Discrete Mathematics (for Computer Science)

Part II: Computability

Zeyu Mi

TA: Yuqi Hu

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SHANGHAI JIAO TONG UNIVERSITY

IPADS
INSTITUTE OF PARALLEL AND DISTRIBUTED SYSTEMS

What problems can a computer solve?

Is a number prime?

Is a number prime?

Easy!

A prime number is a natural number greater than 1 whose only factors are 1 and itself

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Easy!

A prime number is a natural number greater than 1 whose only factors are 1 and itself

We only need to verify there are no its factors from 2 to itseft-1

```
bool isPrime(int n){
    for(int i=2;i<n;i++){
        if(n%i==0) return false;
    }
    return true;
}
```

Is a person good programmer?

Is a person good programmer?

What is a good programmer?

Well-known reality:
Produce 1000 lines of bug-free code per hour.
Sleep less than 5 hours.
Have less hair than 99% people.

Is a person good programmer?

What is a good programmer?

Well-known reality:
Produce 1000 lines of bug-free code per hour.
Sleep less than 5 hours.
Have less hair than 99% people.

We need to analyze his/her code data, sleep data and hair data to say if he/she is qualified.

```
bool isGoodProgrammer
(codedata,sleepdata,hairdata){
        ......
}
```

Does god exist?

Does god exist?

Emmmm

What if god exists? And what if not?
What should we take into computation.

I can't tell. I don't think a computer can solve this problem.

Input

Is a number prime?

Easy!

Output Constraints

A prime number is n natural number greater than 1 whose only factors are 1 and itself

We only need to verify there are no its factors from 2 to itseft-1

Solution

Code

```
bool isPrime(int n){
    for(int i=2;i<n;i++){
        if(n%i==0) return false;
    }
    return true;
}
```

What's a good programmer?

Output Constraints

Produce 1000 lines of bug-free code per hour.
Sleep less than 5 hours.
Have less hair than 99% person.

"Person" is too abstract for computation

Is a person good programmer?

Input

We need to analyze his/her code data, sleep data and hair data to say if he/she is qualified.

Solution

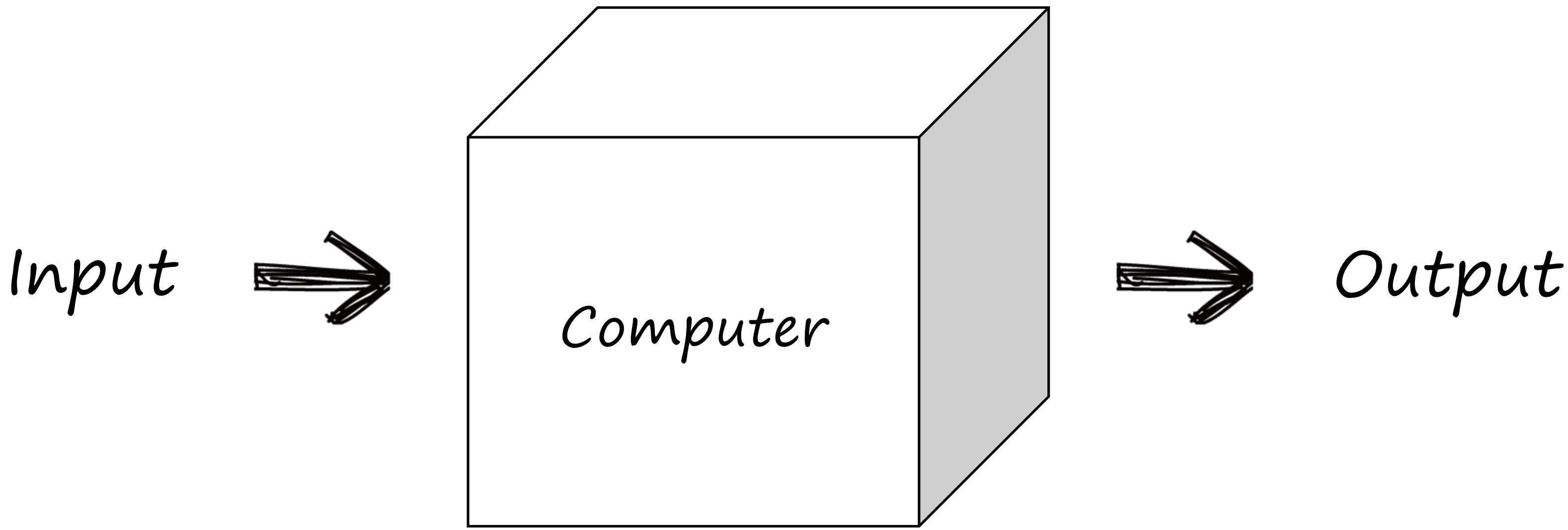
Code

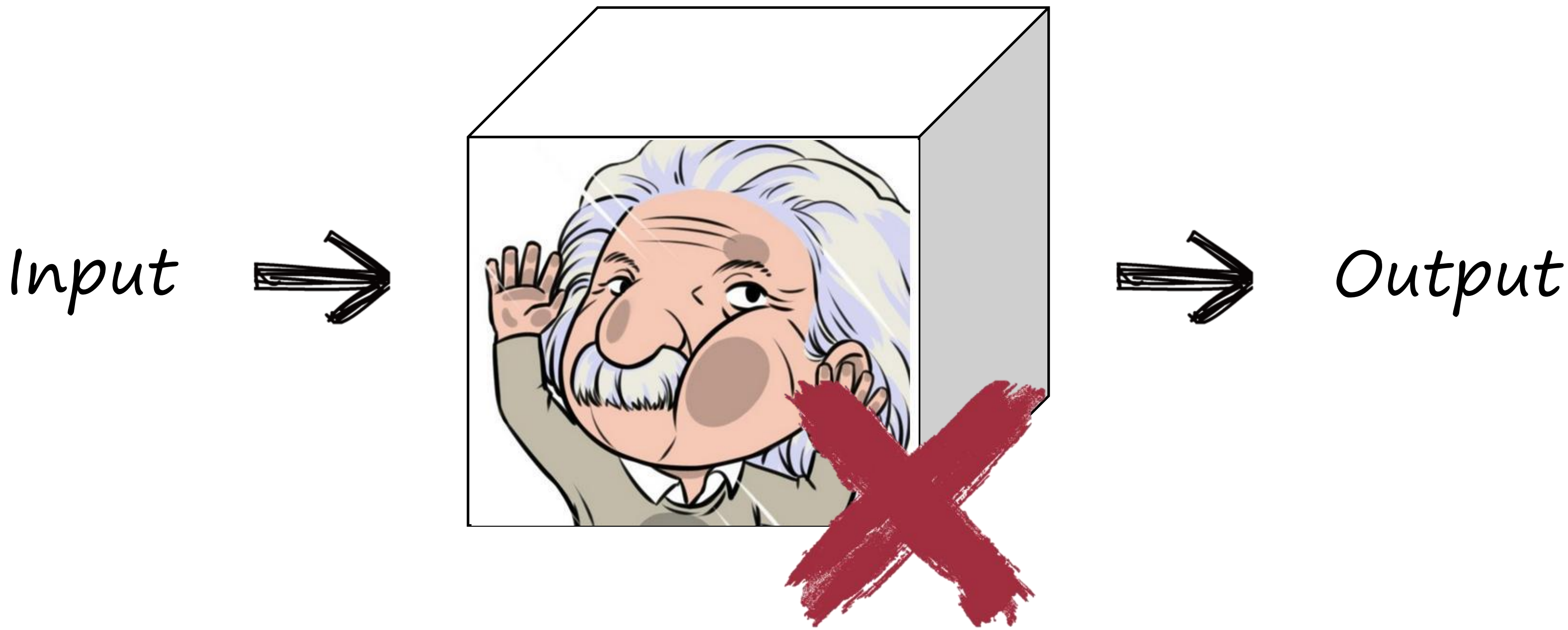
```
bool isGoodProgrammer
(codedata,sleepdata,hair data){
        ......
}
```

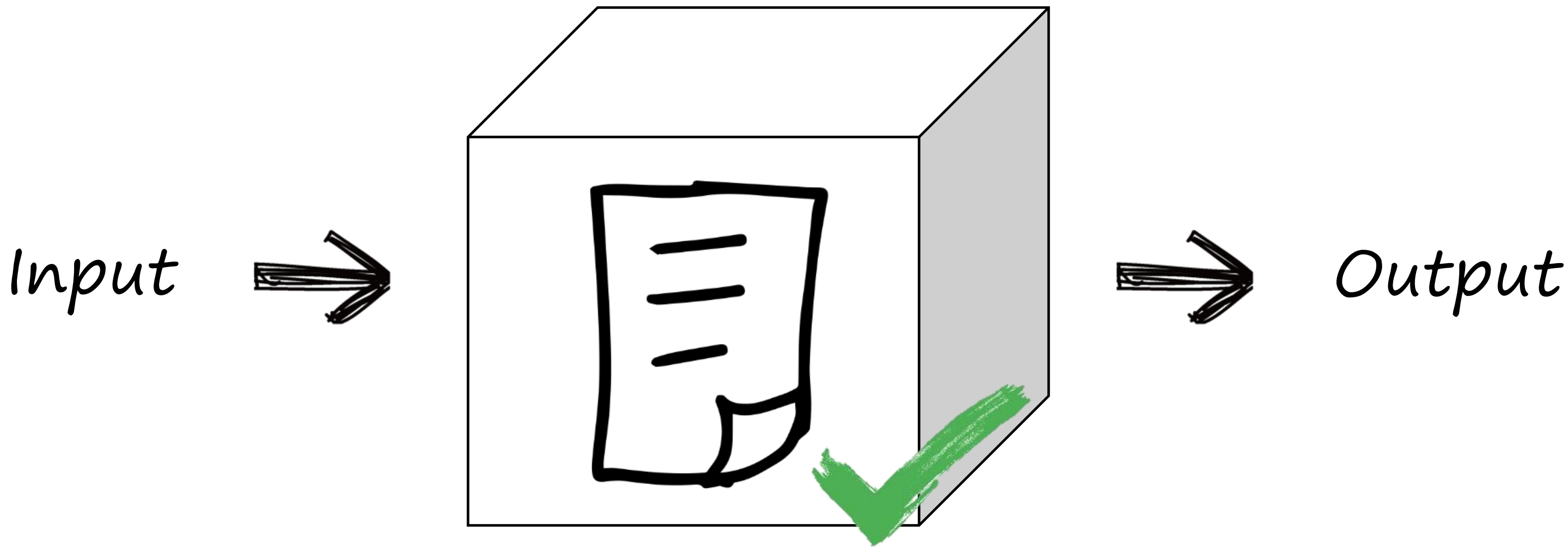
Computational Problem（计算问题）

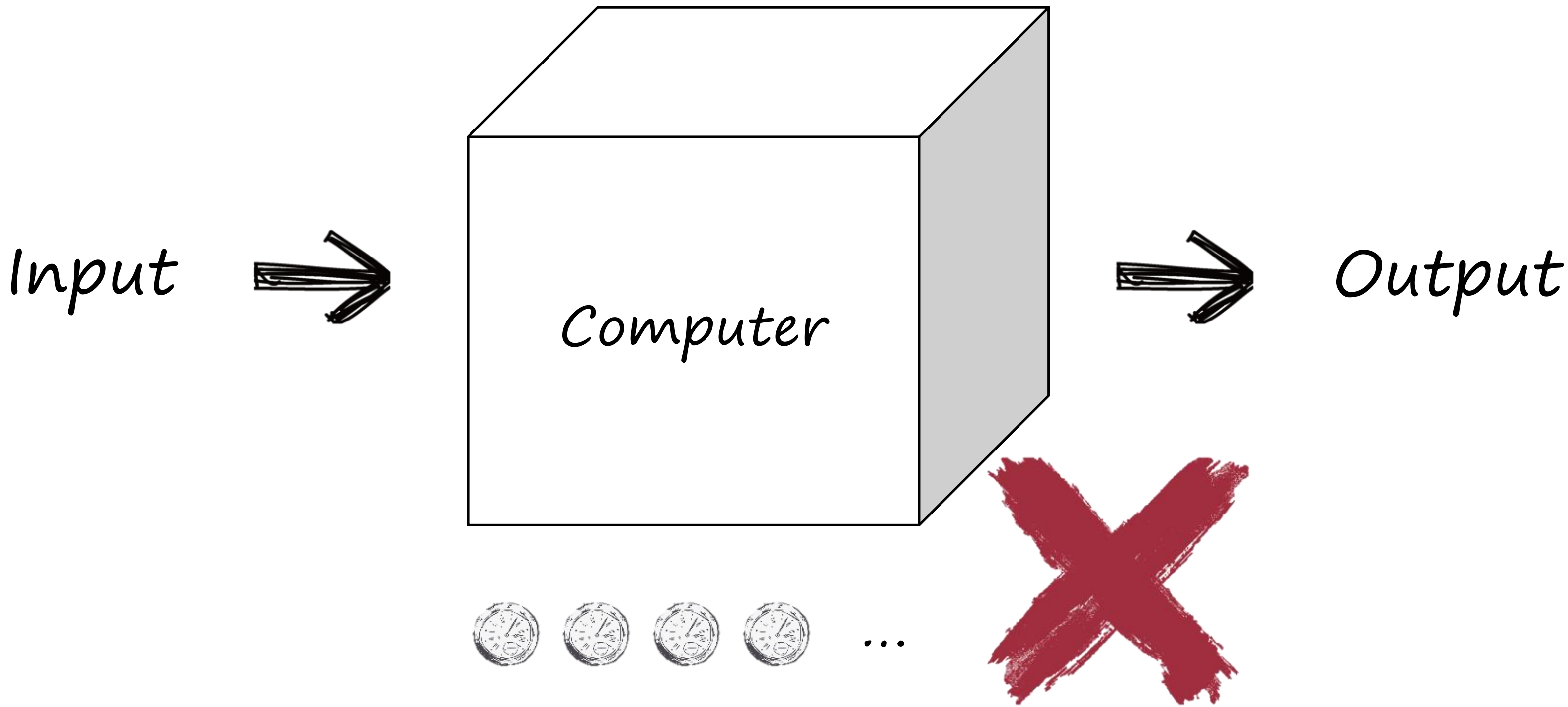
- **Problems for computation usually have:**
  - Well-defined input
  - Constraints that the output must satisfy.
- **Some problems are hard because they are hard to converted into computational problems**
  - Existence of god
- **We only focus on the computational problems**

