

# Introduction to Operator Algebras

Alcides Buss

Notes by: Linus Mußmächer

2336440

Summer 2023

# Contents

<b>1</b>	<b>Algebras</b>	<b>4</b>
<b>2</b>	<b>Normed algebras</b>	<b>5</b>
<b>3</b>	<b>Algebras</b>	<b>6</b>
<b>4</b>	<b>Homomorphisms of algebras</b>	<b>8</b>
<b>5</b>	<b>Spectral theory</b>	<b>9</b>

The set of all linear bounded operators  $\mathcal{L}(H) = \mathcal{B}(H)$  on a given Banach space  $H$  is a (Banach) algebra with  $S \cdot T = S \circ T$ .  $M \subseteq \mathcal{L}$  is a Subalgebra such that  $M^* \subseteq M$  where  $T^*$  is the adjoint of  $T$ . This is also a closed subspace with respect to the strong topology. This is equivalent to  $M = M''$  (when  $X \subseteq \mathcal{B}(H)$ ,  $X' = \{T \in \mathcal{B}(H) \mid TS = ST \ \forall S \in X\}$ )

## Some topological basics

### Definition 0.1

- *Topology, Open*
- *Hausdorff, locally Hausdorff*
- *compact*

**Definition 0.2** A topological space  $X$  is **locally Hausdorff** if every  $x \in X$  admits a compact neighborhood basis, that is for every  $x \in X$  and every open set  $U \ni x$  there exists an open set  $V \ni x$  with  $\bar{V}$  is compact.

**Corollary 0.3** If a set  $V$  is compact in any subset  $U \subseteq X$ , it is also compact in  $X$ .

**Example 0.4 (Snake with two heads)** Consider  $I = [0, 1]$  with the standard topology and extend the set with an element  $1^+$  such that  $I \cup 1^+ \setminus 1$  is isomorphic to  $I$ . Then  $I \cup 1^+$  is locally Hausdorff and compact, but not Hausdorff.

## Some results about locally compact Hausdorff spaces

**Lemma 0.5 (Uryson's Lemma)** Let  $X$  be locally compact and Hausdorff. For all  $F \subseteq X$  closed and  $K \subseteq X$  compact with  $F \cap K = \emptyset$ , there exists an  $f : X \rightarrow [0, 1]$  continuous such that  $f|_K \equiv 1$  and  $f|_F \equiv 0$ .

**Theorem 0.6 (Tietze's extension theorem)** Let  $X$  be locally compact,  $K \subseteq X$  compact and  $f : K \rightarrow \mathbb{C}$  continuous. Then there exists a continuous  $\tilde{f} : X \rightarrow \mathbb{C}$  such that  $\tilde{f}|_K = f$ .

**Theorem 0.7 (Alexandroff's compactification)** If  $X$  is locally compact and Hausdorff, then  $\tilde{X} \sqcup \{\infty\}$  is a compact Hausdorff space  $\mathcal{O}(\tilde{X}) = \mathcal{O}(X) \cup \{K^c \cup \{\infty\} \mid K \text{ compact}\}$ .

**Example 0.8** Compacting the real line  $\mathbb{R}$  yields the space  $\tilde{\mathbb{R}}$ , which is isomorphic to the unit circle  $\Pi = \mathbb{S}^1$ .

**Theorem 0.9** Conversely, if  $Y$  is a compact Hausdorff space, then for all  $y_0 \in Y$ ,  $X := Y \setminus \{y_0\}$  is locally compact (in respect to the subspace topology).

More generally, if  $Y$  is locally compact and Hausdorff, and  $Z \subseteq Y$  is a difference of open and closed subsets, of  $Y$  (i.e.  $Z = U \setminus F$ , where  $U$  is open in  $Y$  and  $F$  is closed in  $Y$ ), then  $Z$  is locally compact.

# 1 Algebras

**Definition 1.1** An **algebra** is a (complex) vector space  $\mathcal{A}$  endowed with a bilinear and associative multiplication:  $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, (a, b) \mapsto a \cdot b$ . So

- (i)  $(a + \alpha b) \cdot (c + \beta d) = ac + \alpha bc + \beta ad + \alpha \beta bd$
- (ii)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

for all  $a, b, c \in \mathcal{A}$  and  $\alpha, \beta \in \mathbb{C}$ . We say that  $\mathcal{A}$  is

- (i) **commutative**, if  $ab = ba$  for all  $a, b \in \mathcal{A}$  and
- (ii) **unital**, if there exists  $1 = 1_{\mathcal{A}} \in \mathcal{A}$  such that  $1 \cdot a = a \cdot 1 = a$  for all  $a \in \mathcal{A}$ .

## Example 1.2

- (i)  $\mathbb{C}$ , or more generally  $\mathbb{C}^n = \mathbb{C} \oplus \dots \oplus \mathbb{C}$ , is an algebra.
- (ii) Say  $X$  is any set; let  $\mathbb{C}^X = \{f : X \rightarrow \mathbb{C}\}$  with point wise multiplication  $(f \cdot g)(x) = f(x) \cdot g(x)$ . These are commutative unital algebras (with  $1(x) = 1 \in \mathbb{C}$ ).
- (iii) Consider the polynomials  $\mathbb{C}[X] = \{\sum_{i=0}^n \lambda_i x^i \mid \lambda_i \in \mathbb{C}, n \in \mathbb{N}\}$  with the usual operations. This is a commutative unital algebra.
- (iv) Let  $X$  be a topological space and  $C(X) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is continuous}\} \subseteq \mathbb{C}^X$  the set of continuous functions on  $X$ . This is a commutative unital (sub)algebra (of  $\mathbb{C}^X$ ).
- (v) Take any vector space  $A$  define a (trivial) multiplication  $a \cdot b := 0$ . This is a commutative Algebra (that is not unital unless  $A = 0$ ).
- (vi)  $M_n(\mathbb{C})$  (the complex  $n \times n$  matrices) with the usual multiplication are a non-commutative (unless  $n = 1$ ) unital algebra.
- (vii) Let  $V$  be any (complex) vector space. The set of all linear operators  $L(V) := \{T : V \rightarrow V \mid T \text{ linear operator}\}$  is a unital (non-commutative for  $\dim V > 1$ ). We observe  $\mathcal{L}(\mathbb{C}^n) \simeq M_n(\mathbb{C})$ .
- (viii) Let  $S$  be a semigroup (i.e. a set with an associative operation  $S \times S \rightarrow S$ , e.g.  $(\mathbb{N}, +)$ ). Then  $\mathbb{C}[S] = \{\sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{C}, |\{s : \lambda_s \neq 0\}| < \infty\}$  (the finite formal sums of elements of  $S$ ) with the following product

