

CSE 331 Final Exam Preparation

This is in no way a substitute for exam preparation, merely a compilation of all the key talking points.

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Counter Example

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Ex: Every day is a Wednesday, where a counter example would be Monday is not Wednesday.

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Assume what you want to prove is false, then show this leads to a contradiction.



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Assume what you want to prove is false, then show this leads to a contradiction.

Therefore, the original assumption has to be true.



Figure: Sourced from Google

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This is especially useful if the **scope** of F is smaller than the scope of E .

Direct Proof

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Remember though, that you must maintain *W.L.O.G*, that your proof can never be too specific and must be arbitrary.

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If proof needs to be correct for all numbers $\in \mathbb{N}$, and each step is dependant on the previous step, then *every* step can be reduced to a definitive base case that is easy to directly prove.

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Note: This isn't a runtime analysis, rather a proof that the algorithm terminates.

Greedy Stays Ahead

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HW4 “Attack on Alarms” and Interval Scheduling are examples of problems with greedy solutions.



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Introduction

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Moreover, what is a **stable** matching?

Perfect Matchings

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With n members in each group, there are $n!$ perfect matchings.

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If m prefers n over their current matching **and** n prefers m over their current matching, then (m, n) is an instability to the perfect matching.

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With the right data structures, the runtime can be reduced to $O(n^2)$.

Even though the runtime isn't linear, because the input size is $2n^2 \rightarrow \Theta(n^2)$ ¹, the runtime **with respect** to the input size is $O(N)$, or linear time.

¹This comes from n Group A members and n Group B members with their $2n$ preference lists

Stable Matching

Code:

```
Initially all  $m \in M$  and  $w \in W$  are free
While there is a man  $m$  who is free and hasn't proposed to every
woman
    Choose such a man  $m$ 
    Let  $w$  be the highest-ranked woman in  $m$ 's
        preference list to whom  $m$  has not yet proposed
    If  $w$  is free then
         $(m, w)$  become engaged
    Else  $w$  is currently engaged to  $m'$ 
        If  $w$  prefers  $m'$  to  $m$  then
             $m$  remains free
        Else  $w$  prefers  $m$  to  $m'$ 
             $(m, w)$  become engaged
             $m'$  becomes free
        Endif
    Endif
Endwhile
Return the set  $S$  of engaged pairs.
```

What are graphs?

Graphs are a way to represent relations between data points.

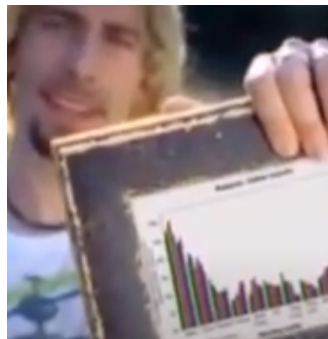


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An edge exists between two or more vertices *iff* there is a connection between them.



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Graph Representation

Adjacency List: Keep a n length array. Vertices are indices of the array where each element of the array contains a pointer to a list of adjacent vertices to the index/vertex. This takes up $\Theta(nm)$ space for m edges.

Adjacency Matrix: Keep a $n \times n$ matrix. Rows and columns are vertices. An edge (u, v) exists if the matrix at row u and column v is not null This takes up $\Theta(n^2)$ space.

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A graph is undirected if edges go both ways. If for any vertices u , v , vertex u connects to v , v connects to u .

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Bus transportation routes could be classified as an undirected graph, while airports and flights could be modeled as a directed graph.

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A cycle needs **at least** 4 elements. The **maximum length** of a simple path for a graph with n vertices is $n - 1$ ¹.

¹Pigeonhole Principle

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A directed acyclic graph \Leftrightarrow topological ordering, meaning the vertices can be ordered in a way that all edges point in the **same** direction.

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See **the greedy stays ahead proof** for details on proving these types of algorithms.



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Interval Scheduling Problem

Given a set of tasks to do over a period in time, how do you pick the **maximum** number of tasks that aren't in conflict?

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Refer to the sources for proof of correctness and runtime analysis.

Shortest Path Problem

Given a weighted graph with **no** negative weights, how do you find the shortest path?