CSE 331 Final Exam Preparation

This is in no way a substitute for exam preperation, mearly a compilation of all the key talking points.

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May 4, 2021

Counter Example

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Ex: Every day is a Wednesday, where a counter example would be Monday is not Wednesday.

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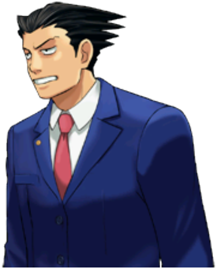

Figure: Sourced from Google

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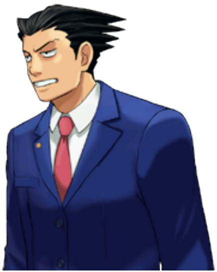

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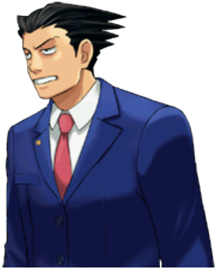
Therefore, the original assumption has to be true.



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If you want to prove that $E \to F$, it might be more doable to prove $\neg F \to \neg E$, as they are both logically equivalent.

This is especially useful if the **scope** of F is smaller than the scope of E.

Direct Proof

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Remember though, that you must maintain *W.L.O.G*, that your proof can never be too specific and must be arbitrary.

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If proof needs to be correct for all numbers $\in \mathbb{N}$, and each step is dependant on the previous step, then *every* step can be reduced to a definitive base case that is easy to directly prove.

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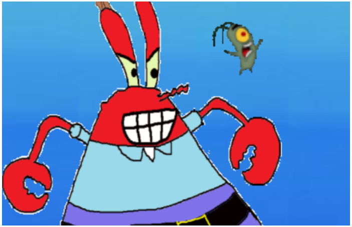
Note: This isn't a runtime analysis, rather a proof that the algorithm terminates.

Greedy Stays Ahead

This technique is used to prove that a greedy algorithm returns an optimal solution.



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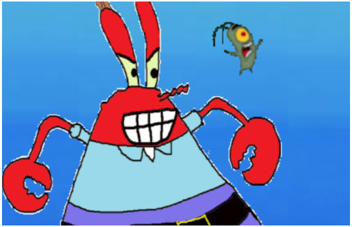

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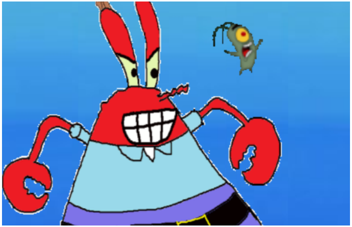
HW4 "Attack on Alarms" and Interval Scheduling are examples of problems with greedy solutions.



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Introduction

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Moreover, what is a **stable** matching?

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With n members in each group, there are $n!$ perfect matchings.

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If m prefers n over their current matching **and** n prefers m over their current matching, then (m, n) is an instability to the perfect matching.

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With the right data structures, the runtime can be reduced to $O(n^2)$.

Even though the runtime isn't linear, because the input size is $2n^2 \to \Theta(n^2)$ [1], the runtime **with respect** to the input size is $O(N)$, or linear time.

[1] This comes from n Group A members and n Group B members with their $2n$ preference lists

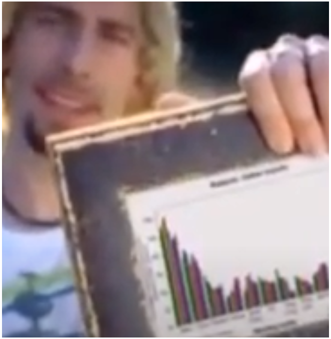
Stable Matching

Code:

```
Initially all m ∈ M and w ∈ W are free
While there is a man m who is free and hasn't proposed to every woman
      Choose such a man m
      Let w be the highest-ranked woman in m's
            preference list to whom m has not yet proposed
      If w is free then
            (m, w) become engaged
      Else w is currently engaged to m'
            If w prefers m' to m then
                  m remains free
            Else w prefers m to m'
                  (m, w) become engaged
                  m' becomes free
            Endif
      Endif
Endwhile
Return the set S of engaged pairs.
```

What are graphs?

Graphs are a way to represent relations between data points.



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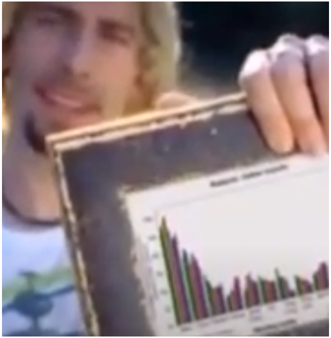
Graphs have a set of vertices which represent data points and a set of edges to model the **relation** between vertices.



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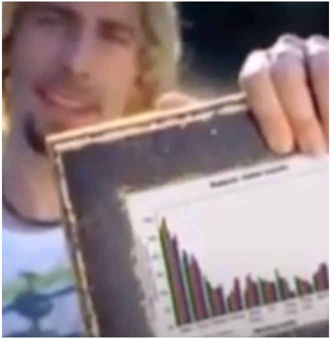
An edge exists between two or more vertices *iff* there is a connection between them.



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Graph Representation

Adjacency List: Keep a n length array. Vertices are indices of the array where each element of the array contains a pointer to a list of adjacent vertices to the index/vertex. This takes up $\Theta(n + m)$ space for m edges.

Adjacency Matrix: Keep a n X n matrix. Rows and columns are vertices. An edge (u, v) exists if the matrix at row u and column v is not null This takes up $\Theta(n^2)$ space.

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Bus transportation routes could be classified as an undirected graph, while airports and flights could be modeled as a directed graph.

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A cycle needs **at least** 4 elements. The **maximum length** of a simple path for a graph with n vertices is $n - 1$ [1].

[1] Pigeonhole Principle

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A directed acyclic graph $\Leftrightarrow$ topological ordering, meaning the vertices can be ordered in a way that all edges point in the **same** direction.

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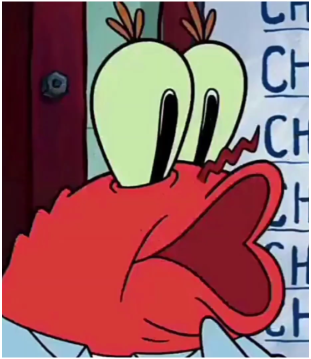

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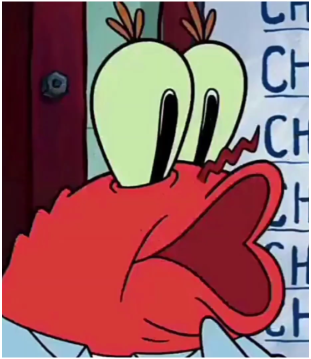

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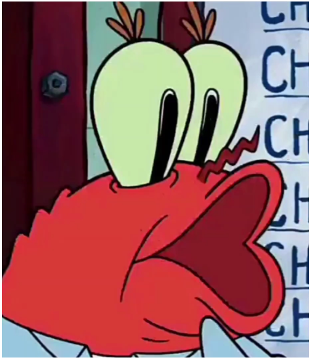
See **the greedy stays ahead proof** for details on proving these types of algorithms.



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Interval Scheduling Problem

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One greedy solution is to sort the set of tasks by their earliest $\rightarrow$ latest finishing time.

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Refer to the sources for proof of correctness and runtime analysis.

Shortest Path Problem

Given a weighted graph with **no** negative weights, how do you find the shortest path?