

# Exercises to Introduction to Operator Algebras

Alcides Buss

Notes by: Linus Mußmächer

2336440

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# 1 Topological Basics

Let  $X$  be a topological space, that is there exists a subset  $\mathcal{O}(X) \in \mathbb{P}(X)$ .

**Definition 1.1**  $X$  is **Hausdorff** if for all  $x, y \in X$  there exist open sets  $U, V \in \mathcal{O}(X)$  such that  $x \in U, y \in V$  and  $U \cap V = \emptyset$ .

$X$  is **locally Hausdorff** if for all  $x \in X$  there exists an open neighborhood  $U \in \mathcal{O}(X)$  such that  $U$  is Hausdorff with the relative topology from  $X$ .

**Example 1.2 (Snake with two heads)** We consider the space  $[0, 1] \cup \{1^+\}$  equipped with a topology such that both the subspace  $[0, 1]$  and  $[0, 1] \setminus \{1\} \cup \{1^+\}$  are isomorphic to  $[0, 1]$ . Then  $X$  is compact, locally Hausdorff but not Hausdorff.

**Definition 1.3**  $X$  is compact if for every open cover  $(U_i)_{i \in I}$  there exists a finite open subcover.

$X$  is locally compact if for every  $x \in X$  there exists a neighborhood basis of  $x$  consisting of open relatively compact subsets of  $X$ , that is for every open neighborhood  $U$  of  $x$  there exists an open neighborhood  $V$  of  $x$  such that  $\bar{V}$  is compact and  $\bar{V} \subset U$ .

Observation: For a locally Hausdorff  $X$ ,  $X$  is locally compact if and only if for all  $x \in X$  there exists an open neighborhood  $U$  of  $x$  such that  $\bar{U}$  is compact.

## 1.1 Results about locally compact Hausdorff spaces

Let  $X$  be Hausdorff and locally compact.

**Proposition 1.4 (Uryson's Lemma)** For all closed  $F \subset X$  and all compact  $K \subseteq X$  with  $F \cap K = \emptyset$ , there is a continuous function  $f : X \rightarrow [0, 1]$  such that  $f|_K \equiv 1$  and  $f|_F \equiv 0$ .

**Proposition 1.5 (Tietze's extension theorem)** For all  $K \subseteq X$  compact and  $f : K \rightarrow \mathbb{C}$  continuous, there exists an  $\tilde{f} : X \rightarrow \mathbb{C}$  continuous such that  $\tilde{f}|_K \equiv f$ .

**Proposition 1.6 (Alexandroff's compactification theorem)**  $\tilde{X} = X \cup \{\infty\}$  ( $\infty \notin K$ ) is a compact Hausdorff space with  $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{K^c \cup \{\infty\} \mid K \subseteq X \text{ compact}\}$ .

For example, compactifying  $\mathbb{R}$  yields the unit circle  $\mathbb{S}^1$ .

**Proposition 1.7** Conversely, if  $Y$  is a compact Hausdorff space, then for all  $y_0 \in Y$  the space  $X = Y \setminus \{y_0\}$  is a locally compact Hausdorff space.

**Proposition 1.8** More generally, if  $Y$  is a locally compact Hausdorff space and  $Z \subseteq Y$  is a difference of open and closed subsets of  $Y$  (i.e.  $Z = U \setminus F$  or  $Z = F \setminus U$  where  $U \subseteq Y$  is open and  $F \subseteq Y$  is closed) then  $Z$  is locally compact.

**Exercise 1.1** Let  $X$  be a locally compact Hausdorff space. The following are equivalent:

- (1)  $X$  is compact.
- (2)  $C(X) = C_0(X)$  ( $= C_b(X)$ ).
- (3)  $C_0(X)$  is unital.
- (4)  $1 \in C_0(X)$  where  $1(x) = 1 \in \mathbb{C}$  for all  $x \in X$ .

PROOF:

- (1)  $\Rightarrow$  (2): Recall:

$$C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$$

If  $X$  is compact, then every closed subset of  $X$  is compact, so all sets of form  $\{x \in X \mid |f(x)| \geq \varepsilon\}$  are compact, and we have  $C(X) = C_0(X)$ .

- (2)  $\Rightarrow$  (3): This is trivial because  $C(X)$  is always unital.
- (3)  $\Rightarrow$  (4): Suppose  $C_0(X)$  is unital and let  $f \in C_0(X)$  be the unit. Then  $f \cdot g = g$  for all  $g \in C_0(X)$ , that is  $f(x)g(x) = 1$  for all  $x \in X, g \in C_0(X)$ . By Uryson's Lemma, given  $x_0 \in X$ , there exists a  $g \in C_0(X)$  with  $g(x_0) = 1$  (by looking at  $K = \{x_0\}$ , take any precompact open neighborhood  $U$  of  $x$  and look at  $F := U^c \subseteq X$ ). Then we have  $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$ . As this is possible for every  $x_0 \in X$ , we have  $f \equiv 1$ .
- (4)  $\Rightarrow$  (1): Suppose  $f = 1 \in C_0(X)$ . Then choosing  $\varepsilon = \frac{1}{2}$  shows that  $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$  is compact.  $\square$

**Exercise 1.2** Let  $X$  be a locally compact Hausdorff space. Prove that  $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$ .

## 2 Exercise sheet 1

### Exercise 2.1 (1.1)

**PROOF: Case 1:** If  $b_1, b_2 \in A$ , then  $b_i = \alpha_i a$  for certain  $\alpha_i \in \mathbb{C}$ . Thus,  $b_1 \cdot b_2 = \alpha_1 \alpha_2 a^2 = 0$ . Thus, the multiplication is trivial. From this, it immediately follows that  $\varphi : \mathcal{A} \rightarrow \mathcal{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$  is an isomorphism.

**Case 2:**  $\lambda \neq 0$ , and  $a^2 = \lambda a$ . Let  $b = \frac{1}{\lambda} a$ , then  $b \cdot a = a = a \cdot b$ . But then, for any  $c = \mu a \in \mathcal{A}$ , we have  $bc = \mu ba = \mu a = c = cb$ , so the algebra is unital and isomorphic to  $\mathbb{C}$ .  $\square$

**Exercise 2.2 (1.2)** We consider pathological examples for  $C_0(X)$ .

Let  $X = \{x_0\}$ , e.g.  $x_0 \in X$  with  $\mathcal{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$ .  $X$  is highly non-Hausdorff unless we already have  $X = \{x_0\}$ . In this space, the constant sequence  $(x_0)$  converges to any  $x \in X$ .

For a continuous function  $f : X \rightarrow \mathbb{C}$ , this implies  $f(x_0) \rightarrow f(x)$  for all  $x \in X$ , so every continuous function must already be constant. It follows that  $C(X) \simeq \mathbb{C}$ .

We now look at  $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$ . But since all functions are constant, we can use  $f(x_0)$  instead of  $X$  and  $\{x \in X \mid |f(x)| \geq \varepsilon\}$  is either empty or the whole space.  $X$  is compact if and only if  $X$  is finite. From here on, assume  $X$  to be infinite. Then, only the finite subsets are compact. Thus, if we now have  $f \neq 0$ , there exists an  $|f(x_0)| > \varepsilon > 0$  and thus  $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$  is not compact. This implies  $C_0(X) = \{0\}$ .

To find a non-compact topological space that has non-zero unital  $C_0(X)$ , consider  $X = X_0 \sqcup X_1$  with  $X_0$  as before and  $X_1$  compact.

**Theorem 2.1** Let  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  be a  $*$ -homomorphism between  $C^*$ -algebras. Then we already have  $\|\varphi(a)\| \leq \|a\|$  for all  $a \in \mathcal{A}$ .