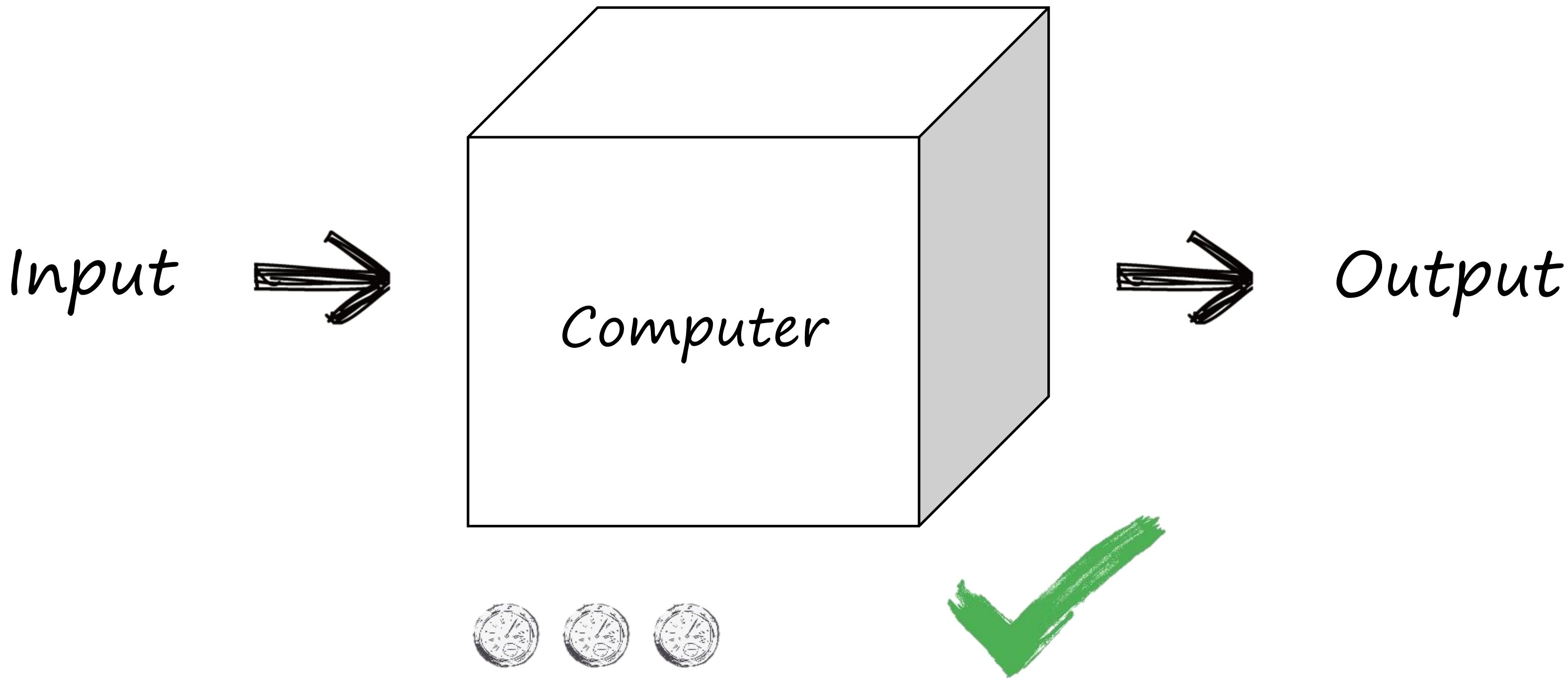
What about “computable”?


Input
Computer
Output


Input
Output


Input
Output


Input
Computer
Output
...


Input
Computer
Output

Computable（可计算的）

- **A computational problem is computable if there is a procedure (or algorithm) for it which:**
  - Rigorously follows some finite instructions, requires no ingenuity.
  - Always finishes (terminates) after a finite number of steps.

Are all the computational problem computable?

Entscheidungsproblem (判定问题)

All axioms of math(Finite)

There is no natural number whose successor is 0

...

First-order logic （谓词逻辑）

$\forall x \in N(0 \neq S(x))$

...

A statement of first-order logic

Input

Prover

True. The statement can be deduced from the axioms

False. Otherwise.

How do we investigate **computability**?

Problem has large varieties.

Graph Theory
A web server
A recommendation system
....

Unify $\Rightarrow$ **How to represent a problem?**

Computer has many details.

Instruction set architecture
Memory management
Device management
...

Simplify $\Rightarrow$ **How to represent computation?**

How do we investigate computability?

2.1 Discussion on Problems

Part1. Intro & Set theory(I) : Basics & Formal Language.

Part2. Set Theory(II): Axiom system & Cardinality.

Part3. Capture Structures: Binary Relation & Function

2.2 Discussion on Computation

Part1. Turing Machine Basics.

Part2. Variants of Turing Machine. Church-Turing Thesis.

2.3 Discussion on computability

Part1. The Language of Turing Machine. R & RE

Part2. Undecidability.

How do we investigate computability?

《数理逻辑与集合论》9.1-9.5
Stranford CS103 Finite Atomota

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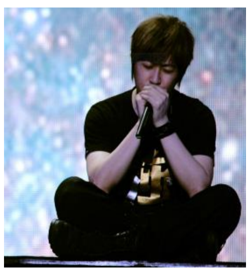
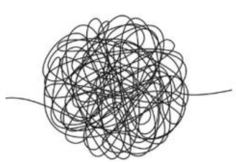
What is a set?

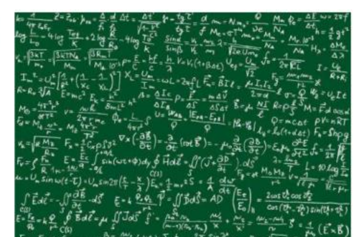
Emojis: 

The hobbies of a talented man.




Or a bunch of things that even have no exact meaning





Definition

Set

A Set is an unordered collection of distinct objects, which may be anything (including other sets).

If a is an item of set A, we say that a belongs to A, writing as :

$$\boldsymbol{a} \in \boldsymbol{A}$$

If b is not an item of set A, we say that b doesn't belongs to A, writing as:

$$\boldsymbol{b} \notin \boldsymbol{A}$$

Representation

- **For simple sets, we can enumerate all the objects wrapped by curly braces**
  - {sing, jump, rap, basketball}
- **For some common sets, we use specific notations for them**

$\mathbb{N}$:All the natural number:$0, 1, 2, 3, \ldots$

$\mathbb{Z}$:All the integer:$\ldots, -2, -1, 0, 1, 2, \ldots$

$\mathbb{Q}$:All the rational number

$\mathbb{R}$:All the real number

$\mathbb{C}$:All the complex number

$\varnothing$:Empty set, contains nothing.$\{\}$

U:The universal set, contains all the objects under *consideration*

Representation

For sets whose object has a certain property, we'll use set-builder notation

Set-Builder Notation

$$A = \{x|P(x)\} \text{ or } A = \{x: P(x)\}$$

Where P is a predicate, and A is the set of all elements which makes P true

$$\{x|x \text{ is a prime number}\} = \{2,3,5,7,11, \ldots\}$$

Unorder（无序性）

The order of objects in a set doesn't matter

{ , , ,  }

and

{ , , ,  }

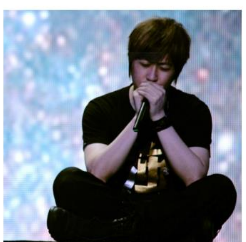
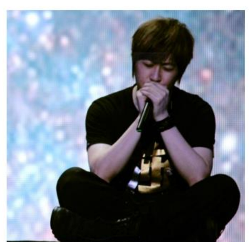
are identical

Distinction（相异性）

The objects in a set are distincted, or the repeated elements are ignored.

{ ,  }

and

{ , , ,  }

are identical

Deterministic（确定性）

For any set A and any object a, whether a belongs to A is determinate and mutually exclusive.

$$\boldsymbol{either\ a \in A\ or\ a \notin A}$$

Relations Between Sets

Equal

A and B are equal iff they have the same elements

$$A = B \Leftrightarrow (\forall x)(x \in A \leftrightarrow x \in B)$$

Subset

A is subset of B iff every element of A is element of B

$$A \subseteq B \Leftrightarrow (\forall x)(x \in A \rightarrow x \in B)$$

Proper Subset

A is proper subset of B iff A is subset of B and A is not equal to B

$$A \subset B \Leftrightarrow (A \subseteq B \wedge A \neq B)$$

Relations Between Sets

$$1 \in \{1,2,3\}$$

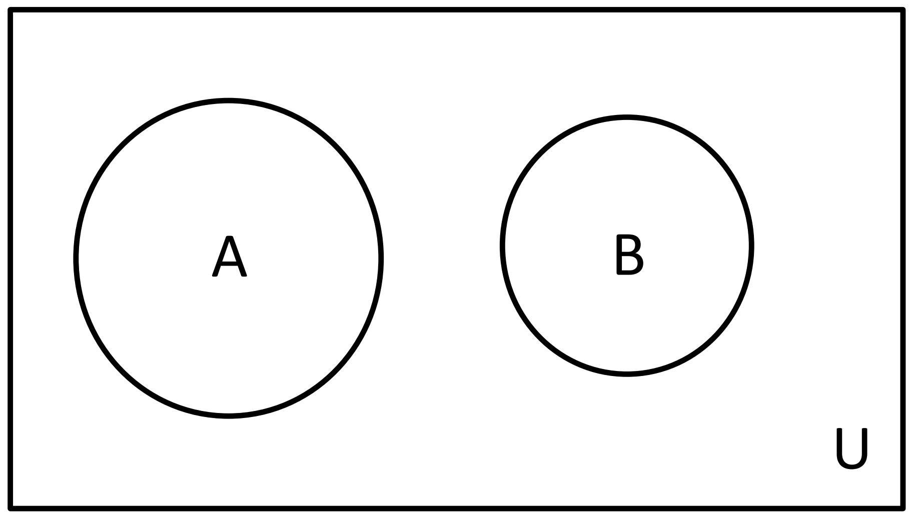
$$\{1\} \subset \{1,2,3\}$$

$$\varnothing \subset \{1,2,3\}$$

$\varnothing \in \{\varnothing\}$ and $\varnothing \subset \{\varnothing\}$

For any set A, $\varnothing \subseteq A$

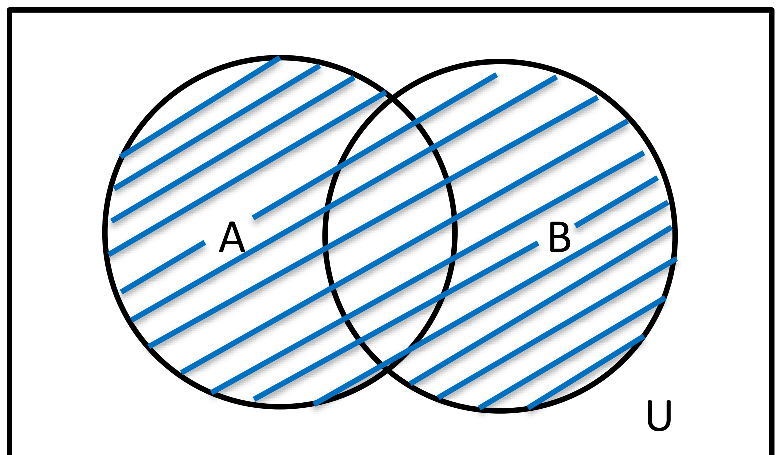
Venn Diagrams



Set Operation

Union(并)

