# Introduction to Operator Algebras

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Summer 2023

# Contents

The set of all linear bounded operators $\mathscr{L}(H) = \mathscr{B}(H)$ on a given Banach space $H$ is a (Banach) algebra with $S \cdot T = S \circ T$. $M \subseteq \mathscr{L}$ is a Subalgebra such that $M^* \subseteq M$ where $T^*$ is the adjoint of $T$. This is also a closed subspace with respect to the strong topology. This is equivalent to $M = M''$ (when $X \subseteq \mathscr{B}(H), X' = \{T \in \mathscr{B}(H) \mid TS = ST \ \forall_{S \in X}\}$)

## Some topological basics

**Definition 0.1**

- *Topology, Open*
- *Hausdorff, locally Hausdorff*
- *compact*

**Definition 0.2** *A topological space $X$ is **locally Hausdorff** if every $x \in X$ admits a compact neighborhood basis, that is for every $x \in X$ and every open set $U \ni x$ there exists an open set $V \ni x$ with $\overline{V}$ is compact.*

**Corollary 0.3** *If a set $V$ is compact in any subset $U \subseteq X$, it is also compact in $X$.*

**Example 0.4 (Snake with two heads)** Consider $I = [0, 1]$ with the standard topology and extend the set with an element $1^+$ such that $I \cup 1^+ \setminus 1$ is isomorphic to $I$. Then $I \cup 1^+$ is locally Hausdorff and compact, but not Hausdorff.

## Some results about locally compact Hausdorff spaces

**Lemma 0.5 (Uryson's Lemma)** *Let $X$ be locally compact and Hausdorff. For all $F \subseteq X$ closed and $K \subseteq X$ compact with $F \cap K = \emptyset$, there exists an $f : X \to [0, 1]$ continuous such that $f|_K \equiv 1$ and $f|_F \equiv 0$.*

**Theorem 0.6 (Tietze's extension theorem)** *Let $X$ be locally compact, $K \subseteq X$ compact and $f : K \to \mathbb{C}$ continuous. Then there exists a continuous $\tilde{f} : X \to \mathbb{C}$ such that $\tilde{f}|_K = f$.*

**Theorem 0.7 (Alexandroff's conpactification)** *If $X$ is locally compact and Hausdorff, then $\tilde{X} \sqcup \{\infty\}$ is a compact Hausdorff space $\mathscr{O}(\tilde{X}) = \mathscr{O}(X) \cup \{K^{\complement} \cup \{\infty\} \mid K \text{ compact}\}$.*

**Example 0.8** Compacting the real line $\mathbb{R}$ yields the space $\tilde{\mathbb{R}}$, which is isomorphic to the unit circle $\Pi = \mathbb{S}^1$.

**Theorem 0.9** *Conversely, if $Y$ is a compact Hausdorff space, then for all $y_0 \in Y$, $X := Y \setminus \{y_0\}$ is locally compact (in respect to the subspace topology).*

*More generally, if $Y$ is locally compact and Hausdorff, and $Z \subseteq Y$ is a difference of open and closed subsets, of $Y$ (i.e. $Z = U \setminus F$, where $U$ is open in $Y$ and $F$ is closed in $Y$), then $Z$ is locally compact.*

# 1 Algebras

**Definition 1.1** *An **algebra** is a (complex) vector space $\mathscr{A}$ endowed with a bilinear and associative multiplication: $\mathscr{A} \times \mathscr{A} \to \mathscr{A}, (a,b) \mapsto a \cdot b$. So*

*(i)* $(a + \alpha b) \cdot (c + \beta d) = ac + \alpha bc + \beta ad + \alpha\beta bd$

*(ii)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

*for all $a, b, c \in \mathscr{A}$ and $\alpha, \beta \in \mathbb{C}$. We say that $\mathscr{A}$ is*

*(i)* ***commutative**, if $ab = ba$ for all $a, b \in \mathscr{A}$ and*

*(ii)* ***unital**, if there exists $1 = 1_{\mathscr{A}} \in \mathscr{A}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathscr{A}$.*

**Example 1.2**

(i) $\mathbb{C}$, or more generally $\mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, is an algebra.

(ii) Say $X$ is any set; let $\mathbb{C}^X = \{f : X \to \mathbb{C}\}$ with point wise multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$. These are commutative unital algebras (with $1(x) = 1 \in \mathbb{C}$).

(iii) Consider the polynomials $\mathbb{C}[X] = \{\sum_{i=0}^n \lambda_i x^i \mid \lambda_i \in \mathbb{C}, n \in \mathbb{N}\}$ with the usual operations. This is a commutative unital algebra.

(iv) Let $X$ be a topological space and $C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continouus}\} \subseteq \mathbb{C}^X$ the set of continuous functions on $X$. This is a commutative unital (sub)algebra (of $\mathbb{C}^X$).

(v) Take any vector space $A$ define a (trivial) multiplication $a \cdot b := 0$. This is a commutative Algebra (that is not unital unless $A = 0$).

(vi) $M_n(\mathbb{C})$ (the complex $n \times n$ matrices) with the usual multiplication are a non-commutative (unless $n = 1$) unital algebra.

(vii) Let $V$ be any (complex) vector space. The set of all linear operators $L(V) := \{T : V \to VT \text{ linear operator}\}$ is a unital (non-commutative for $\dim V > 1$). We observe $\mathscr{L}(\mathbb{C}^n) \simeq M_n(\mathbb{C})$.

(viii) Let $S$ be a semigroup (i.e. a set with an associative operation $S \times S \to S$, e.g. $(\mathbb{N}, +)$). Then $\mathbb{C}[S] = \{\sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{C}, |\{s : \lambda_s \neq 0\}| < \infty\}$ (the finite formal sums of elements of $S$) with the following product

$$\left(\sum_{s \in S'} \lambda_s s\right) \cdot \left(\sum_{t \in S} \lambda'_t t\right) := \sum_{s,t \in S} (\lambda_s \cdot \lambda'_t)(s \cdot t) \in S$$

Observe: As a vector space: $\mathbb{C}[S] \subseteq \mathbb{C}^S$. In general, this is neither commutative nor unital.

# 2 Normed algebras

**Definition 2.1** *An algebra $\mathscr{A}$ is **normed**, if it is endowed with a (vector space) norm $\|\cdot\|: \mathscr{A} \to [0, \infty)$ satisfying $\|a \cdot b\| \leq \|a\| \cdot \|b\|$. If $\mathscr{A}$ is unital with unit $1_{\mathscr{A}}$, we usually assume $\|1_{\mathscr{A}}\| = 1$ except for $\mathscr{A} = 0$.*

**Definition 2.2** *A **Banach algebra** is a normed algebra that is also complete (as a metric space with respect to the distance $d(a, b) := \|a - b\|$), i.e. every Cauchy sequence converges.*

**Example 2.3** (i) If $X$ is a compact space then $C(X)$ is a commutative unital Banach algebra with respect to the norm $\|f\|_\infty := \sup_{x \in X} |f(x)| < \infty$ (since $X$ is compact).

(ii) If $V$ is a normed (respectively Banach) vector space, e.g. $\mathbb{C}^n$ or $\ell^p(\mathbb{N})$, then $\mathscr{L}(V) = \{T \in L(V) \mid T$ is bounded/continouus$\}$ with $\|T\| := \sup_{\|v\| \leq 1} \|T(v)\| < \infty$ is a normed Banach algebra.

(iii) If $X$ is a topological space, then $C_b(X) = \{f \in C(X) \mid \|f\|_\infty < \infty\}$ (bounded continuous functions) is a Banach space.

(iv) Let $X$ again be a topological space. Then the set of all functions **vanishing at** $\infty$,

$$\begin{aligned} C_0(X) &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \exists_{K \subseteq X, K \text{ compact}} \forall_{x \notin K} |f(x)| < \varepsilon\} \\ &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\} \subseteq C_b(X), \end{aligned}$$

is also a Banach algebra.

**Exercise 2.1** Assume $X$ is locally compact and Hausdorff. Prove the following are equivalent:

(1) $X$ is compact.

(2) $C(X) = C_0(X)$

(3) $C_0(X)$ is unital.

(4) The unit function $1 \in C_b(X)$ belongs to $C_0(X)$.

PROOF:
- (1) $\Rightarrow$ (2): Recall the definition of $C_0(X)$. If $X$ is compact, every closed subset (especially every $\{x : |f(x)| \geq \varepsilon\}$) is compact, so the condition of $C_0(X)$ is trivial.
- (2) $\Rightarrow$ (3): Since $C(X)$ is unital, $C_0(X)$ is as well.
- (3) $\Rightarrow$ (4): Suppose $C_0$ is unital, and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, i.e. $f(x)g(x) = g(x) \forall_{x \in X} \forall_{g \in C_0(X)}$. By Uryson's lemma, given any $x_0 \in X$, there exists $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$ and taking $F$ as the complement of any relatively compact environment of $x_0$.). Then $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. Doing this for every $x_0 \in X$ yields $f \equiv 1$.
- (4) $\Rightarrow$ (1): Since $1 \in C_0(X)$, for every $\varepsilon > 0$ the set $\{x \mid |f(x)| \geq \varepsilon\}$ is compact. Choose $\varepsilon = \frac{1}{2}$. Then, $\{x \mid |f(x)| = |1| \geq \frac{1}{2}\} = X$ is compact. $\square$

**Exercise 2.2** Let $X$ be a locally compact Hausdorff space. Prove that $C_0(X) \cong \{f \in C(X) \mid f(\infty) = 0\}$

# 3 Algebras

**Definition 3.1** *A* $^*$***-algebra*** *is a complex algebra* $\mathscr{A}$ *with an* ***involution*** $^* : \mathscr{A} \to \mathscr{A}$ *satisfying*

*(i)* $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$

*(ii)* $(a^*)^* = a$

*(iii)* $(ab)^* = b^* a^*$

*for all* $a, b \in \mathscr{A}$ *and all* $\lambda \in \mathbb{C}$.

**Definition 3.2** *A* ***normed*** $^*$***-algebra*** *is a normed algebra* $\mathscr{A}$ *with an involution (such that* $\mathscr{A}$ *is a* $^*$*-algebra) also satisfying* $\|a^*\| = \|a\|$ *for all* $a \in \mathscr{A}$.
*A* ***Banach-***$^*$***-algebra*** *is a complete normed* $^*$*-algebra.*

**Definition 3.3** *A* $C^*$*-algebra is a Banach-*$^*$*-algebra satisfying* $\|a^* \cdot a\| = \|a\|^2$.

Observation: Recall that $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ in all normed algebras. Applying this to a $C^*$-algebra we get $\|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$. If $\mathscr{A}$ is a $C^*$-algebra, then $\|a\|^2 = \|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$, so $\|a\| = \|a^*\|$.

**Example 3.4**

(i) If $X$ is a set, then $\mathbb{C}^X$ is a $^*$-algebra with $f^* = \overline{f}$ and $\mathscr{C}^\infty(X)$ is a $C^*$-algebra.

(ii) If $X$ is a topological space, then $C(X) \subseteq \mathbb{C}^X$ is also a $^*$-subalgebra and for $\{f \in C(X) \mid \operatorname{supp}(f) = \overline{\{x \in X \mid |f(x)| \neq 0\}} \text{ compact}\}$ we have

$$C_c(X) =\subseteq C_0(X) \subseteq C_b(X) \subseteq C(X) \subseteq C^\infty(X)$$

and $C^\infty$ is a $C^*$-algebra. $C_c$ is a $^*$-algebra, but not Banach in general.

If $X$ is compact, it follows $C_c(X) = C_0(X) = C_b(X)$.

Observation: If $X$ is locally compact and Hausdorff, then $\overline{C_c(X)} = C_0(X)$.

(iii) Let $X$ be a measured space ($X$ is endowed with a $\sigma$-algebra). Then $B_\infty(X) = \{f \in C^\infty \mid f \text{ is measurable}\}$ is a $C^*$-algebra. If $\mu$ is a measure on $X$ (e.g. $X = \mathbb{R}^n$ and $\mu$ the Lebesgue measure) then $L^\infty(X, \mu)$ are the essentially bounded functions and

$$L^\infty(X) = \{f : X \to \mathbb{C} \mid \|f\| := \inf\{c \geq 0 \mid \mu(\{x \mid |f(x)| > c\}) = 0\}\}$$

is also a $C^*$-algebra.

Observation: $L^2(X, \mu) =$ "$\mu$-separable function", $L^\infty(X, \mu) \xrightarrow{\mu} B(L^2(X, \mu)), f \mapsto \mu_f = \{g \mapsto f \cdot g\}$

(iv) A non-example: Let $\mathbb{D}$ be the unit disk and $\mathscr{A}(\mathrm{d}) = \{f \in \mathbb{C}(\mathbb{D}) \mid \text{ analytic in } \mathbb{D}^\circ\}$

**Morera's Theorem** from complex analysis states that $f \in C(\mathbb{D})$ is analytic if and only if $\int_\gamma f(z)dz = 0$ for all closed and piece wise smooth paths in $\mathbb{D}^\circ$. From this, it follows that $\mathscr{A}(\mathbb{D})$ is closed in $C(\mathbb{D})$, therefore a Banach algebra. It is also a Banach-$^*$-algebra with, but $f^* = \overline{f}$ (point wise) is not possible, as $z \mapsto \overline{z}$ is not analytic. Thus, we have to choose $f^*(z) = f(\overline{z})$. But $\mathscr{A}(\mathbb{D})$ is not a $C^*$-algebra, as $\|f^* f\|_\infty \neq \|f\|_\infty^2$ for some $f \in \mathscr{A}(\mathbb{D})$.

(v) A non-commutative example: Let $H$ be a Hilbert space and $B(H) = \mathscr{L}(H) = \{T : H \to H \mid T\text{bounded, continuous, linear}\}$ and $\|H\| := \sup_{\|z\|<1} \|T(z)\| < \infty$. This is a $C^*$-algebra where $T^*$ is the adjoint of $T$, that is $\langle T^*z, w\rangle = \langle z, Tw\rangle$ for all $z, w \in H$.

$C^*$-axiom: $\|T^* \cdot T\| \le \|T\|^2$ since $\mathscr{L}(H)$ is a Banach algebra, and we also have

$$\begin{aligned}\|T\|^2 &= \sup_{\|z\|<1} \|T(z)\|^2 = \sup_{\|z\|<1} \langle Tz, Tz\rangle = \sup_{\|z\|<1} \langle z, T^*Tz\rangle \\ &\le \sup_{\|z\|<1} \|z\|\|T^*Tz\| \le \sup_{\|z\|<1} \|z\|\|T^*T\| \le \|T^*T\|\end{aligned}$$

In particular, $M_n(\mathbb{C}) \simeq \mathscr{L}(\mathbb{C}^n)$ is a unital $C^*$-algebra.

(vi) To produce more examples, take any subset $S \subseteq \mathscr{L}(H)$ and take $C^*(S) \subseteq \mathscr{L}(H) = \overline{\text{span}\{S_i \mid S_i \in S \cup S^*, i \le n \in \mathbb{N}\}}$.

**Example 3.5** Let $s \in \mathscr{L}(\ell^2(\mathbb{N}))$. The shift $s$, defined by $s(e_i) = e_{i+1}$ for all $i \in \mathbb{N}$ (where $\{e_i\}$ is the canonical basis of the sequence space), is an isometry, that is $s^* \cdot s = \text{id}$. Since $s \cdot s^* \neq \text{id}$, it is not surjective and not a proper isometry. We define

$$T = C^*(s) = \overline{\text{span}\{s^n(s^*)^m \mid m, n \in \mathbb{N}_0\}} \subseteq \mathscr{L}(\ell^2(\mathbb{N}))$$

as the **Toeplitz algebra**.

**Example 3.6** Let $H$ be a Hilbert space and $S$ the set of all finite rank operators on $H$.

**Example 3.7**

(i) **Commutative**: $C_0(X)$ for a locally Hausdorff space $X$.

(ii) **Non-commutative**: $\mathcal{L}(\mathfrak{H}) = \mathcal{B}(\mathfrak{H})$ for any Hilbert space $\mathfrak{H}$ (with dimension greater 1).

(iii) **More generally**: Take any subset $S \subseteq \mathcal{L}(\mathfrak{H})$ and construct $C^*(S) \subseteq \mathcal{L}(H)$ as

$$\overline{\text{span}}\{S_1, \dots, S_n \mid S_i \in S \cap S^*\}$$

**Example 3.8 (Cuntz algebras)** Take again $\mathfrak{H} = \ell^2\mathbb{N} = \{(\lambda_n)_{n\in\mathbb{N}_0} \mid \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$ where $\langle \lambda, \lambda'\rangle = \sum_{i\in\mathbb{N}_0} \overline{\lambda_i}\lambda_i'$ and which has the orthonormal base $(e_n)_{n\in\mathbb{N}}$ where $(e_n) = (\delta_{in})_{i\in\mathbb{N}_0}$.
On this algebra, define

- $S_1(e_n) = e_{2n}$.
- $S_2(e_n) = e_{2n+1}$.

We have partitioned the natural numbers into evens and odds. This defines two (proper) isometries $S_1, S_2 \in \mathscr{L}(\mathfrak{H})$, that is $S_i^*S_i = \text{id}_{\mathfrak{H}}$, to subspaces of $\mathfrak{H}$. Notice: $S_i^*S_j = 0$ for $i \neq j$ as well as $S_1S_1^* + S_2S_2^* = \text{id}_{\mathfrak{H}}$. Define $\mathcal{O}_2 = C^*(S_1, S_2) = \overline{\text{span}}\{S_\alpha S_\beta^* \mid \alpha, \beta \text{ finite words in } \{1, 2\}\}$. For example, for $\alpha = 121211$ we have $S_\alpha = S_1S_2S_1S_2S_1^2$. $\mathcal{O}_2$ is called the **Cuntz algebra**. More generally, one can define $\mathcal{O}_3, \mathcal{O}_4$, ... Cuntz algebras. Joachim Cuntz proved that these are simple $C^*$-algebras with additional interesting properties we will see later.

**Example 3.9 (Rotation algebras)** Let $\mathfrak{H} = \ell^2(\mathbb{Z})$ (bi-infinite sequences) with basis $(e_n)_{n\in\mathbb{Z}}$ Define:

- $U(e_n) := e_{n+1}$ (bilateral shift)

- $V(e_n) := \lambda^n e_n$ where $\lambda \in \mathbb{C}$ is some fixed number $|\lambda| = 1$.

This defines two *unitary* operators: $UU^* = 1 = U^*U$ and $V^*V = 1 = V^*V$. If $\exp(2\pi i\theta), \theta \in \mathbb{R}$ define $A_\theta := C^*(U, V) \subseteq \mathscr{L}(\ell^2\mathbb{N})$.

There is a special relation between $U$ and $V$ where $UV = \lambda VU = \exp(2\pi i\theta)VU$. From this relation, we can describe $A_\theta = \overline{\operatorname{span}}\{\sum_{n,m\in\mathbb{Z}}^{\text{finite}} a_{n,m}U^nV^m \mid a_{n,m} \in \mathbb{C}\}$.

Furthermore, if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $A_\theta$ is simple.

**Example 3.10 ($C^*$-algebras of groups)** Let $G$ be a (discrete) group. Look at $\mathfrak{H} = \ell^2(G) = \{(a_g)_{g\in G} \mid \sum_{g\in G} |a_g|^2 < \infty\}$ (Note: This limit will only converge if there are countably (or finitely) many non-zero parts) with ONB $(\delta_g)_{g\in G}$ where $\delta_g(h) = \delta_{gh}$. Define for each $g \in G$ an operator $\lambda_g \in \mathscr{L}(\ell^2G)$ by $\lambda_g(\delta_h) = \delta_{gh}$. Notice that $h \mapsto gh$ is a bijection, and thus $\lambda_g$ is a unitary operator with $\lambda_g^* = \lambda_{g^{-1}}$. We can now define the **reduced $C^*$-algebra** of the group:

$$C_R^*(G) := C_\lambda^*(G) \subseteq \mathscr{L}(\ell^2G) = C^*(\lambda_g \mid g \in G)$$

Here, we have the relation $\lambda_g \cdot \lambda_h = \lambda_{gh}$ and thus $C_R^*(G) = \{\sum a_g\lambda_g \mid a_g \in \mathbb{C}\}$.

In general, take $U : G \to \mathscr{L}(H), g \mapsto U_g$ a **unitary representation of** $G$ with $U_gU_h = U_{gh}$ and $U_1 = \mathrm{id}$ as well as $U_g^{-1} = U_{g^{-1}}$. Then $C_U^*(G) := \{\sum_{g\in G} a_gU_g \mid a_g \in \mathbb{C}\} \subseteq \mathscr{L}(H)$. There exists a **universal unitary representation** $C_{\max}^*(G)$, a full $C^*$-algebra of $G$.

**Remark 3.11**

(i) If $G$ is Abelian, then $C_U^*(G)$ is also abelian (commutative). In particular, $C_\lambda^*$ is abelian. Later, we will prove $C_\lambda^*(G) \simeq C(\hat{G})$ where $\hat{G}$ is the dual of $G$, i.e. $\{X : G \to \mathbb{C} \text{ characters}\}$.

(ii) For many groups, like $G = \mathbb{F}_n$ (the free groups) the reduced $C^*$-algebra $C_\lambda^*(G)$ is simple.

# 4 Homomorphisms of algebras

**Definition 4.1** *If $\mathscr{A}, \mathscr{B}$ are algebras, a **homomorphism** from $\mathscr{A}$ to $\mathscr{B}$ is a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in \mathscr{A}$.*

*If $\mathscr{A}$ and $\mathscr{B}$ are $^*$-algebras, a **$^*$-homomorphism** is a homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathscr{A}$.*

*If $\mathscr{A}, \mathscr{B}$ are Banach algebras, then usually we want to have **continuous** homomorphisms. Even more, we usually ask for **contractive** homomorphisms $\varphi : \mathscr{A} \to \mathscr{B}$, (that is $\|\varphi\| \leq 1$).*

We will be especially interested in **characters**:

**Definition 4.2** *A **character** of an algebra $\mathscr{A}$ is a non-zero homomorphism $\chi : \mathscr{A} \to \mathbb{C}$.*

**Example 4.3** Take any subalgebra $\mathscr{A} \subseteq \mathbb{C}^X$. Take $x_0 \in X$ and set $\chi_{x_0} := \mathrm{ev}_{x_0} : \mathscr{A} \to \mathbb{C}, f \mapsto f(x_0)$. This is not necessarily a character, but it is for example, if $\mathscr{A} = C(X)$ or $C_b(X)$ or $C_0(X)$ (if $X$ is "nice", like Hausdorff).

**Definition 4.4** *A ($^*$)-isomorphism between two ($^*$)-algebras $\mathscr{A}$ and $\mathscr{B}$ is a bijective ($^*$)- homomorphism $\varphi : \mathscr{A} \xrightarrow{\sim} \mathscr{B}$.*

**Definition 4.5** *A ($^*$)-**ideal** of a $^*$-algebra $\mathscr{A}$ is a subspace $I \subset A$ such that $I \cdot A \subseteq I$, $A \cdot I \subseteq I$ (if only one condition applies, we call this a **left ideal** or **right ideal**). For $^*$-ideals, we also want $I^* = I$. We notate this as $I \trianglelefteq \mathscr{A}$.*

**Example 4.6** If $\varphi : \mathscr{A} \to \mathscr{B}$ is a $(^*)$-homomorphism, then $\ker \varphi \trianglelefteq \mathscr{A}$.

**Example 4.7** If $I \trianglelefteq \mathscr{A}$ for $\mathscr{A}$ a $(^*)$-algebra

$$\mathscr{A}/I = \{a + I \mid a \in \mathscr{A}\}$$

with $(a + I) \cdot (b + I) := ab + I$ and $(a + I)^* = a^* + I$ is a $(^*)$-algebra.

**Theorem 4.8** *If $\mathscr{A}$ is a Banach-$^*$-algebra, then $I \trianglelefteq \mathscr{A}$ is a closed ideal, then the quotient $I/\mathscr{A}$ is also a Banach-$^*$-algebra.*

Proof: Later. □

# 5 Spectral theory

**Notation 5.1** If $\mathscr{A}$ is a unital algebra, we write

$$\operatorname{inv}(\mathscr{A}) = \{a \in \mathscr{A} \mid a \text{ is invertible in } \mathscr{A}\} = \{a \in \mathscr{A} \mid \exists_{a^{-1} \in \mathscr{A}} aa^{-1} = 1 = a^{-1}a\}$$

This is a group. Sometimes we also write $GL(\mathscr{A})$.

**Definition 5.2** *Given a unital algebra $\mathscr{A}$ and $a \in \mathscr{A}$, we define its* ***spectrum*** *(in $\mathscr{A}$) as*

$$\sigma_{\mathscr{A}}(a) = \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \notin \operatorname{inv}(\mathscr{A})\}$$

*and the resolvent of $a$ (in $\mathscr{A}$) as*

$$\rho_{\mathscr{A}}(a) = \rho(a) = \mathscr{A} \setminus \sigma_{\mathscr{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \operatorname{inv}(\mathscr{A})\}$$

**Example 5.3 (Linear Algebra)** Let $\mathscr{A} = M_m(\mathbb{C})$ and $a \in \mathscr{A}$. Then we have

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \operatorname{inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \det(\lambda - a) = 0\}$$

and these are the roots of the characteristic polynomial $\det(\lambda - a)$. This is exactly the usual spectrum from linear algebra.

**Example 5.4 (Functional Analysis)** Let $\mathscr{A} = \mathscr{L}(\mathfrak{H})$ – where $\mathfrak{H}$ is any Hilbert- or Banach space – and $T \in \mathscr{A}$. Then $\sigma_{\mathscr{A}}(T)$ is exactly the spectrum as defined in functional analysis.

If $S$ is the shift in $\mathscr{L}(\ell^2\mathbb{N})$, then we have $\sigma(S) = \mathbb{D}$.