**Exercise 2.3 (1.4 - Products)** Let  $(A_i)_{i \in I}$  be a family of  $C^*$ -algebras and define

$$\prod_{i \in I} A_i = \{a = (a_i)_{i \in I} \mid a_i \in A_i \forall i \in I \text{ and } \|a\| := \sup_{i \in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the  $C^*$ -axiom.

**Remark 2.2 (Differences between product and direct sum)**

In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall \varepsilon > 0 \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of  $\prod_{i \in I} A_i$  as the closure of  $\bigoplus_{i \in I}^{alg} A_i$ , where

$$\bigoplus_{i \in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| = 0 \right\}.$$

For finite  $I$ , these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective)  $*$ -homomorphisms  $\pi_j : \prod_{i \in I} A_i \rightarrow A_j$  for all  $j \in I$ . If  $B$  is any  $C^*$ -algebra with  $*$ -homomorphisms  $\varphi_j : B \rightarrow A_j$  for every  $j \in I$ , there is a unique  $*$ -homomorphism  $\varphi : B \rightarrow \prod_{i \in I} A_i$  such that  $\pi_j \circ \varphi = \varphi_j$ . This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi_j} & A_j \\ \downarrow \varphi & \nearrow \pi_j & \\ A & & \end{array}$$

**Exercise 2.4 (1.5)**  $X$  is a locally compact Hausdorff space that can be written as  $X = U \cup V$  with open and disjoint  $U, V$  (so  $U, V$  are clopen). We want to prove  $C_0(X) \simeq C_0(U) \oplus C_0(V)$ . To build this map, we map  $f \mapsto (f|_U, f|_V)$ . We check that this is well-defined and a  $*$ -isomorphism.

**Exercise 2.5 (2.6)** Let  $X$  be a locally compact Hausdorff space and  $\widetilde{C_0(X)} \simeq C(\tilde{X})$  with  $\tilde{X} := X \sqcup \{\infty\}$  with the topology  $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{\tilde{X} \setminus K \mid K \subseteq X \text{ kompakt}\}$ .

Observation: If  $X$  is already compact, then  $\infty$  is an isolated point of  $\tilde{X}$  (i.e.  $\{\infty\}$  is clopen).

If  $\mathcal{A}$  is a  $C^*$ -algebra, then  $\tilde{\mathcal{A}}$  (this is not the same  $\sim$  as on the  $X$ !) is a  $C^*$ -algebra with

$$\|a + \lambda 1\|_{C^*} := \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|ab + \lambda b\|_{\mathcal{A}}$$

We check that  $\tilde{\mathcal{A}}$  is a  $C^*$ -algebra.

- $C^*$ -axiom:  $\|a + \lambda 1\|_{C^*}^2 = \|(a + \lambda 1)^*(a + \lambda 1)\|_{C^*}$ . We have

$$\|a + \lambda \cdot 1\|_{C^*} = \|(a^*a + \bar{\lambda}a + \lambda a^*) + |\lambda|^2 \cdot 1\|_{C^*}$$

$$= \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 \cdot b\|_{\mathcal{A}}$$

On the other hand:

$$\begin{aligned} \|a + \lambda \cdot 1\|_{C^*}^2 &:= \sup_{\|b\| \leq 1} \|ab + \lambda b\|_{\mathcal{A}}^2 \\ &= \sup_{\|b\| \leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\|_{\mathcal{A}} \\ &= \sup_{\|b\| \leq 1} \|b^*a^*ab + \bar{\lambda}b^*ab + \lambda b^*a^*b + |\lambda|^2 b^*b\|_{\mathcal{A}} \\ &\leq \sup_{\|b\| \leq 1} \|b^*\|_{\mathcal{A}} \cdot \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathcal{A}} \\ &\leq \sup_{\|b\| \leq 1} \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathcal{A}} \\ &= \|a + \lambda \cdot 1\|_{C^*}^2 \end{aligned}$$

- The other conditions are easy to check and are left for the student.

We still want to prove  $\varphi : \widetilde{C_0(X)} \rightarrow C(\tilde{X}), f + \lambda \cdot 1 \mapsto f_\lambda$  with  $f_\lambda(x) := \lambda$  for  $x = \infty$  and  $f_\lambda(x) = f(x) + \lambda$  otherwise. Note that once again these are not the same  $\tilde{\phantom{x}}$ .

- $f$  is well-defined: We have to check that  $f_\lambda$  is continuous in  $\tilde{X}$ . Take any sequence  $X \ni x_i \rightarrow \infty$  in  $\tilde{X}$ . We have to show  $f_\lambda(x_i) \rightarrow f_\lambda(\infty) = \lambda$ . Since  $f_\lambda(x_i) = f(x_i) + \lambda$  this is equivalent to  $f(x_i) \rightarrow 0$ . But as  $f \in C_0(X)$ , we have that for every  $\varepsilon > 0$  the set  $K_\varepsilon(f) = \{x \mid |f(x)| \geq \varepsilon\}$  is compact. Since  $x_i$  will eventually leave this compact set (or it would not diverge to  $\infty$ ), we know that  $f(x_i)$  eventually becomes smaller than (any)  $\varepsilon$ . So we have  $f(x_i) \rightarrow 0$  and thus  $f_\lambda(x_i) \rightarrow f_\lambda(\infty)$ . So  $f_\lambda$  is continuous in  $\infty$ . The continuity on every other point follows immediately from the continuity of  $f$ .
- $\varphi$  is a \*-isomorphism:

- Linearity:  $\varphi$  is clearly linear as we can check component-wise:

$$(f_1 + f_2)_\lambda = (f_1)_\lambda + (f_2)_\lambda$$

- Homomorphism: For every  $x \in X$  we have

$$\begin{aligned} \varphi((f + \lambda \cdot 1) \cdot (g + \lambda' \cdot 1))(x) &= \varphi((fg + \lambda'f + \lambda g) + \lambda\lambda' \cdot 1)(x) \\ &= (fg + \lambda'f + \lambda g)(x) + \lambda\lambda' \\ &= (fg)(x) + \lambda'f(x) + \lambda g(x) + \lambda\lambda' \\ &= (f(x) + \lambda) \cdot (g(x) + \lambda') \\ &= (\varphi(f + \lambda) \cdot \varphi(g + \lambda'))(x). \end{aligned}$$

In the case of  $x = \infty$ , this equality of course also holds. Thus we have  $\varphi((f + \lambda)(g + \lambda')) = \varphi(f + \lambda)\varphi(g + \lambda')$ .

- \*-homomorphism:

$$\varphi(f + \lambda)^*(x) = \varphi(f^* + \bar{\lambda} \cdot 1)(x)$$

For  $x \in X$  this follows by  $\overline{f(x)} + \bar{\lambda} = f^*(x) + \bar{\lambda}$ , for  $x = \infty$  we have  $\bar{\lambda} = \bar{\lambda}$ .

- Injective:  $f_\lambda(0)$  leads to  $f_\lambda(x) = 0$  for all  $x \in \tilde{X}$ , since if  $x = \infty$  then  $\lambda$  must be 0 and  $f(x) = 0$  for all  $x \in X$ . Thus  $f = 0$  and  $\lambda = 0$ .
- Surjective: Take  $g \in C(\tilde{X})$  and choose  $\lambda = g(\infty)$  and  $f(x) := g(x) - \lambda$ . and check  $f \in C_0(X)$ .