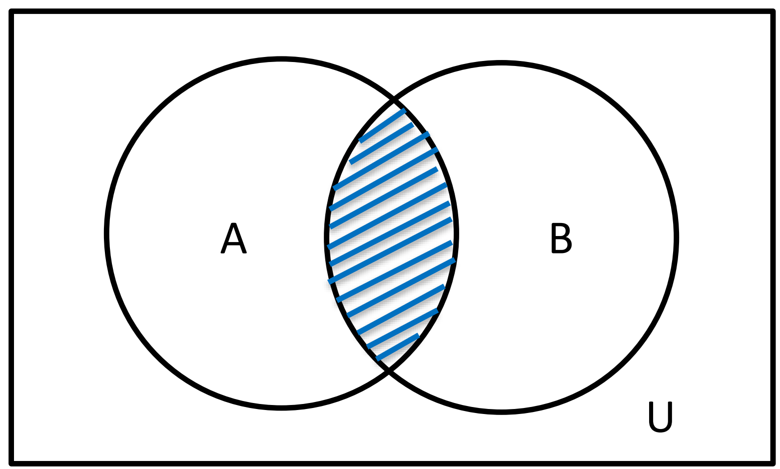
$$A \cup B = \{x | x \in A \lor x \in B\}$$



Intersection

Intersection(交)

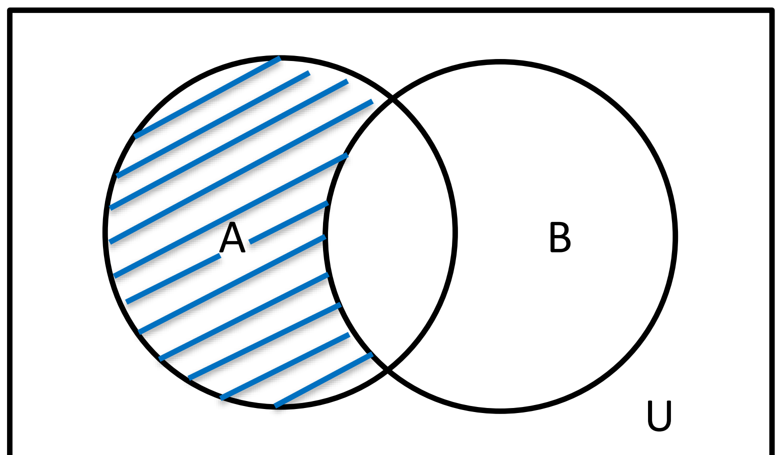
$$A \cap B = \{x | x \in A \land x \in B\}$$



Difference

Difference(差)

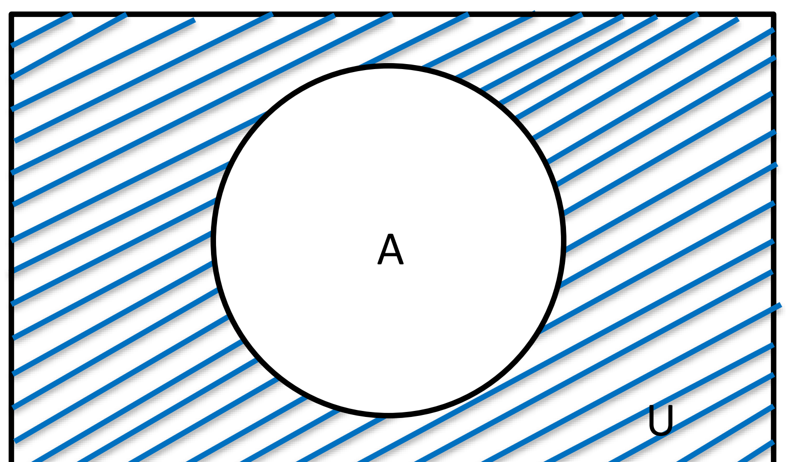
$$A - B = \{x | x \in A \land x \notin B\}$$



Complement

Complement(补)

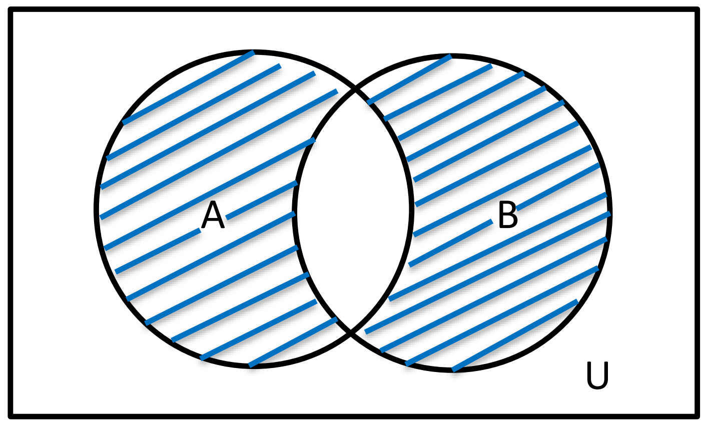
$$-A \ or \ \bar{A} = \{x | x \in U \wedge x \notin A\}$$



Symmetric Difference

Symmetric Difference(对称差)

$$A \oplus B = \{x | x \in A \ \overline{\vee}\ x \in B\} = (A - B) \cup (B - A)$$



Generalized Union/Intersection

Generalized Union/Intersection(广义并/交)

$$\bigcup A = \{x \mid (\exists z)(z \in A \land x \in z)\}$$
$$\bigcap A = \{x \mid (\forall z)(z \in A \to x \in z)\}$$

$A = \{\{a,b,c\},\{a,b\},\{b,c,d\}\}$

Then $\bigcup A = \{a,b,c,d\}$, $\bigcap A = \{b\}$

Define that $\bigcup \emptyset = \emptyset$, $\bigcap \emptyset$ is undefined!

Set Identities（集合恒等式）

Identity laws (同一律)

$$A \cap U = A$$
$$A \cup \varnothing = A$$

Commutative laws (交换律)

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Domination laws (零律)

$$A \cup U = U$$
$$A \cap \varnothing = \varnothing$$

Associative laws (结合律)

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Idempotent laws (幂等律)

$$A \cup A = A$$
$$A \cap A = A$$

Distributive laws (分配律)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Set Identities

Complementation law (双补律)

$$\overline{(\overline{A})} = A$$

Complement laws (补余律)

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \varnothing$$

Absorption laws (吸收律)

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

De Morgan's laws (摩根律)

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$
$$A - (B \cup C) = (A - B) \cap (A - C)$$

Proof of Set Identities

Method 1: Using definition

Example: $A \cup B = B \Rightarrow A \subseteq B$

Analysis: We need to prove $A \subseteq B \Leftrightarrow \forall x(x \in A \rightarrow x \in B)$, so the proof structure is

for any x, assume $x \in A$,{...bunch of inference...}, we get $x \in B$

Proof:

For any x, assume $x \in A$, according to the definition of set union, $x \in A \cup B$. And $A \cup B = B$, so we get $x \in B$. Q.E.D.

Proof of Set Identities

Method 2: Using logical equivalent operation

Example: $A-(B\cup C)=(A-B)\cap(A-C)$

Proof:

$$
\begin{aligned}
x\in A-(B\cup C) &\Leftrightarrow (x\in A)\wedge(x\notin(B\cup C))\\
&\Leftrightarrow x\in A\wedge x\notin B\wedge x\notin C\\
&\Leftrightarrow (x\in A\wedge x\notin B)\wedge(x\in A\wedge x\notin C)\\
&\Leftrightarrow (x\in(A-B))\wedge(x\in(A-C))\\
&\Leftrightarrow x\in((A-B)\cap(A-C))
\end{aligned}
$$

Proof of Set Identities

Method 3: Deriving a set algebra using known identities or equations

Example: $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

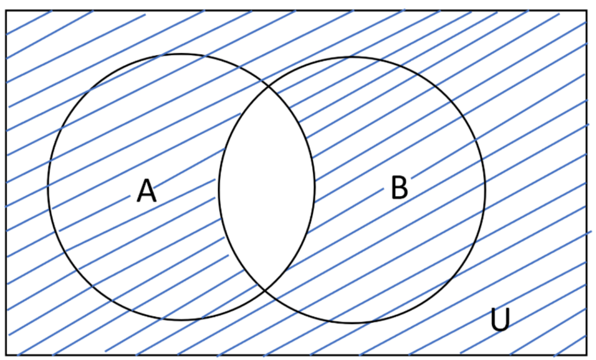
Proof:

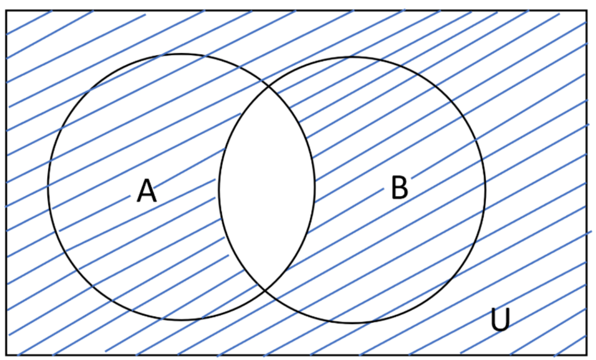
$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the second De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the first De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for intersections}
\end{aligned}$$

Proof of Set Identities

We can use venn diagram to help anlyzing.

$\overline{A \cap B}$ $\overline{A} \cup \overline{B}$





This is not a proof!!

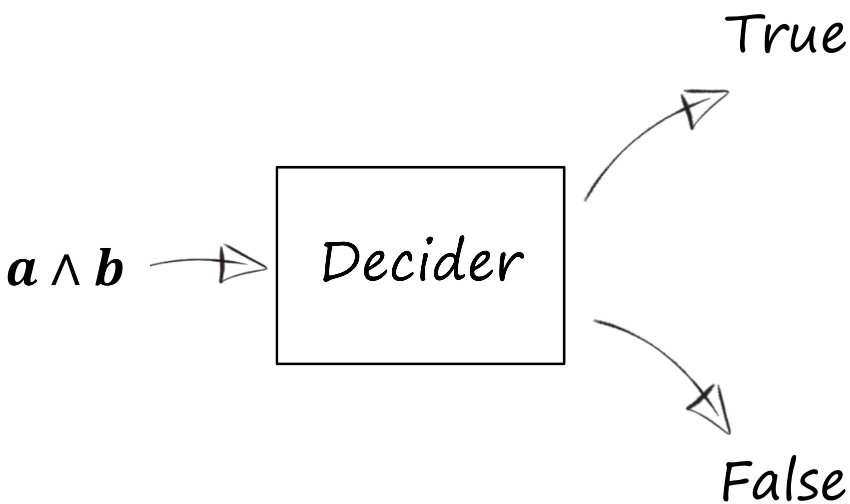
Wait a minute,
What is the relationship between set theory and computational problems?

Types of Computational Problem

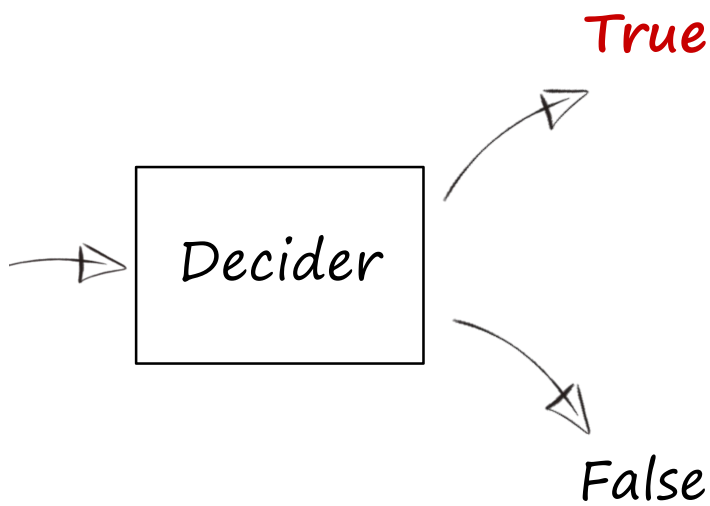
Decision Problem	**Search Problem**
It is satisfiable? (True or False)	*Find a satisfying assignment if it is satisfiable.*
Find a satisfying assignment so that it has the minimal count of true variables if it is satisfiable.	*Count the number of satisfying assignments so that they have minimal count of true variables if it is satisfiable.*
Optimizing Problem	**Counting Problem**

