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Steady laminar incompressible flow between adjacent discs:

- *Continuity:*

$$\frac{1}{\mathrm{r}}\frac{\partial(\mathrm{rv_r})}{\partial\,\mathrm{r}}+\frac{1}{\mathrm{r}}\frac{\partial\,\mathrm{v}_\theta}{\partial\theta}+\frac{\partial\,\mathrm{v_z}}{\partial\,\mathrm{z}}=0$$

- *Radial direction momentum:*
- $v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}+v_z\frac{\partial v_r}{\partial z}-\frac{{v_\theta}^2}{r}=-\frac{1}{\rho}\left(\frac{\partial P}{\partial r}\right)+\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right\}+f_r$
- *Theta direction momentum:*

$$v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_\theta v_r}{r}$$
$$=-\frac{1}{\rho r}\left(\frac{\partial P}{\partial\theta}\right)+\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{v_\theta}{r^2}-\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right\}+f_\theta$$

- *Z direction momentum:*

$$v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho}\left(\frac{\partial P}{\partial z}\right)+\nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right\}+f_z$$

Assumptions:

- *Flow is taken to be two-dimensional: $v_z=0$, v_r and v_θ are taken to be constant across the channel between rotor plates, and hence, are mean flow velocities at each r and θ.*
- *The flow field with v_r and v_θ velocity components is treated as begin inviscid, with a body-force representation of the wall shear effects. The viscous drag exerted on the flow by the side walls of the channel between the rotos is modelled as a body force acting on the flow at each (r, θ) location.*
- *The flow field is radially symmetric. The inlet flow at the rotor outer edge is assumed uniform, resulting in a flow field that is the same at any angle θ. All θ derivatives of the flow quantities are therefore zero.*
- *Radial velocity derivatives and divisions in radial and theta directions are negligible as compared to Axial case.*
- *Constant friction factor characteristics within the flow passage.*
- *Axial direction effects must not be ignored.*

Simplified version:

- *Continuity:*

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}=0$$

- *Radial direction momentum:*

$$v_r\frac{\partial v_r}{\partial r}-\frac{{v_\theta}^2}{r}=-\frac{1}{\rho}\left(\frac{\partial P}{\partial r}\right)+\nu\left\{\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}\right\}$$

- *Theta direction momentum:*

$$v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta v_r}{r}=f_\theta$$

- *Z direction momentum:*

$$-\frac{1}{\rho}\left(\frac{\partial P}{\partial z}\right)\;=\;0$$

Velocity definition:

$$v_r(r,z) = \bar{v}_r(r)\phi(z)$$

$$v_\theta(r,z) = \bar{v}_\theta(r)\phi(z) + \mathrm{U}(r)$$

Where:

$$\phi(z) = \left(\frac{n+1}{n}\right)\left[1 - \left(\frac{2z}{b}\right)^n\right]$$

$$\bar{v}_r(r) = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} v_r(r,z)dz$$

$$\bar{v}_\theta(r) = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} v_\theta(r,z)dz$$

$$\mathrm{U}(r) = \frac{r}{r_0} U_0$$

Integral simplifications for phi:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \phi(z)dz = 2\int_0^{\frac{b}{2}} \phi(z)dz = \mathrm{b}$$

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \phi(z)^2 dz = 2\int_0^{\frac{b}{2}} \phi(z)^2 dz = \frac{2(n+1)}{2n+1}\mathrm{b}$$

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \frac{d^2\phi(z)}{dz^2} dz = 2\int_0^{\frac{b}{2}} \frac{d^2\phi(z)}{dz^2} dz = -\frac{4(n+1)}{b}$$

Equation 5 yields that $\mathrm{r}v_r = C_r$*, and integrating it over the channel:*

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} r v_r dz = \int_{-\frac{b}{2}}^{\frac{b}{2}} r\bar{v}_r(r)\phi(z)dz = \mathrm{r}\bar{v}_r(r)\mathrm{b} = \mathrm{b}C_r$$

mass conservation requires that $-2\pi r_0\rho\int_{-b/2}^{b/2} v_r dz = \dot{m}_c$*, where* $\dot{m}_c$ *is the mass flow rate per channel:*

$$-2\pi r_0\rho\int_{-b/2}^{b/2} v_r dz = -2\pi r_0\rho\bar{v}_r(r_0)\int_{-b/2}^{b/2} \phi(z)dz = -2\pi r_0\rho\bar{v}_r(r_0)\mathrm{b} = \dot{m}_c$$

