

# Introduction to Operator Algebras

Alcides Buss

Notes by: Linus Mußmächer

2336440

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The set of all linear bounded operators  $\mathcal{L}(H) = \mathcal{B}(H)$  on a given Banach space  $H$  is a (Banach) algebra with  $S \cdot T = S \circ T$ .  $M \subseteq \mathcal{L}$  is a Subalgebra such that  $M^* \subseteq M$  where  $T^*$  is the adjoint of  $T$ . This is also a closed subspace with respect to the strong topology. This is equivalent to  $M = M''$  (when  $X \subseteq \mathcal{B}(H)$ ,  $X' = \{T \in \mathcal{B}(H) \mid TS = ST \ \forall S \in X\}$ )

## Some topological basics

### Definition 0.1

- *Topology, Open*
- *Hausdorff, locally Hausdorff*
- *compact*

**Definition 0.2** A topological space  $X$  is **locally Hausdorff** if every  $x \in X$  admits a compact neighborhood basis, that is for every  $x \in X$  and every open set  $U \ni x$  there exists an open set  $V \ni x$  with  $\bar{V}$  is compact.

**Corollary 0.3** If a set  $V$  is compact in any subset  $U \subseteq X$ , it is also compact in  $X$ .

**Example 0.4 (Snake with two heads)** Consider  $I = [0, 1]$  with the standard topology and extend the set with an element  $1^+$  such that  $I \cup 1^+ \setminus 1$  is isomorphic to  $I$ . Then  $I \cup 1^+$  is locally Hausdorff and compact, but not Hausdorff.

## Some results about locally compact Hausdorff spaces

**Lemma 0.5 (Uryson's Lemma)** Let  $X$  be locally compact and Hausdorff. For all  $F \subseteq X$  closed and  $K \subseteq X$  compact with  $F \cap K = \emptyset$ , there exists an  $f : X \rightarrow [0, 1]$  continuous such that  $f|_K \equiv 1$  and  $f|_F \equiv 0$ .

**Theorem 0.6 (Tietze's extension theorem)** Let  $X$  be locally compact,  $K \subseteq X$  compact and  $f : K \rightarrow \mathbb{C}$  continuous. Then there exists a continuous  $\tilde{f} : X \rightarrow \mathbb{C}$  such that  $\tilde{f}|_K = f$ .

**Theorem 0.7 (Alexandroff's compactification)** If  $X$  is locally compact and Hausdorff, then  $\tilde{X} \sqcup \{\infty\}$  is a compact Hausdorff space  $\mathcal{O}(\tilde{X}) = \mathcal{O}(X) \cup \{K^c \cup \{\infty\} \mid K \text{ compact}\}$ .

**Example 0.8** Compacting the real line  $\mathbb{R}$  yields the space  $\tilde{\mathbb{R}}$ , which is isomorphic to the unit circle  $\Pi = \mathbb{S}^1$ .

**Theorem 0.9** Conversely, if  $Y$  is a compact Hausdorff space, then for all  $y_0 \in Y$ ,  $X := Y \setminus \{y_0\}$  is locally compact (in respect to the subspace topology).

More generally, if  $Y$  is locally compact and Hausdorff, and  $Z \subseteq Y$  is a difference of open and closed subsets, of  $Y$  (i.e.  $Z = U \setminus F$ , where  $U$  is open in  $Y$  and  $F$  is closed in  $Y$ ), then  $Z$  is locally compact.

# 1 Algebras

**Definition 1.1** An **algebra** is a (complex) vector space  $\mathcal{A}$  endowed with a bilinear and associative multiplication:  $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, (a, b) \mapsto a \cdot b$ . So

- (i)  $(a + \alpha b) \cdot (c + \beta d) = ac + \alpha bc + \beta ad + \alpha \beta bd$
- (ii)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

for all  $a, b, c \in \mathcal{A}$  and  $\alpha, \beta \in \mathbb{C}$ . We say that  $\mathcal{A}$  is

- (i) **commutative**, if  $ab = ba$  for all  $a, b \in \mathcal{A}$  and
- (ii) **unital**, if there exists  $1 = 1_{\mathcal{A}} \in \mathcal{A}$  such that  $1 \cdot a = a \cdot 1 = a$  for all  $a \in \mathcal{A}$ .

## Example 1.2

- (i)  $\mathbb{C}$ , or more generally  $\mathbb{C}^n = \mathbb{C} \oplus \dots \oplus \mathbb{C}$ , is an algebra.
- (ii) Say  $X$  is any set; let  $\mathbb{C}^X = \{f : X \rightarrow \mathbb{C}\}$  with point wise multiplication  $(f \cdot g)(x) = f(x) \cdot g(x)$ . These are commutative unital algebras (with  $1(x) = 1 \in \mathbb{C}$ ).
- (iii) Consider the polynomials  $\mathbb{C}[X] = \{\sum_{i=0}^n \lambda_i x^i \mid \lambda_i \in \mathbb{C}, n \in \mathbb{N}\}$  with the usual operations. This is a commutative unital algebra.
- (iv) Let  $X$  be a topological space and  $C(X) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is continuous}\} \subseteq \mathbb{C}^X$  the set of continuous functions on  $X$ . This is a commutative unital (sub)algebra (of  $\mathbb{C}^X$ ).
- (v) Take any vector space  $A$  define a (trivial) multiplication  $a \cdot b := 0$ . This is a commutative Algebra (that is not unital unless  $A = 0$ ).
- (vi)  $M_n(\mathbb{C})$  (the complex  $n \times n$  matrices) with the usual multiplication are a non-commutative (unless  $n = 1$ ) unital algebra.
- (vii) Let  $V$  be any (complex) vector space. The set of all linear operators  $L(V) := \{T : V \rightarrow V \mid T \text{ linear operator}\}$  is a unital (non-commutative for  $\dim V > 1$ ). We observe  $\mathcal{L}(\mathbb{C}^n) \simeq M_n(\mathbb{C})$ .
- (viii) Let  $S$  be a semigroup (i.e. a set with an associative operation  $S \times S \rightarrow S$ , e.g.  $(\mathbb{N}, +)$ ). Then  $\mathbb{C}[S] = \{\sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{C}, |\{s : \lambda_s \neq 0\}| < \infty\}$  (the finite formal sums of elements of  $S$ ) with the following product

$$\left( \sum_{s \in S'} \lambda_s s \right) \cdot \left( \sum_{t \in S} \lambda'_t t \right) := \sum_{s, t \in S} (\lambda_s \cdot \lambda'_t)(s \cdot t) \in S$$

Observe: As a vector space:  $\mathbb{C}[S] \subseteq \mathbb{C}^S$ . In general, this is neither commutative nor unital.

## 2 Normed algebras

**Definition 2.1** An algebra  $\mathcal{A}$  is **normed**, if it is endowed with a (vector space) norm  $\|\cdot\|: \mathcal{A} \rightarrow [0, \infty)$  satisfying  $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ . If  $\mathcal{A}$  is unital with unit  $1_{\mathcal{A}}$ , we usually assume  $\|1_{\mathcal{A}}\| = 1$  except for  $\mathcal{A} = 0$ .

**Definition 2.2** A **Banach algebra** is a normed algebra that is also complete (as a metric space with respect to the distance  $d(a, b) := \|a - b\|$ ), i.e. every Cauchy sequence converges.

**Example 2.3** (i) If  $X$  is a compact space then  $C(X)$  is a commutative unital Banach algebra with respect to the norm  $\|f\|_{\infty} := \sup_{x \in X} |f(x)| < \infty$  (since  $X$  is compact).

(ii) If  $V$  is a normed (respectively Banach) vector space, e.g.  $\mathbb{C}^n$  or  $\ell^p(\mathbb{N})$ , then  $\mathcal{L}(V) = \{T \in L(V) \mid T \text{ is bounded/continuous}\}$  with  $\|T\| := \sup_{\|v\| \leq 1} \|T(v)\| < \infty$  is a normed Banach algebra.

(iii) If  $X$  is a topological space, then  $C_b(X) = \{f \in C(X) \mid \|f\|_{\infty} < \infty\}$  (bounded continuous functions) is a Banach space.

(iv) Let  $X$  again be a topological space. Then the set of all functions **vanishing at  $\infty$** ,

$$\begin{aligned} C_0(X) &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \exists K \subseteq X, K \text{ compact} \forall_{x \notin K} |f(x)| < \varepsilon\} \\ &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\} \subseteq C_b(X), \end{aligned}$$

is also a Banach algebra.

**Exercise 2.1** Assume  $X$  is locally compact and Hausdorff. Prove the following are equivalent:

- (1)  $X$  is compact.
- (2)  $C(X) = C_0(X)$
- (3)  $C_0(X)$  is unital.
- (4) The unit function  $1 \in C_b(X)$  belongs to  $C_0(X)$ .

PROOF: • (1)  $\Rightarrow$  (2): Recall the definition of  $C_0(X)$ . If  $X$  is compact, every closed subset (especially every  $\{x \mid |f(x)| \geq \varepsilon\}$ ) is compact, so the condition of  $C_0(X)$  is trivial.

• (2)  $\Rightarrow$  (3): Since  $C(X)$  is unital,  $C_0(X)$  is as well.

• (3)  $\Rightarrow$  (4): Suppose  $C_0$  is unital, and let  $f \in C_0(X)$  be the unit. Then  $f \cdot g = g$  for all  $g \in C_0(X)$ , i.e.  $f(x)g(x) = g(x) \forall_{x \in X} \forall_{g \in C_0(X)}$ . By Uryson's lemma, given any  $x_0 \in X$ , there exists  $g \in C_0(X)$  with  $g(x_0) = 1$  (by looking at  $K = \{x_0\}$  and taking  $F$  as the complement of any relatively compact environment of  $x_0$ ). Then  $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$ . Doing this for every  $x_0 \in X$  yields  $f \equiv 1$ .

• (4)  $\Rightarrow$  (1): Since  $1 \in C_0(X)$ , for every  $\varepsilon > 0$  the set  $\{x \mid |f(x)| \geq \varepsilon\}$  is compact. Choose  $\varepsilon = \frac{1}{2}$ . Then,  $\{x \mid |f(x)| = |1| \geq \frac{1}{2}\} = X$  is compact.  $\square$

**Exercise 2.2** Let  $X$  be a locally compact Hausdorff space. Prove that  $C_0(X) \cong \{f \in C(X) \mid f(\infty) = 0\}$

### 3 Algebras

**Definition 3.1** A *\*-algebra* is a complex algebra  $\mathcal{A}$  with an *involution*  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$  satisfying

- (i)  $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$
- (ii)  $(a^*)^* = a$
- (iii)  $(ab)^* = b^*a^*$

for all  $a, b \in \mathcal{A}$  and all  $\lambda \in \mathbb{C}$ .

**Definition 3.2** A *normed \*-algebra* is a normed algebra  $\mathcal{A}$  with an involution (such that  $\mathcal{A}$  is a \*-algebra) also satisfying  $\|a^*\| = \|a\|$  for all  $a \in \mathcal{A}$ .

A *Banach\*-algebra* is a complete normed \*-algebra.

**Definition 3.3** A *C\*-algebra* is a Banach\*-algebra satisfying  $\|a^* \cdot a\| = \|a\|^2$ .

Observation: Recall that  $\|a \cdot b\| \leq \|a\| \cdot \|b\|$  in all normed algebras. Applying this to a C\*-algebra we get  $\|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$ . If  $\mathcal{A}$  is a C\*-algebra, then  $\|a\|^2 = \|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$ , so  $\|a\| = \|a^*\|$ .

#### Example 3.4

- (i) If  $X$  is a set, then  $\mathbb{C}^X$  is a \*-algebra with  $f^* = \bar{f}$  and  $\mathcal{C}^\infty(X)$  is a C\*-algebra.
- (ii) If  $X$  is a topological space, then  $C(X) \subseteq \mathbb{C}^X$  is also a \*-subalgebra and for  $\{f \in C(X) \mid \text{supp}(f) = \overline{\{x \in X \mid |f(x)| \neq 0\}} \text{ compact}\}$  we have

$$C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X) \subseteq C^\infty(X)$$

and  $C^\infty$  is a C\*-algebra.  $C_c$  is a \*-algebra, but not Banach in general.

If  $X$  is compact, it follows  $C_c(X) = C_0(X) = C_b(X)$ .

Observation: If  $X$  is locally compact and Hausdorff, then  $\overline{C_c(X)} = C_0(X)$ .

- (iii) Let  $X$  be a measured space ( $X$  is endowed with a  $\sigma$ -algebra). Then  $B_\infty(X) = \{f \in C^\infty \mid f \text{ is measurable}\}$  is a C\*-algebra. If  $\mu$  is a measure on  $X$  (e.g.  $X = \mathbb{R}^n$  and  $\mu$  the Lebesgue measure) then  $L^\infty(X, \mu)$  are the essentially bounded functions and

$$L^\infty(X) = \{f : X \rightarrow \mathbb{C} \mid \|f\| := \inf\{c \geq 0 \mid \mu(\{x \mid |f(x)| > c\}) = 0\}\}$$

is also a C\*-algebra.

Observation:  $L^2(X, \mu) = \{\mu\text{-separable function}\}$ ,  $L^\infty(X, \mu) \xrightarrow{\mu} B(L^2(X, \mu)), f \mapsto \mu_f = \{g \mapsto f \cdot g\}$

- (iv) A non-example: Let  $\mathbb{D}$  be the unit disk and  $\mathcal{A}(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid \text{analytic in } \mathbb{D}^\circ\}$

**Morera's Theorem** from complex analysis states that  $f \in C(\mathbb{D})$  is analytic if and only if  $\int_\gamma f(z)dz = 0$  for all closed and piece wise smooth paths in  $\mathbb{D}^\circ$ . From this, it follows that  $\mathcal{A}(\mathbb{D})$  is closed in  $C(\mathbb{D})$ , therefore a Banach algebra. It is also a Banach\*-algebra with, but  $f^* = \bar{f}$  (point wise) is not possible, as  $z \mapsto \bar{z}$  is not analytic. Thus, we have to choose  $f^*(z) = f(\bar{z})$ . But  $\mathcal{A}(\mathbb{D})$  is not a C\*-algebra, as  $\|f^*f\|_\infty \neq \|f\|_\infty^2$  for some  $f \in \mathcal{A}(\mathbb{D})$ .

- (v) A non-commutative example: Let  $H$  be a Hilbert space and  $B(H) = \mathcal{L}(H) = \{T : H \rightarrow H \mid T \text{ bounded, continuous, linear}\}$  and  $\|H\| := \sup_{\|z\| < 1} \|T(z)\| < \infty$ . This is a  $C^*$ -algebra where  $T^*$  is the adjoint of  $T$ , that is  $\langle T^*z, w \rangle = \langle z, Tw \rangle$  for all  $z, w \in H$ .

$C^*$ -axiom:  $\|T^* \cdot T\| \leq \|T\|^2$  since  $\mathcal{L}(H)$  is a Banach algebra, and we also have

$$\begin{aligned} \|T\|^2 &= \sup_{\|z\| < 1} \|T(z)\|^2 = \sup_{\|z\| < 1} \langle Tz, Tz \rangle = \sup_{\|z\| < 1} \langle z, T^*Tz \rangle \\ &\leq \sup_{\|z\| < 1} \|z\| \|T^*Tz\| \leq \sup_{\|z\| < 1} \|z\| \|T^*T\| \leq \|T^*T\| \end{aligned}$$

In particular,  $M_n(\mathbb{C}) \simeq \mathcal{L}(\mathbb{C}^n)$  is a unital  $C^*$ -algebra.

- (vi) To produce more examples, take any subset  $S \subseteq \mathcal{L}(H)$  and take  $C^*(S) \subseteq \mathcal{L}(H) = \overline{\text{span}\{S_i \mid S_i \in S \cup S^*, i \leq n \in \mathbb{N}\}}$ .

**Example 3.5** Let  $s \in \mathcal{L}(\ell^2(\mathbb{N}))$ . The shift  $s$ , defined by  $s(e_i) = e_{i+1}$  for all  $i \in \mathbb{N}$  (where  $\{e_i\}$  is the canonical basis of the sequence space), is an isometry, that is  $s^* \cdot s = \text{id}$ . Since  $s \cdot s^* \neq \text{id}$ , it is not surjective and not a proper isometry. We define

$$T = C^*(s) = \overline{\text{span}\{s^n(s^*)^m \mid m, n \in \mathbb{N}_0\}} \subseteq \mathcal{L}(\ell^2(\mathbb{N}))$$

as the **Toeplitz algebra**.

**Example 3.6** Let  $H$  be a Hilbert space and  $S$  the set of all finite rank operators on  $H$ .

**Example 3.7**

- (i) **Commutative:**  $C_0(X)$  for a locally Hausdorff space  $X$ .
- (ii) **Non-commutative:**  $\mathcal{L}(\mathfrak{H}) = \mathcal{B}(\mathfrak{H})$  for any Hilbert space  $\mathfrak{H}$  (with dimension greater 1).
- (iii) **More generally:** Take any subset  $S \subseteq \mathcal{L}(\mathfrak{H})$  and construct  $C^*(S) \subseteq \mathcal{L}(H)$  as

$$\overline{\text{span}\{S_1, \dots, S_n \mid S_i \in S \cap S^*\}}$$

**Example 3.8 (Cuntz algebras)** Take again  $\mathfrak{H} = \ell^2\mathbb{N} = \{(\lambda_n)_{n \in \mathbb{N}_0} \mid \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$  where  $\langle \lambda, \lambda' \rangle = \sum_{i \in \mathbb{N}_0} \overline{\lambda_i} \lambda'_i$  and which has the orthonormal base  $(e_n)_{n \in \mathbb{N}}$  where  $(e_n) = (\delta_{in})_{i \in \mathbb{N}_0}$ .

On this algebra, define

- $S_1(e_n) = e_{2n}$ .
- $S_2(e_n) = e_{2n+1}$ .

We have partitioned the natural numbers into evens and odds. This defines two (proper) isometries  $S_1, S_2 \in \mathcal{L}(\mathfrak{H})$ , that is  $S_i^* S_i = \text{id}_{\mathfrak{H}}$ , to subspaces of  $\mathfrak{H}$ . Notice:  $S_i^* S_j = 0$  for  $i \neq j$  as well as  $S_1 S_1^* + S_2 S_2^* = \text{id}_{\mathfrak{H}}$ . Define  $\mathcal{O}_2 = C^*(S_1, S_2) = \overline{\text{span}\{S_\alpha S_\beta^* \mid \alpha, \beta \text{ finite words in } \{1, 2\}\}}$ . For example, for  $\alpha = 121211$  we have  $S_\alpha = S_1 S_2 S_1 S_2 S_1^2$ .  $\mathcal{O}_2$  is called the **Cuntz algebra**. More generally, one can define  $\mathcal{O}_3, \mathcal{O}_4, \dots$  Cuntz algebras. Joachim Cuntz proved that these are simple  $C^*$ -algebras with additional interesting properties we will see later.

**Example 3.9 (Rotation algebras)** Let  $\mathfrak{H} = \ell^2(\mathbb{Z})$  (bi-infinite sequences) with basis  $(e_n)_{n \in \mathbb{Z}}$ . Define:

- $U(e_n) := e_{n+1}$  (bilateral shift)

- $V(e_n) := \lambda^n e_n$  where  $\lambda \in \mathbb{C}$  is some fixed number  $|\lambda| = 1$ .

This defines two *unitary* operators:  $UU^* = 1 = U^*U$  and  $V^*V = 1 = V^*V$ . If  $\exp(2\pi i\theta), \theta \in \mathbb{R}$  define  $A_\theta := C^*(U, V) \subseteq \mathcal{L}(\ell^2\mathbb{N})$ .

