

Exercises to Introduction to Operator Algebras

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1 Topological Basics

Let X be a topological space, that is there exists a subset $\mathcal{O}(X) \in \mathbb{P}(X)$.

Definition 1.1 X is **Hausdorff** if for all $x, y \in X$ there exist open sets $U, V \in \mathcal{O}(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

X is **locally Hausdorff** if for all $x \in X$ there exists an open neighborhood $U \in \mathcal{O}(X)$ such that U is Hausdorff with the relative topology from X .

Example 1.2 (Snake with two heads) We consider the space $[0, 1] \cup \{1^+\}$ equipped with a topology such that both the subspace $[0, 1]$ and $[0, 1] \setminus \{1\} \cup \{1^+\}$ are isomorphic to $[0, 1]$. Then X is compact, locally Hausdorff but not Hausdorff.

Definition 1.3 X is compact if for every open cover $(U_i)_{i \in I}$ there exists a finite open subcover.

X is locally compact if for every $x \in X$ there exists a neighborhood basis of x consisting of open relatively compact subsets of X , that is for every open neighborhood U of x there exists an open neighborhood V of x such that \bar{V} is compact and $\bar{V} \subset U$.

Observation: For a locally Hausdorff X , X is locally compact if and only if for all $x \in X$ there exists an open neighborhood U of x such that \bar{U} is compact.

1.1 Results about locally compact Hausdorff spaces

Let X be Hausdorff and locally compact.

Proposition 1.4 (Uryson's Lemma) For all closed $F \subset X$ and all compact $K \subseteq X$ with $F \cap K = \emptyset$, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f|_K \equiv 1$ and $f|_F \equiv 0$.

Proposition 1.5 (Tietze's extension theorem) For all $K \subseteq X$ compact and $f : K \rightarrow \mathbb{C}$ continuous, there exists an $\tilde{f} : X \rightarrow \mathbb{C}$ continuous such that $\tilde{f}|_K \equiv f$.

Proposition 1.6 (Alexandroff's compactification theorem) $\tilde{X} = X \cup \{\infty\}$ ($\infty \notin K$) is a compact Hausdorff space with $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{K^c \cup \{\infty\} \mid K \subseteq X \text{ compact}\}$.

For example, compactifying \mathbb{R} yields the unit circle \mathbb{S}^1 .

Proposition 1.7 Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$ the space $X = Y \setminus \{y_0\}$ is a locally compact Hausdorff space.

Proposition 1.8 More generally, if Y is a locally compact Hausdorff space and $Z \subseteq Y$ is a difference of open and closed subsets of Y (i.e. $Z = U \setminus F$ or $Z = F \setminus U$ where $U \subseteq Y$ is open and $F \subseteq Y$ is closed) then Z is locally compact.

Exercise 1.1 Let X be a locally compact Hausdorff space. The following are equivalent:

- (1) X is compact.
- (2) $C(X) = C_0(X)$ ($= C_b(X)$).
- (3) $C_0(X)$ is unital.
- (4) $1 \in C_0(X)$ where $1(x) = 1 \in \mathbb{C}$ for all $x \in X$.

PROOF:

- (1) \Rightarrow (2): Recall:

$$C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$$

If X is compact, then every closed subset of X is compact, so all sets of form $\{x \in X \mid |f(x)| \geq \varepsilon\}$ are compact, and we have $C(X) = C_0(X)$.

- (2) \Rightarrow (3): This is trivial because $C(X)$ is always unital.
- (3) \Rightarrow (4): Suppose $C_0(X)$ is unital and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, that is $f(x)g(x) = 1$ for all $x \in X, g \in C_0(X)$. By Uryson's Lemma, given $x_0 \in X$, there exists a $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$, take any precompact open neighborhood U of x and look at $F := U^c \subseteq X$). Then we have $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. As this is possible for every $x_0 \in X$, we have $f \equiv 1$.
- (4) \Rightarrow (1): Suppose $f = 1 \in C_0(X)$. Then choosing $\varepsilon = \frac{1}{2}$ shows that $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$ is compact. \square

Exercise 1.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$.

