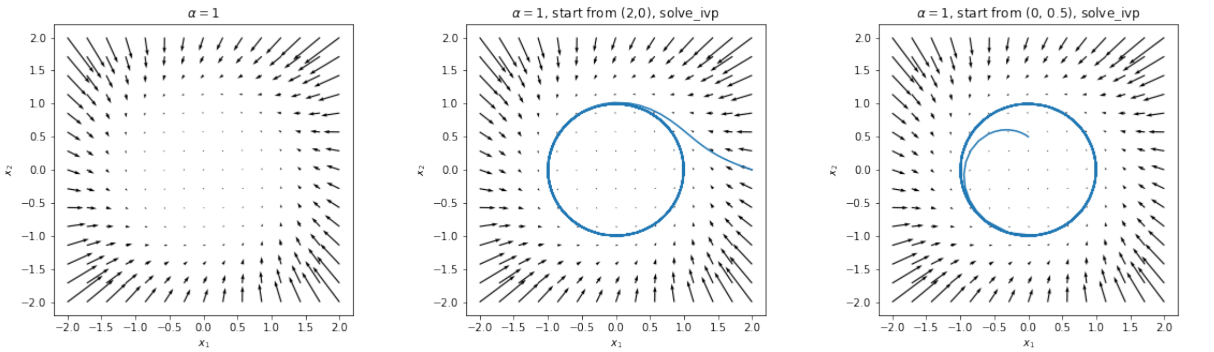
Now we would like to plot some representative trajectories on the vector field of the system at $\alpha = 1$. As mentioned in report on task 1, we did implemented a framework for Euler's Algorithm; therefore, we tried it out here and compared the simulated results solved by Euler's Algorithm and solve_ivp() method. For codes details please check our jupyter notebook at the root of our repo.

Figure 11 shows the simulated trajectories starting from $(2, 0)$ and $(0, 0.5)$ using Euler's Algorithm, and figure 12 shows the simulated trajectories using solve_ivp() method. At least in this problem, Euler's Algorithm did not show significantly different results to the ones by solve_ivp(). As we expected, starting from $(2, 0)$, the trajectory will travel from outside the circle and end up on the limit circle and loop forever; starting from $(0, 0.5)$, the trajectory will travel from inner side of the circle and end up on the limit circle and loop forever.



(a) The vector field plotted by quiver(). (b) The trajectory starts from (2,0). (c) The trajectory starts from (0, 0.5).

Figure 11: Simulation of trajectories using Euler's Algorithm (T=100, step=0.01).



(a) The vector field plotted by quiver(). (b) The trajectory starts from (2,0). (c) The trajectory starts from (0, 0.5).

Figure 12: Simulation of trajectories using solve_ivp (T=100).

For the system (6) In this sub-task, we are required to plot the bifurcation surface for the system (6). Notice that it is an implicit function in terms of the z-axis (x in the system). To better visualize such a 3D surface of implicit function, we adopted @Paul's codes[https://stackoverflow.com/questions/4680525/plotting-implicit-equations-in-3d] and modify it for our plotting. Figure 13 shows the results.

By observing figure 13, we can find a critical point, or the cusp, at $(\alpha_1, \alpha_2) = (0, 0)$, where the surface starts to bend and fold gently. For $\alpha_1 > 0, \alpha_2 > 0$, multiple fixed points appear. This is why it is called cusp bifurcation.

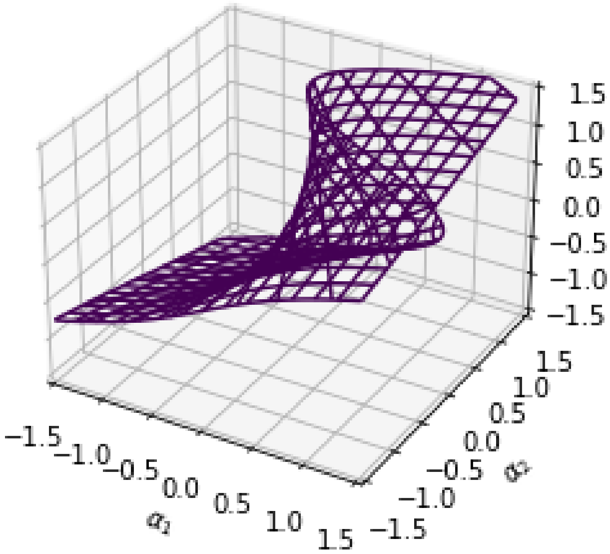


Figure 13: Bifurcation surface for system (6)

Report on task 4, Chaotic dynamics

Task description Dynamical systems can behave in very irregular ways, and changes in their parameters can lead to very drastic changes in their behavior. Consider the discrete map

$$x_{n+1} = rx_n(1 - x_n), n \in N$$

with the parameter $r \in (0, 4]$. The system described above is well studied. It is called the "logistic map", and is a good example of chaos in discrete maps on one-dimensional spaces.