**Example 5.5** Let $\mathscr{A} = \mathbb{C}[X]$. Here we have $\operatorname{inv}(\mathscr{A}) = \{a_0 X^0 \mid a_0 \in \mathbb{C} \setminus \{0\}\}$ the constant non-zero polynomials. If $a = \sum_{k=0}^N a_k x^k \in \mathscr{A}$, then we have two cases:

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \mathbb{C} & \text{otherwise} \end{cases}$$

**Example 5.6** Let $\mathscr{A} = \mathbb{C}(X) = \{p, q \mid p, q \in \mathbb{C}[X], q \neq 0\}$. Now we have $\operatorname{inv}(\mathscr{A}) = \mathscr{A} \setminus \{0\}$. If $a \in \mathscr{A}$, then

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \emptyset & \text{otherwise} \end{cases}$$

**Example 5.7** Let $\mathscr{A} = C(X)$ for any topological space $X$. Then

$$\text{inv}(\mathscr{A}) = \{f \in C(X) \mid \forall_{x\in X} f(x) \neq 0\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{x\in X} f(x) = \lambda\} = \text{im}(f) = f(X).$$

**Example 5.8** Let $X$ be any topological space and consider $\mathscr{A} = C_b(X)$. Then

$$\text{inv}(C_b(X)) = \{f \in C_b(X) \mid \exists_{\varepsilon>0}\forall_{x\in X}|f(x)| \geq \varepsilon\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{(x_n)} f(x_n) \to \lambda\} = \overline{\text{im}(f)} = \overline{f(X)}.$$

This is a compact subset of $\mathbb{C}$.

**Theorem 5.9 (Algebraic spectral mapping theorem)** *Let $\mathscr{A}$ be an algebra, $a \in \mathscr{A}$ and $p \in \mathbb{C}[X], p(X) = \sum_{k=0}^{n} \lambda_k X^k$ and define $p(a) = \sum_{k=0}^{n} \lambda_k a^k$. Recall that the mapping $\mathbb{C}[X] \to \mathscr{A}, p \mapsto p(a)$ is a unital homomorphism.*

*Then $\sigma(p(a)) = p(\sigma(a))$ assuming $\sigma(a) \neq \emptyset$.*

Proof: If $p(X) = \lambda_0$ constant, this is clear (the spectrum is exactly $\lambda_0$ on both sides). Assume $p(x)$ is not constant. Fix $\mu \in \mathbb{C}$ and write

$$\mu - p(x) = \lambda_0(x - \lambda_1)\cdots(x - \lambda_n)$$

as per the fundamental theorem of algebra (note that these are not the same $\lambda$ as before) with $\lambda_0 \neq 0$. Then $\mu - p(a) = \lambda_0(a - \lambda_1)\cdots(a - \lambda_n)$. Since these expressions commute, this product is invertible if and only if $(a - \lambda_i)$ is invertible for every $i$. So $\mu \in \sigma(p(a)) \Leftrightarrow \mu - p(a)$ is not invertible if and only if there exists an $i$ for which $\lambda_i - a$ is not invertible, so $\lambda_i \in \sigma(a)$. But the $\lambda_i$ are exactly the numbers satisfying $p(\lambda) = \mu$. Thus, $\mu$ is in $\sigma(p(a))$ if it is in the image of $\sigma(a)$ under $p$. Therefore, we conclude $\sigma(p(a)) = p(\sigma(a))$. □

We now focus on invertible elements in **Banach algebras**.

**Theorem 5.10** *If $\mathscr{A}$ is a unital Banach algebra and $a \in \mathscr{A}$ with $\|a\| < 1$ then $1 - a$ is invertible and $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$.*

Proof: Observe that, since $\|a\| < 1$, we have $\sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1-\|a\|} < \infty$. This implies the (absolute) convergence of $\sum_{n=0}^{\infty}$ by the characteristic property of Banach spaces. Hence, $b := \lim_{N\to\infty} \sum_{n=0}^{N} a^n \in \mathscr{A}$. No, if $N \in \mathbb{N}$, then

$$(1-a)\left(\sum_{n=0}^{N} a^n\right) = \left(\sum_{n=0}^{N} a^n\right) - \left(\sum_{n=1}^{N+1} a^n\right) = 1 - a^{N+1} \to 1$$

because of $\|a\| < 1$. This yields $(1-a)b = 1$. □

**Theorem 5.11** *Let $\mathscr{A}$ be a non-empty, non-zero unital Banach algebra. Then $\text{inv}(\mathscr{A})$ is an open subset of $\mathscr{A}$ and the function $f : \text{inv}(\mathscr{A}) \to \mathscr{A}, a \mapsto a^{-1}$ is Frechet-differentiable and in particular continuous as well as $f'(a)b = -a^{-1}ba^{-1}$.*

Recall from calculus that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$. Also recall that $f : U \overset{\text{open}}{\subseteq} X \to Y$ with $X, Y$ Banach spaces is **differentiable** at $x_0 \in U$ there exists an operator $D_{x_0} = f'(x_0) \in \mathscr{L}(X,Y)$ such that

$$\lim_{h\to 0} \frac{f\|(x_0+h) - f(x_0) - D_{x_0}(h)\|}{\|h\|} = 0$$

PROOF: Take $a \in \operatorname{inv}(\mathscr{A})$. If $b \in \mathscr{A}$ such that $\|a-b\| < \|a^{-1}\|^{-1}$. From this, we have $\|ba^{-1}-1\| = \|ba^{-1} - aa^{-1}\| = \|(b-a)a^{-1}\| \leq \|b-a\| \cdot \|a^{-1}\| < 1$. Per the previous theorem, $ba^{-1} \in \operatorname{inv}(\mathscr{A})$. This implies that $b$ is also invertible. This shows that $\operatorname{inv}(\mathscr{A})$ is open.

Furthermore, if $\|b\| < 1$, then also $(\| - b\| < 1)$. Thus, $1 + b \in \operatorname{inv}(\mathscr{A})$ and $(1+b)^{-1} = \sum_{n=0}^{\infty}(-1)^n b^n$. Thus,

$$\|(1+b)^{-1} - 1 + b\| = \left\| \sum_{n=0}^{\infty} (-1)^n b^n - 1 + b \right\| \leq \left\| \sum_{n=2}^{\infty} (-1)^n b^n \right\| \leq \sum_{n=2}^{\infty} \|b^n\| \leq \sum_{n=2}^{\infty} \|b\|^n = \frac{\|b\|^2}{1 - \|b\|}$$

Now let $a \in \inf(\mathscr{A})$ and $c \in \mathscr{A}$ such that $\|c\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Then $\|a^-1c\| \leq \|a^{-1}\|\|c\| \leq \frac{1}{2}$. So if $b = a^{-1}$, then

$$\|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| =\leq \frac{\|a^{-1}c\|^2}{1 = \|a^{-1}c\|} < 2\|a^{-1}c\|^2$$

Now, define $U : \mathscr{A} \to \mathscr{A}, b \mapsto -a^{-1}ba^{-1}$. Then this is a linear odd operation with $\|U\| \leq \|a^{-1}\|^2$, and we have

$$\begin{aligned} \|(a+c)^{-1} - a^{-1} - U(c)\| &= \|(a+c)^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &= \|(1 + a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &\leq \|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \cdot \|a^{-1}\| \\ &\leq 2\|a^{-1}c\|^2\|a^{-1}\| \leq 2\|a^{-1}\|^3\|c\|^2 \end{aligned}$$

and thus

$$\lim_{c\to 0} \frac{\|(a+c)^{-1} - a^{-1} - U(c)\|}{\|c\|} = 0 \qquad \square$$

**Example 5.12** If we choose $\mathscr{A} = \mathbb{C}[X]$ and the norm $\|p\| = \sup_{\lambda\in[0,1]} |p(x)|$. Then $(\mathscr{A}, \|\cdot\|)$ is a normed (but not Banach) algebra. For example, we see that $\lim_{m\to 0} 1 + X/m = 1 \in \operatorname{inv}(\mathscr{A})$, but $1 + X/m \notin \operatorname{inv}(\mathscr{A})$ and thus $\operatorname{inv}(\mathscr{A})$ is not open (because the complement is not closed).

**Theorem 5.13** *If $\mathscr{A}$ is a Banach algebra with unit 1, then for all $a \in \mathscr{A}$ the spectrum $\sigma(a) \subseteq \mathbb{C}$ is closed and $\sigma(a) \subseteq \overline{B(0, \|a\|)} = D(0, \|a\|) := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$. Therefore, $\sigma(a)$ is compact by the Heine-Borell theorem.*

PROOF: By definition

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \operatorname{inv}(\mathscr{A})\}$$

is the inverse image of the closed subset $\mathscr{A} \setminus \operatorname{inv}(\mathscr{A}) \subseteq \mathscr{A}$ by the continuous function $\lambda \mapsto \lambda - a$. Therefore, $\sigma(a)$ is closed.

Now if $|\lambda| \leq \|a\|$ then $\|\lambda^{-1}a\| < 1$. Then $1 - \lambda^{-1}a \in \operatorname{inv}(\mathscr{A})$. Multiplying by $\lambda$ yields $\lambda - a \in \operatorname{inv}(\mathscr{A})$. Thus, $\{\lambda \in \mathbb{C} \mid |\lambda| > \|a\|\} \subseteq \rho(a)$ and thus $\sigma(a) \subseteq D(0, \|a\|)$. $\square$

**Lemma 5.14** *Let $\mathscr{A}$ be a unital Banach algebra and $a \in \mathscr{A}$. Then, the map $R_a : \rho(a) \subseteq \mathbb{C} \to \mathscr{A}, \lambda \mapsto (a - \lambda)^{-1}$ is Frechet-differentiable.*

PROOF: This follows from the following general result:

If $g : U \overset{\text{open}}{\subseteq} X \to Y$ and $f : V \overset{\text{open}}{\subseteq} Y \to Z$ for Banach spaces $X, Y, Z$ with $g(U) \subseteq V$ are differentiable at $x_0 \in U$ or respectively $y_0 = g(x_0) \in V$, then $f \circ g$ is differentiable and $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$. □

Observation: For $R_a(\lambda) = (a-\lambda)^{-1}$ we get $R_a'(\lambda) = (a-\lambda)^{-2}$. We have $\mathcal{L}(\mathbb{C}, \mathscr{A}) \simeq \mathscr{A}$ by $T \mapsto T(1)$. Recall that if $f(a) = a^{-1}$ yields $f'(a)b = -a^{-1}ba^{-1}$.

**Theorem 5.15 (Gelfand)** *If $\mathscr{A} \neq 0$ is a unital Banach algebra and $a \in \mathscr{A}$ then $\sigma(a) \neq \emptyset$.*

PROOF: Suppose $\sigma(a) = \emptyset$. Idea: Show that $R_a : \rho(a) \subseteq \mathbb{C} \to \mathscr{A}, \lambda \mapsto (a-\lambda)^{-1} = \frac{1}{a-\lambda}$ is bounded and differentiable and achieve a contradiction by Liouville's theorem.

Claim: $\|(a-\lambda)^{-1}\| < \|a\|^{-1}$ if $|\lambda| > 2\|a\|$. Indeed, if $|\lambda| > 2\|a\|$ then $\|\lambda^{-1}a\| < \frac{1}{2}$, and in particular $1 - \lambda^{-1}a \in \operatorname{inv}(\mathscr{A})$ and

$$\left\|(1-\lambda^{-1}a)^{-1} - 1\right\| = \left\|\sum_{n=1}^{\infty}(\lambda^{-1}a)^{-1}\right\| \leq \sum_{n=1}^{\infty}\|\lambda^{-1}a\|^n = \frac{\|\lambda^{-1}a\|}{1-\|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

From here we deduce that $\|(1-\lambda^{-1}a)^{-1}\| < 2$ and thus

$$\|(a-\lambda)^{-1}\| < \|\lambda^{-1}(\lambda^{-1}a-1)^{-1}\| = \frac{\|(1-\lambda^{-1}a)^{-1}\|}{|\lambda|} < \frac{2}{\lambda} < \frac{1}{\|\lambda\|}.$$

So $R_a : \mathbb{C} \to \mathscr{A}$ is bounded outside $\overline{B(0, 2\|a\|}$. Since $R_a$ is continuous, it is bounded on $\mathbb{C} \to \mathscr{A}$. Let $\varphi \in \mathscr{A}^*$ be a bounded linear functional in $\mathcal{L}(\mathscr{A}, \mathbb{C})$. Thus, $\varphi$ is differentiable with $\varphi'(a) = \varphi$ for all $a \in \mathscr{A}$. Then $\varphi \circ R_a$ is differentiable and bounded, so it is an "integer" function. By Liouville's theorem, $\varphi \circ R_a$ is constant. Therefore, $\varphi \circ R_a(x) = \varphi \circ R_a(y)$ for all $x, y \in \mathscr{A}$. Especially, we have $\varphi((a-\lambda)^{-1}) = \varphi(a^{-1})$ for all $\varphi$. Hahn-Banach shows $(a-\lambda)^{-1} = a^{-1}$ for all $\lambda$, proving $a - \lambda = a$ for all $a, \lambda$. This is a contradiction. □

**Theorem 5.16 (Gelfand-Mazur)** *If $\mathscr{A}$ is a unital Banach algebra and every $a \neq 0$ admits an inverse ($\mathscr{A}$ is a field), then $\mathscr{A} = \mathbb{C} \cdot 1$.*

PROOF: By the assumption, $\operatorname{inv}(\mathscr{A}) = \mathscr{A} \setminus \{0\}$. By the previous theorem, if $a \in \mathscr{A}$ there exists some $\lambda \in \sigma(a)$, so $a - \lambda \notin \operatorname{inv}(\mathscr{A})$, so $a - \lambda = 0$ and thus $a = \lambda \cdot 1$. □

**Corollary 5.17** *Let $\mathbb{C}(X) = \left\{\frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{Q}[X]\right\}$ is a field, but it cannot be turned into a Banach algebra.*

**Theorem 5.18 (Adjointing units - unitization of algebras)** *Let $\mathscr{A}$ be any algebra. Consider $\tilde{\mathscr{A}} = \mathscr{A} \oplus \mathbb{C}$ as a vector space. We write elements of $\tilde{\mathscr{A}}$ as $a + \lambda \cdot 1 := (a, \lambda)$. Think of $a = (a, 0)$ and $\lambda = (a, \lambda)$. Define*

$$(a + \lambda 1)(b + \lambda' 1) = (ab + \lambda' a + \lambda b) + \lambda \cdot \lambda'.$$

*Ten (exercise $\tilde{\mathscr{A}}$) becomes a unital algebra with $1_{\tilde{\mathscr{A}}} = 1 = (0, 1)$.*

*Notice that $\mathscr{A}$ is an ideal in $\tilde{\mathscr{A}}$.*

*Moreover, we get a short exact sequence*

$$0 \to \mathscr{A} \hookrightarrow \tilde{\mathscr{A}} \to \mathbb{C} \to 0$$

*so* $1 + \lambda \mapsto \lambda$.

*If* $\mathscr{A}$ *is a normed algebra, then* $\tilde{\mathscr{A}}$ *is normed by* $\|a + \lambda \cdot 1\| := \|a\| + |\lambda|$

*If* $\mathscr{A}$ *is Banach and closed, then so is* $\tilde{\mathscr{A}}$.

*If* $\mathscr{A}$ *is a* $^*$*-algebra, then so is* $\tilde{\mathscr{A}}$ *with* $(a + \lambda 1)^*$.

*If* $\mathscr{A}$ *is a (Banach) normed* $^*$*-algebra, then so is* $\tilde{A}$.

*If* $\mathscr{A}$ *is a* $C^*$*-algebra, in general the norm given above is not a Norm on* $\mathscr{A}$*, but* $\|a + \lambda \cdot 1\| := \sup_{b \in \mathscr{A}, b \in \mathscr{B}, b \leq 1} \|ab + \lambda b\|$ *is.*

**Exercise 5.1** If $\mathscr{A}$ is already unital, then $\tilde{A} \simeq A \oplus \mathbb{C}$ as algebras by $a + \lambda \cdot 1 \mapsto (a + \lambda 1_{\mathscr{A}}, -\lambda)$.

**Definition 5.19** *Re-Definition: If* $\mathscr{A}$ *is non-unital, then* $\tilde{A} + \mathbb{C} \cdot 1$ *is a* $(*-)$*Banach algebra, and we define* $\sigma_A(a) := \sigma_{\tilde{\mathscr{A}}}(a)$.

Observation: If $\mathscr{A}$ is already unital, then for $\tilde{A} \simeq \mathscr{A} \oplus \mathbb{C}$ we have $\sigma_{\tilde{\mathscr{A}}}(a) = \sigma_{\mathscr{A}}(a) \cup \{0\}$.

**Remark 5.20** If $\mathscr{A}$ is a $C^*$-algebra, then $\tilde{\mathscr{A}}$ is a $C^*$-algebra.

(i) If $\mathscr{A}$ is unital, then $\tilde{\mathscr{A}} \simeq \mathscr{A} \oplus \mathbb{C}$ and $\|a + \lambda \cdot 1\| = \max\{\|a + \lambda \cdot 1\|, |\lambda|\}$.

(ii) If $\mathscr{A}$ is not unital, then $\|a + \lambda \cdot 1\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$.

# 6 Spectral Radius

**Definition 6.1** *Let* $\mathscr{A}$ *be an algebra. Given* $a \in \mathscr{A}$*, we define:*

$$r(a) := \sup\{|\lambda| \mid \lambda \in \sigma_{\mathscr{A}}(a)\}$$

*as the* ***spectral radius*** *of* $a$ *if* $\emptyset \neq \sigma_{\mathscr{A}}(a)$ *is bounded (e.g. if* $\mathscr{A}$ *is Banach).*

Observation: In a Banach algebra, we have $0 \leq r(a) \leq \|a\|$.

**Example 6.2**

(i) Let $f \in \mathscr{A} = C_0(X)$ using $\sigma_A(f) = \overline{f(X)}$. Thus,

$$r(f) = \sup\{|\lambda| \mid \lambda \in \overline{f(X)} = \sup_{x \in X} |f(x)| = \|f\|_{C_0(X)}$$

(ii) Let $\mathscr{A} = M_2(\mathbb{C})$ and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\sigma_{\mathscr{A}} = \{0\}$ and $r(a) = 0$, but $\|a\| = 1 \neq 0$.

**Theorem 6.3 (Beurling-Gelfand)** *Let* $\mathscr{A}$ *be a Banach algebra, then*

$$r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}}$$

Proof: We may assume $\mathscr{A}$ is unital (otherwise we consider $\tilde{\mathscr{A}}$). If $\lambda \in \sigma(a)$, then

$$\lambda^n \in \sigma(a^n) \Rightarrow |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a\|^{\frac{1}{n}} \quad \forall_{n \in \mathbb{N}}$$

and therefore

$$r(a) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n \to \infty} \|a^n\|^{\frac{1}{n}}.$$

We prove now that $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$. Set $\Delta := B\Big(0, \frac{1}{r(a)}\Big)$. Where per convention we set $\frac{1}{r(a)} = \infty$ if $r(a) = 0$. If $\lambda \in \Delta$, then $1 - \lambda a \in \operatorname{inv}(\mathscr{A})$ (because $|\lambda| < \frac{1}{r(a)}$ implies $|\lambda^{-1}| > r(a)$ and therefore $\lambda^{-1} \notin \sigma(a) \Rightarrow \lambda^{-1} - a \in \operatorname{inv} A \Rightarrow 1 - \lambda a \in \operatorname{inv}(A)$).

Now fix $\varphi \in \mathscr{A}^*$. Then $f : \Delta \to \mathbb{C}, \lambda \mapsto \varphi((1-\lambda a)^{-1})$ is analytic, so it can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n \lambda^n, a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{C}, \lambda \in \Delta.$$

On the other hand, if

$$|\lambda| < \frac{1}{\|a\|} \leq \frac{1}{r(a)}$$

then $\|\lambda a\| < 1$, so

$$(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n \Rightarrow f(\lambda) = \varphi((1-\lambda)^{-1}) = \sum_{k=0}^{\infty} \varphi(a^n)\lambda^n$$

for $|\lambda| < \frac{1}{\|\lambda\|}$.

By uniqueness of the Taylor series expansion, it follows that

$$a_n = \varphi(a^n) \forall_{n\in\mathbb{N}}.$$

In particular, $(\varphi(a^n)\lambda^n)$ converges to zero for all $\lambda \in \Delta$ and thus $(\varphi(a^n)\lambda^n)$ is bounded for all $\lambda \in \Delta$.

From the principle of uniform convergence, it follows that $(a^n\lambda^n)$ is bounded. So there exists an $M = M_\lambda$ such that

$$\begin{aligned} &\|\lambda^n a^n\| \leq M \forall_{n\in\mathbb{N}} \\ \Rightarrow &\|\lambda^n\|^{\frac{1}{n}} \leq \frac{M^{\frac{1}{n}}}{|\lambda|} \forall_{n\in\mathbb{N}}, \forall_{\lambda\in\Delta,\lambda\neq 0} \\ \Rightarrow &\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \frac{1}{\lambda} \forall_{\lambda\in\Delta \text{ i.e. } |\lambda|<\frac{1}{r(a)}} \end{aligned}$$

Letting $\lambda < \frac{1}{r(a)}$ yields $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$. $\square$

**Example 6.4** Let $A = C^1([0,1]) = \{I \in C[0,1] \mid \exists_{f'(t)} \forall_{t\in[0,1]}, t \mapsto f'(t) \text{ continuous}\}$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

Then $\mathscr{A}$ is unital, commutative and a Banach algebra. Consider $x \in \mathscr{A}$, $x(t) = t$. We have $x^n(t) = t^n$ and

$$\begin{aligned} \|x^n\| &= \sup_{t\in[0,1]} |t^n| + \sup_{t\in[0,1]} |nt^{n-1}| = 1 + n \\ r(x) &= \lim_{n\to\infty} (1+n)^{\frac{1}{n}} = 1 \\ \|x\| &= 2 \end{aligned}$$

Observation: $\sigma(x) = \operatorname{im}(x) = [0,1]$.

**Theorem 6.5** *Let $\mathscr{B} \nsubseteq \mathscr{A}$ be an inclusion of unital Banach algebras with $1 = 1_{\mathscr{A}} = 1_{\mathscr{B}}$. Then $\sigma_{\mathscr{A}}(b) \subseteq \sigma_{\mathscr{B}}(b)$ for all $b \in \mathscr{B}$ and the inclusion may be proper. If $\sigma_{\mathscr{A}}(b)$ is simply connected (not holes), then $\sigma_{\mathscr{A}}(b) = \sigma_{\mathscr{B}}(b)$.*
*The holes of a compact subset $K \subseteq \mathbb{C}$ are the bounded connected components of $\mathbb{C} \setminus K$. So saying that $K$ has no holes means that $\mathbb{C} \setminus K$ is connected.*

Proof: See Murphy, 1.2.8. □

**Example 6.6** Let $\mathscr{B} := A(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid f \text{ analytic on } \mathbb{D}^\circ\}$ and $\mathscr{A} = C(\mathbb{S}^1)$. Then we have an embedding by $\iota : \mathscr{B} \hookrightarrow \mathscr{A}, f \mapsto f|_{\mathbb{S}^1}$.

By the principle of maximum modules, $\iota$ is an embedding of (unital) Banach algebras. Consider: $f(z) = z$ for $z \in \mathbb{D}$. (Observation: $\overline{Alg}(1, z) = A(\mathbb{D})$) Then:

$$\sigma_{A(\mathbb{D})}(f) = f(\mathbb{D}) = \mathbb{D}$$

and $\sigma_{C(\mathbb{S}^1)}(f|_{\mathbb{S}^1}) = \mathbb{S}^1$.

**Definition 6.7 (Exponentials)** *Let $\mathscr{A}$ be a unital Banach algebra, given $a \in \mathscr{A}$ we define*

$$e^a = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

*Note $\left\|\frac{a^n}{n!}\right\| \leq \frac{\|a\|^n}{n!}$, so the series converges and $\|\exp(a)\| \leq \exp(\|a\|)$.*

**Theorem 6.8**

*(i) Let $\mathscr{A}$ be a unital Banach algebra. If $a \in \mathscr{A}$, then $f : \mathbb{R} \to \mathscr{A}, t \mapsto \exp(ta)$ is the unique solution of*

$$\begin{cases} f'(t) &= af(t) \\ f(0) &= 1 \end{cases}$$

*(ii) $e^a \in \operatorname{inv}(\mathscr{A})$ and $(e^a)^{-1} = e^{-a}$.*

*(iii) If $a, b \in \mathscr{A}$ then $e^{a+b} = e^a \cdot e^b$ (here some commutativity is neccessary).*

Proof: See Murphy, 1.2.9. □

# 7 Gelfand Representation for commutative Banach algebras

Idea: Given a commutative algebra $\mathscr{A}$, we want to represent $\mathscr{A}$ by a homomorphism $\varphi : \mathscr{A} \to C_0(X)$ for $X$ some locally compact Hausdorff space. We hope that $\varphi$ is injective, or even isometric, on an isomorphism. But what is $X$, and what is $\varphi$?

Notice that, if $\mathscr{A} = C_0(X)$ already, then for each $x \in X$ we get a character $\mathrm{ev}_x : \mathscr{A} \to \mathbb{C}, f \mapsto f(x)$.

**Definition 7.1** *Given an algebra $\mathscr{A}$, we define*

$$\hat{\mathscr{A}} = \Omega(\mathscr{A}) := \{\chi : \mathscr{A} \to \mathbb{C} \mid \chi \textit{ non-zero homomorphism}\}.$$

**Example 7.2**

(i) For $\mathscr{A} = C_0(X)$ we get a map

$$X \to \Omega(\mathscr{A}), x \mapsto \mathrm{ev}_x$$

that is a bijection. After we give $\Omega(\mathscr{A})$ an appropriate topology, it will also be a homomorphism.

