# Introduction to Operator Algebras

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The set of all linear bounded operators $\mathscr{L}(H) = \mathscr{B}(H)$ on a given Banach space $H$ is a (Banach) algebra with $S \cdot T = S \circ T$. $M \subseteq \mathscr{L}$ is a Subalgebra such that $M^* \subseteq M$ where $T^*$ is the adjoint of $T$. This is also a closed subspace with respect to the strong topology. This is equivalent to $M = M''$ (when $X \subseteq \mathscr{B}(H), X' = \{T \in \mathscr{B}(H) \mid TS = ST \ \forall_{S \in X}\}$)

## Some topological basics

**Definition 0.1**

- *Topology, Open*
- *Hausdorff, locally Hausdorff*
- *compact*

**Definition 0.2** *A topological space $X$ is* ***locally Hausdorff*** *if every $x \in X$ admits a compact neighborhood basis, that is for every $x \in X$ and every open set $U \ni x$ there exists an open set $V \ni x$ with $\overline{V}$ is compact.*

**Corollary 0.3** *If a set $V$ is compact in any subset $U \subseteq X$, it is also compact in $X$.*

**Example 0.4 (Snake with two heads)** Consider $I = [0, 1]$ with the standard topology and extend the set with an element $1^+$ such that $I \cup 1^+ \setminus 1$ is isomorphic to $I$. Then $I \cup 1^+$ is locally Hausdorff and compact, but not Hausdorff.

## Some results about locally compact Hausdorff spaces

**Lemma 0.5 (Uryson's Lemma)** *Let $X$ be locally compact and Hausdorff. For all $F \subseteq X$ closed and $K \subseteq X$ compact with $F \cap K = \emptyset$, there exists an $f : X \to [0, 1]$ continuous such that $f|_K \equiv 1$ and $f|_F \equiv 0$.*

**Theorem 0.6 (Tietze's extension theorem)** *Let $X$ be locally compact, $K \subseteq X$ compact and $f : K \to \mathbb{C}$ continuous. Then there exists a continuous $\tilde{f} : X \to \mathbb{C}$ such that $\tilde{f}|_K = f$.*

**Theorem 0.7 (Alexandroff's conpactification)** *If $X$ is locally compact and Hausdorff, then $\tilde{X} \sqcup \{\infty\}$ is a compact Hausdorff space $\mathscr{O}(\tilde{X}) = \mathscr{O}(X) \cup \{K^{\complement} \cup \{\infty\} \mid K$ compact$\}$.*

**Example 0.8** Compacting the real line $\mathbb{R}$ yields the space $\tilde{\mathbb{R}}$, which is isomorphic to the unit circle $\Pi = \mathbb{S}^1$.

**Theorem 0.9** *Conversely, if $Y$ is a compact Hausdorff space, then for all $y_0 \in Y$, $X \coloneqq Y \setminus \{y_0\}$ is locally compact (in respect to the subspace topology).*

*More generally, if $Y$ is locally compact and Hausdorff, and $Z \subseteq Y$ is a difference of open and closed subsets, of $Y$ (i.e. $Z = U \setminus F$, where $U$ is open in $Y$ and $F$ is closed in $Y$), then $Z$ is locally compact.*

# 1 Algebras

**Definition 1.1** *An* ***algebra*** *is a (complex) vector space $\mathscr{A}$ endowed with a bilinear and associative multiplication: $\mathscr{A} \times \mathscr{A} \to \mathscr{A}, (a,b) \mapsto a \cdot b$. So*

*(i)* $(a + \alpha b) \cdot (c + \beta d) = ac + \alpha bc + \beta ad + \alpha\beta bd$

*(ii)* $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

*for all $a, b, c \in \mathscr{A}$ and $\alpha, \beta \in \mathbb{C}$. We say that $\mathscr{A}$ is*

*(i)* ***commutative****, if $ab = ba$ for all $a, b \in \mathscr{A}$ and*

*(ii)* ***unital****, if there exists $1 = 1_{\mathscr{A}} \in \mathscr{A}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathscr{A}$.*

**Example 1.2**

(i) $\mathbb{C}$, or more generally $\mathbb{C}^n = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, is an algebra.

(ii) Say $X$ is any set; let $\mathbb{C}^X = \{f : X \to \mathbb{C}\}$ with point wise multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$. These are commutative unital algebras (with $1(x) = 1 \in \mathbb{C}$).

(iii) Consider the polynomials $\mathbb{C}[X] = \{\sum_{i=0}^{n} \lambda_i x^i \mid \lambda_i \in \mathbb{C}, n \in \mathbb{N}\}$ with the usual operations. This is a commutative unital algebra.

(iv) Let $X$ be a topological space and $C(X) = \{f : X \to \mathbb{C} \mid f \text{ is continouus}\} \subseteq \mathbb{C}^X$ the set of continuous functions on $X$. This is a commutative unital (sub)algebra (of $\mathbb{C}^X$).

(v) Take any vector space $A$ define a (trivial) multiplication $a \cdot b := 0$. This is a commutative Algebra (that is not unital unless $A = 0$).

(vi) $M_n(\mathbb{C})$ (the complex $n \times n$ matrices) with the usual multiplication are a non-commutative (unless $n = 1$) unital algebra.

(vii) Let $V$ be any (complex) vector space. The set of all linear operators $L(V) := \{T : V \to VT \text{ linear operator}\}$ is a unital (non-commutative for $\dim V > 1$). We observe $\mathscr{L}(\mathbb{C}^n) \simeq M_n(\mathbb{C})$.

(viii) Let $S$ be a semigroup (i.e. a set with an associative operation $S \times S \to S$, e.g. $(\mathbb{N}, +)$). Then $\mathbb{C}[S] = \{\sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{C}, |\{s : \lambda_s \neq 0\}| < \infty\}$ (the finite formal sums of elements of $S$) with the following product

$$\left(\sum_{s \in S'} \lambda_s s\right) \cdot \left(\sum_{t \in S} \lambda'_t t\right) := \sum_{s,t \in S} (\lambda_s \cdot \lambda'_t)(s \cdot t) \in S$$

Observe: As a vector space: $\mathbb{C}[S] \subseteq \mathbb{C}^S$. In general, this is neither commutative nor unital.

# 2 Normed algebras

**Definition 2.1** *An algebra $\mathscr{A}$ is **normed**, if it is endowed with a (vector space) norm $\|\cdot\|\colon \mathscr{A} \to [0,\infty)$ satisfying $\|a\cdot b\| \le \|a\|\cdot\|b\|$. If $\mathscr{A}$ is unital with unit $1_{\mathscr{A}}$, we usually assume $\|1_{\mathscr{A}}\| = 1$ except for $\mathscr{A} = 0$.*

**Definition 2.2** *A **Banach algebra** is a normed algebra that is also complete (as a metric space with respect to the distance $d(a,b) := \|a-b\|$), i.e. every Cauchy sequence converges.*

**Example 2.3** (i) If $X$ is a compact space then $C(X)$ is a commutative unital Banach algebra with respect to the norm $\|f\|_\infty := \sup_{x\in X}|f(x)| < \infty$ (since $X$ is compact).

(ii) If $V$ is a normed (respectively Banach) vector space, e.g. $\mathbb{C}^n$ or $\ell^p(\mathbb{N})$, then $\mathscr{L}(V) = \{T \in L(V) \mid T \text{ is bounded/continouus}\}$ with $\|T\| := \sup_{\|v\|\le 1}\|T(v)\| < \infty$ is a normed Banach algebra.

(iii) If $X$ is a topological space, then $C_b(X) = \{f \in C(X) \mid \|f\|_\infty < \infty\}$ (bounded continuous functions) is a Banach space.

(iv) Let $X$ again be a topological space. Then the set of all functions **vanishing at** $\infty$,

$$\begin{aligned} C_0(X) &= \{f \in C(X) \mid \forall_{\varepsilon>0}\exists_{K\subseteq X, K \text{ compact}}\forall_{x\notin K}|f(x)| < \varepsilon\} \\ &= \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \ge \varepsilon\} \text{ is compact}\} \subseteq C_b(X), \end{aligned}$$

is also a Banach algebra.

