Exercises to Introduction to Operator Algebras

Alcides Buss
Notes by: Linus Mußmächer
2336440

Summer 2023

Contents

1 Topological Basics

Let X be a topological space, that is there exists a subset $\mathcal{O}(X) \in \mathbb{P}(X)$.

Definition 1.1 *X is **Hausdorff** if for all $x, y \in X$ there exist open sets $U, V \in \mathcal{O}(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.*

*X is **locally Hausdorff** if for all $x \in X$ there exists an open neighborhood $U \in \mathcal{O}(X)$ such that U is Hausdorff with the relative topology from X.*

Example 1.2 (Snake with two heads) We consider the space $[0,1] \cup \{1^+\}$ equipped with a topology such that both the subspace $[0,1]$ and $[0,1] \setminus \{1\} \cup \{1^+\}$ are isomorphic to $[0,1]$. Then X is compact, locally Hausdorff but not Hausdorff.

Definition 1.3 *X is compact if for every open cover $(U_i)_{i \in I}$ there exists a finite open subcover.*

X is locally compact if for every $x \in X$ there exists a neighborhood basis of x consisting of open relatively compact subsets of X, that is for every open neighborhood U of x there exists and open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Observation: For a locally Hausdorff X, X is locally compact if and only if for all $x \in X$ there exists an open neighborhood U of x such that $\overline{U}$ is compact.

1.1 Results about locally compact Hausdorff spaces

Let X be Hausdorff and locally compact.

Proposition 1.4 (Uryson's Lemma) *For all closed $F \subset X$ and all compact $K \subseteq X$ with $F \cap K = \emptyset$, there is a continuous function $f : X \to [0,1]$ such that $f|_K \equiv 1$ and $f|_F \equiv 0$.*

Proposition 1.5 (Tietze's extension theorem) *For all $K \subseteq X$ compact and $f : K \to \mathbb{C}$ continuous, there exists and $\tilde{f} : X \to \mathbb{C}$ continuous such that $\tilde{f}|_K \equiv f$.*

Proposition 1.6 (Alexandroff's compactification theorem) *$\tilde{X} = X \cup \{\infty\}$ ($\infty \notin K$) is a compact Hausdorff space with $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{K^{\complement} \cup \{\infty\} \mid K \subseteq X \, compact\}$.*

For example, compactifying $\mathbb{R}$ yields the unit circle $\mathbb{S}^1$.

Proposition 1.7 *Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$ the space $X = Y \setminus \{y_0\}$ is a locally compact Hausdorff space.*

Proposition 1.8 *More generally, if Y is a locally compact Hausdorff space and $Z \subseteq Y$ is a difference of open and closed subsets of Y (i.e. $Z = U \setminus F$ or $Z = F \setminus U$ where $U \subseteq Y$ is open and $F \subseteq Y$ is closed) then Z is locally compact.*

Exercise 1.1 Let X be a locally compact Hausdorff space. The following are equivalent:

(1) X is compact.

(2) $C(X) = C_0(X)$ $(= C_b(X))$.

(3) $C_0(X)$ is unital.

(4) $1 \in C_0(X)$ where $1(x) = 1 \in \mathbb{C}$ for all $x \in X$.

PROOF:

- (1) $\Rightarrow$ (2): Recall:

$$C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$$

  If X is compact, then every closed subset of X is compact, so all sets of form $\{x \in X \mid |f(x)| \geq \varepsilon\}$ are compact, and we have $C(X) = C_0(X)$.

- (2) $\Rightarrow$ (3): This is trivial because $C(X)$ is always unital.

- (3) $\Rightarrow$ (4): Suppose $C_0(X)$ is unital and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, that is $f(x)g(x) = 1$ for all $x \in X, g \in C_0(X)$. By Uryson's Lemma, given $x_0 \in X$, there exists a $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$, take any precompact open neighborhood U of x and look at $F := U^{\complement} \subseteq X$). Then we have $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. As this is possible for every $x_0 \in X$, we have $f \equiv 1$.

- (4) $\Rightarrow$ (1): Suppose $f = 1 \in C_0(X)$. Then choosing $\varepsilon = \frac{1}{2}$ shows that $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$ is compact. $\square$

Exercise 1.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$.

2 Exercise sheet 1

Exercise 2.1 (1.1)

Proof: **Case 1**: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1\alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathscr{A} \to \mathscr{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda}a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathscr{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to $\mathbb{C}$. $\square$

Exercise 2.2 (1.2) We consider pathological examples for $C_0(X)$.

Let $X = \overline{\{x_0\}}$, e.g. $x_0 \in X$ with $\mathscr{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f : X \to \mathbb{C}$, this implies $f(x_0) \to f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.$\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \not\equiv 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 2.1 *Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a *-homomorphism between C^*-algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathscr{A}$.*

Exercise 2.3 (1.4 - Products) Let $(A_i)_{i\in I}$ be a family of C^*-algebras and define

$$\prod_{i\in I} A_i = \{a = (a_i)_{i\in I} \mid a_i \in A_i \forall_{i\in I} \text{ and } \|a\| := \sup_{i\in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^*-axiom.

Remark 2.2 (Differences between product and direct sum)
In addition to the product space, we define

$$\bigoplus_{i\in I} A_i = \left\{(a_i) \in \prod_{i\in I} A_i \mid \forall_{\varepsilon>0} \exists_{\text{finite } F\subseteq I} \forall_{i\notin F} \|a_i\| < \varepsilon\right\}.$$

This is a closed subspace of $\prod_{i\in I} A_i$ as the closure of $\bigoplus_{i\in I}^{alg} A_i$, where

$$\bigoplus_{i\in I}^{alg} A_i = \left\{(a_i) \in \prod_{i\in I} A_i \mid \exists_{\text{finite } F\subseteq I} \forall_{i\notin F} \|a_i\| = 0\right\}.$$

For finite I, these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) *-homomorphisms $\pi_j : \prod_{i\in I} A_i \to A_j$ for all $j \in I$. If B is any C^*-algebra with *-homomorphisms $\varphi_j \to A_j$ for every $j \in I$, there is a unique *-homomorphism $\varphi : B \to \prod_{i\in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} B \arrow[r, "\varphi_j"] \arrow[d, "\varphi"'] & A_j \\ A \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Exercise 2.4 (1.5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a *-isomorphism.

Exercise 2.5 (2.6) Let X be a locally compact Hausdorff space and $\widetilde{C_0(X)} \simeq C(\tilde{X})$ with $\tilde{X} := X \sqcup \{\infty\}$ with the topology $\mathscr{O}_{\tilde{X}} = \mathscr{O}_X \cup \{\tilde{X} \setminus K \mid K \subseteq X \text{ kompakt}\}$.

Observation: If X is already compact, then ∞ is an isolated point of $\tilde{X}$ (i.e. $\{\infty\}$ is clopen).

If $\mathscr{A}$ is a C^*-algebra, then $\tilde{\mathscr{A}}$ (this is not the same ˜ as on the X!) is a C^*-algebra with

$$\|a + \lambda 1\|_{C^*} := \sup_{b\in\mathscr{A}, \|b\|\leq 1} \|ab + \lambda b\|_{\mathscr{A}}$$

We check that $\tilde{\mathscr{A}}$ is a C^*-algebra.

