1] schmidt numbers

$|0\rangle\otimes|0\rangle + |0\rangle\otimes|1\rangle + |1\rangle\otimes|1\rangle = |0\rangle\otimes(|0\rangle+|1\rangle) + |1\rangle\otimes|1\rangle$

density matrix

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ dim $= 2$
schmidt $= 2$

$|0\rangle\otimes|0\rangle - |0\rangle\otimes|1\rangle - |1\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle$

density matrix:

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ dim $= 1 =$ schmidt number

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2] H_1, H_2 are f.d Hilbert

$L(H)_{sa}$ means $A \in L(H)$, $A = A^*$, $\otimes_{\mathbb{R}}$ means we have a real vecter space (no complex values). Wan to show a linear isomorphism between $L(H_1)_{sa} \otimes_{\mathbb{R}} L(H_2)$ and $L(H_1 \otimes H_2)_{sa}$ that is we have a function T taking inputs from $L(H_1)_{sa} \otimes_{\mathbb{R}} L(H_2)_{sa}$ and an output in $L(H_1 \otimes H_2)_{sa}$ this function should be 1-1 and linear. (preserve structure)

Let A be a real matrix in $L(H_1)_{sa}$
B —— " —— $L(H_2)_{sa}$

$A \otimes B$ then takes a vector in $H_1 \otimes H_2$ and returns a vector in $H_1 \otimes H_2$
so $A \otimes B \in L(H_1 \otimes H_2)$
also if $A_1, A_2 \in L(H_1)$, $B_1, B_2 \in L(H_2)$
$a, b \in \mathbb{R}$ it is not very difficult to see that

$$(aA_1 + A_2) \otimes (bB_1 \otimes B_2) = abA_1 \otimes B_1 + aA_1 \otimes B_2 + bA_2 \otimes B_1 + A_2 \otimes B_2$$

so it is linear.
Also if $(A \otimes B)(|v_1\rangle \otimes |v_2\rangle) = 0$ for all $|v_1\rangle \in H_1$, $|v_2\rangle \in H_2$ then either $A = 0$ or $B = 0$ implying $A \otimes B = 0$ so the kernel is 0 and the mapping is therefore injective. Last we need to check that $A \otimes B$ is self adjoint from the definition

$$\langle \phi | (A \otimes B)^* | \psi \rangle = \overline{\langle \psi | A \otimes B | \phi \rangle}$$

this means we need to check that

$$\langle \phi | A \otimes B | \psi \rangle = \langle \psi | A \otimes B | \phi \rangle$$

(since we are in a real vector space)

A and B are both self-adjoint thus we can let $\{|\psi_1\rangle, \dots, |\psi_n\rangle\}$ be an orthonormal basis for H_1 of eigenvectors of A with eigen values a_j. Similarly $\{|\phi_1\rangle, \dots, |\phi_m\rangle\}$ and $\{b_1, \dots, b_m\}$ is an orthonormal basis for H_2 of eigenvectors for B with their corresponding eigenvalues

then if $|\eta_1\rangle, |\eta_2\rangle \in H_1 \otimes H_2$ we can write

$$|\eta_i\rangle = \sum_{j=1}^{n} \sum_{k=1}^{m} c_{kj}^{(i)} |\psi_j\rangle \otimes |\phi_m\rangle$$

then

$$\langle \eta_1 | A \otimes B | \eta_2 \rangle = \langle \eta_1 | \sum_{j=1}^{m} \sum_{k=1}^{m} c_{kj}^{(2)} a_j b_k |\psi_j\rangle \otimes |\phi_k\rangle$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{m} c_{kj}^{(1)} c_{kj}^{(2)} a_j b_k = \langle \eta_2 | \sum_{j} \sum_{k} c_{kj}^{(1)} a_j b_k |\psi_j\rangle \otimes |\phi_k\rangle$$

$$= \langle \eta_2 | A \otimes B | \eta_1 \rangle \blacksquare$$

Kan skrives kortere men skrev det ut eksplisitt her.

4] we have $\eta = \sum |\psi_i\rangle \otimes |\phi_i\rangle$

orthonormal basis

we have

$\varphi_1(A) = \varphi(A \otimes I)$

$\varphi = \langle \eta | \cdot | \eta \rangle$

$\varphi_1(A) = \langle \eta | A \otimes I | \eta \rangle$

$= \sum_i \langle \psi_i | \otimes \langle \phi_i | \, (A \otimes I) \left(\sum_j |\psi_j\rangle \otimes |\phi_k\rangle \right)$

$= \sum_i \sum_j \langle \psi_i | A | \psi_j \rangle \langle \phi_i | \phi_j \rangle$

orthonormal

$= \sum_i \langle \psi_i | A | \psi_i \rangle$

so $\varphi_1 = \sum \langle \psi_i | \cdot | \psi_i \rangle$

to conclude φ_1 is pure $\iff$ φ is separable.

$\Longrightarrow$

"$\Rightarrow$"

assume φ_1 is pure. then

$\varphi_1 = \langle v | \cdot | v \rangle$ for some unit $|v\rangle \in H_1$

then $|\eta\rangle = |v\rangle \otimes \left(\sum c_i |\phi_i\rangle \right) = |v\rangle \otimes |\phi\rangle$

$|\phi\rangle \in H_2$ so φ is separable

"$\Leftarrow$"

assume $|\eta\rangle = |v\rangle \otimes |\phi\rangle$, $|v\rangle$ and $|\phi\rangle$ are unit vectors in H_1 and H_2 respectively

$\varphi_1(A) = \langle v | A | v \rangle \langle \phi | \phi \rangle = \langle v | A | v \rangle$

$\varphi_1 = \langle v | \cdot | v \rangle$ so φ_1 is pure $\blacksquare$