Exercises to Introduction to Operator Algebras

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1 Topological Basics

Let X be a topological space, that is there exists a subset $\mathcal{O}(X) \in \mathbb{P}(X)$.

Definition 1.1 X is **Hausdorff** if for all $x, y \in X$ there exist open sets $U, V \in \mathcal{O}(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

X is **locally Hausdorff** if for all $x \in X$ there exists an open neighborhood $U \in \mathcal{O}(X)$ such that U is Hausdorff with the relative topology from X.

Example 1.2 (Snake with two heads) We consider the space $[0,1] \cup \{1^+\}$ equipped with a topology such that both the subspace [0,1] and $[0,1] \setminus \{1\} \cup \{1^+\}$ are isomorphic to [0,1]. Then X is compact, locally Hausdorff but not Hausdorff.

Definition 1.3 X is compact if for every open cover $(U_i)_{i\in I}$ there exists a finite open subcover. X is locally compact if for every $x\in X$ there exists a neighborhood basis of x consisting of open relatively compact subsets of X, that is for every open neighborhood U of x there exists and open neighborhood V of x such that \overline{V} is compact and $\overline{V}\subset U$.

Observation: For a locally Hausdorff X, X is locally compact if and only if for all $x \in X$ there exists an open neighborhood U of x such that \overline{U} is compact.

1.1 Results about locally compact Hausdorff spaces

Let X be Hausdorff and locally compact.

Proposition 1.4 (Uryson's Lemma) For all closed $F \subset X$ and all compact $K \subseteq X$ with $F \cap K = \emptyset$, there is a continuous function $f: X \to [0,1]$ such that $f|_K \equiv 1$ and $f|_F \equiv 0$.

Proposition 1.5 (Tietze's extension theorem) For all $K \subseteq X$ compact and $f : K \to \mathbb{C}$ continuous, there exists and $\tilde{f} : X \to \mathbb{C}$ continuous such that $\tilde{f}|_K \equiv f$.

Proposition 1.6 (Alexandroff's compactification theorem) $\tilde{X} = X \cup \{\infty\}$ $(\infty \notin K)$ is a compact Hausdorff space with $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{K^{\complement} \cup \{\infty\} \mid K \subseteq X compact\}.$ For example, compactifying \mathbb{R} yields the unit circle \mathbb{S}^1 .

Proposition 1.7 Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$ the space $X = Y \setminus \{y_0\}$ is a locally compact Hausdorff space.

Proposition 1.8 More generally, if Y is a locally compact Hausdorff space and $Z \subseteq Y$ is a difference of open and closed subsets of Y (i.e. $Z = U \setminus F$ or $Z = F \setminus U$ where $U \subseteq Y$ is open and $F \subseteq Y$ is closed) then Z is locally compact.

Exercise 1.1 Let X be a locally compact Hausdorff space. The following are equivalent:

- (1) X is compact.
- (2) $C(X) = C_0(X) (= C_b(X)).$
- (3) $C_0(X)$ is unital.
- (4) $1 \in C_0(X)$ where $1(x) = 1 \in \mathbb{C}$ for all $x \in X$.

Proof:

• $(1) \Rightarrow (2)$: Recall:

$$C_0(X) = \{ f \in C(X) \mid \forall_{\varepsilon > 0} \{ x \in X \mid |f(x)| \ge \varepsilon \} \text{ is compact} \}$$

If X is compact, then every closed subset of X is compact, so all sets of form $\{x \in X \mid |f(x)| \geq \varepsilon\}$ are compact, and we have $C(X) = C_0(X)$.