- C^*-axiom: $\|a + \lambda 1\|_{C^*}^2 ! = \|(a + \lambda 1)^*(a + \lambda 1)\|_{C^*}$. We have

$$\|a + \lambda \cdot 1\|_{C^*} = \|(a^*a + \overline{\lambda}a + \lambda a^*) + |\lambda|^2 \cdot 1\|_{C^*}$$

$$= \sup_{b \in \mathscr{A}, \|b\| \leq 1} \|a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 \cdot b\|_{\mathscr{A}}$$

On the other hand:

$$\begin{aligned}
\|a + \lambda \cdot 1\|_{C^*}^2 &\coloneqq \sup_{\|b\| \leq 1} \|ab + \lambda b\|_{\mathscr{A}}^2 \\
&= \sup_{\|b\| \leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\|_{\mathscr{A}} \\
&= \sup_{\|b\| \leq 1} \|b^*a^*ab + \overline{\lambda}b^*ab + \lambda b^*a^*b + |\lambda|^2 b^*b\|_{\mathscr{A}} \\
&\leq \sup_{\|b\| \leq 1} \|b^*\|_{\mathscr{A}} \cdot \|a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathscr{A}} \\
&\leq \sup_{\|b\| \leq 1} \|a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathscr{A}} \\
&= \|a + \lambda \cdot 1\|_{C^*}^2
\end{aligned}$$

- The other conditions are easy to check and are left for the student.

We still want to prove $\varphi : \widetilde{C_0(X)} \to C(\tilde{X}), f + \lambda \cdot 1 \mapsto f_\lambda$ with $f_\lambda(x) \coloneqq \lambda$ for $x = \infty$ and $f_\lambda(x) = f(x) + \lambda$ otherwise. Nothe that once again these are not the same $\tilde{}$.

- f is well-defined: We have to check that f_λ is continuous in $\tilde{X}$. Take any sequence $X \ni x_i \to \infty$ in $\tilde{X}$. We have to show $f_\lambda(x_i) \to f_\lambda(\infty) = \lambda$. Since $f_\lambda(x_i) = f(x_i) + \lambda$ this is equivalent to $f(x_i) \to 0$. But as $f \in C_0(X)$, we have that for every $\varepsilon > 0$ the set $K_\varepsilon(f) = \{x \mid |f(x)| \geq \varepsilon\}$ is compact. Since x_i will eventually leave this compact set (or it would not diverge to ∞), we know that $f(x_i)$ eventually becomes smaller than (any) ε. So we have $f(x_i) \to 0$ and thus $f_\lambda(x_i) \to f_\lambda(\infty)$. So f_λ is continuous in ∞. The continuity on every other point follows immediately from the continuity of f.
- φ is a *-isomorphism:
  - Linearity: φ is clearly linear as we can check component-wise:

    $$(f_1 + f_2)_\lambda = (f_1)_\lambda + (f_2)_\lambda$$

  - Homomorphism: For every $x \in X$ we have

    $$\begin{aligned}
    \varphi((f + \lambda \cdot 1) \cdot (g + \lambda' \cdot 1))(x) &= \varphi((fg + \lambda' f + \lambda g) + \lambda\lambda' \cdot 1)(x) \\
    &= (fg + \lambda' f + \lambda g)(x) + \lambda\lambda' \\
    &= (fg)(x) + \lambda' f(x) + \lambda g(x) + \lambda\lambda' \\
    &= (f(x) + \lambda) \cdot (g(x) + \lambda') \\
    &= (\varphi(f + \lambda) \cdot \varphi(g + \lambda'))(x).
    \end{aligned}$$

    In the case of $x = \infty$, this equality of course also holds. Thus we have $\varphi((f + \lambda)(g + \lambda')) = \varphi(f + \lambda)\varphi(g + \lambda')$.
  - *-homomorphism:

    $$\varphi(f + \lambda)^*(x) = \varphi(f^* + \overline{\lambda} \cdot 1)(x)$$

    For $x \in X$ this follows by $\overline{f(x)} + \overline{\lambda} = f^*(x) + \overline{\lambda}$, for $x = \infty$ we have $\overline{\lambda} = \overline{\lambda}$.

- Injective: $f_\lambda(0)$ leads to $f_\lambda(x) = 0$ for all $x \in \tilde{X}$, since if $x = \infty$ then λ must be 0 and $f(x) = 0$ for all $x \in X$. Thus $f = 0$ and $\lambda = 0$.
- Surjective: Take $g \in C(\tilde{X})$ and choose $\lambda = g(\infty)$ and $f(x) \coloneqq g(x) - \lambda$. and check $f \in C_0(X)$.

- We can also prove that φ is isometric for the C^*-norm:

$$\|f + \lambda \cdot 1\| \coloneqq \sup_{g \in C_0(X), \|g\| \leq 1} \|fg + \lambda g\|_\infty$$

Look at

$$\begin{aligned}\|\varphi(f + \lambda 1)\| &= \sup_{x \in \tilde{X}} |f_\lambda(x)| = \max\{|\lambda|, \sup_{x \in X} |f(x) + \lambda|\} \\ &\overset{(*)}{=} \sup_{x \in X} |f(x) + \lambda|\end{aligned}$$

and

$$\begin{aligned}\|f + \lambda \cdot 1\|_{C^*} &= \sup_{\|g\| \leq 1} \|fg + \lambda g\|_{C_0(X)} \\ &= \sup_{|g(x)| \leq 1 \forall_x} \sup_{x \in X} \|f(x)g(x) + \lambda g(x)\| \\ &= \sup_{x \in X} |f(x) + \lambda| \\ &\overset{(**)}{=} \sup_{x \in X} |f(x) + \lambda|\end{aligned}$$

This proof may need to be divided into two cases:

- X is not compact: We can find a net $(x_i) \subseteq X$ with $f(x_i) \to 0$ and $(*)$ follows and use a $g(x) \approx 1$ for $(**)$.
- X is compact: Choose $g \equiv 1$ for $(**)$ and think about $(*)$ later.

Exercise 2.6 (1.8) It is difficult to prove $I^* = I$. The idea is to prove $I = C_0(U)$ where $C_0(U) = \{f \in C_0(X) \mid f|_{U^\complement} \equiv 0\}$.

One can also prove $C_0(X)/C_0(U) \simeq C_0(F)$ (as C_0 of the subspace) where $F = U^\complement$.

Exercise 2.7 Prove that $\mathscr{A}/\mathsf{I}$ is normed algebra, and

(i) if $\mathscr{A}$ is Banach and $I \trianglelefteq A$ is closed, then $\mathscr{A}/\mathsf{I}$ is Banach.

(ii) if $\mathscr{A}$ is unital and Banach, then $\mathscr{A}/\mathsf{I}$ is unital.

unital if $\mathscr{A}$ is, Banach if $\mathscr{A}$ is and $\mathsf{I} \trianglelefteq \mathscr{A}$ closed.

Proof: Consider $\mathscr{A}/\mathsf{I}$ with $(a + \mathsf{I})(b + \mathsf{I}) = ab + \mathsf{I}$. For the norm, use $\|a + \mathsf{I}\| = \operatorname{dist}(a, \mathsf{I}) = \inf_{x \in \mathsf{I}} \|a - x\|$. This is submultiplicative. For every $\varepsilon > 0$, there exist $x, y \in \mathsf{I}$ for which we have

$$(\varepsilon + \|a + \mathsf{I}\|) \cdot (\varepsilon + \|b + \mathsf{I}\|) \geq \|a + x\| \cdot \|b + y\| \geq \|(a + x)(b + y)\| \geq \|ab + \underbrace{ay + xb + xy}_{\in \mathsf{I}}\| \geq \|ab + \mathsf{I}\|$$

and taking the limit yields the desired result.

