Exercises to Introduction to Operator Algebras

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1 Topological Basics

Let X be a topological space, that is there exists a subset $\mathcal{O}(X) \in \mathbb{P}(X)$.

Definition 1.1 *X is **Hausdorff** if for all $x, y \in X$ there exist open sets $U, V \in \mathcal{O}(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.*

*X is **locally Hausdorff** if for all $x \in X$ there exists an open neighborhood $U \in \mathcal{O}(X)$ such that U is Hausdorff with the relative topology from X.*

Example 1.2 (Snake with two heads) We consider the space $[0,1] \cup \{1^+\}$ equipped with a topology such that both the subspace $[0,1]$ and $[0,1] \setminus \{1\} \cup \{1^+\}$ are isomorphic to $[0,1]$. Then X is compact, locally Hausdorff but not Hausdorff.

Definition 1.3 *X is compact if for every open cover $(U_i)_{i\in I}$ there exists a finite open subcover.*

X is locally compact if for every $x \in X$ there exists a neighborhood basis of x consisting of open relatively compact subsets of X, that is for every open neighborhood U of x there exists and open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Observation: For a locally Hausdorff X, X is locally compact if and only if for all $x \in X$ there exists an open neighborhood U of x such that $\overline{U}$ is compact.

1.1 Results about locally compact Hausdorff spaces

Let X be Hausdorff and locally compact.

Proposition 1.4 (Uryson's Lemma) *For all closed $F \subset X$ and all compact $K \subseteq X$ with $F \cap K = \emptyset$, there is a continuous function $f : X \to [0,1]$ such that $f|_K \equiv 1$ and $f|_F \equiv 0$.*

Proposition 1.5 (Tietze's extension theorem) *For all $K \subseteq X$ compact and $f : K \to \mathbb{C}$ continuous, there exists and $\tilde{f} : X \to \mathbb{C}$ continuous such that $\tilde{f}|_K \equiv f$.*

Proposition 1.6 (Alexandroff's compactification theorem) *$\tilde{X} = X \cup \{\infty\}$ ($\infty \notin K$) is a compact Hausdorff space with $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{K^{\complement} \cup \{\infty\} \mid K \subseteq X \, compact\}$.*

For example, compactifying $\mathbb{R}$ yields the unit circle $\mathbb{S}^1$.

Proposition 1.7 *Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$ the space $X = Y \setminus \{y_0\}$ is a locally compact Hausdorff space.*

Proposition 1.8 *More generally, if Y is a locally compact Hausdorff space and $Z \subseteq Y$ is a difference of open and closed subsets of Y (i.e. $Z = U \setminus F$ or $Z = F \setminus U$ where $U \subseteq Y$ is open and $F \subseteq Y$ is closed) then Z is locally compact.*

Exercise 1.1 Let X be a locally compact Hausdorff space. The following are equivalent:

(1) X is compact.

(2) $C(X) = C_0(X)$ $(= C_b(X))$.

(3) $C_0(X)$ is unital.

(4) $1 \in C_0(X)$ where $1(x) = 1 \in \mathbb{C}$ for all $x \in X$.

PROOF:

- (1) $\Rightarrow$ (2): Recall:

$$C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$$

  If X is compact, then every closed subset of X is compact, so all sets of form $\{x \in X \mid |f(x)| \geq \varepsilon\}$ are compact, and we have $C(X) = C_0(X)$.

- (2) $\Rightarrow$ (3): This is trivial because $C(X)$ is always unital.

- (3) $\Rightarrow$ (4): Suppose $C_0(X)$ is unital and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, that is $f(x)g(x) = 1$ for all $x \in X, g \in C_0(X)$. By Uryson's Lemma, given $x_0 \in X$, there exists a $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$, take any precompact open neighborhood U of x and look at $F := U^{\complement} \subseteq X$). Then we have $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. As this is possible for every $x_0 \in X$, we have $f \equiv 1$.

- (4) $\Rightarrow$ (1): Suppose $f = 1 \in C_0(X)$. Then choosing $\varepsilon = \frac{1}{2}$ shows that $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$ is compact. $\square$

Exercise 1.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$.

2 Exercise sheet 1

Exercise 2.1 (1.1)

PROOF: **Case 1**: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1\alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathscr{A} \to \mathscr{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda}a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathscr{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to $\mathbb{C}$. $\square$

Exercise 2.2 (1.2) We consider pathological examples for $C_0(X)$.