- (2) \Rightarrow (3): This is trivial because C(X) is always unital.
- (3) \Rightarrow (4): Suppose $C_0(X)$ is unital and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, that is f(x)g(x) = 1 for all $x \in X, g \in C_0(X)$. By Uryson's Lemma, given $x_0 \in X$, there exists a $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$, take any precompact open neighborhood U of x and look at $F := U^{\complement} \subseteq X$). Then we have $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. As this is possible for every $x_0 \in X$, we have $f \equiv 1$.
- (4) \Rightarrow (1): Suppose $f = 1 \in C_0(X)$. Then choosing $\varepsilon = \frac{1}{2}$ shows that $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$ is compact.

Exercise 1.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$.

2 Exercise sheet 1

Exercise 2.1 (1.1)

PROOF: Case 1: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1 \alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathcal{A} \to \mathcal{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda}a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathcal{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to \mathbb{C} .

Exercise 2.2 (1.2) We consider pathological examples for $C_0(X)$.

Let $X = \overline{\{x_0\}}$, e.g. $x_0 \in X$ with $\mathcal{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f: X \to \mathbb{C}$, this implies $f(x_0) \to f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact.} \}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \not\equiv 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 2.1 Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a *-homomorphism between C^* -algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.

Exercise 2.3 (1.4 - Products) Let $(A_i)_{i\in I}$ be a family of C^* -algebras and define

$$\prod_{i \in I} A_i = \{ a = (a_i)_{i \in I} \mid a_i \in A_i \forall_{i \in I} \text{ and } ||a|| := \sup_{i \in I} ||a_i|| < \infty \}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^* -axiom.

Remark 2.2 (Differences between product and direct sum)

In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall_{\varepsilon > 0} \exists_{\text{finite } F \subseteq I} \forall_{i \notin F} ||a_i|| < \varepsilon \right\}.$$

This is a closed subspace of $\prod_{i \in I} A_i$ as the closure of $\bigoplus_{i \in I}^{alg} A_i$, where

$$\bigoplus_{i \in I}^{alg} A_i = \bigg\{ (a_i) \in \prod_{i \in I} A_i \mid \exists_{\text{finite } F \subseteq I} \forall_{i \notin F} \|a_i\| = 0 \bigg\}.$$

For finite I, these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) *-homomorphisms $\pi_j: \prod_{i\in I} A_i \to A_j$ for all $j\in I$. If B is any C^* -algebra with *-homomorphisms $\varphi_j \to A_j$ for every $j\in I$, there is a unique *-homomorphism $\varphi: B\to \prod_{i\in I} A_i$ such that $\pi_j\circ\varphi=\varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{array}{c}
B \xrightarrow{\varphi_j} A_j \\
\downarrow^{\varphi} \xrightarrow{\pi_j} \\
A
\end{array}$$

Exercise 2.4 (1.5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a *-isomorphism.

Exercise 2.5 (2.6) Let X be a locally compact Hausdorff space and $\widetilde{C_0(X)} \simeq C(\tilde{X})$ with $\tilde{X} := X \sqcup \{\infty\}$ with the topology $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{\tilde{X} \setminus K \mid K \subseteq X \text{ kompakt}\}.$

Observation: If X is already compact, then ∞ is an isolated point of \tilde{X} (i.e. $\{\infty\}$ is clopen). If \mathscr{A} is a C^* -algebra, then $\tilde{\mathscr{A}}$ (this is not the same $\tilde{}$ as on the X!) is a C^* -algebra with

$$||a + \lambda 1||_{C^*} := \sup_{b \in \mathcal{A}, ||b|| \le 1} ||ab + \lambda b||_{\mathcal{A}}$$

We check that $\tilde{\mathcal{A}}$ is a C^* -algebra.