There is a special relation between  $U$  and  $V$  where  $UV = \lambda VU = \exp(2\pi i\theta)VU$ . From this relation, we can describe  $A_\theta = \overline{\text{span}}\{\sum_{n,m \in \mathbb{Z}}^{\text{finite}} a_{n,m} U^n V^m \mid a_{n,m} \in \mathbb{C}\}$ .

Furthermore, if  $\theta \in \mathbb{R} \setminus \mathbb{Q}$ ,  $A_\theta$  is simple.

**Example 3.10 ( $C^*$ -algebras of groups)** Let  $G$  be a (discrete) group. Look at  $\mathfrak{H} = \ell^2(G) = \{(a_g)_{g \in G} \mid \sum_{g \in G} |a_g|^2 < \infty\}$  (Note: This limit will only converge if there are countably (or finitely) many non-zero parts) with ONB  $(\delta_g)_{g \in G}$  where  $\delta_g(h) = \delta_{gh}$ . Define for each  $g \in G$  an operator  $\lambda_g \in \mathcal{L}(\ell^2 G)$  by  $\lambda_g(\delta_h) = \delta_{gh}$ . Notice that  $h \mapsto gh$  is a bijection, and thus  $\lambda_g$  is a unitary operator with  $\lambda_g^* = \lambda_{g^{-1}}$ . We can now define the **reduced  $C^*$ -algebra** of the group:

$$C_R^*(G) := C_\lambda^*(G) \subseteq \mathcal{L}(\ell^2 G) = C^*(\lambda_g \mid g \in G)$$

Here, we have the relation  $\lambda_g \cdot \lambda_h = \lambda_{gh}$  and thus  $C_R^*(G) = \{\sum a_g \lambda_g \mid a_g \in \mathbb{C}\}$ .

In general, take  $U : G \rightarrow \mathcal{L}(H), g \mapsto U_g$  a **unitary representation of  $G$**  with  $U_g U_h = U_{gh}$  and  $U_1 = \text{id}$  as well as  $U_g^{-1} = U_{g^{-1}}$ . Then  $C_U^*(G) := \{\sum_{g \in G} a_g U_g \mid a_g \in \mathbb{C}\} \subseteq \mathcal{L}(H)$ . There exists a **universal unitary representation**  $C_{\max}^*(G)$ , a full  $C^*$ -algebra of  $G$ .

**Remark 3.11**

- (i) If  $G$  is Abelian, then  $C_U^*(G)$  is also abelian (commutative). In particular,  $C_\lambda^*$  is abelian. Later, we will prove  $C_\lambda^*(G) \simeq C(\hat{G})$  where  $\hat{G}$  is the dual of  $G$ , i.e.  $\{X : G \rightarrow \mathbb{C} \text{ characters}\}$ .
- (ii) For many groups, like  $G = \mathbb{F}_n$  (the free groups) the reduced  $C^*$ -algebra  $C_\lambda^*(G)$  is simple.

## 4 Homomorphisms of algebras

**Definition 4.1** If  $\mathcal{A}, \mathcal{B}$  are algebras, a **homomorphism** from  $\mathcal{A}$  to  $\mathcal{B}$  is a linear map  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\varphi(ab) = \varphi(a)\varphi(b)$  for any  $a, b \in \mathcal{A}$ .

If  $\mathcal{A}$  and  $\mathcal{B}$  are  $*$ -algebras, a  **$*$ -homomorphism** is a homomorphism  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  such that  $\varphi(a^*) = \varphi(a)^*$  for all  $a \in \mathcal{A}$ .

If  $\mathcal{A}, \mathcal{B}$  are Banach algebras, then usually we want to have **continuous** homomorphisms. Even more, we usually ask for **contractive** homomorphisms  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ , (that is  $\|\varphi\| \leq 1$ ).

We will be especially interested in **characters**:

**Definition 4.2** A **character** of an algebra  $\mathcal{A}$  is a non-zero homomorphism  $\chi : \mathcal{A} \rightarrow \mathbb{C}$ .

**Example 4.3** Take any subalgebra  $\mathcal{A} \subseteq \mathbb{C}^X$ . Take  $x_0 \in X$  and set  $\chi_{x_0} := \text{ev}_{x_0} : \mathcal{A} \rightarrow \mathbb{C}, f \mapsto f(x_0)$ . This is not necessarily a character, but it is for example, if  $\mathcal{A} = C(X)$  or  $C_b(X)$  or  $C_0(X)$  (if  $X$  is “nice”, like Hausdorff).

**Definition 4.4** A  $(*)$ -isomorphism between two  $(*)$ -algebras  $\mathcal{A}$  and  $\mathcal{B}$  is a bijective  $(*)$ -homomorphism  $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$ .

**Definition 4.5** A  $(*)$ -ideal of a  $*$ -algebra  $\mathcal{A}$  is a subspace  $I \subset \mathcal{A}$  such that  $I \cdot \mathcal{A} \subseteq I$ ,  $\mathcal{A} \cdot I \subseteq I$  (if only one condition applies, we call this a **left ideal** or **right ideal**). For  $*$ -ideals, we also want  $I^* = I$ . We notate this as  $I \trianglelefteq \mathcal{A}$ .



**Example 4.6** If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a  $(*)$ -homomorphism, then  $\ker \varphi \trianglelefteq \mathcal{A}$ .

**Example 4.7** If  $I \trianglelefteq \mathcal{A}$  for  $\mathcal{A}$  a  $(*)$ -algebra

$$\mathcal{A}/I = \{a + I \mid a \in \mathcal{A}\}$$

with  $(a + I) \cdot (b + I) := ab + I$  and  $(a + I)^* = a^* + I$  is a  $(*)$ -algebra.

**Theorem 4.8** If  $\mathcal{A}$  is a Banach- $*$ -algebra, then  $I \trianglelefteq \mathcal{A}$  is a closed ideal, then the quotient  $I/\mathcal{A}$  is also a Banach- $*$ -algebra.

PROOF: Later. □

## 5 Spectral theory

**Notation 5.1** If  $\mathcal{A}$  is a unital algebra, we write

$$\text{inv}(\mathcal{A}) = \{a \in \mathcal{A} \mid a \text{ is invertible in } \mathcal{A}\} = \{a \in \mathcal{A} \mid \exists a^{-1} \in \mathcal{A} \, aa^{-1} = 1 = a^{-1}a\}$$

This is a group. Sometimes we also write  $GL(\mathcal{A})$ .

**Definition 5.2** Given a unital algebra  $\mathcal{A}$  and  $a \in \mathcal{A}$ , we define its **spectrum** (in  $\mathcal{A}$ ) as

$$\sigma_{\mathcal{A}}(a) = \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \notin \text{inv}(\mathcal{A})\}$$

and the resolvent of  $a$  (in  $\mathcal{A}$ ) as

$$\rho_{\mathcal{A}}(a) = \rho(a) = \mathcal{A} \setminus \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \text{inv}(\mathcal{A})\}$$

**Example 5.3 (Linear Algebra)** Let  $\mathcal{A} = M_m(\mathbb{C})$  and  $a \in \mathcal{A}$ . Then we have

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \det(\lambda - a) = 0\}$$

and these are the roots of the characteristic polynomial  $\det(\lambda - a)$ . This is exactly the usual spectrum from linear algebra.

**Example 5.4 (Functional Analysis)** Let  $\mathcal{A} = \mathcal{L}(\mathfrak{H})$  – where  $\mathfrak{H}$  is any Hilbert- or Banach space – and  $T \in \mathcal{A}$ . Then  $\sigma_{\mathcal{A}}(T)$  is exactly the spectrum as defined in functional analysis.

If  $S$  is the shift in  $\mathcal{L}(\ell^2\mathbb{N})$ , then we have  $\sigma(S) = \mathbb{D}$ .

**Example 5.5** Let  $\mathcal{A} = \mathbb{C}[X]$ . Here we have  $\text{inv}(\mathcal{A}) = \{a_0 X^0 \mid a_0 \in \mathbb{C} \setminus \{0\}\}$  the constant non-zero polynomials. If  $a = \sum_{k=0}^N a_k x^k \in \mathcal{A}$ , then we have two cases:

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \mathbb{C} & \text{otherwise} \end{cases}$$

**Example 5.6** Let  $\mathcal{A} = \mathbb{C}(X) = \{p/q \mid p, q \in \mathbb{C}[X], q \neq 0\}$ . Now we have  $\text{inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$ . If  $a \in \mathcal{A}$ , then

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \emptyset & \text{otherwise} \end{cases}$$

**Example 5.7** Let  $\mathcal{A} = C(X)$  for any topological space  $X$ . Then

$$\text{inv}(\mathcal{A}) = \{f \in C(X) \mid \forall_{x \in X} f(x) \neq 0\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{x \in X} f(x) = \lambda\} = \text{im}(f) = f(X).$$

**Example 5.8** Let  $X$  be any topological space and consider  $\mathcal{A} = C_b(X)$ . Then

$$\text{inv}(C_b(X)) = \{f \in C_b(X) \mid \exists_{\varepsilon > 0} \forall_{x \in X} |f(x)| \geq \varepsilon\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{(x_n)} f(x_n) \rightarrow \lambda\} = \overline{\text{im}(f)} = \overline{f(X)}.$$

This is a compact subset of  $\mathbb{C}$ .

**Theorem 5.9 (Algebraic spectral mapping theorem)** *Let  $\mathcal{A}$  be an algebra,  $a \in \mathcal{A}$  and  $p \in \mathbb{C}[X]$ ,  $p(X) = \sum_{k=0}^n \lambda_k X^k$  and define  $p(a) = \sum_{k=0}^n \lambda_k a^k$ . Recall that the mapping  $\mathbb{C}[X] \rightarrow \mathcal{A}$ ,  $p \mapsto p(a)$  is a unital homomorphism.*

*Then  $\sigma(p(a)) = p(\sigma(a))$  assuming  $\sigma(a) \neq \emptyset$ .*

PROOF: If  $p(X) = \lambda_0$  constant, this is clear (the spectrum is exactly  $\lambda_0$  on both sides). Assume  $p(x)$  is not constant. Fix  $\mu \in \mathbb{C}$  and write

$$\mu - p(x) = \lambda_0(x - \lambda_1) \cdots (x - \lambda_n)$$

as per the fundamental theorem of algebra (note that these are not the same  $\lambda$  as before) with  $\lambda_0 \neq 0$ . Then  $\mu - p(a) = \lambda_0(a - \lambda_1) \cdots (a - \lambda_n)$ . Since these expressions commute, this product is invertible if and only if  $(a - \lambda_i)$  is invertible for every  $i$ . So  $\mu \in \sigma(p(a)) \Leftrightarrow \mu - p(a)$  is not invertible if and only if there exists an  $i$  for which  $\lambda_i - a$  is not invertible, so  $\lambda_i \in \sigma(a)$ . But the  $\lambda_i$  are exactly the numbers satisfying  $p(\lambda) = \mu$ . Thus,  $\mu$  is in  $\sigma(p(a))$  if it is in the image of  $\sigma(a)$  under  $p$ . Therefore, we conclude  $\sigma(p(a)) = p(\sigma(a))$ .  $\square$

We now focus on invertible elements in **Banach algebras**.

**Theorem 5.10** *If  $\mathcal{A}$  is a unital Banach algebra and  $a \in \mathcal{A}$  with  $\|a\| < 1$  then  $1 - a$  is invertible and  $(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n$ .*

PROOF: Observe that, since  $\|a\| < 1$ , we have  $\sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} < \infty$ . This implies the (absolute) convergence of  $\sum_{n=0}^{\infty} a^n$  by the characteristic property of Banach spaces. Hence,  $b := \lim_{N \rightarrow \infty} \sum_{n=0}^N a^n \in \mathcal{A}$ . No, if  $N \in \mathbb{N}$ , then

$$(1 - a) \left( \sum_{n=0}^N a^n \right) = \left( \sum_{n=0}^N a^n \right) - \left( \sum_{n=1}^{N+1} a^n \right) = 1 - a^{N+1} \rightarrow 1$$

because of  $\|a\| < 1$ . This yields  $(1 - a)b = 1$ .  $\square$

**Theorem 5.11** *Let  $\mathcal{A}$  be a non-empty, non-zero unital Banach algebra. Then  $\text{inv}(\mathcal{A})$  is an open subset of  $\mathcal{A}$  and the function  $f : \text{inv}(\mathcal{A}) \rightarrow \mathcal{A}$ ,  $a \mapsto a^{-1}$  is Frechet-differentiable and in particular continuous as well as  $f'(a)b = -a^{-1}ba^{-1}$ .*

Recall from calculus that  $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$ . Also recall that  $f : U \xrightarrow{\text{open}} X \rightarrow Y$  with  $X, Y$  Banach spaces is **differentiable** at  $x_0 \in U$  there exists an operator  $D_{x_0} = f'(x_0) \in \mathcal{L}(X, Y)$  such that

$$\lim_{h \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - D_{x_0}(h)\|}{\|h\|} = 0$$

PROOF: Take  $a \in \text{inv}(\mathcal{A})$ . If  $b \in \mathcal{A}$  such that  $\|a - b\| < \|a^{-1}\|^{-1}$ . From this, we have  $\|ba^{-1} - 1\| = \|ba^{-1} - aa^{-1}\| = \|(b - a)a^{-1}\| \leq \|b - a\| \cdot \|a^{-1}\| < 1$ . Per the previous theorem,  $ba^{-1} \in \text{inv}(\mathcal{A})$ . This implies that  $b$  is also invertible. This shows that  $\text{inv}(\mathcal{A})$  is open.

Furthermore, if  $\|b\| < 1$ , then also  $\| -b \| < 1$ . Thus,  $1 + b \in \text{inv}(\mathcal{A})$  and  $(1 + b)^{-1} = \sum_{n=0}^{\infty} (-1)^n b^n$ . Thus,

$$\|(1 + b)^{-1} - 1 + b\| = \left\| \sum_{n=0}^{\infty} (-1)^n b^n - 1 + b \right\| \leq \left\| \sum_{n=2}^{\infty} (-1)^n b^n \right\| \leq \sum_{n=2}^{\infty} \|b^n\| \leq \sum_{n=2}^{\infty} \|b\|^n = \frac{\|b\|^2}{1 - \|b\|}$$

Now let  $a \in \text{inv}(\mathcal{A})$  and  $c \in \mathcal{A}$  such that  $\|c\| < \frac{1}{2} \|a^{-1}\|^{-1}$ . Then  $\|a^{-1}c\| \leq \|a^{-1}\| \|c\| \leq \frac{1}{2}$ . So if  $b = a^{-1}c$ , then

$$\|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \leq \frac{\|a^{-1}c\|^2}{1 - \|a^{-1}c\|} < 2\|a^{-1}c\|^2$$

Now, define  $U : \mathcal{A} \rightarrow \mathcal{A}, b \mapsto -a^{-1}ba^{-1}$ . Then this is a linear odd operation with  $\|U\| \leq \|a^{-1}\|^2$ , and we have

$$\begin{aligned} \|(a + c)^{-1} - a^{-1} - U(c)\| &= \|(a + c)^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &= \|(1 + a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &\leq \|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \cdot \|a^{-1}\| \\ &\leq 2\|a^{-1}c\|^2 \|a^{-1}\| \leq 2\|a^{-1}\|^3 \|c\|^2 \end{aligned}$$

and thus

$$\lim_{c \rightarrow 0} \frac{\|(a + c)^{-1} - a^{-1} - U(c)\|}{\|c\|} = 0 \quad \square$$

**Example 5.12** If we choose  $\mathcal{A} = \mathbb{C}[X]$  and the norm  $\|p\| = \sup_{\lambda \in [0,1]} |p(\lambda)|$ . Then  $(\mathcal{A}, \|\cdot\|)$  is a normed (but not Banach) algebra. For example, we see that  $\lim_{m \rightarrow 0} 1 + X/m = 1 \in \text{inv}(\mathcal{A})$ , but  $1 + X/m \notin \text{inv}(\mathcal{A})$  and thus  $\text{inv}(\mathcal{A})$  is not open (because the complement is not closed).

**Theorem 5.13** *If  $\mathcal{A}$  is a Banach algebra with unit 1, then for all  $a \in \mathcal{A}$  the spectrum  $\sigma(a) \subseteq \mathbb{C}$  is closed and  $\sigma(a) \subseteq \overline{B(0, \|a\|)} = D(0, \|a\|) := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$ . Therefore,  $\sigma(a)$  is compact by the Heine-Borell theorem.*

PROOF: By definition

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \text{inv}(\mathcal{A})\}$$

is the inverse image of the closed subset  $\mathcal{A} \setminus \text{inv}(\mathcal{A}) \subseteq \mathcal{A}$  by the continuous function  $\lambda \mapsto \lambda - a$ . Therefore,  $\sigma(a)$  is closed.

Now if  $|\lambda| \leq \|a\|$  then  $\|\lambda^{-1}a\| < 1$ . Then  $1 - \lambda^{-1}a \in \text{inv}(\mathcal{A})$ . Multiplying by  $\lambda$  yields  $\lambda - a \in \text{inv}(\mathcal{A})$ . Thus,  $\{\lambda \in \mathbb{C} \mid |\lambda| > \|a\|\} \subseteq \rho(a)$  and thus  $\sigma(a) \subseteq D(0, \|a\|)$ .  $\square$

**Lemma 5.14** *Let  $\mathcal{A}$  be a unital Banach algebra and  $a \in \mathcal{A}$ . Then, the map  $R_a : \rho(a) \subseteq \mathbb{C} \rightarrow \mathcal{A}, \lambda \mapsto (a - \lambda)^{-1}$  is Frechet-differentiable.*

PROOF: This follows from the following general result:

If  $g : U \xrightarrow{\text{open}} X \rightarrow Y$  and  $f : V \xrightarrow{\text{open}} Y \rightarrow Z$  for Banach spaces  $X, Y, Z$  with  $g(U) \subseteq V$  are differentiable at  $x_0 \in U$  or respectively  $y_0 = g(x_0) \in V$ , then  $f \circ g$  is differentiable and  $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$ .  $\square$

Observation: For  $R_a(\lambda) = (a - \lambda)^{-1}$  we get  $R'_a(\lambda) = (a - \lambda)^{-2}$ . We have  $\mathcal{L}(\mathbb{C}, \mathcal{A}) \simeq \mathcal{A}$  by  $T \mapsto T(1)$ . Recall that if  $f(a) = a^{-1}$  yields  $f'(a)b = -a^{-1}ba^{-1}$ .

**Theorem 5.15 (Gelfand)** *If  $\mathcal{A} \neq 0$  is a unital Banach algebra and  $a \in \mathcal{A}$  then  $\sigma(a) \neq \emptyset$ .*

PROOF: Suppose  $\sigma(a) = \emptyset$ . Idea: Show that  $R_a : \rho(a) \subseteq \mathbb{C} \rightarrow \mathcal{A}, \lambda \mapsto (a - \lambda)^{-1} = \frac{1}{a - \lambda}$  is bounded and differentiable and achieve a contradiction by Liouville's theorem.