Note: $\bar{v}_r(r_0) = \bar{v}_{r0}$

$$\bar{v}_{r0} = \frac{\dot{m}_c}{2\pi r_0 b\rho}$$

Hence:

$$\bar{v}_r = -\frac{r_0}{r}\bar{v}_{r0}$$

Theta direction wall friction force for fluid element in channel between rotors with volume V_e is given by:

$$F_\theta = \tau_w A_w = \frac{4\tau_w V_e}{D_H}$$

where D_H is the hydraulic diameter of the channel ($D_H = 4V_e/A_w$).For parallel plates, $D_H = 2\mathrm{b}$. For Newtonian fluid, it follows that:

$$\tau_w = \mathrm{f}\frac{\rho\bar{v}_\theta^2}{2}$$

$$\mathrm{f} = \frac{\tau_w}{\rho\bar{v}_\theta^2/2} = \frac{2\mu\left[\frac{\partial(v_\theta(r,z) - U)}{\partial \mathrm{z}}\right]_{z=\frac{b}{2}}}{\rho\bar{v}_\theta^2}$$

For the purpose of this analysis, tangential shear interaction o the flow within the disk surface is postulated

$$\mathrm{f} = \frac{Po}{\mathrm{Re}_c}$$

Where:

$$\mathrm{Re}_c = \frac{\rho\bar{v}_\theta D_H}{\mu}$$

For laminar flow, $\mathrm{Po} = 24$. For flow with roughened surface of turbulence action, an enhancement number F_{Po} is defined as such:

$$F_{Po} = \frac{Po}{24}$$

Relationship between n, Fpo and Po

$$\mathrm{f} = \frac{2\mu\left[\frac{\partial(v_\theta(r,z) - U)}{\partial z}\right]_{z=\frac{b}{2}}}{\rho\bar{v}_\theta^2}$$

$$= \frac{2\mu\bar{v}_\theta}{\rho\bar{v}_\theta^2}\left[\frac{\partial\phi(z)}{\partial z}\right]_{z=\frac{b}{2}}$$

$$= \frac{2\mu}{\rho\bar{v}_\theta}\left[-\frac{n+1}{n}\left(\frac{2}{b}\right)\left(n\left(\frac{2z}{b}\right)^{n-1}\right)\right]_{z=\frac{b}{2}}$$

$$= \frac{2\mu}{\rho\bar{v}_\theta}\left[-\frac{2(n+1)}{b}\right]$$

Combining with the remaining two equations (absolute terms):

$$\frac{Po}{\mathrm{Re}_c} = \frac{2\mu}{\rho\bar{v}_\theta}\left[\frac{2(n+1)}{b}\right]$$

$$\frac{\mu Po}{\rho\bar{v}_\theta D_H} = \frac{2\mu}{\rho\bar{v}_\theta}\left[\frac{2(n+1)}{b}\right]$$

$$\mathrm{Po} = D_H\left[\frac{4(n+1)}{b}\right]$$

$$\mathrm{Po} = 8(n+1)$$

$$\mathrm{Po}/8 = (n+1) = 3F_{Po}$$

Wall Shear Stress

$$\tau_w = \mathrm{f}\frac{\rho\bar{v}_\theta^2}{2}$$

$$\tau_w = \frac{2\mu}{\rho\bar{v}_\theta}\left[-\frac{2(n+1)}{b}\right]\frac{\rho\bar{v}_\theta^2}{2}$$

$$\tau_w = -\frac{2\mu(n+1)\bar{v}_\theta}{b}$$

Wall Friction Term

$$f_\theta = \frac{F_\theta}{\rho V_e} = \frac{4\tau_w}{\rho D_H} = -\frac{4}{\rho D_H}\frac{2\mu(n+1)\bar{v}_\theta}{b} = -\frac{16\mu(n+1)\bar{v}_\theta}{\rho D_H^2}$$

Dimensionless Variables:

$$\xi = \frac{r}{r_0}$$

$$\widehat{W} = \frac{\bar{v}_\theta}{U_0}$$

$$\hat{P} = \frac{P - P_0}{\rho U_0^2/2}$$

$$V_{r0} = \frac{\bar{v}_{r0}}{U_0}$$

$$\varepsilon = 2\mathrm{b}/r_0$$

$$\mathrm{Re}_m^* = \frac{D_H}{r_0}\frac{\dot{m}_c D_H}{2\pi r_0 b\mu} = \frac{\dot{m}_c D_H}{{r_0}^2\mu} = 4V_{r0}\frac{b^2 U_0}{r_0 \nu}$$