$$\left( \sum_{s \in S'} \lambda_s s \right) \cdot \left( \sum_{t \in S} \lambda'_t t \right) := \sum_{s, t \in S} (\lambda_s \cdot \lambda'_t)(s \cdot t) \in S$$

Observe: As a vector space:  $\mathbb{C}[S] \subseteq \mathbb{C}^S$ . In general, this is neither commutative nor unital.

## 2 Normed algebras

**Definition 2.1** An algebra  $\mathcal{A}$  is **normed**, if it is endowed with a (vector space) norm  $\|\cdot\|: \mathcal{A} \rightarrow [0, \infty)$  satisfying  $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ . If  $\mathcal{A}$  is unital with unit  $1_{\mathcal{A}}$ , we usually assume  $\|1_{\mathcal{A}}\| = 1$  except for  $\mathcal{A} = 0$ .

**Definition 2.2** A **Banach algebra** is a normed algebra that is also complete (as a metric space with respect to the distance  $d(a, b) := \|a - b\|$ ), i.e. every Cauchy sequence converges.

**Example 2.3** (i) If  $X$  is a compact space then  $C(X)$  is a commutative unital Banach algebra with respect to the norm  $\|f\|_{\infty} := \sup_{x \in X} |f(x)| < \infty$  (since  $X$  is compact).

(ii) If  $V$  is a normed (respectively Banach) vector space, e.g.  $\mathbb{C}^n$  or  $\ell^p(\mathbb{N})$ , then  $\mathcal{L}(V) = \{T \in L(V) \mid T \text{ is bounded/continuous}\}$  with  $\|T\| := \sup_{\|v\| \leq 1} \|T(v)\| < \infty$  is a normed Banach algebra.

(iii) If  $X$  is a topological space, then  $C_b(X) = \{f \in C(X) \mid \|f\|_{\infty} < \infty\}$  (bounded continuous functions) is a Banach space.

(iv) Let  $X$  again be a topological space. Then the set of all functions **vanishing at  $\infty$** ,

$$\begin{aligned} C_0(X) &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \exists K \subseteq X, K \text{ compact} \forall_{x \notin K} |f(x)| < \varepsilon\} \\ &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\} \subseteq C_b(X), \end{aligned}$$

is also a Banach algebra.

**Exercise 2.1** Assume  $X$  is locally compact and Hausdorff. Prove the following are equivalent:

- (1)  $X$  is compact.
- (2)  $C(X) = C_0(X)$
- (3)  $C_0(X)$  is unital.
- (4) The unit function  $1 \in C_b(X)$  belongs to  $C_0(X)$ .

PROOF: • (1)  $\Rightarrow$  (2): Recall the definition of  $C_0(X)$ . If  $X$  is compact, every closed subset (especially every  $\{x \mid |f(x)| \geq \varepsilon\}$ ) is compact, so the condition of  $C_0(X)$  is trivial.

• (2)  $\Rightarrow$  (3): Since  $C(X)$  is unital,  $C_0(X)$  is as well.

• (3)  $\Rightarrow$  (4): Suppose  $C_0$  is unital, and let  $f \in C_0(X)$  be the unit. Then  $f \cdot g = g$  for all  $g \in C_0(X)$ , i.e.  $f(x)g(x) = g(x) \forall_{x \in X} \forall_{g \in C_0(X)}$ . By Uryson's lemma, given any  $x_0 \in X$ , there exists  $g \in C_0(X)$  with  $g(x_0) = 1$  (by looking at  $K = \{x_0\}$  and taking  $F$  as the complement of any relatively compact environment of  $x_0$ ). Then  $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$ . Doing this for every  $x_0 \in X$  yields  $f \equiv 1$ .

• (4)  $\Rightarrow$  (1): Since  $1 \in C_0(X)$ , for every  $\varepsilon > 0$  the set  $\{x \mid |f(x)| \geq \varepsilon\}$  is compact. Choose  $\varepsilon = \frac{1}{2}$ . Then,  $\{x \mid |f(x)| = |1| \geq \frac{1}{2}\} = X$  is compact.  $\square$

**Exercise 2.2** Let  $X$  be a locally compact Hausdorff space. Prove that  $C_0(X) \cong \{f \in C(X) \mid f(\infty) = 0\}$

### 3 Algebras

**Definition 3.1** A *\*-algebra* is a complex algebra  $\mathcal{A}$  with an *involution*  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$  satisfying

- (i)  $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$
- (ii)  $(a^*)^* = a$
- (iii)  $(ab)^* = b^*a^*$

for all  $a, b \in \mathcal{A}$  and all  $\lambda \in \mathbb{C}$ .

**Definition 3.2** A *normed \*-algebra* is a normed algebra  $\mathcal{A}$  with an involution (such that  $\mathcal{A}$  is a \*-algebra) also satisfying  $\|a^*\| = \|a\|$  for all  $a \in \mathcal{A}$ .

A *Banach\*-algebra* is a complete normed \*-algebra.