• C^* -axiom: $||a + \lambda 1||_{C^*}^2! = ||(a + \lambda 1)^*(a + \lambda 1)||_{C^*}$. We have

$$||a + \lambda \cdot 1||_{C^*} = ||(a^*a + \overline{\lambda}a + \lambda a^*) + |\lambda|^2 \cdot 1||_{C^*}$$

$$= \sup_{b \in \mathcal{A}, ||b|| \le 1} ||a^*ab + \overline{\lambda}ab + \lambda a^*b + |\lambda|^2 \cdot b||_{\mathcal{A}}$$

On the other hand:

$$\begin{split} \|a + \lambda \cdot 1\|_{C^*}^2 &\coloneqq \sup_{\|b\| \le 1} \|ab + \lambda b\|_{\mathscr{A}}^2 \\ &= \sup_{\|b\| \le 1} \|(ab + \lambda b)^* (ab + \lambda b)\|_{\mathscr{A}} \\ &= \sup_{\|b\| \le 1} \|b^* a^* a b + \overline{\lambda} b^* a b + \lambda b^* a^* b + |\lambda|^2 b^* b\|_{\mathscr{A}} \\ &\le \sup_{\|b\| \le 1} \|b^*\|_{\mathscr{A}} \cdot \|a^* a b + \overline{\lambda} a b + \lambda a^* b + |\lambda|^2 b\|_{\mathscr{A}} \\ &\le \sup_{\|b\| \le 1} \|a^* a b + \overline{\lambda} a b + \lambda a^* b + |\lambda|^2 b\|_{\mathscr{A}} \\ &= \|a + \lambda \cdot 1\|_{C^*}^2 \end{split}$$

• The other conditions are easy to check and are left for the student.

We still want to prove $\varphi: C_0(X) \to C(\tilde{X}), f + \lambda \cdot 1 \mapsto f_{\lambda}$ with $f_{\lambda}(x) := \lambda$ for $x = \infty$ and $f_{\lambda}(x) = f(x) + \lambda$ otherwise. Nother that once again these are not the same \tilde{x} .

- f is well-defined: We have to check that f_{λ} is continuous in \tilde{X} . Take any sequence $X\ni x_i\to\infty$ in \tilde{X} . We have to show $f_{\lambda}(x_i)\to f_{\lambda}(\infty)=\lambda$. Since $f_{\lambda}(x_i)=f(x_i)+\lambda$ this is equivalent to $f(x_i)\to 0$. But as $f\in C_0(X)$, we have that for every $\varepsilon>0$ the set $K_{\varepsilon}(f)=\{x\mid |f(x)|\geq \varepsilon\}$ is compact. Since x_i will eventually leave this compact set (or it would not diverge to ∞), we know that $f(x_i)$ eventually becomes smaller than (any) ε . So we have $f(x_i)\to 0$ and thus $f_{\lambda}(x_i)\to f_{\lambda}(\infty)$. So f_{λ} is continuous in ∞ . The continuity on every other point follows immediately from the continuity of f.
- φ is a *-isomorphism:
 - Linearity: φ is clearly linear as we can check component-wise:

$$(f_1 + f_2)_{\lambda} = (f_1)_{\lambda} + (f_2)_{\lambda}$$

– Homomorphism: For every $x \in X$ we have

$$\begin{split} \varphi((f+\lambda\cdot 1)\cdot (g+\lambda'\cdot 1))(x) &= \varphi((fg+\lambda'f+\lambda g)+\lambda\lambda'\cdot 1)(x) \\ &= (fg+\lambda'f+\lambda g)(x)+\lambda\lambda' \\ &= (fg)(x)+\lambda'f(x)+\lambda g(x)+\lambda\lambda' \\ &= (f(x)+\lambda)\cdot (g(x)+\lambda') \\ &= (\varphi(f+\lambda)\cdot \varphi(g+\lambda'))(x). \end{split}$$

In the case of $x = \infty$, this equality of course also holds. Thus we have $\varphi((f + \lambda)(g + \lambda')) = \varphi(f + \lambda)\varphi(g + \lambda')$.

- *-homomorphism:

$$\varphi(f+\lambda)^*(x) = \varphi(f^* + \overline{\lambda} \cdot 1)(x)$$

For $x \in X$ this follows by $\overline{f(x)} + \overline{\lambda} = f^*(x) + \overline{\lambda}$, for $x = \infty$ we have $\overline{\lambda} = \overline{\lambda}$.