Result (i) follows from functional analysis, that a space is Banach if and only if the convergence of $\sum_{k=0}^{\infty} a_n$ is equivalent to the convergence of $\sum_{k=0}^{\infty} \|a_n\|$.

Now let $\mathscr{A}$ also be unital, then $\mathscr{A}/\mathsf{I}$ is unital. If $\mathsf{I} = \mathscr{A}$, the algebra is the zero-algebra. Thus, let I be a proper ideal. The fact that $1 = 1_{\mathscr{A}} + \mathsf{I}$ is a unit is clear, but we need to prove $\|1_\mathsf{A} + \mathsf{I}\| = 1$. Observe that, if $x \in \mathsf{I} \triangleleft \mathscr{A}$ then $x \notin \mathrm{inv}(\mathscr{A})$ and $\|1_{\mathscr{A}} + x\| \geq 1$. Because otherwise, we have $\|1_{\mathscr{A}} + x\| < 1$ and then (because $\mathscr{A}$ is Banach) $x = a - 1_{\mathscr{A}} \in \mathrm{inv}(\mathscr{A})$. Hence $\|1_{\mathscr{A}}\| = \inf_{x \in \mathsf{I}} \|1_{\mathscr{A}} + x\| \geq 1$. In addition, we have $1 \leq \|1_{\mathscr{A}} + \mathsf{I}\| = \inf_{x\in\mathsf{I}} \|1_{\mathscr{A}} - x\| \leq \|1_{\mathscr{A}} + 0\| \leq 1$. This proves $\|1\| = \|1_{\mathscr{A}} + \mathsf{I}\| = 1$. □

In the following, $\mathbb{D}$ is the **closed** unit circle.

Exercise 2.8 Consider $\chi \in \Omega(\mathscr{A})$. We have proved $\|\chi\| \leq 1$. It may happen that $\|\chi\| < 1$. We need a non-unital algebra for this, because we have $\|\chi\| = 1$ if $1 \in \mathscr{A}$.

Consider $S = (\mathbb{N}, +)$ as an additive semigroup. Then

$$\ell^1(S) = \{(a_n)_{n\in\mathbb{N}} \mid \sum_{n=0}^{\infty} |a_n| < \infty\}$$

is a unital Banach algebrea with $\delta_n \cdot \delta_m = \delta_{n+m}$ for all $n, m \in \mathbb{N}$ where

$$\delta_n(m) = \begin{cases} 1 & m = n \\ 0 & m \neq 0 \end{cases}$$

Observe $\ell^1(S) = \overline{\mathrm{alg}}\{\delta_0, \delta_1\}$ because of $\delta_1^n = \delta_n$. The unit of the algebra is δ_0. What are the characters of $\ell^1(S)$?

We can write any $a \in \ell^1(S)$ as $a = \sum_{n=0}^{\infty} a_n\delta_n$. So if $\chi \in \Omega(\ell^1(S))$ then

$$\chi(a) = \sum_{n=0}^{\infty} a_n \chi(\delta_n) \in \mathbb{C}.$$

In particular, $\chi(1) = 1$ so $\chi(\delta_0) = 1$. This leads to $\chi(\delta_n) = \chi(\delta_1^n) = \chi(\delta_1)^n = \chi(\delta_1)^n$. So if we set $z := \chi(\delta_1) \in \mathbb{C}$, we have $\chi(a) = \sum_{n=0}^{\infty} a_n z^n$. Observe $|z| = |\chi(\delta_1)| \leq \|\delta_1\| = 1$ (because the Image of a character is a subset of the spectrum, which is bounded by the norm) so z must be in $\mathbb{D}$. By conventioning $z^0 = 1$ for every $z \in \mathbb{C}$, we can even choose $z = 0$.

Conversely, if $z \in \mathbb{D}$, we define $\chi_z(a) := \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}$. Then $\chi_z(\delta_n) = z^n$ and

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n \cdot z^m = \chi_z(\delta_n) \cdot \chi_z(\delta_m)$$

So we get a map $\mathbb{D} \to \Omega(\ell^1(S)) \subseteq \ell^1(S)^*, z \mapsto \chi_z$ that is bijective and continuous. If $z_i \to z$ in $\mathbb{D}$, we need to prove $\chi_{z_i} \to \chi_z$ in respect to the weak *-topology. So we need to evaluate and prove $\chi_{z_i}(a) \to \chi_z(a)$, or $\sum_{n=0}^{\infty} a_n z_i^n \to \sum_{n=0}^{\infty} a_m z^m$. Partial sums would obviously converge, so χ_{z_i} converges on a dense subspace of $\ell^1(S)$. The uniform boundedness principle (if a bounded set of operators converge on a dense subset $T_i \to T$, $\sup_i \|T_i\| < \infty$, they converge everywhere) shows that the infinite sums also converge. In general, showing that an operator converges on a dense set of an algebra always shows the convergence on any point of the algebra.

Observe $\sigma(\delta_1) = \{\chi(\delta_1) \mid \chi \in \Omega(\ell^1(S))\} = \mathbb{D}$ and $\sigma(\delta_1) = \mathbb{D}$ as well.

Concerning the norm, we know that

$$|\chi_z(a)| = \left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \sum_{n=0}^{\infty} |a_n||z|^n \leq \sum_{n=0}^{\infty} |a_n| = \|a\|$$

for all $a \in \ell^1(S)$, so $\|\chi_z\| \leq 1$. For $a = (a_0, 0, 0, \dots)$ we have $|\chi_z(a)| = |a_0| = \|a\|$, so $\|\chi_z\| = 1$ for any $z \in \mathbb{C}$ (and thus for any $\chi = \chi_z \in \Omega(\ell^1(S))$).

Remark 2.3 (Gelfand-Representation) In general, we seek a mapping $\mathscr{A} \to C_0(X), a \mapsto \hat{a}$, taking $X = \hat{\mathscr{A}} = \Omega(\mathscr{A})$ and $\hat{a}(\chi) = \chi(a)$.

If we apply the Gelfand representation here, we have

$$\ell^1(S) \to C(\mathbb{D}), a \mapsto \hat{a} \text{ where } \hat{a}(z) = \chi_z(a) = \sum_{n=0}^{\infty} a_n z^n$$

Example 2.4 (Norms < 1) Consider

$$\ell_0^1(S) = \overline{\mathrm{alg}}(\delta_1) = \left\{ \sum_{n=1}^{\infty} a_n \delta_n \mid a_n \in \mathbb{C} \right\} \lhd \ell^1(S)$$

Observe $\widetilde{\ell_0^1(S)} \simeq \ell^1(S)$. Recall $\Omega(\tilde{\mathscr{A}} = \Omega(\mathscr{A}) \sqcup \{\chi_\infty\}$. So we are looking for our χ_∞, which is $\chi_\infty(a_0, a_1, \dots) = a_0$ – that is χ_0 and corresponds to $z = 0$ in the unit circle. It follows $\Omega(\ell_0^1(S)) \simeq \mathbb{D} \setminus \{0\}$ and $\chi_0 \in \Omega(\ell^1(S)) \setminus \Omega(\ell_0^1(S))$.

We compute $\|\chi_z\| = \sup_{\|a\|_1 \le 1} |\chi_z(a)|$. Consider:

$$|\chi_z(a)| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| z \left(\sum_{n=1}^{\infty} a_n z^{n-1} \right) \right| \le |z| \cdot \|a\|_1$$

so because of $\chi_z(\delta_1) = z$, we have $\|\chi_z\| = |z|$, which can be smaller than 1.