**Definition 3.3** A *C\*-algebra* is a Banach\*-algebra satisfying  $\|a^* \cdot a\| = \|a\|^2$ .

Observation: Recall that  $\|a \cdot b\| \leq \|a\| \cdot \|b\|$  in all normed algebras. Applying this to a C\*-algebra we get  $\|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$ . If  $\mathcal{A}$  is a C\*-algebra, then  $\|a\|^2 = \|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$ , so  $\|a\| = \|a^*\|$ .

#### Example 3.4

- (i) If  $X$  is a set, then  $\mathbb{C}^X$  is a \*-algebra with  $f^* = \bar{f}$  and  $\mathcal{C}^\infty(X)$  is a C\*-algebra.
- (ii) If  $X$  is a topological space, then  $C(X) \subseteq \mathbb{C}^X$  is also a \*-subalgebra and for  $\{f \in C(X) \mid \text{supp}(f) = \overline{\{x \in X \mid |f(x)| \neq 0\}} \text{ compact}\}$  we have

$$C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X) \subseteq C^\infty(X)$$

and  $C^\infty$  is a C\*-algebra.  $C_c$  is a \*-algebra, but not Banach in general.

If  $X$  is compact, it follows  $C_c(X) = C_0(X) = C_b(X)$ .

Observation: If  $X$  is locally compact and Hausdorff, then  $\overline{C_c(X)} = C_0(X)$ .

- (iii) Let  $X$  be a measured space ( $X$  is endowed with a  $\sigma$ -algebra). Then  $B_\infty(X) = \{f \in C^\infty \mid f \text{ is measurable}\}$  is a C\*-algebra. If  $\mu$  is a measure on  $X$  (e.g.  $X = \mathbb{R}^n$  and  $\mu$  the Lebesgue measure) then  $L^\infty(X, \mu)$  are the essentially bounded functions and

$$L^\infty(X) = \{f : X \rightarrow \mathbb{C} \mid \|f\| := \inf\{c \geq 0 \mid \mu(\{x \mid |f(x)| > c\}) = 0\}\}$$

is also a C\*-algebra.

Observation:  $L^2(X, \mu) = \{\mu\text{-separable function}\}$ ,  $L^\infty(X, \mu) \xrightarrow{\mu} B(L^2(X, \mu)), f \mapsto \mu_f = \{g \mapsto f \cdot g\}$

- (iv) A non-example: Let  $\mathbb{D}$  be the unit disk and  $\mathcal{A}(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid \text{analytic in } \mathbb{D}^\circ\}$

**Morera's Theorem** from complex analysis states that  $f \in C(\mathbb{D})$  is analytic if and only if  $\int_\gamma f(z)dz = 0$  for all closed and piece wise smooth paths in  $\mathbb{D}^\circ$ . From this, it follows that  $\mathcal{A}(\mathbb{D})$  is closed in  $C(\mathbb{D})$ , therefore a Banach algebra. It is also a Banach\*-algebra with, but  $f^* = \bar{f}$  (point wise) is not possible, as  $z \mapsto \bar{z}$  is not analytic. Thus, we have to choose  $f^*(z) = f(\bar{z})$ . But  $\mathcal{A}(\mathbb{D})$  is not a C\*-algebra, as  $\|f^*f\|_\infty \neq \|f\|_\infty^2$  for some  $f \in \mathcal{A}(\mathbb{D})$ .

- (v) A non-commutative example: Let  $H$  be a Hilbert space and  $B(H) = \mathcal{L}(H) = \{T : H \rightarrow H \mid T \text{ bounded, continuous, linear}\}$  and  $\|H\| := \sup_{\|z\| < 1} \|T(z)\| < \infty$ . This is a  $C^*$ -algebra where  $T^*$  is the adjoint of  $T$ , that is  $\langle T^*z, w \rangle = \langle z, Tw \rangle$  for all  $z, w \in H$ .

$C^*$ -axiom:  $\|T^* \cdot T\| \leq \|T\|^2$  since  $\mathcal{L}(H)$  is a Banach algebra, and we also have

$$\begin{aligned} \|T\|^2 &= \sup_{\|z\| < 1} \|T(z)\|^2 = \sup_{\|z\| < 1} \langle Tz, Tz \rangle = \sup_{\|z\| < 1} \langle z, T^*Tz \rangle \\ &\leq \sup_{\|z\| < 1} \|z\| \|T^*Tz\| \leq \sup_{\|z\| < 1} \|z\| \|T^*T\| \leq \|T^*T\| \end{aligned}$$

In particular,  $M_n(\mathbb{C}) \simeq \mathcal{L}(\mathbb{C}^n)$  is a unital  $C^*$ -algebra.

- (vi) To produce more examples, take any subset  $S \subseteq \mathcal{L}(H)$  and take  $C^*(S) \subseteq \mathcal{L}(H) = \overline{\text{span}\{S_i \mid S_i \in S \cup S^*, i \leq n \in \mathbb{N}\}}$ .

**Example 3.5** Let  $s \in \mathcal{L}(\ell^2(\mathbb{N}))$ . The shift  $s$ , defined by  $s(e_i) = e_{i+1}$  for all  $i \in \mathbb{N}$  (where  $\{e_i\}$  is the canonical basis of the sequence space), is an isometry, that is  $s^* \cdot s = \text{id}$ . Since  $s \cdot s^* \neq \text{id}$ , it is not surjective and not a proper isometry. We define

$$T = C^*(s) = \overline{\text{span}\{s^n(s^*)^m \mid m, n \in \mathbb{N}_0\}} \subseteq \mathcal{L}(\ell^2(\mathbb{N}))$$

as the **Toeplitz algebra**.

