Exercise Sheet 03
Operator Algebras

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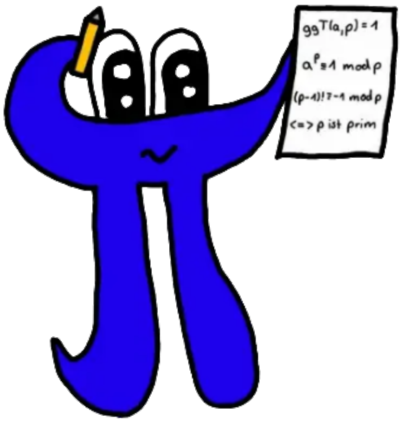

3.2

The C^*-property shows $\|a^2\| = \|a^*a\| = \|a\|^2$, and by using this as well as the C^* property again, we have for $n = 4$ that $\|a^4\| = \|a^*a^*aa\| = \|(a^2)^*(a^2)\| = \|a^2\|^2 = \|a^4\|$. Inductively, we can likewise prove $\|a^{2^k}\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $n + m = 2^k$ for some $k \in \mathbb{N}$. Then we have

$$\|a\|^{2^k} = \|a^{2^n}\| = \|a^n a^m\| \leq \|a^n\| \cdot \|a^m\| \leq \|a\|^n \cdot \|a^m\| \leq \|a\|^{n+m} = \|a\|^{2^k}$$

and because the first and last element are equal, we must have equality in every intermediate step. This especially proves $\|a^n\| = \|a\|^n$.

Let now $a \in \mathscr{A}$ be an arbitrary element. Then $\|a^*a \dots a^*a\| = \|(a^*a)^{\frac{n}{2}}\| = \|a^*a\|^{\frac{n}{2}} = \|a\|^n$ as proven above, because (a^*a) is self-adjunct. For non-even n (and thus even $n+1$) we can once again calculate

$$\|a\|^{n+1} = \|a^*aa^* \dots a^*a\| \leq \|a\| \cdot \|aa^* \dots a^*\| \leq \|a\| \cdot \|a\|^n = \|a\|^{n+1}$$

and therefore $\|aa^* \dots a^*\| = \|a\|^n$ by the same argument as above.

Now, for a normal $a \in \mathscr{A}$ (that is, $a^*a = aa^*$) we have

$$\|a^n\|^{\frac{1}{n}} = \left(\|a^n\|^2\right)^{\frac{1}{2n}} = \|(a^n) * a^n\|^{\frac{1}{2n}} = \|aa^*a \dots a^*\|^{\frac{1}{2n}} = \left(\|a\|^{2n}\right)^{\frac{1}{2n}} = \|a\|$$

and therefore $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \|a\|$.

Finally, we can use the fundamental theorem of continuous functional calculus. Consider for the moment $\mathscr{A}$ to be unital (if it is not, consider $\tilde{\mathscr{A}}$). Then $a \in \mathscr{A}$ is normal and $f : \mathbb{C} \to \mathbb{C}, x \mapsto |x|^2 = x \cdot \overline{x}$. f is continuous on $\mathbb{C}$ and therefore especially on $\sigma(a)$. Thus we have

$$\sigma(aa^*) = \sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\} = \{|\lambda|^2 \mid \lambda \in \sigma(a)\}.$$

As a is normal, we also have $a^*a = aa^*$ and therefore $\sigma(aa^*) = \sigma(a^*a)$.

3.6

First, to prove that $M(\mathscr{A})$ fulfills the given property. We already know that $\mathscr{A}$ is a closed, two-sided and essential ideal in $M(\mathscr{A})$. Consider the following morphism:

$$\varphi : \mathscr{B} \to M(\mathscr{A}), b \mapsto (L_b, R_b)$$

where

$$\begin{aligned} L_b &: \mathscr{A} \to \mathscr{A} a \mapsto b \cdot a \\ R_b &: \mathscr{A} \to \mathscr{A} a \mapsto a \cdot b \end{aligned}$$

defined via the multiplication in $\mathscr{B}$. Because $\mathscr{A} \trianglelefteq \mathscr{B}$, we actually have $a \cdot b, b \cdot a \in \mathscr{A}$ for all a, b and L_b, R_b are well-defined and, as they are clearly linear, φ is also well-defined. Because of $L_{ab} = L_a \circ L_b$ and $R_{ab} = R_b \circ R_a$, we have $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ with the multiplication as defined in the lecture. Furthermore, $\varphi(1) = (L_1, R_1) = (\mathrm{id}, \mathrm{id})$ and φ is therefore a homomorphism. Lastly we have $\varphi(b^*) = (L_{b^*}, R_{b^*})$ and

$$\begin{aligned} L_{b^*}(a) &= b^*a = (a^*b)^* = R_b(a^*)^* = (R_b)^*(a) \\ R_{b^*}(a) &= ab^* = (ba^*)^* = L_b(a^*)^* = (L_b)^*(a) \\ \Rightarrow \varphi(b^*) &= (R_b^*, L_b^*) = (L_b, R_b)^* \end{aligned}$$

so φ is indeed a *-homomorphism. Since $\varphi|_{\mathscr{A}}$ reduces to the normal left- and right-multiplication on $\mathscr{A}$, it coincides with canonical inclusion map as defined in the lecture. φ therefore fulfills all conditions as given.

To conclude that the universal property is indeed correct, we need to consider the case that $\mathscr{A} \trianglelefteq \mathscr{B}$ is an essential ideal. In this case, $b\mathscr{A} = 0$ implies $b = 0$ for any $b \in \mathscr{B}$. Assume $\varphi(b) = \varphi(c)$ for any two $b, c \in \mathscr{B}$. Then we have $(L_b, R_b) = (L_c, R_c)$ and thus $ba = ca$ and $ab = ac$ for all $a \in \mathscr{A}$. This is equivalent to $b\mathscr{A} = c\mathscr{A}$ and $\mathscr{A}b = \mathscr{A}c$ or, stated differently, $(b-c)\mathscr{A} = 0$ and $\mathscr{A}(b-c) = 0$. As stated above, this implies $(b-c) = 0 \Leftrightarrow b = c$ and thus proves that φ is injective.

Next, we want to prove that any algebra $D \trianglerighteq \mathscr{A}$ that fulfills the above property (and where $\mathscr{A}$ is a closed, two-sided essential ideal in D) is already equal to $M(\mathscr{A})$.

We already know that $\mathscr{A}$ is an essential ideal in $M(\mathscr{A})$, so if D also fulfills the property above the therefore existent morphism $\varphi_D : M(\mathscr{A}) \to D$ must be injective. We may thus treat $M(\mathscr{A})$ as a subalgebra of D. In parallel, since $\mathscr{A}$ is also an essential ideal of D, the morphism $\varphi_M : D \to M(\mathscr{A})$ is also injective and we may consider $M(\mathscr{A})$ as a subalgebra of D. But then these two algebras are isomorphic to subalgebras of each other, so they must already be equal.

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