2 Exercise sheet 1

Exercise 2.1 (1.1)

PROOF: Case 1: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1 \alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathcal{A} \rightarrow \mathcal{M}, \lambda a \mapsto$

$\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda} a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathcal{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to \mathbb{C} . \square

Exercise 2.2 (1.2) We consider pathological examples for $C_0(X)$.

Let $X = \{x_0\}$, e.g. $x_0 \in X$ with $\mathcal{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f : X \rightarrow \mathbb{C}$, this implies $f(x_0) = f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \neq 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 2.1 Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a $*$ -homomorphism between C^* -algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.

Exercise 2.3 (1.4 - Products) Let $(A_i)_{i \in I}$ be a family of C^* -algebras and define

$$\prod_{i \in I} A_i = \{a = (a_i)_{i \in I} \mid a_i \in A_i \forall i \in I \text{ and } \|a\| := \sup_{i \in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^* -axiom.

Remark 2.2 (Differences between product and direct sum)

In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall \varepsilon > 0 \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of $\prod_{i \in I} A_i$ as the closure of $\bigoplus_{i \in I}^{alg} A_i$, where

$$\bigoplus_{i \in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| = 0 \right\}.$$

For finite I , these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) $*$ -homomorphisms $\pi_j : \prod_{i \in I} A_i \rightarrow A_j$ for all $j \in I$. If B is any C^* -algebra with $*$ -homomorphisms $\varphi_j : B \rightarrow A_j$ for every $j \in I$, there is a unique $*$ -homomorphism $\varphi : B \rightarrow \prod_{i \in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi_j} & A_j \\ \downarrow \varphi & \nearrow \pi_j & \\ A & & \end{array}$$

Exercise 2.4 (1.5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a $*$ -isomorphism.

Exercise 2.5 (2.6) Let X be a locally compact Hausdorff space and $\widetilde{C_0(X)} \simeq C(\tilde{X})$ with $\tilde{X} := X \sqcup \{\infty\}$ with the topology $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{\tilde{X} \setminus K \mid K \subseteq X \text{ kompakt}\}$.

Observation: If X is already compact, then ∞ is an isolated point of \tilde{X} (i.e. $\{\infty\}$ is clopen).

If \mathcal{A} is a C^* -algebra, then $\tilde{\mathcal{A}}$ (this is not the same $\tilde{}$ as on the X !) is a C^* -algebra with

$$\|a + \lambda 1\|_{C^*} := \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|ab + \lambda b\|_{\mathcal{A}}$$

We check that $\tilde{\mathcal{A}}$ is a C^* -algebra.

- C^* -axiom: $\|a + \lambda 1\|_{C^*}^2 = \|(a + \lambda 1)^*(a + \lambda 1)\|_{C^*}$. We have

$$\|a + \lambda \cdot 1\|_{C^*} = \|(a^*a + \bar{\lambda}a + \lambda a^*) + |\lambda|^2 \cdot 1\|_{C^*}$$

$$= \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 \cdot b\|_{\mathcal{A}}$$

On the other hand:

$$\begin{aligned} \|a + \lambda \cdot 1\|_{C^*}^2 &:= \sup_{\|b\| \leq 1} \|ab + \lambda b\|_{\mathcal{A}}^2 \\ &= \sup_{\|b\| \leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\|_{\mathcal{A}} \\ &= \sup_{\|b\| \leq 1} \|b^*a^*ab + \bar{\lambda}b^*ab + \lambda b^*a^*b + |\lambda|^2 b^*b\|_{\mathcal{A}} \\ &\leq \sup_{\|b\| \leq 1} \|b^*\|_{\mathcal{A}} \cdot \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathcal{A}} \\ &\leq \sup_{\|b\| \leq 1} \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathcal{A}} \\ &= \|a + \lambda \cdot 1\|_{C^*}^2 \end{aligned}$$

- The other conditions are easy to check and are left for the student.