**Example 3.6** Let  $H$  be a Hilbert space and  $S$  the set of all finite rank operators on  $H$ .

**Example 3.7**

- (i) **Commutative:**  $C_0(X)$  for a locally Hausdorff space  $X$ .
- (ii) **Non-commutative:**  $\mathcal{L}(\mathfrak{H}) = \mathcal{B}(\mathfrak{H})$  for any Hilbert space  $\mathfrak{H}$  (with dimension greater 1).
- (iii) **More generally:** Take any subset  $S \subseteq \mathcal{L}(\mathfrak{H})$  and construct  $C^*(S) \subseteq \mathcal{L}(H)$  as

$$\overline{\text{span}\{S_1, \dots, S_n \mid S_i \in S \cap S^*\}}$$

**Example 3.8 (Cuntz algebras)** Take again  $\mathfrak{H} = \ell^2\mathbb{N} = \{(\lambda_n)_{n \in \mathbb{N}_0} \mid \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$  where  $\langle \lambda, \lambda' \rangle = \sum_{i \in \mathbb{N}_0} \overline{\lambda_i} \lambda'_i$  and which has the orthonormal base  $(e_n)_{n \in \mathbb{N}}$  where  $(e_n) = (\delta_{in})_{i \in \mathbb{N}_0}$ .

On this algebra, define

- $S_1(e_n) = e_{2n}$ .
- $S_2(e_n) = e_{2n+1}$ .

We have partitioned the natural numbers into evens and odds. This defines two (proper) isometries  $S_1, S_2 \in \mathcal{L}(\mathfrak{H})$ , that is  $S_i^* S_i = \text{id}_{\mathfrak{H}}$ , to subspaces of  $\mathfrak{H}$ . Notice:  $S_i^* S_j = 0$  for  $i \neq j$  as well as  $S_1 S_1^* + S_2 S_2^* = \text{id}_{\mathfrak{H}}$ . Define  $\mathcal{O}_2 = C^*(S_1, S_2) = \overline{\text{span}\{S_\alpha S_\beta^* \mid \alpha, \beta \text{ finite words in } \{1, 2\}\}}$ . For example, for  $\alpha = 121211$  we have  $S_\alpha = S_1 S_2 S_1 S_2 S_1^2$ .  $\mathcal{O}_2$  is called the **Cuntz algebra**. More generally, one can define  $\mathcal{O}_3, \mathcal{O}_4, \dots$  Cuntz algebras. Joachim Cuntz proved that these are simple  $C^*$ -algebras with additional interesting properties we will see later.

**Example 3.9 (Rotation algebras)** Let  $\mathfrak{H} = \ell^2(\mathbb{Z})$  (bi-infinite sequences) with basis  $(e_n)_{n \in \mathbb{Z}}$ . Define:

- $U(e_n) := e_{n+1}$  (bilateral shift)

- $V(e_n) := \lambda^n e_n$  where  $\lambda \in \mathbb{C}$  is some fixed number  $|\lambda| = 1$ .

This defines two *unitary* operators:  $UU^* = 1 = U^*U$  and  $V^*V = 1 = V^*V$ . If  $\exp(2\pi i\theta), \theta \in \mathbb{R}$  define  $A_\theta := C^*(U, V) \subseteq \mathcal{L}(\ell^2\mathbb{N})$ .

There is a special relation between  $U$  and  $V$  where  $UV = \lambda VU = \exp(2\pi i\theta)VU$ . From this relation, we can describe  $A_\theta = \overline{\text{span}}\{\sum_{n,m \in \mathbb{Z}}^{\text{finite}} a_{n,m} U^n V^m \mid a_{n,m} \in \mathbb{C}\}$ .

Furthermore, if  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ ,  $A_\theta$  is simple.

**Example 3.10 ( $C^*$ -algebras of groups)** Let  $G$  be a (discrete) group. Look at  $\mathfrak{H} = \ell^2(G) = \{(a_g)_{g \in G} \mid \sum_{g \in G} |a_g|^2 < \infty\}$  (Note: This limit will only converge if there are countably (or finitely) many non-zero parts) with ONB  $(\delta_g)_{g \in G}$  where  $\delta_g(h) = \delta_{gh}$ . Define for each  $g \in G$  an operator  $\lambda_g \in \mathcal{L}(\ell^2 G)$  by  $\lambda_g(\delta_h) = \delta_{gh}$ . Notice that  $h \mapsto gh$  is a bijection, and thus  $\lambda_g$  is a unitary operator with  $\lambda_g^* = \lambda_{g^{-1}}$ . We can now define the **reduced  $C^*$ -algebra** of the group:

$$C_R^*(G) := C_\lambda^*(G) \subseteq \mathcal{L}(\ell^2 G) = C^*(\lambda_g \mid g \in G)$$

Here, we have the relation  $\lambda_g \cdot \lambda_h = \lambda_{gh}$  and thus  $C_R^*(G) = \{\sum a_g \lambda_g \mid a_g \in \mathbb{C}\}$ .

In general, take  $U : G \rightarrow \mathcal{L}(H), g \mapsto U_g$  a **unitary representation of  $G$**  with  $U_g U_h = U_{gh}$  and  $U_1 = \text{id}$  as well as  $U_g^{-1} = U_{g^{-1}}$ . Then  $C_U^*(G) := \{\sum_{g \in G} a_g U_g \mid a_g \in \mathbb{C}\} \subseteq \mathcal{L}(H)$ . There exists a **universal unitary representation**  $C_{\max}^*(G)$ , a full  $C^*$ -algebra of  $G$ .

**Remark 3.11**

- (i) If  $G$  is Abelian, then  $C_U^*(G)$  is also abelian (commutative). In particular,  $C_\lambda^*$  is abelian. Later, we will prove  $C_\lambda^*(G) \simeq C(\hat{G})$  where  $\hat{G}$  is the dual of  $G$ , i.e.  $\{X : G \rightarrow \mathbb{C} \text{ characters}\}$ .
- (ii) For many groups, like  $G = \mathbb{F}_n$  (the free groups) the reduced  $C^*$ -algebra  $C_\lambda^*(G)$  is simple.