Claim:  $\|(a - \lambda)^{-1}\| < \|a\|^{-1}$  if  $|\lambda| > 2\|a\|$ . Indeed, if  $|\lambda| > 2\|a\|$  then  $\|\lambda^{-1}a\| < \frac{1}{2}$ , and in particular  $1 - \lambda^{-1}a \in \text{inv}(\mathcal{A})$  and

$$\|(1 - \lambda^{-1}a)^{-1} - 1\| = \left\| \sum_{n=1}^{\infty} (\lambda^{-1}a)^{-1} \right\| \leq \sum_{n=1}^{\infty} \|\lambda^{-1}a\|^n = \frac{\|\lambda^{-1}a\|}{1 - \|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

From here we deduce that  $\|(1 - \lambda^{-1}a)^{-1}\| < 2$  and thus

$$\|(a - \lambda)^{-1}\| < \|\lambda^{-1}(\lambda^{-1}a - 1)^{-1}\| = \frac{\|(1 - \lambda^{-1}a)^{-1}\|}{|\lambda|} < \frac{2}{\lambda} < \frac{1}{\|\lambda\|}.$$

So  $R_a : \mathbb{C} \rightarrow \mathcal{A}$  is bounded outside  $\overline{B(0, 2\|a\|)}$ . Since  $R_a$  is continuous, it is bounded on  $\mathbb{C} \rightarrow \mathcal{A}$ . Let  $\varphi \in \mathcal{A}^*$  be a bounded linear functional in  $\mathcal{L}(\mathcal{A}, \mathbb{C})$ . Thus,  $\varphi$  is differentiable with  $\varphi'(a) = \varphi$  for all  $a \in \mathcal{A}$ . Then  $\varphi \circ R_a$  is differentiable and bounded, so it is an “integer” function. By Liouville's theorem,  $\varphi \circ R_a$  is constant. Therefore,  $\varphi \circ R_a(x) = \varphi \circ R_a(y)$  for all  $x, y \in \mathcal{A}$ . Especially, we have  $\varphi((a - \lambda)^{-1}) = \varphi(a^{-1})$  for all  $\varphi$ . Hahn-Banach shows  $(a - \lambda)^{-1} = a^{-1}$  for all  $\lambda$ , proving  $a - \lambda = a$  for all  $a, \lambda$ . This is a contradiction.  $\square$

**Theorem 5.16 (Gelfand-Mazur)** *If  $\mathcal{A}$  is a unital Banach algebra and every  $a \neq 0$  admits an inverse ( $\mathcal{A}$  is a field), then  $\mathcal{A} = \mathbb{C} \cdot 1$ .*

PROOF: By the assumption,  $\text{inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$ . By the previous theorem, if  $a \in \mathcal{A}$  there exists some  $\lambda \in \sigma(a)$ , so  $a - \lambda \notin \text{inv}(\mathcal{A})$ , so  $a - \lambda = 0$  and thus  $a = \lambda \cdot 1$ .  $\square$

**Corollary 5.17** *Let  $\mathbb{C}(X) = \left\{ \frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{Q}[X] \right\}$  is a field, but it cannot be turned into a Banach algebra.*

**Theorem 5.18 (Adjoining units - unitization of algebras)** *Let  $\mathcal{A}$  be any algebra. Consider  $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$  as a vector space. We write elements of  $\tilde{\mathcal{A}}$  as  $a + \lambda \cdot 1 := (a, \lambda)$ . Think of  $a = (a, 0)$  and  $\lambda = (0, \lambda)$ . Define*

$$(a + \lambda 1)(b + \lambda' 1) = (ab + \lambda'a + \lambda b) + \lambda \cdot \lambda'.$$

*Ten (exercise  $\tilde{\mathcal{A}}$ ) becomes a unital algebra with  $1_{\tilde{\mathcal{A}}} = 1 = (0, 1)$ .*

*Notice that  $\mathcal{A}$  is an ideal in  $\tilde{\mathcal{A}}$ .*

*Moreover, we get a short exact sequence*

$$0 \rightarrow \mathcal{A} \hookrightarrow \tilde{\mathcal{A}} \rightarrow \mathbb{C} \rightarrow 0$$

so  $1 + \lambda \mapsto \lambda$ .

If  $\mathcal{A}$  is a normed algebra, then  $\tilde{\mathcal{A}}$  is normed by  $\|a + \lambda \cdot 1\| := \|a\| + |\lambda|$

If  $\mathcal{A}$  is Banach and closed, then so is  $\tilde{\mathcal{A}}$ .

If  $\mathcal{A}$  is a  $*$ -algebra, then so is  $\tilde{\mathcal{A}}$  with  $(a + \lambda 1)^*$ .

If  $\mathcal{A}$  is a (Banach) normed  $*$ -algebra, then so is  $\tilde{\mathcal{A}}$ .

If  $\mathcal{A}$  is a  $C^*$ -algebra, in general the norm given above is not a Norm on  $\mathcal{A}$ , but  $\|a + \lambda \cdot 1\| := \sup_{b \in \mathcal{A}, b \in \mathcal{B}, b \leq 1} \|ab + \lambda b\|$  is.

**Exercise 5.1** If  $\mathcal{A}$  is already unital, then  $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$  as algebras by  $a + \lambda \cdot 1 \mapsto (a + \lambda 1_{\mathcal{A}}, -\lambda)$ .

**Definition 5.19** *Re-Definition:* If  $\mathcal{A}$  is non-unital, then  $\tilde{\mathcal{A}} + \mathbb{C} \cdot 1$  is a  $(*-)$ Banach algebra, and we define  $\sigma_A(a) := \sigma_{\tilde{\mathcal{A}}}(a)$ .

Observation: If  $\mathcal{A}$  is already unital, then for  $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$  we have  $\sigma_{\tilde{\mathcal{A}}}(a) = \sigma_{\mathcal{A}}(a) \cup \{0\}$ .

**Remark 5.20** If  $\mathcal{A}$  is a  $C^*$ -algebra, then  $\tilde{\mathcal{A}}$  is a  $C^*$ -algebra.

- (i) If  $\mathcal{A}$  is unital, then  $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$  and  $\|a + \lambda \cdot 1\| = \max\{\|a + \lambda \cdot 1\|, |\lambda|\}$ .
- (ii) If  $\mathcal{A}$  is not unital, then  $\|a + \lambda \cdot 1\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$ .

## 6 Spectral Radius

**Definition 6.1** Let  $\mathcal{A}$  be an algebra. Given  $a \in \mathcal{A}$ , we define:

$$r(a) := \sup\{|\lambda| \mid \lambda \in \sigma_{\mathcal{A}}(a)\}$$

as the **spectral radius** of  $a$  if  $\emptyset \neq \sigma_{\mathcal{A}}(a)$  is bounded (e.g. if  $\mathcal{A}$  is Banach).

Observation: In a Banach algebra, we have  $0 \leq r(a) \leq \|a\|$ .

**Example 6.2**

- (i) Let  $f \in \mathcal{A} = C_0(X)$  using  $\sigma_A(f) = \overline{f(X)}$ . Thus,

$$r(f) = \sup\{|\lambda| \mid \lambda \in \overline{f(X)}\} = \sup_{x \in X} |f(x)| = \|f\|_{C_0(X)}$$

- (ii) Let  $\mathcal{A} = M_2(\mathbb{C})$  and  $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Then  $\sigma_{\mathcal{A}} = \{0\}$  and  $r(a) = 0$ , but  $\|a\| = 1 \neq 0$ .

**Theorem 6.3 (Beurling-Gelfand)** Let  $\mathcal{A}$  be a Banach algebra, then

$$r(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$$

PROOF: We may assume  $\mathcal{A}$  is unital (otherwise we consider  $\tilde{\mathcal{A}}$ ). If  $\lambda \in \sigma(a)$ , then

$$\lambda^n \in \sigma(a^n) \Rightarrow |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a\|^{\frac{1}{n}} \quad \forall n \in \mathbb{N}$$

and therefore

$$r(a) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}.$$

We prove now that  $\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$ . Set  $\Delta := B\left(0, \frac{1}{r(a)}\right)$ . Where per convention we set  $\frac{1}{r(a)} = \infty$  if  $r(a) = 0$ . If  $\lambda \in \Delta$ , then  $1 - \lambda a \in \text{inv}(\mathcal{A})$  (because  $|\lambda| < \frac{1}{r(a)}$  implies  $|\lambda^{-1}| > r(a)$ ) and therefore  $\lambda^{-1} \notin \sigma(a) \Rightarrow \lambda^{-1} - a \in \text{inv } A \Rightarrow 1 - \lambda a \in \text{inv}(A)$ .

Now fix  $\varphi \in \mathcal{A}^*$ . Then  $f : \Delta \rightarrow \mathbb{C}, \lambda \mapsto \varphi((1 - \lambda a)^{-1})$  is analytic, so it can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n \lambda^n, a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{C}, \lambda \in \Delta.$$

On the other hand, if

$$|\lambda| < \frac{1}{\|a\|} \leq \frac{1}{r(a)}$$

then  $\|\lambda a\| < 1$ , so

$$(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n \Rightarrow f(\lambda) = \varphi((1 - \lambda a)^{-1}) = \sum_{k=0}^{\infty} \varphi(a^k) \lambda^k$$

for  $|\lambda| < \frac{1}{\|a\|}$ .

By uniqueness of the Taylor series expansion, it follows that

$$a_n = \varphi(a^n) \forall n \in \mathbb{N}.$$

In particular,  $(\varphi(a^n) \lambda^n)$  converges to zero for all  $\lambda \in \Delta$  and thus  $(\varphi(a^n) \lambda^n)$  is bounded for all  $\lambda \in \Delta$ .

From the principle of uniform convergence, it follows that  $(a^n \lambda^n)$  is bounded. So there exists an  $M = M_\lambda$  such that

$$\begin{aligned} \|\lambda^n a^n\| &\leq M \forall n \in \mathbb{N} \\ \Rightarrow \|\lambda^n\|^{\frac{1}{n}} &\leq \frac{M^{\frac{1}{n}}}{|\lambda|} \forall n \in \mathbb{N}, \forall \lambda \in \Delta, \lambda \neq 0 \\ \Rightarrow \limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} &\leq \frac{1}{\lambda} \forall \lambda \in \Delta \text{ i.e. } |\lambda| < \frac{1}{r(a)} \end{aligned}$$

Letting  $\lambda < \frac{1}{r(a)}$  yields  $\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq r(a)$ . □

**Example 6.4** Let  $A = C^1([0, 1]) = \{I \in C[0, 1] \mid \exists f'(t) \forall t \in [0, 1], t \mapsto f'(t) \text{ continuous}\}$  with  $\|f\| = \|f\|_\infty + \|f'\|_\infty$ .

Then  $\mathcal{A}$  is unital, commutative and a Banach algebra. Consider  $x \in \mathcal{A}, x(t) = t$ . We have  $x^n(t) = t^n$  and

$$\begin{aligned} \|x^n\| &= \sup_{t \in [0, 1]} |t^n| + \sup_{t \in [0, 1]} |nt^{n-1}| = 1 + n \\ r(x) &= \lim_{n \rightarrow \infty} (1 + n)^{\frac{1}{n}} = 1 \\ \|x\| &= 2 \end{aligned}$$

Observation:  $\sigma(x) = \text{im}(x) = [0, 1]$ .

**Theorem 6.5** Let  $\mathcal{B} \not\subseteq \mathcal{A}$  be an inclusion of unital Banach algebras with  $1 = 1_{\mathcal{A}} = 1_{\mathcal{B}}$ . Then  $\sigma_{\mathcal{A}}(b) \subseteq \sigma_{\mathcal{B}}(b)$  for all  $b \in \mathcal{B}$  and the inclusion may be proper. If  $\sigma_{\mathcal{A}}(b)$  is simply connected (not holes), then  $\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b)$ .

The holes of a compact subset  $K \subseteq \mathbb{C}$  are the bounded connected components of  $\mathbb{C} \setminus K$ . So saying that  $K$  has no holes means that  $\mathbb{C} \setminus K$  is connected.

PROOF: See Murphy, 1.2.8. □

**Example 6.6** Let  $\mathcal{B} := A(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid f \text{ analytic on } \mathbb{D}^\circ\}$  and  $\mathcal{A} = C(\mathbb{S}^1)$ . Then we have an embedding by  $\iota : \mathcal{B} \hookrightarrow \mathcal{A}, f \mapsto f|_{\mathbb{S}^1}$ .

By the principle of maximum modules,  $\iota$  is an embedding of (unital) Banach algebras. Consider:  $f(z) = z$  for  $z \in \mathbb{D}$ . (Observation:  $\overline{Alg}(1, z) = A(\mathbb{D})$ ) Then:

$$\sigma_{A(\mathbb{D})}(f) = f(\mathbb{D}) = \mathbb{D}$$

and  $\sigma_{C(\mathbb{S}^1)}(f|_{\mathbb{S}^1}) = \mathbb{S}^1$ .

**Definition 6.7 (Exponentials)** Let  $\mathcal{A}$  be a unital Banach algebra, given  $a \in \mathcal{A}$  we define

$$e^a = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

Note  $\left\| \frac{a^n}{n!} \right\| \leq \frac{\|a\|^n}{n!}$ , so the series converges and  $\|\exp(a)\| \leq \exp(\|a\|)$ .

**Theorem 6.8**

(i) Let  $\mathcal{A}$  be a unital Banach algebra. If  $a \in \mathcal{A}$ , then  $f : \mathbb{R} \rightarrow \mathcal{A}, t \mapsto \exp(ta)$  is the unique solution of

$$\begin{cases} f'(t) &= af(t) \\ f(0) &= 1 \end{cases}$$

(ii)  $e^a \in \text{inv}(\mathcal{A})$  and  $(e^a)^{-1} = e^{-a}$ .

(iii) If  $a, b \in \mathcal{A}$  then  $e^{a+b} = e^a \cdot e^b$  (here some commutativity is necessary).

PROOF: See Murphy, 1.2.9. □

## 7 Gelfand Representation for commutative Banach algebras

Idea: Given a commutative algebra  $\mathcal{A}$ , we want to represent  $\mathcal{A}$  by a homomorphism  $\varphi : \mathcal{A} \rightarrow C_0(X)$  for  $X$  some locally compact Hausdorff space. We hope that  $\varphi$  is injective, or even isometric, or an isomorphism. But what is  $X$ , and what is  $\varphi$ ?

Notice that, if  $\mathcal{A} = C_0(X)$  already, then for each  $x \in X$  we get a character  $\text{ev}_x : \mathcal{A} \rightarrow \mathbb{C}, f \mapsto f(x)$ .

**Definition 7.1** Given an algebra  $\mathcal{A}$ , we define

$$\hat{\mathcal{A}} = \Omega(\mathcal{A}) := \{\chi : \mathcal{A} \rightarrow \mathbb{C} \mid \chi \text{ non-zero homomorphism}\}.$$

**Example 7.2**

(i) For  $\mathcal{A} = C_0(X)$  we get a map

$$X \rightarrow \Omega(\mathcal{A}), x \mapsto \text{ev}_x$$

that is a bijection. After we give  $\Omega(\mathcal{A})$  an appropriate topology, it will also be a homomorphism.

- (ii) Let  $\mathcal{A} = M_2(\mathbb{C})$  (or any  $M_n(\mathbb{C})$ ). This is a simple algebra, so non-zero homomorphisms  $\chi : \mathcal{A} \rightarrow \mathbb{C}$  do not exist (same for any  $\mathcal{A}$  with dimension  $> 1$ ).

So in this case we have  $\Omega(\mathcal{A}) = \emptyset$ . This can also happen in commutative algebras.

- (iii) Consider

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\}$$

Then for all  $a \in \mathcal{A}$  we have  $a^2 = 0$ , so if  $\chi : \mathcal{A} \rightarrow \mathbb{C}$  is an homomorphism, then  $\chi(a)^2 = \chi(a^2) = 0$ , so  $\chi(a) = 0$  for all  $a \in \mathcal{A}$ . So again,  $\Omega(\mathcal{A}) = \emptyset$  (and  $\mathcal{A}$  is commutative with  $\dim \mathcal{A} = 1$ ).

Question: Given an abstract algebra  $\mathcal{A}$  how do we possibly find its characters?

Idea: Assume that  $\mathfrak{l} \triangleleft \mathcal{A}$  is a maximal ideal and  $\mathcal{A}$  is a unital Banach algebra. Then  $\mathcal{A}/\mathfrak{l} \simeq \mathbb{C}$  and  $\chi \in \Omega(\mathcal{A})$ .

**Theorem 7.3** *Let  $\mathcal{A}$  be a unital non-zero Banach algebra. If  $\chi \in \Omega(\mathcal{A})$  then  $\|\chi\| = \sup_{\|a\|=1} |\chi(a)| = 1$  and  $\ker(\chi) \triangleleft \mathcal{A}$ . So  $\chi \in \mathcal{A}^*$  (the topological dual of  $\Omega(\mathcal{A}) \subseteq D_{\mathcal{A}^*}(0, 1)$ ).*

*Moreover, if  $\mathcal{A}$  is a unital Banach commutative algebra, then  $\Omega(\mathcal{A}) \ni \chi \mapsto \ker(\chi) \triangleleft \mathcal{A}$  is a bijection between of characters of  $\mathcal{A}$  and maximal ideals of  $\mathcal{A}$ .*

PROOF: If  $a \in \mathcal{A}$  and  $\chi$  a character, then  $\chi(a) \in \sigma(\mathcal{A})$ , because  $\chi(a - \chi(a) \cdot 1) = \chi(a) - \chi(a) \cdot \chi(1) = 0$ , so  $a - \chi(a) \cdot 1 \in \ker(\chi) \triangleleft \mathcal{A}$  and thus  $a - \chi(a) \cdot 1 \notin \text{inv}(\mathcal{A})$ .

Therefore:  $|\chi(a)| \leq r(a) \leq \|a\|$ . So  $\|\chi\| \leq 1$ . Since  $\chi(1) = 1$  and  $\|1\| = 1$  we have  $\|\chi\| = 1$ .

Now, apply linear algebra. Then  $\ker(\chi)$  is a maximal proper subspace, in particular a maximal ideal. And  $\ker(\chi)$  is closed, because  $\chi$  is continuous. Now assume that  $\mathcal{A}$  is commutative (in addition to unital and Banach). Then we have the mapping

$$\varphi : \Omega(\mathcal{A}) \rightarrow \text{MaxIdeals}(\mathcal{A}), \chi \mapsto \ker(\chi).$$

- $\varphi$  is injective. If  $\ker(\chi_1) = \ker(\chi_2)$  for  $\chi_1, \chi_2 \in \Omega(\mathcal{A})$ , then for every  $a \in \mathcal{A}$  we have  $a - \chi_1(a) \cdot 1 \in \ker(\chi_1) = \ker(\chi_2)$ . Thus,  $\chi_2(a - \chi_1(a) \cdot 1) = 0$  and therefore  $\chi_2(a) = \chi_1(a)$  for every  $a \in \mathcal{A}$ .
- $\varphi$  is surjective. Take  $\mathfrak{l} \triangleleft \mathcal{A}$  a maximal ideal. Then  $\mathfrak{l} = \bar{\mathfrak{l}}$  because  $\bar{\mathfrak{l}} \neq \mathcal{A}$ , otherwise  $1 \in \bar{\mathfrak{l}}$  and since  $\text{inv}(\mathcal{A})$  is open in  $\mathcal{A}$ , we get  $\mathfrak{l} \cap \text{inv}(\mathcal{A}) \neq \emptyset$ . But then we have an invertible element in the ideal  $\mathfrak{l}$  already, but this implies the contradiction  $\mathfrak{l} = \mathcal{A}$ . Therefore,  $\mathcal{A}/\mathfrak{l}$  is a commutative, unital Banach algebra which is simple ( $\mathfrak{l}$  is maximal).

Exercise: If  $\mathfrak{l} \triangleleft \mathcal{A}$ , then  $\mathcal{A}/\mathfrak{l}$  is field if and only if there exists no  $\mathfrak{j} \triangleleft \mathcal{A}$  such that  $\mathfrak{l} \triangleleft \mathfrak{j}$ .