Remark 2.5 Do we have $\ell^1(S) \hookrightarrow A(\mathbb{D}), a \mapsto \hat{a}$ where $\hat{a}(z) \coloneqq \sum_{n=0}^{\infty} a_n z^n$?

Exercise 2.9 (02-03) Is $\mathscr{A}(\mathbb{D})$ a C^*-algebra? Consider $f(z) = \exp(iz)$, $f \in \mathscr{A}$ and notice $z^* = z$. But we have $\|f^* f\|_\infty \neq \|f\|_\infty^2$, because $f^* f = 1$ and because $f(-i) = e$, we have $\|f\|_\infty \ge e$ and $\|f\|_\infty^2 \ge e^2 > 1 = \|f^* f\|_\infty$. Since the *-property is not fulfilled.

Remark 2.6 Talk about functoriality. If X, Y are compact Hausdorff spaces and $f : X \to Y$ is continuous then

$$f_* : C(Y) \to C(X), g \mapsto g \circ f$$

You can check that f_* is a unital *-homomorphism. So we receive a functor from the compact spaces to the unital commutative C^*-algebras:

$$\text{Comp. Spaces} \to \text{unital abelian } C^*, X \mapsto C(X)$$
$$Hom(X, Y) \to Hom(C(Y), C(X)), f \mapsto f_*$$

This is a contravariant function because for $f : X \to Y, g : Y \to Z$ we have $(g \circ f)_* = f_* \circ g_*$. It is also natural. If $\varphi : C(Y) \to C(X)$ is a unital *-homomorphism, we get a continuous map $f : X \to Y$ by duality.

3 More multiplier algebra

We continue to look at the multiplier algebra.

$$M(\mathscr{A}) = \{\mu = (L, R) \in L(\mathscr{A}) \times L(\mathscr{A}) \mid aL(b) = R(a)b, L(ab) = L(a)b, R(ab) = aR(b)\}$$

If $\mathscr{A}$ is a C^*-algebra, $\mathscr{A}$ embeds into $M(\mathscr{A})$ as an essential ideal. If A embeds into a C^*-algebra B as an essential ideal, then $B \to M(\mathscr{A}), b \to \mu_b$ with $(a \mapsto ba, a \mapsto ab)$ is an isomorphism.

We also define the **strict topology** on $M(\mathscr{A})$ as the smallest topology that makes the map $a \mapsto \mu \cdot a, a \mapsto \mu \cdot a$ norm-continuous on $\mathscr{A} \to \mathscr{A}$. So if $(\mu_i) \subseteq M(\mathscr{A})$ is a net, then $\mu_i \to \mu \in M(\mathscr{A})$ if and only if $\mu_i a \to \mu a$ and $a \mu_i \to a\mu$ for all $a \in \mathscr{A}$.

Remark 3.1 Writing $0 \leq a \leq 1$ in a C^*-algbra means $a \geq 0$, so $\sigma(a) \subseteq [0, \infty)$ and $a \leq 1$ means $(1 - a)$ is positive in $\tilde{\mathscr{A}}$ or $M(\mathscr{A})$ which is equivalent to $\|a\| \leq 1$.

All of this is equivalent to $\sigma(a) \subseteq [0, 1]$.

Relation with approximate units: If $(e_i) \subseteq A_{+,1}$ is an increasing net $(0 \leq e_i \leq 1)$ then (e_i) is an approximate unit iff $e_i \to 1$ (strictly) in $M(\mathscr{A})$.

By definition this means $e_i a \xrightarrow{\|\cdot\|} a$, $a \cdot e_i \xrightarrow{\|\cdot\|} a$.

3.1 Non-degeneratre *-homomorphisms

Definition 3.2 *Let $\pi : \mathscr{A} \to M(\mathscr{B})$ a *-homomorphism. We say that π is* ***non-degenerate*** *if* $\overline{\operatorname{span} \pi(\mathscr{A}) \cdot B} = B$.

Lemma 3.3 *Let $\pi : \mathscr{A} \to M(\mathscr{B})$ be a *-homomorphism. The following are equivalent:*

(i) π is non-degenerate.

(ii) $\pi(e_i) \to 1$ (strictly) in $M(B)$ if (e_i) is some approximate unit in $\mathscr{A}$.

(iii) π extends to a strictly continuous unital *-homomorphism $\tilde{\pi} : M(\mathscr{A}) \to M(\mathscr{B})$.

Proof:

(i) $\Rightarrow$ (ii): Let (e_i) be an approximate unit. Prove $\pi(e_i) \to 1$ (strictly) in $M(\mathscr{B})$, that is $\pi(e_i)b \to b$ for all $b \in \mathscr{B}$. Since (e_i) is bounded, it is enough to show that $\pi(e_i)b \to b$ for all $b \in \pi(A)B$ as this is dense in B.B ut if $b = \pi(a)c$ for $a \in \mathscr{A}, c \in \mathscr{B}$, then $\pi(e_i)b = \pi(e_i)\pi(a)c = \pi(e_i a)c \to \pi(a)c = b$ because π is norm-continuous and $e_i a \to a$.

(ii) $\Rightarrow$ (iii): We want to extend π to $\tilde{\pi} : M(A) \to M(B)$. We need $\tilde{\pi}$ such that

$$\tilde{\pi}(\mu)\pi(a) = \tilde{\pi}(\mu)\tilde{\pi}(a)\tilde{\pi}(\mu \cdot a) = \pi(\mu \cdot a).$$

Therefore, define the multiplier $\tilde{\pi}(\mu)$ just on $\pi(\mathscr{A})\mathscr{B}$ by the mappings

$$L(\pi(a)b) = \tilde{\pi}(\mu) \cdot (\pi(a)b) = \pi(\mu a) \cdot b \in \mathscr{B}$$
$$R(b\pi(a)) = (b\pi(a)) \cdot \tilde{\pi}(\mu) = b\pi(a\mu) \in \mathscr{B}.$$

These morphisms are certainly linear. By (ii), notice that $\overline{\pi(\mathscr{A})\mathscr{B}} = \mathscr{B}$. So the above defines morphism on all of $\mathscr{B}$ by continuous extension.

We need to prove: L, R are well-defined and extend to $\mathscr{B}$ and $\mu = (L, R)$ is a multiplier of $\mathscr{B}$.

Claim:

$$\|\sum_{i=0}^{n} \pi(\mu \cdot a_i) \cdot b_i\| \leq \|\mu\| \cdot \|\sum_{i=0}^{n} \pi(a_i)b_i\|$$

for all sequences $(a_i) \subseteq \mathscr{A}$ and $(b_i) \subseteq \mathscr{B}$. To prove that L is well-defined compute $\tilde{\pi}\pi(a)b = \pi(\mu a)b$.

$$\pi(\mu a)b = \lim_i \pi(\mu e_i a)b = \lim_i \pi(\mu e_i)\pi(a)b$$

So L is well-defined, because we can write $L(\pi(a)b) = \lim_i \pi(\mu e_i)(\pi(a)b)$ as directly dependent on $\pi(a)b$ (so different representations of $\pi(a)b$ will yield the same result). Analogously, we have $R(b\pi(a)) = b\pi(a) \lim_i \pi(e_i \mu)$. Now we can compute

$$\|\tilde{\pi}(\mu)\pi(a)b\|^2 = \|b^*\pi(\mu a)^*\pi(\mu a)b\| = \|b^*\pi(a^*\mu^*\mu a)b\| \leq \|b^*\pi(\|\mu\|^2 a^* a)b\|$$

$$= \|\mu\|^2 \|b^* \pi(a)^* \pi(a) b\| = \|\mu\|^2 \|\pi(a)b\|^*$$

Now we show that $\tilde{\pi}$ is actually a multiplier, let $x = b_1\pi(a_1)$ and $y = \pi(a_2)b_2$ and consider

$$xL(y) = b_1\pi(a_1)\pi(\mu a_2)b_2 = b_1\pi(a_1\mu a_2)b_2 = b_1\pi(a_1\mu)\pi(a_2)b_2 = R(x)y$$

This proves the last multiplier property, the others follow.