## 4 Homomorphisms of algebras

**Definition 4.1** If  $\mathcal{A}, \mathcal{B}$  are algebras, a **homomorphism** from  $\mathcal{A}$  to  $\mathcal{B}$  is a linear map  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\varphi(ab) = \varphi(a)\varphi(b)$  for any  $a, b \in \mathcal{A}$ .

If  $\mathcal{A}$  and  $\mathcal{B}$  are  $*$ -algebras, a  **$*$ -homomorphism** is a homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\varphi(a^*) = \varphi(a)^*$  for all  $a \in \mathcal{A}$ .

If  $\mathcal{A}, \mathcal{B}$  are Banach algebras, then usually we want to have **continuous** homomorphisms. Even more, we usually ask for **contractive** homomorphisms  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ , (that is  $\|\varphi\| \leq 1$ ).

We will be especially interested in **characters**:

**Definition 4.2** A **character** of an algebra  $\mathcal{A}$  is a non-zero homomorphism  $\chi : \mathcal{A} \rightarrow \mathbb{C}$ .

**Example 4.3** Take any subalgebra  $\mathcal{A} \subseteq \mathbb{C}^X$ . Take  $x_0 \in X$  and set  $\chi_{x_0} := \text{ev}_{x_0} : \mathcal{A} \rightarrow \mathbb{C}, f \mapsto f(x_0)$ . This is not necessarily a character, but it is for example, if  $\mathcal{A} = C(X)$  or  $C_b(X)$  or  $C_0(X)$  (if  $X$  is “nice”, like Hausdorff).

**Definition 4.4** A  $(*)$ -isomorphism between two  $(*)$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  is a bijective  $(*)$ -homomorphism  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ .

**Definition 4.5** A  $(*)$ -ideal of a  $*$ -algebra  $\mathcal{A}$  is a subspace  $I \subset \mathcal{A}$  such that  $I \cdot \mathcal{A} \subseteq I$ ,  $\mathcal{A} \cdot I \subseteq I$  (if only one condition applies, we call this a **left ideal** or **right ideal**). For  $*$ -ideals, we also want  $I^* = I$ . We notate this as  $I \trianglelefteq \mathcal{A}$ .



**Example 4.6** If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a  $(*)$ -homomorphism, then  $\ker \varphi \trianglelefteq \mathcal{A}$ .

**Example 4.7** If  $I \trianglelefteq \mathcal{A}$  for  $\mathcal{A}$  a  $(*)$ -algebra

$$\mathcal{A}/I = \{a + I \mid a \in \mathcal{A}\}$$

with  $(a + I) \cdot (b + I) := ab + I$  and  $(a + I)^* = a^* + I$  is a  $(*)$ -algebra.

**Theorem 4.8** If  $\mathcal{A}$  is a Banach- $*$ -algebra, then  $I \trianglelefteq \mathcal{A}$  is a closed ideal, then the quotient  $I/\mathcal{A}$  is also a Banach- $*$ -algebra.

PROOF: Later. □

## 5 Spectral theory

**Notation 5.1** If  $\mathcal{A}$  is a unital algebra, we write

$$\text{inv}(\mathcal{A}) = \{a \in \mathcal{A} \mid a \text{ is invertible in } \mathcal{A}\} = \{a \in \mathcal{A} \mid \exists_{a^{-1} \in \mathcal{A}} aa^{-1} = 1 = a^{-1}a\}$$

This is a group. Sometimes we also write  $GL(\mathcal{A})$ .

**Definition 5.2** Given a unital algebra  $\mathcal{A}$  and  $a \in \mathcal{A}$ , we define its **spectrum** (in  $\mathcal{A}$ ) as

$$\sigma_{\mathcal{A}}(a) = \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \notin \text{inv}(\mathcal{A})\}$$

and the resolvent of  $a$  (in  $\mathcal{A}$ ) as

$$\rho_{\mathcal{A}}(a) = \rho(a) = \mathcal{A} \setminus \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \text{inv}(\mathcal{A})\}$$

**Example 5.3 (Linear Algebra)** Let  $\mathcal{A} = M_m(\mathbb{C})$  and  $a \in \mathcal{A}$ . Then we have

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \det(\lambda - a) = 0\}$$

and these are the roots of the characteristic polynomial  $\det(\lambda - a)$ . This is exactly the usual spectrum from linear algebra.

**Example 5.4 (Functional Analysis)** Let  $\mathcal{A} = \mathcal{L}(\mathfrak{H})$  – where  $\mathfrak{H}$  is any Hilbert- or Banach space – and  $T \in \mathcal{A}$ . Then  $\sigma_{\mathcal{A}}(T)$  is exactly the spectrum as defined in functional analysis.

If  $S$  is the shift in  $\mathcal{L}(\ell^2\mathbb{N})$ , then we have  $\sigma(S) = \mathbb{D}$ .

**Example 5.5** Let  $\mathcal{A} = \mathbb{C}[X]$ . Here we have  $\text{inv}(\mathcal{A}) = \{a_0 X^0 \mid a_0 \in \mathbb{C} \setminus \{0\}\}$  the constant non-zero polynomials. If  $a = \sum_{k=0}^N a_k x^k \in \mathcal{A}$ , then we have two cases:

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \mathbb{C} & \text{otherwise} \end{cases}$$

**Example 5.6** Let  $\mathcal{A} = \mathbb{C}(X) = \{p/q \mid p, q \in \mathbb{C}[X], q \neq 0\}$ . Now we have  $\text{inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$ . If  $a \in \mathcal{A}$ , then

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \emptyset & \text{otherwise} \end{cases}$$

**Example 5.7** Let  $\mathcal{A} = C(X)$  for any topological space  $X$ . Then

$$\text{inv}(\mathcal{A}) = \{f \in C(X) \mid \forall_{x \in X} f(x) \neq 0\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{x \in X} f(x) = \lambda\} = \text{im}(f) = f(X).$$

**Example 5.8** Let  $X$  be any topological space and consider  $\mathcal{A} = C_b(X)$ . Then

$$\text{inv}(C_b(X)) = \{f \in C_b(X) \mid \exists_{\varepsilon > 0} \forall_{x \in X} |f(x)| \geq \varepsilon\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \in \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{(x_n)} f(x_n) \rightarrow \lambda\} = \overline{\text{im}(f)} = \overline{f(X)}.$$

This is a compact subset of  $\mathbb{C}$ .