**Exercise 2.1** Assume $X$ is locally compact and Hausdorff. Prove the following are equivalent:

(1) $X$ is compact.

(2) $C(X) = C_0(X)$

(3) $C_0(X)$ is unital.

(4) The unit function $1 \in C_b(X)$ belongs to $C_0(X)$.

PROOF:

- (1) $\Rightarrow$ (2): Recall the definition of $C_0(X)$. If $X$ is compact, every closed subset (especially every $\{x : |f(x)| \ge \varepsilon\}$) is compact, so the condition of $C_0(X)$ is trivial.
- (2) $\Rightarrow$ (3): Since $C(X)$ is unital, $C_0(X)$ is as well.
- (3) $\Rightarrow$ (4): Suppose $C_0$ is unital, and let $f \in C_0(X)$ be the unit. Then $f\cdot g = g$ for all $g \in C_0(X)$, i.e. $f(x)g(x) = g(x)\forall_{x\in X}\forall_{g\in C_0(X)}$. By Uryson's lemma, given any $x_0 \in X$, there exists $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$ and taking $F$ as the complement of any relatively compact environment of $x_0$.). Then $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. Doing this for every $x_0 \in X$ yields $f \equiv 1$.
- (4) $\Rightarrow$ (1): Since $1 \in C_0(X)$, for every $\varepsilon > 0$ the set $\{x \mid |f(x)| \ge \varepsilon\}$ is compact. Choose $\varepsilon = \frac{1}{2}$. Then, $\{x \mid |f(x)| = |1| \ge \frac{1}{2}\} = X$ is compact. $\square$

**Exercise 2.2** Let $X$ be a locally compact Hausdorff space. Prove that $C_0(X) \cong \{f \in C(X) \mid f(\infty) = 0\}$

# 3 Algebras

**Definition 3.1** *A* $^*$***-algebra*** *is a complex algebra* $\mathscr{A}$ *with an* ***involution*** $^*: \mathscr{A} \to \mathscr{A}$ *satisfying*

*(i)* $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$

*(ii)* $(a^*)^* = a$

*(iii)* $(ab)^* = b^* a^*$

*for all* $a, b \in \mathscr{A}$ *and all* $\lambda \in \mathbb{C}$.

**Definition 3.2** *A* ***normed*** $^*$***-algebra*** *is a normed algebra* $\mathscr{A}$ *with an involution (such that* $\mathscr{A}$ *is a* $^*$*-algebra) also satisfying* $\|a^*\| = \|a\|$ *for all* $a \in \mathscr{A}$.

*A* ***Banach-***$^*$***-algebra*** *is a complete normed* $^*$*-algebra.*

**Definition 3.3** *A* $C^*$*-algebra is a Banach-*$^*$*-algebra satisfying* $\|a^* \cdot a\| = \|a\|^2$.

Observation: Recall that $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ in all normed algebras. Applying this to a $C^*$-algebra we get $\|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$. If $\mathscr{A}$ is a $C^*$-algebra, then $\|a\|^2 = \|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$, so $\|a\| = \|a^*\|$.

**Example 3.4**

(i) If $X$ is a set, then $\mathbb{C}^X$ is a $^*$-algebra with $f^* = \overline{f}$ and $\mathscr{C}^\infty(X)$ is a $C^*$-algebra.

(ii) If $X$ is a topological space, then $C(X) \subseteq \mathbb{C}^X$ is also a $^*$-subalgebra and for $\{f \in C(X) \mid \operatorname{supp}(f) = \overline{\{x \in X \mid |f(x)| \neq 0\}} \text{ compact}\}$ we have

$$C_c(X) = \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X) \subseteq C^\infty(X)$$

and $C^\infty$ is a $C^*$-algebra. $C_c$ is a $^*$-algebra, but not Banach in general.

If $X$ is compact, it follows $C_c(X) = C_0(X) = C_b(X)$.

Observation: If $X$ is locally compact and Hausdorff, then $\overline{C_c(X)} = C_0(X)$.

(iii) Let $X$ be a measured space ($X$ is endowed with a $\sigma$-algebra). Then $B_\infty(X) = \{f \in C^\infty \mid f \text{ is measurable}\}$ is a $C^*$-algebra. If $\mu$ is a measure on $X$ (e.g. $X = \mathbb{R}^n$ and $\mu$ the Lebesgue measure) then $L^\infty(X, \mu)$ are the essentially bounded functions and

$$L^\infty(X) = \{f : X \to \mathbb{C} \mid \|f\| := \inf\{c \geq 0 \mid \mu(\{x \mid |f(x)| > c\}) = 0\}\}$$

is also a $C^*$-algebra.

Observation: $L^2(X, \mu) =$ "$\mu$-separable function", $L^\infty(X, \mu) \xrightarrow{\mu} B(L^2(X, \mu)), f \mapsto \mu_f = \{g \mapsto f \cdot g\}$

(iv) A non-example: Let $\mathbb{D}$ be the unit disk and $\mathscr{A}(\mathrm{d}) = \{f \in \mathbb{C}(\mathbb{D}) \mid \text{ analytic in } \mathbb{D}^\circ\}$

**Morera's Theorem** from complex analysis states that $f \in C(\mathbb{D})$ is analytic if and only if $\int_\gamma f(z) dz = 0$ for all closed and piece wise smooth paths in $\mathbb{D}^\circ$. From this, it follows that $\mathscr{A}(\mathbb{D})$ is closed in $C(\mathbb{D})$, therefore a Banach algebra. It is also a Banach-$^*$-algebra with, but $f^* = \overline{f}$ (point wise) is not possible, as $z \mapsto \overline{z}$ is not analytic. Thus, we have to choose $f^*(z) = f(\overline{z})$. But $\mathscr{A}(\mathbb{D})$ is not a $C^*$-algebra, as $\|f^* f\|_\infty \neq \|f\|_\infty^2$ for some $f \in \mathscr{A}(\mathbb{D})$.

(v) A non-commutative example: Let $H$ be a Hilbert space and $B(H) = \mathscr{L}(H) = \{T : H \to H \mid T \text{bounded, continuous, linear}\}$ and $\|H\| := \sup_{\|z\|<1} \|T(z)\| < \infty$. This is a $C^*$-algebra where $T^*$ is the adjoint of $T$, that is $\langle T^* z, w\rangle = \langle z, Tw\rangle$ for all $z, w \in H$.

$C^*$-axiom: $\|T^* \cdot T\| \leq \|T\|^2$ since $\mathscr{L}(H)$ is a Banach algebra, and we also have

$$\begin{aligned}\|T\|^2 &= \sup_{\|z\|<1} \|T(z)\|^2 = \sup_{\|z\|<1} \langle Tz, Tz\rangle = \sup_{\|z\|<1} \langle z, T^*Tz\rangle \\ &\leq \sup_{\|z\|<1} \|z\|\|T^*Tz\| \leq \sup_{\|z\|<1} \|z\|\|T^*T\| \leq \|T^*T\|\end{aligned}$$

In particular, $M_n(\mathbb{C}) \simeq \mathscr{L}(\mathbb{C}^n)$ is a unital $C^*$-algebra.

(vi) To produce more examples, take any subset $S \subseteq \mathscr{L}(H)$ and take $C^*(S) \subseteq \mathscr{L}(H) = \overline{\text{span}\{S_i \mid S_i \in S \cup S^*, i \leq n \in \mathbb{N}\}}$.

**Example 3.5** Let $s \in \mathscr{L}(\ell^2(\mathbb{N}))$. The shift $s$, defined by $s(e_i) = e_{i+1}$ for all $i \in \mathbb{N}$ (where $\{e_i\}$ is the canonical basis of the sequence space), is an isometry, that is $s^* \cdot s = \text{id}$. Since $s \cdot s^* \neq \text{id}$, it is not surjective and not a proper isometry. We define

$$T = C^*(s) = \overline{\text{span}\{s^n(s^*)^m \mid m, n \in \mathbb{N}_0\}} \subseteq \mathscr{L}(\ell^2(\mathbb{N}))$$

as the **Toeplitz algebra**.