SAT

Solve Decision Problem

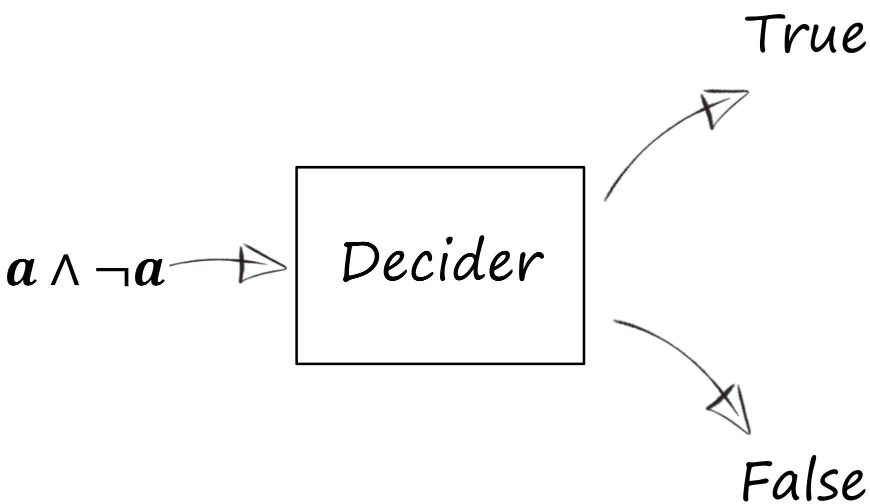


Solve Decision Problem



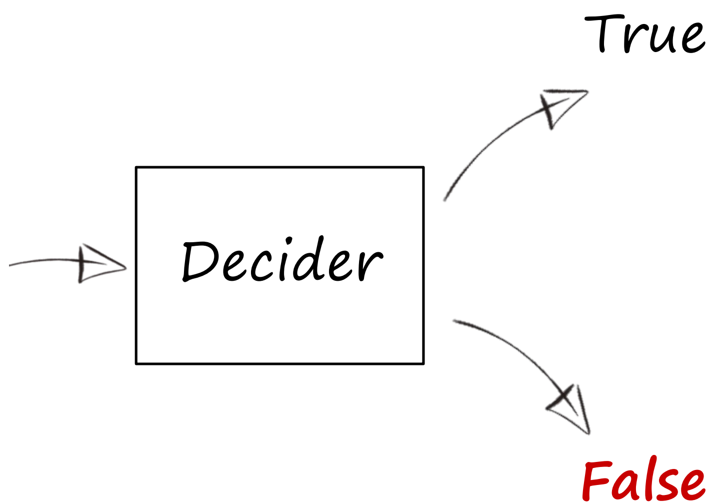
$$\boldsymbol{a} \wedge \boldsymbol{b}$$

Solve Decision Problem



$\boldsymbol{a} \wedge \boldsymbol{b}$

Solve Decision Problem



$$a \wedge b$$

$$a \wedge \neg a$$

Solve Other Problems

Decision Problems

Results are only true or false.

$a \wedge b$ → Decider → True / False

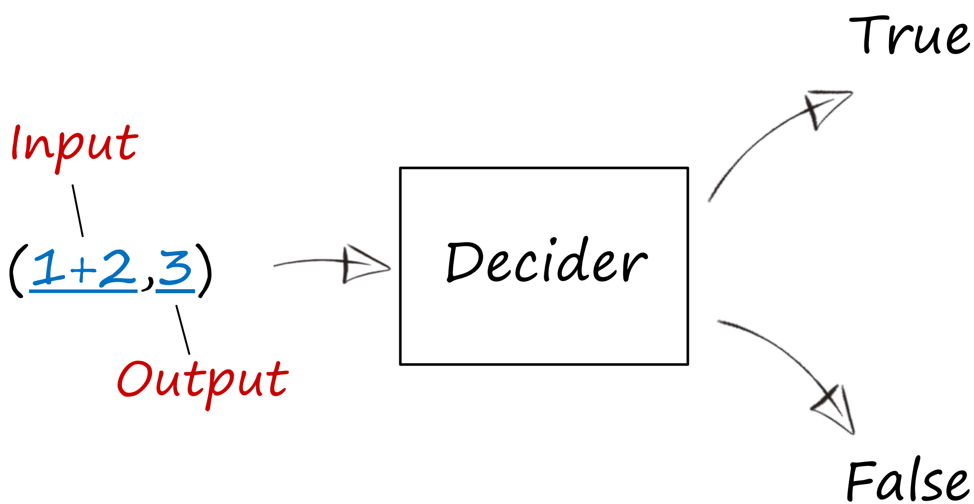
Other Problems

Results are not only true or false.

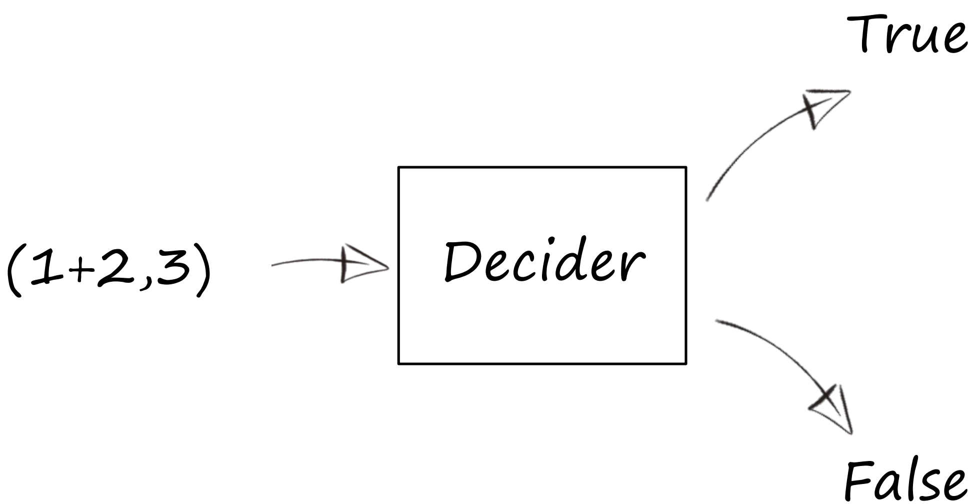
$1+2$ → Solver → 3

Solve Other Problems

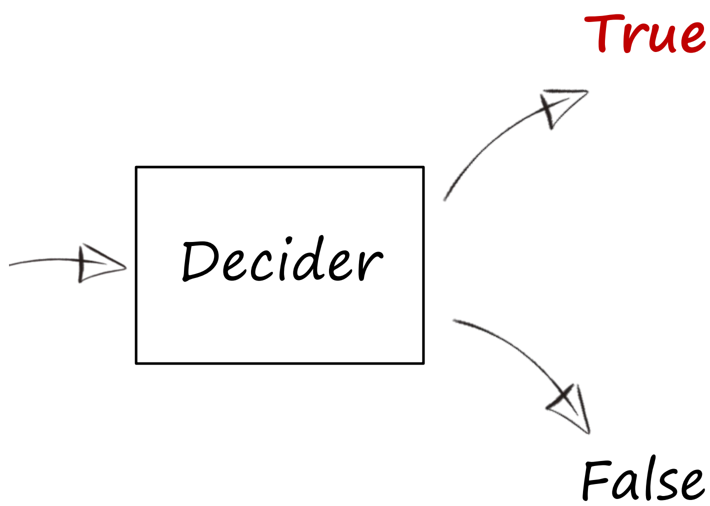
We can build a decider to judge whether an input/output pair is correct or not



Solve Other Problems

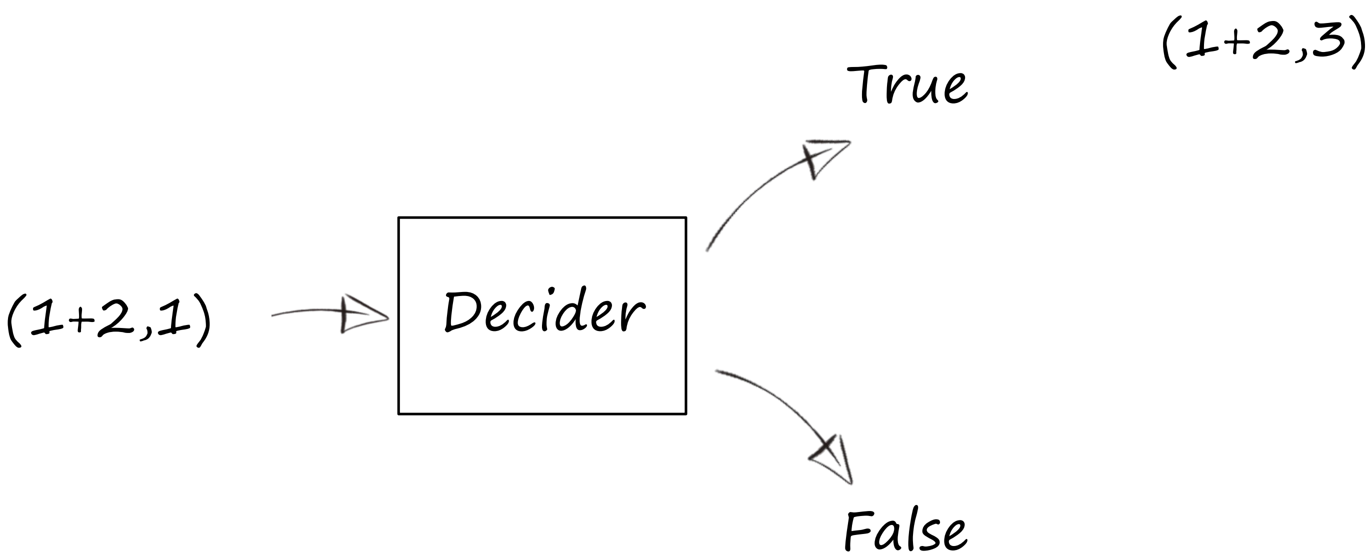


Solve Other Problems

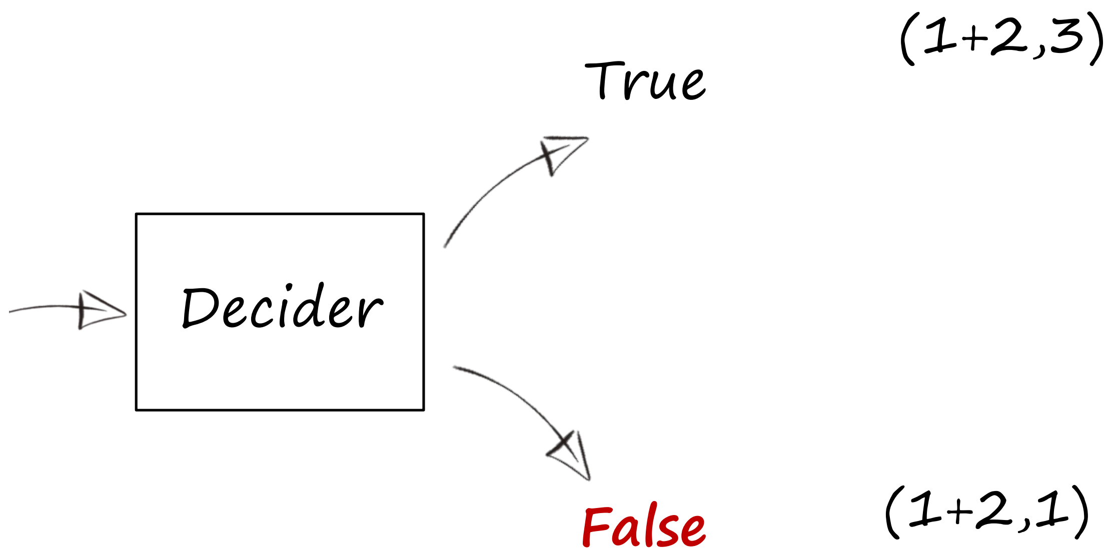


(1+2,3)

