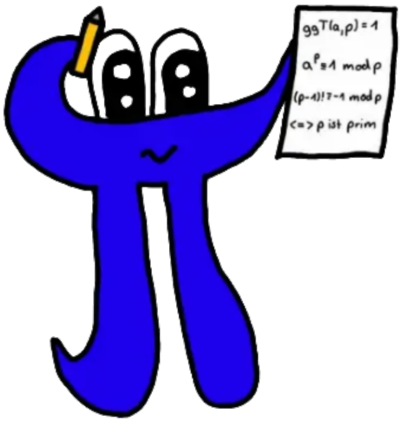
Exercise Sheet 02
Operator Algebras

Contributors: **Valentin Hock, Linus Mußmächer, Minona Schäfer**

May 19, 2023



2.9 Topological zero divisors

We consider two cases:

- First, let X be non-compact. Then we have $\inf_{x\in X} |f(x)| = 0$, because if it were $\varepsilon > 0$ we would have $X = \{x \in X \mid |f(x)| \geq \varepsilon\}$ non-compact and thus $f \notin C_0(X)$. Therefore, we need only prove $\zeta(f) = 0$.

  Choose any $\varepsilon > 0$ define $K \coloneqq \{x \in X \mid |f(x)| \geq \varepsilon\}$. Because $\inf_{x\in X} |f(x)| = 0$, there exists an $x_0 \in X$ for which $|f(x_0)| < \varepsilon$ holds (and thus $x_0 \notin K$). Because K is compact, it is closed and thus $X \setminus K$ is open. Choose an open, pre-compact neighborhood U_0 of x_0 in $X \setminus K$ and set $K' = X \setminus U_0$. The set $\{x_0\}$ is compact, and K' is closed, so Uryson's Lemma yields the existence of a function $b : X \to \mathbb{C}$ (with $\|b\| = 1$) (in $C_0(X)$) with $b(x_0) = 1$ and $b|_{K'} \equiv 0$. Then for $x \in K \subseteq K'$, we have $|(fb)(x)| = |f(x)| \cdot |b(x)| = |f(x) \cdot 0 < \varepsilon$. For $x \in K^{\complement}$, it follows that $|(fb)(x)| = |f(x) \cdot |b(x)| < \varepsilon \cdot 1 = \varepsilon$ and thus $\|fb\| < \varepsilon$. This shows $\zeta(f) = \inf_{b\in C_0(X),\|b\|=1} \|fb\| = 0$.

  So if X is not compact, $\zeta(f) = \inf_{x\in X} |f(x)| = 0$ holds and every $f \in C_0(X)$ is a topological zero divisor.

- Now, let X be a compact Hausdorff space and $f \in C_0(X)$. If f is non-invertible, we have $0 \in f(X)$ and thus $\inf_{x\in X} |f(x)| = 0$. In this case, we can argue as we did in the first point and thusly show $\zeta(f) = 0$ in much the same way.

Consider now an invertible f with $\inf_{x\in X} |f(x)| = k > 0$. We conclude

$$\left\|\frac{1}{f}\right\| = \sup_{x\in X} \frac{1}{|f(x)|} = \frac{1}{\inf_{x\in X} |f(x)|} = \frac{1}{k},$$

so for any $b \in C_0(X)$ with $\|b\| = 1$ we have $\|f \cdot b\| \cdot \|\frac{1}{f}\| \geq \|f \cdot b \cdot \frac{1}{f}\| = \|b\| = 1$, so $\|f \cdot b\| \geq k$ and therefore $\zeta(f) \geq k$.

Choose now any $\varepsilon > 0$. Then $K \coloneqq \{x \in X \mid |f(x)| \geq k + \varepsilon\}$ is compact and $K \neq X$ (or k would not be the infimum of $|f(x)|$). Just like in the first bullet point, we can choose $x_0 \in X \setminus K$ and fitting neighborhoods to get the existence of a function b fulfilling $\|b\| = 1$, $\|bf\| < k + \varepsilon$ and $b \in C_0(X)$. Therefore, $\zeta(f) \leq k$ and thus $\zeta(f) = k > 0$. This also shows that the (invertible) element f is not a topological zero divisor.

To summarize, we have proven $\zeta(f) = \inf_{x\in X} f(x)$ for any $f \in C_0(X)$, that f is a topological zero divisor in a compact space always and in a non-compact space if and only if it is invertible. It remains to show that in a commutative C^*-algebra $\mathscr{A}$, $f \in \mathscr{A}$ is a topological zero divisor if and only if $0 \in \sigma(f)$. As $\mathscr{A}$ is commutative, we can employ the Gelfand Representation (1.3.6) and conclude that $\mathscr{A}$ can be embedded in the algebra $C_0(\Omega(\mathscr{A}))$ by Γ, and $\sigma(f) = \operatorname{im} \hat{f}$ (because $\mathscr{A}$ is unital, or $\sigma(f)$ would not be defined). Then $0 \in \sigma(f) \Leftrightarrow 0 \in \operatorname{im} \hat{f} \Leftrightarrow f$ is non-invertible, and because $\Omega(\mathscr{A})$ is compact (as $\mathscr{A}$ is unital), this is equivalent to f being a topological zero divisor.