Vary r from 0 to 2 and from 2 to 4 respectively. Observe which bifurcations occur and compute at which numerical values do you find steady states and limit cycles of the system.

Plot a bifurcation diagram for r between 0 and 4 (horizontal axis), x between 0 and 1 (vertical axis), indicating the positions of steady states and limit cycles.

Dynamical systems in continuous time cannot have smooth evolution operators that produce chaotic dynamics if the dimension of the state space is smaller than three. The Lorenz attractor is a famous example for a system in three-dimensional space that forms a strange attractor, a fractal set on which the dynamics are chaotic. The Lorenz equations are known as:

$$\frac{dx}{dt} = \sigma(y - x), \tag{7}$$

$$\frac{dy}{dt} = x(\rho - z) - y, \tag{8}$$

$$\frac{dz}{dt} = xy - z\beta. \tag{9}$$

The chaotic nature of the system implies that small perturbations in the initial condition will grow larger at an exponential rate, until the error is as large as the diameter of the attractor.

Visualize a single trajectory of the Lorenz system starting at $x_0 = (10, 10, 10)$, until you reach the end time of $T_{end} = 1000$, at the parameter values $\sigma = 10, \beta = 8/3, and \rho = 28$. Observe what the attractor looks like.

Test this by plotting another trajectory from $x_0 = (10^{-8} + 10, 10, 10)$. Calculate at what time is the difference between the points on the trajectory larger than 1.

Now, change the parameter ρ to the value 0.5 and again compute and plot the two trajectories. Determine the difference in terms of the sensitivity to the initial conditions. And determine whether there is a bifurcation (or multiple ones) between the value 0.5 and 28 and give a brief argument and reason.

Solutions and Results Consider $x_{n+1} = rx_n(1 - x_n)$; $x_n \in [0, 1]$ and $r \in (0, 4]$.

Vary r from 0 to 2 For $r \in (0, 1)$, $x_n \to 0$ as $n \to \infty$, $x^* = 0$ is stable. At $r = 1$, x^* bifurcates from the origin in a transcritical bifurcation (figure 14).