You can repeat this for R. So we have a well-defined map $\tilde{\pi} : M(\mathscr{A}) \to M(\mathscr{B})$. We need to prove that this is a strictly continuous *-homomorphism extending π. Calculate:

$$\tilde{\pi}(1)\pi(a)b = \pi(1 \cdot a)b = \pi(a)b$$

so $\tilde{\pi}(1) = 1$ and

$$\tilde{\pi}(\mu) \cdot \tilde{\pi}(\nu)\pi(a)b = \tilde{(\pi)}(\mu)\pi(\nu a)b = \pi(\mu \cdot \nu \cdot a)b = \tilde{\pi}(\mu \cdot \nu)\pi(a)b$$

so $\tilde{\pi}(\mu) \cdot \tilde{\pi}(\nu) = \tilde{\pi}(\mu \cdot \nu)$. To prove that $\tilde{\pi}$ is strictly continuous, take $\mu_i \to \mu$ (strictly) and show

$$\tilde{\pi}(\mu_i)\pi(a)b = \pi(\mu_i a)b \xrightarrow{\|\cdot\|} \pi(\mu a)b = \tilde{\pi}(\mu)\pi(a)b$$

Now use the Cohen-Hilbert Factorization theorem:

> If E is a Banach right (or left) module over some Banach algebra $\mathscr{A}$ and $\mathscr{A}$ has an approximate unit $(e_i) \subseteq A_1$, $\|e_i\| \leq 1$, then $\overline{\operatorname{span} E \cdot \mathscr{A}} = E \cdot \mathscr{A} = \{x \cdot a \mid x \in E, a \in \mathscr{A}$.

Apply this for $E = \mathscr{B}$ (view it as a left $\mathscr{A}$-module via π): $a \cdot b := \pi(a)b$. This shows $\overline{\operatorname{span}}\pi(\mathscr{A})\mathscr{B} = \pi(\mathscr{A})\mathscr{B}$.

(iii) $\Rightarrow$ (i): Suppose $\mathscr{A} \to M(\mathscr{B})$ extends to $\tilde{\pi} : M(\mathscr{A}) \to M(\mathscr{B})$ (strictly continuous * homomorphism).

(iv) But then $\tilde{\pi}(1) = 1$, so

$$1 = \tilde{\pi}(1) = \tilde{\pi}(\operatorname{strict} \lim_i e_i) = \operatorname{strict} \lim_i \pi(e_i)$$

so $\pi(e_i)$ strictly converges to 1. $\square$

Exercise 3.1 (03-01) For any C^*-algebra $\mathscr{A}$ and $a \in \mathscr{A}$ we have $a(a^*a)^{\frac{1}{n}} \to a$ (and $(aa^*)^{\frac{1}{n}} a \to a$).

Proof: If $\mathscr{A} = C_0(X)$ then $a = f$ and we have $f(x)|f(x)|^{\frac{2}{n}} \to f(x)$ as $|f(x)|^{\frac{2}{n}} \to 1$ on the non-zero points of f and $\to 0$ otherwise. On a compact set, this then converges uniformly.

In general: Compute

$$\begin{aligned}
\|a(a^*a)^{\frac{1}{n}} - a\|^2 &= \|(a(a^*a)^{\frac{1}{n}})^* \cdot (a(a^*a)^{\frac{1}{n}} - a)\| \\
&= \|(a^* - (a^*a)^{\frac{1}{n}} a^*) \cdot (a^*a)^{\frac{1}{n}} - a\| \\
&= \|a^*a(a^*a)^{\frac{1}{n}} - a^*a - (a^*a)^{\frac{1}{n}} a^*a(a^*a)^{\frac{1}{n}} + (a^*a)^{\frac{1}{n}} a^*a\| \\
&= \|bb^{\frac{1}{n}} - b - b^{\frac{1}{n}} bb^{\frac{1}{n}} + b^{\frac{1}{n}} b\|
\end{aligned}$$

with $b = a^*a$ self-adjoint. We may then replace $\mathscr{A}$ by $C^*(b)$, which is commutative. The above case then applies by considering b as a function via the Gelfand-Representation. $\square$

Exercise 3.2 (04-10) $C_0(X)$ is σ-unital (there exists a countable approximate unit) if and only if X is σ-compact (X is the union of countably many compact sets).

Example 3.4 Take ω as the first uncountable ordinal. Then $X = [0, \omega)$ is not σ-compact.

If X is metrizable and locally compact, there exists a metric $d : X \times X \to \mathbb{R}$ such that the topology on X is induced by d.

If X is separable, then X is σ-compact.

Proof: $\Rightarrow$: Assume $C_0(X)$ is σ-unital. Then we have a countable approximate unit $(e_n)_{n\in\mathbb{N}}$. Define $f = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \in C_0(X)$ as $0 \leq e_n \leq 1$. Then f is positive ($f \geq 0$) and bounded ($f \leq 1$). f is even strictly positive ($f > 0$), that is $f(x) > 0$ for all $x \in X$ because $e_n(x) \to 1$ uniformly on compacts and $f \geq \frac{e_n}{2^n}$ for all n.

Then, defining $K_n := \{x \in X \mid f(x) \geq \frac{1}{n}\} \subseteq X$ is compact. Notice $\bigcup_{n=1}^{\infty} K_n = X$ and $K_n \subseteq K_{n+1}$.

$\Leftarrow$: Assume $X = \bigcup_{n=1}^{\infty} K_n$ where $K_n \subseteq X$ are compact. We may assume $K_n \subseteq K_{n+1}$. From σ-compactness we may find an open, countable cover of $X = \bigcup_{n=1}^{\infty} U_n$ such that U_n is open and $\overline{U_n}$ is compact for every $n \in \mathbb{N}$. This cover can be found by covering every K_n with (finitely many) open sets and taking the union. Now we use Uryson's Lemma to find a sequence $(f_n)_{n\in\mathbb{N}}$ where $0 \leq f \leq 1$ and $f_n|_{\overline{U}_n} \equiv 1$ and $f_n \in C_0(X)$.

Take now a compact $K \subseteq X = \bigcup_{n=1}^{\infty} U_n$. As K is compact, there exists an n, m with $K \subseteq U_n \subseteq U_m$ and $f_m|_K \equiv 1$ and $m \geq n$. This shows $f_n(x) \to 1$ on compacts.

Observe: (f_n) may not be increasing. To get this, choose two covers $(U_n), (V_n)$ of X of open sets with compact closures where $U_n \subseteq \overline{U_n} \subseteq V_n \subseteq \overline{V_n} \subseteq U_{n+1}$. Now choose by Uryson $(f_n)_{n=1}^{\infty}$ with $f_n|_{\overline{U_n}} \equiv 1, f_n|_{V_n^\complement} \equiv 0$. Now f_n is increasing. □

Remark 3.5 The following are equivalent:

(i) The C^*-algebra $\mathscr{A}$ is σ-unital (there exists a countable approximate unit $(e_n)_{n\in\mathbb{N}}$).