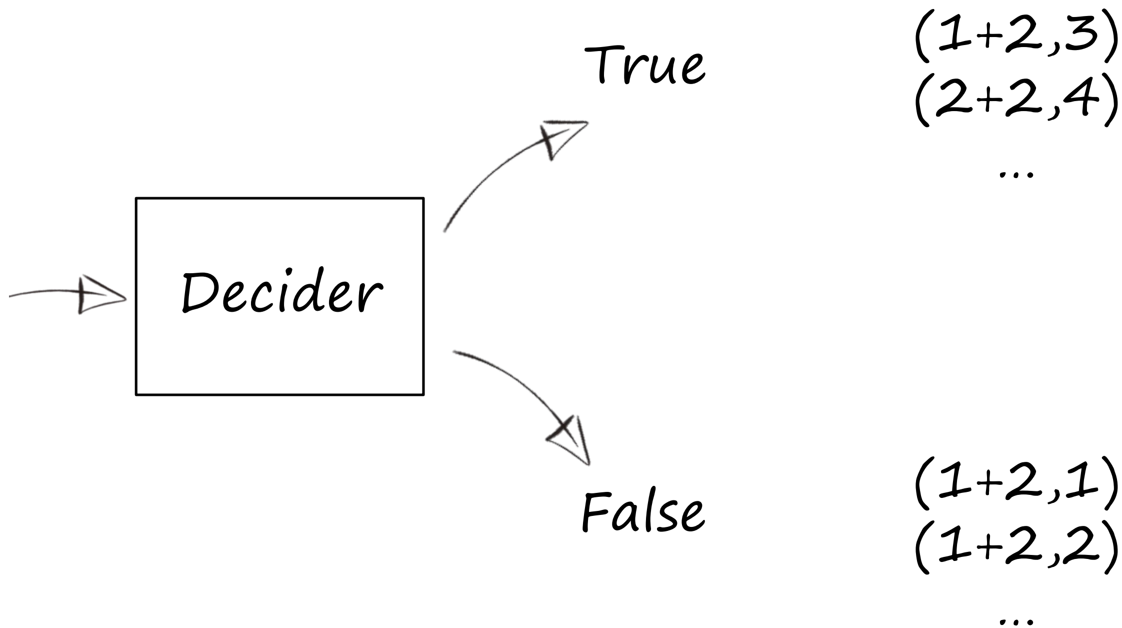
Solve Other Problems



Solve Other Problems



Solve Other Problems



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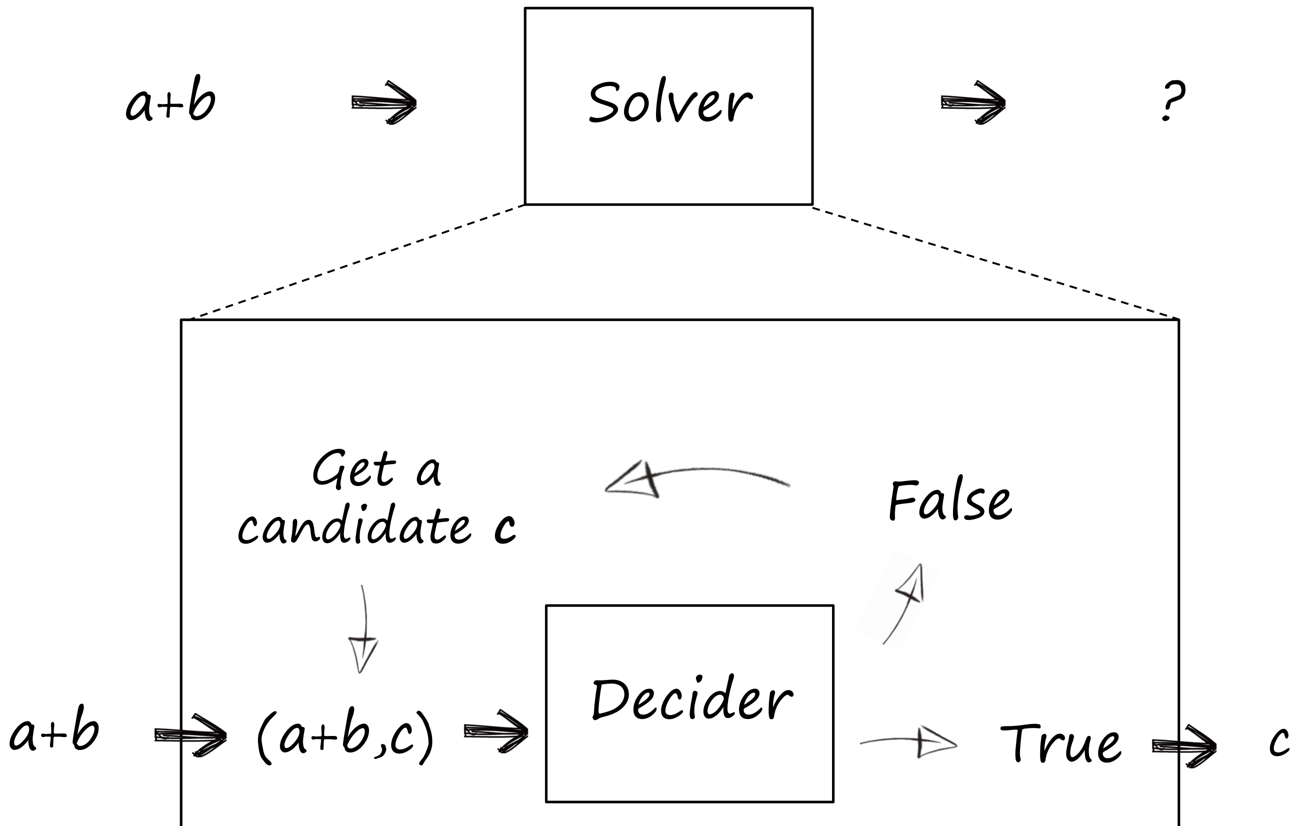

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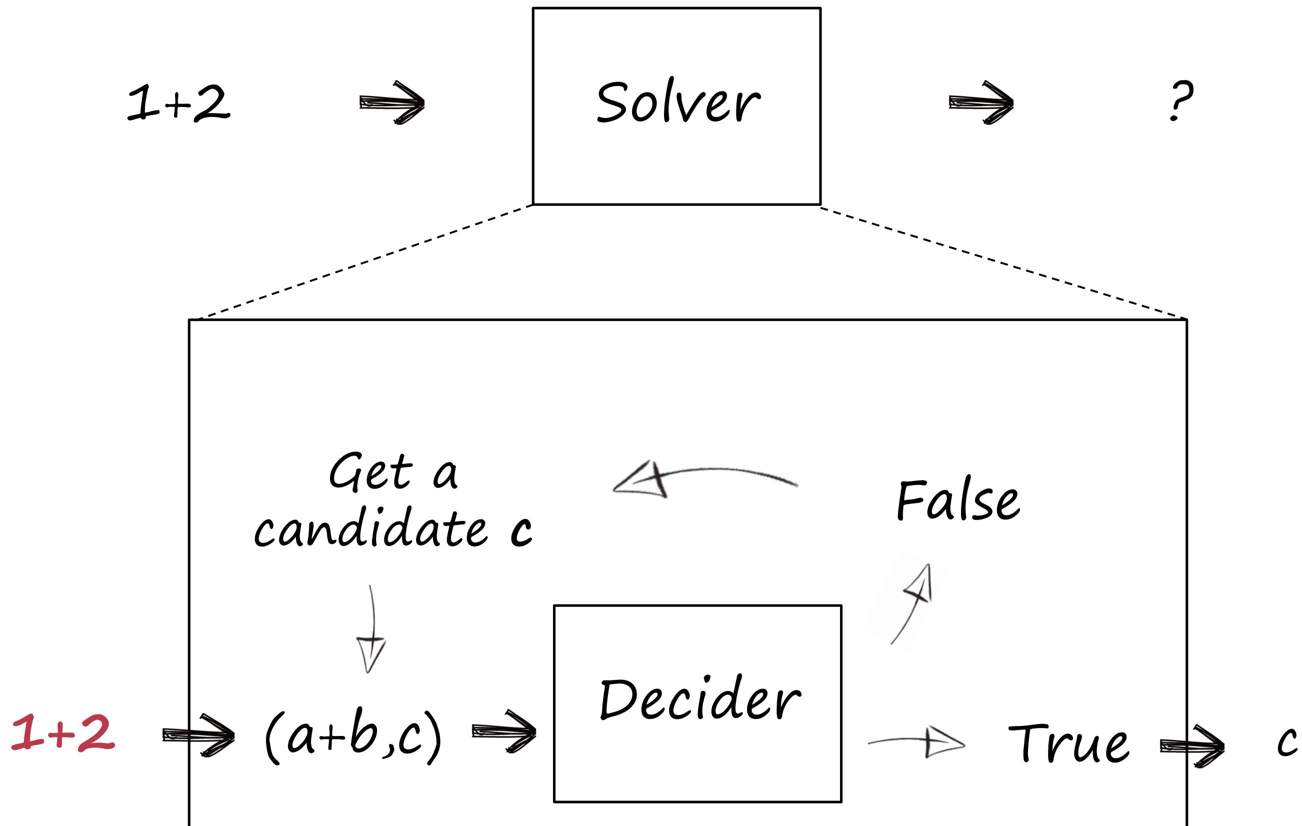

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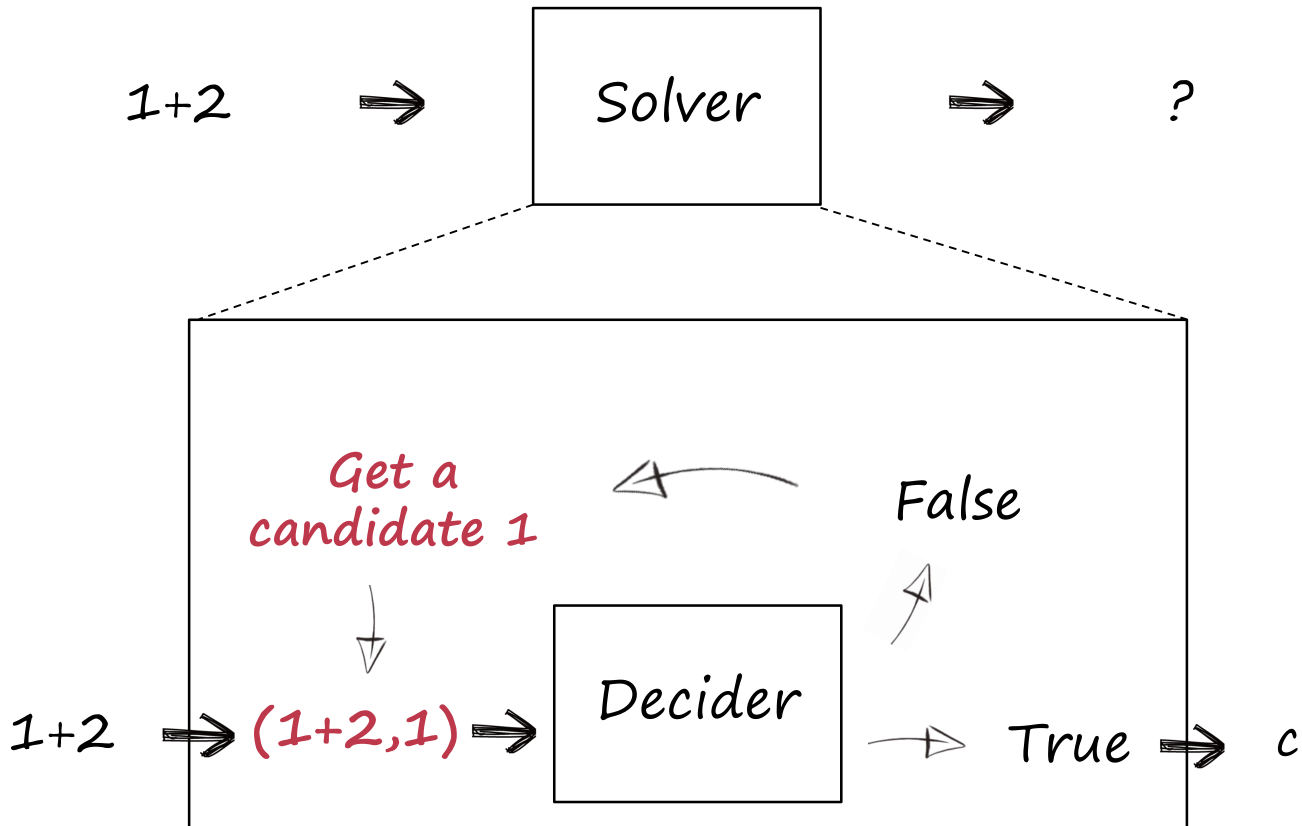

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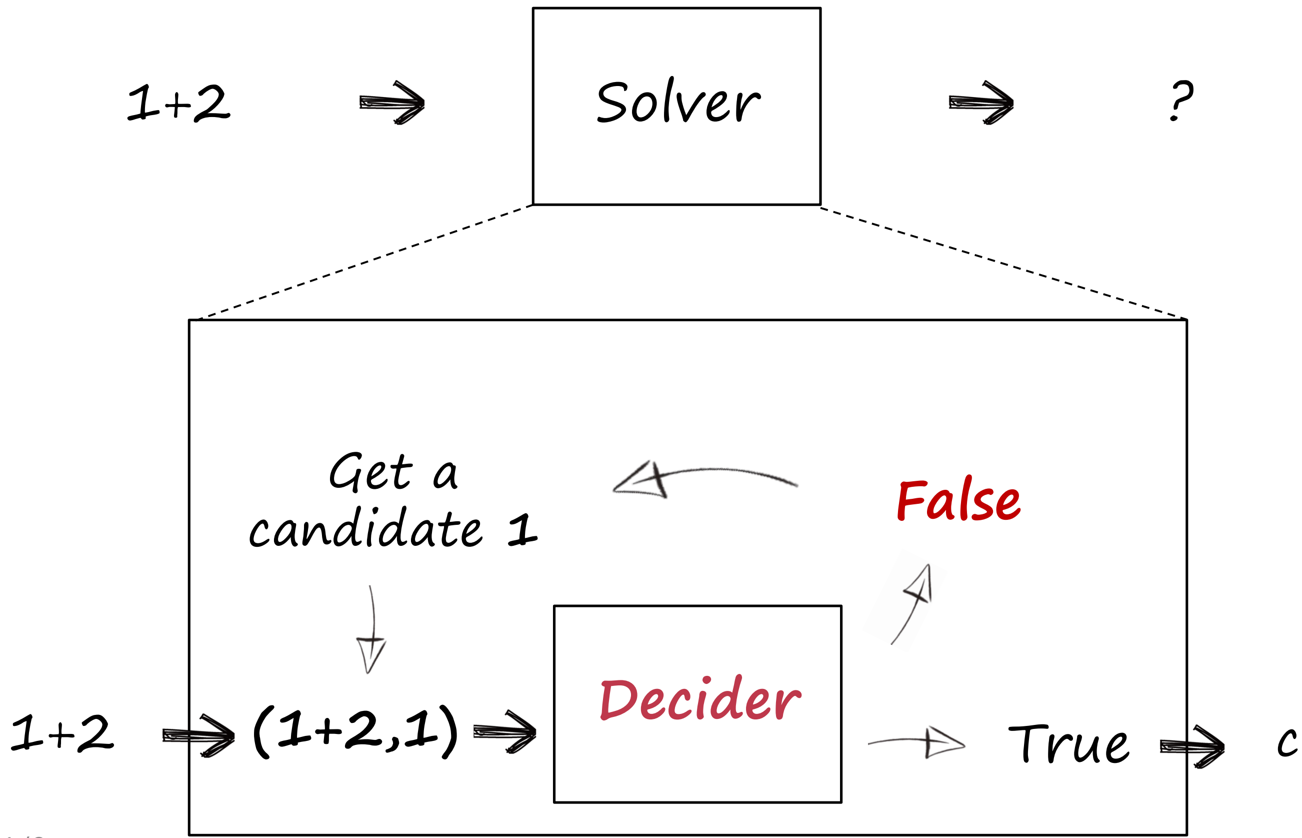

