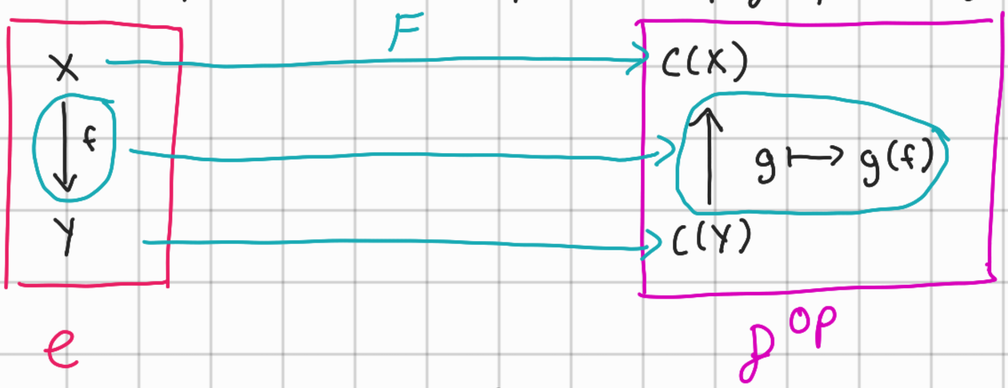
2.5 Let $\mathcal{C}$ be the category of compact Hausdorff spaces with morphisms the continuous maps and let $\mathcal{D}$ be the category of unital commutative C^*-algebras with morphisms the $*$-homomorphisms.

We now define a functor $F: \mathcal{C} \to \mathcal{D}^{op}$, $X \mapsto C(X)$, which maps a compact Hausdorff space X to the set of continuous functions on X. $C(X)$ is a C^*-algebra (one could either verify this or argue that $C(X)$ inherits all its properties from the C^*-algebra $\mathbb{C}$ since all functions in $C(X)$ have $\mathbb{C}$ as their co-domain).

F is defined on $Morph(\mathcal{C})$ as the following:

$X \to Y$ $\quad$ $Y \to \mathbb{C}$ $\quad$ $X \to Y \to \mathbb{C}$

For all $f \in Morph(\mathcal{C})$ we have $F: Morph(\mathcal{C}) \longrightarrow Morph(\mathcal{D}^{op})$, $f \longmapsto (g \mapsto g(f))$

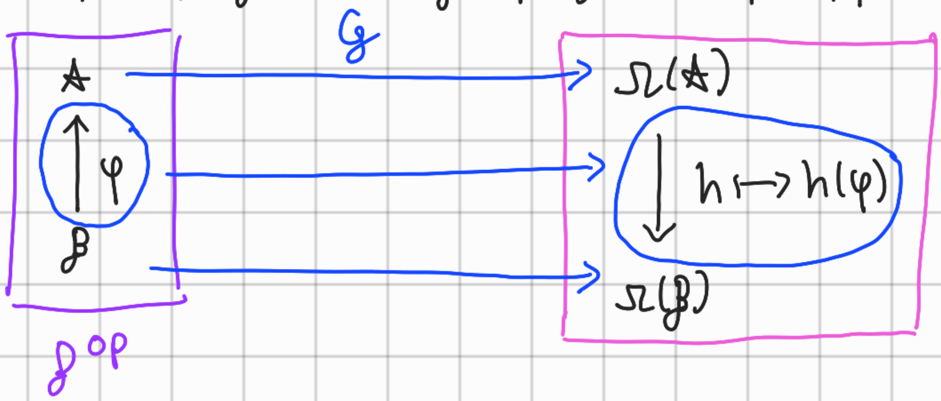


We now define a functor $G: \mathcal{D}^{op} \longrightarrow \mathcal{C}$, $\mathcal{A} \mapsto \Omega(\mathcal{A})$, which maps a unital commutative C^*-algebra to its Gelfand spectrum. Since $\mathcal{A}$ is a Banach algebra (as it is a C^*-algebra), $\Omega(\mathcal{A})$ is a compact Hausdorff space according to the lecture.

G is defined on $Morph(\mathcal{D}^{op})$ as the following:

$\mathcal{B} \to \mathcal{A}$ $\quad$ $\mathcal{A} \to \mathbb{C}$ $\quad$ $\mathcal{B} \to \mathcal{A} \to \mathbb{C}$

For all $\varphi \in Morph(\mathcal{D}^{op})$ we have $G: Morph(\mathcal{D}^{op}) \longrightarrow Morph(\mathcal{C})$, $\varphi \longmapsto (h \mapsto h(\varphi))$



We know from the lecture that $\iota: X \longrightarrow \Omega(C(X))$, $x \mapsto ev_x$ with $ev_x: f \mapsto f(x)$ is a bijection and with the weak*-topology on $\Omega(C(X))$ a homeomorphism of compact Hausdorff spaces. So we have $id_{\mathcal{C}} \cong G \circ F$

According to the Gelfand representation theorem we have a $*$-homomorphism $\Gamma: \mathcal{A} \longrightarrow C(\Omega(\mathcal{A}))$, $a \mapsto \hat{a}$ with $\hat{a}: \varphi \longmapsto \varphi(a)$ and since $\mathcal{A}$ is unital, Γ is an isomorphism.

So we have $id_{\mathcal{D}^{op}} \cong F \circ G$.

Therefore we have a contravariant equivalence of categories.