Note

From the above definitions:

$$v_r = -\frac{\nu\phi(z)\mathrm{Re}_m^*}{D_H\xi\varepsilon} = -\frac{r_0}{r}\phi(z)\bar{v}_{r0} = -\frac{r_0}{r}\phi(z)V_{r0}U_0$$

$$\frac{\partial}{\partial \mathrm{r}} = \frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial \mathrm{r}} = \frac{1}{r_0}\frac{\partial}{\partial\xi}$$

Tangential ODE

$$v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta v_r}{r} = -\frac{16\mu(n+1)\bar{v}_\theta}{\rho D_H^2}$$

$$\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = -\frac{1}{v_r}\frac{16\mu(n+1)\bar{v}_\theta}{\rho D_H^2}$$

$$\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = \frac{D_H\xi\varepsilon}{\nu\phi(z)\mathrm{Re}_m^*}\frac{16\mu(n+1)\bar{v}_\theta}{\rho D_H^2}$$

$$\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} = \frac{\xi\varepsilon}{\phi(z)\mathrm{Re}_m^*}\frac{16(n+1)\bar{v}_\theta}{D_H}$$

$$\phi(z)\left[\frac{\partial\bar{v}_\theta}{\partial \mathrm{r}} + \frac{\bar{v}_\theta}{r}\right] + \frac{2U_0}{r_0} = \frac{\xi\varepsilon}{\phi(z)\mathrm{Re}_m^*}\frac{16(n+1)\bar{v}_\theta}{D_H}$$

$$\phi(z)^2\left[\frac{\partial\bar{v}_\theta}{\partial \mathrm{r}} + \frac{\bar{v}_\theta}{r}\right] + \phi(z)\frac{2U_0}{r_0} = \frac{\xi\varepsilon}{\mathrm{Re}_m^*}\frac{16(n+1)\bar{v}_\theta}{D_H}$$

Integrating both side across the channel $\left(\frac{1}{b}\int_{-b/2}^{b/2} dz\right)$*:*

$$\frac{2(n+1)}{2n+1}\left[\frac{\partial\bar{v}_\theta}{\partial \mathrm{r}} + \frac{\bar{v}_\theta}{r}\right] + \frac{2U_0}{r_0} = \frac{\xi\varepsilon}{\mathrm{Re}_m^*}\frac{16(n+1)\bar{v}_\theta}{D_H}$$

$$\frac{2(n+1)}{2n+1}U_0\left[\frac{\partial\widehat{W}}{\partial \mathrm{r}} + \frac{\widehat{W}}{r}\right] + \frac{2U_0}{r_0} = \frac{U_0\varepsilon}{\mathrm{Re}_m^*}\frac{16\xi(n+1)\widehat{W}}{D_H}$$

$$\frac{2(n+1)}{2n+1}\left(\frac{U_0}{r_0}\right)\left[\frac{\partial\widehat{W}}{\partial\xi}+\frac{\widehat{W}}{\xi}\right]+2\left(\frac{U_0}{r_0}\right)=\left(\frac{U_0}{r_0}\right)\frac{16\xi(n+1)\widehat{W}}{\mathrm{Re}_m^*}$$

Divide both side by $\left(\frac{U_0}{r_0}\right)$:

$$\frac{2(n+1)}{2n+1}\left[\frac{\partial\widehat{W}}{\partial\xi}+\frac{\widehat{W}}{\xi}\right]=\frac{16\xi(n+1)\widehat{W}}{\mathrm{Re}_m^*}-2$$

$$\left[\frac{\partial\widehat{W}}{\partial\xi}+\frac{\widehat{W}}{\xi}\right]=\frac{2n+1}{n+1}\left(\frac{8\xi(n+1)\widehat{W}}{\mathrm{Re}_m^*}\right)-\frac{2n+1}{n+1}$$

$$\left[\frac{\partial\widehat{W}}{\partial\xi}+\frac{\widehat{W}}{\xi}\right]=\frac{8(2n+1)\xi\widehat{W}}{\mathrm{Re}_m^*}-\frac{2n+1}{n+1}$$

$$-\frac{2n+1}{n+1}=\left[\frac{1}{\xi}-\frac{8(2n+1)\xi}{\mathrm{Re}_m^*}\right]\widehat{W}+\frac{\partial\widehat{W}}{\partial\xi}$$