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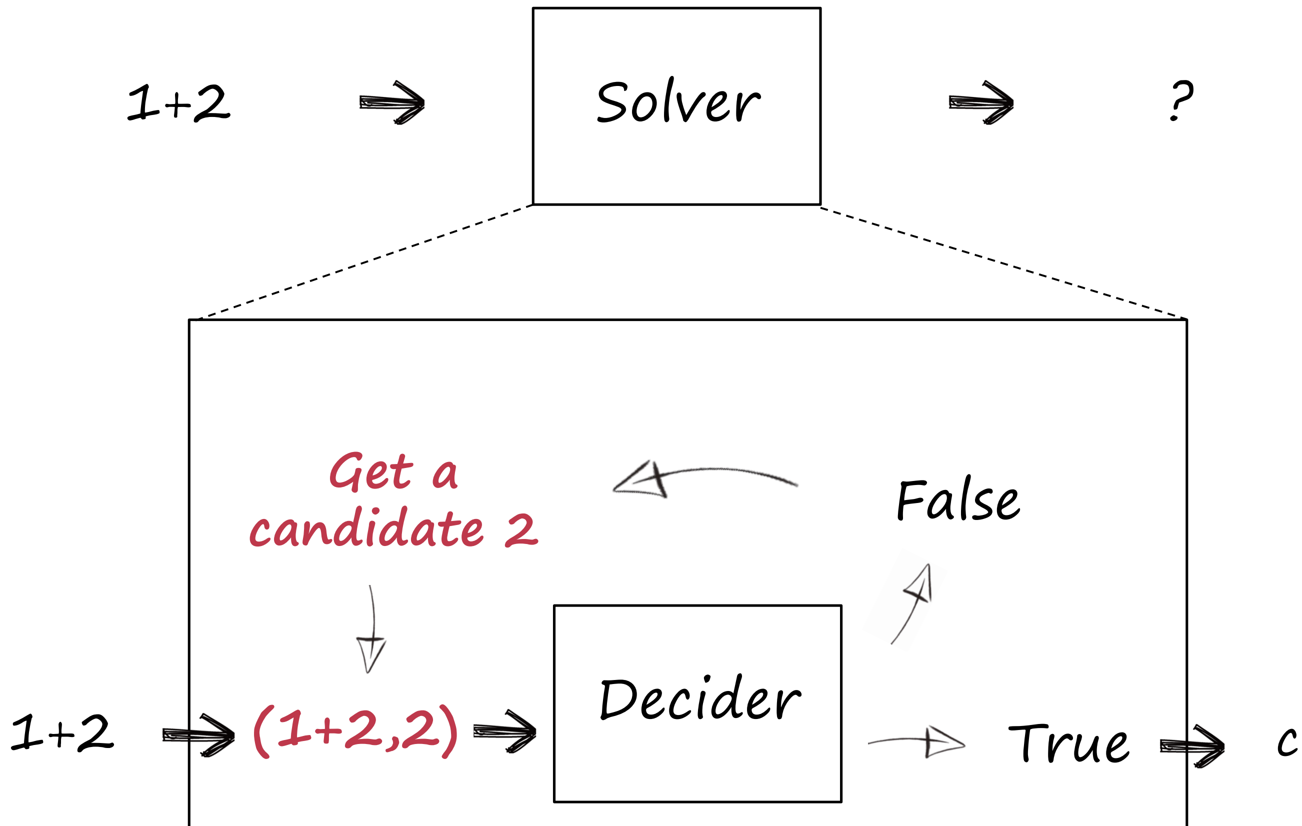
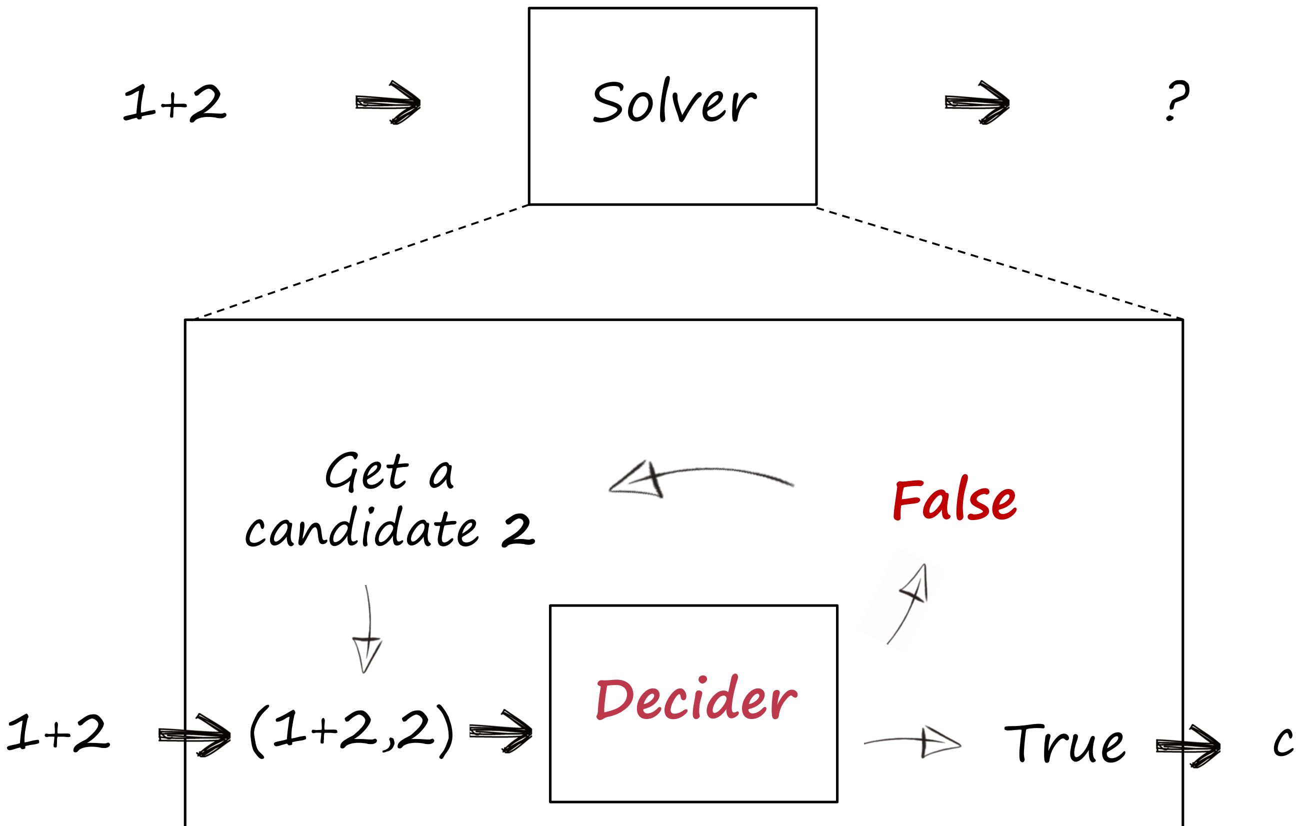
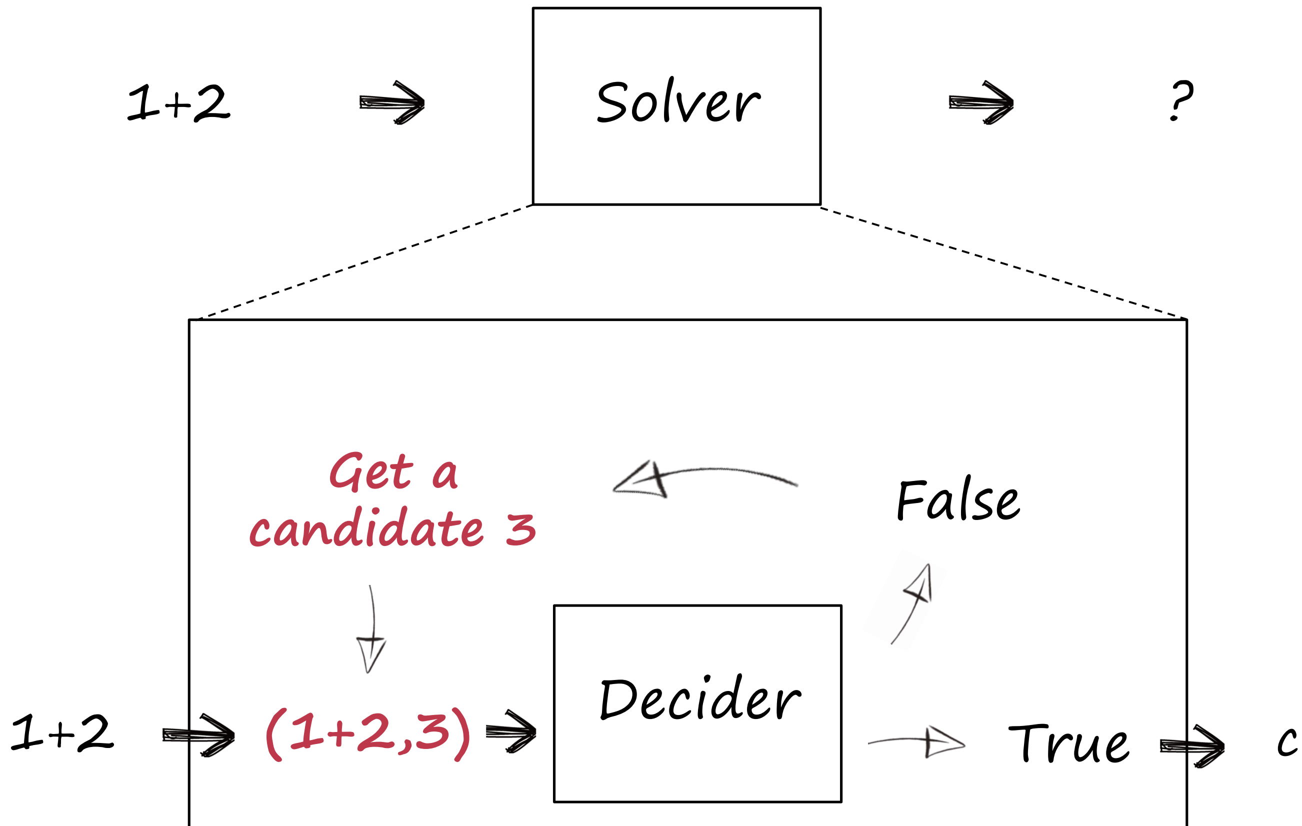