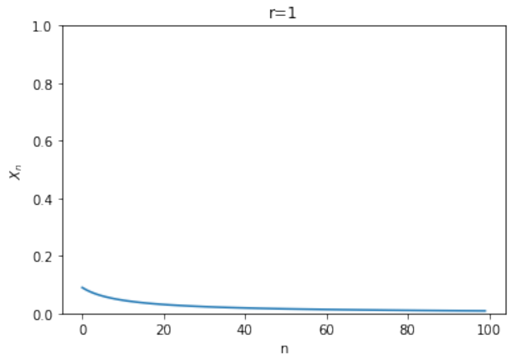
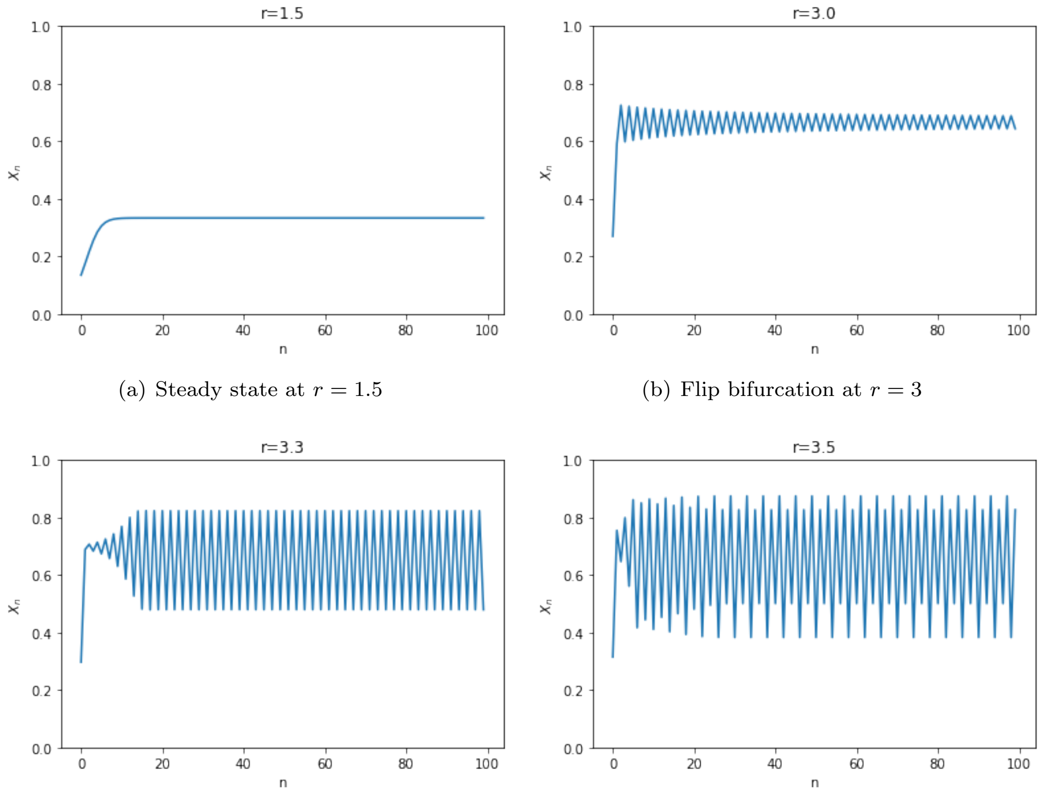


Figure 14: Transcritical bifurcation at r = 1

For $r \in (1, 2)$, x_n grows as n increases, reaching a non-zero stable steady state $x^* = 1 - \frac{1}{r}$. For example, Figure 15(a) illustrates that at $r = 1.5$, as n increases, x_n reaches a steady state with value $\frac{1}{3}$.



(a) Steady state at $r = 1.5$

(b) Flip bifurcation at $r = 3$

(c) Oscillation at $r = 3.3$

(d) Limit cycles exist at $r = 3.5$

Figure 15: Explore different values of r

Vary r from 2 to 4 For $r \in (2, 3)$, $x^* = 1 - \frac{1}{r}$ is also a stable steady state. The critical slope is attained when $r = 3$, the resulting bifurcation is called a flip bifurcation, where limit cycles exist.

For $r \in (3, 4)$, x_n eventually oscillates around the steady state we mentioned above (e.g. r = 3.3, figure 15(c)). A 2-cycle exists for all $r > 3$.

At larger r (e.g. $r = 3.5$), x_n approaches a cycle which repeats every larger generations. Further period doubling cycles of period would occur as r increases, i.e. a period-doubling bifurcation would be observed.

Bifurcation diagram The successive bifurcations come faster and faster as r increases. For most values of r, the sequence x_n will never settle to a fixed point or a periodic orbit. The long-term behaviour of bifurcation is aperiodic. Figure 16 shows the long-term behaviour for $r \in (0, 4)$ at once.

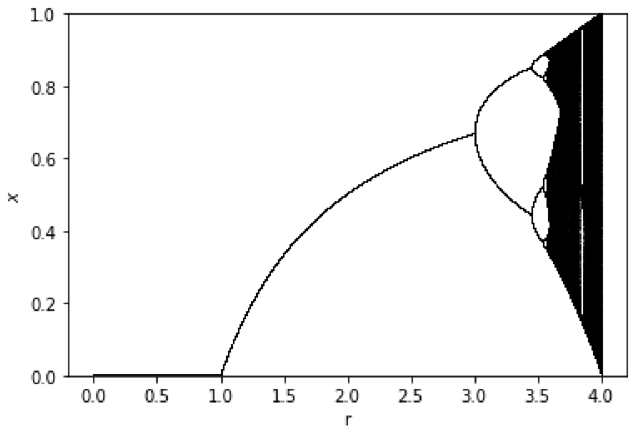
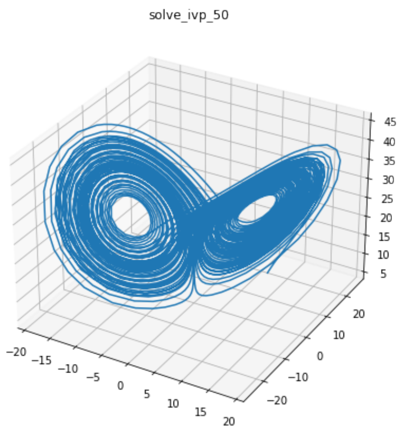


