1 Banach Algebras

Definition 1.1 *Algebra, Subalgebra, Norm, Complete, Banach algebra, unital, homomorphisms*

Theorem 1.2 *Closed subspace of Banach is Banach.*

Theorem 1.3 *I closed ideal $\Rightarrow$ A/I normed algebra with norm $\|a+I\| = \inf_{b\in I}\|a+b\|$.*

2 Spectrum and Spectral Radius

Considering unital normed algebras.

Definition 2.1 *Invertible elements, spectrum, spectral radius.*

Remark 2.2 $1-ab$ invertible iff $1-ba$ invertible. $\sigma(ba)\setminus 0 = \sigma(ba)\setminus 0$.

Theorem 2.3 *$\sigma(a)$ non-empty and $p \in \mathbb{C}[z] \Rightarrow \sigma(p(a)) = p(\sigma(a))$.*

Theorem 2.4 *$\|a\| < 1 \Rightarrow 1-a \in \mathrm{inv}(A), (1-a)^{-1} = \sum_{n=0}^{\infty} a^n$.*

Theorem 2.5 *$\mathrm{inv}(A)$ open and $a \mapsto a^{-1}$ differentiable.*

Theorem 2.6 *$\sigma(a)$ non-empty, closed and $\subseteq \overline{K_{\|a\|}(0)}$, $\mathbb{C}\setminus\sigma(a) \to A, \lambda \mapsto (a-\lambda)^{-1}$ differentiable.*

Theorem 2.7 *A unital, Banach and $\mathrm{inv}(A) = A\setminus\{0\} \Rightarrow A = \mathbb{C}1$.*

Theorem 2.8 *$r(a) = \lim_{n\to\infty}\|a^n\|^{1/n} = \inf_{n\geq 1}\|a^n\|^{1/n}$.*

Theorem 2.9 *$1 \in B \leq A$ closed, A Banach. Then $\mathrm{inv}(B) = B \cap \mathrm{inv}(A)$ closed. $\sigma_A(b) \subseteq \sigma_B(b)$, also for boundaries. Equality if $\sigma_A(b)$ has no holes or both are C^*.*

3 Gelfand Representation

Definition 3.1 *Ideal, characters, character space.*

Theorem 3.2 *A Banach. Proper ideals have proper closure. Maximal ideals are closed. If A abelian, unital: Quotients of maximal ideals are fields.*

Theorem 3.3 *A Banach, abelian, unital. If $r \in \Omega(A) \Rightarrow \|r\| = 1$. $\Omega(A)$ non-empty and $r \mapsto \ker(r)$ is a bijection between $\Omega(A)$ and the maximal ideals in A.*

Theorem 3.4 *A Banach, abelian. A unital $\Rightarrow \sigma(a) = \Omega(A)(a)$. A non-unital $\Rightarrow \sigma(a) = \Omega(A)(a) \cup \{0\}$.*

Theorem 3.5 *A Banach, abelian $\Rightarrow \Omega(A)$ locally compact Hausdorff space. A unital $\Rightarrow \Omega(A)$ compact.*

Theorem 3.6 *A Banach, abelian, $\Omega(A) \neq \emptyset$.*

$$\Phi : A \to C_0(\Omega(A)), a \mapsto (\hat{a} : \Omega(A) \to \mathbb{C}, r \mapsto r(a))$$

norm-decreasing homomorphism and $r(a) = \|\hat{a}\|_\infty$. A unital $\Rightarrow \sigma(a) = \hat{a}(\Omega(A))$. A non-unital $\Rightarrow \sigma(a) = \hat{a}(\Omega(A)) \cup \{0\}$.
A Banach, $A = (1,a) \Rightarrow A$ abelian and $\hat{a}$ homeomorphism.
A $C^* \Rightarrow \Phi$ isometric isomorphism with weak-*-topology.

4 C^*-algebras

Definition 4.1 *Involution, *-algebra, C^*-algebra, self- adjoint, unital (isometry, co-isometry), normal, projection.*

Theorem 4.2 *$a = b + ic\frac{1}{2}(a+a^*) + i\frac{1}{2i}(a-a^*)$ with b,c self-adjoint.*

From now on: C^*-algebras, so $\|aa^*\| = \|a\|^2$ ($\geq$ enough).