(ii) Let $\mathscr{A} = M_2(\mathbb{C})$ (or any $M_n(\mathbb{C})$). This is a simple algebra, so non-zero homomorphisms $\chi : \mathscr{X} \to \mathbb{C}$ do not exist (same for any $\mathscr{A}$ with dimension $> 1$).

So in this case we have $\Omega(\mathscr{A}) = \emptyset$. This can also happen in commutative algebras.

(iii) Consider

$$\mathscr{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\}$$

Then for all $a \in \mathscr{A}$ we have $a^2 = 0$, so if $\chi : \mathscr{A} \to \mathbb{C}$ is an homomorphism, then $\chi(a)^2 = \chi(a^2) = 0$, so $\chi(a) = 0$ for all $a \in \mathscr{A}$. So again, $\Omega(\mathscr{A}) = \emptyset$ (and $\mathscr{A}$ is commutative with $\dim \mathscr{A} = 1$).

Question: Given an abstract algebra $\mathscr{A}$ how do we possibly find its characters?

Idea: Assume that $\mathsf{I} \lhd \mathscr{A}$ is a maximal ideal and $\mathscr{A}$ is a unital Banach algebra. Then $\mathscr{A}/\mathsf{I} \simeq \mathbb{C}$ and $\chi \in \Omega(\mathscr{A})$.

**Theorem 7.3** *Let $\mathscr{A}$ be a unital non-zero Banach algebra. If $\chi \in \Omega(\mathscr{A})$ then $\|\chi\| = \sup_{\|a\|=1} |\chi(a)| = 1$ and $\ker(\chi) \lhd \mathscr{A}$. So $\chi \in \mathscr{A}^*$ (the topological dual of $\Omega(\mathscr{A}) \subseteq D_{\mathscr{A}^*}(0,1)$).*

*Moreover, if $\mathscr{A}$ is a unital Banach commutative algebra, then $\Omega(\mathscr{A}) \ni \chi \mapsto \ker(\chi) \lhd \mathscr{A}$ is a bijection between of characters of $\mathscr{A}$ and maximal ideals of $\mathscr{A}$.*

Proof: If $a \in \mathscr{A}$ and $\chi$ a character, then $\chi(a) \in \sigma(\mathscr{A})$, because $\chi(a - \chi(a) \cdot 1) = \chi(a) - \chi(a) \cdot \chi(1) = 0$, so $a - \chi(a) \cdot 1 \in \ker(\chi) \lhd \mathscr{A}$ and thus $a - \chi(a) \cdot 1 \notin \mathrm{inv}(\mathscr{A})$.

Therefore: $|\chi(a)| \leq r(a) \leq \|a\|$. So $\|\chi\| \leq 1$. Since $\chi(1) = 1$ and $\|1\| = 1$ we have $\|\chi\| = 1$.

Now, apply linear algebra. Then $\ker(\chi)$ is a maximal proper subspace, in particular a maximal ideal. And $\ker(\chi)$ is closed, because $\chi$ is continuous. Now assume that $\mathscr{A}$ is commutative (in addition to unital and Banach). Then we have the mapping

$$\varphi : \Omega(\mathscr{A}) \to \mathrm{MaxIdeals}(\mathscr{A}), \chi \to \ker(\chi).$$

- $\varphi$ is injective. If $\ker(\chi_1) = \ker(\chi_2)$ for $\chi_1, \chi_2 \in \mathscr{A}$, then for every $a \in \mathscr{A}$ we have $a - \chi_1(a) \cdot 1 \in \ker(\chi_1) = \ker(\chi_2)$. Thus, $\chi_2(a = \chi_1(a) \cdot 1) = 0$ and therefore $\chi_2(a) = \chi_1(a)$ for every $\mathscr{A}$.

- $\varphi$ is surjective. Take $\mathsf{I} \lhd \mathscr{A}$ a maximal ideal. Then $\mathsf{I} = \bar{\mathsf{I}}$ because $\bar{\mathsf{I}} \neq \mathscr{A}$, otherwise $1 \in \bar{\mathsf{I}}$ and since $\mathrm{inv}(\mathscr{A})$ is open in $\mathscr{A}$, we get $\mathsf{I} \cap \mathrm{inv}(\mathscr{A}) \neq \emptyset$. But then we have an invertible element in the ideal $\mathsf{I}$ already, but this implies the contradiction $\mathsf{I} = \mathscr{A}$. Therefore, $\mathscr{A}/\mathsf{I}$ is a commutative, unital Banach algebra which is simple ($\mathsf{I}$ is maximal).

  Exercise: If $\mathsf{I} \lhd \mathscr{A}$, then $\mathscr{A}/\mathsf{I}$ is field if and only if there exists no $\mathsf{J} \lhd \mathscr{A}$ such that $\mathsf{I} \lhd \mathsf{J}$.

  Thus, $\mathscr{A}/\mathsf{I}$ is a field and $\mathscr{A}/\mathsf{I} \simeq \mathbb{C}$. Then the composition

  $$\mathscr{A} \xrightarrow{q} \mathscr{A}/\mathsf{I} \simeq \mathbb{C}$$

  is a character with $\ker(\chi) = \mathsf{I}$. □

**Exercise 7.1** An application of Zorn's Lemma. Show that every ideal $I \lhd \mathscr{A}$ in a unital algebra $\mathscr{A}$ is contained in a maximal ideal.

In particular, we can apply this to $\mathsf{I} = 0$ in $\mathscr{A} \neq 0$ (with $\mathscr{A}$ is unital and commutative) and thus $\Omega(\mathscr{A}) \neq \emptyset$.

## Topology on $\Omega(\mathscr{A})$

We have for $\mathscr{A}$ a Banach algebra. We can add a unit to receive $\tilde{\mathscr{A}}$, which is a Banach algebra.

Observe: If $\chi \in \Omega(\mathscr{A})$, then there exists a unique $\tilde{X} \in \Omega(\tilde{\mathscr{A}})$ via $\tilde{X}(a + \lambda \cdot 1) = \chi(a) + \lambda$. Thus, $\|\chi\| \leq \|\tilde{X}\| = 1$ (Note that it may still be smaller than 1. See exercises 2023-05-09).

In any case,

$$\Omega(\mathscr{A}) \subseteq D_{\mathscr{A}^*}(0,1) = \{\varphi \in \mathscr{A}^* \mid \|\varphi\| \leq 1\}$$

and $\mathscr{A}^*$ carries the weak $^*$-topology (the smallest topology to make all point-evaluations continuous, that is for a net $(\varphi_i) \subset {}^*$ weakly converging to $\varphi \in \mathscr{A}^*$ if and only if $\varphi_i(a) \to \varphi(a)$ for all $a \in A$).

**Definition 7.4** *Given a Banach algebra $\mathscr{A}$, we endow $\Omega(\mathscr{A})$ with the weak $^*$-topology and call this the **Gelfand spectrum** of $\mathscr{A}$.*

**Proposition 7.5** *$\Omega(\mathscr{A})$ is a locally compact Hausdorff space. If $\mathscr{A}$ is unital, then $\Omega(\mathscr{A})$ is compact.*

Proof: By Banach-Alaoglu-Theorem, $D_{\mathscr{A}^*}(0,1)$ is compact and Hausdorff with the weak $^*$-topology. Let

$$\begin{aligned} S &:= \{\chi : A \to \mathbb{C} \mid \chi \text{ hom.}\} \\ &= \Omega(\mathscr{A}) \cup \{0\} \end{aligned}$$

Then $S \subseteq D_{\mathscr{A}^*}(0,1)$. So $\chi(ab) = \lim_{i\to\infty} K_i = \lim_{i\to\infty} \chi_i(a)\chi_i(b) = \chi(a)\chi(b)$ and therefore $x \in S$. Thus, $S$ is a compact Hausdorff space and $\Omega(\mathscr{A}) = S \setminus \{0\}$ is relatively compact.

If $\mathscr{A}$ is unital, then $\Omega(\mathscr{A}) \subseteq D_{\mathscr{A}^*}(0,1)$ is closed. Then we have $(X_i) \subseteq \Omega(\mathscr{A})$ and $X_i \to X \in \mathscr{A}^*$ and thus $X \in S = \hom(\mathscr{A}, \mathbb{C})$. $\square$

Observation: Given a Banach algebra $\mathscr{A}$, we have an isomorphism

$$\Omega(\tilde{\mathscr{A}}) \to \Omega(\mathscr{A}) \sqcup \{\chi_\infty\}, \varphi \mapsto \begin{cases} \varphi|_{\mathscr{A}} & \varphi|_{\mathscr{A}} \neq 0 \\ \chi_\infty & \varphi|_{\mathscr{A}} = 0 \end{cases},$$

where $\chi_\infty(a + \lambda \cdot 1) = \lambda$. Thus, $\Omega(\mathscr{A}) \sqcup \{\chi_\infty\}$ is already the unitization of $\Omega(\mathscr{A})$.

**Theorem 7.6** *Let $\mathscr{A}$ be a Banach algebra. Then for every $a \in \mathscr{A}$.*

$$\{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \subseteq \sigma(a)$$

*If $\mathscr{A}$ is commutative, then*

- $\{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} = \sigma(a)$ *in case $\mathscr{A}$ is unital.*
- $\{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\} = \sigma_{\mathscr{A}}(a)$.

Proof:

- $\mathscr{A}$ is unital and $a \in \mathscr{A}$. $\chi(a - \chi(a) \cdot 1) = 0$, so $\chi(a) \in \sigma(a)$, so $\{\chi(a) \mid x \in \Omega(a)\} \subseteq \sigma(a)$.

  Now if $\lambda \in \sigma(a)$, consider $\mathsf{I} := (a - \lambda \cdot 1)\mathscr{A} \lhd \mathscr{A}$ if $\mathscr{A}$ is commutative. By Zorns Lemma, we get $I \subseteq J \lhd \mathscr{A}$ with $J = \ker(\chi)$ for some $\chi \in \Omega(\mathscr{A})$. Thus we have $a - \lambda \cdot 1 \in \mathsf{I} \subseteq J = \ker(\chi)$ so $\chi(a) = \lambda$.

- $\mathscr{A}$ is not unital. Consider $\tilde{\mathscr{A}}$. By the first part,

$$\sigma_{\mathscr{A}}(a) = \sigma_{\tilde{\mathscr{A}}}(a) \supseteq \{\chi(a) \mid \chi \in \Omega(\tilde{\mathscr{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\}$$

If $\mathscr{A}$ is commutative, by the first part again:

$$\sigma_{\mathscr{A}}(a) = \sigma_{\tilde{\mathscr{A}}}(a) = \{\chi(a) \mid \chi \in \Omega(\tilde{\mathscr{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\} \qquad \square$$

## 7.1 Gelfand-Transformation

**Definition 7.7** *Given a Banach algebra $\mathscr{A}$ and $a \in \mathscr{A}$, we define $\hat{a} : \Omega(\mathscr{A}) \to \mathbb{C}, \chi \mapsto \chi(a)$.*

Observe that $\hat{a} \in C(\Omega(\mathscr{A}))$, because if $\chi_i \to \chi$ then we have $\hat{a}(\chi_i) = \chi_i(a) \to \chi(a) = \hat{a}(\chi)$. So we have a map $\Gamma : \mathscr{A} \to C(\Omega(\mathscr{A}))$. This map is called the **Gelfand transform** of $\mathscr{A}$.

**Theorem 7.8 (Gelfand Representation)** $\mathrm{im}(\Gamma) \subseteq C_0(\Omega(\mathscr{A}))$ *and* $\Gamma : \mathscr{A} \to C_0(\Omega(\mathscr{A}))$ *is a contractive homomorphism, i.e.* $\|\Gamma(a)\| \le r(a) \le \|a\|$ *for every Banach algebra* $\mathscr{A}$*. If moreover* $\mathscr{A}$ *is commutative, then* $\|\Gamma(a)\| = r(a)$*. Also, for all* $a \in \mathscr{A}$*, we have*

$$\sigma(a) = \begin{cases} \mathrm{im}(\hat{a}) & \mathscr{A} \text{ unital} \\ \mathrm{im}(\hat{a}) \cup \{0\} & \text{otherwise} \end{cases}.$$

Proof: If $\mathscr{A}$ is unital, then $\Omega(\mathscr{A})$ is compact so $\mathrm{im}(\Gamma) \subseteq C(\Omega(\mathscr{A})) = C_0(\Omega(\mathscr{A}))$. If $\mathscr{A}$ is not unital, we use observation 7. Then we have $\Omega(\tilde{\mathscr{A}}) \simeq \Omega(\mathscr{A}) \cup \{\chi_\infty\}$ so that

$$C_0(\Omega(\mathscr{A})) \simeq \{f \in C(\Omega(\tilde{\mathscr{A}})) \mid f(x_\infty) = 0\}.$$

Now if $a \in \mathscr{A}$, then $\hat{a}(\chi_\infty) = \chi_\infty(a) = 0$.

<u>$\Gamma$ is a homomorphism</u>: The linearity is obvious, as is the homomorphism property:

$$(\Gamma(a)\Gamma(b))(\chi) = (\hat{a} \cdot \hat{b})(\chi) = \hat{a}(\chi)\hat{b}(\chi) = \chi(a)\chi(b) = \chi(ab) = \hat{ab}(\chi) = \Gamma(ab)(\chi).$$

<u>$\Gamma$ is contractive</u>: Given $a \in \mathscr{A}$, $\chi \in \Omega(\mathscr{A})$, we have $\hat{a}(\chi) = \chi(a) \in \sigma(a)$, so $\|\hat{a}(\chi)\| \le r(a)$ yielding $\|\Gamma(a)\|_\infty = \|\hat{a}\|_\infty \le r(a) \le \|a\|$. If $\mathscr{A}$ is commutative, we have

$$\sigma(a) = \begin{cases} \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} & 1 \in \mathscr{A} \\ \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\} & \text{otherwise} \end{cases} = \begin{cases} \{\hat{a}(\chi) \mid \chi \in \Omega(\mathscr{A})\} & 1 \in \mathscr{A} \\ \{\hat{a}(\chi) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\} & \text{otherwise} \end{cases}$$

and thus

$$\|\Gamma(a)\| = \|\hat{a}\|_\infty = \sup_{\chi \in \Omega(\mathscr{A})} |\chi(a)| = \sup_{\lambda \in \sigma(a)} |\lambda| = r(a) \qquad \square$$

As a convention, if $\Gamma(\mathscr{A}) =$, then $C_0(\Omega(\mathscr{A})) = \{0\}$ and thus $\hat{a} = 0$ for all $a \in \mathscr{A}$.

**Example 7.9**

(i) If $\mathscr{A} = M_n(\mathbb{C})$ with $n > 1$ or $\mathscr{A}$ is any unital simple Banach algebra with $\dim \mathscr{A} > 1$, then $\Omega(\mathscr{A}) = \emptyset$ so $\Gamma \equiv 0$.

(ii) Take the commutative subalgebra

$$\mathscr{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

then $\mathscr{A}$ is not unital, commutative, Banach and $\dim \mathscr{A} = 1$. Once again, $\Omega(\mathscr{A}) = \emptyset$ and thus $\Gamma \equiv 0$.

(iii) Take

$$\mathscr{A} = \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mid \lambda, \alpha \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

is a unital, commutative Banach algebra with $\dim \mathscr{A} = 2$. We have

$$\Omega(\mathscr{A}) = \{\chi_\infty\} \qquad \chi_\infty : \mathscr{A} \to \mathbb{C}, \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mapsto \lambda$$

and thus

$$\Gamma : \mathscr{A} \to C_0(\Omega(\mathscr{A})) = C_0(\{\chi_\infty\}) \simeq \mathbb{C}, a = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mapsto \hat{a} \equiv \lambda$$

This shows that $\Gamma$ is not injective, as $\dim \mathscr{A} = 2$ but $\dim \Gamma(\mathscr{A}) = 1$.

**Definition 7.10** *Let $\mathscr{A}$ be a Banach algebra. We say that $a \in \mathscr{A}$ is quasi-nilpotent if $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = 0$. Sometimes, you will read*

$$\operatorname{Rad}(\mathscr{A}) = \{a \in \mathscr{A} \mid a \text{ quasi-nilpotent}\}$$

*If $\operatorname{Rad}(\mathscr{A}) = 0$, we say that $\mathscr{A}$ is* ***semi-simple****. Notice that if $a \in \mathscr{A}$ is quasi-nilpotent, then $\Gamma(a) = \hat{a} = 0$ because $\Gamma(a) \le r(a) = 0$. If $\mathscr{A}$ is commutative, then $\ker(\Gamma) = \operatorname{Rad}(\mathscr{A})$.*

**Example 7.11**

(iv) $\mathscr{A} = \ell^1(\mathbb{Z}) = \{(a_n)_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} |a_n| < \infty$.

Recall from exercises, that $\Omega(\ell^1(\mathbb{Z})) \simeq \mathbb{D}$ with $\mathbb{D} \to \Omega(\ell^1(\mathbb{Z})), z \mapsto \chi_z$ defined as $\chi_z(a) = \hat{a}(z) = \sum_{n=0}^\infty a_n z^n$.
We define a multiplication $\delta_m \cdot \delta_n = \delta_{n+m}$. Then $\delta_0$ is the unit and $\delta_1$ is a generator of $\mathscr{A} = \ell^1(\mathbb{Z})$.
The elements $\delta_m - (\dots, 0, 1, 0, \dots)$ form a basis for $\mathscr{A}$. We have $a = \sum_{n\in\mathbb{Z}} a_n \delta_n$ and for $\chi \in \mathscr{A}^*$ it follows $\chi(a) = \sum_{n\in\mathbb{Z}} a_n \chi(\delta_n)$.

We now want to calculate the spectrum. We have seen that $\chi(\delta_0) = \chi(1_{\mathscr{A}}) = 1$ and $\chi(\delta_n) = \chi(\delta_1^n)\chi(\delta_1)^n$. Therefore, $\chi$ is determined by $z = \chi(\delta_1) \in \mathbb{C}$. We know at least that $|z| = |\chi(\delta_1)| \le \|\delta_1\| = 1$, so $z \in \mathbb{D}$. Claim: $z \in \Pi = \mathbb{S}^1$.

General fact: If $a \in \operatorname{inv} \mathscr{A}$ for $\mathscr{A}$ a unital Banach algebra, then $\sigma(a^{-1}) = \sigma(a)^{-1} = \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$.

Observe that $\mathbb{S}^1 = \operatorname{inv}(\mathscr{A})$ with $\delta_1^{-1} = \delta_{-1}$. So $\sigma(\delta) \subseteq \mathbb{D}$ and $\sigma(\delta_1)^{-1} = \sigma(\delta_{-1}) \subseteq \mathbb{D}$, so $\sigma(\delta_1) \subseteq \mathbb{S}^1$. So $z = \chi(\delta_1) \in \sigma(\delta_1) \subseteq \mathbb{S}^1$. Conversely, if $z \in \mathbb{S}^1$, then $\chi_z : \mathscr{A} \to \mathbb{C}, \chi_z(a) = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathbb{C}$ is well-defined (as the sum converges) and is a character, as

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n z^m = \chi_z(\delta_n) \cdot \chi + z(\delta_m)$$

and checking in the basis also proves the homomorphism property for all of $\mathscr{A}$. Notice that $z = \chi_z(\delta_1)$. This shows the injectivity of

$$\Pi \simeq \Omega(\mathscr{A}) = \Omega(\ell^1(\mathbb{Z})), z \mapsto \chi_z, \chi(\delta_1) \leftarrow \chi$$

which is continuous and therefore a homeomorphism (isomorphism), as both spaces are compact. Notice

$$\sigma(\delta_1) = \{\chi(\delta_1) \mid \chi \in \Omega(\mathscr{A})\} = \{\chi_z(\delta_1) \mid z \in \mathbb{S}^1\} = \mathbb{S}^1$$

The Gelfand transformation is now

$$\Gamma : \mathscr{A} = \ell^1(\mathbb{Z}) \to C(\Omega(\mathscr{A})) \simeq C(\mathbb{S}^1), a \mapsto \left(\hat{a} : z \mapsto \sum_{n\in\mathbb{Z}} a_n z^n\right)$$

$\Gamma$ is always a contractive algebra homomorphism, as $\|\hat{a}\|_\infty \le \|a\|_1$. $\Gamma$ is a $^*$-homomorphism where $\ell^1(\mathbb{Z})$ carries the involution $a^* = \left(\sum_{n\in\mathbb{Z} a_n \delta_n}\right) = \sum_{n\in\mathbb{Z}} \overline{a}_n \delta_{-n}$ because of $\delta_n^* = \delta_{-n}$. The involution of $C(\mathbb{S}^1)$ is complex conjugation. But on the unit circle, $\overline{z} = z^{-1}$, so we have a $^*$-homomorphism.

$\Gamma$ is <u>injective</u>. If $f \in C(\mathbb{S}^1)$, we can define its "inverse Fourier-Transform"

$$\check{f}(n) = \int_{\mathbb{S}^1} f(z) z^{-n} dz = \frac{1}{2\pi} \int_0^{2\pi} f(\exp(it)) \exp(-int) dt$$

This is **not** the line integral from functional analysis, as the derivative of the path is not included. You can now check that $(\hat{a})\check{}(n) = a_n$. $g \mapsto \int_{\mathbb{S}^1} g$ is a continuous function on $C(\mathbb{S}^1)$ and we have

$$\hat{a}(z) = \sum_{n\in\mathbb{Z}} a_n z^n = \lim_{F\subseteq\mathbb{Z} \text{ finite}} \sum_{n\in F} a_n z^n = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n z^n$$

so

$$(\hat{a})\check{}(n) = \sum_{m\in\mathbb{Z}} a_m (\hat{\delta_m})\check{}(n)$$

Because of $\int_{\mathbb{S}^1} z^k = \delta_{k,0}$, we have

$$\int_{\mathbb{S}^1} z^m z^n dz = \delta_{n,m}$$

and using $\hat{\delta_m}(z) = z^m$ we can show $(\hat{\delta_m})\check{}(n) = \delta_{n,m}$ and thus

$$(\hat{a})\check{}(n) = \sum_{m\in\mathbb{Z}} a_m (\hat{\delta_m})\check{}(n) = \sum_{m\in\mathbb{Z}} a_m \delta_{m,n} = a_m$$

This shows that we can re-gain the elements of the sequence from $\hat{a}$, so $\Gamma : (a_n) \mapsto \hat{a}$ must be injective.

$\Gamma$ has dense range because the polynomials are dense in $C(\mathbb{S}^1)$ because of Stone-Weierstraß theorem.

$\Gamma$ is <u>not isometric</u>. If $\Gamma$ was isometric, then $\Gamma$ were an isometric $^*$-homomorphism with dense range. Since isometric homomorphisms have closed image, $\Gamma$ were surjective and thus an isometric $^*$-isomorphism $\ell^1(\mathbb{Z}) = C(\mathbb{S}^1)$. Then $\ell^1(\mathbb{Z})$ would be a $C^*$-algebra with the $\ell^1(\mathbb{Z})$-norm, and thus $\|a^*a\|_1 = \|a\|_1^2$ (with the involution as described above). Then, using the $C^*$-property of $C(\mathbb{S}^1)$ and isometry of $\Gamma$, we have

$$\|a^*a\|_1 = \|\Gamma(a^*a)\|_\infty = \|\Gamma(a)^*\Gamma(a)\|_\infty = \|\Gamma(a)\|_\infty^2 - \|a\|_1^2.$$

Now we only need to find $a \in \ell^1(\mathbb{Z})$ with $\|a^*a\|_1 \neq \|a\|_1^2$. Choose $a = \alpha\delta_0 + \beta\delta_1 + \gamma\delta_{-1} = \alpha + \beta\delta_1 + \gamma\delta_{-1}$ (not writing $\delta_0$ as it is the unit).

$$a^*a = (\overline{\alpha} + \overline{\beta}\delta_{-1} + \overline{\gamma}\delta_1)(\alpha + \beta\delta_1 + \gamma\delta_{-1}) = \dots$$

and thus

$$\|a^*a\|_1 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2|\overline{\alpha}\beta + \alpha\overline{\gamma}| + 2|\gamma\beta|$$

while

$$\|a\|_1^2 = (|\alpha| + |\beta| + |\gamma|)^2.$$

Now choosing $\alpha = i$ and $\beta = \gamma = 1$ yields $\|a^*a\|_1 = 5$ and $\|a\|_1^2 = 9$. This shows that $\ell^1(\mathbb{Z})$ does not fulfil the $^*$-property and cannot be a $C^*$-algebra. This is a contradiction, so $\Gamma$ cannot be isometric.

This is also a valid counterexample for the isometry directly, because $a$ has Norm 3, but $\Gamma(a) = (z \mapsto \frac{1}{z} + i + z = 2\Re(z) + i)$ has maximum $2 + i$ with Norm $\sqrt{5} < 3$ on the unit circle. $\Gamma$ is <u>not surjective</u>. This is complicated.

Recall: For $\mathscr{A}$ a Banach algebra, we have a Gelfand representation

$$\Gamma : \mathscr{A} \to C_0(\Omega(\mathscr{A})), a \mapsto (\hat{a} : \Omega(\mathscr{A}) \to \mathbb{C}, \chi \mapsto \chi(a))$$

where $\Omega(\mathscr{A}) = \{\chi : \mathscr{A} \to \mathbb{C} \mid \text{ non-zero hom}\} \subseteq D_{\mathscr{A}^*}(0,1)$ with the weak $^*$-topology. $\Gamma$ is a contractive homomorphism, and if $\mathscr{A}$ is commutative $\|\Gamma(a)\| = r(a) \leq \|a\|$ for all $a \in \mathscr{A}$.

We now want to consider commutative $C^*$-algebras.

**Theorem 7.12 (Gelfand)** *If $\mathscr{A}$ is a commutative $C^*$-algebra, then $\Gamma : \mathscr{A} \to C_0(\Omega(\mathscr{A}))$ is an isometric $^*$-isomorphism.*

For this proof we require a set of lemmas.