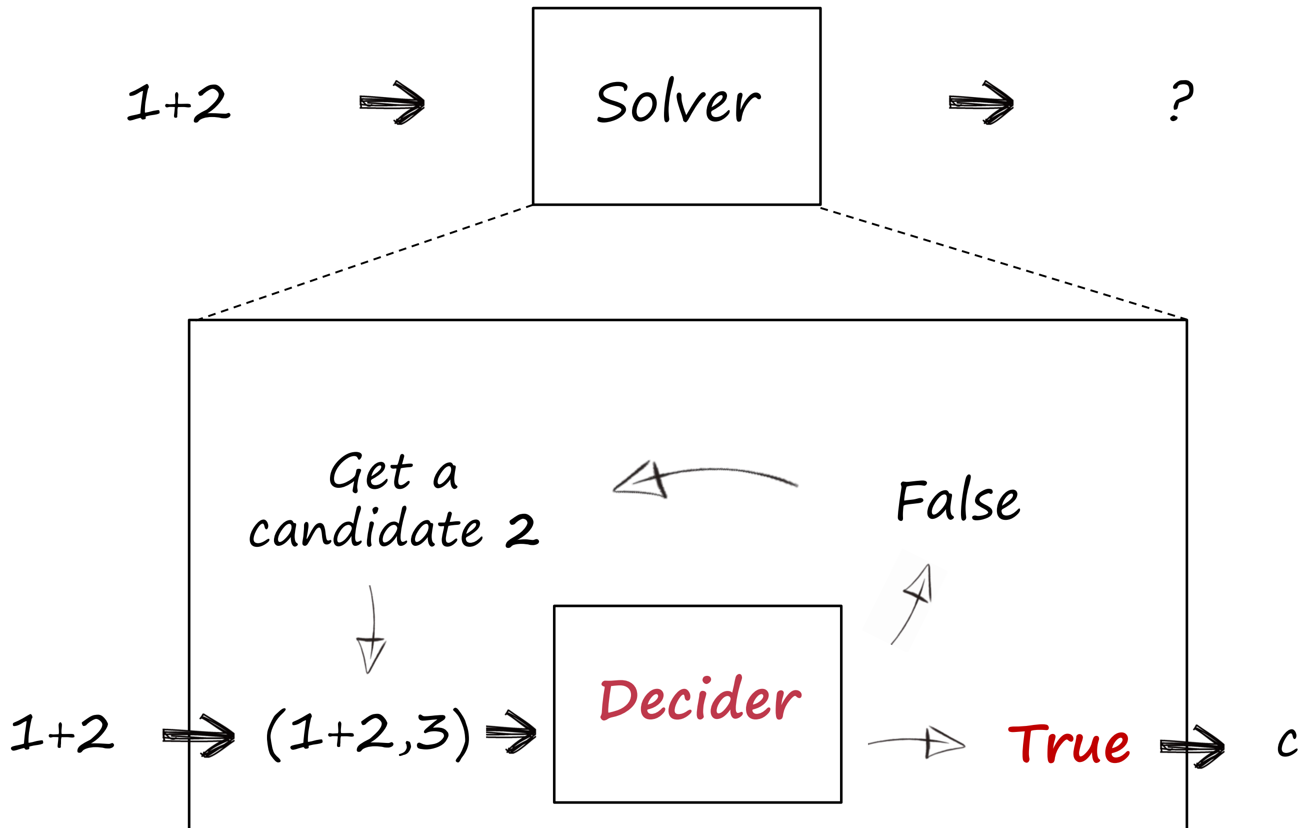
Solve Other Problems



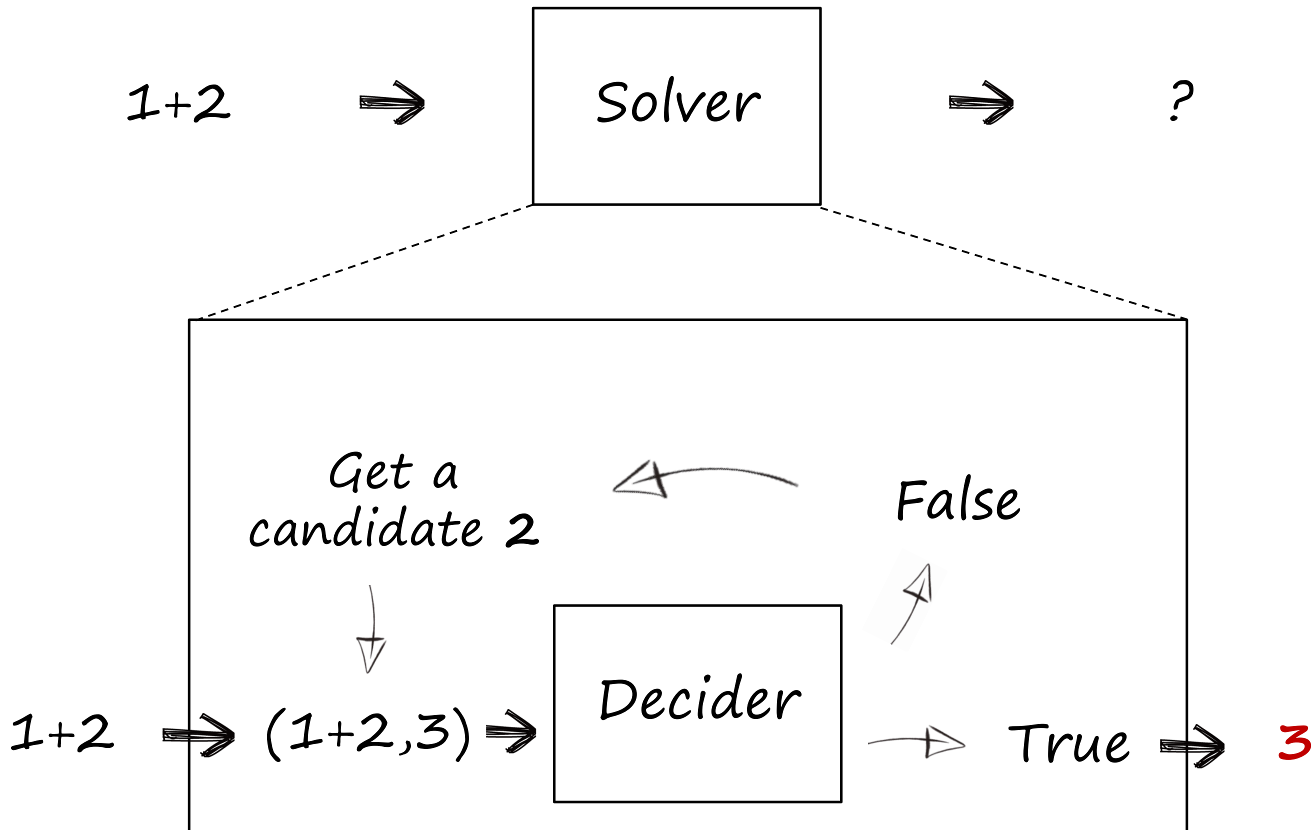
Solve Other Problems



Solve Other Problems

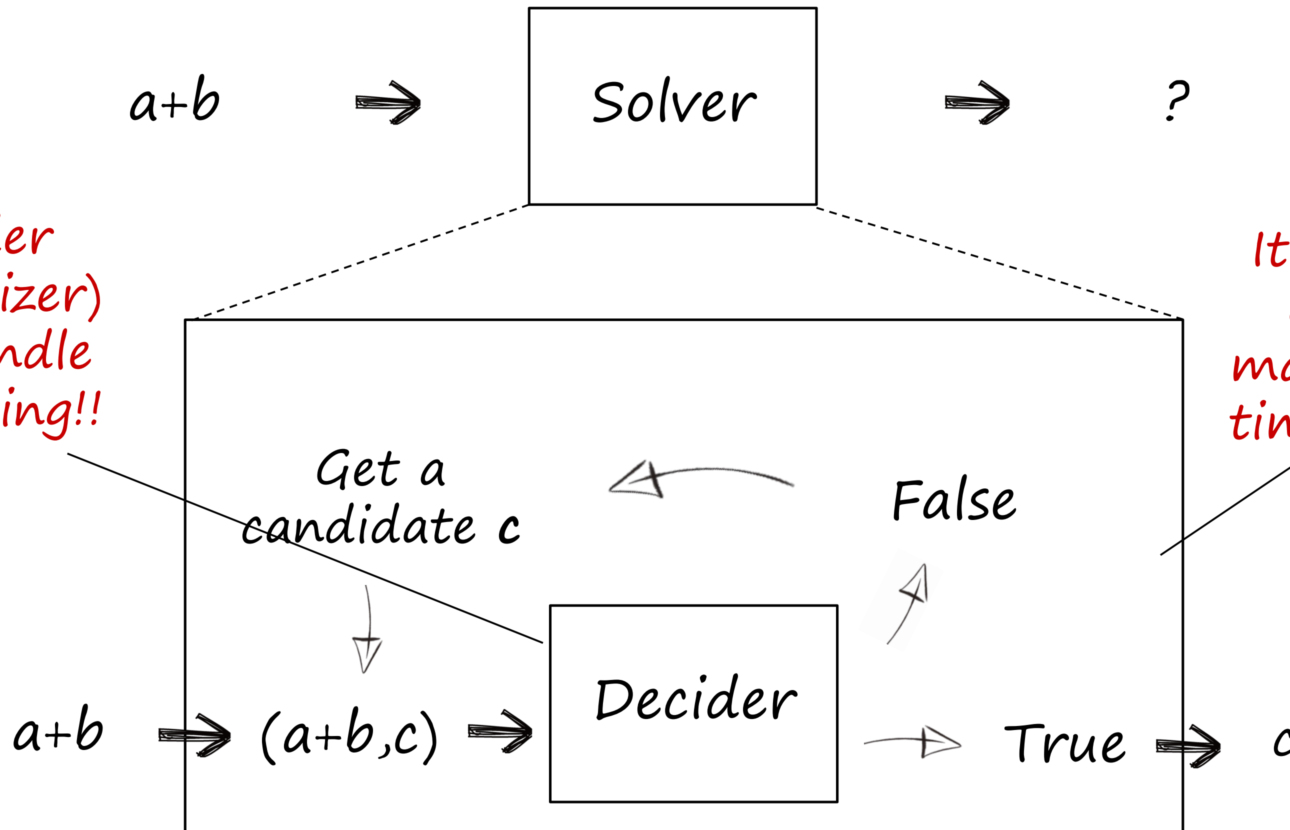


Solve Other Problems



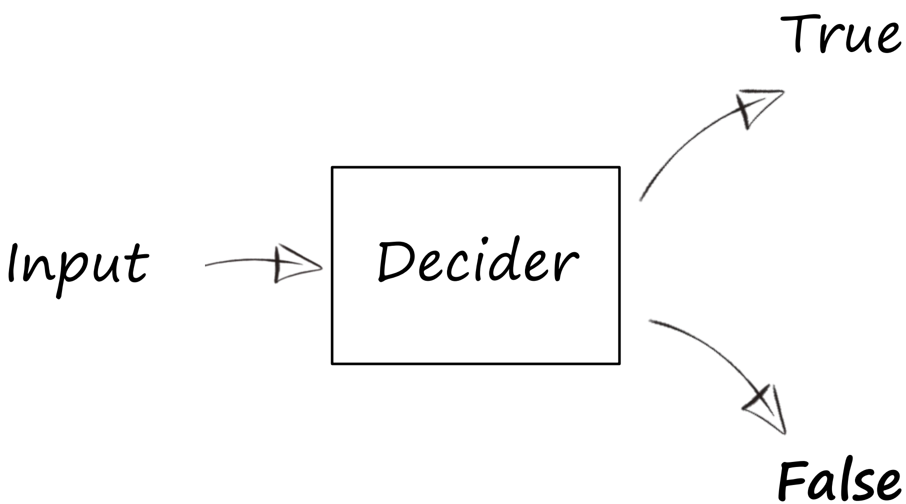
Solve Other Problems

Decider (recognizer) can handle everything!!



It just shows the capability, no matter how much time it consumes!!

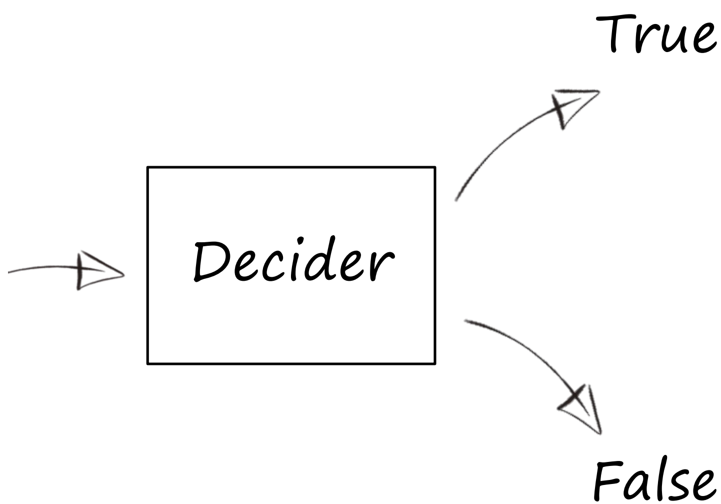
Solve Decision Problem



$\boldsymbol{a \wedge b}$

$\boldsymbol{a \wedge \neg a}$

Solve Decision Problem



$$\boldsymbol{a} \wedge \boldsymbol{b}$$
$$(\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\boldsymbol{a} \vee \boldsymbol{c})$$
$$(\boldsymbol{a} \vee \neg \boldsymbol{b}) \wedge (\boldsymbol{b} \vee \neg \boldsymbol{c}) \wedge (\boldsymbol{c} \vee \neg \boldsymbol{a})$$
$$\dots$$

$$\boldsymbol{a} \wedge \neg \boldsymbol{a}$$
$$(\boldsymbol{a} \vee \neg \boldsymbol{b}) \wedge (\neg \boldsymbol{a}) \wedge \boldsymbol{b}$$
$$\dots$$

Solve Decision Problem

Imaging that there is a god who knows all the Boolean formulas which are satisfiable, and he puts them all into a set

Set of all the Boolean formulas which are satisfiable, mark as S

$a \wedge b$

a

b

$(a \vee b) \wedge (a \vee c)$

$(a \vee \neg b) \wedge (b \vee \neg c) \wedge (c \vee \neg a)$

…

$a \wedge \neg a$

Solve Decision Problem

A Boolean formula t is satisfiable? $\longleftrightarrow$ Is t in S?