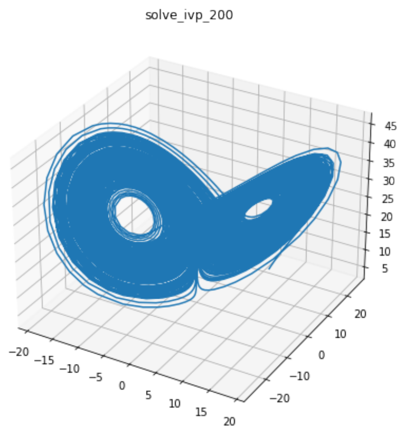
Figure 16: Bifurcation diagram for the discrete map.

A single trajectory of the Lorenz system In order to better display the trajectory, we extracted 50 simulated points, 200 simulated points and 1000 simulated points respectively, and compared them.

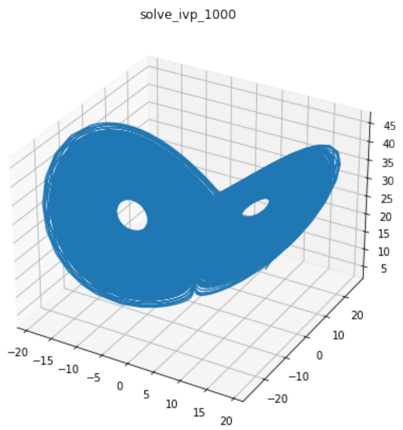
The shape of the attractor looks like a butterfly. The "butterfly effect" stems from the influence of the Lorentz attractor on the real world, that is, in any physical system, in the absence of a perfect understanding of the initial conditions (even the tiny air disturbance caused by the butterfly flapping its wings), the ability to predict its future direction will always fail. This shows that the physical system can be completely deterministic, even in the absence of quantum effects, it is still inherently unpredictable.



(a) Simulated time = 50



(b) Simulated time = 200



(c) Simulated time = 1000

Figure 17: Three trajectories with different amounts of simulated time

Another trajectory from a new starting point The chaotic nature of the system implies that small perturbations in the initial condition will grow larger at an exponential rate, until the error is as large as the diameter of the attractor. Test this by plotting another trajectory from $x_0 = (10^{-8} + 10, 10, 10)$.

Change the parameter ρ Now, change the parameter ρ to the value 0.5 and again compute and plot the two trajectories.

By observing the figures, we can judge that the larger ρ (ρ=28) is more sensitive to the initial conditions, even if the starting point position changes slightly, the trajectory will also change greatly.

For small values of ρ, the system is stable and evolves to one of two fixed point attractors. When ρ reaches 28.0, the fixed points become repulsors and the trajectory is repelled by them in a very complex way. Therefore, there is a bifurcation between the value 0.5 and 28. We cannot find a continuous and invertible map h that takes the orbits from the smaller value to the larger value and back.