**Example 3.6** Let $H$ be a Hilbert space and $S$ the set of all finite rank operators on $H$.

**Example 3.7**

(i) **Commutative**: $C_0(X)$ for a locally Hausdorff space $X$.

(ii) **Non-commutative**: $\mathcal{L}(\mathfrak{H}) = \mathcal{B}(\mathfrak{H})$ for any Hilbert space $\mathfrak{H}$ (with dimension greater 1).

(iii) **More generally**: Take any subset $S \subseteq \mathcal{L}(\mathfrak{H})$ and construct $C^*(S) \subseteq \mathcal{L}(H)$ as

$$\overline{\text{span}}\{S_1, \ldots, S_n \mid S_i \in S \cap S^*\}$$

**Example 3.8 (Cuntz algebras)** Take again $\mathfrak{H} = \ell^2\mathbb{N} = \{(\lambda_n)_{n\in\mathbb{N}_0} \mid \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$ where $\langle \lambda, \lambda'\rangle = \sum_{i\in\mathbb{N}_0} \overline{\lambda_i}\lambda_i'$ and which has the orthonormal base $(e_n)_{n\in\mathbb{N}}$ where $(e_n) = (\delta_{in})_{i\in\mathbb{N}_0}$.
On this algebra, define

- $S_1(e_n) = e_{2n}$.
- $S_2(e_n) = e_{2n+1}$.

We have partitioned the natural numbers into evens and odds. This defines two (proper) isometries $S_1, S_2 \in \mathscr{L}(\mathfrak{H})$, that is $S_i^* S_i = \text{id}_{\mathfrak{H}}$, to subspaces of $\mathfrak{H}$. Notice: $S_i^* S_j = 0$ for $i \neq j$ as well as $S_1 S_1^* + S_2 S_2^* = \text{id}_{\mathfrak{H}}$. Define $\mathcal{O}_2 = C^*(S_1, S_2) = \overline{\text{span}}\{S_\alpha S_\beta^* \mid \alpha, \beta \text{ finite words in } \{1, 2\}\}$. For example, for $\alpha = 121211$ we have $S_\alpha = S_1 S_2 S_1 S_2 S_1^2$. $\mathcal{O}_2$ is called the **Cuntz algebra**. More generally, one can define $\mathcal{O}_3, \mathcal{O}_4$, ... Cuntz algebras. Joachim Cuntz proved that these are simple $C^*$-algebras with additional interesting properties we will see later.

**Example 3.9 (Rotation algebras)** Let $\mathfrak{H} = \ell^2(\mathbb{Z})$ (bi-infinite sequences) with basis $(e_n)_{n\in\mathbb{Z}}$ Define:

- $U(e_n) := e_{n+1}$ (bilateral shift)

- $V(e_n) := \lambda^n e_n$ where $\lambda \in \mathbb{C}$ is some fixed number $|\lambda| = 1$.

This defines two *unitary* operators: $UU^* = 1 = U^*U$ and $V^*V = 1 = V^*V$. If $\exp(2\pi i\theta), \theta \in \mathbb{R}$ define $A_\theta := C^*(U, V) \subseteq \mathscr{L}(\ell^2\mathbb{N})$.

There is a special relation between $U$ and $V$ where $UV = \lambda VU = \exp(2\pi i\theta)VU$. From this relation, we can describe $A_\theta = \overline{\operatorname{span}}\{\sum_{n,m\in\mathbb{Z}}^{\text{finite}} a_{n,m}U^nV^m \mid a_{n,m} \in \mathbb{C}\}$.

Furthermore, if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, $A_\theta$ is simple.

**Example 3.10 ($C^*$-algebras of groups)** Let $G$ be a (discrete) group. Look at $\mathfrak{H} = \ell^2(G) = \{(a_g)_{g\in G} \mid \sum_{g\in G} |a_g|^2 < \infty\}$ (Note: This limit will only converge if there are countably (or finitely) many non-zero parts) with ONB $(\delta_g)_{g\in G}$ where $\delta_g(h) = \delta_{gh}$. Define for each $g \in G$ an operator $\lambda_g \in \mathscr{L}(\ell^2G)$ by $\lambda_g(\delta_h) = \delta_{gh}$. Notice that $h \mapsto gh$ is a bijection, and thus $\lambda_g$ is a unitary operator with $\lambda_g^* = \lambda_{g^{-1}}$. We can now define the **reduced $C^*$-algebra** of the group:

$$C_R^*(G) := C_\lambda^*(G) \subseteq \mathscr{L}(\ell^2G) = C^*(\lambda_g \mid g \in G)$$

Here, we have the relation $\lambda_g \cdot \lambda_h = \lambda_{gh}$ and thus $C_R^*(G) = \{\sum a_g\lambda_g \mid a_g \in \mathbb{C}\}$.

In general, take $U : G \to \mathscr{L}(H), g \mapsto U_g$ a **unitary representation of** $G$ with $U_gU_h = U_{gh}$ and $U_1 = \mathrm{id}$ as well as $U_g^{-1} = U_{g^{-1}}$. Then $C_U^*(G) := \{\sum_{g\in G} a_gU_g \mid a_g \in \mathbb{C}\} \subseteq \mathscr{L}(H)$. There exists a **universal unitary representation** $C_{\max}^*(G)$, a full $C^*$-algebra of $G$.

**Remark 3.11**

(i) If $G$ is Abelian, then $C_U^*(G)$ is also abelian (commutative). In particular, $C_\lambda^*$ is abelian. Later, we will prove $C_\lambda^*(G) \simeq C(\hat{G})$ where $\hat{G}$ is the dual of $G$, i.e. $\{X : G \to \mathbb{C} \text{ characters}\}$.

(ii) For many groups, like $G = \mathbb{F}_n$ (the free groups) the reduced $C^*$-algebra $C_\lambda^*(G)$ is simple.

# 4 Homomorphisms of algebras

**Definition 4.1** *If $\mathscr{A}, \mathscr{B}$ are algebras, a **homomorphism** from $\mathscr{A}$ to $\mathscr{B}$ is a linear map $\varphi : \mathscr{A} \to \mathscr{B}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in \mathscr{A}$.*

*If $\mathscr{A}$ and $\mathscr{B}$ are $^*$-algebras, a $^*$**-homomorphism** is a homomorphism $\varphi : \mathscr{A} \to \mathscr{B}$ such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathscr{A}$.*

*If $\mathscr{A}, \mathscr{B}$ are Banach algebras, then usually we want to have **continuous** homomorphisms. Even more, we usually ask for **contractive** homomorphisms $\varphi : \mathscr{A} \to \mathscr{B}$, (that is $\|\varphi\| \leq 1$).*

We will be especially interested in **characters**:

**Definition 4.2** *A **character** of an algebra $\mathscr{A}$ is a non-zero homomorphism $\chi : \mathscr{A} \to \mathbb{C}$.*

**Example 4.3** Take any subalgebra $\mathscr{A} \subseteq \mathbb{C}^X$. Take $x_0 \in X$ and set $\chi_{x_0} := \mathrm{ev}_{x_0} : \mathscr{A} \to \mathbb{C}, f \mapsto f(x_0)$. This is not necessarily a character, but it is for example, if $\mathscr{A} = C(X)$ or $C_b(X)$ or $C_0(X)$ (if $X$ is "nice", like Hausdorff).

**Definition 4.4** *A ($^*$)-isomorphism between two ($^*$)-algebras $\mathscr{A}$ and $\mathscr{B}$ is a bijective ($^*$)- homomorphism $\varphi : \mathscr{A} \xrightarrow{\sim} \mathscr{B}$.*

**Definition 4.5** *A ($^*$)**-ideal** of a $^*$-algebra $\mathscr{A}$ is a subspace $I \subset A$ such that $I \cdot A \subseteq I$, $A \cdot I \subseteq I$ (if only one condition applies, we call this a **left ideal** or **right ideal**). For $^*$-ideals, we also want $I^* = I$. We notate this as $I \trianglelefteq \mathscr{A}$.*

**Example 4.6** If $\varphi : \mathscr{A} \to \mathscr{B}$ is a $(^*)$-homomorphism, then $\ker \varphi \trianglelefteq \mathscr{A}$.