- We can also prove that  $\varphi$  is isometric for the  $C^*$ -norm:

$$\|f + \lambda \cdot 1\| := \sup_{g \in C_0(X), \|g\| \leq 1} \|fg + \lambda g\|_\infty$$

Look at

$$\begin{aligned} \|\varphi(f + \lambda 1)\| &= \sup_{x \in \tilde{X}} |f_\lambda(x)| = \max\{|\lambda|, \sup_{x \in X} |f(x) + \lambda|\} \\ &\stackrel{(*)}{=} \sup_{x \in X} |f(x) + \lambda| \end{aligned}$$

and

$$\begin{aligned} \|f + \lambda \cdot 1\|_{C^*} &= \sup_{\|g\| \leq 1} \|fg + \lambda g\|_{C_0(X)} \\ &= \sup_{|g(x)| \leq 1 \forall x} \sup_{x \in X} \|f(x)g(x) + \lambda g(x)\| \\ &= \sup_{x \in X} |f(x) + \lambda| \\ &\stackrel{(**)}{=} \sup_{x \in X} |f(x) + \lambda| \end{aligned}$$

This proof may need to be divided into two cases:

- $X$  is not compact: We can find a net  $(x_i) \subseteq X$  with  $f(x_i) \rightarrow 0$  and  $(*)$  follows and use a  $g(x) \approx 1$  for  $(**)$ .
- $X$  is compact: Choose  $g \equiv 1$  for  $(**)$  and think about  $(*)$  later.

**Exercise 2.6 (1.8)** It is difficult to prove  $I^* = I$ . The idea is to prove  $I = C_0(U)$  where  $C_0(U) = \{f \in C_0(X) \mid f|_U \equiv 0\}$ .

One can also prove  $C_0(X)/C_0(U) \simeq C_0(F)$  (as  $C_0$  of the subspace) where  $F = U^c$ .

**Exercise 2.7** Prove that  $\mathcal{A}/\mathfrak{I}$  is normed algebra, and

- (i) if  $\mathcal{A}$  is Banach and  $\mathfrak{I} \trianglelefteq \mathcal{A}$  is closed, then  $\mathcal{A}/\mathfrak{I}$  is Banach.
- (ii) if  $\mathcal{A}$  is unital and Banach, then  $\mathcal{A}/\mathfrak{I}$  is unital.

unital if  $\mathcal{A}$  is, Banach if  $\mathcal{A}$  is and  $\mathfrak{I} \trianglelefteq \mathcal{A}$  closed.

PROOF: Consider  $\mathcal{A}/\mathfrak{I}$  with  $(a + \mathfrak{I})(b + \mathfrak{I}) = ab + \mathfrak{I}$ . For the norm, use  $\|a + \mathfrak{I}\| = \text{dist}(a, \mathfrak{I}) = \inf_{x \in \mathfrak{I}} \|a - x\|$ . This is submultiplicative. For every  $\varepsilon > 0$ , there exist  $x, y \in \mathfrak{I}$  for which we have

$$(\varepsilon + \|a + \mathfrak{I}\|) \cdot (\varepsilon + \|b + \mathfrak{I}\|) \geq \|a + x\| \cdot \|b + y\| \geq \|(a + x)(b + y)\| \geq \|ab + \underbrace{ay + xb + xy}_{\in \mathfrak{I}}\| \geq \|ab + \mathfrak{I}\|$$

and taking the limit yields the desired result.

Result (i) follows from functional analysis, that a space is Banach if and only if the convergence of  $\sum_{k=0}^{\infty} a_n$  is equivalent to the convergence of  $\sum_{k=0}^{\infty} \|a_n\|$ .

Now let  $\mathcal{A}$  also be unital, then  $\mathcal{A}/\mathfrak{I}$  is unital. If  $\mathfrak{I} = \mathcal{A}$ , the algebra is the zero-algebra. Thus, let  $\mathfrak{I}$  be a proper ideal. The fact that  $1 = 1_{\mathcal{A}} + \mathfrak{I}$  is a unit is clear, but we need to prove  $\|1_{\mathcal{A}} + \mathfrak{I}\| = 1$ . Observe that, if  $x \in \mathfrak{I} \triangleleft \mathcal{A}$  then  $x \notin \text{inv}(\mathcal{A})$  and  $\|1_{\mathcal{A}} + x\| \geq 1$ . Because otherwise, we have  $\|1_{\mathcal{A}} + x\| < 1$  and then (because  $\mathcal{A}$  is Banach)  $x = a - 1_{\mathcal{A}} \in \text{inv}(\mathcal{A})$ . Hence  $\|1_{\mathcal{A}}\| = \inf_{x \in \mathfrak{I}} \|1_{\mathcal{A}} + x\| \geq 1$ . In addition, we have  $1 \leq \|1_{\mathcal{A}} + \mathfrak{I}\| = \inf_{x \in \mathfrak{I}} \|1_{\mathcal{A}} - x\| \leq \|1_{\mathcal{A}} + 0\| \leq 1$ . This proves  $\|1\| = \|1_{\mathcal{A}} + \mathfrak{I}\| = 1$ .  $\square$

In the following,  $\mathbb{D}$  is the **closed** unit circle.

**Exercise 2.8** Consider  $\chi \in \Omega(\mathcal{A})$ . We have proved  $\|\chi\| \leq 1$ . It may happen that  $\|\chi\| < 1$ . We need a non-unital algebra for this, because we have  $\|\chi\| = 1$  if  $1 \in \mathcal{A}$ .

Consider  $S = (\mathbb{N}, +)$  as an additive semigroup. Then

$$\ell^1(S) = \{(a_n)_{n \in \mathbb{N}} \mid \sum_{n=0}^{\infty} |a_n| < \infty\}$$

is a unital Banach algebra with  $\delta_n \cdot \delta_m = \delta_{n+m}$  for all  $n, m \in \mathbb{N}$  where

$$\delta_n(m) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Observe  $\ell^1(S) = \overline{\text{alg}}\{\delta_0, \delta_1\}$  because of  $\delta_1^n = \delta_n$ . The unit of the algebra is  $\delta_0$ . What are the characters of  $\ell^1(S)$ ?

We can write any  $a \in \ell^1(S)$  as  $a = \sum_{n=0}^{\infty} a_n \delta_n$ . So if  $\chi \in \Omega(\ell^1(S))$  then

$$\chi(a) = \sum_{n=0}^{\infty} a_n \chi(\delta_n) \in \mathbb{C}.$$

In particular,  $\chi(1) = 1$  so  $\chi(\delta_0) = 1$ . This leads to  $\chi(\delta_n) = \chi(\delta_1^n) = \chi(\delta_1)^n = \chi(\delta_1)^n$ . So if we set  $z := \chi(\delta_1) \in \mathbb{C}$ , we have  $\chi(a) = \sum_{n=0}^{\infty} a_n z^n$ . Observe  $|z| = |\chi(\delta_1)| \leq \|\delta_1\| = 1$  (because the Image of a character is a subset of the spectrum, which is bounded by the norm) so  $z$  must be in  $\mathbb{D}$ . By conventioning  $z^0 = 1$  for every  $z \in \mathbb{C}$ , we can even choose  $z = 0$ .

Conversely, if  $z \in \mathbb{D}$ , we define  $\chi_z(a) := \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}$ . Then  $\chi_z(\delta_n) = z^n$  and

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n \cdot z^m = \chi_z(\delta_n) \cdot \chi_z(\delta_m)$$

So we get a map  $\mathbb{D} \rightarrow \Omega(\ell^1(S)) \subseteq \ell^1(S)^*$ ,  $z \mapsto \chi_z$  that is bijective and continuous. If  $z_i \rightarrow z$  in  $\mathbb{D}$ , we need to prove  $\chi_{z_i} \rightarrow \chi_z$  in respect to the weak \*-topology. So we need to evaluate and prove  $\chi_{z_i}(a) \rightarrow \chi_z(a)$ , or  $\sum_{n=0}^{\infty} a_n z_i^n \rightarrow \sum_{n=0}^{\infty} a_n z^n$ . Partial sums would obviously converge, so  $\chi_{z_i}$  converges on a dense subspace of  $\ell^1(S)$ . The uniform boundedness principle (if a bounded set of operators converge on a dense subset  $T_i \rightarrow T$ ,  $\sup_i \|T_i\| < \infty$ , they converge everywhere) shows that the infinite sums also converge. In general, showing that an operator converges on a dense set of an algebra always shows the convergence on any point of the algebra.