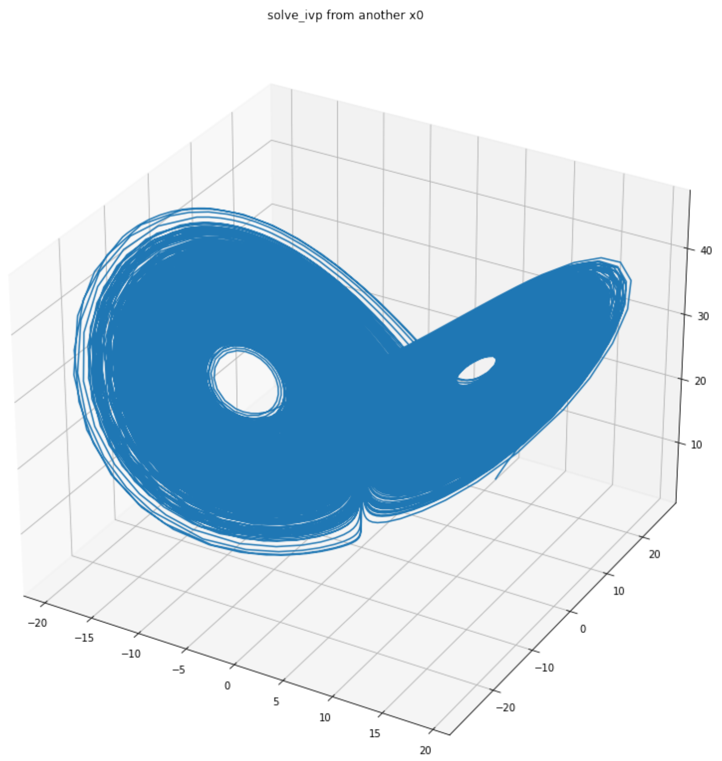
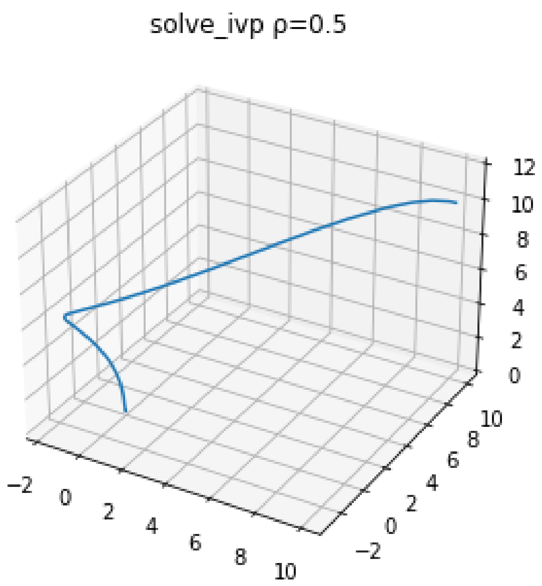
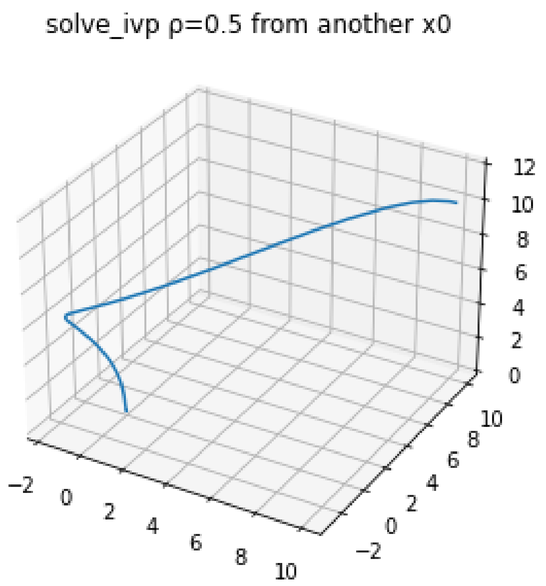


Figure 18: Plot another trajectory from new starting point $x_0 = (10^{-8} + 10, 10, 10)$



(a) From original starting point $x_0 = (10, 10, 10)$



(b) From new starting point $x_0 = (10^{-8} + 10, 10, 10)$

Figure 19: Change the parameter ρ from 28.0 to 0.5

Report on task 5, Bifurcations in crowd dynamics

Task description In this task, you have to apply your knowledge about bifurcation theory to analyze and describe a given SIR model. Download the example Jupyter notebook from Moodle and try to run it. It