Theorem 4.3 *If A is self-adjoint then $\sigma(a) \subseteq \mathbb{R}$ and $r(a) = \|a\|$. On every *-algebra, there is at most one norm to make it C^*.*

Theorem 4.4 *Multiplier-algebra of C^*: Largest unitization, $\|L\| = \|R\|$. Extension of norm of C^* makes $\tilde{A}$ into C^*.*

Theorem 4.5 **-hom between *-alg and C^* are norm-decreasing. *-hom between C^* are isometric if injective and the image is a C^*-subalgebra.*

Theorem 4.6 *Characters on C^* preserve adjoints.*

Theorem 4.7 *B C^*-subalgebra. $\sigma_B(b) = \sigma_A(a)$.*

Theorem 4.8 *a normal in unital C^* $A \Rightarrow$ exists $\varphi : C(\sigma(a)) \to C^*(1,a)$ unital isometric *-iso with $\varphi(\mathrm{id}) = a$. Write $f(a) \in A$ for $\varphi(f)$.*

Theorem 4.9 *a normal, $f \in C(\sigma(a)) \Rightarrow f(\sigma(a)) = \sigma(f(a))$. If $g \in C(\sigma(f(a))) \Rightarrow (g\circ f)(a) = g(f(a))$.*

Theorem 4.10 *X compact Hausdorff. $X \simeq \Omega(C(X))$.*

5 Positive Elements in C^*

Definition 5.1 *Positive elements (hermitsch und $\sigma(a) \subseteq \mathbb{R}_0^+$), ordered elements*

Theorem 5.2 *$B^+ = A^+ \cap B$. $A^+ \subseteq A_{sa}$. $A^+ = \{a^*a \mid a \in A\}$. Conjugation self-adjoint elements keeps their order. $a \leq b \Rightarrow \|a\| \leq \|b\|$ Inverting invertes order, square roots keep it (and square roots exist).*

6 Ideals in C^*

Definition 6.1 *Approximate units (increasing net of positive elements), essential ideals.*

Theorem 6.2 *C^*-algebras have approximate units (take A^+ with $\|a\| < 1$.)*

Theorem 6.3 *Quotients and approximate units. Quotient of closed ideal is C^*-algebra. If B is a C^*-subalgebra and I a closed ideal, then $B+I$ is a C^*-subalgebra.*

Theorem 6.4 *I closed in C^* $A \Rightarrow \exists$ unique *-extension $A \to M(I)$ of $I \to M(I)$, injective if I essential.*

7 Positive linear functionals

Definition 7.1 *Positive maps, positive linear functionals, states*

Theorem 7.2 **-homs are positive. $\varphi(A_{sa}) \subseteq \varphi(B_{sa})$ and $\varphi|_{A_{sa}}$ is increasing.*

Theorem 7.3 *PLFs are bounded and* $r(a^*) = r(a)^-$ *and* $|r(a)|^2 \leq \|r\| r(a^*a)$. $\|r + r'\| = \|r\| + \|r'\|$. $r(a^*a) = 0 \Leftrightarrow r(ba) = 0$ *for all* $b \in A$. $r(b^*a^*ab) \leq \|a^*a\| r(b^*b)$. $a \in A^+ \Leftrightarrow r(a) \geq 0$ *for all PLFs.*

Theorem 7.4 *For a bounded linear functional* r*, these are equivalent:* r *is positive for each/some approx. unit we have* $\|r\| = \lim_\lambda r(e_\lambda)$*. If* A *is unital,* r *is positive iff* $r(1) = \|r\|$

Theorem 7.5 *There exists a state* r *of* A *such that* $\|a\| = |r(a)|$.

Theorem 7.6 *You can extend linear functionals on* C^**-subalgebras to the whole algebra while keeping the norm.*

Theorem 7.7 *Self-adjoint bounded linear functionals can be decomposed to positive linear functionals with* $r = r_+ = r_-$ *and* $\|r\| = \|r_+\| + \|r_-\|$.

8 Gelfand-Neymark-Representation

Definition 8.1 *Representation, faithful, direct sums*

Theorem 8.2 *Hilbert space completion. Linears functionals induce representations: Take a positive linear functional, set* $N_r = \{a \in A \mid r(a^*a) = 0\}$ *(closed left ideal) and perform the Hilbert space completion* H_r *of* A/N_r *with innter product* $(\overline{a}, \overline{b}) \mapsto r(b^*a)$*. Then define an operator* $\varphi(a) \in B(A/N_r)$ *as* $\varphi(a)(\overline{b}) = \overline{ab}$ *and uniquely extend to* H_r*. Then* φ *is a* **-homomorphism.*

Doing this for all states and taking the direct product yields the univesal representation.

Any C^**-algebra has a faithful representation and its universal representation is faithful.*

Theorem 8.3 *A* $C^* \Rightarrow$ *exists unique norm on* $M_n(A)$ *to make it* C^*.