Let $X = \overline{\{x_0\}}$, e.g. $x_0 \in X$ with $\mathscr{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f : X \to \mathbb{C}$, this implies $f(x_0) \to f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.$\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \not\equiv 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 2.1 *Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a *-homomorphism between C^*-algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathscr{A}$.*

Exercise 2.3 (1.4 - Products) Let $(A_i)_{i\in I}$ be a family of C^*-algebras and define

$$\prod_{i\in I} A_i = \{a = (a_i)_{i\in I} \mid a_i \in A_i \forall_{i\in I} \text{ and } \|a\| := \sup_{i\in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^*-axiom.

Remark 2.2 (Differences between product and direct sum)
In addition to the product space, we define

$$\bigoplus_{i\in I} A_i = \left\{ (a_i) \in \prod_{i\in I} A_i \mid \forall_{\varepsilon>0} \exists_{\text{finite } F\subseteq I} \forall_{i\notin F} \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of $\prod_{i\in I} A_i$ as the closure of $\bigoplus_{i\in I}^{alg} A_i$, where

$$\bigoplus_{i\in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i\in I} A_i \mid \exists_{\text{finite } F\subseteq I} \forall_{i\notin F} \|a_i\| = 0 \right\}.$$

For finite I, these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) *-homomorphisms $\pi_j : \prod_{i\in I} A_i \to A_j$ for all $j \in I$. If B is any C^*-algebra with *-homomorphisms $\varphi_j \to A_j$ for every $j \in I$, there is a unique *-homomorphism $\varphi : B \to \prod_{i\in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} B \arrow[r, "\varphi_j"] \arrow[d, "\varphi"] & A_j \\ A \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Exercise 2.4 (1.5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a *-isomorphism.

Exercise 2.5 (2.6) Let X be a locally compact Hausdorff space and $\widetilde{C_0(X)} \simeq C(\tilde{X})$ with $\tilde{X} := X \sqcup \{\infty\}$ with the topology $\mathscr{O}_{\tilde{X}} = \mathscr{O}_X \cup \{\tilde{X} \setminus K \mid K \subseteq X \text{ kompakt}\}$.

Observation: If X is already compact, then ∞ is an isolated point of $\tilde{X}$ (i.e. $\{\infty\}$ is clopen).

If $\mathscr{A}$ is a C^*-algebra, then $\tilde{\mathscr{A}}$ (this is not the same $\tilde{}$ as on the X!) is a C^*-algebra with

$$\|a + \lambda 1\|_{C^*} := \sup_{b\in\mathscr{A}, \|b\|\leq 1} \|ab + \lambda b\|_{\mathscr{A}}$$

We check that $\tilde{\mathscr{A}}$ is a C^*-algebra.

- C^*-axiom: $\|a + \lambda 1\|_{C^*}^2 ! = \|(a + \lambda 1)^*(a + \lambda 1)\|_{C^*}$. We have

$$\|a + \lambda \cdot 1\|_{C^*} = \|(a^*a + \overline{\lambda}a + \lambda a^*) + |\lambda|^2 \cdot 1\|_{C^*}$$

$$= \sup_{b\in\mathscr{A}, \|b\|\leq 1} \|a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 \cdot b\|_{\mathscr{A}}$$

On the other hand:

$$\begin{aligned}
\|a + \lambda \cdot 1\|_{C^*}^2 &:= \sup_{\|b\|\leq 1} \|ab + \lambda b\|_{\mathscr{A}}^2 \\
&= \sup_{\|b\|\leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\|_{\mathscr{A}} \\
&= \sup_{\|b\|\leq 1} \|b^*a^*ab + \overline{\lambda}b^*ab + \lambda b^*a^*b + |\lambda|^2 b^*b\|_{\mathscr{A}} \\
&\leq \sup_{\|b\|\leq 1} \|b^*\|_{\mathscr{A}} \cdot \|a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathscr{A}} \\
&\leq \sup_{\|b\|\leq 1} \|a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathscr{A}} \\
&= \|a + \lambda \cdot 1\|_{C^*}^2
\end{aligned}$$

- The other conditions are easy to check and are left for the student.

We still want to prove $\varphi : \widetilde{C_0(X)} \to C(\tilde{X}), f + \lambda \cdot 1 \mapsto f_\lambda$ with $f_\lambda(x) := \lambda$ for $x = \infty$ and $f_\lambda(x) = f(x) + \lambda$ otherwise. Nothe that once again these are not the same $\tilde{}$.