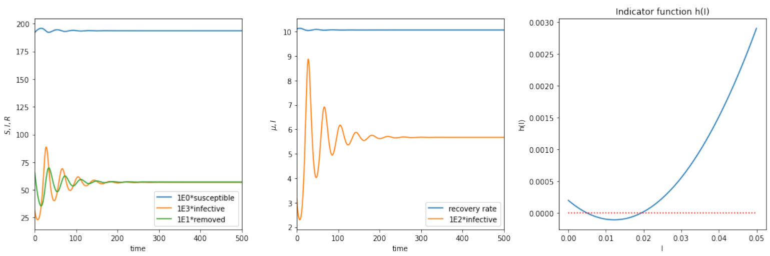


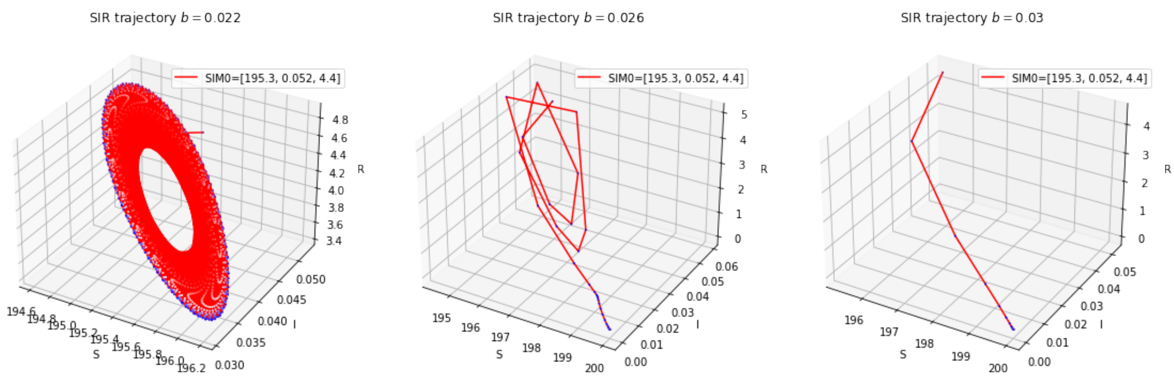
Figure 20: Changes of various parameters over time

implements parts of the model, and also has the proper setup for its numerical integration. Complete the unfinished model and document the notebook well, ideally moving the documented methods to another file.You may have to fix some imports that are missing (it should only need numpy, scipy, and matplotlib), and you have to actually implement the SIR model equations that are missing in the model method.

Change the parameter b from 0.01 to 0.03 in very small increments (e.g. 0.001) and observe what happens from the starting points (S0, I0, R0) = (195.3, 0.052, 4.4), (195.7, 0.03, 3.92), and (193, 0.08, 6.21). Illustrate what happens by plotting either the 3D picture, or just the 2D projection in the (S, I) plane.

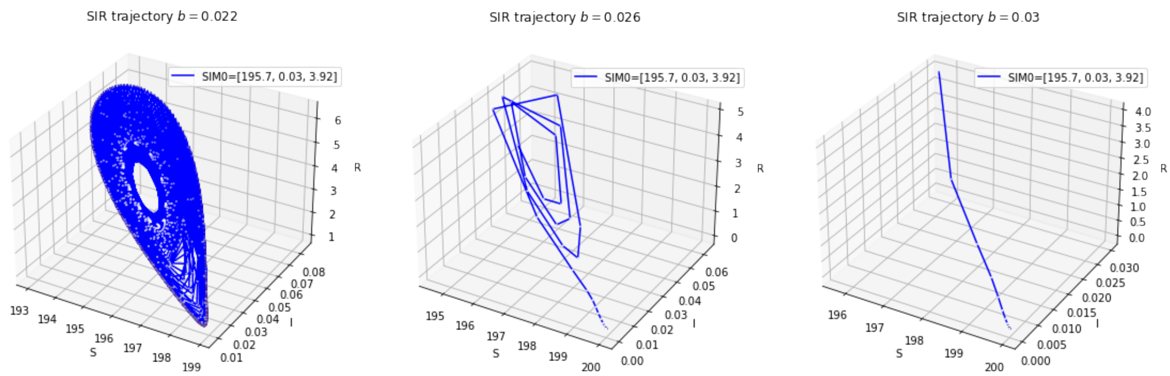
Solutions and Results

sub-problem 3 We studied the qualitative changes of the dynamic system from three different starting points over changes of the parameter b. Figure 21, 22 and 23 illustrate the bifurcations. Take figure 21 as an example. The trajectory is trapped into a ring at $b = 0.022$, then it manages to escape the ring as b increases and attracted by the final steady state $(S, I) = (200, 0)$. Such bifurcation is somehow similar to the Andronov-Hopf bifurcation we examined in task 3, but in a 3D manner.