**Lemma 7.13** *If $a \in \mathscr{A}$, $\mathscr{A}$ a $C^*$-algebra, with $a = a^*$ then $r(a) = \|a\|$.*

Proof: Use $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$. Notice $\|a^2\| = \|a^*a\| = \|a\|^2$ and $\|a^4\| = \|(a^2)^*a^2\| = \|a^2\|^2 = \|a\|^4$ and likewise for all powers that are powers of 2 we have $\|a^{2^n}\| = \|a\|^{2^n}$. So $r(a) = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|$ is the limit of the subsequence and therefore the limit of the sequence. ☐

**Remark 7.14** In general, $\|a\| \neq r(a)$ if $a \neq a^*$ in a $C^*$-algebra, e.g. $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$. But if $a^*a = aa^*$ ($a$ is normal), then $\|a\| = r(a)$.

PROOF: Exercise. □

**Corollary 7.15** *There exists at most one norm that makes a $^*$-algebra $\mathscr{A}$ into a $C^*$-algebra.*

PROOF: If $\mathscr{A}$ is a $C^*$-algebra with norm $\|\cdot\|$, then for all $a \in \mathscr{A}$ we have $\|a\| = \|aa^*\|^{\frac{1}{2}}$. Note that $a^*a$ is self-adjoint, so by the previous lemma we have

$$\|a\| = \|aa^*\|^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}} = \sup_{\lambda\in\sigma(a^*a)} |\lambda|^{\frac{1}{2}}$$

and this only depends on the algebra structure, not its norm. □

**Corollary 7.16** *If $\varphi : \mathscr{A} \to \mathscr{B}$ is a $^*$-homomorphism from a Banach-$^*$-algebra $\mathscr{A}$ into a $C^*$-algebra $\mathscr{B}$ then $\varphi$ is contractive, i.e. $\|\varphi(a)\|_{\mathscr{B}} \le \|a\|_{\mathscr{A}}$ for all $a \in \mathscr{A}$*

PROOF: Replacing $\mathscr{A}, \mathscr{B}$ by their unitizations $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{B}}$ and extending $\varphi$ to $\tilde{\varphi} : \tilde{A} \to \tilde{B}, a+\lambda 1_{\mathscr{B}} \mapsto \varphi(a) + \lambda 1_{\mathscr{B}}$ shows that we can just assume $\mathscr{A}, \mathscr{B}, \varphi$ to be unital.

Now, if $a \in \operatorname{inv}(\mathscr{A})$, then $\varphi(a) \in \operatorname{inv}(\mathscr{B})$, so it follows

$$\lambda \in \rho_{\mathscr{A}}(a) \Leftrightarrow a - \lambda \in \operatorname{inv}(\mathscr{A}) \Leftrightarrow \varphi(a) - \lambda \in \operatorname{inv}(\mathscr{B}) \Leftrightarrow \lambda \in \rho_{\mathscr{B}}(\varphi(a))$$

so $\rho_{\mathscr{A}}(a) \subseteq \rho_{\mathscr{B}}(\varphi(a))$ and $\sigma_{\mathscr{A}}(a) \supseteq \sigma_{\mathscr{B}}(\varphi(a))$. It follows for the spectral radius: $r(\varphi(a)) \le r(a)$. As $\mathscr{B}$ is a $C^*$-algebra, this implies

$$\begin{aligned}\|\varphi(a)\|_{\mathscr{B}}^2 &= \|\varphi(a)^*\varphi(a)\|_{\mathscr{B}} = \|\varphi(a^*a)\|_{\mathscr{B}} = r(\varphi(a^*a)) \\ &\le r(a^*a) \le \|a^*a\|_{\mathscr{A}} \le \|a^*\|_{\mathscr{A}} \cdot \|a\|_{\mathscr{A}} = \|a\|_{\mathscr{A}}^2\end{aligned}$$

and therefore $\|\varphi(a)\|_{\mathscr{B}} \le \|a\|_{\mathscr{A}}$. □

**Lemma 7.17** *If $\mathscr{A}$ is a $C^*$-algebra and $a \in \mathscr{A}$, then*

*(i) If $a$ is self-adjoint, $\sigma(a) \subseteq \mathbb{R}$.*

*(ii) If $\mathscr{A}$ is unital and $u \in \mathcal{U}(\mathscr{A})$ is unitary (that is, $u^*u = uu^* = 1$) then $\sigma(u) \subseteq \mathbb{S}^1$.*

*(iii) If $a \in \operatorname{inv}(\mathscr{A})$, then $\sigma(a^{-1}) = \sigma(a)^{-1} = \{z^{-1} \mid z \in \sigma(a)\}$.*

*(iv) $\sigma(a^*) = \overline{\sigma(a)}$.*

PROOF: (iii) If $\lambda \in \mathbb{C}, \lambda \neq 0$ and $\lambda - a \notin \operatorname{inv}(\mathscr{A})$. Because $\lambda - a$ is not invertible, $\lambda^{-1}(\lambda - a) = 1 - \lambda^{-1}a$ and $a^{-1}(1-\lambda^{-1}a) = a^{-1} - \lambda^{-1}$ is also not invertible. So we have $\lambda^{-1} - a^{-1} \notin \operatorname{inv}(\mathscr{A})$ and therefore $\sigma(a^{-1}) \subseteq \sigma(a)^{-1}$. The result follows by symmetry.

(iv) Similarly, you can prove (iv).

(ii) If $u \in \mathcal{U}(\mathscr{A})$, then $\sigma(a) \subseteq \mathbb{D} = \{z \in \mathbb{C} \mid |z| \le 1\}$ because

$$\|u\| = \|u^*u\|^{\frac{1}{2}} = \|1\|^{\frac{1}{2}} = 1.$$

So, since $u \in \mathcal{U}(\mathscr{A})$, $u^{-1} = u^* \in \mathcal{U}(\mathscr{A})$ and therefore $\sigma(u)^{-1} = \sigma(u^{-1}) \subseteq \mathbb{D}$. This implies $\|\lambda\| = 1$ for all $\lambda \in \sigma(u)$ and thus $\sigma(u) \subseteq \mathbb{S}^1$.

(i) Assume that $\mathscr{A}$ is unital, otherwise work in $\tilde{\mathscr{A}}$. If $a$ is self-adjoint then $u = \exp(ia) = \sum_{n=0}^{\infty} \frac{i^n a^n}{n!} \in \mathcal{U}(\mathscr{A})$ because $\exp(ia)^* = \exp(-ia)$ and therefore $u^*u = \exp(-ia)\exp(ia) = \exp(0) = 1 = uu^*$. Because of (i) we know $\sigma(u) \subseteq \mathbb{S}^1$. Now, let $\lambda \in \sigma(u)$ and define $b = \sum_{n=1}^{\infty} \frac{i^n(a-\lambda)^n}{n!} = \exp(i(a-\lambda)) - 1$ as well as $c = \sum_{n=1}^{\infty} \frac{i^n(a-\lambda)^{n-1}}{n!} \in \mathscr{A}$ . Consider

$$\begin{aligned}\exp(ia) - \exp(i\lambda 1) &= (\exp(i(a-\lambda)) - 1)\exp(i\lambda) = b\exp(i\lambda) \\ &= \left(\sum_{n=1}^{\infty} \frac{i^n(a-\lambda)^n}{n!}\right)\exp(i\lambda) \\ &= (a-\lambda)\left(\sum_{n=1}^{\infty} \frac{i^n(a-\lambda)^{n-1}}{n!}\right)\exp(i\lambda) \\ &= (a-\lambda)c\exp(i\lambda).\end{aligned}$$

Since $\lambda \in \sigma(a)$ and $c, (a-\lambda)$ commute, $\exp(ia) - \exp(i\lambda)$ is not invertible (or $a - \lambda$ would also be invertible) and we have $\exp(i\lambda) \in \sigma(u) \subseteq \mathbb{S}^1$. But for this to happen, we require $\lambda \in \mathbb{R}$. □

**Corollary 7.18** *If $\mathscr{A}$ is a $C^*$-algebra and $\chi \in \Omega(\mathscr{A})$, then $\chi(a^*) = \overline{\chi(a)}$ for all $a \in \mathscr{A}$. So $\chi$ is a $^*$-homomorphism.*

Proof: If $a \in \mathscr{A}$ is self-adjoint, then $\chi(a) \in \sigma(a) \subseteq \mathbb{R}$ so $\overline{\chi(a)} = \chi(a) = \chi(a^*)$.

Now, if $a \in \mathscr{A}$ is any element we can write it as $a = b + ic$ where $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$ so that $b, c$ are self-adjoint. Now $\chi(b), \chi(c) \in \mathbb{R}$ so

$$\chi(a^*) = \chi(b - ic) = \chi(b) - i \cdot \chi(c) = \overline{\chi(b) + i\chi(c)} = \overline{\chi(b + ic)} = \overline{\chi(a)}$$ □

**Corollary 7.19** *If $\mathscr{A}$ is a commutative $C^*$-algebra and $\mathscr{A} \neq 0$, then $\Omega(\mathscr{A}) \neq \emptyset$.*

Proof: If $\mathscr{A} \neq 0$ there is some self-adjoint non-zero element $a \in \mathscr{A}$ so that $r(a) = \|a\| \neq 0$. But $\sigma(a) \subseteq \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\}$. But for this to be true there must exist a character $\chi \in \Omega(\mathscr{A})$, so $\Omega(\mathscr{A}) \neq \emptyset$. □

Proof (Gelfand):

- **$\Gamma$ is a $^*$-homomorphism**: Consider

  $$\Gamma(a)^*(\chi) = \hat{a}^*(x) = \overline{\hat{a}(\chi)} = \overline{\chi(a)} = \chi(a^*) = \hat{a^*}(\chi) = \Gamma(a^*)(\chi)$$

  so $\Gamma(a)^* = \Gamma(a^*)$.

- **$\Gamma$ is isometric**: We have

  $$\|\Gamma(a)\|^2 = \|\Gamma(a)^*\Gamma(a)\| = \|\Gamma(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$$

  using our lemmas and the $C^*$-property.

- **$\Gamma$ is surjective**: Let $\mathscr{B} := \operatorname{im}(\Gamma) \subseteq C_0(\mathscr{A})$. Then $\mathscr{B}$ is a $C^*$-subalgebra of $C_0(\Omega(\mathscr{A}))$. Then

  - $\mathscr{B}$ does not vanish at any point, i.e. for every point $\chi \in \Omega(\mathscr{A})$ there is a $b \in \mathscr{B}$ with $\chi(b) \neq 0$.

    As $\chi \in \Omega(\mathscr{A})$ means $\chi \neq 0$, there exists an $a \in \mathscr{A}$ with $\chi(a) \neq 0$. But we can rewrite this as $b(\chi) = \hat{a}(\chi) = \chi(a) \neq 0$ for $b = \hat{a}$.

- $\mathscr{B}$ sperates points in $\Omega(\mathscr{A})$, i.e. for every $\chi_1 \neq \chi_2$ in $\Omega(\mathscr{A})$ there exists $b \in \mathscr{B}$ with $b(\chi_1) \neq b(\chi_2)$.
  If $\chi_1 \neq \chi_2$ there exists $a \in \mathscr{A}$ with $\chi_1(a) \neq \chi_2(a)$. Taking $b = \hat{a}$ yields the result.

The result $\mathscr{B} = C_0(\Omega(\mathscr{A}))$ follows from the Stone-Weierstraß-theorem:

> If $X$ is a locally compact Hausdorff space and $B \subseteq C_0(X)$ is a $^*$-subalgebra satisfying
> - $B$ does not vanish on any point of $X$
> - $B$ separates points of $\mathscr{A}$
>
> then $B$ is dense in $C_0(X)$.

So $\operatorname{im}(\Gamma)$ is dense and closed in $C_0(\Omega(\mathscr{A}))$, so $\Gamma$ is surjective. □

**Proposition 7.20** *Conclusion: Every commutative $C^*$-algebra is (up to $^*$-isomorphism) of the form $C_0(X)$ for a locally compact Hausdorff space $X$. Let $\mathscr{A} = C_0(X)$ for a locally compact Hausdorff space $X$. Then $\Omega(\mathscr{A}) \simeq X$ with isomorphism*

$$\varphi : X \to \Omega(C_0(X)), x \mapsto (\mathrm{ev}_x : C_0(X) \to \mathbb{C}, f \mapsto f(x)).$$

Proof:

- $\varphi$ is **well-defined**, because characters are never zero.
- $\varphi$ is **continuous**. Take $x_i \to x$ in $X$. Then, for all $f \in C_0(X)$ we have $\mathrm{ev}_{x_i}(f) \to \mathrm{ev}_x(f)$ because $f$ is continuous and therefore $f(x_i) \to f(x)$. This shows $\mathrm{ev}_{x_i} \to \mathrm{ev}_x$ in the weak $^*$-topology.
- $\varphi$ is **injective**. I f $x_1 \neq x_2$ there exists a function $f \in C_0(X)$ that separates them, but then $\mathrm{ev}_{x_1}(f) \neq \mathrm{ev}_{x_2}(f)$, so $\mathrm{ev}_{x_1} \neq \mathrm{ev}_{x_2}$.
- $\varphi$ is **surjective**. Prove that every $\chi \in \Omega(\mathscr{A})$ is $\chi = \mathrm{ev}_x$ for some $x \in X$.

  We know that the characters of $\mathscr{A}$ are equivalent to the ideals in $C_0(X)$, so this is equivalent to: Every maximal ideal $I \triangleleft C_0(X)$ is of the form $I = C_0(X \setminus \{x_0\}) = \{f \in C_0(X) \mid f(x_0) = 0\}$.

  In Exercise 01-08 we have proven that every closed (2-sided) ideal $I \triangleleft C_0(X)$ has the form $I = C_0(U) := \{f \in C_0(X) \mid f|_{X \setminus U} \equiv 0\}$ for some open $U \subseteq X$.

  > See 01-08 for more details.
  > Take any $f \in I \triangleleft C_0(X)$. First, prove $I^* = I$. Consider $f \in I$ and
  >
  > $$f_n := \sqrt[n]{f^* f} = (\overline{f} f)^{\frac{1}{n}} = |f|^{\frac{2}{n}}.$$
  >
  > We have $f_n \in I$ for all $n$, because $g := f^* f \in I$ and $t \mapsto \sqrt[n]{t}$ is a continuous function that can be uniformly approximated by polynomials on the compact sets. It follows that $f_n = \lim g_n$ where $g_n$ is a polynomial in $g \in I$, so $f_n \in I$. So $f^* f_n \in I$ for all $n$. Then
  >
  > $$\begin{aligned} \|f^* = f_n f^*\|_\infty^2 &= \|(f^* - f_n f^*)(f^* - f_n f^*)\|_\infty = \|(f = f_n f)(f^* - f_n f^*)\|_\infty \\ &= \|f^* f - 2 f^* f f_n + f_n^2 f^* f\|_\infty \end{aligned}$$

$$\leq \|g - g\sqrt[n]{g}\| + \|g - g\sqrt[n]{g}\|\|f_n\| \to 0,$$

because $|g(x) - g(x)\sqrt[n]{g(x)}| \to 0$ pointwise (as the $n$-th square root converges to the 1 on the support and 0 elsewhere) and $|g(x)| \leq \varepsilon$ everywhere except a compact set $K$, and on that $K$ we have $\sup_{x\in K} |g(x)||1-\sqrt[n]{g(x)}| = |g(x_0)||1-\sqrt[n]{g(x_0)}| < \varepsilon$ for some $n \in \mathbb{N}$. We therefore have $f^* = \lim_{n\to\infty} f^* f_n \in I$ and thus $f^* = \lim_{n\to\infty} f_n f^*$. Now let $I \lhd C_0(X)$ closed, so $I^* = I$ and $I$ is a $C^*$-subalgebra of $X$.

Define $U^{\complement} := \{x \in X \mid f(x) = 0 \forall_{f\in I}\}$. This is closed (because for $x_i \to x$ in $X$, $x_i \in U^{\complement}$, we have $0 = f(x_i) \to f(x)$), so $U$ is open. We claim $I = C_0(U)$.

If $f \in I$, $f|_{U^{\complement}} \equiv 0$ per Definition, so $f \in C_0(U)$. Therefore, $I$ is a closed subideal of $C_0(U)$.

$I$ does not vanish on $U$, because if there was an $x \in U$ with $f(x) = 0$ for all $f \in I$, we would have $x \in U^{\complement}$.

$I$ separates the points of $U$. Take $x_1 \neq x_2$. We can choose $h \in C_0(X)$ with $h(x_1) = 1$ and $h(x_2) = 0$ (Uryson) as well as $g \in I$ with $g(x_1) \neq 0$, then $f = g \cdot h \in I$ separates $x_1$ from $x_2$.

Stone-Weierstraß now proves $I = C_0(U)$.

Notice $U \subseteq V \subseteq X$ (open) iff $C_0(U) \subseteq C_0(V) \trianglelefteq C_0(X)$ (see exercise 08-01). So we have a bijection between the opens of $X$ and the ideals of $C_0(X)$. Especially, the maximal ideals of $C_0(X)$ correspond to the maximal open sets, that is the sets of form $X \setminus \{x_0\}$ for some $x_0$, of $X$.

Therefore, if $\chi \in \Omega(C_0(X))$ we have $\ker\chi = C_0(X \setminus \{x_0\})$, so $\chi$ maps a function to 0 if and only if $f$ is zero on $x$. This proves and $\chi = \mathrm{ev}_x$.

- $\varphi$ is **open**. If $X$ is compact, this is clear because $C_0(X) = C(X)$ and unital, so $\Omega(C_0(X))$ is compact and we have a bijection between two compact sets. In general, consider $\tilde{X}$ (the compactification) and use $\widetilde{C_0(X)} \simeq C(\tilde{X})$. So we have a homeomorphism

$$\tilde{X} \to \Omega(C(\tilde{x})) = \Omega(\widetilde{C_0(X)}) \simeq \Omega(C_0(X)) \sqcup \{\chi_\infty\}$$

where $\infty \mapsto \chi_\infty$, so we can restrict the homeomorphism to $X$ and are done. □

**Theorem 7.21 (Spectral inclusion for $C^*$-algebras)** *Let $\mathscr{A} \subseteq \mathscr{B}$ be an inclusion of unital $C^*$-algebras with $1 = 1_{\mathscr{A}} = 1_{\mathscr{B}}$. Then for all $a \in \mathscr{A}$ we have $\sigma_{\mathscr{A}}(a) = \sigma_{\mathscr{B}}(a)$, so $\mathrm{inv}(\mathscr{A}) = \mathrm{inv}(\mathscr{B}) \cap \mathscr{A}$.*

Proof: If $a$ is self-adjoint, that is $a^* = a$, then $\sigma_{\mathscr{A}}(a)\backslash\mathbb{R}$, so $\sigma_{\mathscr{A}}$ has no holes, i.e. the complement $\mathbb{C} \subseteq \sigma_{\mathscr{A}}(a)$ is connected in $\mathbb{C}$. By the general result on Banach algebras $\sigma_{\mathscr{A}}(a) = \sigma_{\mathscr{B}}(a)$. In particular, this implies $a \in \mathrm{inv}(\mathscr{A}) \Leftrightarrow a \in \mathrm{inv}(\mathscr{B})$ for all self-adjoint $a \in \mathscr{A}$.

We now prove that this holds for all $a \in \mathscr{A}$. Of course, $\mathrm{inv}(\mathscr{A}) \subseteq \mathrm{inv}(\mathscr{B}) \cap \mathscr{A}$. Let $a \in \mathscr{A}$ such that $a \in \mathrm{inv}(\mathscr{B})$. Then there exists $b \in \mathscr{B}$ such that $ab = ba = 1$ and $b^*a^* = a^*b^* = 1 \Leftrightarrow bb^*a^*a = 1 = a^*abb^*$. Therefore, $a^*a \in \mathrm{inv}\,\mathscr{B} \cap \mathscr{A} \subseteq \mathrm{inv}(\mathscr{A})$ because $a^*a$ is self adjoint. So there exists $c \in albebraA$ with $ca^*a = 1 = a^*ac$ and thus $ca^*ab = ca^* = b$, so $b \in \mathscr{A}$ as it is the product of two elements $a^*, c \in \mathscr{A}$. This concludes the proof, as $a$ is now invertible in $\mathscr{A}$. □

**Definition 7.22** *We say $a \in \mathscr{A}$ (for $\mathscr{A}$ a $C^*$-algebra) is* ***normal*** *if $a^*a = aa^*$. This means $C^*(a)$ (the $C^*$-subalgebra of $\mathscr{A}$ generated by $a$) is commutative. Then $C^*(a) \simeq C_0(X)$.*

**Lemma 7.23** *Let $a \in \mathscr{A}$ ($C^*$-algebra) be a normal element. Assume that $1 \in \mathscr{A}$ (unital). Then $\Omega(C^*(a,1)) \simeq \sigma(a)$ by homeomorphism $\chi \mapsto \chi(a)$. In general, if $\mathscr{A}$ is possibly not unital, then $\Omega(C^*(a)) \simeq \sigma(a) \setminus \{0\}$. In particular, $\chi(a) = 0$ only if $a = 0$ but then $C^*(a)$ is just the zero space.*

Proof: It is enough to consider the unital case.

Consider $\varphi : \Omega(C^*(a,1)) \to \sigma(a), \chi \to \chi(a)$ which is well-defined because $\chi(a) \in \sigma(a)$.

- $\varphi$ is **continuous**. If $\chi_i \to \chi$ in $\Omega(C^*(a,1))$ then this also converges point wise, so $\chi_i(a) \to \chi(a)$.
- $\varphi$ is **injective**. Take $\chi_1, \chi_2 \in \Omega(C^*(a,1))$ with $\chi_1(a) = \chi_2(a)$. Since $\chi_1(1) = 1 = \chi_2(1)$, so the two characters coincide on the generators and are thus equal by linearity and continuity.
- $\varphi$ is **surjective**. We know that $\sigma(a) = \{\chi(a) \mid \chi \in \Omega(B)\}$ for all commutative unital Banach algebras $B$, in particular for $B = C^*(a,1)$.

Because both spaces are compact this concludes the proof. □

**Theorem 7.24 (Fundamental theorem of continuous functional calculus)**

*Let $\mathscr{A}$ be a unital $C^*$-algebra and $a \in \mathscr{A}$ normal. Then there exists a unique unital $^*$-homomorphism $\varphi : C(\sigma(a)) \to \mathscr{A}$ such that $\mathrm{id}_{\sigma(a)} \mapsto a$.*

*In general, if $\mathscr{A}$ is possibly not unital, there exists a unique $^*$-homomorphism $\varphi : C_0(\sigma(a)) \to \mathscr{A}$ where $C_0(\sigma(a)) \coloneqq \{f \in C(\sigma(a)) \mid f(0) = 0\}$.*

*Both of these morphisms are also isometric.*

Notation: If $f \in C(\sigma(a))$ we write $f(a) \coloneqq \varphi(a)$. Notice: If $f$ is a polynomial in $z, \overline{z}$ then $f(a) = \varphi(a)$ as usual.

Proof: Consider $1 \in \mathscr{A}$ and let $\mathscr{B} = C^*(a,1) \subseteq \mathscr{A}$. Then $\mathscr{B}$ is commutative because $a$ is normal (i.e. commutes with its adjoint). By Gelfand, we get an isometric $^*$-isomorphism $T : \mathscr{B} \to C(\Omega(\mathscr{B})), b \mapsto \hat{b}$. By the Lemma, $\Omega(\mathscr{B}) \equiv \sigma(a), \chi \mapsto \chi(a)$. Via this identification (homeomorphism), we have $\hat{b}(\chi) = \chi(b)$ and $\hat{a}(\chi) = \chi(a)$. So $\hat{a}$ corresponds to $z \in C(\sigma(a)) \simeq C(\Omega(\mathscr{B}))$. Therefore, considering the inverse of $T$ and identifying $\Omega(\mathscr{B}) \simeq \sigma(a)$ we get an isometric

$$C(\sigma(a)) \simeq C(\Omega(C^*(a,1))) \simeq C^*(a,1) \simeq \mathscr{A}.$$

This gives $\varphi$ as defined.

The **non-unital case**: Just consider $\tilde{\mathscr{A}}$. □

**Example 7.25** Let $f(z) = \exp(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. $f$ is a continuous function on the whole plane. If $a \in \mathscr{A}$ is normal, then $f(a) = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$. In general, $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ (or $f(z) = \sum_{n=0}^{\infty} \lambda_n (z - z_0)^n$), so $f(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ if $\sigma(a) \subseteq \mathrm{Domain}(f)$.

**Theorem 7.26** *Let $\mathscr{A}$ be unital $C^*$-algebra and $a \in \mathscr{A}$ be normal. If $f \in C(\sigma(a))$, then $\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\}$.*

*Moreover, if $g \in C(\sigma(f(a)))$, then $g(f(a)) = (g \circ f)(a)$.*

Proof: Let $\mathscr{B} = C^*(a,1) \subseteq \mathscr{A}$. $\mathscr{B}$ is commutative and unital. Then $f(a) \in \mathscr{B}$ and $\sigma(f(a)) = \sigma_{\mathscr{B}}(f(a))$. Now notice $\chi(f(a)) = f(\chi(a))$ since both maps

$$f \mapsto \chi(f(a))$$

$$f \mapsto f(\chi(a))$$

are unital $^*$-homomorphisms that coincide on $z$. Therefore,

$$\sigma(f(a)) = \{\chi(f(a)) \mid \chi \in \Omega(\mathscr{B})\} = \{f(\chi(a)) \mid \chi \in \Omega(\mathscr{B})\} = f(\sigma(a)).$$

Now to prove $(g \circ f)(a) = g(f(a))$. Let $C = C^*(1, f(a)) \subseteq \mathscr{B} = C^*(1, a) \subseteq \mathscr{A}$. Let $\chi \in \Omega(\mathscr{B})$. Then $\chi_C := \chi|_C \in \Omega(C)$. So $(g \circ f)(a)$ is sensibly defined and an element of $\mathscr{B}$, so we can apply a character:

$$\begin{aligned}\chi((g \circ f)(a)) &= (g \circ f)(\chi(a)) = g(f(\chi(a))) = g(\chi(f(a))) = g(\chi_C(f(a))) \\ &= \chi_C(g(f(a))) = \chi(\underbrace{g(f(a))}_{\in \mathscr{B}})\end{aligned}$$

Because the Gelfand-transform is injective, this implies $(g \circ f)(a) = g(f(a))$. □

**Proposition 7.27** *Let $\mathscr{A}$ be a unital $C^*$-algebra and $u \in \mathcal{U}(\mathscr{A}) = \{u \in \mathscr{A} \mid u^*u = 1 = uu^*\}$. If $\sigma(u) \neq \mathbb{S}^1$ there exists a self-adjoint $a \in \mathscr{A}$ with $u = \exp(ia)$.*

Proof: The idea is to take $\log \approx \exp^{-1}$. Problem: exp is not invertible as a complex function, because it is $2\pi i$-periodic. We will need to restrict it. Consider the principal branch of the logarithm, $\log(z) = \log|z| + i\arg(z)$.