**Example 4.7** If $I \trianglelefteq \mathscr{A}$ for $\mathscr{A}$ a $(^*)$-algebra

$$\mathscr{A}/I = \{a + I \mid a \in \mathscr{A}\}$$

with $(a+I) \cdot (b+I) := ab + I$ and $(a+I)^* = a^* + I$ is a $(^*)$-algebra.

**Theorem 4.8** *If $\mathscr{A}$ is a Banach-$^*$-algebra, then $I \trianglelefteq \mathscr{A}$ is a closed ideal, then the quotient $I/\mathscr{A}$ is also a Banach-$^*$-algebra.*

Proof: Later. □

# 5 Spectral theory

**Notation 5.1** If $\mathscr{A}$ is a unital algebra, we write

$$\operatorname{inv}(\mathscr{A}) = \{a \in \mathscr{A} \mid a \text{ is invertible in } \mathscr{A}\} = \{a \in \mathscr{A} \mid \exists_{a^{-1} \in \mathscr{A}} aa^{-1} = 1 = a^{-1}a\}$$

This is a group. Sometimes we also write $GL(\mathscr{A})$.

**Definition 5.2** *Given a unital algebra $\mathscr{A}$ and $a \in \mathscr{A}$, we define its* ***spectrum*** *(in $\mathscr{A}$) as*

$$\sigma_{\mathscr{A}}(a) = \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \notin \operatorname{inv}(\mathscr{A})\}$$

*and the resolvent of $a$ (in $\mathscr{A}$) as*

$$\rho_{\mathscr{A}}(a) = \rho(a) = \mathscr{A} \setminus \sigma_{\mathscr{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \operatorname{inv}(\mathscr{A})\}$$

**Example 5.3 (Linear Algebra)** Let $\mathscr{A} = M_m(\mathbb{C})$ and $a \in \mathscr{A}$. Then we have

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \operatorname{inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \det(\lambda - a) = 0\}$$

and these are the roots of the characteristic polynomial $\det(\lambda - a)$. This is exactly the usual spectrum from linear algebra.

**Example 5.4 (Functional Analysis)** Let $\mathscr{A} = \mathscr{L}(\mathfrak{H})$ – where $\mathfrak{H}$ is any Hilbert- or Banach space – and $T \in \mathscr{A}$. Then $\sigma_{\mathscr{A}}(T)$ is exactly the spectrum as defined in functional analysis.

If $S$ is the shift in $\mathscr{L}(\ell^2\mathbb{N})$, then we have $\sigma(S) = \mathbb{D}$.

**Example 5.5** Let $\mathscr{A} = \mathbb{C}[X]$. Here we have $\operatorname{inv}(\mathscr{A}) = \{a_0 X^0 \mid a_0 \in \mathbb{C} \setminus \{0\}\}$ the constant non-zero polynomials. If $a = \sum_{k=0}^{N} a_k x^k \in \mathscr{A}$, then we have two cases:

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \mathbb{C} & \text{otherwise} \end{cases}$$

**Example 5.6** Let $\mathscr{A} = \mathbb{C}(X) = \{p, q \mid p, q \in \mathbb{C}[X], q \neq 0\}$. Now we have $\operatorname{inv}(\mathscr{A}) = \mathscr{A} \setminus \{0\}$. If $a \in \mathscr{A}$, then

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \emptyset & \text{otherwise} \end{cases}$$

**Example 5.7** Let $\mathscr{A} = C(X)$ for any topological space $X$. Then

$$\mathrm{inv}(\mathscr{A}) = \{f \in C(X) \mid \forall_{x\in X} f(x) \neq 0\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \mathrm{inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{x\in X} f(x) = \lambda\} = \mathrm{im}(f) = f(X).$$

**Example 5.8** Let $X$ be any topological space and consider $\mathscr{A} = C_b(X)$. Then

$$\mathrm{inv}(C_b(X)) = \{f \in C_b(X) \mid \exists_{\varepsilon>0}\forall_{x\in X}|f(x)| \geq \varepsilon\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \in \mathrm{inv}(\mathscr{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{(x_n)} f(x_n) \to \lambda\} = \overline{\mathrm{im}(f)} = \overline{f(X)}.$$

This is a compact subset of $\mathbb{C}$.

**Theorem 5.9 (Algebraic spectral mapping theorem)** *Let $\mathscr{A}$ be an algebra, $a \in \mathscr{A}$ and $p \in \mathbb{C}[X], p(X) = \sum_{k=0}^{n} \lambda_k X^k$ and define $p(a) = \sum_{k=0}^{n} \lambda_k a^k$. Recall that the mapping $\mathbb{C}[X] \to \mathscr{A}, p \mapsto p(a)$ is a unital homomorphism.*

*Then $\sigma(p(a)) = p(\sigma(a))$ assuming $\sigma(a) \neq \emptyset$.*

Proof: If $p(X) = \lambda_0$ constant, this is clear (the spectrum is exactly $\lambda_0$ on both sides). Assume $p(x)$ is not constant. Fix $\mu \in \mathbb{C}$ and write

$$\mu - p(x) = \lambda_0(x - \lambda_1)\cdots(x - \lambda_n)$$

as per the fundamental theorem of algebra (note that these are not the same $\lambda$ as before) with $\lambda_0 \neq 0$. Then $\mu - p(a) = \lambda_0(a - \lambda_1)\cdots(a - \lambda_n)$. Since these expressions commute, this product is invertible if and only if $(a - \lambda_i)$ is invertible for every $i$. So $\mu \in \sigma(p(a)) \Leftrightarrow \mu - p(a)$ is not invertible if and only if there exists an $i$ for which $\lambda_i - a$ is not invertible, so $\lambda_i \in \sigma(a)$. But the $\lambda_i$ are exactly the numbers satisfying $p(\lambda) = \mu$. Thus, $\mu$ is in $\sigma(p(a))$ if it is in the image of $\sigma(a)$ under $p$. Therefore, we conclude $\sigma(p(a)) = p(\sigma(a))$. □

We now focus on invertible elements in **Banach algebras**.

**Theorem 5.10** *If $\mathscr{A}$ is a unital Banach algebra and $a \in \mathscr{A}$ with $\|a\| < 1$ then $1 - a$ is invertible and $(1-a)^{-1} = \sum_{n=0}^{\infty} a^n$.*

Proof: Observe that, since $\|a\| < 1$, we have $\sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1-\|a\|} < \infty$. This implies the (absolute) convergence of $\sum_{n=0}^{\infty}$ by the characteristic property of Banach spaces. Hence, $b := \lim_{N\to\infty} \sum_{n=0}^{N} a^n \in \mathscr{A}$. No, if $N \in \mathbb{N}$, then

$$(1-a)\left(\sum_{n=0}^{N} a^n\right) = \left(\sum_{n=0}^{N} a^n\right) - \left(\sum_{n=1}^{N+1} a^n\right) = 1 - a^{N+1} \to 1$$

because of $\|a\| < 1$. This yields $(1-a)b = 1$. □

**Theorem 5.11** *Let $\mathscr{A}$ be a non-empty, non-zero unital Banach algebra. Then $\mathrm{inv}(\mathscr{A})$ is an open subset of $\mathscr{A}$ and the function $f : \mathrm{inv}(\mathscr{A}) \to \mathscr{A}, a \mapsto a^{-1}$ is Frechet-differentiable and in particular continuous as well as $f'(a)b = -a^{-1}ba^{-1}$.*

Recall from calculus that $\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$. Also recall that $f : U \overset{\text{open}}{\subseteq} X \to Y$ with $X, Y$ Banach spaces is **differentiable** at $x_0 \in U$ there exists an operator $D_{x_0} = f'(x_0) \in \mathscr{L}(X, Y)$ such that

$$\lim_{h\to 0} \frac{f\|(x_0+h) - f(x_0) - D_{x_0}(h)\|}{\|h\|} = 0$$

Proof: Take $a \in \operatorname{inv}(\mathscr{A})$. If $b \in \mathscr{A}$ such that $\|a-b\| < \|a^{-1}\|^{-1}$. From this, we have $\|ba^{-1}-1\| = \|ba^{-1} - aa^{-1}\| = \|(b-a)a^{-1}\| \leq \|b-a\| \cdot \|a^{-1}\| < 1$. Per the previous theorem, $ba^{-1} \in \operatorname{inv}(\mathscr{A})$. This implies that $b$ is also invertible. This shows that $\operatorname{inv}(\mathscr{A})$ is open.