Thus,  $\mathcal{A}/\mathfrak{l}$  is a field and  $\mathcal{A}/\mathfrak{l} \simeq \mathbb{C}$ . Then the composition

$$\mathcal{A} \xrightarrow{q} \mathcal{A}/\mathfrak{l} \simeq \mathbb{C}$$

is a character with  $\ker(\chi) = \mathfrak{l}$ . □

**Exercise 7.1** An application of Zorn's Lemma. Show that every ideal  $I \triangleleft \mathcal{A}$  in a unital algebra  $\mathcal{A}$  is contained in a maximal ideal.

In particular, we can apply this to  $\mathfrak{l} = 0$  in  $\mathcal{A} \neq 0$  (with  $\mathcal{A}$  is unital and commutative) and thus  $\Omega(\mathcal{A}) \neq \emptyset$ .



## Topology on $\Omega(\mathcal{A})$

We have for  $\mathcal{A}$  a Banach algebra. We can add a unit to receive  $\tilde{\mathcal{A}}$ , which is a Banach algebra.

Observe: If  $\chi \in \Omega(\mathcal{A})$ , then there exists a unique  $\tilde{\chi} \in \Omega(\tilde{\mathcal{A}})$  via  $\tilde{\chi}(a + \lambda \cdot 1) = \chi(a) + \lambda$ . Thus,  $\|\chi\| \leq \|\tilde{\chi}\| = 1$  (Note that it may still be smaller than 1. See exercises 2023-05-09).

In any case,

$$\Omega(\mathcal{A}) \subseteq D_{\mathcal{A}^*}(0, 1) = \{\varphi \in \mathcal{A}^* \mid \|\varphi\| \leq 1\}$$

and  $\mathcal{A}^*$  carries the weak \*-topology (the smallest topology to make all point-evaluations continuous, that is for a net  $(\varphi_i) \subset \mathcal{A}^*$  weakly converging to  $\varphi \in \mathcal{A}^*$  if and only if  $\varphi_i(a) \rightarrow \varphi(a)$  for all  $a \in \mathcal{A}$ ).

**Definition 7.4** Given a Banach algebra  $\mathcal{A}$ , we endow  $\Omega(\mathcal{A})$  with the weak \*-topology and call this the **Gelfand spectrum** of  $\mathcal{A}$ .

**Proposition 7.5**  $\Omega(\mathcal{A})$  is a locally compact Hausdorff space. If  $\mathcal{A}$  is unital, then  $\Omega(\mathcal{A})$  is compact.

PROOF: By Banach-Alaoglu-Theorem,  $D_{\mathcal{A}^*}(0, 1)$  is compact and Hausdorff with the weak \*-topology. Let

$$\begin{aligned} S &:= \{\chi : \mathcal{A} \rightarrow \mathbb{C} \mid \chi \text{ hom.}\} \\ &= \Omega(\mathcal{A}) \cup \{0\} \end{aligned}$$

Then  $S \subseteq D_{\mathcal{A}^*}(0, 1)$ . So  $\chi(ab) = \lim_{i \rightarrow \infty} K_i = \lim_{i \rightarrow \infty} \chi_i(a)\chi_i(b) = \chi(a)\chi(b)$  and therefore  $x \in S$ . Thus,  $S$  is a compact Hausdorff space and  $\Omega(\mathcal{A}) = S \setminus \{0\}$  is relatively compact.

If  $\mathcal{A}$  is unital, then  $\Omega(\mathcal{A}) \subseteq D_{\mathcal{A}^*}(0, 1)$  is closed. Then we have  $(X_i) \subseteq \Omega(\mathcal{A})$  and  $X_i \rightarrow X \in \mathcal{A}^*$  and thus  $X \in S = \text{hom}(\mathcal{A}, \mathbb{C})$ .  $\square$

Observation: Given a Banach algebra  $\mathcal{A}$ , we have an isomorphism

$$\Omega(\tilde{\mathcal{A}}) \rightarrow \Omega(\mathcal{A}) \sqcup \{\chi_\infty\}, \varphi \mapsto \begin{cases} \varphi|_{\mathcal{A}} & \varphi|_{\mathcal{A}} \neq 0 \\ \chi_\infty & \varphi|_{\mathcal{A}} = 0 \end{cases},$$

where  $\chi_\infty(a + \lambda \cdot 1) = \lambda$ . Thus,  $\Omega(\mathcal{A}) \sqcup \{\chi_\infty\}$  is already the unitization of  $\Omega(\mathcal{A})$ .

**Theorem 7.6** Let  $\mathcal{A}$  be a Banach algebra. Then for every  $a \in \mathcal{A}$ .

$$\{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \subseteq \sigma(a)$$

If  $\mathcal{A}$  is commutative, then

- $\{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} = \sigma(a)$  in case  $\mathcal{A}$  is unital.
- $\{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} = \sigma_{\mathcal{A}}(a)$ .

PROOF:

- $\mathcal{A}$  is unital and  $a \in \mathcal{A}$ .  $\chi(a - \chi(a) \cdot 1) = 0$ , so  $\chi(a) \in \sigma(a)$ , so  $\{\chi(a) \mid x \in \Omega(a)\} \subseteq \sigma(a)$ .

Now if  $\lambda \in \sigma(a)$ , consider  $\mathfrak{l} := (a - \lambda \cdot 1)\mathcal{A} \triangleleft \mathcal{A}$  if  $\mathcal{A}$  is commutative. By Zorns Lemma, we get  $I \subseteq J \triangleleft \mathcal{A}$  with  $J = \ker(\chi)$  for some  $\chi \in \Omega(\mathcal{A})$ . Thus we have  $a - \lambda \cdot 1 \in \mathfrak{l} \subseteq J = \ker(\chi)$  so  $\chi(a) = \lambda$ .

- $\mathcal{A}$  is not unital. Consider  $\tilde{\mathcal{A}}$ . By the first part,

$$\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(a) \supseteq \{\chi(a) \mid \chi \in \Omega(\tilde{\mathcal{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\}$$

If  $\mathcal{A}$  is commutative, by the first part again:

$$\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(a) = \{\chi(a) \mid \chi \in \Omega(\tilde{\mathcal{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} \quad \square$$

## 7.1 Gelfand-Transformation

**Definition 7.7** Given a Banach algebra  $\mathcal{A}$  and  $a \in \mathcal{A}$ , we define  $\hat{a} : \Omega(\mathcal{A}) \rightarrow \mathbb{C}, \chi \mapsto \chi(a)$ .

Observe that  $\hat{a} \in C(\Omega(\mathcal{A}))$ , because if  $\chi_i \rightarrow \chi$  then we have  $\hat{a}(\chi_i) = \chi_i(a) \rightarrow \chi(a) = \hat{a}(\chi)$ . So we have a map  $\Gamma : \mathcal{A} \rightarrow C(\Omega(\mathcal{A}))$ . This map is called the **Gelfand transform** of  $\mathcal{A}$ .

**Theorem 7.8 (Gelfand Representation)**  $\text{im}(\Gamma) \subseteq C_0(\Omega(\mathcal{A}))$  and  $\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A}))$  is a contractive homomorphism, i.e.  $\|\Gamma(a)\| \leq r(a) \leq \|a\|$  for every Banach algebra  $\mathcal{A}$ . If moreover  $\mathcal{A}$  is commutative, then  $\|\Gamma(a)\| = r(a)$ . Also, for all  $a \in \mathcal{A}$ , we have

$$\sigma(a) = \begin{cases} \text{im}(\hat{a}) & \mathcal{A} \text{ unital} \\ \text{im}(\hat{a}) \cup \{0\} & \text{otherwise} \end{cases}.$$

PROOF: If  $\mathcal{A}$  is unital, then  $\Omega(\mathcal{A})$  is compact so  $\text{im}(\Gamma) \subseteq C(\Omega(\mathcal{A})) = C_0(\Omega(\mathcal{A}))$ . If  $\mathcal{A}$  is not unital, we use observation 7. Then we have  $\Omega(\tilde{\mathcal{A}}) \simeq \Omega(\mathcal{A}) \cup \{\chi_\infty\}$  so that

$$C_0(\Omega(\mathcal{A})) \simeq \{f \in C(\Omega(\tilde{\mathcal{A}})) \mid f(x_\infty) = 0\}.$$

Now if  $a \in \mathcal{A}$ , then  $\hat{a}(\chi_\infty) = \chi_\infty(a) = 0$ .

$\Gamma$  is a homomorphism: The linearity is obvious, as is the homomorphism property:

$$(\Gamma(a)\Gamma(b))(\chi) = (\hat{a} \cdot \hat{b})(\chi) = \hat{a}(\chi)\hat{b}(\chi) = \chi(a)\chi(b) = \chi(ab) = \hat{ab}(\chi) = \Gamma(ab)(\chi).$$

$\Gamma$  is contractive: Given  $a \in \mathcal{A}$ ,  $\chi \in \Omega(\mathcal{A})$ , we have  $\hat{a}(\chi) = \chi(a) \in \sigma(a)$ , so  $\|\hat{a}(\chi)\| \leq r(a)$  yielding  $\|\Gamma(a)\|_\infty = \|\hat{a}\|_\infty \leq r(a) \leq \|a\|$ . If  $\mathcal{A}$  is commutative, we have

$$\sigma(a) = \begin{cases} \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} & 1 \in \mathcal{A} \\ \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} & \text{otherwise} \end{cases} = \begin{cases} \{\hat{a}(\chi) \mid \chi \in \Omega(\mathcal{A})\} & 1 \in \mathcal{A} \\ \{\hat{a}(\chi) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} & \text{otherwise} \end{cases}$$

and thus

$$\|\Gamma(a)\| = \|\hat{a}\|_\infty = \sup_{\chi \in \Omega(\mathcal{A})} |\chi(a)| = \sup_{\lambda \in \sigma(a)} |\lambda| = r(a) \quad \square$$

As a convention, if  $\Gamma(\mathcal{A}) = \{0\}$ , then  $C_0(\Omega(\mathcal{A})) = \{0\}$  and thus  $\hat{a} = 0$  for all  $a \in \mathcal{A}$ .

### Example 7.9

- (i) If  $\mathcal{A} = M_n(\mathbb{C})$  with  $n > 1$  or  $\mathcal{A}$  is any unital simple Banach algebra with  $\dim \mathcal{A} > 1$ , then  $\Omega(\mathcal{A}) = \emptyset$  so  $\Gamma \equiv 0$ .

(ii) Take the commutative subalgebra

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

then  $\mathcal{A}$  is not unital, commutative, Banach and  $\dim \mathcal{A} = 1$ . Once again,  $\Omega(\mathcal{A}) = \emptyset$  and thus  $\Gamma \equiv 0$ .

(iii) Take

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mid \lambda, \alpha \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

is a unital, commutative Banach algebra with  $\dim \mathcal{A} = 2$ . We have

$$\Omega(\mathcal{A}) = \{\chi_\infty\} \quad \chi_\infty : \mathcal{A} \rightarrow \mathbb{C}, \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mapsto \lambda$$

and thus

$$\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A})) = C_0(\{\chi_\infty\}) \simeq \mathbb{C}, a = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mapsto \hat{a} \equiv \lambda$$

This shows that  $\Gamma$  is not injective, as  $\dim \mathcal{A} = 2$  but  $\dim \Gamma(\mathcal{A}) = 1$ .

**Definition 7.10** Let  $\mathcal{A}$  be a Banach algebra. We say that  $a \in \mathcal{A}$  is *quasi-nilpotent* if  $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0$ . Sometimes, you will read

$$\text{Rad}(\mathcal{A}) = \{a \in \mathcal{A} \mid a \text{ quasi-nilpotent}\}$$

If  $\text{Rad}(\mathcal{A}) = 0$ , we say that  $\mathcal{A}$  is **semi-simple**. Notice that if  $a \in \mathcal{A}$  is quasi-nilpotent, then  $\Gamma(a) = \hat{a} = 0$  because  $\Gamma(a) \leq r(a) = 0$ . If  $\mathcal{A}$  is commutative, then  $\ker(\Gamma) = \text{Rad}(\mathcal{A})$ .

**Example 7.11**

(iv)  $\mathcal{A} = \ell^1(\mathbb{Z}) = \{(a_n)_{n \in \mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} |a_n| < \infty\}$ .

Recall from exercises, that  $\Omega(\ell^1(\mathbb{Z})) \simeq \mathbb{D}$  with  $\mathbb{D} \rightarrow \Omega(\ell^1(\mathbb{Z})), z \mapsto \chi_z$  defined as  $\chi_z(a) = \hat{a}(z) = \sum_{n=0}^{\infty} a_n z^n$ .

We define a multiplication  $\delta_m \cdot \delta_n = \delta_{n+m}$ . Then  $\delta_0$  is the unit and  $\delta_1$  is a generator of  $\mathcal{A} = \ell^1(\mathbb{Z})$ .

The elements  $\delta_m - (\dots, 0, 1, 0, \dots)$  form a basis for  $\mathcal{A}$ . We have  $a = \sum_{n \in \mathbb{Z}} a_n \delta_n$  and for  $\chi \in \mathcal{A}^*$  it follows  $\chi(a) = \sum_{n \in \mathbb{Z}} a_n \chi(\delta_n)$ .

We now want to calculate the spectrum. We have seen that  $\chi(\delta_0) = \chi(1_{\mathcal{A}}) = 1$  and  $\chi(\delta_n) = \chi(\delta_1^n) \chi(\delta_1)^n$ . Therefore,  $\chi$  is determined by  $z = \chi(\delta_1) \in \mathbb{C}$ . We know at least that  $|z| = |\chi(\delta_1)| \leq \|\delta_1\| = 1$ , so  $z \in \mathbb{D}$ . Claim:  $z \in \Pi = \mathbb{S}^1$ .

General fact: If  $a \in \text{inv } \mathcal{A}$  for  $\mathcal{A}$  a unital Banach algebra, then  $\sigma(a^{-1}) = \sigma(a)^{-1} = \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$ .

Observe that  $\mathbb{S}^1 = \text{inv}(\mathcal{A})$  with  $\delta_1^{-1} = \delta_{-1}$ . So  $\sigma(\delta) \subseteq \mathbb{D}$  and  $\sigma(\delta_1)^{-1} = \sigma(\delta_{-1}) \subseteq \mathbb{D}$ , so  $\sigma(\delta_1) \subseteq \mathbb{S}^1$ . So  $z = \chi(\delta_1) \in \sigma(\delta_1) \subseteq \mathbb{S}^1$ . Conversely, if  $z \in \mathbb{S}^1$ , then  $\chi_z : \mathcal{A} \rightarrow \mathbb{C}, \chi_z(a) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{C}$  is well-defined (as the sum converges) and is a character, as

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n z^m = \chi_z(\delta_n) \cdot \chi_z(\delta_m)$$

and checking in the basis also proves the homomorphism property for all of  $\mathcal{A}$ . Notice that  $z = \chi_z(\delta_1)$ . This shows the injectivity of

$$\Pi \simeq \Omega(\mathcal{A}) = \Omega(\ell^1(\mathbb{Z})), z \mapsto \chi_z, \chi(\delta_1) \leftarrow \chi$$

which is continuous and therefore a homeomorphism (isomorphism), as both spaces are compact. Notice

$$\sigma(\delta_1) = \{\chi(\delta_1) \mid \chi \in \Omega(\mathcal{A})\} = \{\chi_z(\delta_1) \mid z \in \mathbb{S}^1\} = \mathbb{S}^1$$

The Gelfand transformation is now

$$\Gamma : \mathcal{A} = \ell^1(\mathbb{Z}) \rightarrow C(\Omega(\mathcal{A})) \simeq C(\mathbb{S}^1), a \mapsto \left( \hat{a} : z \mapsto \sum_{n \in \mathbb{Z}} a_n z^n \right)$$

$\Gamma$  is always a contractive algebra homomorphism, as  $\|\hat{a}\|_\infty \leq \|a\|_1$ .  $\Gamma$  is a  $*$ -homomorphism where  $\ell^1(\mathbb{Z})$  carries the involution  $a^* = (\sum_{n \in \mathbb{Z}} a_n \delta_n)^* = \sum_{n \in \mathbb{Z}} \bar{a}_n \delta_{-n}$  because of  $\delta_n^* = \delta_{-n}$ . The involution of  $C(\mathbb{S}^1)$  is complex conjugation. But on the unit circle,  $\bar{z} = z^{-1}$ , so we have a  $*$ -homomorphism.

$\Gamma$  is injective. If  $f \in C(\mathbb{S}^1)$ , we can define its “inverse Fourier-Transform”

$$\check{f}(n) = \int_{\mathbb{S}^1} f(z) z^{-n} dz = \frac{1}{2\pi} \int_0^{2\pi} f(\exp(it)) \exp(-int) dt$$

This is **not** the line integral from functional analysis, as the derivative of the path is not included. You can now check that  $(\hat{a})^\sim(n) = a_n$ .  $g \mapsto \int_{\mathbb{S}^1} g$  is a continuous function on  $C(\mathbb{S}^1)$  and we have

$$\hat{a}(z) = \sum_{n \in \mathbb{Z}} a_n z^n = \lim_{F \subseteq \mathbb{Z} \text{ finite}} \sum_{n \in F} a_n z^n = \lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n z^n$$

so

$$(\hat{a})^\sim(n) = \sum_{m \in \mathbb{Z}} a_m (\hat{\delta}_m)^\sim(n)$$

Because of  $\int_{\mathbb{S}^1} z^k = \delta_{k,0}$ , we have

$$\int_{\mathbb{S}^1} z^m z^n dz = \delta_{n,m}$$

and using  $\hat{\delta}_m(z) = z^m$  we can show  $(\hat{\delta}_m)^\sim(n) = \delta_{n,m}$  and thus

$$(\hat{a})^\sim(n) = \sum_{m \in \mathbb{Z}} a_m (\hat{\delta}_m)^\sim(n) = \sum_{m \in \mathbb{Z}} a_m \delta_{m,n} = a_n$$

This shows that we can re-gain the elements of the sequence from  $\hat{a}$ , so  $\Gamma : (a_n) \mapsto \hat{a}$  must be injective.

$\Gamma$  has dense range because the polynomials are dense in  $C(\mathbb{S}^1)$  because of Stone-Weierstraß theorem.

$\Gamma$  is not isometric. If  $\Gamma$  was isometric, then  $\Gamma$  were an isometric  $*$ -homomorphism with dense range. Since isometric homomorphisms have closed image,  $\Gamma$  were surjective and thus an isometric  $*$ -isomorphism  $\ell^1(\mathbb{Z}) = C(\mathbb{S}^1)$ . Then  $\ell^1(\mathbb{Z})$  would be a  $C^*$ -algebra with the  $\ell^1(\mathbb{Z})$ -norm, and thus  $\|a^*a\|_1 = \|a\|_1^2$  (with the involution as described above). Then, using the  $C^*$ -property of  $C(\mathbb{S}^1)$  and isometry of  $\Gamma$ , we have

$$\|a^*a\|_1 = \|\Gamma(a^*a)\|_\infty = \|\Gamma(a)^*\Gamma(a)\|_\infty = \|\Gamma(a)\|_\infty^2 = \|a\|_1^2.$$

Now we only need to find  $a \in \ell^1(\mathbb{Z})$  with  $\|a^*a\|_1 \neq \|a\|_1^2$ . Choose  $a = \alpha\delta_0 + \beta\delta_1 + \gamma\delta_{-1} = \alpha + \beta\delta_1 + \gamma\delta_{-1}$  (not writing  $\delta_0$  as it is the unit).

$$a^*a = (\bar{\alpha} + \bar{\beta}\delta_{-1} + \bar{\gamma}\delta_1)(\alpha + \beta\delta_1 + \gamma\delta_{-1}) = \dots$$

and thus

$$\|a^*a\|_1 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2|\bar{\alpha}\beta + \alpha\bar{\gamma}| + 2|\gamma\beta|$$

while

$$\|a\|_1^2 = (|\alpha| + |\beta| + |\gamma|)^2.$$

Now choosing  $\alpha = i$  and  $\beta = \gamma = 1$  yields  $\|a^*a\|_1 = 5$  and  $\|a\|_1^2 = 9$ . This shows that  $\ell^1(\mathbb{Z})$  does not fulfil the  $*$ -property and cannot be a  $C^*$ -algebra. This is a contradiction, so  $\Gamma$  cannot be isometric.