**Theorem 5.9 (Algebraic spectral mapping theorem)** *Let  $\mathcal{A}$  be an algebra,  $a \in \mathcal{A}$  and  $p \in \mathbb{C}[X]$ ,  $p(X) = \sum_{k=0}^n \lambda_k X^k$  and define  $p(a) = \sum_{k=0}^n \lambda_k a^k$ . Recall that the mapping  $\mathbb{C}[X] \rightarrow \mathcal{A}$ ,  $p \mapsto p(a)$  is a unital homomorphism.*

*Then  $\sigma(p(a)) = p(\sigma(a))$  assuming  $\sigma(a) \neq \emptyset$ .*

PROOF: If  $p(X) = \lambda_0$  constant, this is clear (the spectrum is exactly  $\lambda_0$  on both sides). Assume  $p(x)$  is not constant. Fix  $\mu \in \mathbb{C}$  and write

$$\mu - p(x) = \lambda_0(x - \lambda_1) \cdots (x - \lambda_n)$$

as per the fundamental theorem of algebra (note that these are not the same  $\lambda$  as before) with  $\lambda_0 \neq 0$ . Then  $\mu - p(a) = \lambda_0(a - \lambda_1) \cdots (a - \lambda_n)$ . Since these expressions commute, this product is invertible if and only if  $(a - \lambda_i)$  is invertible for every  $i$ . So  $\mu \in \sigma(p(a)) \Leftrightarrow \mu - p(a)$  is not invertible if and only if there exists an  $i$  for which  $\lambda_i - a$  is not invertible, so  $\lambda_i \in \sigma(a)$ . But the  $\lambda_i$  are exactly the numbers satisfying  $p(\lambda) = \mu$ . Thus,  $\mu$  is in  $\sigma(p(a))$  if it is in the image of  $\sigma(a)$  under  $p$ . Therefore, we conclude  $\sigma(p(a)) = p(\sigma(a))$ .  $\square$

We now focus on invertible elements in **Banach algebras**.

**Theorem 5.10** *If  $\mathcal{A}$  is a unital Banach algebra and  $a \in \mathcal{A}$  with  $\|a\| < 1$  then  $1 - a$  is invertible and  $(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n$ .*

PROOF: Observe that, since  $\|a\| < 1$ , we have  $\sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} < \infty$ . This implies the (absolute) convergence of  $\sum_{n=0}^{\infty} a^n$  by the characteristic property of Banach spaces. Hence,  $b := \lim_{N \rightarrow \infty} \sum_{n=0}^N a^n \in \mathcal{A}$ . No, if  $N \in \mathbb{N}$ , then

$$(1 - a) \left( \sum_{n=0}^N a^n \right) = \left( \sum_{n=0}^N a^n \right) - \left( \sum_{n=1}^{N+1} a^n \right) = 1 - a^{N+1} \rightarrow 1$$

because of  $\|a\| < 1$ . This yields  $(1 - a)b = 1$ .  $\square$

**Theorem 5.11** *Let  $\mathcal{A}$  be a non-empty, non-zero unital Banach algebra. Then  $\text{inv}(\mathcal{A})$  is an open subset of  $\mathcal{A}$  and the function  $f : \text{inv}(\mathcal{A}) \rightarrow \mathcal{A}$ ,  $a \mapsto a^{-1}$  is Frechet-differentiable and in particular continuous as well as  $f'(a)b = -a^{-1}ba^{-1}$ .*

Recall from calculus that  $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$ . Also recall that  $f : U \xrightarrow{\text{open}} X \rightarrow Y$  with  $X, Y$  Banach spaces is **differentiable** at  $x_0 \in U$  there exists an operator  $D_{x_0} = f'(x_0) \in \mathcal{L}(X, Y)$  such that

$$\lim_{h \rightarrow 0} \frac{f\|(x_0 + h) - f(x_0) - D_{x_0}(h)\|}{\|h\|} = 0$$

PROOF: Take  $a \in \text{inv}(\mathcal{A})$ . If  $b \in \mathcal{A}$  such that  $\|a - b\| < \|a^{-1}\|^{-1}$ . From this, we have  $\|ba^{-1} - 1\| = \|ba^{-1} - aa^{-1}\| = \|(b - a)a^{-1}\| \leq \|b - a\| \cdot \|a^{-1}\| < 1$ . Per the previous theorem,  $ba^{-1} \in \text{inv}(\mathcal{A})$ . This implies that  $b$  is also invertible. This shows that  $\text{inv}(\mathcal{A})$  is open.

Furthermore, if  $\|b\| < 1$ , then also  $\| -b \| < 1$ . Thus,  $1 + b \in \text{inv}(\mathcal{A})$  and  $(1 + b)^{-1} = \sum_{n=0}^{\infty} (-1)^n b^n$ . Thus,

$$\|(1 + b)^{-1} - 1 + b\| = \left\| \sum_{n=0}^{\infty} (-1)^n b^n - 1 + b \right\| \leq \left\| \sum_{n=2}^{\infty} (-1)^n b^n \right\| \leq \sum_{n=2}^{\infty} \|b^n\| \leq \sum_{n=2}^{\infty} \|b\|^n = \frac{\|b\|^2}{1 - \|b\|}$$

Now let  $a \in \text{inv}(\mathcal{A})$  and  $c \in \mathcal{A}$  such that  $\|c\| < \frac{1}{2} \|a^{-1}\|^{-1}$ . Then  $\|a^{-1}c\| \leq \|a^{-1}\| \|c\| \leq \frac{1}{2}$ . So if  $b = a^{-1}c$ , then

$$\|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \leq \frac{\|a^{-1}c\|^2}{1 - \|a^{-1}c\|} < 2\|a^{-1}c\|^2$$

