Exercise Sheet 01
Operator Algebras

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- First, we notice that if p is idempotent we have $(1-p)^2 = 1 - 2p + p^2 = 1 - 2p + p = 1 - p$, so $1-p$ is also idempotent. Now consider the following two functions:

$$\begin{aligned}\varphi :&\mathscr{A} \to p\mathscr{A} \oplus (1-p)\mathscr{A}, a \mapsto pa \oplus (1-p)a \\ \psi :&p\mathscr{A} \oplus (1-p)\mathscr{A} \to \mathscr{A}, pa \oplus (1-p)b \mapsto pa + b - pb\end{aligned}$$

  Concatenation of these two functions yields

$$\begin{aligned}\psi(\varphi(a)) &= \psi(pa \oplus (1-p)a) = pa + a - pa = a \\ \varphi(\psi(pa \oplus (1-p)b)) &= phi(pa + b - pb) = p(pa + b - pb) \oplus (1-p)(pa + b - pb) \\ &= p^2a + pb - p^2b \oplus pa - p^2a + b - pb - pb + p^2b \\ &= pa + pb - pb \oplus pa - pa + b - pb - pb + pb \\ &= pa \oplus (1-p)b\end{aligned}$$

  showing that φ is a bijective mapping. Furthermore, φ we have

$$\begin{aligned}\varphi(a \cdot b) &= pab \oplus (1-p)ab = p^2ab \oplus (1-p)^2ab = (pa)(pb) \oplus ((1-p)a)((1-p)b) \\ &= (pa \oplus (1-p)a)(pb \oplus (1-p)b) = \varphi(a)\varphi(b)\end{aligned}$$

  and φ is a homomorphism. Because $p \oplus (1-p)$ is the unit in $p\mathscr{A} \oplus (1-p)\mathscr{A}$ and $\varphi(1) = p \oplus (1-p)$, φ is also compatible with the unit.

- Consider the functions c_U and c_V, where $c_U|_U \equiv 1$ and $c_U|_V \equiv 0$ and likewise for c_V. These are continuous, idempotent and $c_U = 1 - c_V$ also holds. Note that these are in fact **not** elements of $C_0(X)$ as U, V need not necessarily be compact. However, above we have not used $p \in \mathscr{A}$ except for the fact that p admits a well-defined multiplication with elements of $\mathscr{A}$ yielding only elements of $\mathscr{A}$. Since c_U, c_V are still continuous the multiplication of $C(X)$ can be used here. By the argument above we then have $C_0(X) \simeq c_U C_0(X) \oplus c_V C_0(X) \simeq C_0(U) \oplus C_0(V)$ with

$$\varphi : C_0(X) \to C_0(U) \oplus C_0(V), f \mapsto (f \cdot c_U)|_U \oplus (f \cdot c_V)|_V = f|_U \oplus f|_V$$

  an isomorphism of unital algebras. To show that this is also an isomorphism of C^*-algebras, we also have to show that φ is continuous and commutes with *. For the continuity, consider

$$\|\varphi(f)\| = \max\{\|f|_U\|, \|f|_V\|\} = \max\{\sup_{x \in U} |f(x)|, \sup_{x \in V} |f(x)|\} = \sup_{x \in X} |f(x)| = \|f\|$$

  so φ is in fact even isometric (and thus bounded and continuous). Furthermore, we have

$$\varphi(f)^* = \overline{f|_U \oplus f|_V} = \overline{f|_U} \oplus \overline{f|_V} = \overline{f}|_U \oplus \overline{f}|_V = \varphi(\overline{f})$$

  and φ is an isomorphism of C^*-algebras.

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- Assume U is dense in X and consider $a \in C_0(X)$ with $aI = 0$ (and thus $Ia = 0$ since $C_0(X)$ is commutative). Let $x_0 \in U$ be any point in U. We want to prove that $a(x_0) = 0$. Since U is open, its complement $U^\complement$ is closed. Applying Uryson's Lemma to the compact set $\{x_0\}$ and the closed set $U^\complement$ (these sets are disjunct because of $x_0 \in U$) yields the existence of a function $f : X \to \mathbb{C}$ with $f(x_0) = 1$ and $f|_{U^\complement} \equiv 0$. Since we have $f \in C_0(U) = I$ because of the latter condition and because ideals are strongly closed with respect to multiplication, we have $af \in aI = 0$, so af is the zero function and in particular $a(x_0)f(x_0) = 0 \Rightarrow a(x_0) = 0$. Repeating this chain of reasoning for any $x_0 \in U$ shows that $a|_U \equiv 0$ and since U is dense in X and a is continuous, we have $a \equiv 0$. This shows that I is an essential ideal.

- Proof by contraposition. Let U be non-dense in X, i.e. there exists a point $x_0 \in X$ admitting an open neighborhood $V \subset X \setminus U$. We once again use Uryson's Lemma, this time for the two disjunct sets $\{x_0\}$ (compact) and $V^\complement$ (closed), proving the existence of a function $a : X \to \mathbb{C}$ with $a|_{V^\complement} \equiv 0$ and $a(x_0) = 1$, which is therefore not equivalent to the zero function. However, for any $f \in I$ we have $a \cdot f \equiv 0$ since f is zero on $U^\complement$ and a is zero on $U \subseteq V^\complement$. Therefore, we have $aI = 0$ but $a \neq 0$ and I cannot be an essential ideal of $C_0(X)$.

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