This is also a valid counterexample for the isometry directly, because  $a$  has Norm 3, but  $\Gamma(a) = (z \mapsto \frac{1}{z} + i + z = 2\Re(z) + i)$  has maximum  $2 + i$  with Norm  $\sqrt{5} < 3$  on the unit circle.  $\Gamma$  is not surjective. This is complicated.

Recall: For  $\mathcal{A}$  a Banach algebra, we have a Gelfand representation

$$\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A})), a \mapsto (\hat{a} : \Omega(\mathcal{A}) \rightarrow \mathbb{C}, \chi \mapsto \chi(a))$$

where  $\Omega(\mathcal{A}) = \{\chi : \mathcal{A} \rightarrow \mathbb{C} \mid \text{non-zero hom}\} \subseteq D_{\mathcal{A}^*}(0, 1)$  with the weak  $*$ -topology.  $\Gamma$  is a contractive homomorphism, and if  $\mathcal{A}$  is commutative  $\|\Gamma(a)\| = r(a) \leq \|a\|$  for all  $a \in \mathcal{A}$ .

We now want to consider commutative  $C^*$ -algebras.

**Theorem 7.12 (Gelfand)** *If  $\mathcal{A}$  is a commutative  $C^*$ -algebra, then  $\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A}))$  is an isometric  $*$ -isomorphism.*

For this proof we require a set of lemmas.

**Lemma 7.13** *If  $a \in \mathcal{A}$ ,  $\mathcal{A}$  a  $C^*$ -algebra, with  $a = a^*$  then  $r(a) = \|a\|$ .*

PROOF: Use  $r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$ . Notice  $\|a^2\| = \|a^*a\| = \|a\|^2$  and  $\|a^4\| = \|(a^2)^*a^2\| = \|a^2\|^2 = \|a\|^4$  and likewise for all powers that are powers of 2 we have  $\|a^{2^n}\| = \|a\|^{2^n}$ . So  $r(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|$  is the limit of the subsequence and therefore the limit of the sequence.  $\square$

**Remark 7.14** In general,  $\|a\| \neq r(a)$  if  $a \neq a^*$  in a  $C^*$ -algebra, e.g.  $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ .

But if  $a^*a = aa^*$  ( $a$  is normal), then  $\|a\| = r(a)$ .

PROOF: Exercise. □

**Corollary 7.15** *There exists at most one norm that makes a  $*$ -algebra  $\mathcal{A}$  into a  $C^*$ -algebra.*

PROOF: If  $\mathcal{A}$  is a  $C^*$ -algebra with norm  $\|\cdot\|$ , then for all  $a \in \mathcal{A}$  we have  $\|a\| = \|aa^*\|^{\frac{1}{2}}$ . Note that  $a^*a$  is self-adjoint, so by the previous lemma we have

$$\|a\| = \|aa^*\|^{\frac{1}{2}} = r(a^*a)^{\frac{1}{2}} = \sup_{\lambda \in \sigma(a^*a)} |\lambda|^{\frac{1}{2}}$$

and this only depends on the algebra structure, not its norm. □

**Corollary 7.16** *If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a  $*$ -homomorphism from a Banach- $*$ -algebra  $\mathcal{A}$  into a  $C^*$ -algebra  $\mathcal{B}$  then  $\varphi$  is contractive, i.e.  $\|\varphi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$  for all  $a \in \mathcal{A}$*

PROOF: Replacing  $\mathcal{A}, \mathcal{B}$  by their unitizations  $\tilde{\mathcal{A}}$  and  $\tilde{\mathcal{B}}$  and extending  $\varphi$  to  $\tilde{\varphi} : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}}, a + \lambda 1_{\mathcal{B}} \mapsto \varphi(a) + \lambda 1_{\mathcal{B}}$  shows that we can just assume  $\mathcal{A}, \mathcal{B}, \varphi$  to be unital.

Now, if  $a \in \text{inv}(\mathcal{A})$ , then  $\varphi(a) \in \text{inv}(\mathcal{B})$ , so it follows

$$\lambda \in \rho_{\mathcal{A}}(a) \Leftrightarrow a - \lambda \in \text{inv}(\mathcal{A}) \Leftrightarrow \varphi(a) - \lambda \in \text{inv}(\mathcal{B}) \Leftrightarrow \lambda \in \rho_{\mathcal{B}}(\varphi(a))$$

so  $\rho_{\mathcal{A}}(a) \subseteq \rho_{\mathcal{B}}(\varphi(a))$  and  $\sigma_{\mathcal{A}}(a) \supseteq \sigma_{\mathcal{B}}(\varphi(a))$ . It follows for the spectral radius:  $r(\varphi(a)) \leq r(a)$ . As  $\mathcal{B}$  is a  $C^*$ -algebra, this implies

$$\begin{aligned} \|\varphi(a)\|_{\mathcal{B}}^2 &= \|\varphi(a)^* \varphi(a)\|_{\mathcal{B}} = \|\varphi(a^*a)\|_{\mathcal{B}} = r(\varphi(a^*a)) \\ &\leq r(a^*a) \leq \|a^*a\|_{\mathcal{A}} \leq \|a^*\|_{\mathcal{A}} \cdot \|a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}^2 \end{aligned}$$

and therefore  $\|\varphi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$ . □

**Lemma 7.17** *If  $\mathcal{A}$  is a  $C^*$ -algebra and  $a \in \mathcal{A}$ , then*

- (i) *If  $a$  is self-adjoint,  $\sigma(a) \subseteq \mathbb{R}$ .*
- (ii) *If  $\mathcal{A}$  is unital and  $u \in \mathcal{U}(\mathcal{A})$  is unitary (that is,  $u^*u = uu^* = 1$ ) then  $\sigma(u) \subseteq \mathbb{S}^1$ .*
- (iii) *If  $a \in \text{inv}(\mathcal{A})$ , then  $\sigma(a^{-1}) = \sigma(a)^{-1} = \{z^{-1} \mid z \in \sigma(a)\}$ .*
- (iv)  *$\sigma(a^*) = \overline{\sigma(a)}$ .*

PROOF: (iii) If  $\lambda \in \mathbb{C}, \lambda \neq 0$  and  $\lambda - a \notin \text{inv}(\mathcal{A})$ . Because  $\lambda - a$  is not invertible,  $\lambda^{-1}(\lambda - a) = 1 - \lambda^{-1}a$  and  $a^{-1}(1 - \lambda^{-1}a) = a^{-1} - \lambda^{-1}$  is also not invertible. So we have  $\lambda^{-1} - a^{-1} \notin \text{inv}(\mathcal{A})$  and therefore  $\sigma(a^{-1}) \subseteq \sigma(a)^{-1}$ . The result follows by symmetry.

(iv) Similarly, you can prove (iv).

(ii) If  $u \in \mathcal{U}(\mathcal{A})$ , then  $\sigma(u) \subseteq \mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$  because

$$\|u\| = \|u^*u\|^{\frac{1}{2}} = \|1\|^{\frac{1}{2}} = 1.$$

So, since  $u \in \mathcal{U}(\mathcal{A})$ ,  $u^{-1} = u^* \in \mathcal{U}(\mathcal{A})$  and therefore  $\sigma(u)^{-1} = \sigma(u^{-1}) \subseteq \mathbb{D}$ . This implies  $\|\lambda\| = 1$  for all  $\lambda \in \sigma(u)$  and thus  $\sigma(u) \subseteq \mathbb{S}^1$ .

- (i) Assume that  $\mathcal{A}$  is unital, otherwise work in  $\tilde{\mathcal{A}}$ . If  $a$  is self-adjoint then  $u = \exp(ia) = \sum_{n=0}^{\infty} \frac{i^n a^n}{n!} \in \mathcal{U}(\mathcal{A})$  because  $\exp(ia)^* = \exp(-ia)$  and therefore  $u^*u = \exp(-ia)\exp(ia) = \exp(0) = 1 = uu^*$ . Because of (i) we know  $\sigma(u) \subseteq \mathbb{S}^1$ . Now, let  $\lambda \in \sigma(u)$  and define  $b = \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^n}{n!} = \exp(i(a-\lambda)) - 1$  as well as  $c = \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^{n-1}}{n!} \in \mathcal{A}$ . Consider

$$\begin{aligned} \exp(ia) - \exp(i\lambda 1) &= (\exp(i(a-\lambda)) - 1) \exp(i\lambda) = b \exp(i\lambda) \\ &= \left( \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^n}{n!} \right) \exp(i\lambda) \\ &= (a-\lambda) \left( \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^{n-1}}{n!} \right) \exp(i\lambda) \\ &= (a-\lambda) c \exp(i\lambda). \end{aligned}$$

Since  $\lambda \in \sigma(a)$  and  $c, (a-\lambda)$  commute,  $\exp(ia) - \exp(i\lambda)$  is not invertible (or  $a-\lambda$  would also be invertible) and we have  $\exp(i\lambda) \in \sigma(u) \subseteq \mathbb{S}^1$ . But for this to happen, we require  $\lambda \in \mathbb{R}$ .  $\square$

**Corollary 7.18** *If  $\mathcal{A}$  is a  $C^*$ -algebra and  $\chi \in \Omega(\mathcal{A})$ , then  $\chi(a^*) = \overline{\chi(a)}$  for all  $a \in \mathcal{A}$ . So  $\chi$  is a  $*$ -homomorphism.*

PROOF: If  $a \in \mathcal{A}$  is self-adjoint, then  $\chi(a) \in \sigma(a) \subseteq \mathbb{R}$  so  $\overline{\chi(a)} = \chi(a) = \chi(a^*)$ .

Now, if  $a \in \mathcal{A}$  is any element we can write it as  $a = b + ic$  where  $b = \frac{a+a^*}{2}$  and  $c = \frac{a-a^*}{2i}$  so that  $b, c$  are self-adjoint. Now  $\chi(b), \chi(c) \in \mathbb{R}$  so

$$\chi(a^*) = \chi(b - ic) = \chi(b) - i \cdot \chi(c) = \overline{\chi(b) + i\chi(c)} = \overline{\chi(b + ic)} = \overline{\chi(a)} \quad \square$$

**Corollary 7.19** *If  $\mathcal{A}$  is a commutative  $C^*$ -algebra and  $\mathcal{A} \neq 0$ , then  $\Omega(\mathcal{A}) \neq \emptyset$ .*

PROOF: If  $\mathcal{A} \neq 0$  there is some self-adjoint non-zero element  $a \in \mathcal{A}$  so that  $r(a) = \|a\| \neq 0$ . But  $\sigma(a) \subseteq \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\}$ . But for this to be true there must exist a character  $\chi \in \Omega(\mathcal{A})$ , so  $\Omega(\mathcal{A}) \neq \emptyset$ .  $\square$

PROOF (GELFAND):

- **$\Gamma$  is a  $*$ -homomorphism:** Consider

$$\Gamma(a)^*(\chi) = \hat{a}^*(\chi) = \overline{\hat{a}(\chi)} = \overline{\chi(a)} = \chi(a^*) = \hat{a}^*(\chi) = \Gamma(a^*)(\chi)$$

so  $\Gamma(a)^* = \Gamma(a^*)$ .

- **$\Gamma$  is isometric:** We have

$$\|\Gamma(a)\|^2 = \|\Gamma(a)^*\Gamma(a)\| = \|\Gamma(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$$

using our lemmas and the  $C^*$ -property.

- **$\Gamma$  is surjective:** Let  $\mathcal{B} := \text{im}(\Gamma) \subseteq C_0(\Omega(\mathcal{A}))$ . Then  $\mathcal{B}$  is a  $C^*$ -subalgebra of  $C_0(\Omega(\mathcal{A}))$ . Then

- $\mathcal{B}$  does not vanish at any point, i.e. for every point  $\chi \in \Omega(\mathcal{A})$  there is a  $b \in \mathcal{B}$  with  $\chi(b) \neq 0$ .

As  $\chi \in \Omega(\mathcal{A})$  means  $\chi \neq 0$ , there exists an  $a \in \mathcal{A}$  with  $\chi(a) \neq 0$ . But we can rewrite this as  $b(\chi) = \hat{a}(\chi) = \chi(a) \neq 0$  for  $b = \hat{a}$ .

- $\mathcal{B}$  separates points in  $\Omega(\mathcal{A})$ , i.e. for every  $\chi_1 \neq \chi_2$  in  $\Omega(\mathcal{A})$  there exists  $b \in \mathcal{B}$  with  $b(\chi_1) \neq b(\chi_2)$ .

If  $\chi_1 \neq \chi_2$  there exists  $a \in \mathcal{A}$  with  $\chi_1(a) \neq \chi_2(a)$ . Taking  $b = \hat{a}$  yields the result.

The result  $\mathcal{B} = C_0(\Omega(\mathcal{A}))$  follows from the Stone-Weierstraß-theorem:

If  $X$  is a locally compact Hausdorff space and  $B \subseteq C_0(X)$  is a  $*$ -subalgebra satisfying

- $B$  does not vanish on any point of  $X$
- $B$  separates points of  $\mathcal{A}$

then  $B$  is dense in  $C_0(X)$ .

So  $\text{im}(\Gamma)$  is dense and closed in  $C_0(\Omega(\mathcal{A}))$ , so  $\Gamma$  is surjective.  $\square$

**Proposition 7.20** *Conclusion: Every commutative  $C^*$ -algebra is (up to  $*$ -isomorphism) of the form  $C_0(X)$  for a locally compact Hausdorff space  $X$ . Let  $\mathcal{A} = C_0(X)$  for a locally compact Hausdorff space  $X$ . Then  $\Omega(\mathcal{A}) \simeq X$  with isomorphism*

$$\varphi : X \rightarrow \Omega(C_0(X)), x \mapsto (\text{ev}_x : C_0(X) \rightarrow \mathbb{C}, f \mapsto f(x)).$$

PROOF:

- $\varphi$  is **well-defined**, because characters are never zero.
- $\varphi$  is **continuous**. Take  $x_i \rightarrow x$  in  $X$ . Then, for all  $f \in C_0(X)$  we have  $\text{ev}_{x_i}(f) \rightarrow \text{ev}_x(f)$  because  $f$  is continuous and therefore  $f(x_i) \rightarrow f(x)$ . This shows  $\text{ev}_{x_i} \rightarrow \text{ev}_x$  in the weak  $*$ -topology.
- $\varphi$  is **injective**. If  $x_1 \neq x_2$  there exists a function  $f \in C_0(X)$  that separates them, but then  $\text{ev}_{x_1}(f) \neq \text{ev}_{x_2}(f)$ , so  $\text{ev}_{x_1} \neq \text{ev}_{x_2}$ .
- $\varphi$  is **surjective**. Prove that every  $\chi \in \Omega(\mathcal{A})$  is  $\chi = \text{ev}_x$  for some  $x \in X$ .

We know that the characters of  $\mathcal{A}$  are equivalent to the ideals in  $C_0(X)$ , so this is equivalent to: Every maximal ideal  $I \triangleleft C_0(X)$  is of the form  $I = C_0(X \setminus \{x_0\}) = \{f \in C_0(X) \mid f(x_0) = 0\}$ .

In Exercise 01-08 we have proven that every closed (2-sided) ideal  $I \triangleleft C_0(X)$  has the form  $I = C_0(U) := \{f \in C_0(X) \mid f|_{X \setminus U} \equiv 0\}$  for some open  $U \subseteq X$ .

See 01-08 for more details.

Take any  $f \in I \triangleleft C_0(X)$ . First, prove  $I^* = I$ . Consider  $f \in I$  and

$$f_n := \sqrt[n]{f^* f} = (\overline{f} f)^{\frac{1}{n}} = |f|^{\frac{2}{n}}.$$

We have  $f_n \in I$  for all  $n$ , because  $g := f^* f \in I$  and  $t \mapsto \sqrt[n]{t}$  is a continuous function that can be uniformly approximated by polynomials on the compact sets. It follows that  $f_n = \lim g_n$  where  $g_n$  is a polynomial in  $g \in I$ , so  $f_n \in I$ . So  $f^* f_n \in I$  for all  $n$ . Then

$$\begin{aligned} \|f^* = f_n f^*\|_\infty^2 &= \|(f^* - f_n f^*)(f^* - f_n f^*)\|_\infty = \|(f = f_n f)(f^* - f_n f^*)\|_\infty \\ &= \|f^* f - 2f^* f f_n + f_n^2 f^* f\|_\infty \end{aligned}$$



$$\leq \|g - g \sqrt[n]{g}\| + \|g - g \sqrt[n]{g}\| \|f_n\| \rightarrow 0,$$

because  $|g(x) - g(x) \sqrt[n]{g(x)}| \rightarrow 0$  pointwise (as the  $n$ -th square root converges to the 1 on the support and 0 elsewhere) and  $|g(x)| \leq \varepsilon$  everywhere except a compact set  $K$ , and on that  $K$  we have  $\sup_{x \in K} |g(x)| |1 - \sqrt[n]{g(x)}| = |g(x_0)| |1 - \sqrt[n]{g(x_0)}| < \varepsilon$  for some  $n \in \mathbb{N}$ . We therefore have  $f^* = \lim_{n \rightarrow \infty} f^* f_n \in I$  and thus  $f^* = \lim_{n \rightarrow \infty} f_n f^*$ . Now let  $I \triangleleft C_0(X)$  closed, so  $I^* = I$  and  $I$  is a  $C^*$ -subalgebra of  $X$ .

Define  $U^c := \{x \in X \mid f(x) = 0 \forall f \in I\}$ . This is closed (because for  $x_i \rightarrow x$  in  $X$ ,  $x_i \in U^c$ , we have  $0 = f(x_i) \rightarrow f(x)$ ), so  $U$  is open. We claim  $I = C_0(U)$ .

If  $f \in I$ ,  $f|_{U^c} \equiv 0$  per Definition, so  $f \in C_0(U)$ . Therefore,  $I$  is a closed subideal of  $C_0(U)$ .

$I$  does not vanish on  $U$ , because if there was an  $x \in U$  with  $f(x) = 0$  for all  $f \in I$ , we would have  $x \in U^c$ .

$I$  separates the points of  $U$ . Take  $x_1 \neq x_2$ . We can choose  $h \in C_0(X)$  with  $h(x_1) = 1$  and  $h(x_2) = 0$  (Uryson) as well as  $g \in I$  with  $g(x_1) \neq 0$ , then  $f = g \cdot h \in I$  separates  $x_1$  from  $x_2$ .

Stone-Weierstraß now proves  $I = C_0(U)$ .

Notice  $U \subseteq V \subseteq X$  (open) iff  $C_0(U) \subseteq C_0(V) \trianglelefteq C_0(X)$  (see exercise 08-01). So we have a bijection between the opens of  $X$  and the ideals of  $C_0(X)$ . Especially, the maximal ideals of  $C_0(X)$  correspond to the maximal open sets, that is the sets of form  $X \setminus \{x_0\}$  for some  $x_0$ , of  $X$ .