Observe  $\sigma(\delta_1) = \{\chi(\delta_1) \mid \chi \in \Omega(\ell^1(S))\} = \mathbb{D}$  and  $\sigma(\delta_1) = \mathbb{D}$  as well.

Concerning the norm, we know that

$$|\chi_z(a)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| \leq \sum_{n=0}^{\infty} |a_n| |z|^n \leq \sum_{n=0}^{\infty} |a_n| = \|a\|$$

for all  $a \in \ell^1(S)$ , so  $\|\chi_z\| \leq 1$ . For  $a = (a_0, 0, 0, \dots)$  we have  $|\chi_z(a)| = |a_0| = \|a\|$ , so  $\|\chi_z\| = 1$  for any  $z \in \mathbb{C}$  (and thus for any  $\chi = \chi_z \in \Omega(\ell^1(S))$ ).

**Remark 2.3 (Gelfand-Representation)** In general, we seek a mapping  $\mathcal{A} \rightarrow C_0(X), a \mapsto \hat{a}$ , taking  $X = \hat{\mathcal{A}} = \Omega(\mathcal{A})$  and  $\hat{a}(\chi) = \chi(a)$ .

If we apply the Gelfand representation here, we have

$$\ell^1(S) \rightarrow C(\mathbb{D}), a \mapsto \hat{a} \text{ where } \hat{a}(z) = \chi_z(a) = \sum_{n=0}^{\infty} a_n z^n$$

**Example 2.4 (Norms < 1)** Consider

$$\ell_0^1(S) = \overline{\text{alg}}(\delta_1) = \left\{ \sum_{n=1}^{\infty} a_n \delta_n \mid a_n \in \mathbb{C} \right\} \triangleleft \ell^1(S)$$

Observe  $\widehat{\ell_0^1(S)} \simeq \ell^1(S)$ . Recall  $\Omega(\tilde{\mathcal{A}}) = \Omega(\mathcal{A}) \sqcup \{\chi_{\infty}\}$ . So we are looking for our  $\chi_{\infty}$ , which is  $\chi_{\infty}(a_0, a_1, \dots) = a_0$  – that is  $\chi_0$  and corresponds to  $z = 0$  in the unit circle. It follows  $\Omega(\ell_0^1(S)) \simeq \mathbb{D} \setminus \{0\}$  and  $\chi_0 \in \Omega(\ell^1(S)) \setminus \Omega(\ell_0^1(S))$ .

We compute  $\|\chi_z\| = \sup_{\|a\|_1 \leq 1} |\chi_z(a)|$ . Consider:

$$|\chi_z(a)| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| z \left( \sum_{n=1}^{\infty} a_n z^{n-1} \right) \right| \leq |z| \cdot \|a\|_1$$

so because of  $\chi_z(\delta_1) = z$ , we have  $\|\chi_z\| = |z|$ , which can be smaller than 1.

**Remark 2.5** Do we have  $\ell^1(S) \hookrightarrow A(\mathbb{D}), a \mapsto \hat{a}$  where  $\hat{a}(z) := \sum_{n=0}^{\infty} a_n z^n$ ?

**Exercise 2.9 (02-03)** Is  $\mathcal{A}(\mathbb{D})$  a  $C^*$ -algebra? Consider  $f(z) = \exp(iz)$ ,  $f \in \mathcal{A}$  and notice  $z^* = z$ . But we have  $\|f^* f\|_{\infty} \neq \|f\|_{\infty}^2$ , because  $f^* f = 1$  and because  $f(-i) = e$ , we have  $\|f\|_{\infty} \geq e$  and  $\|f\|_{\infty}^2 \geq e^2 > 1 = \|f^* f\|_{\infty}$ . Since the  $*$ -property is not fulfilled.

**Remark 2.6** Talk about functoriality. If  $X, Y$  are compact Hausdorff spaces and  $f : X \rightarrow Y$  is continuous then

$$f_* : C(Y) \rightarrow C(X), g \mapsto g \circ f$$

You can check that  $f_*$  is a unital  $*$ -homomorphism. So we receive a functor from the compact spaces to the unital commutative  $C^*$ -algebras:

$$\begin{aligned} \text{Comp. Spaces} &\rightarrow \text{unital abelian } C^*, X \mapsto C(X) \\ \text{Hom}(X, Y) &\rightarrow \text{Hom}(C(Y), C(X)), f \mapsto f_* \end{aligned}$$

This is a contravariant function because for  $f : X \rightarrow Y, g : Y \rightarrow Z$  we have  $(g \circ f)_* = f_* \circ g_*$ . It is also natural. If  $\varphi : C(Y) \rightarrow C(X)$  is a unital  $*$ -homomorphism, we get a continuous map  $f : X \rightarrow Y$  by duality.

### 3 More multiplier algebra

We continue to look at the multiplier algebra.

$$M(\mathcal{A}) = \{\mu = (L, R) \in L(\mathcal{A}) \times L(\mathcal{A}) \mid aL(b) = R(a)b, L(ab) = L(a)b, R(ab) = aR(b)\}$$

If  $\mathcal{A}$  is a  $C^*$ -algebra,  $\mathcal{A}$  embeds into  $M(\mathcal{A})$  as an essential ideal. If  $A$  embeds into a  $C^*$ -algebra  $B$  as an essential ideal, then  $B \rightarrow M(\mathcal{A}), b \mapsto \mu_b$  with  $(a \mapsto ba, a \mapsto ab)$  is an isomorphism.

We also define the **strict topology** on  $M(\mathcal{A})$  as the smallest topology that makes the map  $a \mapsto \mu \cdot a, a \mapsto \mu \cdot a$  norm-continuous on  $\mathcal{A} \rightarrow \mathcal{A}$ . So if  $(\mu_i) \subseteq M(\mathcal{A})$  is a net, then  $\mu_i \rightarrow \mu \in M(\mathcal{A})$  if and only if  $\mu_i a \rightarrow \mu a$  and  $a \mu_i \rightarrow a \mu$  for all  $a \in \mathcal{A}$ .



**Remark 3.1** Writing  $0 \leq a \leq 1$  in a  $C^*$ -algebra means  $a \geq 0$ , so  $\sigma(a) \subseteq [0, \infty)$  and  $a \leq 1$  means  $(1 - a)$  is positive in  $\tilde{\mathcal{A}}$  or  $M(\mathcal{A})$  which is equivalent to  $\|a\| \leq 1$ .

All of this is equivalent to  $\sigma(a) \subseteq [0, 1]$ .

Relation with approximate units: If  $(e_i) \subseteq A_{+,1}$  is an increasing net ( $0 \leq e_i \leq 1$ ) then  $(e_i)$  is an approximate unit iff  $e_i \rightarrow 1$  (strictly) in  $M(\mathcal{A})$ .

By definition this means  $e_i a \xrightarrow{\|\cdot\|} a$ ,  $a \cdot e_i \xrightarrow{\|\cdot\|} a$ .

### 3.1 Non-degenerate $*$ -homomorphisms

**Definition 3.2** Let  $\pi : \mathcal{A} \rightarrow M(\mathcal{B})$  a  $*$ -homomorphism. We say that  $\pi$  is **non-degenerate** if  $\text{span } \pi(\mathcal{A}) \cdot B = B$ .

**Lemma 3.3** Let  $\pi : \mathcal{A} \rightarrow M(\mathcal{B})$  be a  $*$ -homomorphism. The following are equivalent:

- (i)  $\pi$  is non-degenerate.
- (ii)  $\pi(e_i) \rightarrow 1$  (strictly) in  $M(B)$  if  $(e_i)$  is some approximate unit in  $\mathcal{A}$ .
- (iii)  $\pi$  extends to a strictly continuous unital  $*$ -homomorphism  $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ .

PROOF:

(i)  $\Rightarrow$  (ii): Let  $(e_i)$  be an approximate unit. Prove  $\pi(e_i) \rightarrow 1$  (strictly) in  $M(\mathcal{B})$ , that is  $\pi(e_i)b \rightarrow b$  for all  $b \in \mathcal{B}$ . Since  $(e_i)$  is bounded, it is enough to show that  $\pi(e_i)b \rightarrow b$  for all  $b \in \pi(A)B$  as this is dense in  $B$ . But if  $b = \pi(a)c$  for  $a \in \mathcal{A}, c \in \mathcal{B}$ , then  $\pi(e_i)b = \pi(e_i)\pi(a)c = \pi(e_i a)c \rightarrow \pi(a)c = b$  because  $\pi$  is norm-continuous and  $e_i a \rightarrow a$ .

(ii)  $\Rightarrow$  (iii): We want to extend  $\pi$  to  $\tilde{\pi} : M(A) \rightarrow M(B)$ . We need  $\tilde{\pi}$  such that

$$\tilde{\pi}(\mu)\pi(a) = \tilde{\pi}(\mu)\tilde{\pi}(a)\tilde{\pi}(\mu \cdot a) = \pi(\mu \cdot a).$$

Therefore, define the multiplier  $\tilde{\pi}(\mu)$  just on  $\pi(\mathcal{A})\mathcal{B}$  by the mappings

$$\begin{aligned} L(\pi(a)b) &= \tilde{\pi}(\mu) \cdot (\pi(a)b) = \pi(\mu a) \cdot b \in \mathcal{B} \\ R(b\pi(a)) &= (b\pi(a)) \cdot \tilde{\pi}(\mu) = b\pi(a\mu) \in \mathcal{B}. \end{aligned}$$

These morphisms are certainly linear. By (ii), notice that  $\overline{\pi(\mathcal{A})\mathcal{B}} = \mathcal{B}$ . So the above defines morphism on all of  $\mathcal{B}$  by continuous extension.

We need to prove:  $L, R$  are well-defined and extend to  $\mathcal{B}$  and  $\mu = (L, R)$  is a multiplier of  $\mathcal{B}$ .

Claim:

$$\left\| \sum_{i=0}^n \pi(\mu \cdot a_i) \cdot b_i \right\| \leq \|\mu\| \cdot \left\| \sum_{i=0}^n \pi(a_i) b_i \right\|$$

for all sequences  $(a_i) \subseteq \mathcal{A}$  and  $(b_i) \subseteq \mathcal{B}$ . To prove that  $L$  is well-defined compute  $\tilde{\pi}\pi(a)b = \pi(\mu a)b$ .

$$\pi(\mu a)b = \lim_i \pi(\mu e_i a)b = \lim_i \pi(\mu e_i)\pi(a)b$$

So  $L$  is well-defined, because we can write  $L(\pi(a)b) = \lim_i \pi(\mu e_i)(\pi(a)b)$  as directly dependent on  $\pi(a)b$  (so different representations of  $\pi(a)b$  will yield the same result). Analogously, we have  $R(b\pi(a)) = b\pi(a) \lim_i \pi(e_i \mu)$ . Now we can compute

$$\|\tilde{\pi}(\mu)\pi(a)b\|^2 = \|b^* \pi(\mu a)^* \pi(\mu a)b\| = \|b^* \pi(a^* \mu^* \mu a)b\| \leq \|b^* \pi(\|\mu\|^2 a^* a)b\|$$

$$= \|\mu\|^2 \|b^* \pi(a)^* \pi(a) b\| = \|\mu\|^2 \|\pi(a) b\|^*$$

Now we show that  $\tilde{\pi}$  is actually a multiplier, let  $x = b_1 \pi(a_1)$  and  $y = \pi(a_2) b_2$  and consider

$$xL(y) = b_1 \pi(a_1) \pi(\mu a_2) b_2 = b_1 \pi(a_1 \mu a_2) b_2 = b_1 \pi(a_1 \mu) \pi(a_2) b_2 = R(x)y$$

This proves the last multiplier property, the others follow.

You can repeat this for  $R$ . So we have a well-defined map  $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ . We need to prove that this is a strictly continuous  $*$ -homomorphism extending  $\pi$ . Calculate:

$$\tilde{\pi}(1) \pi(a) b = \pi(1 \cdot a) b = \pi(a) b$$

so  $\tilde{\pi}(1) = 1$  and

$$\tilde{\pi}(\mu) \cdot \tilde{\pi}(\nu) \pi(a) b = \tilde{\pi}(\pi)(\mu) \pi(\nu a) b = \pi(\mu \cdot \nu \cdot a) b = \tilde{\pi}(\mu \cdot \nu) \pi(a) b$$

so  $\tilde{\pi}(\mu) \cdot \tilde{\pi}(\nu) = \tilde{\pi}(\mu \cdot \nu)$ . To prove that  $\tilde{\pi}$  is strictly continuous, take  $\mu_i \rightarrow \mu$  (strictly) and show

$$\tilde{\pi}(\mu_i) \pi(a) b = \pi(\mu_i a) b \xrightarrow{\|\cdot\|} \pi(\mu a) b = \tilde{\pi}(\mu) \pi(a) b$$

Now use the Cohen-Hilbert Factorization theorem:

If  $E$  is a Banach right (or left) module over some Banach algebra  $\mathcal{A}$  and  $\mathcal{A}$  has an approximate unit  $(e_i) \subseteq A_1$ ,  $\|e_i\| \leq 1$ , then  $\overline{\text{span } E \cdot \mathcal{A}} = E \cdot \mathcal{A} = \{x \cdot a \mid x \in E, a \in \mathcal{A}\}$ .

Apply this for  $E = \mathcal{B}$  (view it as a left  $\mathcal{A}$ -module via  $\pi$ ):  $a \cdot b := \pi(a) b$ . This shows  $\overline{\text{span } \pi(\mathcal{A}) \mathcal{B}} = \pi(\mathcal{A}) \mathcal{B}$ .

(iii)  $\Rightarrow$  (i): Suppose  $\mathcal{A} \rightarrow M(\mathcal{B})$  extends to  $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$  (strictly continuous  $*$  homomorphism).

(iv) But then  $\tilde{\pi}(1) = 1$ , so

$$1 = \tilde{\pi}(1) = \tilde{\pi}(\text{strict } \lim_i e_i) = \text{strict } \lim_i \pi(e_i)$$

so  $\pi(e_i)$  strictly converges to 1. □

**Exercise 3.1 (03-01)** For any  $C^*$ -algebra  $\mathcal{A}$  and  $a \in \mathcal{A}$  we have  $a(a^* a)^{\frac{1}{n}} \rightarrow a$  (and  $(aa^*)^{\frac{1}{n}} a \rightarrow a$ ).

PROOF: If  $\mathcal{A} = C_0(X)$  then  $a = f$  and we have  $f(x)|f(x)|^{\frac{2}{n}} \rightarrow f(x)$  as  $|f(x)|^{\frac{2}{n}} \rightarrow 1$  on the non-zero points of  $f$  and  $\rightarrow 0$  otherwise. On a compact set, this then converges uniformly.