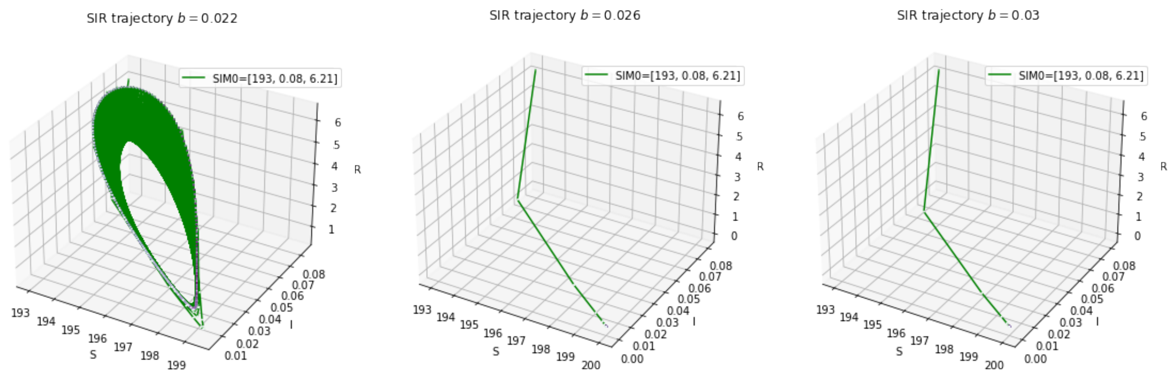
(a) b=0.022 (b) b=0.026 (c) b=0.03

Figure 21: Three trajectories with different b from starting pointP_0(195.3, 0.052, 4.4)



(a) b=0.022 (b) b=0.026 (c) b=0.03

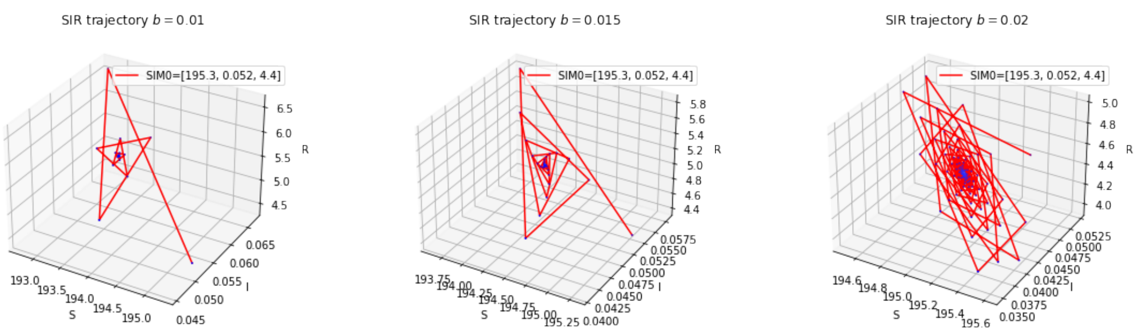
Figure 22: Three trajectories with different b from starting point $P_0(195.7,\ 0.03,\ 3.92)$



(a) b=0.022 (b) b=0.026 (c) b=0.03

Figure 23: Three trajectories with different b from starting point $P_0(193,\ 0.08,\ 6.21)$

Sub-problem 4. Bifurcation around b=0.02 Figure 24(c) shows the trajectory starts from (195.3, 0.052, 4.4) at $b = 0.020$, in which the trajectory has a spiral-like development. Therefore, there may exist an unstable focus fixed point here. As we examined the values of b, we found that for $0 < b < 0.021$, the trajectories would have such spiral-like manner. In particular, the trajectory might be absorbed by steady states during the spiral-like development, especially for $b < 0.015$.



(a) $b = 0.010$ (b) $b = 0.015$ (c) $b = 0.020$

Figure 24: Caption

Sub-problem 6. Attracting Node Based on the many plots we produced above, we got a taste of what this attracting node means. Except for $b = 0.022$ where trajectories shown as rings, trajectories in most other plots tend to end up or point to $(S, I, R) = (200, 0, 0)$. For figure 21, 22 and 23, if we look at the escape behaviours from the ring at $b = 0.026$, we can observe that escape behaviours happen when the trajectory reach a point on the ring that is close enough to $(200, 0, 0)$. Therefore, we can conclude that the attracting node is probably a stable focus or a stable node. Since such steady state has less potential in the vector field, values of (S, I, R) close to E_0 will be updated toward and absorbed by E_0.