Given that $\sigma(a) \neq \mathbb{S}^1$, there exists an $\lambda \in \mathbb{S}^1 \setminus \sigma(a)$ and therefore also an $f_\lambda \in C(\mathbb{S}^1 \setminus \{\lambda\})$ (so some form of argument-mapping of $z$) such that $\exp(if_\lambda(z)) = z$. This $f_\lambda$ is real-valued, continuous and analytical. Now use functional calculus: Let $a := f_\lambda(u) \in \mathscr{A}$. Since $f_\lambda$ is real-valued, it is self-adjoint in the algebra, so $a$ is also self-adjoint. By the previous theorem $\exp(ia) = \exp(if_\lambda(u)) = (\exp \circ i f_\lambda)(u) = u$.

### Multiplier Algebras

This is another kind of unitization. We will consider $\mathscr{A} \to M(\mathscr{A}) \ni \mu$ such that $\mu \cdot a \in \mathscr{A} \ni a \cdot \mu$ so $\mathscr{A} \trianglelefteq M(\mathscr{A})$. Remember that this was the case for the usual unitization, with Quotient $\mathbb{C}$. Here, the multiplier is usually much bigger, so the quotient is as well. In fact, $\mathscr{A} \times \mathbb{C}$ is the 'smallest' unitization while $M(\mathscr{A})$ is the 'largest' one.

**Definition 7.28 (Multiplier, see Murphy)** *Let $\mathscr{A}$ be an algebra. A* ***multiplier*** *of $\mathscr{A}$ is a pair $\mu = (L, R)$ where $L, R : \mathscr{A} \to \mathscr{A}$ are linear maps such that*

*(i)* $L(ab) = L(a) \cdot b$ *or* $\mu(ab) = (\mu a)b$

*(ii)* $R(ab) = a \cdot R(b)$ *or* $(ab)\mu = a(b\mu)$

*(iii)* $a \cdot L(b) = R(a) \cdot b$ *or* $a(\mu b) = (a\mu)b$.

*To simplify this, use the notation $\mu \cdot a := L(a)$ and $a \cdot \mu := R(a)$.*

For the space of all multipliers we write $M(A) = \{\mu = (L, R) \mid \mu \text{ multiplier}\}$. This is a $\mathbb{C}$-vector space with

$$(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2) \qquad \lambda(L_1, R_1) = (\lambda L_1, \lambda R_2)$$

and an algebra with

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 \cdot L_2, R_2 \cdot R_1).$$

If $\mathscr{A}$ is a ${}^*$-algebra, we further define

$$(L,R)^* = (R^*, L^*) \text{ where } L^*(a) \coloneqq L(a^*)^* \text{ and } R^*(a) \coloneqq R(a^*)^*$$

Moreover, we have a canonical $({}^*)$-homomorphism $\iota : \mathscr{A} \to M(\mathscr{A}), a \mapsto (L_a, R_a)$ where $L_a(b) = ab$ and $R_a(b) = ba$. Note: $\iota$ is always a $({}^*)$-homomorphism but injective if and only if

$$\forall_{a\in\mathscr{A}} \;\; \begin{aligned} a\cdot b = 0 \;\forall_b \Rightarrow a = 0 \\ b \cdot a = 0 \;\forall_b \Rightarrow a = 0 \end{aligned}$$

i.e. $\mathscr{A}$ is an essential ideal of itself. This is not always true for a general algebra, consider the algebra with the 0-product $a \cdot b = 0$, but it always holds for $C^*$-algebras or if $\mathscr{A}$ is unital already.

More generally this holds if $\mathscr{A}$ is a Banach algebra with an **approximate unit**, a net $e_i \subseteq \mathscr{A}$ such that $e_i a \to a$ and $a \cdot e_i \to a$ for any $a \in \mathscr{A}$ as well as $\|e_i\|$. This is always the case for unital and $C^*$-algebras.

Assume $\iota$ is injective. Then $\mathscr{A}$ is identified with an essential $({}^*)$-ideal of $M(\mathscr{A})$.

**Remark 7.29 (Norms on the multiplier)** If $\mathscr{A}$ is a Banach algebra with an approximate unit, we define for $\mu = (L,R) \in M(A)$ the norm

$$\|\mu\| \coloneqq \|L\| = \|R\| < \infty.$$

Proof: To show $\|L\|, \|R\| < \infty$ we use the Closed Graph Theorem. Say we have $(a_n) \subseteq \mathscr{A}$ with $a_n \to a$ and $L(a_n) \to b$. Take $c \in \mathscr{A}$ and consider

$$c \cdot L(a) = R(c) \cdot a = \lim_{n\to\infty} R(c) \cdot a_n = \lim_{n\to\infty} c \cdot L(a_n) = c \cdot b.$$

Because of the approximate unit (or $\iota$ injective) we have $L(a) = b$. This shows that $L$ (and, analogously, $R$) are bounded. Now to prove $\|L\| = \|R\|$. Take any $a \in \mathscr{A}$ and consider

$$\|L(a)\| \overset{\text{approx. unit}}{=} \sup_{\|b\|\leq 1} \|bL(a)\| = \sup_{\|b\|\leq 1} \|R(b)a\| \leq \sup_{\|b\|\leq 1} \|R(b)\|\|a\| \leq \|R\| \cdot \|a\|$$

which implies $\|L\| \leq \|R\|$. By symmetry of the situation, we have $\|L\| = \|R\|$. $\square$

With the norm above, $M(\mathscr{A})$ becomes a Banach algebra.

**Proposition 7.30** *If $\mathscr{A}$ is a $C^*$-algebra then $M(\mathscr{A})$ is too.*

Proof: Write $\mu = (L,R)$. We compute $\mu^*\mu = (R^*, L^*) \cdot (L,R) = (R^*L, RL^*)$. So $\|\mu\mu^*\| = \|R^*L\|$. Take $a \in \mathscr{A}$ with $\|a\| \leq 1$. Then

$$\|L(a)\|^2 = \|L(a)L(a)^*\| = \|L(a)L^*(a^*)\| = \|R^*(L(a))a^*\| \leq \|R^*(L(a))\| \leq \|R^*L\|$$

This shows $\|L\|^2 \leq \|R^*L\|$ and therefore $\|\mu\|^2 = \|L\|^2 \leq \|R^*L\| = \|\mu^*\mu\|$. Because $\|\mu\|^2 \geq \|\mu\mu^*\|$ is clear by submultiplicativity, the $C^*$-property follows. $\square$

Compare now $\tilde{\mathscr{A}}$ and $M(\mathscr{A})$. We have $\mathscr{A} \trianglelefteq \tilde{A}$ and $\mathscr{A} \trianglelefteq M(\mathscr{A})$. When are these ideals essential?

**Lemma 7.31** *Let $\mathscr{A}$ be a $C^*$-algebra or Banach algebra with approximate unit. $\mathscr{A} \trianglelefteq \tilde{\mathscr{A}}$ if and only if $\mathscr{A}$ is not unital.*

PROOF: Suppose that $\mathscr{A}$ is unital with $1_{\mathscr{A}}$ as the unit. In this case, take $p = 1 - 1_{\mathscr{A}} \in \tilde{\mathscr{A}}$ (where $1 = (0, 1)$ is the unit in $\tilde{\mathscr{A}}$). Notice that $p \cdot \mathscr{A} = 0$, but $p \neq 0$. So $\mathscr{A}$ is not essential in $\tilde{\mathscr{A}}$.

Suppose that $\mathscr{A}$ is not unital. To prove: For $a + \lambda \cdot 1 \in \tilde{\mathscr{A}}$ and $(a + \lambda \cdot 1)\mathscr{A} = 0$ we have $a = 0$, $\lambda = 0$. So take any $(a + \lambda \cdot 1) \cdot b = 0$ for all $b \in \mathscr{A}$, that is $ab + \lambda b = 0$. This means $L_a(b) = -\lambda b$, that is $L_a = -\lambda \operatorname{id}_{\mathscr{A}}$. Notice $L : \mathscr{A} \to \mathcal{L}(\mathscr{A})$, a unital algbra with unit $\operatorname{id}_{\mathscr{A}}$, is an injective (because $\iota$ is injective) algebra homomorphism. If $\lambda \neq 0$, then division by $\lambda$ implies $\operatorname{id}_{\mathscr{A}} \in \operatorname{im}(L) \simeq \mathscr{A}$. But then $\mathscr{A}$ has a unit, a contradiction. So $\lambda = 0$. Then $a \cdot b = 0$ for every $b$, so $a = 0$ as well. This shows that $\mathscr{A}$ is an essential ideal of $\tilde{\mathscr{A}}$. $\square$

**Remark 7.32** Let $\mathscr{A}$ be a $C^*$-algebra or Banach algebra with approximate unit. Then $\mathscr{A}$ is unital if and only if $M(\mathscr{A}) = \mathscr{A}$.

PROOF: One direction is simple: $M(\mathscr{A})$ is always unital, so $\mathscr{A} \simeq M(\mathscr{A})$ implies that $\mathscr{A}$ is unital.

Let now $\mathscr{A}$ be unital and prove that every multiplier is of the form $(L_a, R_a)$. Let $\mu = (L, R) \in M(\mathscr{A})$ and define $a := L(1_{\mathscr{A}})$. Then $L_a(b) = ab = L(1_{\mathscr{A}})b = L(b)$, so $L = L_a$. Analogously we can prove $R = R_a$. This shows that $\iota$ is surjective, and since it is already injective (because $\mathscr{A}$ is either $C^*$ or has an approximate unit) it is an isomorphism. $\square$

Say $\mathscr{A}$ is a $C^*$-algebra (or a Banach algebra with an approximate unit) and not unital. Then $\iota : \mathscr{A} \to M(\mathscr{A}), a \mapsto \mu_a = (L_a, R_a)$ extends to a (*)-embedding

$$\tilde{\iota} : \tilde{\mathscr{A}} \to M(\mathscr{A}), a + \lambda \cdot 1 \mapsto \iota(a) + \lambda \cdot \underbrace{(\operatorname{id}, \operatorname{id})}_{= \operatorname{id}_{M(\mathscr{A})}}.$$

More generally: If $\mathscr{B}$ is any $C^*$-algebra that contains $\mathscr{A}$ as an essential ideal (closed), then $\mathscr{B}$ embeds in the multiplier algebra via the following map:

$$\lambda : \mathscr{B} \to M(\mathscr{A}), b \mapsto (L_b, R_b)$$

where $L_b, R_b$ are the usual left and right multiplication. We have $L_b(a), R_b(a) \in \mathscr{A}$ for any $a \in \mathscr{A}$ because $\mathscr{A}$ is an ideal. The above is a universal property of the multiplier algebra. $M(\mathscr{A})$ is the largest unital $C^*$-algebra that contains $\mathscr{A}$ as an essential ideal.

**Example 7.33** Take $\mathscr{A} = C_0(X)$ (for a locally compact Hausdorff-space, so a commutative $C^*$-algebra). Then $\tilde{\mathscr{A}} = C(\tilde{X})$ where $\tilde{X} = X \sqcup \{\infty\}$. One can now show $M(\mathscr{A}) \simeq C(\beta X)$ where $\beta X$ is the Stone-Cech-compactification of $X$. This can be proven using the universal property and the universal property of $\beta X$: $\beta X$ is a compact Hausdorff space such that $X \hookrightarrow \beta X$ as a dense open topological subspace and for every other compact Hausdorff space $K$ such that $X \to K$ via a continuous function $f$ there exists a unique continuous extension $betaf : \beta X \to K$.

First: Prove that $M(\mathscr{A})$ is even commutative. Then it is the continuous functions on some space, use the spectrum and compare the universal properties. For commutativity, one can show $M(C_0(X)) \simeq C_b(X)$ via the universal property.

## 7.2 Positive Elements of $C^*$-algebras

**Definition 7.34** *Let $\mathscr{A}$ be a $C^*$-algebra. We say that $a \in \mathscr{A}$ is positive (and write $a \geq 0$) if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$.*

*The set of all positive elements of a given algebra we notate as $\mathscr{A}_+$.*

**Example 7.35** Let $A = \underline{C_0(X)}$ (commutative) and $f \in \mathscr{A}$. Then $f = f^*$ iff $f$ is real (that is $f : X \to \mathbb{R}$). Since $\sigma(f) = \overline{\operatorname{im}(f)}$ we see that $f \geq 0$ iff $f(x) \geq 0$ for all $x \in X$.

**Theorem 7.36** *If $a \in \mathscr{A}$ for $\mathscr{A}$ a $C^*$-algebra and $a \geq 0$ then there exists a unique $b \in \mathscr{A}_+$ such that $b^2 = a$. We sometimes notate this as $b = \sqrt{a} = a^{\frac{1}{2}}$.*

Proof: Since $a$ is positive, it is self-adjoint and therefore normal. Continuous functional calculus:

$$\varphi : C_0(\sigma(a)) \to \mathscr{A}, f \mapsto f(a)$$

Apply this to $f(x) = \sqrt{x}$. Notice that $f \in C_0(\sigma(a))$ because $\sigma(a) \subseteq [0, \infty)$. Now simply choose $b = f(a) = \sqrt{a}$. Since $\varphi$ is a $^*$-homomorphism, we have $b^2 = \varphi(f)^2 = \varphi(f^2) = \varphi(id) = a$.

Reminder: Writing '$f(a)$' does not mean to imply that $a \in \mathscr{A}$ can simply be plugged into the function $f : \sigma(\mathscr{A}) \to \mathbb{C}$ but is simply a different way of writing $\varphi(f) \in \mathscr{A}$.

**Uniqueness**: Suppose $c \in \mathscr{A}_+$ such that $c^2 = a$. Then $c$ commutes with $c^2 = a$ and therefore $c$ commutes with $b = \sqrt{a}$ since $b = \lim_{n\to\infty} p_n(a)$ (polynomial approximation). Then $B := C^*(b, c) \subseteq \mathscr{A}$ is a commutative $C^*$-algebra so $B \simeq C_0(X)$ for some locally compact Hausdorff space $X$. Since $a, b, c \in B = C_0(X)$ we have $a \simeq f, b \simeq g, c \simeq h \in C_0(X)$ with $f = g^2 = h^2$ where all these functions are positive. But then $f(x) = g(x)^2 = h(x)^2$ for all $x$. Because $g(x), h(x) \geq 0$ for all $x$, this shows $g(x) = h(x)$ for all $x$ and therefore $g = h$ and $b = c$.□

**Remark 7.37** Given any self-adjoint element $a \in \mathscr{A}$ $(a^* = a)$ we can write it as $a^+ - a^-$ where $a^+, a^- \geq 0$ and $a^+ \cdot a^- = 0$. Just define $f(x) = \frac{|x|+x}{2}$ and $g(x) = \frac{|x|-x}{2}$. Both are positive functions with $f \cdot g = 0$. Define $a^+ = f(a)$ and $a^- = g(a)$ (once again per continuous functional calculus), transferring the neccessary properties:

$$\begin{aligned} f(a) - g(a) &= \varphi(f) - \varphi(g) = \varphi(f - g) = \varphi(\mathrm{id}) = a \\ f(a) \cdot g(a) &= \varphi(f) \cdot \varphi(g) = \varphi(f \cdot g) = \varphi(0) = 0 \\ \sigma(f(a)) &= \sigma(\varphi(f)) \subseteq \sigma(f) = \overline{\mathrm{im}(f)} = [0, \infty) \end{aligned}$$

**Remark 7.38** If $\mathscr{A}$ is unital $C^*$-algebra and $a \in \mathscr{A}$ is self-adjoint with $\|a\| \leq 1$, so $\sigma(a) \subseteq [-1, 1]$. Define

$$f(x) = x + i\sqrt{1 - x^2} \qquad g(x) = x - i\sqrt{1 - x^2}$$

This means that $f, g \in \mathcal{U}C(\sigma(a))$ (Recall that unitaries of $\mathcal{U}(\mathscr{A}) = \{u \in \mathscr{A} \mid u^*u = 1 = uu^*\}$) and $\frac{f+g}{2} = \mathrm{id}_{\sigma(a)}$. So if we now define $u := f(a), v := g(a) \in C^*(a, 1) \subseteq \mathscr{A}$ we have $\frac{u+v}{2} = a$. In particular $\mathscr{A} = \mathrm{span}(\mathcal{U}(\mathscr{A}))$.

**Lemma 7.39** *Let $\mathscr{A}$ be a unital $C^*$-algebra, $a \in \mathscr{A}$ self-adjoint and $t \in \mathbb{R}_+$.*

*(i) If $a \geq 0$ and $\|a\| \leq t$ then $\|a - t\| \leq t$.*

*(ii) Conversely, if $\|a = t\| \leq t$ then $a \geq 0$.*

Proof: Replace $\mathscr{A}$ by $C^*(a, 1)$ we may assume that $\mathscr{A} = C(X)$ is commutative and $X$ compact. Let $a = f \in C_(X)$ be a self-adjoint, real function and $t \geq 0$ a real number.

(i) $f \geq 0$ and $\|f\|_\infty \leq t$ and thus $f(x) - t \in [-t, 0]$ for all $x \in X$, so $\|f - t\| \leq t$.

(ii) Let $f \in C(X)$ be a self-adjoint real function with $\|f - t\| \leq t$, so $|f(x) - t| \leq t$ for every $x$. But if $f(x) < 0$ for any $x \in X$ we have $f(x) - t < t$ and thus $|f(x) - t| > t$, a contradiction. So $f$ must be positive. □

**Corollary 7.40** *If $\mathscr{A}$ is a $C^*$-algebra, then $\mathscr{A}_+$ is a closed subset (but not subspace!) of $\mathscr{A}$.*

PROOF: Taking unitization, we may assume that $\mathscr{A}$ is unital. Let $(a_n) \subseteq \mathscr{A}_+$ and $a_n \to a \in \mathscr{A}$. Then $a_n^* = a_n$ for all $n \in \mathbb{N}$ and therefore $a$ is also self-adjoint. There also exists $t \geq 0$ with $\|a_n\| \leq t$ for all $n \in \mathbb{N}$ and by the Lemma $\|a_n - t\| \leq t$ and therefore $\|a - t\| \leq t$. Again by the Lemma $a \geq 0$. □

**Corollary 7.41** *If $\mathscr{A}$ is a $C^*$-algebra and $a, b \in \mathscr{A}_+$ then $a + b \in \mathscr{A}_+$.*

PROOF: Taking unitization, we may assume that $\mathscr{A}$ is unital. Since $a, b \geq 0$ by $t = \|a\|, \|b\|$ we have $\|a - \|a\|\| \leq \|a\|$ and $\|b - \|b\|\| \leq \|b\|$. Then

$$\|(a + b) - (\|a\| + \|b\|)\| = \|(a - \|a\|) + (b - \|b\|)\| \leq \|(a - \|a\|)\| + \|(b - \|b\|)\| \leq \|a\| + \|b\|$$

and $a + b$ is positive by the lemma. □

**Theorem 7.42** *If $\mathscr{A}$ is a $C^*$-algebra and $a \in \mathscr{A}$ then $a^*a \geq 0$.*

PROOF: First, we prove that if $-a^*a \geq 0$ then $a = 0$. For this we use the following observation $\sigma(bc) \setminus \{0\} = \sigma(cb) \setminus \{0\}$ (the two sets are equal except for the zero, which may be contained in one but not the other) because for $b, c$ in a unital algebra and $1 - bc \in \operatorname{inv} \mathscr{A}$ iff $1 - cb \in \operatorname{inv}(\mathscr{A})$ and if $d := (1 - bc)^{-1}$ then $(1 - cb)^{-1} = 1 + cdb$.

Therefore, if $-a^*a \in \mathscr{A}_+$ then also $-a^*a \in \mathscr{A}_+$ (notice that $a, a^*$ are self-adjoint). Then write $a = b + c$ with $b, c \in \mathscr{A}$ self-adjoint. Then

$$a^*a + aa^* = (b - ic)(b + ic) + (b + ic)(b - ic) = b^2 + c^2 + ibc - icb + b^2 + c^2 + icb - icb = 2b^2 + 2c^2.$$

and we can write $a^*a = 2b^2 + 2c^2 - aa^*$. The squares are certainly positive and we have assumed $-aa^* \geq 0$, but then $a^*a \geq 0$. We see that $aa^* \geq 0$ as well, so the spectrum has to be zero.

Now suppose that $a \in \mathscr{A}$ arbitrarily. We show that $a^*a \geq 0$. Let $b := a^*a$. Then $b \in \mathscr{A}$ is self adjoint with $b = b^+ - b^-$ where $b^+, b^- \geq 0$. Let $c := ab^-$. Then

$$-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \geq 0$$

and $c$ must be 0 by our first result. This implies $(b^-)^3 = 0$ so $b^- = 0$. It follows that $b = b^+ \geq 0$.□

**Definition 7.43** *Let $\mathscr{A}$ be a self-adjoint algebra and $a, b \in \mathscr{A}$. We write $a \leq b$ if $b - a \geq 0$. This turns $\mathscr{A}$ into a poset. Because $A_+$ is a cone, that is $A_+ + A_+ \subseteq A_+$ and $\mathbb{R}_+ \cdot \mathscr{A}_+ \subseteq \mathscr{A}$ as well as $A_{\text{self-adjoint}} = A_+ - A_+$ and $A_+ \cap -A_+ = \{\}$.*

**Theorem 7.44** *Let $\mathscr{A}$ be a $C^*$-algebra.*

*(i)* $A_+ = \{a^*a \mid a \in \mathscr{A}\}$

*(ii)* *$a, b$ self-adjoint and $c \in \mathscr{A}$. Then $a \leq b$ imples $c^*ac \leq c^*bc$.*

*(iii)* *$0 \leq a \leq b$ implies $\|a\| \leq |b\|$*

*(iv)* *If $\mathscr{A}$ is unital and $a, b \geq 0$ with $a \leq b$ and $a, b \in \operatorname{inv}(\mathscr{A})$ then $b^{-1} \leq a^{-1}$.*

PROOF:

(i) It follows from the previous theorem. The fact that $a \in \mathscr{A}_+$ has a square root $a = b^2 = b^*b$ with $b \geq 0$.

(ii) $c^*bc - c^*ac = c^*(b - a)c$ and if we set $b - a = d^*d$ for a $d \in \mathscr{A}$ we receive $c^*(b - a)c = c^*d^*dc = (dc)^*dc \geq 0$.

(iii) We may assume $1 \in \mathscr{A}$. Notice that $b \leq \|b\| \cdot 1$ (consider the commutative case). So wie have $a \leq b \leq \|b\| \cdot 1$ and therefore $a \leq \|b\| \cdot 1$ so $\|a\| \leq \|b\|$.

(iv) Let $a, b \in \operatorname{inv} \mathscr{A}$, $a, b \geq 0$ and $a \leq b$. We know that $\sigma(b^{-1}) = \sigma(b)^{-1} \subseteq \mathbb{R}_+$ and thus $b^{-1} \geq 0$ and Similarly $a^{-1} \geq 0$. Notice that if $c \geq 1$ (in $\mathscr{A}$) then $c \in \operatorname{inv} \mathscr{A}$ (as $\sigma(c-1) \subseteq [0, \infty)$ and thus $\sigma(c) \subseteq [1, \infty)$) and $c^{-1} \leq 1$ (think once again commutative).

Now we have $a \leq b$. Then $1 = a^{-\frac{1}{2}} a a^{-\frac{1}{2}} \leq a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$. Then $(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) = (a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}) \leq 1$ by the above, so conjugation yields $b^{-1} \leq a^{-1}$. □

## 7.3 Approximate units

**Definition 7.45** *Let $\mathscr{A}$ be a Banach algebra. An* ***approximate unit*** *for $\mathscr{A}$ is a net $(e_i)_{i \in I} \subseteq \mathscr{A}$ such that $\|e_i\| \leq 1$ and $e_i a \to a, a e_i \to a$ for all $a \in \mathscr{A}$. If $\mathscr{A}$ is a $C^*$-algebra, then we (usually) also assume that $e_i \geq 0$ and $(e_i)$ is increasing.*

**Example 7.46** Let $\mathscr{A} = C_0(X)$ be a commutative $C^*$-algebra ($X$ locally compact and Hausdorff). Then a net $(f_i)_{i \in I}$ is an approximate unit if and only if $1 \geq f_i(x) \geq f_j(x) \geq 0$ for all $x \in X$ and $j \leq i$ and $f_i g \to g$ for al $g \in C_0(X)$, that is $f_i(x) g(x) \to g(x)$ uniformly on $X$. This is equivalent to $f_i(x) \to 1$ uniformly on compacts.