(ii) There exist $a \in \mathscr{A}_{+,1}$ (write: $a > 0$), meaning that $\varphi(a) > 0$ for all $\varphi \in \mathscr{A}_+^*$ (positive linear functions) with $\|\varphi\| = 1$.

(iii) $\overline{\mathscr{A} \cdot a} = \mathscr{A}$.

(iv) $\overline{a\mathscr{A}} = \mathscr{A}$.

Proof: See Diximiere (important guy). □

Exercise 3.3 (04-04) Let $\mathscr{A}$ be a C^*-algebra, $a \in \mathscr{A}$ and $L_a := \overline{\mathscr{A}a}$.

Then L_a is the closed left ideal generated by a, and if a is a projection (self-adjoint and idempotent) we even have $L_a = \mathscr{A}a$.

Furthermore, $L_a = L_{aa^*}$.

Lastly, if $\mathscr{A}$ is unital and $aa^* = 1$ for some a and $L_a = \mathscr{A}$. Then a is unitary.

Proof: We prove this is the closed left ideal generated by a, that is, the smallest closed ideal containing a.

First, observer that $L_a = \overline{\mathscr{A}a}$ is a closed left ideal, because $\mathscr{A} \cdot a = \{xa \mid x \in \mathscr{A}\}$ is a left ideal. We also have $a \in L_a$, because we can use the approximate unit for x and that L_a is closed. Now if $L \subseteq \mathscr{A}$ is a closed left ideal with $a \in L$, we must have $\overline{\mathscr{A}a} \subseteq L$. This concludes the proof.

To prove that if a is a projection (idempotent would be enough), $\mathscr{A}a$ is closed, take a sequence $x_n a \to b \in \mathscr{A}$ and therefore $x_n a = x_n a^2 \to ba$. So $b = ba \in \mathscr{A}a$. Thus, we could write $\mathscr{A}a = \{xa \mid x \in \mathscr{A}\} = \{x \mid x \in \mathscr{A}, xa = x\}$.

To $L_a = L_{aa^*}$, i.e. $\overline{\mathscr{A}a} = \overline{\mathscr{A}a^*a}$. Using exercise 03-01, we see $a = \lim_n a(a^*a)^{\frac{1}{n}} \in \overline{\mathscr{A}a^*a}$ (because $a^*a \in L_{a^*a}$ and therefore $(a^*a)^{\frac{1}{n}} \in L_{a^*a}$ because $(a^*a)^{\frac{1}{n}}$ is approximated by a sequence of polynomials and L_{a^*a} is closed, see Stone-Weierstraß), so $L_a \subseteq L_{a^*a}$. The other inclusion is obvious, as $a^*a \in L_a$.

For the last point, we need to prove $a^*a = 1$. We have $L_a = \overline{\mathscr{A}a} = \overline{\mathscr{A}a^*a} \overset{!}{=} \mathscr{A}$, i.e. $\mathscr{A}a$ must be dense in $\mathscr{A}$. But then the claim is already $\mathscr{A}a = \mathscr{A}$ as $\mathscr{A}$ contains a unit 1 that is also in $\mathscr{A}a$. Otherwise, $1 \notin \mathscr{A}a$ and also $\mathscr{A}a$ does not contain any invertible elements. But the set of invertibles is non-trivially open in $\mathscr{A}$, a contradiction to the denseness.

So we have $\mathscr{A}a = \mathscr{A}$, meaning there exists $b \in \mathscr{A}$ with $ba = 1$, so a is left invertible. But by the assumption, there is also a right inverse. So a is invertible with inverse a^*, so it is unitary.□

3.2 Somethings about irreducible representations

Let $\pi : \mathscr{A} \to B(H)$ be a representation of $\mathscr{A}$. π is irreducible if for any closed, $\pi(\mathscr{A})$-invariant subspace (that is $\pi(\mathscr{A})K \subseteq K$) we have $K = 0$ or $K = H$ already.

Lemma 3.6 (Schur) *π is irreducible iff $\pi(\mathscr{A})' := \{T \in B(H) \mid T\pi(a) = \pi(a)T \ \forall_{a\in\mathscr{A}}\}$, a strongly closed C^*-algebra, is trivial (equal to $\mathbb{C} \cdot 1$).*

Proof: Requires things about von-Neumann-algebras. □

Exercise 3.4 (04-07) $\mathscr{A}$ is primitive if there exists a faithful and irreducible representation $\pi : \mathscr{A} \to \mathcal{B}(H)$.

Recall that $p \trianglelefteq \mathscr{A}$ is primitive if $p = \ker\rho$ for some $[\rho] \in \hat{\mathscr{A}}$.

So $\mathscr{A}$ is primitive if $\{0\}$ is a primitive ideal.

Example 3.7 Take $\mathscr{A} = K(H)$ (compact operators on some Hilbert space H). This is primitive as the embedding/inclusion in $\mathcal{B}(H)$ is already the representation and $K(H) = \mathbb{C} \cdot 1$, as commuting with the compact operator is the same as commuting with the finite rank operators, or (for some $T \in K(H)'$)

$$\begin{aligned} &T \cdot |\zeta\rangle\langle\eta| = |\zeta\rangle\langle\eta| \cdot T \\ &|T(\zeta)\rangle\langle\eta| = |\zeta\rangle\langle T^*(\eta)| \\ &T(\zeta)\langle\eta|J\rangle = \zeta\langle T^*(\eta)|J\rangle \end{aligned}$$

for all $\zeta, \eta, J \in H$. But then $T \in \mathbb{C} \cdot 1$.

Proof (for 04-07): If $\mathscr{A}$ is a primitive C^*-algebra, then $ZM(\mathscr{A}) = \mathbb{C} \cdot 1$ (the center of the multiplier $\mathscr{A}$).

> In general, we have the Dawns-Hoffman-Theorem: $ZM(A) \simeq C_b(\mathrm{Prim}(\mathscr{A}))$ where $\mathrm{Prim}(\mathscr{A}) = \{p \trianglelefteq \mathscr{A} \mid p \text{ primitive}\}$.
>
> We want to use that if $\mathscr{A}$ is primitive, then $\mathrm{Prim}(\mathscr{A}) = \{\{0\}\}$. This is however not universally true.
>
> Example: Look at $\mathscr{A} = \widetilde{K(H)} \subseteq \mathcal{B}(H)$, $\dim H = \infty$ and thus $\mathrm{Prim}(\mathscr{A}) = \{\{0\} \leq K(H)\}$.

Take $\pi : \mathscr{A} \to \mathcal{B}(H)$ a faithful irreducible representation (assume $\mathscr{A} \neq 0$), in particular π is non-degenerate ($\operatorname{span}\pi(\mathscr{A})H = H$). So π extends (strictly continuous) to a unital representation $\tilde{\pi} : M(\mathscr{A}) \to \mathcal{B}(H)$. For this representation we have $\tilde{\pi}(\mu)\pi(a)\zeta = \pi(\mu a)\zeta$ so $\tilde{\pi}$ is faithful. Now, since π is irreducible by Schur we have $\pi(\mathscr{A})' = \mathscr{C} \cdot 1$. Next we have $\tilde{\pi}(M(\mathscr{A}))' = \pi(\mathscr{A})'$. The inclusion $\subseteq$ is trivial. For $\supseteq$ consider that $T \in \pi(\mathscr{A})'$ implies $T\pi(a) = \pi(a)T$ for all $a \in \mathscr{A}$. Take now $\mu \in M(\mathscr{A})$. Then for all $a \in \mathscr{A}$:

$$\tilde{\pi}(\mu)T\pi(a) = \tilde{\pi}(\mu)\pi(a)T = \pi(\mu \cdot a)T = T\pi(\mu a) = T\tilde{\pi}(\mu)\pi(a)$$

so $\tilde{\pi}(\mu)T = T\tilde{\pi}(\mu)$. Therefore, $\tilde{\pi}(M(A))' = \pi(\mathscr{A})' = \mathbb{C}\cdot 1$ is also trivial. Now notice $\tilde{\pi}(ZM(\mathscr{A})) \subseteq Z(\tilde{\pi}(M(\mathscr{A}))) \subseteq \tilde{\pi}(M(A))' = \mathbb{C} \cdot 1$. Since $\tilde{\pi}$ is faithful and unital, we have $ZM(\mathscr{A}) = \mathbb{C} \cdot 1$ (as the image has only 1 dimension, the preimage must also only have 1 dimension). $\square$

Somewhere along the way we also solved exercise 04-06.