Furthermore, if $\|b\| < 1$, then also $(\| - b\| < 1)$. Thus, $1 + b \in \operatorname{inv}(\mathscr{A})$ and $(1+b)^{-1} = \sum_{n=0}^{\infty}(-1)^n b^n$. Thus,

$$\|(1+b)^{-1} - 1 + b\| = \left\|\sum_{n=0}^{\infty}(-1)^n b^n - 1 + b\right\| \leq \left\|\sum_{n=2}^{\infty}(-1)^n b^n\right\| \leq \sum_{n=2}^{\infty}\|b^n\| \leq \sum_{n=2}^{\infty}\|b\|^n = \frac{\|b\|^2}{1-\|b\|}$$

Now let $a \in \inf(\mathscr{A})$ and $c \in \mathscr{A}$ such that $\|c\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Then $\|a^{-}1c\| \leq \|a^{-1}\|\|c\| \leq \frac{1}{2}$. So if $b = a^{-1}$, then

$$\|(1+a^{-1}c)^{-1} - 1 + a^{-1}c\| =\leq \frac{\|a^{-1}c\|^2}{1 = \|a^{-1}c\|} < 2\|a^{-1}c\|^2$$

Now, define $U : \mathscr{A} \to \mathscr{A}, b \mapsto -a^{-1}ba^{-1}$. Then this is a linear odd operation with $\|U\| \leq \|a^{-1}\|^2$, and we have

$$\begin{aligned}
\|(a+c)^{-1} - a^{-1} - U(c)\| &= \|(a+c)^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\
&= \|(1+a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\
&\leq \|(1+a^{-1}c)^{-1} - 1 + a^{-1}c\| \cdot \|a^{-1}\| \\
&\leq 2\|a^{-1}c\|^2\|a^{-1}\| \leq 2\|a^{-1}\|^3\|c\|^2
\end{aligned}$$

and thus

$$\lim_{c\to 0}\frac{\|(a+c)^{-1} - a^{-1} - U(c)\|}{\|c\|} = 0$$

□

**Example 5.12** If we choose $\mathscr{A} = \mathbb{C}[X]$ and the norm $\|p\| = \sup_{\lambda\in[0,1]}|p(x)|$. Then $(\mathscr{A}, \|\cdot\|)$ is a normed (but not Banach) algebra. For example, we see that $\lim_{m\to 0} 1 + X/m = 1 \in \operatorname{inv}(\mathscr{A})$, but $1 + X/m \notin \operatorname{inv}(\mathscr{A})$ and thus $\operatorname{inv}(\mathscr{A})$ is not open (because the complement is not closed).

**Theorem 5.13** *If $\mathscr{A}$ is a Banach algebra with unit $1$, then for all $a \in \mathscr{A}$ the spectrum $\sigma(a) \subseteq \mathbb{C}$ is closed and $\sigma(a) \subseteq \overline{B(0, \|a\|)} = D(0, \|a\|) := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$. Therefore, $\sigma(a)$ is compact by the Heine-Borell theorem.*

Proof: By definition

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \operatorname{inv}(\mathscr{A})\}$$

is the inverse image of the closed subset $\mathscr{A} \setminus \operatorname{inv}(\mathscr{A}) \subseteq \mathscr{A}$ by the continuous function $\lambda \mapsto \lambda - a$. Therefore, $\sigma(a)$ is closed.

Now if $|\lambda| \leq \|a\|$ then $\|\lambda^{-1}a\| < 1$. Then $1 - \lambda^{-1}a \in \operatorname{inv}(\mathscr{A})$. Multiplying by $\lambda$ yields $\lambda - a \in \operatorname{inv}(\mathscr{A})$. Thus, $\{\lambda \in \mathbb{C} \mid |\lambda| > \|a\|\} \subseteq \rho(a)$ and thus $\sigma(a) \subseteq D(0, \|a\|)$. □

**Lemma 5.14** *Let $\mathscr{A}$ be a unital Banach algebra and $a \in \mathscr{A}$. Then, the map $R_a : \rho(a) \subseteq \mathbb{C} \to \mathscr{A}, \lambda \mapsto (a-\lambda)^{-1}$ is Frechet-differentiable.*

PROOF: This follows from the following general result:

If $g : U \overset{\text{open}}{\subseteq} X \to Y$ and $f : V \overset{\text{open}}{\subseteq} Y \to Z$ for Banach spaces $X, Y, Z$ with $g(U) \subseteq V$ are differentiable at $x_0 \in U$ or respectively $y_0 = g(x_0) \in V$, then $f \circ g$ is differentiable and $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$. □

Observation: For $R_a(\lambda) = (a - \lambda)^{-1}$ we get $R_a'(\lambda) = (a - \lambda)^{-2}$. We have $\mathcal{L}(\mathbb{C}, \mathscr{A}) \simeq \mathscr{A}$ by $T \mapsto T(1)$. Recall that if $f(a) = a^{-1}$ yields $f'(a)b = -a^{-1}ba^{-1}$.

**Theorem 5.15 (Gelfand)** *If $\mathscr{A} \neq 0$ is a unital Banach algebra and $a \in \mathscr{A}$ then $\sigma(a) \neq \emptyset$.*

PROOF: Suppose $\sigma(a) = \emptyset$. Idea: Show that $R_a : \rho(a) \subseteq \mathbb{C} \to \mathscr{A}, \lambda \mapsto (a - \lambda)^{-1} = \frac{1}{a-\lambda}$ is bounded and differentiable and achieve a contradiction by Liouville's theorem.

Claim: $\|(a - \lambda)^{-1}\| < \|a\|^{-1}$ if $|\lambda| > 2\|a\|$. Indeed, if $|\lambda| > 2\|a\|$ then $\|\lambda^{-1}a\| < \frac{1}{2}$, and in particular $1 - \lambda^{-1}a \in \operatorname{inv}(\mathscr{A})$ and

$$\left\|(1 - \lambda^{-1}a)^{-1} - 1\right\| = \left\|\sum_{n=1}^{\infty}(\lambda^{-1}a)^{-1}\right\| \leq \sum_{n=1}^{\infty}\|\lambda^{-1}a\|^n = \frac{\|\lambda^{-1}a\|}{1 - \|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

From here we deduce that $\|(1 - \lambda^{-1}a)^{-1}\| < 2$ and thus

$$\|(a - \lambda)^{-1}\| < \|\lambda^{-1}(\lambda^{-1}a - 1)^{-1}\| = \frac{\|(1 - \lambda^{-1}a)^{-1}\|}{|\lambda|} < \frac{2}{\lambda} < \frac{1}{\|\lambda\|}.$$

So $R_a : \mathbb{C} \to \mathscr{A}$ is bounded outside $\overline{B(0, 2\|a\|}$. Since $R_a$ is continuous, it is bounded on $\mathbb{C} \to \mathscr{A}$. Let $\varphi \in \mathscr{A}^*$ be a bounded linear functional in $\mathcal{L}(\mathscr{A}, \mathbb{C})$. Thus, $\varphi$ is differentiable with $\varphi'(a) = \varphi$ for all $a \in \mathscr{A}$. Then $\varphi \circ R_a$ is differentiable and bounded, so it is an "integer" function. By Liouville's theorem, $\varphi \circ R_a$ is constant. Therefore, $\varphi \circ R_a(x) = \varphi \circ R_a(y)$ for all $x, y \in \mathscr{A}$. Especially, we have $\varphi((a - \lambda)^{-1}) = \varphi(a^{-1})$ for all $\varphi$. Hahn-Banach shows $(a - \lambda)^{-1} = a^{-1}$ for all $\lambda$, proving $a - \lambda = a$ for all $a, \lambda$. This is a contradiction. □