**Example 7.47** Let $\mathscr{A} = \mathcal{K}(H)$, the span of the compact operators on a Hilbert space $H$, and use physics notation: $|\xi\rangle\langle\eta|(\zeta) = \xi\langle\eta, \zeta\rangle$. Let $(\xi_i)_{i \in I} \subseteq H$ be an orthonormal basis. For each $F \subseteq I$ finite we define

$$e_F \coloneqq \sum_{i \in F} |\xi_i\rangle\langle\xi_i| \in \mathcal{K}(H)$$

In particular, $0 \leq e_F \leq 1$ (because $\|e_F\| \leq 1$) and $e_F \leq e_G$ if $F \subseteq G$. Then $(e_F)_{F \subseteq I \text{ finite}}$, if ordered by size, is an approximate unit of for $\mathcal{K}$.

If $H$ is separable, we could also take $e_n = \sum_{i=1}^n |\xi_i\rangle\langle\xi_i|$. Just check that $e_F(\zeta) = \sum_{i \in F} \xi_i \langle \xi_i, \zeta\rangle \to \zeta$, so $e_F \to 1$ strongly in $B(H)$ (the bounded operators). Then it follows $e_F a \to a$ for al $a \in \mathcal{K}(H)$ and $a e_F \to a$ likewise.

**Remark 7.48** If $\mathscr{A}$ already has a unit $1 \in \mathscr{A}$, then $(e_i) \subseteq \mathscr{A}$ is an approximate unit iff $e_i \to 1$ (by the norm) and $0 \leq e_i \leq e_j \leq 1$ for $i \leq j$.

In particular, the constant net (1) is an approximate unit in any unital Banach algebra.

**Theorem 7.49** *Every $C^*$-algebra has an approximate unit. Moreover if $\mathscr{A}$ is a $C^*$-algebra and*

$$\Lambda \coloneqq \{a \in \mathscr{A}_+ \mid \|a\| < 1\}$$

*then $\Lambda$ is directed with the canonical order of $\mathscr{A}_+ \subseteq \mathscr{A}_{\text{self-adjoint}}$ and the canonical net*

$$(e_\lambda)_{\lambda \in \Lambda} e_\lambda = \lambda$$

*is an approximate unit.*

Proof: $\Lambda$ is directed. To prove: For every $a, b \in \Lambda$ there is a $c \in \Lambda$ such that $a, b \leq c$. Indeed, if $a \in \mathscr{A}_+$, then $1 + a \geq 1$ in $\tilde{\mathscr{A}} = \mathscr{A} + \mathbb{C} \cdot 1$. Here, we work in the unitization for a moment, but do not assume we have a unit in $\mathscr{A}$! In particular, $1 + a \in \operatorname{inv}(\tilde{\mathscr{A}})$ and $a \cdot (1+a)^{-1} \in \mathscr{A}$ as $A \trianglelefteq \tilde{A}$.

Notice: $a(1+a)^{-1} = (a + 1 - 1)(1+a)^{-1} = 1 - (1+a)^{-1}$ in the unitization.

Claim: For $a, b \in \mathscr{A}_+$ and $a \le b$ we have $a(1+a)^{-1} \le b(1+b)^{-1}$. This should be true because $a(1+a)^{-1} = f(a)$ where $f : [0,\infty) \to [0,1), x \mapsto \frac{x}{x+1} = x(1+x)^{-1}$ is increasing. $f$ is a homeomorphism with $g = f^{-1} : [0,1) \to [0,\infty)$ given by $g(x) = \frac{x}{x-1}$.

Indeed, take $0 \le a \le b$ then $1+a \le 1+b$ so $(1+b)^{-1} \le (1+a)^{-1}$ and therefore $a(1+a)^{-1} = 1-(1+a)^{-1} \le 1-(1+b)^{-1} = b(1+b)^{-1}$. Now observe that if $a \in \mathscr{A}_+$ then $f(a) = a(1+a)^{-1} \in \Lambda$ because $\|f\|_{\sigma(a)\subseteq[0,\infty)}$ and thus $0 \le f < 1$. So we get an increasing map $\mathscr{A}_+ \to \Lambda, a \mapsto a(1+a)^{-1}$. Now suppose $a, b \in \Lambda$, consider $g = f^{-1} : [0,1) \to [0,\infty), x \mapsto \frac{x}{x-1}$. Define $a' := g(a), b' := g(b)$ and let $c := (a'+b')(1+a'+b')^{-1} = f(a'+b')$. Then $c \in \Lambda$ and since $a' \le a'+b'$ we have $a = f(a') \le f(a'+b') = c$ and likewise $b \le c$. This shows that $\Lambda$ is a directed set.

Now we have to check that $(e_\lambda)_{\lambda\in\Lambda}$ with $e_\lambda = \lambda$ is an approximate unit for $\mathscr{A}$. Notice that $(e_\lambda)$ is increasing and $e_\lambda = \lambda \ge 0$ and $\|e_\lambda\| < 1$ for all $\lambda$. So we need only prove $e_\lambda \cdot a \to a \leftarrow a \cdot e_\lambda$ for every $a \in \mathscr{A}$. But using the involution, these two are equivalent:

$$(e_\lambda a) \to a \Leftrightarrow (e_\lambda a)^* \to a^* \Leftrightarrow a^* e_\lambda \to a \Leftrightarrow a^* e_\lambda \to a^*$$

It is even enough to prove $ae_\lambda \to a$ for only $a \in \Lambda$ because $\operatorname{span}\Lambda = \operatorname{span}(\mathscr{A}_+) = \mathscr{A}$. Let $a \in \Lambda$, in particular $a \in \mathscr{A}_+$. Consider 'its' Gelfand representation $\varphi : C^*(a) \to C_0(X)$ and let $f = \varphi(a) \in C_0(X)$. This function fulfils $0 \le f(x) < 1$ for all $x \in X$ because it comes from $a \in \mathscr{A}_+$.

Let furthermore $\varepsilon > 0$ and $K := \{x \in X \mid |f(x)| \ge \varepsilon\} \subseteq X$ compact. By Uryson's Lemma, we have a $g \in C_0(X), g : X \to [0,1]$ such that $g(x) = 1$ for all $x \in K$. Next, choose $\delta > 0$ with $\delta < 1$ and $1-\delta < \varepsilon$. Then $g_\delta = \delta \cdot g \le \delta$ and therefore

$$\begin{aligned}\|f - g_\delta \cdot f\| &= \|f - \delta g f\| = \sup_{x\in X} \|f(x)\| \cdot \|1-\delta g(x)\| \\ &\le \max\{\sup_{x\in K} \|f(x)\| \cdot \|1-\delta g(x)\|, \sup_{x\notin K} \|f(x)\| \cdot \|1-\delta g(x)\|\} \\ &\le \max\{\varepsilon, 1-\delta\} \le \varepsilon\end{aligned}$$

Now let $b := \varphi^{-1}(g_\delta) \in \mathscr{A}_+$ with $\|b\| < 1$ and $\|a - ba\| < \varepsilon$.

This shows that for any $a \in \Lambda$ we can find $\lambda_0 = b \in \Lambda$ such that $\|a - e_{\lambda_0} a\| < \varepsilon$. If now $\lambda \in \Lambda, \lambda \ge \lambda_0$ we have $e_{\lambda_0} \le e_\lambda$, so $1 - e_\lambda \le 1 - e_{\lambda_0}$ (in $\tilde{\mathscr{A}}$) and therefore $a(1-e_\lambda)a \le a(1-e_{\lambda_0})a$ (*) (by conjugation property and because $a$ is self-adjoint). But then

$$\begin{aligned}\|a - e_\lambda a\|^2 \|(1-e_\lambda a)\|^2 &= \| \overbrace{\underbrace{(1-e_\lambda)^{\frac{1}{2}} \cdot (1-e_\lambda)^{\frac{1}{2}}}_{\in \tilde{\mathscr{A}}} a}^{\in \mathscr{A} \trianglelefteq \tilde{\mathscr{A}}} \| \le \|(1-e_\lambda)^{\frac{1}{2}} a\|^2 \\ &\overset{(*)}{\le} \|a(1-e_\lambda)a\| \le \|a(1-e_{\lambda_0})a\| \overset{\|a\|\le 1}{\le} \|(1-e_{\lambda_0})a\| \\ &= \|a - e_{\lambda_0}\| < \varepsilon\end{aligned}$$

so $e_\lambda a \to a$. □

**Definition 7.50** *In general, $C^*$-algebras do not admit a sequential approximate unit.*

*We say that a $C^*$-algebra $\mathscr{A}$ is $\sigma$-unital if there exists such a sequential approximate unit $(e_n)_{n\in\mathbb{N}}$.*

**Example 7.51** $\mathscr{A} = C_0(X)$ is $\sigma$-unital if and only if $X$ is $\sigma$-compact: $X = \bigcup_{n=1}^\infty K_n$ where $K_n \subseteq X$ are compact spaces.

# 8 Ideals in $C^*$-algebras

**Theorem 8.1** *Let $\mathscr{A}$ be a $C^*$-algebra and $L \subseteq \mathscr{A}$ a left closed ideal. Then there exists a net $(u_\lambda)_{\lambda\in\Lambda} \subseteq A_{+,1} \cap L$ (that is, elements with $0 \le u_\lambda$ and $\|u_\lambda\| \le 1$) such that $a = \lim_\lambda au_\lambda$ for all $a \in L$.*

Proof: Set $B = L \cap L^*$. This is clearly a $C^*$-subalgebra. There is now an approximate unit $(u_\lambda) \subseteq B_{+,1} \subseteq A_{+,1}$ for $B$. Let $a \in L$. Then $a^*a \in L \cap L^* \in B$ and we have $\lim_\lambda a^*au_\lambda = a^*a = \lim_\lambda u_\lambda a$. It follows that

$$\begin{aligned}\lim_\lambda \|a - au_\lambda\|^2 &= \lim_\lambda \|(a-au_\lambda)^*(a-au_\lambda)\| = \lim_\lambda \|a^*a - a^*au_\lambda - u_\lambda a^*a - u_\lambda a^*au_\lambda\| \\ &\le \lim_\lambda \|a^*a - a^*au_\lambda\| + \lim_\lambda \|u_\lambda\| \cdot \|a^*a - a^*au_\lambda\| = 0\end{aligned}$$

□

Let $L \subseteq \mathscr{A}$ be a closed left ideal and $(u_\lambda) \subseteq B = L \cap L^* \subseteq \mathscr{A}$. Then $\lim_\lambda au_\lambda = a$ for all $a \in L$. As a consequence:

**Theorem 8.2** *Every closed two-sided ideal $I \trianglelefteq \mathscr{A}$ of a $C^*$-algebra satiesfies $I^* = I$, so it is a $^*$-ideal and in particular a $C^*$-algebra.*

Proof: By the lemma above, we find a net $(u_\lambda) \subseteq I$, $u_\lambda \ge 0$, such that $a = \lim_\lambda au_\lambda$ Then $a^* = \lim_\lambda u_\lambda a^* \in I$ (because $u_\lambda \in I$). □

**Corollary 8.3** *Let $I \trianglelefteq \mathscr{A}$ be a closed two-sided ideal of a $C^*$-algebra $\mathscr{A}$. Then for all $a \in \mathscr{A}$, $\|a + I\| = \lim_\lambda \|a - u_\lambda a\| = \lim_\lambda \|a - au_\lambda\|$ where $(u_\lambda)$ is an approximate unit for $I$.*

Proof: Let $\varepsilon > 0$ and take $b \in I$ such that $\|a + b\| \le \|a + I\| + \frac{\varepsilon}{2}$. Recall that $\|a + I\| = \mathrm{dist}(a, I) = \inf_{b\in I} \|a + b\|$.

Since $\lim_\lambda u_\lambda b = b$. Then there exists $\lambda_0$ such that $\|b - u_\lambda b\| < \frac{\varepsilon}{2}$ for all $\lambda \ge \lambda_0$. Then

$$\begin{aligned}\|a - u_\lambda a\| &\le \|(1-u_\lambda)(a+b)\| + \|b - u_\lambda b\| \\ &\le \|a + b\| + \|b - u_\lambda b\| \\ &< \|a + I\| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \|a + I\| + \varepsilon\end{aligned}$$

On the other hand, $\|a - u_\lambda a\| \ge \|a + I\|$ for all $\lambda$ and $\|a + I\| = \lim_\lambda \|a + u_\lambda a\| = \inf_\lambda \|a - u_\lambda a\|$. This shows the existence of the limit and therefore that the norm equals the distance. □

**Theorem 8.4** *If $I \trianglelefteq \mathscr{A}$ is a closed $^*$-ideal in a $C^*$-algebra $\mathscr{A}$, then $\mathscr{A}/I$ is itself a $C^*$-algebra.*

Proof: We already know that $\mathscr{A}/I$ is a Banach $^*$-algebra. We only need to show that $\|a + I\| = \|(a + I)^*(a + I)\|$.

Let $(u_\lambda) \subseteq I$ be an approximate unit and take $b \in I$. Then

$$\begin{aligned}\|a + I\|^2 &= \lim_\lambda \|a - au_\lambda\|_A^2 \overset{*}{=} \lim_\lambda \|(1-u_\lambda)a^*a(1-u_\lambda)\| \\ &\le \sup_\lambda \|(1-u_\lambda)(a^*a + b)(1-u_\lambda)\| + \lim_\lambda \|(1-u_\lambda)b(1-u_\lambda)\| \\ &\le \|a^*a + b\|\end{aligned}$$

Where $*$ is because we can use the $C^*$-property of $\mathscr{A}$ and $(1 - u_\lambda)$ is self-adjoint. The last inequality follows because the latter limit tends to 0.

Since $b$ was arbitrary, we get

$$\|a+I\|^2 \leq \inf_{b\in I} \|a^*a+b\|_{\mathscr{A}} = \|a^*a+I\| = \|(a+I)^*(a+I)\| \qquad \square$$

**Theorem 8.5** *If $\varphi : \mathscr{A} \to \mathscr{B}$ (where $\mathscr{A}, \mathscr{B}$ are $C^*$ algebras) is an injective $^*$-homomorphism, then $\varphi$ is isometric, i.e. $\|\varphi(a)\| = \|a\|$ for all $a \in \mathscr{A}$.*

Proof: It suffices to show that $\|\varphi(a)\|^2 = \|a\|^2$ or $\|\varphi(a^*a)\| = \|a^*a\|$.

Replacing $\mathscr{A}$ by the $C^*$-algebra $C^*(a^*a)$ and $B$ by $C^*(\varphi(a^*a)) \subseteq B$ (with $a^*a, \varphi(a^*a) = \varphi(a)^*\varphi(a) \geq 0$) we may assume that $\mathscr{A}, \mathscr{B}$ are commutative. Also by adding units and extending $\varphi$ to the unitization $\tilde{\varphi} : \tilde{\mathscr{A}} \to \tilde{\mathscr{B}}$ we may assume that $\mathscr{A}, \mathscr{B}, \varphi$ are unital. Now given $\chi \in \Omega(\mathscr{B})$ notice that $\chi \circ \varphi \in \Omega(\mathscr{A})$. So we get a map $\varphi_* : \Omega(\mathscr{B}) \to \Omega(\mathscr{A}), \chi \mapsto \chi \circ \varphi$. This is clearly continuous. Since $\Omega(\mathscr{B})$ is compact, $K \coloneqq \varphi_*(\Omega(\mathscr{B}))$ is compact (in particular closed). By Uryson's Lemma, we find some continuous function $f \in C(\Omega(\mathscr{A}))$ such that $f|_K \equiv 0$ and $f \neq 0$ (if we assume $K \neq \Omega(\mathscr{A})$). By Gelfand-Representation we find ($\mathscr{A} \simeq C(\Omega(\mathscr{A}))$) and $a \in \mathscr{A}$ such that $\hat{a} = f$. Then for each $\chi \in \Omega(\mathscr{B})$,

$$\chi(\varphi(a)) = \hat{a}(\chi \circ \varphi) = \underbrace{\hat{a}}_{f}(\underbrace{\varphi_*(\chi)}_{\in K}) = 0 \Rightarrow \varphi(a) = 0$$

and if $f \neq 0$, then $a \neq 0$. But we have $\varphi(a) = 0$ for all $a$, a contradiction. Therefore, $\varphi_*$ is surjective. Now

$$\|a\|_{\mathscr{A}} = \|\hat{a}\|_\infty = \sup_{\chi\in\Omega(\mathscr{A})} |\chi(a)| = \sup_{\chi\in\Omega(\mathscr{B})} |(\chi\circ\varphi)(a)| = \|\widehat{\varphi(a)}\|_\infty = \|\varphi(a)\|_{\mathscr{B}} \qquad \square$$

**Corollary 8.6** *If $\varphi : \mathscr{A} \to \mathscr{B}$ is any $^*$-homomorphism ($\mathscr{A}, \mathscr{B}$ $C^*$-algebras) then $\varphi(\mathscr{A})$ is closed, hence a $C^*$-subalgebra of $\mathscr{B}$.*

Proof: Consider $\psi : \mathscr{A}/_{\ker\varphi} \to \mathscr{B}, a+\ker\varphi \mapsto \varphi(a)$. Then $\psi$ is a well-defined $^*$-homomorphism and $\psi$ is injective and therefore isometric. This shows that $\psi(\mathscr{A}/_{\ker\varphi}) = \varphi(\mathscr{A})$ is closed. $\square$

**Remark 8.7** For some other related consequences, see Murphy's book.

(i) If $\mathscr{A} \subseteq \mathscr{B}$ are $C^*$-algebras and $I \trianglelefteq \mathscr{B}$ is a closed 2-sided ideal then $\mathscr{A}+I$ is a $C^*$-subalgebra of $\mathscr{B}$. In particular, the sum of ideals in $C^*$-algebras are ideals: For any $I, J \trianglelefteq \mathscr{A}$ have that $I + J \trianglelefteq \mathscr{A}$ as well.

(ii) If $I, J \trianglelefteq \mathscr{A}$ then $I \cdot J = I \cap J$. The product here is defined as the linear span of products ($I \cdot J = \overline{\operatorname{span}}\{i \cdot j \mid i \in I, j \in J\}$) but is actually just the products.

Proof (Ideas):

(i) To prove that $\mathscr{A} + I$ is closed, check that $(A+I)/I$ is Banach by identifying it with

$$(A+I)/I \simeq \mathscr{A}/(\mathscr{A}\cap I), a+I \leftarrow a + A\cap I$$

Can also build arbitrary familys of ideals and the sum will be an ideal, also the intersection and product of ideals exist.

(ii) $I \cdot J \subseteq I \cap J$ is clear. To prove the converse, use the approximate unit. $I \cap J$ is clearly a $C^*$-algebra, take an approximate unit $(u_\lambda) \subseteq I \cap J$ and $x \in I \cap J$. Then $x = \lim_\lambda xu_\lambda$ where $xu_\lambda$ is in $I \cdot J$ at all times. $\square$

# 9 Gelfand-Neymark representation

We know for commutative $\mathscr{A}$ that $\mathscr{A} = C_0(\Omega(\mathscr{A}))$. But if $\mathscr{A}$ is not commutative, $\Omega(\mathscr{A}) - \emptyset$ and this is useless. So we want to look at non-homomorphism functionals (the elements of the spectrum are homomorphism functionals) and hope that this is not empty. Hence we want to study positive linear functionals.

**Definition 9.1** *Let $\mathscr{A}, \mathscr{B}$ $C^*$-algebras. A linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is called **positive** if $\varphi(\mathscr{A}_+) = \mathscr{B}_+$. We write $\varphi \geq 0$ for this.*

**Remark 9.2** Let $\mathscr{A}, \mathscr{B}$ $C^*$-algebras and $\varphi \geq 0$.

(i) $\varphi \geq 0$ implies that $\varphi(\mathscr{A}_{sa}) = \mathscr{B}_{sa}$ (self-adjoint to self-adjoint). This follows because for any $a \in \mathscr{A}_{sa}$, we have $a = a^+ = a^-$ and $\varphi(a) = \varphi(a^+) - \varphi(a^-) \in B_{sa}$.

(ii) $a_1 \leq a_2$ in $\mathscr{A}$ yields $\varphi(a_1) \leq \varphi(a_2)$. This is because every $^*$-homomorphism is primitive because $\varphi : \mathscr{A} \to \mathscr{B}$ a $^*$-homomorphism and $a \geq 0$ in $\mathscr{A}$ imply $a = x^*x$ for some $x \in \mathscr{A}$ andthus $\varphi(a) = \varphi(x)^*\varphi(x) \geq 0$.

**Example 9.3** Let $\varphi : M_m(\mathbb{C}) \to M_m(\mathbb{C}), a \mapsto a^T$ is positive but not a homomorphism.
For this, consider $(a^*)^T = (a^T)^*$ and therefore $(a^*a)^T = (a^T)(a^T)^* \geq 0$, but not $(a^*a)^T \neq (a^T)^*(a^T)$.

**Example 9.4** $\mathscr{A} = C_0(X)$. If $B(X)$ are the Borell-subsetes of $X$ $\mu : B(X) \to [0, \infty]$ is a positive bounded measure, then

$$\varphi_\mu : C_0(X) \to \mathbb{C}, f \mapsto \int_X f(x) d\mu(x)$$

is clearly positive, linear but (usually) not a homomorphism. If $\mu$ is a Dirac-measure this is a homomorphism and a character.

# 10 Positive linear maps and functionals

**Definition 10.1** *Let $\mathscr{A}, \mathscr{B}$ be $C^*$-algebras, a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ is called positive if $\varphi(\mathscr{A}_+) \subseteq \mathscr{B}_+$, that is $a \geq 0 \Rightarrow \varphi(a) \geq 0$. We write this as $\varphi \geq 0$.*

**Remark 10.2** Observe that $\varphi \geq 0$ implies $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathscr{A}$ and $\varphi(\mathscr{A}_{sa}) \subseteq \mathscr{B}_{sa}$.
Also, $\varphi$ respects inequality.

PROOF: Just write $a \in \mathscr{A}_{sa}$ as $a = a_+ - a_-$ with $a_+, a_- \in \mathscr{A}_+$. □

**Example 10.3** (i) Let $\mathscr{A} = M_n(\mathbb{C})$ the usual trace $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}, A \mapsto \sum_{i=1}^n a_{ii}$ is a positive linear functional In general a **trace** in a $C^*$-algebra is any positive linear map $\varphi : \mathscr{A} \to \mathbb{C}$ with $\varphi(ab) = \varphi(ba)$.

**Proposition 10.4** *If $\varphi : \mathscr{A} \to \mathscr{B}$ is a positive linear map, then $\varphi$ is bounded (i.e. continuous).*

PROOF: Let $M = \sup_{a \in \mathscr{A}_+} \|\varphi(a)\|$. If we had $M = \infty$ there exists $(a_n) \in \mathscr{A}_{+,1}$ where $\|\varphi(a_n)\| \geq 2^n$ for all $n$. Define $a := \sum_{n=1}^\infty \frac{a_n}{2^n} \in \mathscr{A}_{+,1}$. Since $\varphi \geq 0$ and $\sum_{n=1}^N \frac{a_n}{2^n} \leq a$, we have $\sum_{n=1}^N \frac{\varphi(a_n)}{2^n} \leq \varphi(a)$. Notice that $\varphi(a_n) \geq 2^n$ in $\tilde{\mathscr{B}}$ because whenever $b \in \mathscr{B}_+$ and $\|b\| \geq c \geq 0$ so $b \geq c \cdot 1$. So in conclusion $\varphi(a) \geq \sum_{n=1}^N \frac{\varphi(a_n)}{2^n} \geq N \cdot 1$ (in $\tilde{\mathscr{B}}$), implying $\|\varphi(a)\| \geq N$ for all $N \in \mathbb{N}$, a contradiction.

Now given any $a \in \mathscr{A}$ write it as $a = b + ic$ where $b, c \in \mathscr{A}_{sa}$ where $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$. If $\|a\| \leq 1$ then $\|b\|, \|c\| \leq 1$ and $b = b_+ - b_-$, $c = c_+ - c_-$ so $b_+ = \frac{b+|b|}{2}$, $b_- = \frac{b=|b|}{2}$, $c_+ = \frac{c+|c|}{2i}$ and $c_- = \frac{c-|c|}{2i}$ where $|b| = \sqrt{bb^*}$ so $\|b_+\|^2, \|b_-\| \leq 1$. Then

$$\|\varphi(a)\| = \|\varphi(b) + i\varphi(c)\| = \|\varphi(b_+) + \varphi(b_-) + i\varphi(c_+) + i\varphi(c_-)\| \leq 4M \qquad \square$$

We concentrate from now on positive linear functionals $\varphi : \mathscr{A} \to \mathbb{C}$. The main point is the following observation:

**Remark 10.5** If $\varphi : \mathscr{A} \to \mathbb{C}$ is a positive linear functional, then $\langle a, b\rangle_\varphi := \varphi(a^*b)$ is a semi-inner product on the vector space (fulfilling all requirements of an inner product except for $\langle a, a\rangle_\varphi = 0 \Rightarrow a = 0$). So Cauchy-Schwarz-inequality holds: $|\langle a, b\rangle_\varphi| \leq \|a\|_\varphi \cdot \|b\|_\varphi$ where $\|a\|_\varphi := \langle a, a\rangle_\varphi^{\frac{1}{2}} = \varphi(a^*a)^{\frac{1}{2}}$ is the semi-norm implied by $\langle \cdot, \cdot\rangle_\varphi$. Therefore, $|\varphi(a^*b)|^2 \leq \varphi(a^*a) \cdot \varphi(b^*b)$ for all $a, b \in \mathscr{A}$.