In general: Compute

$$\begin{aligned} \|a(a^* a)^{\frac{1}{n}} - a\|^2 &= \|(a(a^* a)^{\frac{1}{n}})^* \cdot (a(a^* a)^{\frac{1}{n}} - a)\| \\ &= \|(a^* - (a^* a)^{\frac{1}{n}} a^*) \cdot (a^* a)^{\frac{1}{n}} - a\| \\ &= \|a^* a(a^* a)^{\frac{1}{n}} - a^* a - (a^* a)^{\frac{1}{n}} a^* a(a^* a)^{\frac{1}{n}} + (a^* a)^{\frac{1}{n}} a^* a\| \\ &= \|bb^{\frac{1}{n}} - b - b^{\frac{1}{n}} b b^{\frac{1}{n}} + b^{\frac{1}{n}} b\| \end{aligned}$$

with  $b = a^* a$  self-adjoint. We may then replace  $\mathcal{A}$  by  $C^*(b)$ , which is commutative. The above case then applies by considering  $b$  as a function via the Gelfand-Representation. □

**Exercise 3.2 (04-10)**  $C_0(X)$  is  $\sigma$ -unital (there exists a countable approximate unit) if and only if  $X$  is  $\sigma$ -compact ( $X$  is the union of countably many compact sets).

**Example 3.4** Take  $\omega$  as the first uncountable ordinal. Then  $X = [0, \omega)$  is not  $\sigma$ -compact.

If  $X$  is metrizable and locally compact, there exists a metric  $d : X \times X \rightarrow \mathbb{R}$  such that the topology on  $X$  is induced by  $d$ .

If  $X$  is separable, then  $X$  is  $\sigma$ -compact.

PROOF:  $\Rightarrow$ : Assume  $C_0(X)$  is  $\sigma$ -unital. Then we have a countable approximate unit  $(e_n)_{n \in \mathbb{N}}$ . Define  $f = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \in C_0(X)$  as  $0 \leq e_n \leq 1$ . Then  $f$  is positive ( $f \geq 0$ ) and bounded ( $f \leq 1$ ).  $f$  is even strictly positive ( $f > 0$ ), that is  $f(x) > 0$  for all  $x \in X$  because  $e_n(x) \rightarrow 1$  uniformly on compacts and  $f \geq \frac{e_n}{2^n}$  for all  $n$ .

Then, defining  $K_n := \{x \in X \mid f(x) \geq \frac{1}{n}\} \subseteq X$  is compact. Notice  $\bigcup_{n=1}^{\infty} K_n = X$  and  $K_n \subseteq K_{n+1}$ .

$\Leftarrow$ : Assume  $X = \bigcup_{n=1}^{\infty} K_n$  where  $K_n \subseteq X$  are compact. We may assume  $K_n \subseteq K_{n+1}$ . From  $\sigma$ -compactness we may find an open, countable cover of  $X = \bigcup_{n=1}^{\infty} U_n$  such that  $U_n$  is open and  $\overline{U_n}$  is compact for every  $n \in \mathbb{N}$ . This cover can be found by covering every  $K_n$  with (finitely many) open sets and taking the union. Now we use Uryson's Lemma to find a sequence  $(f_n)_{n \in \mathbb{N}}$  where  $0 \leq f \leq 1$  and  $f_n|_{\overline{U_n}} \equiv 1$  and  $f_n \in C_0(X)$ .

Take now a compact  $K \subseteq X = \bigcup_{n=1}^{\infty} U_n$ . As  $K$  is compact, there exists an  $n, m$  with  $K \subseteq U_n \subseteq U_m$  and  $f_m|_K \equiv 1$  and  $m \geq n$ . This shows  $f_n(x) \rightarrow 1$  on compacts.

Observe:  $(f_n)$  may not be increasing. To get this, choose two covers  $(U_n), (V_n)$  of  $X$  of open sets with compact closures where  $U_n \subseteq \overline{U_n} \subseteq V_n \subseteq \overline{V_n} \subseteq U_{n+1}$ . Now choose by Uryson  $(f_n)_{n=1}^{\infty}$  with  $f_n|_{\overline{U_n}} \equiv 1, f_n|_{V_n^c} \equiv 0$ . Now  $f_n$  is increasing.  $\square$

**Remark 3.5** The following are equivalent:

- (i) The  $C^*$ -algebra  $\mathcal{A}$  is  $\sigma$ -unital (there exists a countable approximate unit  $(e_n)_{n \in \mathbb{N}}$ ).
- (ii) There exist  $a \in \mathcal{A}_{+,1}$  (write:  $a > 0$ ), meaning that  $\varphi(a) > 0$  for all  $\varphi \in \mathcal{A}_+^*$  (positive linear functions) with  $\|\varphi\| = 1$ .
- (iii)  $\overline{\mathcal{A}} \cdot a = \mathcal{A}$ .
- (iv)  $\overline{a\mathcal{A}} = \mathcal{A}$ .

PROOF: See Diximiere (important guy).  $\square$

**Exercise 3.3 (04-04)** Let  $\mathcal{A}$  be a  $C^*$ -algebra,  $a \in \mathcal{A}$  and  $L_a := \overline{\mathcal{A}a}$ .

Then  $L_a$  is the closed left ideal generated by  $a$ , and if  $a$  is a projection (self-adjoint and idempotent) we even have  $L_a = \mathcal{A}a$ .

Furthermore,  $L_a = L_{aa^*}$ .

Lastly, if  $\mathcal{A}$  is unital and  $aa^* = 1$  for some  $a$  and  $L_a = \mathcal{A}$ . Then  $a$  is unitary.

PROOF: We prove this is the closed left ideal generated by  $a$ , that is, the smallest closed ideal containing  $a$ .

First, observe that  $L_a = \overline{\mathcal{A}a}$  is a closed left ideal, because  $\mathcal{A} \cdot a = \{xa \mid x \in \mathcal{A}\}$  is a left ideal. We also have  $a \in L_a$ , because we can use the approximate unit for  $x$  and that  $L_a$  is closed. Now if  $L \subseteq \mathcal{A}$  is a closed left ideal with  $a \in L$ , we must have  $\overline{\mathcal{A}a} \subseteq L$ . This concludes the proof.

To prove that if  $a$  is a projection (idempotent would be enough),  $\mathcal{A}a$  is closed, take a sequence  $x_na \rightarrow b \in \mathcal{A}$  and therefore  $x_na = x_na^2 \rightarrow ba$ . So  $b = ba \in \mathcal{A}a$ . Thus, we could write  $\mathcal{A}a = \{xa \mid x \in \mathcal{A}\} = \{x \mid x \in \mathcal{A}, xa = x\}$ .

To  $L_a = L_{aa^*}$ , i.e.  $\overline{\mathcal{A}a} = \overline{\mathcal{A}a^*a}$ . Using exercise 03-01, we see  $a = \lim_n a(a^*a)^{\frac{1}{n}} \in \overline{\mathcal{A}a^*a}$  (because  $a^*a \in L_{a^*a}$  and therefore  $(a^*a)^{\frac{1}{n}} \in L_{a^*a}$  because  $(a^*a)^{\frac{1}{n}}$  is approximated by a sequence of polynomials and  $L_{a^*a}$  is closed, see Stone-Weierstraß), so  $L_a \subseteq L_{a^*a}$ . The other inclusion is obvious, as  $a^*a \in L_a$ .

For the last point, we need to prove  $a^*a = 1$ . We have  $L_a = \overline{\mathcal{A}a} = \overline{\mathcal{A}a^*a} \stackrel{!}{=} \mathcal{A}$ , i.e.  $\mathcal{A}a$  must be dense in  $\mathcal{A}$ . But then the claim is already  $\mathcal{A}a = \mathcal{A}$  as  $\mathcal{A}$  contains a unit 1 that is also in  $\mathcal{A}a$ . Otherwise,  $1 \notin \mathcal{A}a$  and also  $\mathcal{A}a$  does not contain any invertible elements. But the set of invertibles is non-trivially open in  $\mathcal{A}$ , a contradiction to the denseness.