Radial ODE

$$v_r\frac{\partial v_r}{\partial r}-\frac{{v_\theta}^2}{r}=-\frac{1}{\rho}\left(\frac{\partial P}{\partial r}\right)+\nu\left\{\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}\right\}$$

$$\frac{(r_0\phi(z)V_{r0}U_0)^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)-\left[\frac{\left(\bar{v}_\theta(r)\phi(z)+U(r)\right)^2}{r}\right]$$
$$=-\frac{1}{\rho}\left(\frac{\partial P}{\partial r}\right)+\nu\left[\frac{\partial^2}{\partial z^2}\left(-\frac{r_0}{r}\phi(z)V_{r0}U_0\right)-(r_0\phi(z)V_{r0}U_0)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\left(\frac{1}{r}\right)}{\partial r}\right)-\frac{1}{r^3}\right)\right]$$

$$-\frac{(r_0\phi(z)V_{r0}U_0)^2}{r^3}-\left[\frac{\left(\widehat{W}U_0\phi(z)+U(r)\right)^2}{r}\right]$$
$$=-\frac{1}{\rho}\left(\frac{\text{\textbackslash partialP}}{\text{\textbackslash partialr}}\right)+\nu\left[-\left(\frac{r_0V_{r0}U_0}{r}\right)\frac{\partial^2\phi(z)}{\partial z^2}-(r_0\phi(z)V_{r0}U_0)\left(\frac{1}{r}\frac{\partial}{\partial r}\left(-\frac{1}{r}\right)-\frac{1}{r^3}\right)\right]$$

$$-\frac{(r_0\phi(z)V_{r0}U_0)^2}{r^3}-\left[\frac{\left(\widehat{W}U_0\phi(z)\right)^2+2\widehat{W}U_0\mathrm{U}(r)\phi(z)+\mathrm{U}(r)^2}{r}\right]$$
$$=-\frac{1}{\rho}\left(\frac{\partial\mathrm{P}}{\partial\mathrm{r}}\right)+\nu\left[-\left(\frac{r_0V_{r0}U_0}{r}\right)\frac{\partial^2\phi(z)}{\partial z^2}+0\right]$$

Integrating both side across the channel $\left(\frac{1}{b}\int_{-b/2}^{b/2}dz\right)$:

$$-\frac{2(n+1)}{2n+1}\frac{(r_0V_{r0}U_0)^2}{r^3}-\left[\frac{2(n+1)}{2n+1}\frac{\left(\widehat{W}U_0\right)^2}{r}+\frac{2\widehat{W}U_0\mathrm{U}(r)}{r}+\frac{\mathrm{U}(r)^2}{r}\right]$$
$$=-\frac{1}{\rho}\left(\frac{\partial\mathrm{P}}{\partial\mathrm{r}}\right)+\nu\left(\frac{r_0V_{r0}U_0}{r}\right)\left(\frac{4(n+1)}{b^2}\right)$$

$$-\frac{2(n+1)}{2n+1}\frac{(V_{r0}U_0)^2}{r_0}\left(\frac{1}{\xi^3}\right)-\left[\frac{2(n+1)}{2n+1}\frac{\left(\widehat{W}U_0\right)^2}{r_0}+\frac{2\widehat{W}U_0\mathrm{U}(r)}{r_0}+\frac{\mathrm{U}(r)^2}{r_0}\right]\frac{1}{\xi}$$
$$=-\frac{1}{r_0\rho}\left(\frac{\partial \mathrm{P}}{\partial \xi}\right)+\nu\left(\frac{V_{r0}U_0}{\xi}\right)\left(\frac{4(n+1)}{b^2}\right)$$

$$\frac{1}{\rho}\left(\frac{\partial \mathrm{P}}{\partial \xi}\right)=\nu\left(\frac{r_0V_{r0}U_0}{\xi}\right)\left(\frac{4(n+1)}{b^2}\right)+\frac{2(n+1)}{2n+1}\left[\frac{(V_{r0}U_0)^2}{\xi^3}\right]$$
$$+\left[\frac{2(n+1)\left(\widehat{W}U_0\right)^2}{2n+1}+2\widehat{W}U_0\mathrm{U}(r)+\mathrm{U}(r)^2\right]\frac{1}{\xi}$$