Therefore, if  $\chi \in \Omega(C_0(X))$  we have  $\ker \chi = C_0(X \setminus \{x_0\})$ , so  $\chi$  maps a function to 0 if and only if  $f$  is zero on  $x$ . This proves and  $\chi = \text{ev}_x$ .

- $\varphi$  is **open**. If  $X$  is compact, this is clear because  $C_0(X) = C(X)$  and unital, so  $\Omega(C_0(X))$  is compact and we have a bijection between two compact sets. In general, consider  $\tilde{X}$  (the compactification) and use  $\widetilde{C_0(\tilde{X})} \simeq C(\tilde{X})$ . So we have a homeomorphism

$$\tilde{X} \rightarrow \Omega(C(\tilde{x})) = \Omega(\widetilde{C_0(\tilde{X})}) \simeq \Omega(C_0(X)) \sqcup \{\chi_\infty\}$$

where  $\infty \mapsto \chi_\infty$ , so we can restrict the homeomorphism to  $X$  and are done.  $\square$

**Theorem 7.21 (Spectral inclusion for  $C^*$ -algebras)** *Let  $\mathcal{A} \subseteq \mathcal{B}$  be an inclusion of unital  $C^*$ -algebras with  $1 = 1_{\mathcal{A}} = 1_{\mathcal{B}}$ . Then for all  $a \in \mathcal{A}$  we have  $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ , so  $\text{inv}(\mathcal{A}) = \text{inv}(\mathcal{B}) \cap \mathcal{A}$ .*

PROOF: If  $a$  is self-adjoint, that is  $a^* = a$ , then  $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$ , so  $\sigma_{\mathcal{A}}$  has no holes, i.e. the complement  $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$  is connected in  $\mathbb{C}$ . By the general result on Banach algebras  $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$ . In particular, this implies  $a \in \text{inv}(\mathcal{A}) \Leftrightarrow a \in \text{inv}(\mathcal{B})$  for all self-adjoint  $a \in \mathcal{A}$ .

We now prove that this holds for all  $a \in \mathcal{A}$ . Of course,  $\text{inv}(\mathcal{A}) \subseteq \text{inv}(\mathcal{B}) \cap \mathcal{A}$ . Let  $a \in \mathcal{A}$  such that  $a \in \text{inv}(\mathcal{B})$ . Then there exists  $b \in \mathcal{B}$  such that  $ab = ba = 1$  and  $b^*a^* = a^*b^* = 1 \Leftrightarrow bb^*a^*a = 1 = a^*abb^*$ . Therefore,  $a^*a \in \text{inv} \mathcal{B} \cap \mathcal{A} \subseteq \text{inv}(\mathcal{A})$  because  $a^*a$  is self adjoint. So there exists  $c \in \text{algebra } \mathcal{A}$  with  $ca^*a = 1 = a^*ac$  and thus  $ca^*ab = ca^* = b$ , so  $b \in \mathcal{A}$  as it is the product of two elements  $a^*, c \in \mathcal{A}$ . This concludes the proof, as  $a$  is now invertible in  $\mathcal{A}$ .  $\square$

**Definition 7.22** *We say  $a \in \mathcal{A}$  (for  $\mathcal{A}$  a  $C^*$ -algebra) is **normal** if  $a^*a = aa^*$ . This means  $C^*(a)$  (the  $C^*$ -subalgebra of  $\mathcal{A}$  generated by  $a$ ) is commutative. Then  $C^*(a) \simeq C_0(X)$ .*

**Lemma 7.23** Let  $a \in \mathcal{A}$  ( $C^*$ -algebra) be a normal element. Assume that  $1 \in \mathcal{A}$  (unital). Then  $\Omega(C^*(a, 1)) \simeq \sigma(a)$  by homeomorphism  $\chi \mapsto \chi(a)$ . In general, if  $\mathcal{A}$  is possibly not unital, then  $\Omega(C^*(a)) \simeq \sigma(a) \setminus \{0\}$ . In particular,  $\chi(a) = 0$  only if  $a = 0$  but then  $C^*(a)$  is just the zero space.

PROOF: It is enough to consider the unital case.

Consider  $\varphi : \Omega(C^*(a, 1)) \rightarrow \sigma(a), \chi \mapsto \chi(a)$  which is well-defined because  $\chi(a) \in \sigma(a)$ .

- $\varphi$  is **continuous**. If  $\chi_i \rightarrow \chi$  in  $\Omega(C^*(a, 1))$  then this also converges point wise, so  $\chi_i(a) \rightarrow \chi(a)$ .
- $\varphi$  is **injective**. Take  $\chi_1, \chi_2 \in \Omega(C^*(a, 1))$  with  $\chi_1(a) = \chi_2(a)$ . Since  $\chi_1(1) = 1 = \chi_2(1)$ , so the two characters coincide on the generators and are thus equal by linearity and continuity.
- $\varphi$  is **surjective**. We know that  $\sigma(a) = \{\chi(a) \mid \chi \in \Omega(B)\}$  for all commutative unital Banach algebras  $B$ , in particular for  $B = C^*(a, 1)$ .

Because both spaces are compact this concludes the proof.  $\square$

**Theorem 7.24 (Fundamental theorem of continuous functional calculus)**

Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $a \in \mathcal{A}$  normal. Then there exists a unique unital  $*$ -homomorphism  $\varphi : C(\sigma(a)) \rightarrow \mathcal{A}$  such that  $\text{id}_{\sigma(a)} \mapsto a$ .

In general, if  $\mathcal{A}$  is possibly not unital, there exists a unique  $*$ -homomorphism  $\varphi : C_0(\sigma(a)) \rightarrow \mathcal{A}$  where  $C_0(\sigma(a)) := \{f \in C(\sigma(a)) \mid f(0) = 0\}$ .

Both of these morphisms are also isometric.

Notation: If  $f \in C(\sigma(a))$  we write  $f(a) := \varphi(a)$ . Notice: If  $f$  is a polynomial in  $z, \bar{z}$  then  $f(a) = \varphi(a)$  as usual.

PROOF: Consider  $1 \in \mathcal{A}$  and let  $\mathcal{B} = C^*(a, 1) \subseteq \mathcal{A}$ . Then  $\mathcal{B}$  is commutative because  $a$  is normal (i.e. commutes with its adjoint). By Gelfand, we get an isometric  $*$ -isomorphism  $T : \mathcal{B} \rightarrow C(\Omega(\mathcal{B})), b \mapsto \hat{b}$ . By the Lemma,  $\Omega(\mathcal{B}) \equiv \sigma(a), \chi \mapsto \chi(a)$ . Via this identification (homeomorphism), we have  $\hat{b}(\chi) = \chi(b)$  and  $\hat{a}(\chi) = \chi(a)$ . So  $\hat{a}$  corresponds to  $z \in C(\sigma(a)) \simeq C(\Omega(\mathcal{B}))$ . Therefore, considering the inverse of  $T$  and identifying  $\Omega(\mathcal{B}) \simeq \sigma(a)$  we get an isometric

$$C(\sigma(a)) \simeq C(\Omega(C^*(a, 1))) \simeq C^*(a, 1) \simeq \mathcal{A}.$$

This gives  $\varphi$  as defined.

The **non-unital case**: Just consider  $\tilde{\mathcal{A}}$ .  $\square$

**Example 7.25** Let  $f(z) = \exp(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ .  $f$  is a continuous function on the whole plane. If  $a \in \mathcal{A}$  is normal, then  $f(a) = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ . In general,  $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$  (or  $f(z) = \sum_{n=0}^{\infty} \lambda_n (z - z_0)^n$ ), so  $f(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$  if  $\sigma(a) \subseteq \text{Domain}(f)$ .

**Theorem 7.26** Let  $\mathcal{A}$  be unital  $C^*$ -algebra and  $a \in \mathcal{A}$  be normal. If  $f \in C(\sigma(a))$ , then  $\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\}$ .

Moreover, if  $g \in C(\sigma(f(a)))$ , then  $g(f(a)) = (g \circ f)(a)$ .

PROOF: Let  $\mathcal{B} = C^*(a, 1) \subseteq \mathcal{A}$ .  $\mathcal{B}$  is commutative and unital. Then  $f(a) \in \mathcal{B}$  and  $\sigma(f(a)) = \sigma_{\mathcal{B}}(f(a))$ . Now notice  $\chi(f(a)) = f(\chi(a))$  since both maps

$$f \mapsto \chi(f(a))$$

$$f \mapsto f(\chi(a))$$

are unital  $*$ -homomorphisms that coincide on  $z$ . Therefore,

$$\sigma(f(a)) = \{\chi(f(a)) \mid \chi \in \Omega(\mathcal{B})\} = \{f(\chi(a)) \mid \chi \in \Omega(\mathcal{B})\} = f(\sigma(a)).$$

Now to prove  $(g \circ f)(a) = g(f(a))$ . Let  $C = C^*(1, f(a)) \subseteq \mathcal{B} = C^*(1, a) \subseteq \mathcal{A}$ . Let  $\chi \in \Omega(\mathcal{B})$ . Then  $\chi_C := \chi|_C \in \Omega(C)$ . So  $(g \circ f)(a)$  is sensibly defined and an element of  $\mathcal{B}$ , so we can apply a character:

$$\begin{aligned} \chi((g \circ f)(a)) &= (g \circ f)(\chi(a)) = g(f(\chi(a))) = g(\chi(f(a))) = g(\chi_C(f(a))) \\ &= \chi_C(g(f(a))) = \chi(\underbrace{g(f(a))}_{\in \mathcal{B}}) \end{aligned}$$

Because the Gelfand-transform is injective, this implies  $(g \circ f)(a) = g(f(a))$ .  $\square$

**Proposition 7.27** *Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $u \in \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid u^*u = 1 = uu^*\}$ . If  $\sigma(u) \neq \mathbb{S}^1$  there exists a self-adjoint  $a \in \mathcal{A}$  with  $u = \exp(ia)$ .*

PROOF: The idea is to take  $\log \approx \exp^{-1}$ . Problem:  $\exp$  is not invertible as a complex function, because it is  $2\pi i$ -periodic. We will need to restrict it. Consider the principal branch of the logarithm,  $\log(z) = \log|z| + i \arg(z)$ .

Given that  $\sigma(u) \neq \mathbb{S}^1$ , there exists an  $\lambda \in \mathbb{S}^1 \setminus \sigma(u)$  and therefore also an  $f_\lambda \in C(\mathbb{S}^1 \setminus \{\lambda\})$  (so some form of argument-mapping of  $z$ ) such that  $\exp(if_\lambda(z)) = z$ . This  $f_\lambda$  is real-valued, continuous and analytical. Now use functional calculus: Let  $a := f_\lambda(u) \in \mathcal{A}$ . Since  $f_\lambda$  is real-valued, it is self-adjoint in the algebra, so  $a$  is also self-adjoint. By the previous theorem  $\exp(ia) = \exp(if_\lambda(u)) = (\exp \circ if_\lambda)(u) = u$ .

## Multiplier Algebras

This is another kind of unitization. We will consider  $\mathcal{A} \rightarrow M(\mathcal{A}) \ni \mu$  such that  $\mu \cdot a \in \mathcal{A} \ni a \cdot \mu$  so  $\mathcal{A} \trianglelefteq M(\mathcal{A})$ . Remember that this was the case for the usual unitization, with Quotient  $\mathbb{C}$ . Here, the multiplier is usually much bigger, so the quotient is as well. In fact,  $\mathcal{A} \times \mathbb{C}$  is the 'smallest' unitization while  $M(\mathcal{A})$  is the 'largest' one.

**Definition 7.28 (Multiplier, see Murphy)** *Let  $\mathcal{A}$  be an algebra. A **multiplier** of  $\mathcal{A}$  is a pair  $\mu = (L, R)$  where  $L, R : \mathcal{A} \rightarrow \mathcal{A}$  are linear maps such that*

- (i)  $L(ab) = L(a) \cdot b$  or  $\mu(ab) = (\mu a)b$
- (ii)  $R(ab) = a \cdot R(b)$  or  $(ab)\mu = a(b\mu)$
- (iii)  $a \cdot L(b) = R(a) \cdot b$  or  $a(\mu b) = (a\mu)b$ .

To simplify this, use the notation  $\mu \cdot a := L(a)$  and  $a \cdot \mu := R(a)$ .

For the space of all multipliers we write  $M(\mathcal{A}) = \{\mu = (L, R) \mid \mu \text{ multiplier}\}$ . This is a  $\mathbb{C}$ -vector space with

$$(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2) \quad \lambda(L_1, R_1) = (\lambda L_1, \lambda R_2)$$

and an algebra with

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 \cdot L_2, R_2 \cdot R_1).$$

If  $\mathcal{A}$  is a  $*$ -algebra, we further define

$$(L, R)^* = (R^*, L^*) \text{ where } L^*(a) := L(a^*)^* \text{ and } R^*(a) := R(a^*)^*$$

Moreover, we have a canonical  $(*)$ -homomorphism  $\iota : \mathcal{A} \rightarrow M(\mathcal{A}), a \mapsto (L_a, R_a)$  where  $L_a(b) = ab$  and  $R_a(b) = ba$ . Note:  $\iota$  is always a  $(*)$ -homomorphism but injective if and only if

$$\begin{aligned} \forall_{a \in \mathcal{A}} \quad a \cdot b = 0 \quad \forall_b \Rightarrow a = 0 \\ b \cdot a = 0 \quad \forall_b \Rightarrow a = 0 \end{aligned}$$

i.e.  $\mathcal{A}$  is an essential ideal of itself. This is not always true for a general algebra, consider the algebra with the 0-product  $a \cdot b = 0$ , but it always holds for  $C^*$ -algebras or if  $\mathcal{A}$  is unital already.

More generally this holds if  $\mathcal{A}$  is a Banach algebra with an **approximate unit**, a net  $e_i \subseteq \mathcal{A}$  such that  $e_i a \rightarrow a$  and  $a \cdot e_i \rightarrow a$  for any  $a \in \mathcal{A}$  as well as  $\|e_i\|$ . This is always the case for unital and  $C^*$ -algebras.

Assume  $\iota$  is injective. Then  $\mathcal{A}$  is identified with an essential  $(*)$ -ideal of  $M(\mathcal{A})$ .

**Remark 7.29 (Norms on the multiplier)** If  $\mathcal{A}$  is a Banach algebra with an approximate unit, we define for  $\mu = (L, R) \in M(\mathcal{A})$  the norm

$$\|\mu\| := \|L\| = \|R\| < \infty.$$

PROOF: To show  $\|L\|, \|R\| < \infty$  we use the Closed Graph Theorem. Say we have  $(a_n) \subseteq \mathcal{A}$  with  $a_n \rightarrow a$  and  $L(a_n) \rightarrow b$ . Take  $c \in \mathcal{A}$  and consider

$$c \cdot L(a) = R(c) \cdot a = \lim_{n \rightarrow \infty} R(c) \cdot a_n = \lim_{n \rightarrow \infty} c \cdot L(a_n) = c \cdot b.$$

Because of the approximate unit (or  $\iota$  injective) we have  $L(a) = b$ . This shows that  $L$  (and, analogously,  $R$ ) are bounded. Now to prove  $\|L\| = \|R\|$ . Take any  $a \in \mathcal{A}$  and consider

$$\|L(a)\| \stackrel{\text{approx. unit}}{=} \sup_{\|b\| \leq 1} \|bL(a)\| = \sup_{\|b\| \leq 1} \|R(b)a\| \leq \sup_{\|b\| \leq 1} \|R(b)\| \|a\| \leq \|R\| \cdot \|a\|$$

which implies  $\|L\| \leq \|R\|$ . By symmetry of the situation, we have  $\|L\| = \|R\|$ .  $\square$

With the norm above,  $M(\mathcal{A})$  becomes a Banach algebra.

**Proposition 7.30** *If  $\mathcal{A}$  is a  $C^*$ -algebra then  $M(\mathcal{A})$  is too.*

PROOF: Write  $\mu = (L, R)$ . We compute  $\mu^* \mu = (R^*, L^*) \cdot (L, R) = (R^* L, R L^*)$ . So  $\|\mu \mu^*\| = \|R^* L\|$ . Take  $a \in \mathcal{A}$  with  $\|a\| \leq 1$ . Then

$$\|L(a)\|^2 = \|L(a)L(a)^*\| = \|L(a)L^*(a^*)\| = \|R^*(L(a))a^*\| \leq \|R^*(L(a))\| \leq \|R^* L\|$$

This shows  $\|L\|^2 \leq \|R^* L\|$  and therefore  $\|\mu\|^2 = \|L\|^2 \leq \|R^* L\| = \|\mu^* \mu\|$ . Because  $\|\mu\|^2 \geq \|\mu \mu^*\|$  is clear by submultiplicativity, the  $C^*$ -property follows.  $\square$

Compare now  $\tilde{\mathcal{A}}$  and  $M(\mathcal{A})$ . We have  $\mathcal{A} \trianglelefteq \tilde{\mathcal{A}}$  and  $\mathcal{A} \trianglelefteq M(\mathcal{A})$ . When are these ideals essential?

**Lemma 7.31** *Let  $\mathcal{A}$  be a  $C^*$ -algebra or Banach algebra with approximate unit.  $\mathcal{A} \trianglelefteq \tilde{\mathcal{A}}$  if and only if  $\mathcal{A}$  is not unital.*

PROOF: Suppose that  $\mathcal{A}$  is unital with  $1_{\mathcal{A}}$  as the unit. In this case, take  $p = 1 - 1_{\mathcal{A}} \in \tilde{\mathcal{A}}$  (where  $1 = (0, 1)$  is the unit in  $\tilde{\mathcal{A}}$ ). Notice that  $p \cdot \mathcal{A} = 0$ , but  $p \neq 0$ . So  $\mathcal{A}$  is not essential in  $\tilde{\mathcal{A}}$ .

Suppose that  $\mathcal{A}$  is not unital. To prove: For  $a + \lambda \cdot 1 \in \tilde{\mathcal{A}}$  and  $(a + \lambda \cdot 1)\mathcal{A} = 0$  we have  $a = 0$ ,  $\lambda = 0$ . So take any  $(a + \lambda \cdot 1) \cdot b = 0$  for all  $b \in \mathcal{A}$ , that is  $ab + \lambda b = 0$ . This means  $L_a(b) = -\lambda b$ , that is  $L_a = -\lambda \text{id}_{\mathcal{A}}$ . Notice  $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A})$ , a unital algebra with unit  $\text{id}_{\mathcal{A}}$ , is an injective (because  $\iota$  is injective) algebra homomorphism. If  $\lambda \neq 0$ , then division by  $\lambda$  implies  $\text{id}_{\mathcal{A}} \in \text{im}(L) \simeq \mathcal{A}$ . But then  $\mathcal{A}$  has a unit, a contradiction. So  $\lambda = 0$ . Then  $a \cdot b = 0$  for every  $b$ , so  $a = 0$  as well. This shows that  $\mathcal{A}$  is an essential ideal of  $\tilde{\mathcal{A}}$ .  $\square$

**Remark 7.32** Let  $\mathcal{A}$  be a  $C^*$ -algebra or Banach algebra with approximate unit. Then  $\mathcal{A}$  is unital if and only if  $M(\mathcal{A}) = \mathcal{A}$ .

PROOF: One direction is simple:  $M(\mathcal{A})$  is always unital, so  $\mathcal{A} \simeq M(\mathcal{A})$  implies that  $\mathcal{A}$  is unital.