- Injective: $f_{\lambda}(0)$ leads to $f_{\lambda}(x) = 0$ for all $x \in \tilde{X}$, since if $x = \infty$ then λ must be 0 and f(x) = 0 for all $x \in X$. Thus f = 0 and $\lambda = 0$.
- Surjective: Take $g \in C(\tilde{X})$ and choose $\lambda = g(\infty)$ and $f(x) := g(x) \lambda$. and check $f \in C_0(X)$.
- We can also prove that φ is isometric for the C^* -norm:

$$||f + \lambda \cdot 1|| := \sup_{g \in C_0(X), ||g|| \le 1} ||fg + \lambda g||_{\infty}$$

Look at

$$\begin{split} \|\varphi(f+\lambda 1)\| &= \sup_{x \in \tilde{X}} |f_{\lambda}(x)| = \max\{|\lambda|, \sup_{x \in X} |f(x)+\lambda|\} \\ &\stackrel{(*)}{=} \sup_{x \in X} |f(x)+\lambda| \end{split}$$

and

$$\begin{split} \|f + \lambda \cdot 1\|_{C^*} &= \sup_{\|g\| \le 1} \|fg + \lambda g\|_{C_0(X)} \\ &= \sup_{\|g(x)| \le 1 \forall_x} \sup_{x \in X} \|f(x)g(x) + \lambda g(x)\| \\ &= \sup_{x \in X} |f(x) + \lambda| \\ &\stackrel{(**)}{=} \sup_{x \in X} |f(x) + \lambda| \end{split}$$

This proof may need to be divided into two cases:

- X is not compact: We can find a net $(x_i) \subseteq X$ with $f(x_i) \to 0$ and (*) follows and use a $g(x) \approx 1$ for (**).
- X is compact: Choose $g \equiv 1$ for (**) and think about (*) later.

Exercise 2.6 (1.8) It is difficult to prove $I^* = I$. The idea is to prove $I = C_0(U)$ where $C_0(U) = \{ f \in C_0(X) \mid f|_{U^{\complement}} \equiv 0 \}$.

One can also prove $C_0(X)/C_0(U) \simeq C_0(F)$ (as C_0 of the subspace) where $F = U^{\complement}$.

Exercise 2.7 Prove that \mathcal{A}/I is normed algebra, and

- (i) if \mathcal{A} is Banach and $I \subseteq A$ is closed, then \mathcal{A}/I is Banach.
- (ii) if \mathscr{A} is unital and Banach, then \mathscr{A}/I is unital.

unital if \mathscr{A} is, Banach if \mathscr{A} is and $I \subseteq \mathscr{A}$ closed.

PROOF: Consider \mathscr{A}/I with $(a+\mathsf{I})(b+\mathsf{I})=ab+\mathsf{I}$. For the norm, use $||a+\mathsf{I}||=\mathrm{dist}(a,\mathsf{I})=\inf_{x\in\mathsf{I}}||a-x||$. This is submultiplicative. For every $\varepsilon>0$, there exist $x,y\in\mathsf{I}$ for which we have

$$(\varepsilon + \|a + \mathbf{I}\|) \cdot (\varepsilon + \|b + \mathbf{I}\|) \geq \|a + x\| \cdot \|b + y\| \geq \|(a + x)(b + y)\| \geq \|ab + \underbrace{ay + xb + xy}_{\in \mathbf{I}}\| \geq \|ab + \mathbf{I}\|$$

and taking the limit yields the desired result.

Result (i) follows from functional analysis, that a space is Banach if and only if the convergence of $\sum_{k=0}^{\infty} a_n$ is equivalent to the convergence of $\sum_{k=0}^{\infty} ||a_n||$.