So we have  $\mathcal{A}a = \mathcal{A}$ , meaning there exists  $b \in \mathcal{A}$  with  $ba = 1$ , so  $a$  is left invertible. But by the assumption, there is also a right inverse. So  $a$  is invertible with inverse  $a^*$ , so it is unitary.  $\square$

## 3.2 Somethings about irreducible representations

Let  $\pi : \mathcal{A} \rightarrow B(H)$  be a representation of  $\mathcal{A}$ .  $\pi$  is irreducible if for any closed,  $\pi(\mathcal{A})$ -invariant subspace (that is  $\pi(\mathcal{A})K \subseteq K$ ) we have  $K = 0$  or  $K = H$  already.

**Lemma 3.6 (Schur)**  $\pi$  is irreducible iff  $\pi(\mathcal{A})' := \{T \in B(H) \mid T\pi(a) = \pi(a)T \ \forall a \in \mathcal{A}\}$ ,  $a$  strongly closed  $C^*$ -algebra, is trivial (equal to  $\mathbb{C} \cdot 1$ ).

PROOF: Requires things about von-Neumann-algebras.  $\square$

**Exercise 3.4 (04-07)**  $\mathcal{A}$  is primitive if there exists a faithful and irreducible representation  $\pi : \mathcal{A} \rightarrow B(H)$ .

Recall that  $p \trianglelefteq \mathcal{A}$  is primitive if  $p = \ker \rho$  for some  $[\rho] \in \hat{\mathcal{A}}$ .

So  $\mathcal{A}$  is primitive if  $\{0\}$  is a primitive ideal.

**Example 3.7** Take  $\mathcal{A} = K(H)$  (compact operators on some Hilbert space  $H$ ). This is primitive as the embedding/inclusion in  $B(H)$  is already the representation and  $K(H) = \mathbb{C} \cdot 1$ , as commuting with the compact operator is the same as commuting with the finite rank operators, or (for some  $T \in K(H)'$ )

$$\begin{aligned} T \cdot |\zeta\rangle\langle\eta| &= |\zeta\rangle\langle\eta| \cdot T \\ |T(\zeta)\rangle\langle\eta| &= |\zeta\rangle\langle T^*(\eta)| \\ T(\zeta)\langle\eta|J &= \zeta\langle T^*(\eta)|J \end{aligned}$$

for all  $\zeta, \eta, J \in H$ . But then  $T \in \mathbb{C} \cdot 1$ .

PROOF (FOR 04-07): If  $\mathcal{A}$  is a primitive  $C^*$ -algebra, then  $ZM(\mathcal{A}) = \mathbb{C} \cdot 1$  (the center of the multiplier  $\mathcal{A}$ ).

In general, we have the Dawns-Hoffman-Theorem:  $ZM(\mathcal{A}) \simeq C_b(\text{Prim}(\mathcal{A}))$  where  $\text{Prim}(\mathcal{A}) = \{p \trianglelefteq \mathcal{A} \mid p \text{ primitive}\}$ .

We want to use that if  $\mathcal{A}$  is primitive, then  $\text{Prim}(\mathcal{A}) = \{\{0\}\}$ . This is however not universally true.

Example: Look at  $\mathcal{A} = \widehat{K(H)} \subseteq B(H)$ ,  $\dim H = \infty$  and thus  $\text{Prim}(\mathcal{A}) = \{\{0\} \leq K(H)\}$ .

Take  $\pi : \mathcal{A} \rightarrow B(H)$  a faithful irreducible representation (assume  $\mathcal{A} \neq 0$ ), in particular  $\pi$  is non-degenerate ( $\text{span } \pi(\mathcal{A})H = H$ ). So  $\pi$  extends (strictly continuous) to a unital representation  $\tilde{\pi} : M(\mathcal{A}) \rightarrow B(H)$ . For this representation we have  $\tilde{\pi}(\mu)\pi(a)\zeta = \pi(\mu a)\zeta$  so  $\tilde{\pi}$  is faithful. Now,

since  $\pi$  is irreducible by Schur we have  $\pi(\mathcal{A})' = \mathbb{C} \cdot 1$ . Next we have  $\tilde{\pi}(M(\mathcal{A}))' = \pi(\mathcal{A})'$ . The inclusion  $\subseteq$  is trivial. For  $\supseteq$  consider that  $T \in \pi(\mathcal{A})'$  implies  $T\pi(a) = \pi(a)T$  for all  $a \in \mathcal{A}$ . Take now  $\mu \in M(\mathcal{A})$ . Then for all  $a \in \mathcal{A}$ :

$$\tilde{\pi}(\mu)T\pi(a) = \tilde{\pi}(\mu)\pi(a)T = \pi(\mu \cdot a)T = T\pi(\mu a) = T\tilde{\pi}(\mu)\pi(a)$$

so  $\tilde{\pi}(\mu)T = T\tilde{\pi}(\mu)$ . Therefore,  $\tilde{\pi}(M(\mathcal{A}))' = \pi(\mathcal{A})' = \mathbb{C} \cdot 1$  is also trivial. Now notice  $\tilde{\pi}(ZM(\mathcal{A})) \subseteq Z(\tilde{\pi}(M(\mathcal{A}))) \subseteq \tilde{\pi}(M(\mathcal{A}))' = \mathbb{C} \cdot 1$ . Since  $\tilde{\pi}$  is faithful and unital, we have  $ZM(\mathcal{A}) = \mathbb{C} \cdot 1$  (as the image has only 1 dimension, the preimage must also only have 1 dimension).  $\square$

Somewhere along the way we also solved exercise 04-06.

**Example 3.8** Take  $\mathcal{A} \supseteq \widehat{\mathcal{K}(H)} = \mathcal{K}(H) + \mathbb{C} \cdot 1 \subseteq \mathcal{B}(H)$ . Then  $\dim H = \infty$ . This is primitive but not isomorphic to any  $\mathcal{K}(G)$  for another Hilbert space  $G$ .

**Remark 3.9** How to turn everything into a  $C^*$ -algebra.

Take a Banach- $*$ -algebra  $\mathcal{A}$ . There is a (universal)  $C^*$ -algebra  $C_u^*(\mathcal{A})$  such that  $\iota : \mathcal{A} \rightarrow C_u^*(\mathcal{A})$  has dense image and for any  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  there exists  $\tilde{\varphi} : C_u^*(\mathcal{A}) \rightarrow \mathcal{B}$  with  $\varphi = \tilde{\varphi} \circ \iota$ .

Definition of  $C_u^*(\mathcal{A})$ : If  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  is a  $*$ -homomorphism,  $\varphi$  is contractive. Then  $\|\varphi\|_\varphi := \|\varphi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$  is a  $C^*$ -seminorm. Defining

$$\|a\|_{C_u^*} := \sup\{\|a\|_\varphi \mid \varphi : \mathcal{A} \rightarrow \mathcal{B}\} \leq \|a\|_{\mathcal{A}}$$

yields the biggest possible  $C^*$ -seminorm on  $\mathcal{A}$ . Then  $\overline{\mathcal{A}/N_{\|\cdot\|_{C_u^*}}} =: C_u^*(\mathcal{A})$  is a  $C^*$ -algebra as  $\|\cdot\|_{C_u^*}$  becomes a norm.