Let now  $\mathcal{A}$  be unital and prove that every multiplier is of the form  $(L_a, R_a)$ . Let  $\mu = (L, R) \in M(\mathcal{A})$  and define  $a := L(1_{\mathcal{A}})$ . Then  $L_a(b) = ab = L(1_{\mathcal{A}})b = L(b)$ , so  $L = L_a$ . Analogously we can prove  $R = R_a$ . This shows that  $\iota$  is surjective, and since it is already injective (because  $\mathcal{A}$  is either  $C^*$  or has an approximate unit) it is an isomorphism.  $\square$

Say  $\mathcal{A}$  is a  $C^*$ -algebra (or a Banach algebra with an approximate unit) and not unital. Then  $\iota : \mathcal{A} \rightarrow M(\mathcal{A}), a \mapsto \mu_a = (L_a, R_a)$  extends to a  $(*)$ -embedding

$$\tilde{\iota} : \tilde{\mathcal{A}} \rightarrow M(\mathcal{A}), a + \lambda \cdot 1 \mapsto \iota(a) + \lambda \cdot \underbrace{(\text{id}, \text{id})}_{=\text{id}_{M(\mathcal{A})}}.$$

More generally: If  $\mathcal{B}$  is any  $C^*$ -algebra that contains  $\mathcal{A}$  as an essential ideal (closed), then  $\mathcal{B}$  embeds in the multiplier algebra via the following map:

$$\lambda : \mathcal{B} \rightarrow M(\mathcal{A}), b \mapsto (L_b, R_b)$$

where  $L_b, R_b$  are the usual left and right multiplication. We have  $L_b(a), R_b(a) \in \mathcal{A}$  for any  $a \in \mathcal{A}$  because  $\mathcal{A}$  is an ideal. The above is a universal property of the multiplier algebra.  $M(\mathcal{A})$  is the largest unital  $C^*$ -algebra that contains  $\mathcal{A}$  as an essential ideal.

**Example 7.33** Take  $\mathcal{A} = C_0(X)$  (for a locally compact Hausdorff-space, so a commutative  $C^*$ -algebra). Then  $\tilde{\mathcal{A}} = C(\tilde{X})$  where  $\tilde{X} = X \sqcup \{\infty\}$ . One can now show  $M(\mathcal{A}) \simeq C(\beta X)$  where  $\beta X$  is the Stone-Cech-compactification of  $X$ . This can be proven using the universal property and the universal property of  $\beta X$ :  $\beta X$  is a compact Hausdorff space such that  $X \hookrightarrow \beta X$  as a dense open topological subspace and for every other compact Hausdorff space  $K$  such that  $X \rightarrow K$  via a continuous function  $f$  there exists a unique continuous extension  $\beta f : \beta X \rightarrow K$ .

First: Prove that  $M(\mathcal{A})$  is even commutative. Then it is the continuous functions on some space, use the spectrum and compare the universal properties. For commutativity, one can show  $M(C_0(X)) \simeq C_b(X)$  via the universal property.

## 7.2 Positive Elements of $C^*$ -algebras

**Definition 7.34** Let  $\mathcal{A}$  be a  $C^*$ -algebra. We say that  $a \in \mathcal{A}$  is positive (and write  $a \geq 0$ ) if  $a = a^*$  and  $\sigma(a) \subseteq [0, \infty)$ .

The set of all positive elements of a given algebra we notate as  $\mathcal{A}_+$ .

**Example 7.35** Let  $A = C_0(X)$  (commutative) and  $f \in \mathcal{A}$ . Then  $f = f^*$  iff  $f$  is real (that is  $f : X \rightarrow \mathbb{R}$ ). Since  $\sigma(f) = \overline{\text{im}(f)}$  we see that  $f \geq 0$  iff  $f(x) \geq 0$  for all  $x \in X$ .

**Theorem 7.36** If  $a \in \mathcal{A}$  for  $\mathcal{A}$  a  $C^*$ -algebra and  $a \geq 0$  then there exists a unique  $b \in \mathcal{A}_+$  such that  $b^2 = a$ . We sometimes notate this as  $b = \sqrt{a} = a^{\frac{1}{2}}$ .

PROOF: Since  $a$  is positive, it is self-adjoint and therefore normal. Continuous functional calculus:

$$\varphi : C_0(\sigma(a)) \rightarrow \mathcal{A}, f \mapsto f(a)$$

Apply this to  $f(x) = \sqrt{x}$ . Notice that  $f \in C_0(\sigma(a))$  because  $\sigma(a) \subseteq [0, \infty)$ . Now simply choose  $b = f(a) = \sqrt{a}$ . Since  $\varphi$  is a  $*$ -homomorphism, we have  $b^2 = \varphi(f)^2 = \varphi(f^2) = \varphi(\text{id}) = a$ .

Reminder: Writing ' $f(a)$ ' does not mean to imply that  $a \in \mathcal{A}$  can simply be plugged into the function  $f : \sigma(\mathcal{A}) \rightarrow \mathbb{C}$  but is simply a different way of writing  $\varphi(f) \in \mathcal{A}$ .

**Uniqueness:** Suppose  $c \in \mathcal{A}_+$  such that  $c^2 = a$ . Then  $c$  commutes with  $c^2 = a$  and therefore  $c$  commutes with  $b = \sqrt{a}$  since  $b = \lim_{n \rightarrow \infty} p_n(a)$  (polynomial approximation). Then  $B := C^*(b, c) \subseteq \mathcal{A}$  is a commutative  $C^*$ -algebra so  $B \simeq C_0(X)$  for some locally compact Hausdorff space  $X$ . Since  $a, b, c \in B = C_0(X)$  we have  $a \simeq f, b \simeq g, c \simeq h \in C_0(X)$  with  $f = g^2 = h^2$  where all these functions are positive. But then  $f(x) = g(x)^2 = h(x)^2$  for all  $x$ . Because  $g(x), h(x) \geq 0$  for all  $x$ , this shows  $g(x) = h(x)$  for all  $x$  and therefore  $g = h$  and  $b = c$ .  $\square$

**Remark 7.37** Given any self-adjoint element  $a \in \mathcal{A}$  ( $a^* = a$ ) we can write it as  $a^+ - a^-$  where  $a^+, a^- \geq 0$  and  $a^+ \cdot a^- = 0$ . Just define  $f(x) = \frac{|x|+x}{2}$  and  $g(x) = \frac{|x|-x}{2}$ . Both are positive functions with  $f \cdot g = 0$ . Define  $a^+ = f(a)$  and  $a^- = g(a)$  (once again per continuous functional calculus), transferring the necessary properties:

$$\begin{aligned} f(a) - g(a) &= \varphi(f) - \varphi(g) = \varphi(f - g) = \varphi(\text{id}) = a \\ f(a) \cdot g(a) &= \varphi(f) \cdot \varphi(g) = \varphi(f \cdot g) = \varphi(0) = 0 \\ \sigma(f(a)) &= \sigma(\varphi(f)) \subseteq \sigma(f) = \overline{\text{im}(f)} = [0, \infty) \end{aligned}$$

**Remark 7.38** If  $\mathcal{A}$  is unital  $C^*$ -algebra and  $a \in \mathcal{A}$  is self-adjoint with  $\|a\| \leq 1$ , so  $\sigma(a) \subseteq [-1, 1]$ . Define

$$f(x) = x + i\sqrt{1-x^2} \quad g(x) = x - i\sqrt{1-x^2}$$

This means that  $f, g \in \mathcal{UC}(\sigma(a))$  (Recall that unitaries of  $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid u^*u = 1 = uu^*\}$ ) and  $\frac{f+g}{2} = \text{id}_{\sigma(a)}$ . So if we now define  $u := f(a), v := g(a) \in C^*(a, 1) \subseteq \mathcal{A}$  we have  $\frac{u+v}{2} = a$ . In particular  $\mathcal{A} = \text{span}(\mathcal{U}(\mathcal{A}))$ .

**Lemma 7.39** Let  $\mathcal{A}$  be a unital  $C^*$ -algebra,  $a \in \mathcal{A}$  self-adjoint and  $t \in \mathbb{R}_+$ .

- (i) If  $a \geq 0$  and  $\|a\| \leq t$  then  $\|a - t\| \leq t$ .
- (ii) Conversely, if  $\|a - t\| \leq t$  then  $a \geq 0$ .

PROOF: Replace  $\mathcal{A}$  by  $C^*(a, 1)$  we may assume that  $\mathcal{A} = C(X)$  is commutative and  $X$  compact. Let  $a = f \in C(X)$  be a self-adjoint, real function and  $t \geq 0$  a real number.

- (i)  $f \geq 0$  and  $\|f\|_\infty \leq t$  and thus  $f(x) - t \in [-t, 0]$  for all  $x \in X$ , so  $\|f - t\| \leq t$ .
- (ii) Let  $f \in C(X)$  be a self-adjoint real function with  $\|f - t\| \leq t$ , so  $|f(x) - t| \leq t$  for every  $x$ . But if  $f(x) < 0$  for any  $x \in X$  we have  $f(x) - t < -t$  and thus  $|f(x) - t| > t$ , a contradiction. So  $f$  must be positive.  $\square$

**Corollary 7.40** If  $\mathcal{A}$  is a  $C^*$ -algebra, then  $\mathcal{A}_+$  is a closed subset (but not subspace!) of  $\mathcal{A}$ .

PROOF: Taking unitization, we may assume that  $\mathcal{A}$  is unital. Let  $(a_n) \subseteq \mathcal{A}_+$  and  $a_n \rightarrow a \in \mathcal{A}$ . Then  $a_n^* = a_n$  for all  $n \in \mathbb{N}$  and therefore  $a$  is also self-adjoint. There also exists  $t \geq 0$  with  $\|a_n\| \leq t$  for all  $n \in \mathbb{N}$  and by the Lemma  $\|a_n - t\| \leq t$  and therefore  $\|a - t\| \leq t$ . Again by the Lemma  $a \geq 0$ .  $\square$

**Corollary 7.41** *If  $\mathcal{A}$  is a  $C^*$ -algebra and  $a, b \in \mathcal{A}_+$  then  $a + b \in \mathcal{A}_+$ .*

PROOF: Taking unitization, we may assume that  $\mathcal{A}$  is unital. Since  $a, b \geq 0$  by  $t = \|a\|, \|b\|$  we have  $\|a - \|a\|\| \leq \|a\|$  and  $\|b - \|b\|\| \leq \|b\|$ . Then

$$\|(a + b) - (\|a\| + \|b\|)\| = \|(a - \|a\|) + (b - \|b\|)\| \leq \|(a - \|a\|)\| + \|(b - \|b\|)\| \leq \|a\| + \|b\|$$

and  $a + b$  is positive by the lemma.  $\square$

**Theorem 7.42** *If  $\mathcal{A}$  is a  $C^*$ -algebra and  $a \in \mathcal{A}$  then  $a^*a \geq 0$ .*

PROOF: First, we prove that if  $-a^*a \geq 0$  then  $a = 0$ . For this we use the following observation  $\sigma(bc) \setminus \{0\} = \sigma(cb) \setminus \{0\}$  (the two sets are equal except for the zero, which may be contained in one but not the other) because for  $b, c$  in a unital algebra and  $1 - bc \in \text{inv } \mathcal{A}$  iff  $1 - cb \in \text{inv}(\mathcal{A})$  and if  $d := (1 - bc)^{-1}$  then  $(1 - cb)^{-1} = 1 + cdb$ .

Therefore, if  $-a^*a \in \mathcal{A}_+$  then also  $-a^*a \in \mathcal{A}_+$  (notice that  $a, a^*$  are self-adjoint). Then write  $a = b + c$  with  $b, c \in \mathcal{A}$  self-adjoint. Then

$$a^*a + aa^* = (b - ic)(b + ic) + (b + ic)(b - ic) = b^2 + c^2 + ibc - icb + b^2 + c^2 + icb - icb = 2b^2 + 2c^2.$$

and we can write  $a^*a = 2b^2 + 2c^2 - aa^*$ . The squares are certainly positive and we have assumed  $-aa^* \geq 0$ , but then  $a^*a \geq 0$ . We see that  $aa^* \geq 0$  as well, so the spectrum has to be zero.

Now suppose that  $a \in \mathcal{A}$  arbitrarily. We show that  $a^*a \geq 0$ . Let  $b := a^*a$ . Then  $b \in \mathcal{A}$  is self-adjoint with  $b = b^+ - b^-$  where  $b^+, b^- \geq 0$ . Let  $c := ab^-$ . Then

$$-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \geq 0$$

and  $c$  must be 0 by our first result. This implies  $(b^-)^3 = 0$  so  $b^- = 0$ . It follows that  $b = b^+ \geq 0$ .  $\square$

**Definition 7.43** *Let  $\mathcal{A}$  be a self-adjoint algebra and  $a, b \in \mathcal{A}$ . We write  $a \leq b$  if  $b - a \geq 0$ . This turns  $\mathcal{A}$  into a poset. Because  $A_+$  is a cone, that is  $A_+ + A_+ \subseteq A_+$  and  $\mathbb{R}_+ \cdot A_+ \subseteq A_+$  as well as  $A_{\text{self-adjoint}} = A_+ - A_+$  and  $A_+ \cap -A_+ = \{0\}$ .*

**Theorem 7.44** *Let  $\mathcal{A}$  be a  $C^*$ -algebra.*

- (i)  $A_+ = \{a^*a \mid a \in \mathcal{A}\}$
- (ii)  $a, b$  self-adjoint and  $c \in \mathcal{A}$ . Then  $a \leq b$  implies  $c^*ac \leq c^*bc$ .
- (iii)  $0 \leq a \leq b$  implies  $\|a\| \leq \|b\|$
- (iv) If  $\mathcal{A}$  is unital and  $a, b \geq 0$  with  $a \leq b$  and  $a, b \in \text{inv}(\mathcal{A})$  then  $b^{-1} \leq a^{-1}$ .

PROOF:

- (i) It follows from the previous theorem. The fact that  $a \in \mathcal{A}_+$  has a square root  $a = b^2 = b^*b$  with  $b \geq 0$ .
- (ii)  $c^*bc - c^*ac = c^*(b - a)c$  and if we set  $b - a = d^*d$  for a  $d \in \mathcal{A}$  we receive  $c^*(b - a)c = c^*d^*dc = (dc)^*dc \geq 0$ .

(iii) We may assume  $1 \in \mathcal{A}$ . Notice that  $b \leq \|b\| \cdot 1$  (consider the commutative case). So we have  $a \leq b \leq \|b\| \cdot 1$  and therefore  $a \leq \|b\| \cdot 1$  so  $\|a\| \leq \|b\|$ .

(iv) Let  $a, b \in \text{inv } \mathcal{A}$ ,  $a, b \geq 0$  and  $a \leq b$ . We know that  $\sigma(b^{-1}) = \sigma(b)^{-1} \subseteq \mathbb{R}_+$  and thus  $b^{-1} \geq 0$  and Similarly  $a^{-1} \geq 0$ . Notice that if  $c \geq 1$  (in  $\mathcal{A}$ ) then  $c \in \text{inv } \mathcal{A}$  (as  $\sigma(c - 1) \subseteq [0, \infty)$  and thus  $\sigma(c) \subseteq [1, \infty)$ ) and  $c^{-1} \leq 1$  (think once again commutative).

Now we have  $a \leq b$ . Then  $1 = a^{-\frac{1}{2}} a a^{-\frac{1}{2}} \leq a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$ . Then  $(a^{-\frac{1}{2}} b a^{-\frac{1}{2}})^{-1} = (a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})^{-1} \leq 1$  by the above, so conjugation yields  $b^{-1} \leq a^{-1}$ .  $\square$

### 7.3 Approximate units

**Definition 7.45** Let  $\mathcal{A}$  be a Banach algebra. An **approximate unit** for  $\mathcal{A}$  is a net  $(e_i)_{i \in I} \subseteq \mathcal{A}$  such that  $\|e_i\| \leq 1$  and  $e_i a \rightarrow a, a e_i \rightarrow a$  for all  $a \in \mathcal{A}$ . If  $\mathcal{A}$  is a  $C^*$ -algebra, then we (usually) also assume that  $e_i \geq 0$  and  $(e_i)$  is increasing.

**Example 7.46** Let  $\mathcal{A} = C_0(X)$  be a commutative  $C^*$ -algebra ( $X$  locally compact and Hausdorff). Then a net  $(f_i)_{i \in I}$  is an approximate unit if and only if  $1 \geq f_i(x) \geq f_j(x) \geq 0$  for all  $x \in X$  and  $j \leq i$  and  $f_i g \rightarrow g$  for all  $g \in C_0(X)$ , that is  $f_i(x)g(x) \rightarrow g(x)$  uniformly on  $X$ . This is equivalent to  $f_i(x) \rightarrow 1$  uniformly on compacts.

**Example 7.47** Let  $\mathcal{A} = \mathcal{K}(H)$ , the span of the compact operators on a Hilbert space  $H$ , and use physics notation:  $|\xi\rangle\langle\eta|(\zeta) = \xi\langle\eta, \zeta\rangle$ . Let  $(\xi_i)_{i \in I} \subseteq H$  be an orthonormal basis. For each  $F \subseteq I$  finite we define

$$e_F := \sum_{i \in F} |\xi_i\rangle\langle\xi_i| \in \mathcal{K}(H)$$

In particular,  $0 \leq e_F \leq 1$  (because  $\|e_F\| \leq 1$ ) and  $e_F \leq e_G$  if  $F \subseteq G$ . Then  $(e_F)_{F \subseteq I \text{ finite}}$ , if ordered by size, is an approximate unit of for  $\mathcal{K}$ .

If  $H$  is separable, we could also take  $e_n = \sum_{i=1}^n |\xi_i\rangle\langle\xi_i|$ . Just check that  $e_F(\zeta) = \sum_{i \in F} \xi_i \langle \xi_i, \zeta \rangle \rightarrow \zeta$ , so  $e_F \rightarrow 1$  strongly in  $B(H)$  (the bounded operators). Then it follows  $e_F a \rightarrow a$  for all  $a \in \mathcal{K}(H)$  and  $a e_F \rightarrow a$  likewise.

**Remark 7.48** If  $\mathcal{A}$  already has a unit  $1 \in \mathcal{A}$ , then  $(e_i) \subseteq \mathcal{A}$  is an approximate unit iff  $e_i \rightarrow 1$  (by the norm) and  $0 \leq e_i \leq e_j \leq 1$  for  $i \leq j$ .

In particular, the constant net  $(1)$  is an approximate unit in any unital Banach algebra.

**Theorem 7.49** Every  $C^*$ -algebra has an approximate unit. Moreover if  $\mathcal{A}$  is a  $C^*$ -algebra and

$$\Lambda := \{a \in \mathcal{A}_+ \mid \|a\| < 1\}$$

then  $\Lambda$  is directed with the canonical order of  $\mathcal{A}_+ \subseteq \mathcal{A}_{\text{self-adjoint}}$  and the canonical net

$$(e_\lambda)_{\lambda \in \Lambda} e_\lambda = \lambda$$

is an approximate unit.

PROOF:  $\Lambda$  is directed. To prove: For every  $a, b \in \Lambda$  there is a  $c \in \Lambda$  such that  $a, b \leq c$ . Indeed, if  $a \in \mathcal{A}_+$ , then  $1 + a \geq 1$  in  $\tilde{\mathcal{A}} = \mathcal{A} + \mathbb{C} \cdot 1$ . Here, we work in the unitization for a moment, but do not assume we have a unit in  $\mathcal{A}$ ! In particular,  $1 + a \in \text{inv}(\tilde{\mathcal{A}})$  and  $a \cdot (1 + a)^{-1} \in \mathcal{A}$  as  $\mathcal{A} \trianglelefteq \tilde{\mathcal{A}}$ .

Notice:  $a(1 + a)^{-1} = (a + 1 - 1)(1 + a)^{-1} = 1 - (1 + a)^{-1}$  in the unitization.