Example 3.8 Take $\mathscr{A} \supseteq \widetilde{\mathcal{K}(H)} = \mathcal{K}(H) + \mathbb{C} \cdot 1 \subseteq \mathcal{B}(H)$. Then $\dim H = \infty$. This is primitive but not isomorphic to any $\mathcal{K}(G)$ for another Hilbert space G.

Remark 3.9 How to turn everything into a C^*-algebra.

Take a Banach-*-algebra $\mathscr{A}$. There is a (universal) C^*-algebra $C_u^*(\mathscr{A})$ such that $\iota : \mathscr{A} \to C_u(\mathscr{A})$ has dense image and for any $\varphi : \mathscr{A} \to \mathscr{B}$ there exists $\tilde{\varphi} : C_u^*(\mathscr{A}) \to \mathscr{B}$ with $\varphi = \tilde{\varphi} \circ \iota$.

Definition of $C_u^*(\mathscr{A})$: If $\varphi : \mathscr{A} \to \mathscr{B}$ is a *-homomorphism, φ is contractive. Then $\|a\|_\varphi := \|\varphi(a)\|_{\mathscr{B}} \leq \|a\|_{\mathscr{A}}$ is a C^*-seminorm. Defining

$$\|a\|_{C_u^*} := \sup\{\|a\|_\varphi \mid \varphi : \mathscr{A} \to \mathscr{B}\} \leq \|a\|_{\mathscr{A}}$$

yields the biggest possible C^*-seminorm on $\mathscr{A}$. Then $\overline{\mathscr{A}/N_{\|\cdot\|_{C_u^*}}} =: C_u^*(\mathscr{A})$ is a C^*-algebra as $\|\cdot\|_{C_u^*}$ becomes a norm.

Example 3.10 For $\mathscr{A} = A(\mathbb{D})$, C_u^* is commutative and unital (as $A(\mathbb{D})$ is commutative and unital), so we have $C_u^*(\mathscr{A}) \simeq C(X)$ for the spectrum $X = \Omega(C_u^*(A(\mathbb{D})))$. Recall that $\Omega(A(\mathbb{D})) \simeq \mathbb{D}$ via the evaluation functions. Observe also that $A(\mathbb{D}) \subseteq C(\mathbb{D})$. Then

$$\begin{aligned}\Omega(C_u^*(A(\mathbb{D}))) \simeq \Omega_{sym}(A(\mathbb{D})) &:= \{\chi \in \Omega(A(\mathbb{D})) \mid \chi(A(\mathbb{D})_{sa}) \subseteq \mathbb{R}\} \\ &= \{\chi \in \Omega(A(\mathbb{D})) \mid \chi \ ^*\text{-homomorphism}\}\end{aligned}$$

What characters (evaluations) on $A(\mathbb{D})$ fulfil this? We see that $\mathrm{ev}_\lambda(f^*) = \overline{f(\lambda)}$, so $\overline{f(\lambda)} = f^*(\lambda) = f(\overline{\lambda})$ for all f (especially $f = \mathrm{id}$) only for $\lambda \in [-1, 1]$. Therefore, we have $C_u^*(A(\mathbb{D})) \simeq C([-1, 1])$.

Example 3.11 Let S be an inverse semigroup. Then $S \hookrightarrow \mathbb{C}[S] = \left\{\sum_{s\in S}^{fin} a_s\delta_s \mid a_s \in \mathbb{C}\right\}$. This is a *-algebra.

Remark 3.12 We know that for any *-algebra $\mathscr{A}$, $C^*(\mathscr{A})$ is a C^*-algebra with the C^*-norm, so $\overline{\mathscr{A}}^{\|\cdot\|_*}$ is a C^*-algebra. But also $\pi : \mathscr{A} \hookrightarrow \mathcal{B}(H)$ with $\|a\|_\pi = \|\pi(a)\|_{\mathcal{B}(H)}$. We could now take $\|a\|_u = \sup\{\|a\| \mid \|\cdot\| \text{ is a } C^*\text{-(semi)-norm on } \mathscr{A}\}$.

Example 3.13 Take $(\mathbb{N}, +)$ as a *-semigroup with the trivial involution $n^* = n$. Then $\mathbb{C}[\mathbb{N}] = \left\{\sum_{n\in\mathbb{N}}^{f} ina_n\delta_n \mid a_n \in \mathbb{C}\right\}$. Recognize $\delta_n \cdot \delta_m = \delta_{n+m}$ and $\delta_n^* = \delta_m$. This is then isomorphic $\mathbb{C}[\mathbb{N}] \simeq \mathbb{C}[X]$ (identify δ_n with X^n).

We are interested in the representation $\pi : \mathbb{C}[X] \to \mathcal{B}(H)$. For $T = \pi(x) \in \mathcal{B}(H)$, T self adjoint as X is, we have $\pi(\sum a_n X^n) = \sum a_n T^n$. Thus, $\mathrm{Rep}(\mathbb{C}[X])$ equals the set of self-adjoint operators.

Given any $T = T^* \in \mathcal{B}(H)$ and $a = \sum a_n X^n \in \mathbb{C}[X]$, it follows that $\|a\|_T := \|\pi_T(a)\|_{\mathcal{B}(H)} = \|\sum a_n T^n\|_{\mathcal{B}(H)}$ is a C^*-seminorm.

When is $\|\cdot\|_T$ a C^*-norm? We have $\overline{\mathbb{C}[X]/N}^{\|\cdot\|_T} \simeq C^*(T) \subseteq \mathcal{B}(H)$.

We also have $\mathbb{C}[X] \hookrightarrow C^*(T,1) \simeq C(\sigma(T))$. Which T should we choose, is there a canonical or largest choice? The embedding of $\mathbb{C}[X]$ in $C^*(T,1)$ can only be injective if $|\sigma(\lambda)| = \infty$, as $p \hookrightarrow p(T) \simeq (\lambda \mapsto p(\lambda))$.

There is no 'largest' choice! Take any T, then $\|a\|_T \leq r(T)$. So if $r \in \mathbb{R}$ (= self-adjoint operator on $\mathbb{C}$) then $\chi_r(X) = r$ and r specifies a character of $\mathbb{C}[X]$: $\chi_r(\sum a_n X^n) = \sum a_n r^n$. Using this to define a C^*-seminorm $\|a\|_r = \|\chi_r(a)\|$ and $\|x\|_r = |r| \in \mathbb{R}_+$, which is not bounded.