- f is well-defined: We have to check that f_λ is continuous in $\tilde{X}$. Take any sequence $X \ni x_i \to \infty$ in $\tilde{X}$. We have to show $f_\lambda(x_i) \to f_\lambda(\infty) = \lambda$. Since $f_\lambda(x_i) = f(x_i) + \lambda$ this is equivalent to $f(x_i) \to 0$. But as $f \in C_0(X)$, we have that for every $\varepsilon > 0$ the set $K_\varepsilon(f) = \{x \mid |f(x)| \geq \varepsilon\}$ is compact. Since x_i will eventually leave this compact set (or it would not diverge to ∞), we know that $f(x_i)$ eventually becomes smaller than (any) ε. So we have $f(x_i) \to 0$ and thus $f_\lambda(x_i) \to f_\lambda(\infty)$. So f_λ is continuous in ∞. The continuity on every other point follows immediately from the continuity of f.
- φ is a *-isomorphism:
  - Linearity: φ is clearly linear as we can check component-wise:

    $$(f_1 + f_2)_\lambda = (f_1)_\lambda + (f_2)_\lambda$$

  - Homomorphism: For every $x \in X$ we have

    $$\begin{aligned}
    \varphi((f + \lambda \cdot 1) \cdot (g + \lambda' \cdot 1))(x) &= \varphi((fg + \lambda' f + \lambda g) + \lambda\lambda' \cdot 1)(x) \\
    &= (fg + \lambda' f + \lambda g)(x) + \lambda\lambda' \\
    &= (fg)(x) + \lambda' f(x) + \lambda g(x) + \lambda\lambda' \\
    &= (f(x) + \lambda) \cdot (g(x) + \lambda') \\
    &= (\varphi(f + \lambda) \cdot \varphi(g + \lambda'))(x).
    \end{aligned}$$

    In the case of $x = \infty$, this equality of course also holds. Thus we have $\varphi((f + \lambda)(g + \lambda')) = \varphi(f + \lambda)\varphi(g + \lambda')$.
  - *-homomorphism:

    $$\varphi(f + \lambda)^*(x) = \varphi(f^* + \overline{\lambda} \cdot 1)(x)$$

    For $x \in X$ this follows by $\overline{f(x)} + \overline{\lambda} = f^*(x) + \overline{\lambda}$, for $x = \infty$ we have $\overline{\lambda} = \overline{\lambda}$.

- Injective: $f_\lambda(0)$ leads to $f_\lambda(x) = 0$ for all $x \in \tilde{X}$, since if $x = \infty$ then λ must be 0 and $f(x) = 0$ for all $x \in X$. Thus $f = 0$ and $\lambda = 0$.
- Surjective: Take $g \in C(\tilde{X})$ and choose $\lambda = g(\infty)$ and $f(x) \coloneqq g(x) - \lambda$. and check $f \in C_0(X)$.

- We can also prove that φ is isometric for the C^*-norm:

$$\|f + \lambda \cdot 1\| \coloneqq \sup_{g \in C_0(X), \|g\| \leq 1} \|fg + \lambda g\|_\infty$$

Look at

$$\begin{aligned}\|\varphi(f + \lambda 1)\| &= \sup_{x \in \tilde{X}} |f_\lambda(x)| = \max\{|\lambda|, \sup_{x \in X} |f(x) + \lambda|\} \\ &\overset{(*)}{=} \sup_{x \in X} |f(x) + \lambda|\end{aligned}$$

and

$$\begin{aligned}\|f + \lambda \cdot 1\|_{C^*} &= \sup_{\|g\| \leq 1} \|fg + \lambda g\|_{C_0(X)} \\ &= \sup_{|g(x)| \leq 1 \forall_x} \sup_{x \in X} \|f(x)g(x) + \lambda g(x)\| \\ &= \sup_{x \in X} |f(x) + \lambda| \\ &\overset{(**)}{=} \sup_{x \in X} |f(x) + \lambda|\end{aligned}$$

This proof may need to be divided into two cases:

- X is not compact: We can find a net $(x_i) \subseteq X$ with $f(x_i) \to 0$ and $(*)$ follows and use a $g(x) \approx 1$ for $(**)$.
- X is compact: Choose $g \equiv 1$ for $(**)$ and think about $(*)$ later.

Exercise 2.6 (1.8) It is difficult to prove $I^* = I$. The idea is to prove $I = C_0(U)$ where $C_0(U) = \{f \in C_0(X) \mid f|_{U^\complement} \equiv 0\}$.

One can also prove $C_0(X)/C_0(U) \simeq C_0(F)$ (as C_0 of the subspace) where $F = U^\complement$.