Exercises to Introduction to Operator Algebras

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Summer 2023

Contents

1 Topological Basics

Let X be a topological space, that is there exists a subset $\mathscr{O}(X) \in \mathbb{P}(X)$.

Definition 1.1 *X is* ***Hausdorff*** *if for all $x, y \in X$ there exist open sets $U, V \in \mathscr{O}(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.*

X is ***locally Hausdorff*** *if for all $x \in X$ there exists an open neighborhood $U \in \mathscr{O}(X)$ such that U is Hausdorff with the relative topology from X.*

Example 1.2 (Snake with two heads) We consider the space $[0,1] \cup \{1^+\}$ equipped with a topology such that both the subspace $[0,1]$ and $[0,1] \setminus \{1\} \cup \{1^+\}$ are isomorphic to $[0,1]$. Then X is compact, locally Hausdorff but not Hausdorff.

Definition 1.3 *X is compact if for every open cover $(U_i)_{i \in I}$ there exists a finite open subcover.*

X is locally compact if for every $x \in X$ there exists a neighborhood basis of x consisting of open relatively compact subsets of X, that is for every open neighborhood U of x there exists and open neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Observation: For a locally Hausdorff X, X is locally compact if and only if for all $x \in X$ there exists an open neighborhood U of x such that $\overline{U}$ is compact.

1.1 Results about locally compact Hausdorff spaces

Let X be Hausdorff and locally compact.

Proposition 1.4 (Uryson's Lemma) *For all closed $F \subset X$ and all compact $K \subseteq X$ with $F \cap K = \emptyset$, there is a continuous function $f : X \to [0,1]$ such that $f|_K \equiv 1$ and $f|_F \equiv 0$.*

Proposition 1.5 (Tietze's extension theorem) *For all $K \subseteq X$ compact and $f : K \to \mathbb{C}$ continuous, there exists and $\tilde{f} : X \to \mathbb{C}$ continuous such that $\tilde{f}|_K \equiv f$.*

Proposition 1.6 (Alexandroff's compactification theorem) *$\tilde{X} = X \cup \{\infty\}$ $(\infty \notin K)$ is a compact Hausdorff space with $\mathscr{O}_{\tilde{X}} = \mathscr{O}_X \cup \{K^{\complement} \cup \{\infty\} \mid K \subseteq X \, compact\}$.*

For example, compactifying $\mathbb{R}$ yields the unit circle $\mathbb{S}^1$.

Proposition 1.7 *Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$ the space $X = Y \setminus \{y_0\}$ is a locally compact Hausdorff space.*

Proposition 1.8 *More generally, if Y is a locally compact Hausdorff space and $Z \subseteq Y$ is a difference of open and closed subsets of Y (i.e. $Z = U \setminus F$ or $Z = F \setminus U$ where $U \subseteq Y$ is open and $F \subseteq Y$ is closed) then Z is locally compact.*

Exercise 1.1 Let X be a locally compact Hausdorff space. The following are equivalent:

(1) X is compact.

(2) $C(X) = C_0(X)$ $(= C_b(X))$.

(3) $C_0(X)$ is unital.

(4) $1 \in C_0(X)$ where $1(x) = 1 \in \mathbb{C}$ for all $x \in X$.

PROOF:

- (1) ⇒ (2): Recall:

$$C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$$

  If X is compact, then every closed subset of X is compact, so all sets of form $\{x \in X \mid |f(x)| \geq \varepsilon\}$ are compact, and we have $C(X) = C_0(X)$.

- (2) ⇒ (3): This is trivial because $C(X)$ is always unital.

- (3) ⇒ (4): Suppose $C_0(X)$ is unital and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, that is $f(x)g(x) = 1$ for all $x \in X, g \in C_0(X)$. By Uryson's Lemma, given $x_0 \in X$, there exists a $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$, take any precompact open neighborhood U of x and look at $F := U^{\complement} \subseteq X$). Then we have $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. As this is possible for every $x_0 \in X$, we have $f \equiv 1$.

- (4) ⇒ (1): Suppose $f = 1 \in C_0(X)$. Then choosing $\varepsilon = \frac{1}{2}$ shows that $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$ is compact. □

Exercise 1.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$.

2 Exercise sheet 1

Exercise 2.1 (1)

Proof: **Case 1**: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1\alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathscr{A} \to \mathscr{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda}a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathscr{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to $\mathbb{C}$. □

Exercise 2.2 (2) We consider pathological examples for $C_0(X)$.

Let $X = \overline{\{x_0\}}$, e.g. $x_0 \in X$ with $\mathscr{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f : X \to \mathbb{C}$, this implies $f(x_0) \to f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.$\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \not\equiv 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 2.1 *Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a *-homomorphism between C^*-algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathscr{A}$.*

Exercise 2.3 (4 - Products) Let $(A_i)_{i\in I}$ be a family of C^*-algebras and define

$$\prod_{i\in I} A_i = \{a = (a_i)_{i\in I} \mid a_i \in A_i \forall_{i\in I} \text{ and } \|a\| := \sup_{i\in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^*-axiom.

Remark 2.2 (Differences between product and direct sum)
In addition to the product space, we define

$$\bigoplus_{i\in I} A_i = \left\{(a_i) \in \prod_{i\in I} A_i \mid \forall_{\varepsilon>0}\exists_{\text{finite } F\subseteq I}\forall_{i\notin F}\|a_i\| < \varepsilon\right\}.$$

This is a closed subspace of $\prod_{i\in I} A_i$ as the closure of $\bigoplus_{i\in I}^{alg} A_i$, where

$$\bigoplus_{i\in I}^{alg} A_i = \left\{(a_i) \in \prod_{i\in I} A_i \mid \exists_{\text{finite } F\subseteq I}\forall_{i\notin F}\|a_i\| = 0\right\}.$$

For finite I, these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) *-homomorphisms $\pi_j : \prod_{i\in I} A_i \to A_j$ for all $j \in I$. If B is any C^*-algebra with *-homomorphisms $\varphi_j \to A_j$ for every $j \in I$, there is a unique *-homomorphism $\varphi : B \to \prod_{i\in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} B \arrow[r, "\varphi_j"] \arrow[d, "\varphi"] & A_j \\ A \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Exercise 2.4 (5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a *-isomorphism.