Now, define  $U : \mathcal{A} \rightarrow \mathcal{A}, b \mapsto -a^{-1}ba^{-1}$ . Then this is a linear odd operation with  $\|U\| \leq \|a^{-1}\|^2$ , and we have

$$\begin{aligned} \|(a + c)^{-1} - a^{-1} - U(c)\| &= \|(a + c)^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &= \|(1 + a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &\leq \|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \cdot \|a^{-1}\| \\ &\leq 2\|a^{-1}c\|^2 \|a^{-1}\| \leq 2\|a^{-1}\|^3 \|c\|^2 \end{aligned}$$

and thus

$$\lim_{c \rightarrow 0} \frac{\|(a + c)^{-1} - a^{-1} - U(c)\|}{\|c\|} = 0 \quad \square$$

**Example 5.12** If we choose  $\mathcal{A} = \mathbb{C}[X]$  and the norm  $\|p\| = \sup_{\lambda \in [0,1]} |p(\lambda)|$ . Then  $(\mathcal{A}, \|\cdot\|)$  is a normed (but not Banach) algebra. For example, we see that  $\lim_{m \rightarrow 0} 1 + X/m = 1 \in \text{inv}(\mathcal{A})$ , but  $1 + X/m \notin \text{inv}(\mathcal{A})$  and thus  $\text{inv}(\mathcal{A})$  is not open (because the complement is not closed).

**Theorem 5.13** *If  $\mathcal{A}$  is a Banach algebra with unit 1, then for all  $a \in \mathcal{A}$  the spectrum  $\sigma(a) \subseteq \mathbb{C}$  is closed and  $\sigma(a) \subseteq \overline{B(0, \|a\|)} = D(0, \|a\|) := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$ . Therefore,  $\sigma(a)$  is compact by the Heine-Borell theorem.*

PROOF: By definition

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \text{inv}(\mathcal{A})\}$$

is the inverse image of the closed subset  $\mathcal{A} \setminus \text{inv}(\mathcal{A}) \subseteq \mathcal{A}$  by the continuous function  $\lambda \mapsto \lambda - a$ . Therefore,  $\sigma(a)$  is closed.

Now if  $|\lambda| \leq \|a\|$  then  $\|\lambda^{-1}a\| < 1$ . Then  $1 - \lambda^{-1}a \in \text{inv}(\mathcal{A})$ . Multiplying by  $\lambda$  yields  $\lambda - a \in \text{inv}(\mathcal{A})$ . Thus,  $\{\lambda \in \mathbb{C} \mid |\lambda| > \|a\|\} \subseteq \rho(a)$  and thus  $\sigma(a) \subseteq D(0, \|a\|)$ .  $\square$

**Lemma 5.14** *Let  $\mathcal{A}$  be a unital Banach algebra and  $a \in \mathcal{A}$ . Then, the map  $R_a : \rho(a) \subseteq \mathbb{C} \rightarrow \mathcal{A}, \lambda \mapsto (a - \lambda)^{-1}$  is Frechet-differentiable.*

PROOF: This follows from the following general result:

If  $g : U \xrightarrow{\text{open}} X \rightarrow Y$  and  $f : V \xrightarrow{\text{open}} Y \rightarrow Z$  for Banach spaces  $X, Y, Z$  with  $g(U) \subseteq V$  are differentiable at  $x_0 \in U$  or respectively  $y_0 = g(x_0) \in V$ , then  $f \circ g$  is differentiable and  $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$ .  $\square$

Observation: For  $R_a(\lambda) = (a - \lambda)^{-1}$  we get  $R'_a(\lambda) = (a - \lambda)^{-2}$ . We have  $\mathcal{L}(\mathbb{C}, \mathcal{A}) \simeq \mathcal{A}$  by  $T \mapsto T(1)$ . Recall that if  $f(a) = a^{-1}$  yields  $f'(a)b = -a^{-1}ba^{-1}$ .

**Theorem 5.15 (Gelfand)** *If  $\mathcal{A} \neq 0$  is a unital Banach algebra and  $a \in \mathcal{A}$  then  $\sigma(a) \neq \emptyset$ .*

PROOF: Suppose  $\sigma(a) = \emptyset$ . Idea: Show that  $R_a : \rho(a) \subseteq \mathbb{C} \rightarrow \mathcal{A}, \lambda \mapsto (a - \lambda)^{-1} = \frac{1}{a - \lambda}$  is bounded and differentiable and achieve a contradiction by Liouville's theorem.

Claim:  $\|(a - \lambda)^{-1}\| < \|a\|^{-1}$  if  $|\lambda| > 2\|a\|$ . Indeed, if  $|\lambda| > 2\|a\|$  then  $\|\lambda^{-1}a\| < \frac{1}{2}$ , and in particular  $1 - \lambda^{-1}a \in \text{inv}(\mathcal{A})$  and

$$\|(1 - \lambda^{-1}a)^{-1} - 1\| = \left\| \sum_{n=1}^{\infty} (\lambda^{-1}a)^{-1} \right\| \leq \sum_{n=1}^{\infty} \|\lambda^{-1}a\|^n = \frac{\|\lambda^{-1}a\|}{1 - \|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

From here we deduce that  $\|(1 - \lambda^{-1}a)^{-1}\| < 2$  and thus

$$\|(a - \lambda)^{-1}\| < \|\lambda^{-1}(\lambda^{-1}a - 1)^{-1}\| = \frac{\|(1 - \lambda^{-1}a)^{-1}\|}{|\lambda|} < \frac{2}{\lambda} < \frac{1}{\|\lambda\|}.$$