$$\frac{1}{\rho}\left(\frac{\partial \mathrm{P}}{\partial \xi}\right)=\nu\left(\frac{r_0V_{r0}U_0}{\xi}\right)\left(\frac{4(n+1)}{b^2}\right)+\frac{2(n+1)}{2n+1}\left[\frac{(V_{r0}U_0)^2}{\xi^3}\right]$$
$$+\left[\frac{2(n+1)\left(\widehat{W}U_0\right)^2}{2n+1}+2\widehat{W}U_0^2\xi+{U_0}^2\xi^2\right]\frac{1}{\xi}$$

$$\frac{{U_0}^2}{2}\left(\frac{\partial \widehat{P}}{\partial \xi}\right)=\frac{2(n+1)}{2n+1}\left[\frac{(V_{r0}U_0)^2}{\xi^3}+\frac{\left(\widehat{W}U_0\right)^2}{\xi}\right]+2\widehat{W}U_0^2+{U_0}^2\xi+\nu\left(\frac{r_0V_{r0}U_0}{\xi}\right)\left(\frac{4(n+1)}{b^2}\right)$$

$$\frac{\partial \widehat{P}}{\partial \xi}=\frac{4(n+1)}{(2n+1)\xi^3}\left[{V_{r0}}^2+\xi^2\widehat{W}^2\right]+4\widehat{W}+2\xi+\nu\left(\frac{r_0V_{r0}}{\xi U_0}\right)\left(\frac{8(n+1)}{b^2}\right)$$

$$\frac{\partial \widehat{P}}{\partial \xi}=\frac{4(n+1)}{(2n+1)\xi^3}\left[{V_{r0}}^2+\xi^2\widehat{W}^2\right]+4\widehat{W}+2\xi+\left(\frac{1}{4V_{r0}}\right)\left(\frac{\nu r_0}{b^2U_0}\right)\left(\frac{32(n+1)V_{r0}^2}{\xi}\right)$$

$$\frac{\partial \widehat{P}}{\partial \xi}=\frac{4(n+1)}{(2n+1)\xi^3}\left[{V_{r0}}^2+\xi^2\widehat{W}^2\right]+4\widehat{W}+2\xi+\left(\frac{32(n+1)V_{r0}^2}{\mathrm{Re}_m^*\xi}\right)$$

Power Generated Equation

From the definition of torque per rotor disc surface:

$$\mathrm{T}\;=\;2\pi\int_{r_i}^{r_0}\tau_w r^2dr$$

$$\mathrm{T}\;=\;2\pi\int_{r_i}^{r_0}\left(\frac{2\mu(n+1)\bar{v}_\theta}{b}\right)r^2dr$$

$$\mathrm{T}\;=\;2\pi\int_{\xi_i}^{\xi_0=1}\left(\frac{2\mu(n+1)\bar{v}_\theta}{b}\right)(r_0\xi)^2r_0d\xi$$

$$\mathrm{T}\;=\;\left(\frac{4\pi\mu r_0^3(n+1)}{b}\right)\int_{\xi_i}^{1}\bar{v}_\theta\,\xi^2\mathrm{d}\xi$$

$$\mathrm{T}\;=\;\left(\frac{4\pi U_0\mu r_0^3(n+1)}{b}\right)\int_{\xi_i}^{1}\widehat{W}\xi^2d\xi$$

Total Torque:

$$\mathrm{T}_{total} = 2(n_{disc} - 1)\mathrm{T}$$

Total Power:

$$\mathrm{P}_{total} = \omega T_{total}$$

Mechanical Efficiency

$$\eta_{rm} = \frac{v_{\theta 0}U_0 - v_{\theta i}U_i}{v_{\theta 0}U_0} = 1 - \frac{v_{\theta i}U_i}{v_{\theta 0}U_0} = 1 - \frac{(\bar{v}_{\theta i} + U_i)U_i}{(\bar{v}_{\theta 0} + U_0)U_0}$$

Rearranging:

$$\eta_{rm} = 1 - \frac{\left(\widehat{W}_i + \frac{U_i}{U_0}\right)\frac{U_i}{U_0}}{\left(\widehat{W}_0 + 1\right)} = 1 - \frac{\left(\widehat{W}_i + \frac{r_i}{r_0}\right)\frac{r_i}{r_0}}{\left(\widehat{W}_0 + 1\right)} = 1 - \frac{\left(\widehat{W}_i + \xi_i\right)\xi_i}{\left(\widehat{W}_0 + 1\right)}$$

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