**Example 3.10** For  $\mathcal{A} = A(\mathbb{D})$ ,  $C_u^*$  is commutative and unital (as  $A(\mathbb{D})$  is commutative and unital), so we have  $C_u^*(\mathcal{A}) \simeq C(X)$  for the spectrum  $X = \Omega(C_u^*(A(\mathbb{D})))$ . Recall that  $\Omega(A(\mathbb{D})) \simeq \mathbb{D}$  via the evaluation functions. Observe also that  $A(\mathbb{D}) \subseteq C(\mathbb{D})$ . Then

$$\begin{aligned} \Omega(C_u^*(A(\mathbb{D}))) &\simeq \Omega_{sym}(A(\mathbb{D})) := \{\chi \in \Omega(A(\mathbb{D})) \mid \chi(A(\mathbb{D})_{sa}) \subseteq \mathbb{R}\} \\ &= \{\chi \in \Omega(A(\mathbb{D})) \mid \chi \text{ }^*\text{-homomorphism}\} \end{aligned}$$

What characters (evaluations) on  $A(\mathbb{D})$  fulfil this? We see that  $\text{ev}_\lambda(f^*) = \overline{f(\lambda)}$ , so  $\overline{f(\lambda)} = f^*(\lambda) = \overline{f(\overline{\lambda})}$  for all  $f$  (especially  $f = \text{id}$ ) only for  $\lambda \in [-1, 1]$ . Therefore, we have  $C_u^*(A(\mathbb{D})) \simeq C([-1, 1])$ .

**Example 3.11** Let  $S$  be an inverse semigroup. Then  $S \hookrightarrow \mathbb{C}[S] = \left\{ \sum_{s \in S}^{fin} a_s \delta_s \mid a_s \in \mathbb{C} \right\}$ . This is a  $*$ -algebra.

**Remark 3.12** We know that for any  $*$ -algebra  $\mathcal{A}$ ,  $C^*(\mathcal{A})$  is a  $C^*$ -algebra with the  $C^*$ -norm, so  $\overline{\mathcal{A}}^{\|\cdot\|^*}$  is a  $C^*$ -algebra. But also  $\pi : \mathcal{A} \hookrightarrow \mathcal{B}(H)$  with  $\|a\|_\pi = \|\pi(a)\|_{\mathcal{B}(H)}$ . We could now take  $\|a\|_u = \sup\{\|a\| \mid \|\cdot\| \text{ is a } C^*\text{-(semi)-norm on } \mathcal{A}\}$ .

**Example 3.13** Take  $(\mathbb{N}, +)$  as a  $*$ -semigroup with the trivial involution  $n^* = n$ . Then  $\mathbb{C}[\mathbb{N}] = \left\{ \sum_{n \in \mathbb{N}}^f a_n \delta_n \mid a_n \in \mathbb{C} \right\}$ . Recognize  $\delta_n \cdot \delta_m = \delta_{n+m}$  and  $\delta_n^* = \delta_n$ . This is then isomorphic  $\mathbb{C}[\mathbb{N}] \simeq \mathbb{C}[X]$  (identify  $\delta_n$  with  $X^n$ ).

We are interested in the representation  $\pi : \mathbb{C}[X] \rightarrow \mathcal{B}(H)$ . For  $T = \pi(x) \in \mathcal{B}(H)$ ,  $T$  self adjoint as  $X$  is, we have  $\pi(\sum a_n X^n) = \sum a_n T^n$ . Thus,  $\text{Rep}(\mathbb{C}[X])$  equals the set of self-adjoint operators.

Given any  $T = T^* \in \mathcal{B}(H)$  and  $a = \sum a_n X^n \in \mathbb{C}[X]$ , it follows that  $\|a\|_T := \|\pi_T(a)\|_{\mathcal{B}(H)} = \|\sum a_n T^n\|_{\mathcal{B}(H)}$  is a  $C^*$ -seminorm.

When is  $\|\cdot\|_T$  a  $C^*$ -norm? We have  $\overline{\mathbb{C}[X]/N}^{\|\cdot\|_T} \simeq C^*(T) \subseteq \mathcal{B}(H)$ .

We also have  $\mathbb{C}[X] \hookrightarrow C^*(T, 1) \simeq C(\sigma(T))$ . Which  $T$  should we choose, is there a canonical or largest choice? The embedding of  $\mathbb{C}[X]$  in  $C^*(T, 1)$  can only be injective if  $|\sigma(\lambda)| = \infty$ , as  $p \mapsto p(T) \simeq (\lambda \mapsto p(\lambda))$ .

There is no 'largest' choice! Take any  $T$ , then  $\|a\|_T \leq r(T)$ . So if  $r \in \mathbb{R}$  (= self-adjoint operator on  $\mathbb{C}$ ) then  $\chi_r(X) = r$  and  $r$  specifies a character of  $\mathbb{C}[X]$ :  $\chi_r(\sum a_n X^n) = \sum a_n r^n$ . Using this to define a  $C^*$ -seminorm  $\|a\|_r = \|\chi_r(a)\|$  and  $\|x\|_r = |r| \in \mathbb{R}_+$ , which is not bounded.

Therefore, taking the supremum of all  $C^*$ -(semi)-norms does not always yield results. There is no universal  $C^*$ -algebra generated by a self-adjoint element! The problem is that there is no  $C_{\text{univ}}^*(\mathcal{A})$  out of any given  $*$ -algebra. But for a large class of  $*$ -algebras, there is such a universal  $C^*$ -algebra. Specifically, this is the case if  $\mathcal{A}$  as a  $*$ -algebra is generated by **partial isometries**, i.e. there exists  $S \subseteq \mathcal{A}$  where  $\mathcal{A}$  is generated by  $S$  as a  $*$ -algebra and  $ss^*s = s$  for all  $s \in S$ .

If  $\pi : \mathcal{A} \rightarrow \mathcal{B}(H)$  is a representation, then  $T = \pi(s) \in \mathcal{B}(H)$  are partial isometries for all  $s \in S$ . But then  $\|T\| \leq 1$  as  $TT^*$  and  $T^*T$  are projections.

Consequence: If  $S$  is an inverse semigroup, then there exists  $C^*(S) = C_{\text{univ}}^*(S) = C_{\text{univ}}^*(\mathbb{C}[S])$ .

The *universal property* is: Given any  $C^*$ -algebra  $\mathcal{B}$  and any  $*$ -homomorphism  $\pi : S \rightarrow \mathcal{B}$ , there is an extension  $\tilde{\pi} : C^*(S) \rightarrow \mathcal{B}$ ,  $\delta_s \mapsto \pi(s)$ .

Group  $C^*$ -algebras: If  $G$  is a Group there is  $C^*(G) = C_{\text{univ}}^*(\mathbb{C}[G])$ . It has the *universal property* that for all Hilbert spaces  $H$  and all unitary representations  $G \xrightarrow{u} \mathcal{U}(H) := \mathcal{U}(\mathcal{B}(H))$ ,  $u$  extends to  $\tilde{u} : C^*(G) \rightarrow \mathcal{U}(H)$ .

**Example 3.14** Take  $G = (\mathbb{Z}, +) = \langle 1 \rangle$ . For representations  $\mathbb{Z} \rightarrow \mathcal{B}(H)$ , choose any unitary element  $\pi(1) = u \in \mathcal{B}(H)$ . Observe now  $C^*(\mathbb{Z}) = \overline{\mathbb{C}[\mathbb{Z}]} = \overline{\left\{ \sum_{n \in \mathbb{Z}}^{fin} a_n z^n \right\}}$  is a commutative algebra and thus  $C^*(\mathbb{Z}) = C(X)$  and

$$\begin{aligned} X = C^*(\mathbb{Z}) &= \Omega(C^*(\mathbb{Z})) = \{\chi : C^*(\mathbb{Z}) \rightarrow \mathbb{C} \mid \text{non-zero homomorphism}\} \\ &\simeq \{\chi : \mathbb{Z} \rightarrow \mathbb{S}^1 = \mathcal{U}(\mathbb{C}) \mid \text{group homomorphisms}\} \simeq \mathbb{S}^1 \end{aligned}$$

as all these mappings are just  $n \mapsto z^n$  for an arbitrarily chosen  $z \in \mathbb{C}$ . Therefore,  $C^*(\mathbb{Z}) \simeq C(\mathbb{S}^1)$ .