So  $R_a : \mathbb{C} \rightarrow \mathcal{A}$  is bounded outside  $\overline{B(0, 2\|a\|)}$ . Since  $R_a$  is continuous, it is bounded on  $\mathbb{C} \rightarrow \mathcal{A}$ . Let  $\varphi \in \mathcal{A}^*$  be a bounded linear functional in  $\mathcal{L}(\mathcal{A}, \mathbb{C})$ . Thus,  $\varphi$  is differentiable with  $\varphi'(a) = \varphi$  for all  $a \in \mathcal{A}$ . Then  $\varphi \circ R_a$  is differentiable and bounded, so it is an “integer” function. By Liouville's theorem,  $\varphi \circ R_a$  is constant. Therefore,  $\varphi \circ R_a(x) = \varphi \circ R_a(y)$  for all  $x, y \in \mathcal{A}$ . Especially, we have  $\varphi((a - \lambda)^{-1}) = \varphi(a^{-1})$  for all  $\varphi$ . Hahn-Banach shows  $(a - \lambda)^{-1} = a^{-1}$  for all  $\lambda$ , proving  $a - \lambda = a$  for all  $a, \lambda$ . This is a contradiction.  $\square$

**Theorem 5.16 (Gelfand-Mazur)** *If  $\mathcal{A}$  is a unital Banach algebra and every  $a \neq 0$  admits an inverse ( $\mathcal{A}$  is a field), then  $\mathcal{A} = \mathbb{C} \cdot 1$ .*

PROOF: By the assumption,  $\text{inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$ . By the previous theorem, if  $a \in \mathcal{A}$  there exists some  $\lambda \in \sigma(a)$ , so  $a - \lambda \notin \text{inv}(\mathcal{A})$ , so  $a - \lambda = 0$  and thus  $a = \lambda \cdot 1$ .  $\square$

**Corollary 5.17** *Let  $\mathbb{C}(X) = \left\{ \frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{Q}[X] \right\}$  is a field, but it cannot be turned into a Banach algebra.*

**Theorem 5.18 (Adjoining units - unitization of algebras)** *Let  $\mathcal{A}$  be any algebra. Consider  $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$  as a vector space. We write elements of  $\tilde{\mathcal{A}}$  as  $a + \lambda \cdot 1 := (a, \lambda)$ . Think of  $a = (a, 0)$  and  $\lambda = (0, \lambda)$ . Define*

$$(a + \lambda 1)(b + \lambda' 1) = (ab + \lambda' a + \lambda ab + b) + \lambda \cdot \lambda'.$$

*Ten (exercise  $\tilde{\mathcal{A}}$ ) becomes a unital algebra with  $1_{\tilde{\mathcal{A}}} = 1 = (0, 1)$ .*

*Notice that  $\mathcal{A}$  is an ideal in  $\tilde{\mathcal{A}}$ .*

*Moreover, we get a short exact sequence*

$$0 \rightarrow \mathcal{A} \hookrightarrow \tilde{\mathcal{A}} \rightarrow \mathbb{C} \rightarrow 0$$

so  $1 + \lambda \mapsto \lambda$ .

If  $\mathcal{A}$  is a normed algebra, then  $\tilde{\mathcal{A}}$  is normed by  $\|a + \lambda \cdot 1\| := \|a\| + |\lambda|$

If  $\mathcal{A}$  is Banach and closed, then so is  $\tilde{\mathcal{A}}$ .

If  $\mathcal{A}$  is a  $*$ -algebra, then so is  $\tilde{\mathcal{A}}$  with  $(a + \lambda 1)^*$ .

If  $\mathcal{A}$  is a (Banach) normed  $*$ -algebra, then so is  $\tilde{\mathcal{A}}$ .

If  $\mathcal{A}$  is a  $C^*$ -algebra, in general the norm given above is not a Norm on  $\mathcal{A}$ , but  $\|a + \lambda \cdot 1\| := \sup_{b \in \mathcal{A}, b \in \mathcal{B}, b \leq 1} \|ab + \lambda b\|$  is.

**Exercise 5.1** If  $\mathcal{A}$  is already unital, then  $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$  as algebras by  $a + \lambda \cdot 1 \mapsto (a + \lambda 1_{\mathcal{A}}, -\lambda)$ .

**Definition 5.19** If  $\mathcal{A}$  is any algebra, then  $\sigma_{\mathcal{A}}(a) := \sigma_{\tilde{\mathcal{A}}}(a) = \sigma_{\tilde{\mathcal{A}}}(a, 0)$ . Thus, we can identify  $\mathcal{A} \hookrightarrow \tilde{\mathcal{A}}$ .

**Corollary 5.20**

- (i) If  $\mathcal{A}$  is any algebra, then we always have  $0 \in \sigma_{\mathcal{A}}(a)$  because if  $a \in \mathcal{A}$  then  $a \notin \text{inv}(\tilde{\mathcal{A}})$  as  $\mathcal{A} \leq \tilde{\mathcal{A}}$ .
- (ii) Observation: If  $\mathcal{A}$  is unital, then  $\tilde{\mathcal{A}} = \{a + \lambda 1 \mid a \in \mathcal{A}, \lambda \in \mathbb{C}\} \simeq \mathcal{A} \oplus \mathbb{C}$  as algebras via  $a + \lambda 1 \mapsto (a + \lambda 1_{\mathcal{A}}, \lambda)$ . For  $p = 1_{\mathcal{A}} \in Z(\tilde{\mathcal{A}})$ , we have  $\tilde{\mathcal{A}} \simeq p\tilde{\mathcal{A}} \oplus (1 - p)\tilde{\mathcal{A}}$ . This yields  $(a + \lambda 1)(1 - 1_{\mathcal{A}}) = a - a + \lambda(1 - 1_{\mathcal{A}}) = \lambda q$  for  $q := 1 - 1_{\mathcal{A}}$ . Thus, we have  $\sigma_{\tilde{\mathcal{A}}} = \sigma_{\mathcal{A}}(a) \cup \{0\}$ .