**Theorem 5.16 (Gelfand-Mazur)** *If $\mathscr{A}$ is a unital Banach algebra and every $a \neq 0$ admits an inverse ($\mathscr{A}$ is a field), then $\mathscr{A} = \mathbb{C} \cdot 1$.*

PROOF: By the assumption, $\operatorname{inv}(\mathscr{A}) = \mathscr{A} \setminus \{0\}$. By the previous theorem, if $a \in \mathscr{A}$ there exists some $\lambda \in \sigma(a)$, so $a - \lambda \notin \operatorname{inv}(\mathscr{A})$, so $a - \lambda = 0$ and thus $a = \lambda \cdot 1$. □

**Corollary 5.17** *Let $\mathbb{C}(X) = \left\{\frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{Q}[X]\right\}$ is a field, but it cannot be turned into a Banach algebra.*

**Theorem 5.18 (Adjointing units - unitization of algebras)** *Let $\mathscr{A}$ be any algebra. Consider $\tilde{\mathscr{A}} = \mathscr{A} \oplus \mathbb{C}$ as a vector space. We write elements of $\tilde{\mathscr{A}}$ as $a + \lambda \cdot 1 := (a, \lambda)$. Think of $a = (a, 0)$ and $\lambda = (a, \lambda)$. Define*

$$(a + \lambda 1)(b + \lambda' 1) = (ab + \lambda' a + \lambda ab + b) + \lambda \cdot \lambda'.$$

*Ten (exercise $\tilde{\mathscr{A}}$) becomes a unital algebra with $1_{\mathscr{A}} = 1 = (0, 1)$.*

*Notice that $\mathscr{A}$ is an ideal in $\tilde{\mathscr{A}}$.*

*Moreover, we get a short exact sequence*

$$0 \to \mathscr{A} \hookrightarrow \tilde{\mathscr{A}} \to \mathbb{C} \to 0$$

*so* $1+\lambda \mapsto \lambda$.

*If* $\mathscr{A}$ *is a normed algebra, then* $\tilde{\mathscr{A}}$ *is normed by* $\|a+\lambda\cdot 1\| := \|a\|+|\lambda|$

*If* $\mathscr{A}$ *is Banach and closed, then so is* $\mathscr{A}$.

*If* $\mathscr{A}$ *is a* $^*$*-algebra, then so is* $\tilde{\mathscr{A}}$ *with* $(a+\lambda 1)^*$.

*If* $\mathscr{A}$ *is a (Banach) normed* $^*$*-algebra, then so is* $\tilde{A}$.

*If* $\mathscr{A}$ *is a* $C^*$*-algebra, in general the norm given above is not a Norm on* $\mathscr{A}$*, but* $\|a+\lambda\cdot 1\| := \sup_{b\in\mathscr{A}, b\in\mathscr{B}, b\leq 1} \|ab+\lambda b\|$ *is.*

**Exercise 5.1** If $\mathscr{A}$ is already unital, then $\tilde{A} \simeq A \oplus \mathbb{C}$ as algebras by $a+\lambda\cdot 1 \mapsto (a+\lambda 1_{\mathscr{A}}, -\lambda)$.

**Definition 5.19** *Re-Definition: If* $\mathscr{A}$ *is non-unital, then* $\tilde{A}+\mathbb{C}\cdot 1$ *is a* $(*-)$*Banach algebra, and we define* $\sigma_A(a) := \sigma_{\tilde{\mathscr{A}}}(a)$.

Observation: If $\mathscr{A}$ is already unital, then for $\tilde{A} \simeq \mathscr{A} \oplus \mathbb{C}$ we have $\sigma_{\tilde{\mathscr{A}}}(a) = \sigma_{\mathscr{A}}(a) \cup \{0\}$.

**Remark 5.20** If $\mathscr{A}$ is a $C^*$-algebra, then $\tilde{\mathscr{A}}$ is a $C^*$-algebra.

(i) If $\mathscr{A}$ is unital, then $\tilde{\mathscr{A}} \simeq \mathscr{A} \oplus \mathbb{C}$ and $\|a+\lambda\cdot 1\| = \max\{\|a+\lambda\cdot 1\|, |\lambda|\}$.

(ii) If $\mathscr{A}$ is not unital, then $\|a+\lambda\cdot 1\| = \sup_{\|b\|\leq 1}\|ab+\lambda b\|$.

# 6 Spectral Radius

**Definition 6.1** *Let* $\mathscr{A}$ *be an algebra. Given* $a\in\mathscr{A}$*, we define:*

$$\pi(a) := \sup\{|\lambda| \mid \lambda\in\sigma_{\mathscr{A}}(a)\}$$

*as the* ***spectral radius*** *of* $a$ *if* $\emptyset \neq \sigma_{\mathscr{A}}(a)$ *is bounded (e.g. if* $\mathscr{A}$ *is Banach).*

Observation: In a Banach algebra, we have $0 \leq \pi(a) \leq \|a\|$.

**Example 6.2**

(i) Let $f\in\mathscr{A} = C_0(X)$ using $\sigma_A(f) = \overline{f(X)}$. Thus,

$$\pi(f) = \sup\{|\lambda| \mid \lambda\in\overline{f(X)} = \sup_{x\in X}|f(x)| = \|f\|_{C_0(X)}$$

(ii) Let $\mathscr{A} = M_2(\mathbb{C})$ and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\sigma_{\mathscr{A}} = \{0\}$ and $\pi(a)=0$, but $\|a\| = 1 \neq 0$.

**Theorem 6.3 (Beurling-Gelfand)** *Let* $\mathscr{A}$ *be a Banach algebra, then*

$$\pi(a) = \inf_{n\in\mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty}\|a^n\|^{\frac{1}{n}}$$

Proof: We may assume $\mathscr{A}$ is unital (otherwise we consider $\tilde{\mathscr{A}}$). If $\lambda\in\sigma(a)$, then

$$\lambda^n \in \sigma(a^n) \Rightarrow |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a\|^{\frac{1}{n}} \quad \forall_{n\in\mathbb{N}}$$

and therefore

$$\pi(a) \leq \inf_{n\in\mathbb{N}}\|a^n\|^{\frac{1}{n}} \leq \liminf_{n\to\infty}\|a^n\|^{\frac{1}{n}}.$$

We prove now that $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \pi(a)$. Set $\Delta := B\Big(0, \frac{1}{\pi(a)}\Big)$. Where per convention we set $\frac{1}{\pi(a)} = \infty$ if $\pi(a) = 0$. If $\lambda \in \Delta$, then $1 - \lambda a \in \operatorname{inv}(\mathscr{A})$ (because $|\lambda| < \frac{1}{\pi(a)}$ implies $|\lambda^{-1}| > \pi(a)$ and therefore $\lambda^{-1} \notin \sigma(a) \Rightarrow \lambda^{-1} - a \in \operatorname{inv} A \Rightarrow 1 - \lambda a \in \operatorname{inv}(A)$).

Now fix $\varphi \in \mathscr{A}^*$. Then $f : \Delta \to \mathbb{C}, \lambda \mapsto \varphi((1-\lambda a)^{-1})$ is analytic, so it can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n \lambda^n, a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{C}, \lambda \in \Delta.$$

On the other hand, if

$$|\lambda| < \frac{1}{\|a\|} \leq \frac{1}{\pi(a)}$$

then $\|\lambda a\| < 1$, so

$$(1-\lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n \Rightarrow f(\lambda) = \varphi((1-\lambda)^{-1}) = \sum_{k=0}^{\infty} \varphi(a^n)\lambda^n$$

for $|\lambda| < \frac{1}{\|\lambda\|}$.

By uniqueness of the Taylor series expansion, it follows that

$$a_n = \varphi(a^n) \forall_{n \in \mathbb{N}}.$$

In particular, $(\varphi(a^n)\lambda^n)$ converges to zero for all $\lambda \in \Delta$ and thus $(\varphi(a^n)\lambda^n)$ is bounded for all $\lambda \in \Delta$.