We still want to prove $\varphi : \widetilde{C_0(X)} \rightarrow C(\tilde{X}), f + \lambda \cdot 1 \mapsto f_\lambda$ with $f_\lambda(x) := \lambda$ for $x = \infty$ and $f_\lambda(x) = f(x) + \lambda$ otherwise. Note that once again these are not the same $\tilde{}$.

- f is well-defined: We have to check that f_λ is continuous in \tilde{X} . Take any sequence $X \ni x_i \rightarrow \infty$ in \tilde{X} . We have to show $f_\lambda(x_i) \rightarrow f_\lambda(\infty) = \lambda$. Since $f_\lambda(x_i) = f(x_i) + \lambda$ this is equivalent to $f(x_i) \rightarrow 0$. But as $f \in C_0(X)$, we have that for every $\varepsilon > 0$ the set $K_\varepsilon(f) = \{x \mid |f(x)| \geq \varepsilon\}$ is compact. Since x_i will eventually leave this compact set (or it would not diverge to ∞), we know that $f(x_i)$ eventually becomes smaller than (any) ε . So we have $f(x_i) \rightarrow 0$ and thus $f_\lambda(x_i) \rightarrow f_\lambda(\infty)$. So f_λ is continuous in ∞ . The continuity on every other point follows immediately from the continuity of f .
- φ is a *-isomorphism:

- Linearity: φ is clearly linear as we can check component-wise:

$$(f_1 + f_2)_\lambda = (f_1)_\lambda + (f_2)_\lambda$$

- Homomorphism: For every $x \in X$ we have

$$\begin{aligned} \varphi((f + \lambda \cdot 1) \cdot (g + \lambda' \cdot 1))(x) &= \varphi((fg + \lambda'f + \lambda g) + \lambda\lambda' \cdot 1)(x) \\ &= (fg + \lambda'f + \lambda g)(x) + \lambda\lambda' \\ &= (fg)(x) + \lambda'f(x) + \lambda g(x) + \lambda\lambda' \\ &= (f(x) + \lambda) \cdot (g(x) + \lambda') \\ &= (\varphi(f + \lambda) \cdot \varphi(g + \lambda'))(x). \end{aligned}$$

In the case of $x = \infty$, this equality of course also holds. Thus we have $\varphi((f + \lambda)(g + \lambda')) = \varphi(f + \lambda)\varphi(g + \lambda')$.

- *-homomorphism:

$$\varphi(f + \lambda)^*(x) = \varphi(f^* + \bar{\lambda} \cdot 1)(x)$$

For $x \in X$ this follows by $\overline{f(x)} + \bar{\lambda} = f^*(x) + \bar{\lambda}$, for $x = \infty$ we have $\bar{\lambda} = \bar{\lambda}$.

- Injective: $f_\lambda(0)$ leads to $f_\lambda(x) = 0$ for all $x \in \tilde{X}$, since if $x = \infty$ then λ must be 0 and $f(x) = 0$ for all $x \in X$. Thus $f = 0$ and $\lambda = 0$.
- Surjective: Take $g \in C(\tilde{X})$ and choose $\lambda = g(\infty)$ and $f(x) := g(x) - \lambda$. and check $f \in C_0(X)$.

- We can also prove that φ is isometric for the C^* -norm:

$$\|f + \lambda \cdot 1\| := \sup_{g \in C_0(X), \|g\| \leq 1} \|fg + \lambda g\|_\infty$$

Look at

$$\begin{aligned} \|\varphi(f + \lambda 1)\| &= \sup_{x \in \tilde{X}} |f_\lambda(x)| = \max\{|\lambda|, \sup_{x \in X} |f(x) + \lambda|\} \\ &\stackrel{(*)}{=} \sup_{x \in X} |f(x) + \lambda| \end{aligned}$$

and

$$\begin{aligned} \|f + \lambda \cdot 1\|_{C^*} &= \sup_{\|g\| \leq 1} \|fg + \lambda g\|_{C_0(X)} \\ &= \sup_{|g(x)| \leq 1 \forall x} \sup_{x \in X} \|f(x)g(x) + \lambda g(x)\| \\ &= \sup_{x \in X} |f(x) + \lambda| \\ &\stackrel{(**)}{=} \sup_{x \in X} |f(x) + \lambda| \end{aligned}$$