$a \wedge b$

a

b

$(a \vee b) \wedge (a \vee c)$

$(a \vee \neg b) \wedge (b \vee \neg c) \wedge (c \vee \neg a)$

$\ldots$

S

$a \wedge \neg a$

Solve Decision Problem

For any decision problem, we can define a predicate P:

$$P(x): (x\ is\ a\ valid\ input) \wedge (x\ satisfies\ the\ property\ the\ problem\ asks)$$

Then we can build the set:

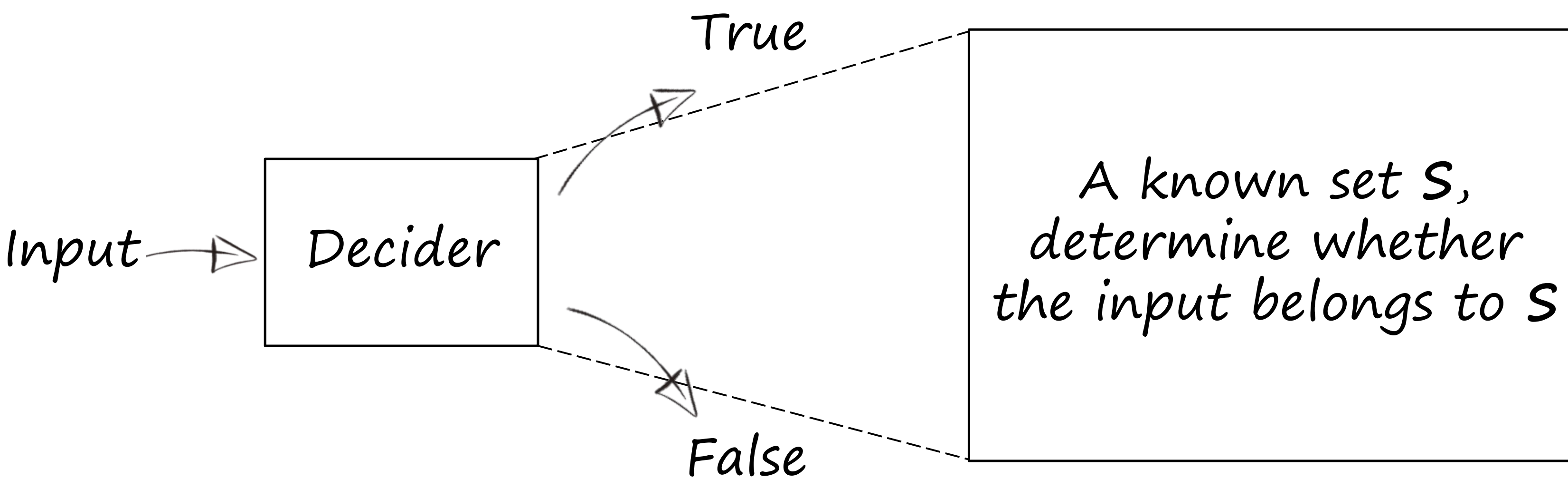
$$S = \{x | P(x)\}$$

Then the decision problem can be converted to:

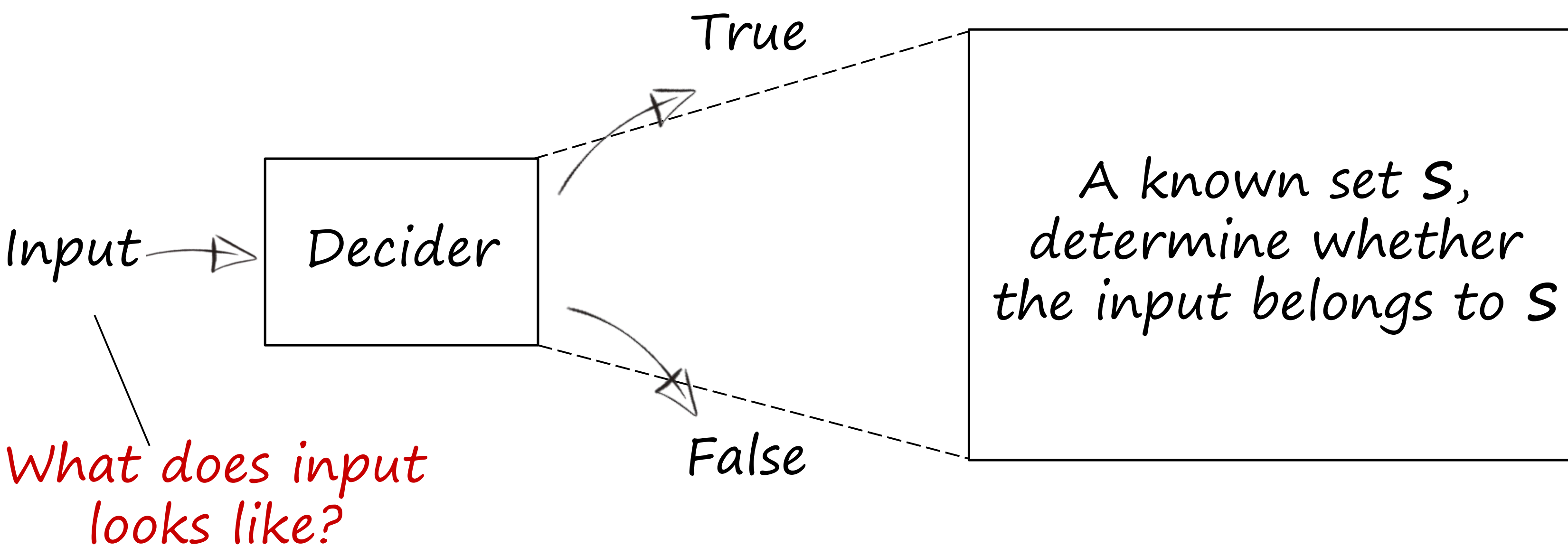
Take x as input and decide whether $x \in S$, that S is a known set.

This is called membership problem.

Solve Decision Problem



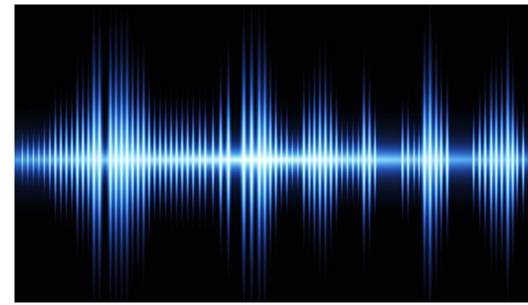
Solve Decision Problem



Input has large varieties

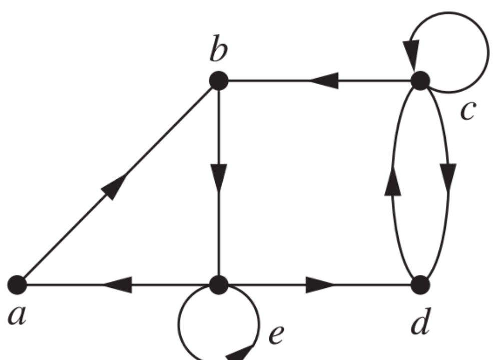
Structural Data

```
{
  "firstName": "Duke",
  "lastName": "Java",
  "streetAddress": "100 Internet Dr",
  "phoneNumbers": [
     { "Mobile": "111-111-1111" },
     { "Home": "222-222-2222" }
  ]
}
```



Sound

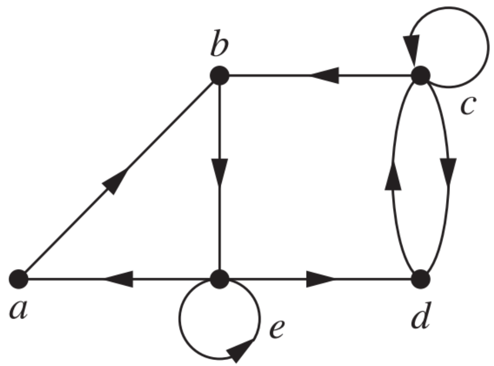
Graph



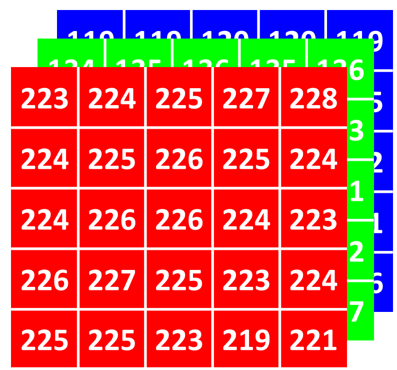
Image

But every input can be encoded as a string

Encode to string

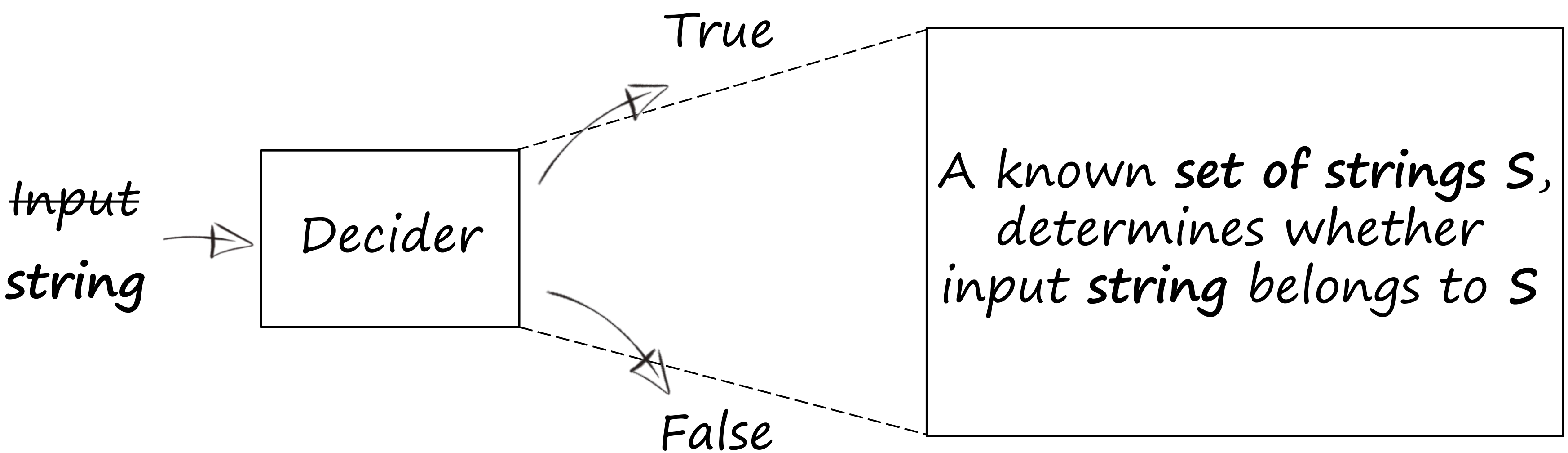


(a|b),(b|e),(c|b,c,d),(d|c),(e|a,d,e)



(5×5),R:223, 224, 225, 227, 228, 224, 225, 226, 225, 224, 224, 226, 226, 224, 223, 226, 227, 225, 223, 224, 225, 225, 223, 219, 221,G:134, 135, 136, 135, 136, 132, 134, 135, 134, 133, 132, 134, 135, 133, 131, 133, 134, 132, 130, 132, 132, 131, 129, 125, 127, B:119, 119, 120, 120, 119, 116, 116, 117, 116, 115, 114, 116, 117, 115, 112, 115, 116, 113, 111, 111, 113, 113, 108, 104, 106

Solve Decision Problem



Formal Language Theory

*Alphabets are **finite, non-empty** sets of characters*

Alphabet: Σ

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ …
$\wedge \vee \neg$ ( )

Characters are individual symbols

finite sequence

A string

$(\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\neg \boldsymbol{b} \vee \boldsymbol{c})$

*Strings are **finite** sequence of characters*

All strings

$\vee\wedge\, \boldsymbol{abc}$
$\boldsymbol{a} \wedge \neg \boldsymbol{a}$
$(\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\neg \boldsymbol{b} \vee \boldsymbol{c})$
…

Set of all string : Σ^*

subset of Σ^*

$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$
$(\boldsymbol{a} \vee \boldsymbol{b}) \wedge (\neg \boldsymbol{b} \vee \boldsymbol{c})$
…

A language

Languages are sets of strings.

Language Recognizing

Throughout the whole course of computability, we'll focus on the membership problem of a language, or language recognizing, that is:

Given a language L and take w as input, determine whether $w \in L$.

~~What problems can a computer solve?~~

What languages can a computer recognize?

Next

The intuitive way to construct a set may lead to paradox.

- Russell's Paradox & ZF Axiom System

What's the relationship between the number of problems and the number of programs?

- Cardinality of a Set and Cantor's Theorem