From the principle of uniform convergence, it follows that $(a^n\lambda^n)$ is bounded. So there exists an $M = M_\lambda$ such that

$$\begin{aligned}
&\|\lambda^n a^n\| \leq M \forall_{n\in\mathbb{N}} \\
\Rightarrow &\|\lambda^n\|^{\frac{1}{n}} \leq \frac{M^{\frac{1}{n}}}{|\lambda|} \forall_{n\in\mathbb{N}}, \forall_{\lambda\in\Delta,\lambda\neq 0} \\
\Rightarrow &\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \frac{1}{\lambda} \forall_{\lambda\in\Delta \text{ i.e. } |\lambda|<\frac{1}{\pi(a)}}
\end{aligned}$$

Letting $\lambda < \frac{1}{\pi(a)}$ yields $\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \pi(a)$. □

**Example 6.4** Let $A = C^1([0,1]) = \{I \in C[0,1] \mid \exists_{f'(t)} \forall_{t\in[0,1]}, t \mapsto f'(t) \text{ continuous}\}$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

Then $\mathscr{A}$ is unital, commutative and a Banach algebra. Consider $x \in \mathscr{A}$, $x(t) = t$. We have $x^n(t) = t^n$ and

$$\begin{aligned}
\|x^n\| &= \sup_{t\in[0,1]} |t^n| + \sup_{t\in[0,1]} |nt^{n-1}| = 1 + n \\
\pi(x) &= \lim_{n\to\infty} (1+n)^{\frac{1}{n}} = 1 \\
\|x\| &= 2
\end{aligned}$$

Observation: $\sigma(x) = \operatorname{im}(x) = [0,1]$.

**Theorem 6.5** *Let $\mathscr{B} \nsubseteq \mathscr{A}$ be an inclusion of unital Banach algebras with $1 = 1_\mathscr{A} = 1_\mathscr{B}$. Then $\sigma_\mathscr{A}(b) \subseteq \sigma_\mathscr{B}(b)$ for all $b \in \mathscr{B}$ and the inclusion may be proper. If $\sigma_\mathscr{A}(b)$ is simply connected (not holes), then $\sigma_\mathscr{A}(b) = \sigma_\mathscr{B}(b)$.*
*The holes of a compact subset $K \subseteq \mathbb{C}$ are the bounded connected components of $\mathbb{C} \setminus K$. So saying that $K$ has no holes means that $\mathbb{C} \setminus K$ is connected.*

Proof: See Murphy, 1.2.8. $\square$

**Example 6.6** Let $\mathscr{B} := A(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid f \text{ analytic on } \mathbb{D}^\circ\}$ and $\mathscr{A} = C(\mathbb{S}^1)$. Then we have an embedding by $\iota : \mathscr{B} \hookrightarrow \mathscr{A}, f \mapsto f|_{\mathbb{S}^1}$.

By the principle of maximum modules, $\iota$ is an embedding of (unital) Banach algebras. Consider: $f(z) = z$ for $z \in \mathbb{D}$. (Observation: $\overline{Alg}(1, z) = A(\mathbb{D})$) Then:

$$\sigma_{A(\mathbb{D})}(f) = f(\mathbb{D}) = \mathbb{D}$$

and $\sigma_{C(\mathbb{S}^1)}(f|_{\mathbb{S}^1}) = \mathbb{S}^1$.

**Definition 6.7 (Exponentials)** *Let $\mathscr{A}$ be a unital Banach algebra, given $a \in \mathscr{A}$ we define*

$$e^a = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

*Note $\left\| \frac{a^n}{n!} \right\| \leq \frac{\|a\|^n}{n!}$, so the series converges and $\|\exp(a)\| \leq \exp(\|a\|)$.*

**Theorem 6.8**

*(i) Let $\mathscr{A}$ be a unital Banach algebra. If $a \in \mathscr{A}$, then $f : \mathbb{R} \to \mathscr{A}, t \mapsto \exp(ta)$ is the unique solution of*

$$\begin{cases} f'(t) &= af(t) \\ f(0) &= 1 \end{cases}$$

*(ii) $e^a \in \operatorname{inv}(\mathscr{A})$ and $(e^a)^{-1} = e^{-a}$.*

*(iii) If $a, b \in \mathscr{A}$ then $e^{a+b} = e^a \cdot e^b$.*

Proof: See Murphy, 1.2.9. $\square$

# 7 Gelfand Representation for commutative Banach algebras

Idea: Given a commutative algebra $\mathscr{A}$, we want to represent $\mathscr{A}$ by a homomorphism $\varphi : \mathscr{A} \to C_0(X)$ for $X$ some locally compact Hausdorff space. We hope that $\varphi$ is injective, or even isometric, on an isomorphism. But what is $X$, and what is $\varphi$?

Notice that, if $\mathscr{A} = C_0(X)$ already, then for each $x \in X$ we get a character $\operatorname{ev}_x : \mathscr{A} \to \mathbb{C}, f \mapsto f(x)$.

**Definition 7.1** *Given an algebra $\mathscr{A}$, we define*

$$\hat{\mathscr{A}} = \Omega(\mathscr{A}) := \{\chi : \mathscr{A} \to \mathbb{C} \mid \chi \text{ non-zero homomorphism}\}.$$

**Example 7.2**

(i) For $\mathscr{A} = C_0(X)$ we get a map

$$X \to \Omega(\mathscr{A}), x \mapsto \operatorname{ev}_x$$

that is a bijection. After we give $\Omega(\mathscr{A})$ an appropriate topology, it will also be a homomorphism.

(ii) Let $\mathscr{A} = M_2(\mathbb{C})$ (or any $M_n(\mathbb{C})$). This is a simple algebra, so non-zero homomorphisms $\chi : \mathscr{X} \to \mathbb{C}$ do not exist (same for any $\mathscr{A}$ with dimension $> 1$).

So in this case we have $\Omega(\mathscr{A}) = \emptyset$. This can also happen in commutative algebras.

(iii) Consider

$$\mathscr{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\}$$

Then for all $a \in \mathscr{A}$ we have $a^2 = 0$, so if $\chi : \mathscr{A} \to \mathbb{C}$ is an homomorphism, then $\chi(a)^2 = \chi(a^2) = 0$, so $\chi(a) = 0$ for all $a \in \mathscr{A}$. So again, $\Omega(\mathscr{A}) = \emptyset$ (and $\mathscr{A}$ is commutative with $\dim \mathscr{A} = 1$).

Question: Given an abstract algebra $\mathscr{A}$ how do we possibly find its characters?

Idea: Assume that $\mathsf{I} \lhd \mathscr{A}$ is a maximal ideal and $\mathscr{A}$ is a unital Banach algebra. Then $\mathscr{A}/\mathsf{I} \simeq \mathbb{C}$ and $\chi \in \Omega(\mathscr{A})$.

**Theorem 7.3** *Let $\mathscr{A}$ be a unital non-zero Banach algebra. If $\chi \in \Omega(\mathscr{A})$ then $\|\chi\| = \sup_{\|a\|=1} |\chi(a)| = 1$ and $\ker(\chi) \lhd \mathscr{A}$. So $\chi \in \mathscr{A}^*$ (the topological dual of $\Omega(\mathscr{A}) \subseteq D_{\mathscr{A}^*}(0,1)$).*

*Moreover, if $\mathscr{A}$ is a unital Banach commutative algebra, then $\Omega(\mathscr{A}) \ni \chi \mapsto \ker(\chi) \lhd \mathscr{A}$ is a bijection between of characters of $\mathscr{A}$ and maximal ideals of $\mathscr{A}$.*

Proof: If $a \in \mathscr{A}$ and $\chi$ a character, then $\chi(a) \in \sigma(\mathscr{A})$, because $\chi(a - \chi(a) \cdot 1) = \chi(a) - \chi(a) \cdot \chi(1) = 0$, so $a - \chi(a) \cdot 1 \in \ker(\chi) \lhd \mathscr{A}$ and thus $a - \chi(a) \cdot 1 \notin \operatorname{inv}(\mathscr{A})$.