**Proposition 10.6** *Let $\mathscr{A}$ be a $C^*$-algebra and $\varphi \in \mathscr{A}^*_+ = \{\varphi : \mathscr{A} \to \mathbb{C} \mid$ positive linear $\}$. Then $|\varphi(a)|^2 \leq \|\varphi\|\varphi(a^*a)$ for all $a \in \mathscr{A}$.*

Proof: Let $(e_\lambda) \subseteq \mathscr{A}_{+,1}$ be an approximate unit. Using CS, we get

$$|\varphi(e_\lambda a)|^2 \leq \varphi(e_\lambda^2) \cdot \varphi(a^*a) \leq \|\varphi\|\varphi(a^*a)$$

and taking the limit yields the statement. $\square$

**Theorem 10.7** *Let $\varphi \in \mathscr{A}^* = \{\varphi : \mathscr{A} \to \mathbb{C} \mid$ bounded linear $\}$. Then the following are equivalent*

*(i)* $\varphi \geq 0$

*(ii) For each approximate unit $(e_\lambda) \subseteq \mathscr{A}_{+,1}$ we have $\|\varphi\| = \lim_\lambda \varphi(e_\lambda) = \sup_\lambda \varphi(e_\lambda)$.*

*(iii) For some approximate unit $(e_\lambda) \subseteq \mathscr{A}_{+,1}$ we have $\|\varphi\| = \lim_\lambda \varphi(e_\lambda) = \sup_\lambda \varphi(e_\lambda)$.*

Proof:

(i) $\Rightarrow$ (ii): B the previous proposition, $|\varphi(a)|^2 \leq \|\varphi\|\varphi(a^*a)$. Applying this for $a = e_\lambda$, we get $|\varphi(e_\lambda)|^2 \leq \|\varphi\|\varphi(e_\lambda)^2$ Notice $e_\lambda^2 = e_\lambda^{\frac{1}{2}} e_\lambda e_\lambda^{\frac{1}{2}} \leq e_\lambda$. Since $\varphi$ preserves inequality, we have $|\varphi(e_\lambda)|^2 \leq \|\varphi\|\varphi(e_\lambda)$, so $\varphi(e_\lambda) \leq \|\varphi\|$ and therefore $\limsup_\lambda \varphi(e_\lambda) \leq \sup_\lambda \varphi(e_\lambda) \leq \|\varphi\|$. We apply CS again: $|\varphi(e_\lambda a)|^2 \leq \varphi(e_\lambda)^2\varphi(a^*a) \leq \varphi(e_\lambda)\varphi(a^*a)$ and hence $|\varphi(a)|^2 = \liminf_\lambda |\varphi(e_\lambda a)|^{\leq} \liminf_\lambda \varphi(e_\lambda)\|a\|^2\|\varphi\|$, as$\varphi(a^*a) \leq \|a\|^2\|\varphi\|$.

Now taking sup over $\|a\| \leq 1$ yields

$$\|\varphi\|^2 \leq \liminf_\lambda \varphi(e_\lambda)\|\varphi\| \Rightarrow \|\varphi\| \leq \liminf_\lambda \varphi(e_\lambda)$$

(ii) $\Rightarrow$ (iii): This is clear, as some linear morhpisms always exist.

(iii) $\Rightarrow$ (i): Let $a \in \mathscr{A}_{sa}$ and $\|a\| \leq 1$. Write $\varphi(a) = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$. We prove that $\beta = 0$, that is $\varphi(a) \in \mathbb{R}$. We may assume $\beta \leq 0$ (or just take $-a$ instead). Let $n \in \mathbb{N}$. Then

$$\|a - ine_\lambda\|^2 = \|(a + ine_\lambda)(a - ine_\lambda)\| = \|a^2n^2e_\lambda^2 - 2n(ae_\lambda - e_\lambda a) \leq 1 + n^2 + n\|ae_\lambda - e_\lambda a\|$$

THen we h and we have

$$\|\varphi(a - ine_\lambda)\|^2 \le \|a - ine_\lambda^2 \le 1 + n^2 + n \underbrace{\|ae_\lambda - e_\lambda a\|}_{\to 0}$$

Taking $\lambda \to \infty$, we get $\varphi(e_\lambda) \le 1 + n^2$. Using $\varphi(a) = \alpha + i\beta$ and we get

$$\|\alpha + i\beta - in\|^2 \le 1 + n^2 \Rightarrow \alpha^2 + \beta^2 - 2n\beta + in^2 \le 1 + n^2 \Rightarrow -2n\beta \le 1 - \alpha^2 - \beta^2$$

. Because $\beta \le 0$, we have to take $\beta = 0$.
Now to prove $\varphi \ge 0$: Take $a \in \mathscr{A}_+$ with $\|a\| \le 1$. Then $e_\lambda - a \in \mathscr{A}_{sa}$ and

$$-1 \le -a \le e_\lambda \le e_\lambda \le$$

So $\|e_\lambda\| \le 1$.

$$\underbrace{\varphi(e_\lambda - a)}_{\in \mathbb{R}} \le |\varphi(e_\lambda)| \le 1$$

Letting $\lambda \to \infty$, then $1 - \varphi(a) \le 1$ so $\varphi(a) \ge 0$.k $\square$

**Corollary 10.8** *If $\mathscr{A}$ is unital and $\varphi \in \mathscr{A}^+$ then $\varphi \ge 0 \Leftrightarrow \varphi(1) = \|\varphi\|$.*

**Corollary 10.9** *If $\mathscr{A}$ is a unital $C^*$-algebra and $\varphi \in \mathscr{A}^*$, then $\varphi \ge 0 \Leftrightarrow \varphi(1) = \|\varphi\|$.*

**Definition 10.10** *A **state** on a $C^*$-algebra $\mathscr{A}$ is a positive linear functional $\varphi \in \mathscr{A}_+^*$ with $\|\varphi\| = 1$.*
*We denote the set of all states by $S(\mathscr{A})$.*

**Example 10.11** If $\mathscr{A} = B(H)$ or $\mathscr{A} = K(H)$ (bounded/compact operators on a hilbert space $\mathscr{A}$) or $\mathscr{A}$ a subalgebra of any of these sets with non-degenerate $e_\lambda \to 1$. Let $\zeta, \eta \in H$ and define $\varphi_{\zeta,\eta}(a) := \langle \zeta, a\eta \rangle$. Then $\varphi_{\zeta,\eta} \in \mathscr{A}^*$ with $\|\varphi_{\zeta,\eta}\| \le \|\zeta\| \cdot \|\eta\|$. If $(e_\lambda) \subseteq \mathscr{A}_{+,1}$ is an approximate unit, then, using $e_\lambda \to 1$ (strictly) shows $\varphi_{\zeta,\eta}(e_\lambda) \to \langle \zeta, \eta \rangle$. If $\zeta = \eta$, then $\varphi_\zeta := \varphi_{\zeta,\zeta}$ is positive and so $\varphi_\zeta(a^*a) = \langle a\zeta, a\zeta \rangle = \|a\zeta\|^2 \ge 0$. By the previous theorem, $\|\varphi_\zeta\| = \lim_\lambda \varphi_\zeta(e_\lambda) = \|\zeta\|^2$. So $\varphi_\zeta$ is a state if and only if $\|\zeta\| = 1$.

Note that there are states that are not of this form at all! The ones presented here are the so-called **pure states**.

**Theorem 10.12** *If $\mathscr{A}$ is a $C^*$-algebra and $a \in \mathscr{A}$ is normal with $\mathscr{A} \ne 0$ there exists a state $\varphi \in S(\mathscr{A})$ with $|\varphi(a)| = \|a\|$*

Proof: We may assume $a \ne 0$ (we would only need to prove that any state exists, but this follows from the construction). Let $\mathscr{B} = C^*(a, 1) \subseteq \tilde{\mathscr{A}}$. $\mathscr{B}$ is abelian, $\hat{a} \in C(X)$ and $X = \Omega(\mathscr{B})$ (compact). Then there exists a $\chi \in \Omega(\mathscr{B}) = X$ (compact) such that $|\hat{a}(\chi)| = |\chi(a)| = \|\hat{a}\|_\infty = \|a\|$. By Hahn-Banach, extend $\chi : \mathscr{B} \to \mathbb{C}$ to $\psi \in (\tilde{\mathscr{A}})^*$ with $\|\psi\| = \|\varphi\| = 1$. So $|\psi(a)| = |\chi(a)| = \|a\|$ and also $|\psi(1)| = |\chi(1)| = 1$. By the corollary, $\psi \ge 0$ and $\psi \in S(\mathscr{A})$. Taking $\varphi := \psi|_\mathscr{A} \in \mathscr{A}_+^*$ shows $\|\varphi\| \le \|\psi\| = 1$ and $|\varphi(a)| = |\psi(a)| = \|a\|$, so $\|\varphi\| \ge 1$, so $\|\varphi\| = 1$ and $\varphi$ is also a state. $\square$

**Theorem 10.13 (Extension of positive linear functionals)** *Let $\mathscr{A} \subseteq \mathscr{B}$ be an inclusion of $C^*$-algebras and $\varphi \in \mathscr{A}_+^*$. Then, there exists $\tilde{\varphi} \in \mathscr{B}_+^*$ with $\tilde{\varphi}|_\mathscr{A} = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$.*

Proof: First consider the case $\mathscr{B} = \tilde{\mathscr{A}}$. In this case, define $\tilde{\varphi} : \tilde{\mathscr{A}} \to \mathbb{C}, a + \lambda \cdot 1 \mapsto \varphi(a) + \lambda\|\varphi\|$. Of course, $\tilde{\varphi}$ is linear and $\tilde{\varphi}|_{\mathscr{A}} = \varphi$. To prove that $\tilde{\varphi}$ is bounded, let $(e_i) \subseteq \mathscr{A}$ be an approximate unit. Then

$$|\tilde{\varphi}(a + \lambda \cdot 1) = |\varphi(a) + \lambda\|\varphi\|| = |\lim_i \varphi(ae_i) + \lambda \lim_i \varphi(e_i)| = \lim_i |\varphi(ae_i + \lambda e_i)|$$
$$= \lim_i |\varphi((a + \lambda 1)e_i)| \le \|\varphi\|\|a + \lambda 1\|\|e_i\| \le \|\varphi\|\|a + \lambda 1\|$$

because $\varphi$ is bounded. So $\tilde{\varphi}$ is also bounded and $\|\tilde{\varphi}\| \le \|\varphi\|$. But $\tilde{\varphi}(1) = \|\varphi\|$, so $\|\tilde{\varphi}\| = \|\varphi\|$ and $\tilde{\varphi}$ is therefore also positive.

Now the general case: Passing to the unitizations, we have an embedding $\tilde{\mathscr{A}} \subseteq \tilde{\mathscr{B}}$ and may assume that both $\mathscr{A}, \mathscr{B}$ are unital with the same unit. By the unital case above, $\varphi$ extends to $\tilde{\mathscr{A}}$ and then also to $\mathscr{A}$ by Hahn-Banach. So there exists $\tilde{\varphi} \in \mathscr{B}^*$ with $\tilde{\varphi}|$. Since $\varphi \ge 0$, we know that $\tilde{\varphi(1)} = \varphi(1) = \|\varphi\| = \|\tilde{\varphi}|$, so $\tilde{\varphi} \ge 0$. □

**Remark 10.14**

(i) In certain cases the extension $\varphi$ to $\tilde{\varphi}$ is unique. This is true if $\mathscr{A} \trianglelefteq \mathscr{B}$ ore more generally if $\mathscr{A} \subseteq \mathscr{B}$ is a hereditary $C^*$-subalgebra (see Murphy: $\mathscr{A}\mathscr{B}\mathscr{A} = \mathscr{B}$ or $\mathscr{A} = L \cap L^*$ for some left-handed ideal $L$). In this case, $\tilde{\varphi}(b) = \lim \varphi(u_\lambda a u_\lambda)$ where $(u_\lambda) \subseteq \mathscr{A}$ where $(u_\lambda)$ is an approximate unit.

(ii) Say $\varphi \in \mathscr{A}^*$ is self-adjoint. If $\varphi^* = \varphi$ where $\varphi^*(a) = \overline{\varphi(a^*)}$ (involution on $\mathscr{A}^*$). We can write $\varphi \in \mathscr{A}^*$ as $\varphi = \Re(\varphi) + i\Im(\varphi)$ where $\Re(\varphi) = \frac{\varphi + \varphi^*}{2}$ and $\Im(\varphi) = \frac{\varphi - \varphi^*}{2i}$ are self-adjoint, contained in $\mathscr{A}^*_{sa}$. Observe that $\mathscr{A}^*_{sa} = (\mathscr{A}_{sa})'$, the topological dual of $\mathscr{A}_{sa}$ as an $\mathbb{R}$-vecotr Banach space.

(iii) Any $\varphi \in \mathscr{A}^*_{sa}$ can be uniquely written as $\varphi = \varphi_+ - \varphi_-$ where $\varphi_+, \varphi_- \in \mathscr{A}^*_+$ and $\|\varphi\| = \|\varphi_+\| + \|\varphi_-\|$.

# 11 The Gelfand-Naimark-Theorem

**Definition 11.1** *Let $\mathscr{A}$ be a $C^*$-algebra. A **representation** of $\mathscr{A}$ is a $^*$-homomorphism $\pi : \mathscr{A} \to \mathcal{L}(H)$ for some Hilbert space $H$.*

*We say that $\pi$ is*

*(i) **faithful** if $\pi$ is injective (and therefore isometric).*

*(ii) **non-degenerate** if $\operatorname{span} \pi(\mathscr{A})H = H$.*

*(iii) **irreducible** if for all closed subspaces $K \subseteq H$ with $\pi(\mathscr{A})K \subseteq K$ ($K$ is $\pi$-invariant) we have $K = 0$ or $K = H$.*

**Remark 11.2** The exercises show that $\pi$ non-degenerate is equivalent to $\pi(e_\lambda) \to 1$ (strongly) for an approximate unit $(e_\lambda) \subseteq \mathscr{A}$

We want to show that there is always a faithful homomorphism.

**Definition 11.3** *Let $\pi : \mathscr{A} \to \mathcal{L}(H)$, $\rho : \mathscr{A} \to \mathcal{L}(K)$ two representations. We say that $\pi, \rho$ are (unitarily) equivalent if there exists a surjective isometry $u : H \to K$ such that $u^*\rho(a)u = \pi(a)$, i.e. $\rho = \operatorname{Ad}_{u^*}\pi$.*

**Definition 11.4 (Spectrum)** *We define*

$$\hat{\mathscr{A}} = \{[\pi] \mid \pi : \mathscr{A} \to \mathcal{L}(H), \pi \neq 0\}$$

*Also define* $\mathrm{Prim}(\mathscr{A}) = \{\ker(\pi) \mid [\pi] \in \hat{\mathscr{A}}\}$ *and* $\hat{\mathscr{A}} \to \mathrm{Prim}(\mathscr{A}), [\pi] \mapsto \ker\pi$ *(primitive ideals). Let* $\chi \in \Omega(\mathscr{A})$ *be a character* $\chi : \mathscr{A} \to \mathbb{C} = \mathcal{L}(\mathbb{C})$*. Then* $[\chi] \in \hat{\mathscr{A}}$ *and* $\ker\chi \in \mathrm{Prim}(\mathscr{A})$*.*

How do we get representations of $\mathscr{A}$?

## Gelfand-Naimark-Siegal-Construction (GNS)

**Theorem 11.5** *Let* $\varphi \in \mathscr{A}_+^*$ *be any positive linear functional. We know that* $\langle a, b\rangle_\varphi \coloneqq \varphi(a^*b)$ *defines a semi-inner-product and* $\|a\|_\varphi = \varphi(a^*a)^{\frac{1}{2}}$ *is a semi-norm.*

*Let* $N_\varphi \coloneqq \{a \in \mathscr{A} \mid \|a\|_\varphi = 0\}$*.*

**Remark 11.6** Notice: $N_\varphi \subseteq \mathscr{A}$ is a closed left ideal.

Proof: From Cauchy-Schwarz:

$$|\varphi(a^*b)|^* \leq \varphi(a^*a)\varphi(b^*b)$$

and therefore

$$N_\varphi = \{b \in \mathscr{A} \mid \varphi(ab) = 0\}$$

Let $H_\varphi^\circ \coloneqq \mathscr{A}/N_\varphi$ the quotient vector space. Then $\langle ,\dot{;}\rangle_\varphi$ factors through an inner product of $H_\varphi^\circ$ and

$$\langle a + N_\varphi, b + N_\varphi\rangle = \langle a, b\rangle = \varphi(a^*b)$$

By completion we get a Hilbert space $H_\varphi = \overline{H_\varphi^\circ}^{\langle\cdot,\cdot,\rangle}$.

Now we define (with $L$ the linear operators)

$$\pi_\varphi^\circ : \mathscr{A} \to L(H_\varphi^\circ)$$

and thus

$$\pi_\varphi^\circ(a)(b + N_\varphi) \coloneqq ab + N_\varphi$$

meaning that $\pi_\varphi^\circ(a) \cdot \pi_\varphi^\circ(b) = \pi_\varphi^\circ(ab)$ and $\pi_\varphi^\circ(a^*) = (\pi_\varphi^\circ(a))^*$. Then

$$\varphi(b^*ac) = \langle \varphi_\varphi^\circ(a^*)(b + N_\varphi), c + N_\varphi\rangle = \langle b + N_\varphi, \pi_\varphi(a)(c + N_\varphi)\rangle. \tag{11.1}$$

We claim now that $\pi_\varphi^\circ$ is bounded for $\|\cdot\|_\varphi$ and therefore show that $\pi_\varphi(a)$ extends to $\pi_\varphi(a) \in \mathcal{L}(H_\varphi)$.

Take

$$\|\pi_\varphi^\circ(a)(b + N_\varphi)\|_\varphi^2 = \|ab + N_\varphi\|_\varphi^2 = \varphi((ab^*ab)) = \varphi(b^*a^*ab) \leq \|a\|^2\varphi(b^*b) \leq \|a\|^2\|b + N_\varphi|$$

Therefore we get a representation: The GNS-Representation associated to $\varphi$.

$$\pi_\varphi : \mathscr{A} \to \mathcal{L}(H_\varphi), a \mapsto \pi_\varphi(a) = [b + N_\varphi \mapsto ab + N_\varphi] \qquad \square$$

If $(\pi_i)_{i\in I}$ is a family of representations $\pi_i : \mathscr{A} \to H$. We define the direct sum $\bigoplus_{i\in I} \pi_i : A \to \mathcal{L}(\bigoplus_{i\in I} H_i), a \mapsto (\pi_i(a))_{i\in I}$ where $(\pi_i(a))_{i\in I} : \zeta \mapsto (\pi_i(a)\zeta)$.

**Theorem 11.7 (Gelfand-Naimar-Representation)** *Let $\mathscr{A}$ be a $C^*$-algebra and define $\pi_U := \bigoplus_{\varphi \in S(\mathscr{A})} \pi_\varphi : \mathscr{A} \to \mathcal{L}(H_U)$ with $H_U = \oplus_{\varphi \in S(\mathscr{A})} H_\varphi$ for $H_\varphi = \overline{\mathscr{A}/N_\varphi}$ with the semi-inner product $\langle \cdot, \cdot \rangle_\varphi$ and $\pi_\varphi(a)(b + N_\varphi) = ab + N_\varphi$. Then $(\pi_U, H_U)$ is* ***faithful****.*

Proof: Suppose $0 \neq a \in \mathscr{A}$, $pi_U(a) = 0$ and $pi_U(a) = 0$. Then there exists $\varphi \in S(\mathscr{A})$ such that $\varphi(a^*a) = \|a^*a\| = \|a\|^2$. We know $\langle a, a \rangle_\varphi = \|a\|_\varphi$. Then $\pi_U(a) = 0$, so $\pi_\varphi(a) = 0$, so $\pi_\varphi(a^*a) = 0$ and therefore $\pi_\varphi(a)(b + N_\varphi) = ab + N_\varphi = 0$. This shows

$$i0 = \langle \pi_\varphi(a)(b + N_\varphi), \pi_\varphi(a)(b + N_\varphi) \rangle = \varphi(b^*a^*ab)$$

for all $b \in \mathscr{A}$, so $b = e_\lambda$ (for $\lambda \to \infty$). But then $\varphi(a^*a) = 0$ and thus $a = 0$. □

Observe that $(\pi_U, H_U)$ is called the universal representation of $\mathscr{A}$. This is always non-degenerate. Indeed, each $(\pi_\varphi, H_\varphi)$ is non-degenerate. Moreover, these are **cyclic representations**:

**Definition 11.8** *A representation $\rho : \mathscr{A} \to L(H)$ is* ***cyclic*** *if there is a $\zeta \in H$, $\|\zeta\| = 1$ such that $\overline{\rho(\mathscr{A})\zeta} = H$. $\zeta$ is called a cyclic vector for $(\rho, H)$.*

Observe: Every non-degenerate representation is a sum of cyclic representations (proof via Zorn's Lemma omitted).

**Proposition 11.9** *Every GNS-representation $(\pi_U, H_U)$ is cyclic.*

Proof: If $\mathscr{A}$ is unital, then $\zeta\varphi := 1 + N_\varphi \in H_\varphi$ is a cyclic vector for $\pi_\varphi$. Then $\pi_\varphi(a)(\zeta\varphi) = a + N_\varphi$ and thus $\pi_\varphi(\mathscr{A})\zeta_\varphi = \mathscr{A}/N_\varphi \subseteq H_\varphi$ (dense). Therefore $\zeta_\varphi$ is cyclic and

$$\|\zeta\varphi\|^2 = \langle 1 + N_\varphi, 1 + N_\varphi \rangle = \varphi(1) == \|\varphi\| = 1$$

so $\varphi \in S(\mathscr{A})$. Moreover: $\langle \zeta_\varphi, \pi_\varphi(a)\zeta_\varphi \rangle = \langle 1 + N_\varphi, 1 + N_\varphi \rangle = \varphi(a)$.

Let us now look at the general case. Consider the linear map $\varphi_0 : \mathscr{A}/N_\varphi \to \mathbb{C}, a + N_\varphi \to \varphi(a)$. This is well-defined and bounded:

$$\|\varphi(a)\|^2 \leq \|\varphi\|\varphi(a^*a) = \varphi(a^*a)$$

as $\varphi$ is a state (and thus $\|\varphi\| = 1$). So $\|\varphi\| \leq 1$. So $\varphi_0$ extends to a bounded linear factorial map on $\tilde{\varphi}_0 H_\varphi :\to \mathbb{C}$. By Riesz-Representation thoerem, we have a $\zeta_\varphi \in_\varphi$ such that $\tilde{\varphi}_0(\eta) = \langle \zeta_\varphi, \eta \rangle$ and $\|\zeta_\varphi\| = \|\varphi_0\| = 1$. In particular $\varphi(a) = \varphi_0(a + N_\varphi) = \langle \zeta_\varphi, a + N_\varphi \rangle$. Now for $a, b \in \mathscr{A}$ we have

$$\langle \pi_\varphi, b + N_\varphi \rangle = \langle \zeta_\varphi, \pi_\varphi(a^*)(b + N\varphi) \rangle = \langle \zeta_\varphi, a^*b \rangle = \varphi(a^*b) = \langle a + N_\varphi, b + N_\varphi \rangle$$

Therefor $\pi_\varphi(a)\zeta_\varphi = a + N_\varphi$ (*) as well as $\overline{\pi_\pi(\mathscr{A})\zeta_\varphi} = H_\varphi$ and $\varphi(a) = \langle \zeta_U, \pi_\varphi(a)\zeta_\varphi \rangle$.

If $(e_\lambda) \subseteq \mathscr{A}$ is an approximate unit so $\pi_\varphi(a^\lambda) \to 1$ strong as $a \to \infty$. Then $\|\varphi\| \leftarrow \varphi(e_\lambda) = \langle \zeta_\varphi, \pi_\varphi(e_\lambda)\zeta_\varphi \rangle \to \|\zeta_\varphi\|^2$, so $1\|\zeta_\varphi\| = 1$ and it is a cyclic representation.

Also, from (*) we know $\zeta_\varphi = \lim_\lambda \pi_\varphi(e_\lambda)\zeta_\varphi = \lim_\lambda e_\lambda + N_\varphi$.

So the GNS-construction gives a triple $(\pi_\varphi, H_\varphi, \zeta_\varphi)$ satisfying our conditions. □

Conversely, if $(\pi, H, \zeta)$ is a cyclic representation of $\mathscr{A}$, then $\varphi(a) := \langle \zeta, \pi(a)\zeta \rangle$ defines a style $\varphi \in S(\mathscr{A})$.

**Remark 11.10 (irreducible representations and pure states)** Notice: $\Omega(\mathscr{A}) \subseteq S(\mathscr{A}) \subseteq \mathscr{A}_1^*$. In particular, we can endow this with the weak $^*$-topology.

This is closed and therefore compact: Take $\varphi_i \in S(\mathscr{A})$ with $\pi_i \to \varphi \in \mathscr{A}_1^*$ ($\|\varphi\| leq 1$). Then

$$1 \xleftarrow{\lambda} \varphi_i(e_\lambda) \xrightarrow{i} \varphi(e_\lambda)$$

with $\|\varphi\| = 1 = \lim_\lambda \varphi(e_\lambda)$.

Moreover, $S(\mathscr{A})$ is convex, so for $\varphi_1, \ldots, \varphi_n \in S(\mathscr{A})$ and $t_1, \ldots, t_n \in \mathbb{R}_+$ with $\sum_{i=1}^n t_i = 1$ we have $\sum_{i=1}^n t_i \varphi_i \in S(\mathscr{A})$.

Recall the Kreim-Milman-Theorem: If $K$ is a compact convex subset of $\mathscr{A}_1^*$, then $K = \overline{\text{conv}(\text{Ext}(K))}$ where conv is the convex hull and Ext are the extremal points, that is all points in $K$ that cannot be reached as linear combinations of other points (e.g. the corners of a closed triangle). In particular, any compact convex set must have extremal points (unless it is empty).

We will apply this to the states $K = S(\mathscr{A})$.