Therefore, taking the supremum of all C^*-(semi)-norms does not always yield results. There is no universal C^*-algebra generated by a self-adjoint element! The problem is that there is no $C^*_{\text{univ}}(\mathscr{A})$ out of any given *-algebra. But for a large class of *-algebras, there is such a universal C^*-algebra. Specifically, this is the case if $\mathscr{A}$ as a *-algebra is generated by **partial isometries**, i.e. there exists $S \subseteq \mathscr{A}$ where $\mathscr{A}$ is generated by S as a *-algebra and $ss^*s = s$ for all $s \in S$.

If $\pi : \mathscr{A} \to \mathcal{B}(H)$ is a representation, then $T = \pi(s) \in \mathcal{B}(H)$ are partial isometries for all $s \in S$. But then $\|T\| \leq 1$ as TT^* and T^*T are projections.

Consequence: If S is an inverse semigroup, then there exists $C^*(S) = C^*_{\text{univ}}(S) = C^*_{\text{univ}}(\mathbb{C}[S])$.

The *universal property* is: Given any C^*-algebra $\mathscr{B}$ and any *-homomorphism $\pi : S \to \mathscr{B}$, there is an extension $\tilde{\pi} : C^*(S) \to \mathscr{B}, \delta_s \mapsto \pi(s)$.

Group C^*-algebras: If G is a Group there is $C^*(G) = C^*_{\text{univ}}(\mathbb{C}[G])$. It has the *universal property* that for all Hilbert spaces H and all unitary representations $G \xrightarrow{u} \mathcal{U}(H) := \mathcal{U}(\mathcal{B}(H))$, u extends to $\tilde{u} : C^*(G) \to \mathcal{U}(H)$.

Example 3.14 Take $G = (\mathbb{Z}, +) = \langle 1 \rangle$. For representations $\mathbb{Z} \to \mathcal{B}(H)$, choose any unitary element $\pi(1) = u \in \mathcal{B}(H)$. Observe now $C^*(\mathbb{Z}) = \overline{\mathbb{C}[\mathbb{Z}]} = \overline{\left\{\sum_{n\in\mathscr{Z}}^{fin} a_n z^n\right\}}$ is a commutative algebra and thus $C^*(\mathbb{Z}) = C(X)$ and

$$\begin{aligned} X = C^*\hat{(}\mathbb{Z}) = \Omega(C^*(\mathbb{Z})) &= \{\chi : C^*(\mathbb{Z}) \to \mathbb{C} \mid \text{non-zero homomorphism}\} \\ &\simeq \{\chi : \mathbb{Z} \to \mathbb{S}^1 = \mathcal{U}(\mathbb{C}) \mid \text{ group homomorphisms}\} \simeq \mathbb{S}^1 \end{aligned}$$

as all these mappings are just $n \mapsto z^n$ for an arbitrarily chosen $z \in \mathbb{C}$. Therefore, $C^*(\mathbb{Z}) \simeq C(\mathbb{S}^1)$.

Take $\hat{\mathscr{A}} = \{[\pi] \mid \pi \text{ immediate representation of } \mathscr{A}\}$ and $\text{Prim}(\mathscr{A}) = \{\ker(\pi) \mid \pi \in \hat{\mathscr{A}}\}$. These are topological spaces by taking the ideals of $\mathscr{A}$ as open sets.

Exercise 3.5 (05-01) $\mathscr{A} \to \tilde{\mathscr{A}}$. Let $\varphi \in S(\mathscr{A})$, then there exists a unique $\tilde{\varphi} \in S(\tilde{\mathscr{A}})$, $\tilde{\varphi}(a+\lambda 1) := \varphi(a) + \lambda$. So $S(\tilde{A}) \simeq \{\tilde{\varphi} \mid \varphi \in S(\mathscr{A})\} \sqcup \{\varphi_\infty\}$ where $\varphi_\infty(a + \lambda 1) = \lambda$.

"$\subseteq$": Take $\psi \in S(\tilde{\mathscr{A}})$ and set $\varphi = \psi|_{\mathscr{A}}$ so $\psi(a + \lambda 1) = \varphi(a) + \lambda$. We have $\varphi \geq 0$ and if $\varphi \equiv 0$, then we would have $\psi = \varphi_\infty$. If φ is not 0, then $\|\varphi\| \leq \|\psi\| = 1$ as $\varphi = \psi|_{\mathscr{A}}$. If now $(e_i) \subseteq \mathscr{A}$ is an approximate unit, and since $\varphi \geq 0$, then $\|\varphi\| = \lim_i \varphi(e_i) = \sup_i \varphi(e_i)$. So $\|\varphi\| = \psi(1)$ and

$$\begin{aligned} |\varphi(e_i(a+\lambda 1))|^2 &\leq \overbrace{\varphi(e_i^2)}^{\to\|\varphi\|} \psi((a+\lambda 1)^*(a+\lambda 1)) \text{ (Cauchy-Schwarz)} \\ |\varphi(e_i(a+\lambda 1))| &= |\varphi(e_i a + \lambda e_i)| = |\varphi(e_i a) + \lambda\varphi(e_i)| \to |\varphi(a) + \lambda\|\varphi\|| \end{aligned}$$

There must exist a $\tilde{\varphi} : \tilde{\mathscr{A}} \to \mathbb{C}$ given by $\tilde{\varphi}(a + \lambda 1) = \varphi(a) + \|\varphi\| \cdot \lambda$.

We know $\psi(1) = 1$ and $\|\psi\| \geq 1$, but is $\|\psi\| \leq 1$? We have $|\psi(a)| = |\varphi(a) + \lambda| \leq \|a\| + |\lambda|$, but this is too much. Let $(e_i) \subseteq \mathscr{A}$ be an approximate unit. Then $|\psi(a + \lambda 1)| = |\varphi(a) + \lambda|$. Can we use $\lim \varphi(e_i) = \|\varphi\| \leq 1$?

Consider the pure states:

$$PS(\tilde{\mathscr{A}}) = \{\tilde{\varphi} \mid \varphi \in PS(\mathscr{A})\} \sqcup \{\varphi_\infty\}$$

"$\supseteq$": φ_∞ is a character and characters are always pure states. Also: $\varphi in PS(\mathscr{A})$ yields $\tilde{\varphi} \in PS(\mathscr{A})$ as $\tilde{\varphi}(a + \lambda 1) = \varphi(a) + \lambda$. Suppose $\tilde{\varphi} = \alpha w_1 + (1-\alpha) w_2$ where $w_1, w_2 \in S(\tilde{\mathscr{A}})$, so $\varphi = \alpha w_1|_{\mathscr{A}} + (1-\alpha) w_2|_{\mathscr{A}}$.

Let now $(e_i) \subseteq \mathscr{A}$ be an approximate unit. Then

$$\varphi(e_i) = \alpha \omega_1(e_i) + (1-\alpha)\omega_2(e_i) \Leftrightarrow 1 = \alpha\|\omega_1\| + (1-\alpha)\|\omega_2\|$$

so $\|\omega_1\| + \|\omega_2\| = 1$ and therefore $\omega_1|_{\mathscr{A}}, \omega_2|_{\mathscr{A}} \in S(\mathscr{A})$. Since φ is pure we have $\omega_1|_{\mathscr{A}} = \omega_2|_{\mathscr{A}} = \varphi|_{\mathscr{A}}$ and thus $\tilde{\varphi} = \omega_1 = \omega_2$. This shows $\tilde{\varphi} \in PS(\mathscr{A})$.

"$\subseteq$": Now if $\psi \in PS(\mathscr{A})$, then $\varphi \coloneqq \psi|_{\mathscr{A}} \in \mathscr{A}^*_+$. If $\varphi \equiv 0$ then $\psi = \varphi_\infty$. If $\varphi \neq 0$ we still need to show $\|\varphi\| = 1$.