Now let \mathcal{A} also be unital, then \mathcal{A}/I is unital. If $I = \mathcal{A}$, the algebra is the zero-algebra. Thus, let I be a proper ideal. The fact that $1 = 1_{\mathscr{A}} + I$ is a unit is clear, but we need to prove $||1_A + I|| = 1$. Observe that, if $x \in I \triangleleft A$ then $x \notin \text{inv}(A)$ and $||1_A + x|| \ge 1$. Because otherwise, we have $||1_{\mathscr{A}} + x|| < 1$ and then (because \mathscr{A} is Banach) $x = a - 1_{\mathscr{A}} \in \operatorname{inv}(\mathscr{A})$. Hence $||1_{\mathscr{A}}|| = \inf_{x \in I} ||1_{\mathscr{A}} + x|| \ge 1$. In addition, we have $1 \le ||1_{\mathscr{A}} + I|| = \inf_{x \in I} ||1_{\mathscr{A}} - x|| \le ||1_{\mathscr{A}} + 0|| \le 1$. This proves $||1|| = ||1_{\mathcal{A}} + I|| = 1$.

In the following, \mathbb{D} is the **closed** unit circle.

Exercise 2.8 Consider $\chi \in \Omega(\mathcal{A})$. We have proved $\|\chi\| \leq 1$. It may happen that $\|\chi\| < 1$. We need a non-unital algebra for this, because we have $\|\chi\| = 1$ if $1 \in \mathcal{A}$.

Consider $S = (\mathbb{N}, +)$ as an additive semigroup. Then

$$\ell^1(S) = \{(a_n)_{n \in \mathbb{N}} \mid \sum_{n=0}^{\infty} |a_n| < \infty\}$$

is a unital Banach algebrea with $\delta_n \cdot \delta_m = \delta_{n+m}$ for all $n, m \in \mathbb{N}$ where

$$\delta_n(m) = \begin{cases} 1 & m = n \\ 0 & m \neq 0 \end{cases}$$

Observe $\ell^1(S) = \overline{\operatorname{alg}}\{\delta_0, \delta_1\}$ because of $\delta_1^n = \delta_n$. The unit of the algebra is δ_0 . What are the characters of $\ell^1(S)$?

We can write any $a \in \ell^1(S)$ as $a = \sum_{n=0}^{\infty} a_n \delta_n$. So if $\chi \in \Omega(\ell^1(S))$ then

$$\chi(a) = \sum_{n=0}^{\infty} a_n \chi(\delta_n) \in \mathbb{C}.$$

In particular, $\chi(1) = 1$ so $\chi(\delta_0) = 1$. This leads to $\chi(\delta_n) = \chi(\delta_1^n) = \chi(\delta_1)^n = \chi(\delta_1)^n$. So if we set $z := \chi(\delta_1) \in \mathbb{C}$, we have $\chi(a) = \sum_{n=0}^{\infty} a_n z^n$. Observe $|z| = |\chi(\delta_1)| \le ||\delta_1|| = 1$ (because the Image of a character is a subset of the spectrum, which is bounded by the norm) so z must be in \mathbb{D} . By conventioning $z^0=1$ for every $z\in\mathbb{C}$, we can even choose z=0. Conversely, if $z\in\mathbb{D}$, we define $\chi_z(a)\coloneqq\sum_{n=0}^\infty a_nz^n\in\mathbb{C}$. Then $\chi_z(\delta_n)=z^n$ and

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n \cdot z^m = \chi_z(\delta_n) \cdot \chi_z(\delta_m)$$

So we get a map $\mathbb{D} \to \Omega(\ell^1(S)) \subseteq \ell^1(S)^*, z \mapsto \chi_z$ that is bijective and continuous. If $z_i \to z$ in D, we need to prove $\chi_{z_i} \to \chi_z$ in respect to the weak *-topology. So we need to evaluate and prove $\chi_{z_i}(a) \to \chi_z(a)$, or $\sum_{n=0}^{\infty} a_n z_i^n \to \sum_{n=0}^{\infty} a_m z^m$. Partial sums would obviously converge, so χ_{z_i} converges on a dense subspace of $\ell^1(S)$. The uniform boundedness principle (if a bounded set of operators converge on a dense subset $T_i \to T$, $\sup_i ||T_i|| < \infty$, they converge everywhere) shows that the infinite sums also converge. In general, showing that an operator converges on a dense set of an algebra always shows the convergence on any point of the algebra.