**Definition 11.11** *Call $PS(\mathscr{A}) := \text{Ext}(S(\mathscr{A}))$ the **pure states** of $\mathscr{A}$.*

**Theorem 11.12** *A state $\varphi \in S(\mathscr{A})$ is pure if and only if $\pi_\varphi \mathscr{A} \to \mathcal{B}(H_\varphi)$ is irreducible if and only iff $\pi_\varphi(\mathscr{A})' := \{T \in \mathcal{B}(H_\varphi) \mid T\pi_\varphi(a) = \pi_\varphi(a)T\} = \mathbb{C} \cdot 1$ by Schur's lemma.*

PROOF: See Murphy. □

**Example 11.13** Let $\mathscr{A} = C_0(X)$. Take $\varphi \in C_0(X)^* \simeq$ Complex bounded Radon measure of $X$. If $\mu : \text{Borells}(X) \to \mathbb{C}, E \to \mu(E)$ has $\mu = \Re\mu + i\Im\mu$. $\Re\mu = \Re(\mu)_+ - \Re(\mu)_-$ is a complex (Radon) measure, then the associated $\varphi = \varphi_\mu \in C_0(X)^*$ is $\varphi_\mu(f) = \int_X f(x)d\mu(x)$.

Moreover, $\varphi_\mu \geq 0 \Leftrightarrow \mu \geq 0$, so $C_0(X)_+^*$ consists of the positive Radon measures on $X$.

Note: Characters correspond to Dirac measures: $\mu_{x_0}(E) = 1$ if $x_0 \in E$ and 0 otherwise. The real measures correspond to the self-adjoint elements and the states correspond to those measures with $\mu(X) = 1$, that is the probability (positive Radon) measures on $X$.

**Remark 11.14** Look at the GNS construction for $\varphi = \varphi_\mu$. Define

$$\langle f, g\rangle_\varphi = \varphi(f^* \cdot g) = \varphi(\overline{f} \cdot g) = \int_X \overline{f(x)} g(x) d\mu(x)$$

Then

$$N_\varphi = \left\{ f \in C_0(X) \mid \varphi(\overline{f} f) = \int_X |f(x)|^2 d\mu(x) = 0 \right\} \trianglelefteq C_0(X)$$

Indeed, $N_\varphi$ corresponds to the support of $\mu$: $\text{supp}(\mu) = \{x \in X \mid \forall_{U \subseteq X \text{ open}} x \in U \Rightarrow \mu(U) > 0\}$ (this is always closed). Now we want to show for $U = \text{supp}(\mu)^\complement$:

$$N_\varphi = C_0(U) = \{f \in C_0(X) \mid f|_{U^\complement} \equiv 0\}$$

"$\supseteq$": If $f \in C_0(U)$, $f|_{\text{supp}(\mu)} \equiv 0$ then $\int_X |f(x)|^2 d\mu(x) = 0$. TODO

Then $H_\varphi = L^2(X, \mu) = \overline{C_0(X)}$ (with closure in respect to $\langle \cdot, \cdot \rangle_{2,\mu}$) and $\pi_\varphi(f)(\zeta + N_\varphi) = f \cdot \zeta + N_\varphi$ (where the added class $N_\varphi$ represents that the functions are equal $\mu$-almost everywhere). These correspond to $M_f(\zeta) = f \cdot \zeta$.

# 12 Inverse Semigroups

Our main results so far:

- Every commutative $C^*$-algebra $\mathscr{A}$ is $\mathscr{A} \simeq C_0(X)$ where $X$ is a locally compact Hausdorff space.
- Every $C^*$-algebra can be embedded into $\mathcal{B}(H)$ for some Hilbert space $H$.

How to now model $C^*$-algebras in general? We look for general constructions of $C^*$-algebras and hope that many $C^*$-algebras in practice are 'part' of this construction. We are going to look at the class of $C^*$-algebras associated to (inverse) semigroups and groupoids. These include, in particular, groups. One of the motivating examples:

**Example 12.1** Recall that the Cuntz-$C^*$-algebra is the (universal, unital) $C^*$-algebra $\mathcal{O}_n$ generated by $n$ isometries $S_1, \dots, S_n \in \mathcal{O}_n$ satisfying the relations $S_i^* S_j = \delta_{ij} \cdot 1$ and $S_1 S_1^* + \cdots + S_n S_n^* = 1$. Then we can look at the set

$$S := \{S_\alpha S_\beta^* \mid \alpha, \beta \text{ finite words in } \{1, 2, \dots, n\}\} \cup \{0\} \subseteq \mathcal{O}_n$$

where for $\alpha = \alpha_1 \alpha_2 \cdots \alpha_k$ with $\alpha_i \in \{1, \dots, n\}$ we have $S_\alpha = S_{\alpha_1} \cdots S_{\alpha_k}$ and we convention that for the empty word $\varepsilon$ we have $S_\varepsilon = 1$.

Note that $\mathcal{O}_n = C^*(S)$ and each non-zero element $s \in S$ is an isometry, and every element is a partial isometry, that is $ss^*s = s$. Also, $S$ is closed under multiplication (of $\mathcal{O}_n$). So, this means that $S$ is a sub semigroup of the multiplicative semigroup of $\mathcal{O}_n$. So $S$ is a $^*$-semigroup of partial isometries. We now consdier only $C^*$-algebras that thusly arise.

**Definition 12.2** *An **inverse semigroup** is a semigroup $S$ (it is endowed with an associative multiplication $S \times S \to S$) which is also a $^*$-semigroup (it is endowed with an involution $^* : S \to S$ sometimes also called a 'pseudo-inverse') satisfying:*

- $(s^*)^* = s$ *(involution)*
- $(st)^* = t^* s^*$ *(antimultiplicative)*
- $ss^*s = s$
- *The elements of $E(S) = \{s^*s \mid s \in S\}$ should commute. These elements are called **idempotents** and $E(S) = \{e \in S \mid e = e^2\}$.*

**Remark 12.3**

- Where the first two properties makes it a $^*$-semigroup and the last makes it a $^*$-semigroup of partial isometries.
- $E(S)$ is a commutative (inverse) subsemigroup of $S$ and $e^* = e$ for all $e \in E(S)$.
- Given $s \in S$, $t = s^* \in S$ is the unique element of $S$ satisfying $sts = s$ and $tst = t$.

**Example 12.4**

(i) Groups are always inverse semigroups with exactly 1 idempotent, that is $E(S) = \{e\}$. Furthermore, we have $E(S) = \{s^*s \mid s \in S\} = \{ss^* \mid s \in S\}$.

If $s^*sess^*$ for all $s$ then $ss^*s = s = ss^*s$.

(ii) **Commutative Inverse**: Semigroups that are exactly the () semilattices, that is partially order set $(E, \leq)$ for which every $e, f \in E$s has $e \wedge f = \in \{e, f\} = e \cdot f = f \cdot e$.

(iii) If $S$ is an inverse semigroup which is commutative, then $S = E(S) =$ is the set of idempotents and this is a semilattice with $e \leq f \Leftrightarrow e \cdot f = e$.

(iv) Let $X$ be any set (with the natural order) and $\mathcal{X}$ the powerset of $X$, then $A \cdot B = A \cap B$ is a commutative ISG.

(v) Consider $(\mathbb{N}_0, +)$. This can be viewed multiplicatively or additively. We will look at the addition. Certainly, this is a semigroup. Can it be an inverse semigroup? No, because we would need an element $n \in \mathbb{N}$ with $n + m + n = 0$ but that would imply $n = 0$. For $(\mathbb{N}, \cdot)$, we have the same problem.

Lets look at $\overline{(\mathbb{N}, \min)}$ with multiplication $n \cdot m = \min(n, m)$ in compliance with the lattice. This is commutative and therefore an Inverse Semigroup.

(vi) Take now $M_n(\mathbb{C})$ with basis $(e_{ij})^n$, then $e_{ij}e_{kl} = \delta_{j,k}e_{i,l}$ and $e_{ij}^* = e_{ji}$. Then $e_{ij} : \mathbb{C} \to \mathbb{C}, e_j \to e_i$ and partial isometries $e_{ij}e_{ij}^*e_{ij} = e_{ij}$. So $S = \{e_{ij} \mid i, j = 1, \ldots, n\} \cup \{0\} \subseteq M_n(\mathbb{C})$ is a $^*$-semigroup of partial isometries, so it is an inverse semigroup. The $C^*$-algebra of this inverse semigroup is $M_n(\mathbb{C})$.

(vii) Let $X$ be any set. Then

$$I(X) = \{f \mid f \text{ is partial bijection between subsets of } X\}$$

i.e. $f : U \to V$ is a bijection where $U, V \subseteq X$ (note that these may be any set, even the empty set and need not be open, as $X$ does not even have a topology). We must still find a suitable product.

Take $f : U \to V$, $g : U' \to V'$. Take $\tilde{U} = f^{-1}(U' \cap V)$ and define $f \cdot g : \tilde{U} \to g(V \cap U'), x \mapsto f(g(x))$.

So $I(X)$ is an inverse semigroup with $f^* = f^{-1}$ and $f(f^{-1}f) = \mathrm{id}_{D(f)} \cdot f = f$.

Additionally, we get $E(I(X)) = \{\mathrm{id}_U \mid U \subseteq X\} = 2^X$.

**Example 12.5 (About $I(X)$)** Take $X = \{1, 2\}$. Then

$$I(X) = \{0 = \emptyset, \mathrm{id}_{\{1\}}, \mathrm{id}_{\{2\}}, \mathrm{id}_X = 1, \{1\} \to \{2\}, \{2\} \to \{1\}, (\{i1\} \to \{2\}, \{2\} \to \{1\}) = (12)\}$$

whereas $\mathrm{Bij}(X) = S_2 = \{\mathrm{id}_X, (12)\}$.

One can also consider $I(\mathscr{A})$

- $\supseteq \mathrm{Aut}(\mathscr{A}) = \{f : \mathscr{A} \to \mathscr{A} : f \ ^*\text{-automorphism}\}$
- $\supseteq \mathrm{pAut}(\mathscr{A}) = \{f : I \to J \mid I, J \trianglelefteq \mathscr{A}, f \ ^*\text{-automorphism}\}$

**Theorem 12.6 (Vagner-Preston-Theorem)** *Every inverse semigroup $S$ can be embedded (as an inverse sub semigroup) into $I(X)$ for some $X$.*

*This is somewhat of a generalization of Caley's theorem.*

Proof (Idea): Take $s \in S$ and $\overline{X} = S$ defines a partial bijection. $f_s(x) = sx$. Take $D_s = \{x \in X \mid s^*sx = x\} \subseteq X$, so $f_s : D_s \to R_s, f_s^{-1} = f_{s^*}$ where $R_s = \{x \mid ss^*x = x\} = D_{s^*}$ is the partial inverse.. □

**Definition 12.7** *Let $S$ be an inverse semigroup (that is, $S$ is a semigroup and for all $s \in S$ we have $s^* \in S$ and $ss^*s = s$). Then $C^*(S)$ is the <u>universal</u> $C^*$-algebra generated by (a 'copy' of) $S$ as a $^*$-semigroup.*

*More precisely: $C^*(S)$ is a $C^*$-algebra endowed with a $^*$-homomorphism $\iota : S \to C^*(S)$ such that for every other $C^*$-algebra $\mathscr{B}$ with a $^*$-homomorphism $\pi : S \to \mathscr{B}$ there exists a unique $^*$-homomorphism $\tilde{\pi} : C^*(S) \to \mathscr{B}$ such that $\tilde{\pi} \circ \iota = \pi$.*

**Remark 12.8**

(i) We are going to prove that the $C^*$-algebra $C^*(S)$ exists.

(ii) An inverse semigroup $S$ might have a unit $1 \in S$ (i.e. $1s = s = s1$). If this is the case, $C^*(S)$ and $\iota : S \to C^*(S)$ will be unital, and in the universal property we may assume $\mathscr{B}$ and $\pi$ to be unital.

Also, you can always formally add such a unit (and only this unit) to any inverse semigroup. Therefore, we will most of the time only consider such unital semigroups.

(iii) An inverse semigroup $S$ might have a zero 0 (i.e. $0s = 0 = s0$). If this is the case, we would like that $0 \in S$ "is" also $0 \in C^*(S)$, that is the embedding $\iota$ is zero-preserving: $\iota(0) = 0$. This is not automatic, but we can change the definition and force this to be true. Formally, we define another $C^*$-algebra $C_0^*(S)$ in a similar way by asking $\iota, \mathscr{B}, \pi$ to be zero-preserving.

(iv) We will proof that $C_0^*(S)$ exists. It is actually $C_0^*(S) = C^*(S)/\langle \iota(0) \rangle$.

**Example 12.9** Let $S = \{s\}$ be a single-element semigroup (with $s = s^* = s^2$). In this case $s = 0 = 1$. Then $C^*(S)$ is the universal $C^*$-algebra generated by a projection. We claim $C^*(S) = \mathbb{C}$. Indeed, $p = 1 \in \mathbb{C}$ is a projection with $\mathbb{C} = C^*(1) = \mathscr{C} \cdot 1$. So this means we have $\iota : S \to \mathbb{C}, s \mapsto 1$. If $\mathscr{B}$ is any algebra with $^*$-homomorphism $\pi : S \to \mathscr{B}$, this just means that $p = \pi(s) \in \mathscr{B}$ is a projection. Then $\tilde{\pi} : \mathbb{C} \to \mathscr{B}, \lambda \mapsto \lambda \cdot p$.

If, however, we treat $s \in S$ as the zero, then $C_0^* * = 0$.

**Example 12.10** Set $S = \{p, q\}$ the inverse semigroup with two elments $qq = q$, and $pp = qq = qp = p$. $C^*(S)$ is the universal $C^*$-algebra generated. So there are two projections $P, Q$ mit $P = P$ ($P \leq Q$). Then the $C^*$-algebra is Commutative!i. Claim: $C^*(S) \simeq C^2 \to \mathbb{C}$. This is indeed that case as $P(1,0) = Q(1,1)$. So we have $C_0^*(S) \simeq \mathbb{C}$.

**Example 12.11** Let $S = \{1, g\}$ and $g = g^* = g^{-1}$ (and $g^2 = 1$). This is a full group. Then $C^*(S)$ is the universal unital $C^*$-algebra generated by a self-adjoint unit, so $C^*(S) = C^*_{univ}(1, u)$ for osme self-adjoint with $u^2 = 1$ and $u^* = u$. Then $C_0^*(S) = \mathbb{C}^2$.

Take $\iota : S \to \mathbb{C} \oplus \mathbb{C}$ where $1 \mapsto (1,1)$ and $g \mapsto u = (\alpha, \beta)$ with $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 = 1 = \beta^2$. Then $\mathbb{C} \oplus \mathbb{C}$

Proof (Existence of $C^*(S)$): First, consider the $^*$-algebra of $S$. Take

$$\mathbb{C}[S] = \left\{ \sum_{s \in S}^{\text{fin}} a_s S_s \mid a_s \in \mathbb{C} \right\}$$

then $\delta_s \cdot \delta_t = \delta(st)$ and $\delta_s^* = \delta_{s^*}$.

The idea is now to complete this to a $C^*$-algebra. To get $C^*(S)$, take $C^*(S) = \overline{C[S]}^{\|\cdot\|}$. For $C^*$ to be 'universal', it must be the largest and its norm must be the largest $\| \cdot \|$ $C^*$-algebra norm. As a $^*$-homomorphism between $C^*$-algebras is automatically contractive, we can define for $a \in \mathbb{C}[S]$. Then $\|a\|_{\max} = \sup\{p(a) \mid p : \mathbb{C}[S] \to [0,\infty), C^*\text{seminorm}\}$. This set is non-empty, but it could be unbounded. We prove that, in the current case of a semigroup construction, this is not the case, and the supremum thus itself defines a $C^*$-seminorm. Write $a = \sum_{s\in S}^{\text{fin}} a_s\delta_s = \sum_{i=1}^m a_{s_i}$. Take $p \in \mathbb{C}[S] \to [0,\infty)$ a $C^*$-seminorm. Idea:

$$p(a) \le \sum_{i=1}^n |a_{s_i}| p(\delta_{s_i})$$

if $s \in S$, $p(\delta_s)^2 = p(\delta_s^*\delta_1) = \dots$.

> Let $\mathscr{A}$ be a $C^*$-algebra, $p : \mathscr{A} \to [0,\infty)$ with a $C^*$-seminorm and $a \in \mathscr{A}$ a partial isometry. Then $p(a) < 1$.
>
> Proof: Omitted.

Define $N_p = \{a \in \mathscr{A} \mid p(a) = 0\} \trianglelefteq \mathscr{A}$. Then

$$\mathscr{A}/N_p \xrightarrow{\|\cdot\|_p} [0,\infty), \|a + N_p\|_p := p(a)$$

is a $C^*$-norm and $C_p^*(\mathscr{A}) = \overline{\mathscr{A}/N_p}^{\|\cdot\|_p}$. Then $\pi : \mathscr{A} \xrightarrow{q} \mathscr{A}$ is a $^*$-homomorphism. Furthermore, $p(a)^2 = \|a + N_p\|^2 = \|\pi(a)\|^2 = \|\pi(a^*a)\| \le 1$. Then $\pi(a)$ is a partial isometry of a $C^*$-algebra.

So $\|\cdot\|_{\max}$ defined by the supremum is a $C^*$-seminorm. As in the lemma, define $C^*(S) = \overline{\mathbb{C}[S]/N_{\|\cdot\|}}^{\|\cdot\|}$. Then $C^*(S)$ is a $C^*$-algebra and we have a $^*$-homomorphism $q : \mathbb{C}[S] \to C^*(S)$ as a quotient map (in completion plus embedding). Concatenating this with $s \mapsto \delta_s$ yields the final $^*$-homomorphism.

Universal property: Take $\mathscr{B}$ any $C^*$-algebra with $^*$-homomorphism $\pi : S \to \mathscr{B}$. This induces a $^*$-homomorphism $\rho : \mathbb{C}[S] \to \mathscr{B}$ by $\rho(\sum a_s\delta_s) = \sum a_s\pi(s)$ and then $\|a\|_\rho := \|\rho(a)\|_{\mathscr{B}}$ defines a $C^*$-seminorm. So $\rho(a) = \|a\|_\rho \le \|a\|_{\max}$. This means that $\rho$ is contractive and continuous (for the max norm). In particular, $\rho$ vanishes on $N_{\|\cdot\|_max}$ and therefore induces a $^*$-homomorphism $\tilde{\pi} : C^*(S) \to \mathscr{B}$.

This also shows the existence of $C_0^*(S)$. $\square$

**Theorem 12.12 (Wordingham's Theorem)** *Let $S$ be an inverse semigroup, then $\lambda : S \to \mathbb{B}(\ell^2(S))$ defined by*

$$\lambda_s(\delta_t) := \begin{cases} \delta_{st} \mid s * st = t \\ 0 \mid \text{otherwise} \end{cases},$$

*where $(\delta_t)_{t\in S} \in \ell^2(S)$ is the orthonormal basis, is a representation, i.e. a $^*$-homomorphism, of $S$. Additionally, it extends to a faithful representation $\tilde{\lambda} : \mathbb{C}[S] \to \mathbb{B}(\ell^2(S))$ given by $\tilde{\lambda}(\sum a_s\delta_s) = \sum a_s\tilde{\lambda}(\delta_s)$.*

*Indeed, $\lambda$ extends faithfully to $\tilde{\lambda} : \ell^1(S) \to \mathbb{B}(\ell^2(S))$. In particular, there exists a $C^*$-norm on $\mathbb{C}[S]$, also on $\ell^1(S)$ namely $\|a\|_\lambda := \|\tilde{\lambda}(a)\|_{\mathbb{B}(\ell^2(S))}$ and therefore $\mathbb{C}[S] \subseteq \ell^1(S) \hookrightarrow C^*(S)$ and $\ell^1(S) \hookrightarrow C_n^*(S) = C_\lambda^*(S) = \overline{\mathbb{C}[S]}^{\|\cdot\|_\lambda}$.*

Proof: See Patterson's Book "Groupoid and inverse Semigroups in $C^*$-algebras".

Maybe at least see that $\lambda : S \to \mathbb{B}(\ell^2(S))$ is injective.

Let $\lambda_s = \lambda_t$ for $s, t \in S$. Then $\lambda_s(\delta_n) = \lambda_t(\delta_n)$ for all $n \in S$. Then

$$\begin{Bmatrix} \delta_{sn} \mid s^*sn = n \\ 0 \end{Bmatrix} = \begin{cases} \delta_{sn} \mid s^*sn = n \\ 0 \end{cases}$$

Take $n = s^*$, then $s^*ss^* = s^*$, then $t^*ts^* = s^*$ (as their images are equal an the left side is non-zero), so $ss^* = tt^*$. This shows $s = ss^*s = ts^*s$. We need that $s^*s = t^*t$. But it follows that $t^*ts^*s = (s^*s)$ and similarly $s^*st^*t - t^*t$. $\square$

**Corollary 12.13** *The image $S$ in $C^*(S)$ is linearly independent, that is $\sum_{s\in S}^{\text{fin}} a_s\iota(s) = 0$, where $\iota : S \to C^*(S)$.*

**Corollary 12.14** *If $S$ is finite then $C^*(S) = \mathbb{C}[S] = C^*_\lambda(S) =$*

k

**Example 12.15** Let $n \in \mathbb{N}$. Then $S = \{kke_{i,j} \mid i,j \in \{0,1,2,\dots,n\}\} \subseteq M_n(\mathbb{C})$ is an ISA. and we have $\|S\| = n^2+1$ and $C^*_0(S)$ is a finite dimensional $C^*$-algebra with dimension $n^2$.

Take $C^*_0(S)$ is the universal $C^*$-algebra genreated by $\{\tilde{e}_{ij} \mid i,j \in \{1,\dots,n\}\}$ satisfying $\tilde{e}_{ij}\tilde{e}_{kl} = \delta_{jk}\tilde{e}_{il}$ and $\tilde{e}^*_{ij} = \tilde{e}_{ij}$ and is therefore isomorphic to $M^*$.

> Observation: With some extreme work one can also analyse to "infinite" on any cardinal. So $C^*_0(S) \simeq N(\ell^2(S^*))$ and thus $\tilde{e}_{ij} = |\delta_i X_\delta|$.

**Example 12.16** Let $S = \{s_i \mid i \in \{1,\dots,n\} \cup \{0\} \cup \{1\}\}$ (for fixed $n \in \mathbb{N}$) be the "Cuntz"-semigroup. Then $s^*is_j = \delta_{ij}$.

Recall that $U_n$, the universal $C^*$-algebra generatd by $\tilde{S}_1$ and $\tilde{S}_n$ satisfying $\tilde{S}^*_i\tilde{S}_j = \delta_{ij}1$ and therefore $\sum_{i=1}^n \tilde{S}_i\tilde{S}^*_i = 1$.

So we get a quotient homomorphism

$$C^*_0(S) \to \mathcal{O}_n,$$

but this is not a $C^*$-homomorphism.

$C^*_0(S)$ is a universal $C^*$-algebra generated by $s_1,\dots,s_n$ satisfying $s^*_is_j = \delta_{ij}1$.

Observe: $C^*_0(S)$ is sometimes called **Toerly**-Cuntz. We have $C^*$-algebra and $\mathcal{T}_n$ and $K \hookrightarrow T_n \hookrightarrow O_n$.

**Example 12.17** Let $n = 1$. Then $S$ is an invertible semigroup generated by $s$ with with $\gamma + 1$ and $s^*s = 1$j. This is ...

... isomorphic to an inverse semigroup generated by the left-shift $s \in \mathbb{B}(\ell^2)$.

... isomorphic $\{s^nt^n \mid n,m \in \mathbb{N}\}$.

Also $C(^S) \simeq T$ (Toeplitz $C^*$-algebra) $= C^*(s) \subseteq \mathbb{B}(\ell^2\mathbb{N})$. with $s$ the Shift.

Observe $O_1$ as the universal $C^*$-algebra generated by $u$ satisfying $u^*u = zuu^* = 1$, so this is the universal $C^*$-algebra gerated by a normal element, so it is abelion and isomorphic to $C^*(Z) \simeq C(\mathbb{S}^1)$. So we get $K \hookrightarrow T \to C(\mathbb{S}^1)$.

**Example 12.18 (Graph algebras)** Let $E = (E^1 \underset{\pi}{\xrightarrow{s}} E^0)$ with vertices $E^0$ and edges $E^1$, where the starting point of edge $e \in E^1$ is $s(e) \in E^0$ and its endpoint is $\pi(e)$.

Then $C^*(E)$ is a universal $C^*$-algebra generated by pairwise orthogonal projections $(p_v)_{v\in E^0}$, so $p_v \cdot p_w = \delta_{v,w}p_v$ and $p^*_v = p_v$, and partial isometries $(S_e)_{e\in E^1}$, so $S_eS^*_eS_e = S_e$, satisfying

(i) $s^*_es_f = \delta_{e,f}p_{\pi(e)}$.

(ii) $p_{s(e)}S_e = S_e$.

(iii) $\sum_{s(e)=v} S_eS^*_e = p_v$ whenever this makes sense, so $0 < |\{s \in C^*(E) \mid s^{-1}\}| < \infty$.

Mini-Example: We have 1 vertex and $n$ self-edges ("petals"). Then $C^*(E_n)$, the universal $C^*$-algebra is generated by the single projections $p_v$ and partial isometries. $S_1, \ldots, S_n$ satisfying $s_i^* s_j = \delta_{ij} p_v$, $p_v \cdot s_i = s_i$ and $\sum s_i s_i^* = p_v$.

So $p_v$ acts as a unit on the $s_i$, and (i) and (ii) imply that $p_v$ is already the unit of $C^*(E_n)$.

Also $C^*(E_n)$ is the universal unital $C^*$-algebra satisfying $s_i^* s_j = \delta_{ij} 1$ and $\sum_{i=1}^n s_i s_i^* = 1$, so it is the Cuntz-algebra.

In general: $C^*(E)$ is a quotient of the inverse semigroup $C^*$-algebra $C^*(S_E)$, where $S_E$ is a $^*$-semigroup of $C^*(E)$ generated by $p_v, s_e$ for $v \in E^0$ and $e \in E^1$.

Observe: $S_E = \{s_\alpha s_\beta^* \mid \alpha, \beta \text{ paths in } E, \pi(\alpha) = \pi(\beta)\} \cup \{0\}$.