Therefore: $|\chi(a)| \leq \pi(a) \leq \|a\|$. So $\|\chi\| \leq 1$. Since $\chi(1) = 1$ and $\|1\| = 1$ we have $\|\chi\| = 1$.

Now, apply linear algebra. Then $\ker(\chi)$ is a maximal proper subspace, in particular a maximal ideal. And $\ker(\chi)$ is closed, because $\chi$ is continuous. Now assume that $\mathscr{A}$ is commutative (in addition to unital and Banach). Then we have the mapping

$$\varphi : \Omega(\mathscr{A}) \to \text{MaxIdeals}(\mathscr{A}), \chi \to \ker(\chi).$$

- $\varphi$ is injective. If $\ker(\chi_1) = \ker(\chi_2)$ for $\chi_1, \chi_2 \in \mathscr{A}$, then for every $a \in \mathscr{A}$ we have $a - \chi_1(a) \cdot 1 \in \ker(\chi_1) = \ker(\chi_2)$. Thus, $\chi_2(a = \chi_1(a) \cdot 1) = 0$ and therefore $\chi_2(a) = \chi_1(a)$ for every $\mathscr{A}$.

- $\varphi$ is surjective. Take $\mathsf{I} \lhd \mathscr{A}$ a maximal ideal. Then $\mathsf{I} = \bar{\mathsf{I}}$ because $\bar{\mathsf{I}} \neq \mathscr{A}$, otherwise $1 \in \bar{\mathsf{I}}$ and since $\operatorname{inv}(\mathscr{A})$ is open in $\mathscr{A}$, we get $\mathsf{I} \cap \operatorname{inv}(\mathscr{A}) \neq \emptyset$. But then we have an invertible element in the ideal $\mathsf{I}$ already, but this implies the contradiction $\mathsf{I} = \mathscr{A}$. Therefore, $\mathscr{A}/\mathsf{I}$ is a commutative, unital Banach algebra which is simple ($\mathsf{I}$ is maximal).

  Exercise: If $\mathsf{I} \lhd \mathscr{A}$, then $\mathscr{A}/\mathsf{I}$ is field if and only if there exists no $\mathsf{J} \lhd \mathscr{A}$ such that $\mathsf{I} \lhd \mathsf{J}$.

  Thus, $\mathscr{A}/\mathsf{I}$ is a field and $\mathscr{A}/\mathsf{I} \simeq \mathbb{C}$. Then the composition

  $$\mathscr{A} \xrightarrow{q} \mathscr{A}/\mathsf{I} \simeq \mathbb{C}$$

  is a character with $\ker(\chi) = \mathsf{I}$. □

**Exercise 7.1** An application of Zorn's Lemma. Show that every ideal $I \lhd \mathscr{A}$ in a unital algebra $\mathscr{A}$ is contained in a maximal ideal.

In particular, we can apply this to $\mathsf{I} = 0$ in $\mathscr{A} \neq 0$ (with $\mathscr{A}$ is unital and commutative) and thus $\Omega(\mathscr{A}) \neq \emptyset$.

## Topology on $\Omega(\mathscr{A})$

We have for $\mathscr{A}$ a Banach algebra. We can add a unit to receive $\tilde{\mathscr{A}}$, which is a Banach algebra.

Observe: If $\chi \in \Omega(\mathscr{A})$, then there exists a unique $\tilde{X} \in \Omega(\tilde{\mathscr{A}})$ via $\tilde{X}(a + \lambda \cdot 1) = \chi(a) + \lambda$. Thus, $\|\chi\| \leq \|\tilde{X}\| = 1$ (Note that it may still be smaller than 1. See exercises 2023-05-09).

In any case,

$$\Omega(\mathscr{A}) = D_{\mathscr{A}^*}(0,1) = \varphi\{\varphi \in \mathscr{A}^* = \{\varphi \in \mathscr{A}^* \mid \|\varphi\| \leq 1\}$$

and $\mathscr{A}^*$ corries the weak $^*$-topology. $\varphi_i \to \varphi$ in $^*$.

**Definition 7.4** *Given a Banach algebra $\mathscr{A}$, we endow $\Omega(\mathscr{A})$ with the weak $^*$-topology and call thus this the* ***Gelfand spectrum*** *of $\mathscr{A}$.*

**Proposition 7.5** $\Omega(\mathscr{A})$ *is a locally compact Hausdorff space. If $\mathscr{A}$ is unital, then $\Omega((\mathscr{A}))$ is compact.*

Proof: By Banach-Alaoglu-Theorem, $D_{\mathscr{A}^*}(0,1)$ is compact and Hausdorff with the weak $^*$-topology. Let

$$\begin{aligned} S &:= \{\chi : A \to \mathbb{C} \mid \chi \text{ hom.}\} \\ &= \Omega(\mathscr{A}) \cup \{0\} \end{aligned}$$

Then $S \subseteq D_{\mathscr{A}^*}(0,1)$. So $\chi(ab) = \lim_{i\to\infty} K_i = \lim_{i\to\infty} \chi_i(a)\chi_i(b) = \chi(a)\chi(b)$ and therefore $x \in S$. Thus, $S$ is a compact Hausdorff space and $\Omega(\mathscr{A}) = S \setminus \{0\}$ is relatively compact.

If $\mathscr{A}$ is unital, then $\Omega(\mathscr{A}) \subseteq D_{\mathscr{A}^*}(0,1)$ is closed. Then we have $(X_i) \subseteq \Omega(\mathscr{A})$ and $X_i \to X \in \mathscr{A}^*$ and thus $X \in S = \hom(\mathscr{A}, \mathbb{C})$. □

Observation: Given a Banach algebra $\mathscr{A}$, we have a homeomorphism

$$\Omega(\tilde{\mathscr{A}}) \to \Omega(\mathscr{A}) \sqcup \{\chi_\infty\}, \varphi \mapsto \begin{cases} \varphi|_{\mathscr{A}} & \varphi|_{\mathscr{A}} \neq 0 \\ \chi_\infty & \varphi|_{\mathscr{A}} = 0 \end{cases},$$

where $\chi_\infty(a + \lambda \cdot 1) = \lambda$. Thus, $\Omega(\mathscr{A}) \sqcup \{\chi_\infty\}$ is already the unitization of $\Omega(\mathscr{A})$.

**Theorem 7.6** *Let $\mathscr{A}$ be a Banach algebra. Then for every $a \in \mathscr{A}$.*

$$\{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \subseteq \sigma(a)$$

*If $\mathscr{A}$ is commutative, then*

- $\{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} = \sigma(a)$ *in case $\mathscr{A}$ is unital.*
- $\{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\} = \sigma_{\mathscr{A}}(a)$.

Proof:

- $\mathscr{A}$ is unital and $a \in \mathscr{A}$. $\chi(a - \chi(a)\cdot 1) = 0$, so $\chi(a) \in \sigma(a)$, so $\{\chi(a) \mid x \in \Omega(a)\} \subseteq \sigma(a)$.

  Now if $\lambda \in \sigma(a)$, consider $\mathsf{I} := (a - \lambda \cdot 1)\mathscr{A} \triangleleft \mathscr{A}$ if $\mathscr{A}$ is commutative. By Zorns Lemma, we get $I \subseteq J \triangleleft \mathscr{A}$ with $J = \ker(\chi)$ for some $\chi \in \Omega(\mathscr{A})$. Thus we have $a - \lambda \cdot 1 \in \mathsf{I} \subseteq J = \ker(\chi)$ so $\chi(a) = \lambda$.

- $\mathscr{A}$ is not unital. Consider $\tilde{\mathscr{A}}$. By the first part,

  $$\sigma_{\mathscr{A}}(a) = \sigma_{\tilde{\mathscr{A}}}(a) \supseteq \{\chi(a) \mid \chi \in \Omega(\tilde{\mathscr{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\}$$

  If $\mathscr{A}$ is commutative, by the first part again:

  $$\sigma_{\mathscr{A}}(a) = \sigma_{\tilde{\mathscr{A}}}(a) = \{\chi(a) \mid \chi \in \Omega(\tilde{\mathscr{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathscr{A})\} \cup \{0\}$$ □