Observe $\sigma(\delta_1) = \{ \chi(\delta_1) \mid \chi \in \Omega(\ell^1(S)) \} = \mathbb{D}$ and $\sigma(\delta_1) = \mathbb{D}$ as well.

Concerning the norm, we know that

$$|\chi_z(a)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| \le \sum_{n=0}^{\infty} |a_n| |z|^n \le \sum_{n=0}^{\infty} |a_n| = ||a||$$

for all $a \in \ell^1(S)$, so $||\chi_z|| \le 1$. For $a = (a_0, 0, 0, \dots)$ we have $|\chi_z(a)| = |a_0| = ||a||$, so $||\chi_z|| = 1$ for any $z \in \mathbb{C}$ (and thus for any $\chi = \chi_z \in \Omega(\ell^1(S))$).

Remark 2.3 (Gelfand-Representation) In general, we seek a mapping $\mathcal{A} \to C_0(X)$, $a \mapsto \hat{a}$, taking $X = \hat{\mathcal{A}} = \Omega(\mathcal{A})$ and $\hat{a}(\chi) = \chi(a)$.

If we apply the Gelfand representation here, we have

$$\ell^1(S) \to C(\mathbb{D}), a \mapsto \hat{a} \text{ where } \hat{a}(z) = \chi_z(a) = \sum_{n=0}^{\infty} a_n z^n$$

Example 2.4 (Norms < 1) Consider

$$\ell_0^1(S) = \overline{\operatorname{alg}}(\delta_1) = \left\{ \sum_{n=1}^{\infty} a_n \delta_n \mid a_n \in \mathbb{C} \right\} \triangleleft \ell^1(S)$$

Observe $\widetilde{\ell_0^1(S)} \simeq \ell^1(S)$. Recall $\Omega(\tilde{\mathcal{A}} = \Omega(\mathcal{A}) \sqcup \{\chi_\infty\}$. So we are looking for our χ_∞ , which is $\chi_\infty(a_0, a_1, \dots) = a_0$ – that is χ_0 and corresponds to z = 0 in the unit circle. It follows $\Omega(\ell_0^1(S)) \simeq \mathbb{D} \setminus \{0\}$ and $\chi_0 \in \Omega(\ell^1(S)) \setminus \Omega(\ell_0^1(S))$.

We compute $\|\chi_z\| = \sup_{\|a\|_1 \le 1} |\chi_z(a)|$. Consider:

$$|\chi_z(a)| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| z \left(\sum_{n=1}^{\infty} a_n z^{n-1} \right) \right| \le |z| \cdot ||a||_1$$

so because of $\chi_z(\delta_1) = z$, we have $||\chi_z|| = |z|$, which can be smaller than 1.

Remark 2.5 Do we have $\ell^1(S) \hookrightarrow A(\mathbb{D}), a \mapsto \hat{a}$ where $\hat{a}(z) := \sum_{n=0}^{\infty} a_n z^n$?

Exercise 2.9 (02-03) Is $\mathscr{A}(\mathbb{D})$ a C^* -algebra? Consider $f(z) = \exp(iz)$, $f \in \mathscr{A}$ and notice $z^* = z$. But we have $||f^*f||_{\infty} \neq ||f||_{\infty}^2$, because $f^*f = 1$ and because f(-i) = e, we have $||f||_{\infty} \geq e$ and $||f||_{\infty}^2 \geq e^2 > 1 = ||f^*f||_{\infty}$. Since the *-property is not fulfilled.