This proof may need to be divided into two cases:

- X is not compact: We can find a net $(x_i) \subseteq X$ with $f(x_i) \rightarrow 0$ and $(*)$ follows and use a $g(x) \approx 1$ for $(**)$.
- X is compact: Choose $g \equiv 1$ for $(**)$ and think about $(*)$ later.

Exercise 2.6 (1.8) It is difficult to prove $I^* = I$. The idea is to prove $I = C_0(U)$ where $C_0(U) = \{f \in C_0(X) \mid f|_U \equiv 0\}$.

One can also prove $C_0(X)/C_0(U) \simeq C_0(F)$ (as C_0 of the subspace) where $F = U^c$.

Exercise 2.7 Prove that \mathcal{A}/\mathfrak{I} is normed algebra, and

- (i) if \mathcal{A} is Banach and $\mathfrak{I} \leq \mathcal{A}$ is closed, then \mathcal{A}/\mathfrak{I} is Banach.
- (ii) if \mathcal{A} is unital and Banach, then \mathcal{A}/\mathfrak{I} is unital.

unital if \mathcal{A} is, Banach if \mathcal{A} is and $\mathfrak{I} \leq \mathcal{A}$ closed.

PROOF: Consider \mathcal{A}/\mathfrak{I} with $(a + \mathfrak{I})(b + \mathfrak{I}) = ab + \mathfrak{I}$. For the norm, use $\|a + \mathfrak{I}\| = \text{dist}(a, \mathfrak{I}) = \inf_{x \in \mathfrak{I}} \|a - x\|$. This is submultiplicative. For every $\varepsilon > 0$, there exist $x, y \in \mathfrak{I}$ for which we have

$$(\varepsilon + \|a + \mathfrak{I}\|) \cdot (\varepsilon + \|b + \mathfrak{I}\|) \geq \|a + x\| \cdot \|b + y\| \geq \|(a + x)(b + y)\| \geq \|ab + \underbrace{ay + xb + xy}_{\in \mathfrak{I}}\| \geq \|ab + \mathfrak{I}\|$$

and taking the limit yields the desired result.

Result (i) follows from functional analysis, that a space is Banach if and only if the convergence of $\sum_{k=0}^{\infty} a_n$ is equivalent to the convergence of $\sum_{k=0}^{\infty} \|a_n\|$.

Now let \mathcal{A} also be unital, then \mathcal{A}/\mathfrak{I} is unital. If $\mathfrak{I} = \mathcal{A}$, the algebra is the zero-algebra. Thus, let \mathfrak{I} be a proper ideal. The fact that $1 = 1_{\mathcal{A}} + \mathfrak{I}$ is a unit is clear, but we need to prove $\|1_{\mathcal{A}} + \mathfrak{I}\| = 1$. Observe that, if $x \in \mathfrak{I} \triangleleft \mathcal{A}$ then $x \notin \text{inv}(\mathcal{A})$ and $\|1_{\mathcal{A}} + x\| \geq 1$. Because otherwise, we have $\|1_{\mathcal{A}} + x\| < 1$ and then (because \mathcal{A} is Banach) $x = a - 1_{\mathcal{A}} \in \text{inv}(\mathcal{A})$. Hence $\|1_{\mathcal{A}}\| = \inf_{x \in \mathfrak{I}} \|1_{\mathcal{A}} + x\| \geq 1$. In addition, we have $1 \leq \|1_{\mathcal{A}} + \mathfrak{I}\| = \inf_{x \in \mathfrak{I}} \|1_{\mathcal{A}} - x\| \leq \|1_{\mathcal{A}} + 0\| \leq 1$. This proves $\|1\| = \|1_{\mathcal{A}} + \mathfrak{I}\| = 1$. \square