Claim: For  $a, b \in \mathcal{A}_+$  and  $a \leq b$  we have  $a(1+a)^{-1} \leq b(1+b)^{-1}$ . This should be true because  $a(1+a)^{-1} = f(a)$  where  $f : [0, \infty) \rightarrow [0, 1)$ ,  $x \mapsto \frac{x}{x+1} = x(1+x)^{-1}$  is increasing.  $f$  is a homeomorphism with  $g = f^{-1} : [0, 1) \rightarrow [0, \infty)$  given by  $g(x) = \frac{x}{x-1}$ .

Indeed, take  $0 \leq a \leq b$  then  $1+a \leq 1+b$  so  $(1+b)^{-1} \leq (1+a)^{-1}$  and therefore  $a(1+a)^{-1} = 1 - (1+a)^{-1} \leq 1 - (1+b)^{-1} = b(1+b)^{-1}$ . Now observe that if  $a \in \mathcal{A}_+$  then  $f(a) = a(1+a)^{-1} \in \Lambda$  because  $\|f\|_{\sigma(a) \subseteq [0, \infty)} < 1$  and thus  $0 \leq f < 1$ . So we get an increasing map  $\mathcal{A}_+ \rightarrow \Lambda$ ,  $a \mapsto a(1+a)^{-1}$ . Now suppose  $a, b \in \Lambda$ , consider  $g = f^{-1} : [0, 1) \rightarrow [0, \infty)$ ,  $x \mapsto \frac{x}{x-1}$ . Define  $a' := g(a)$ ,  $b' := g(b)$  and let  $c := (a' + b')(1 + a' + b')^{-1} = f(a' + b')$ . Then  $c \in \Lambda$  and since  $a' \leq a' + b'$  we have  $a = f(a') \leq f(a' + b') = c$  and likewise  $b \leq c$ . This shows that  $\Lambda$  is a directed set.

Now we have to check that  $(e_\lambda)_{\lambda \in \Lambda}$  with  $e_\lambda = \lambda$  is an approximate unit for  $\mathcal{A}$ . Notice that  $(e_\lambda)$  is increasing and  $e_\lambda = \lambda \geq 0$  and  $\|e_\lambda\| < 1$  for all  $\lambda$ . So we need only prove  $e_\lambda \cdot a \rightarrow a \leftarrow a \cdot e_\lambda$  for every  $a \in \mathcal{A}$ . But using the involution, these two are equivalent:

$$(e_\lambda a) \rightarrow a \Leftrightarrow (e_\lambda a)^* \rightarrow a^* \Leftrightarrow a^* e_\lambda \rightarrow a \Leftrightarrow a^* e_\lambda \rightarrow a^*$$

It is even enough to prove  $ae_\lambda \rightarrow a$  for only  $a \in \Lambda$  because  $\text{span } \Lambda = \text{span}(\mathcal{A}_+) = \mathcal{A}$ . Let  $a \in \Lambda$ , in particular  $a \in \mathcal{A}_+$ . Consider 'its' Gelfand representation  $\varphi : C^*(a) \rightarrow C_0(X)$  and let  $f = \varphi(a) \in C_0(X)$ . This function fulfils  $0 \leq f(x) < 1$  for all  $x \in X$  because it comes from  $a \in \mathcal{A}_+$ .

Let furthermore  $\varepsilon > 0$  and  $K := \{x \in X \mid |f(x)| \geq \varepsilon\} \subseteq X$  compact. By Uryson's Lemma, we have a  $g \in C_0(X)$ ,  $g : X \rightarrow [0, 1]$  such that  $g(x) = 1$  for all  $x \in K$ . Next, choose  $\delta > 0$  with  $\delta < 1$  and  $1 - \delta < \varepsilon$ . Then  $g_\delta = \delta \cdot g \leq \delta$  and therefore

$$\begin{aligned} \|f - g_\delta \cdot f\| &= \|f - \delta g f\| = \sup_{x \in X} \|f(x)\| \cdot \|1 - \delta g(x)\| \\ &\leq \max\left\{\sup_{x \in K} \|f(x)\| \cdot \|1 - \delta g(x)\|, \sup_{x \notin K} \|f(x)\| \cdot \|1 - \delta g(x)\|\right\} \\ &\leq \max\{\varepsilon, 1 - \delta\} \leq \varepsilon \end{aligned}$$

Now let  $b := \varphi^{-1}(g_\delta) \in \mathcal{A}_+$  with  $\|b\| < 1$  and  $\|a - ba\| < \varepsilon$ .

This shows that for any  $a \in \Lambda$  we can find  $\lambda_0 = b \in \Lambda$  such that  $\|a - e_{\lambda_0} a\| < \varepsilon$ . If now  $\lambda \in \Lambda$ ,  $\lambda \geq \lambda_0$  we have  $e_{\lambda_0} \leq e_\lambda$ , so  $1 - e_\lambda \leq 1 - e_{\lambda_0}$  (in  $\tilde{\mathcal{A}}$ ) and therefore  $a(1 - e_\lambda)a \leq a(1 - e_{\lambda_0})a$  (\*) (by conjugation property and because  $a$  is self-adjoint). But then

$$\begin{aligned} \|a - e_\lambda a\|^2 \|(1 - e_\lambda a)\|^2 &= \left\| \overbrace{(1 - e_\lambda)^{\frac{1}{2}} \cdot (1 - e_\lambda)^{\frac{1}{2}} a}^{\substack{\in \mathcal{A} \triangleleft \tilde{\mathcal{A}} \\ \in \tilde{\mathcal{A}}}} \right\|^2 \leq \|(1 - e_\lambda)^{\frac{1}{2}} a\|^2 \\ &\stackrel{(*)}{\leq} \|a(1 - e_\lambda)a\| \leq \|a(1 - e_{\lambda_0})a\| \stackrel{\|a\| \leq 1}{\leq} \|(1 - e_{\lambda_0})a\| \\ &= \|a - e_{\lambda_0} a\| < \varepsilon \end{aligned}$$

so  $e_\lambda a \rightarrow a$ . □

**Definition 7.50** In general,  $C^*$ -algebras do not admit a sequential approximate unit.

We say that a  $C^*$ -algebra  $\mathcal{A}$  is  $\sigma$ -unital if there exists such a sequential approximate unit  $(e_n)_{n \in \mathbb{N}}$ .

**Example 7.51**  $\mathcal{A} = C_0(X)$  is  $\sigma$ -unital if and only if  $X$  is  $\sigma$ -compact:  $X = \bigcup_{n=1}^\infty K_n$  where  $K_n \subseteq X$  are compact spaces.

## 8 Ideals in $C^*$ -algebras

**Theorem 8.1** *Let  $\mathcal{A}$  be a  $C^*$ -algebra and  $L \subseteq \mathcal{A}$  a left closed ideal. Then there exists a net  $(u_\lambda)_{\lambda \in \Lambda} \subseteq A_{+,1} \cap L$  (that is, elements with  $0 \leq u_\lambda$  and  $\|u_\lambda\| \leq 1$ ) such that  $a = \lim_\lambda au_\lambda$  for all  $a \in L$ .*

PROOF: Set  $B = L \cap L^*$ . This is clearly a  $C^*$ -subalgebra. There is now an approximate unit  $(u_\lambda) \subseteq B_{+,1} \subseteq A_{+,1}$  for  $B$ . Let  $a \in L$ . Then  $a^*a \in L \cap L^* \in B$  and we have  $\lim_\lambda a^*au_\lambda = a^*a = \lim_\lambda u_\lambda a$ . It follows that

$$\begin{aligned} \lim_\lambda \|a - au_\lambda\|^2 &= \lim_\lambda \|(a - au_\lambda)^*(a - au_\lambda)\| = \lim_\lambda \|a^*a - a^*au_\lambda - u_\lambda a^*a - u_\lambda a^*au_\lambda\| \\ &\leq \lim_\lambda \|a^*a - a^*au_\lambda\| + \lim_\lambda \|u_\lambda\| \cdot \|a^*a - a^*au_\lambda\| = 0 \end{aligned} \quad \square$$

Let  $L \subseteq \mathcal{A}$  be a closed left ideal and  $(u_\lambda) \subseteq B = L \cap L^* \subseteq \mathcal{A}$ . Then  $\lim_\lambda au_\lambda = a$  for all  $a \in L$ . As a consequence:

**Theorem 8.2** *Every closed two-sided ideal  $I \trianglelefteq \mathcal{A}$  of a  $C^*$ -algebra satisfies  $I^* = I$ , so it is a  $*$ -ideal and in particular a  $C^*$ -algebra.*

PROOF: By the lemma above, we find a net  $(u_\lambda) \subseteq I$ ,  $u_\lambda \geq 0$ , such that  $a = \lim_\lambda au_\lambda$ . Then  $a^* = \lim_\lambda u_\lambda a^* \in I$  (because  $u_\lambda \in I$ ).  $\square$

**Corollary 8.3** *Let  $I \trianglelefteq \mathcal{A}$  be a closed two-sided ideal of a  $C^*$ -algebra  $\mathcal{A}$ . Then for all  $a \in \mathcal{A}$ ,  $\|a + I\| = \lim_\lambda \|a - u_\lambda a\| = \lim_\lambda \|a - au_\lambda\|$  where  $(u_\lambda)$  is an approximate unit for  $I$ .*

PROOF: Let  $\varepsilon > 0$  and take  $b \in I$  such that  $\|a + b\| \leq \|a + I\| + \frac{\varepsilon}{2}$ . Recall that  $\|a + I\| = \text{dist}(a, I) = \inf_{b \in I} \|a + b\|$ .

Since  $\lim_\lambda u_\lambda b = b$ . Then there exists  $\lambda_0$  such that  $\|b - u_\lambda b\| < \frac{\varepsilon}{2}$  for all  $\lambda \geq \lambda_0$ . Then

$$\begin{aligned} \|a - u_\lambda a\| &\leq \|(1 - u_\lambda)(a + b)\| + \|b - u_\lambda b\| \\ &\leq \|a + b\| + \|b - u_\lambda b\| \\ &< \|a + I\| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \|a + I\| + \varepsilon \end{aligned}$$

On the other hand,  $\|a - u_\lambda a\| \geq \|a + I\|$  for all  $\lambda$  and  $\|a + I\| = \lim_\lambda \|a + u_\lambda a\| = \inf_\lambda \|a - u_\lambda a\|$ . This shows the existence of the limit and therefore that the norm equals the distance.  $\square$

**Theorem 8.4** *If  $I \trianglelefteq \mathcal{A}$  is a closed  $*$ -ideal in a  $C^*$ -algebra  $\mathcal{A}$ , then  $\mathcal{A}/I$  is itself a  $C^*$ -algebra.*

PROOF: We already know that  $\mathcal{A}/I$  is a Banach  $*$ -algebra. We only need to show that  $\|a + I\| = \|(a + I)^*(a + I)\|$ .

Let  $(u_\lambda) \subseteq I$  be an approximate unit and take  $b \in I$ . Then

$$\begin{aligned} \|a + I\|^2 &= \lim_\lambda \|a - au_\lambda\|_A^2 \stackrel{*}{=} \lim_\lambda \|(1 - u_\lambda)a^*a(1 - u_\lambda)\| \\ &\leq \sup_\lambda \|(1 - u_\lambda)(a^*a + b)(1 - u_\lambda)\| + \lim_\lambda \|(1 - u_\lambda)b(1 - u_\lambda)\| \\ &\leq \|a^*a + b\| \end{aligned}$$

Where  $*$  is because we can use the  $C^*$ -property of  $\mathcal{A}$  and  $(1 - u_\lambda)$  is self-adjoint. The last inequality follows because the latter limit tends to 0.

Since  $b$  was arbitrary, we get

$$\|a + I\|^2 \leq \inf_{b \in I} \|a^*a + b\|_{\mathcal{A}} = \|a^*a + I\| = \|(a + I)^*(a + I)\| \quad \square$$

**Theorem 8.5** *If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  (where  $\mathcal{A}, \mathcal{B}$  are  $C^*$  algebras) is an injective  $*$ -homomorphism, then  $\varphi$  is isometric, i.e.  $\|\varphi(a)\| = \|a\|$  for all  $a \in \mathcal{A}$ .*

PROOF: It suffices to show that  $\|\varphi(a)\|^2 = \|a\|^2$  or  $\|\varphi(a^*a)\| = \|a^*a\|$ .

Replacing  $\mathcal{A}$  by the  $C^*$ -algebra  $C^*(a^*a)$  and  $\mathcal{B}$  by  $C^*(\varphi(a^*a)) \subseteq \mathcal{B}$  (with  $a^*a, \varphi(a^*a) = \varphi(a)^*\varphi(a) \geq 0$ ) we may assume that  $\mathcal{A}, \mathcal{B}$  are commutative. Also by adding units and extending  $\varphi$  to the unitization  $\tilde{\varphi} : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}}$  we may assume that  $\mathcal{A}, \mathcal{B}, \varphi$  are unital. Now given  $\chi \in \Omega(\mathcal{B})$  notice that  $\chi \circ \varphi \in \Omega(\mathcal{A})$ . So we get a map  $\varphi_* : \Omega(\mathcal{B}) \rightarrow \Omega(\mathcal{A}), \chi \mapsto \chi \circ \varphi$ . This is clearly continuous. Since  $\Omega(\mathcal{B})$  is compact,  $K := \varphi_*(\Omega(\mathcal{B}))$  is compact (in particular closed). By Uryson's Lemma, we find some continuous function  $f \in C(\Omega(\mathcal{A}))$  such that  $f|_K \equiv 0$  and  $f \neq 0$  (if we assume  $K \neq \Omega(\mathcal{A})$ ). By Gelfand-Representation we find  $(\mathcal{A} \simeq C(\Omega(\mathcal{A})))$  and  $a \in \mathcal{A}$  such that  $\hat{a} = f$ . Then for each  $\chi \in \Omega(\mathcal{B})$ ,

$$\chi(\varphi(a)) = \hat{a}(\chi \circ \varphi) = \underbrace{\hat{a}}_f(\underbrace{\varphi_*(\chi)}_{\in K}) = 0 \Rightarrow \varphi(a) = 0$$

and if  $f \neq 0$ , then  $a \neq 0$ . But we have  $\varphi(a) = 0$  for all  $a$ , a contradiction. Therefore,  $\varphi_*$  is surjective. Now

$$\|a\|_{\mathcal{A}} = \|\hat{a}\|_{\infty} = \sup_{\chi \in \Omega(\mathcal{A})} |\chi(a)| = \sup_{\chi \in \Omega(\mathcal{B})} |(\chi \circ \varphi)(a)| = \|\widehat{\varphi(a)}\|_{\infty} = \|\varphi(a)\|_{\mathcal{B}} \quad \square$$

**Corollary 8.6** *If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is any  $*$ -homomorphism ( $\mathcal{A}, \mathcal{B}$   $C^*$ -algebras) then  $\varphi(\mathcal{A})$  is closed, hence a  $C^*$ -subalgebra of  $\mathcal{B}$ .*

PROOF: Consider  $\psi : \mathcal{A}/\ker \varphi \rightarrow \mathcal{B}, a + \ker \varphi \mapsto \varphi(a)$ . Then  $\psi$  is a well-defined  $*$ -homomorphism and  $\psi$  is injective and therefore isometric. This shows that  $\psi(\mathcal{A}/\ker \varphi) = \varphi(\mathcal{A})$  is closed.  $\square$

**Remark 8.7** For some other related consequences, see Murphy's book.

- (i) If  $\mathcal{A} \subseteq \mathcal{B}$  are  $C^*$ -algebras and  $I \trianglelefteq \mathcal{B}$  is a closed 2-sided ideal then  $\mathcal{A} + I$  is a  $C^*$ -subalgebra of  $\mathcal{B}$ . In particular, the sum of ideals in  $C^*$ -algebras are ideals: For any  $I, J \trianglelefteq \mathcal{A}$  have that  $I + J \trianglelefteq \mathcal{A}$  as well.
- (ii) If  $I, J \trianglelefteq \mathcal{A}$  then  $I \cdot J = I \cap J$ . The product here is defined as the linear span of products  $(I \cdot J = \overline{\text{span}}\{i \cdot j \mid i \in I, j \in J\})$  but is actually just the products.

PROOF (IDEAS):

- (i) To prove that  $\mathcal{A} + I$  is closed, check that  $(\mathcal{A} + I)/I$  is Banach by identifying it with

$$(\mathcal{A} + I)/I \simeq \mathcal{A}/(\mathcal{A} \cap I), a + I \leftarrow a + \mathcal{A} \cap I$$

Can also build arbitrary families of ideals and the sum will be an ideal, also the intersection and product of ideals exist.

- (ii)  $I \cdot J \subseteq I \cap J$  is clear. To prove the converse, use the approximate unit.  $I \cap J$  is clearly a  $C^*$ -algebra, take an approximate unit  $(u_\lambda) \subseteq I \cap J$  and  $x \in I \cap J$ . Then  $x = \lim_\lambda xu_\lambda$  where  $xu_\lambda$  is in  $I \cdot J$  at all times.  $\square$

## 9 Gelfand-Neymark representation

We know for commutative  $\mathcal{A}$  that  $\mathcal{A} = C_0(\Omega(\mathcal{A}))$ . But if  $\mathcal{A}$  is not commutative,  $\Omega(\mathcal{A}) = \emptyset$  and this is useless. So we want to look at non-homomorphism functionals (the elements of the spectrum are homomorphism functionals) and hope that this is not empty. Hence we want to study positive linear functionals.

**Definition 9.1** Let  $\mathcal{A}, \mathcal{B}$   $C^*$ -algebras. A linear map  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is called **positive** if  $\varphi(\mathcal{A}_+) = \mathcal{B}_+$ . We write  $\varphi \geq 0$  for this.

**Remark 9.2** Let  $\mathcal{A}, \mathcal{B}$   $C^*$ -algebras and  $\varphi \geq 0$ .

- (i)  $\varphi \geq 0$  implies that  $\varphi(\mathcal{A}_{sa}) = \mathcal{B}_{sa}$  (self-adjoint to self-adjoint). This follows because for any  $a \in \mathcal{A}_{sa}$ , we have  $a = a^+ = a^-$  and  $\varphi(a) = \varphi(a^+) - \varphi(a^-) \in \mathcal{B}_{sa}$ .
- (ii)  $a_1 \leq a_2$  in  $\mathcal{A}$  yields  $\varphi(a_1) \leq \varphi(a_2)$ . This is because every  $*$ -homomorphism is primitive because  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  a  $*$ -homomorphism and  $a \geq 0$  in  $\mathcal{A}$  imply  $a = x^*x$  for some  $x \in \mathcal{A}$  and thus  $\varphi(a) = \varphi(x)^*\varphi(x) \geq 0$ .

**Example 9.3** Let  $\varphi : M_m(\mathbb{C}) \rightarrow M_m(\mathbb{C}), a \mapsto a^T$  is positive but not a homomorphism.

For this, consider  $(a^*)^T = (a^T)^*$  and therefore  $(a^*a)^T = (a^T)(a^T)^* \geq 0$ , but not  $(a^*a)^T = (a^T)^*(a^T)$ .

**Example 9.4**  $\mathcal{A} = C_0(X)$ . If  $B(X)$  are the Borell-subsets of  $X$   $\mu : B(X) \rightarrow [0, \infty]$  is a positive bounded measure, then

$$\varphi_\mu : C_0(X) \rightarrow \mathbb{C}, f \mapsto \int_X f(x) d\mu(x)$$

is clearly positive, linear but (usually) not a homomorphism. If  $\mu$  is a Dirac-measure this is a homomorphism and a character.