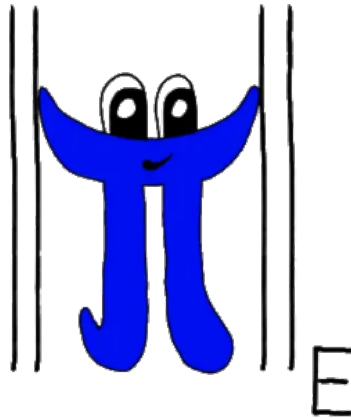


Exercise Sheet 04

Operator Algebras

Contributors: **Linus Mußmächer**, **Minona Schäfer**

June 21, 2023



4.1

The first statement follows immediately from the fact that the canonical inclusion $\mathcal{B} \hookrightarrow \mathcal{A}$ is an injective $*$ -homomorphism, so it is isometric as proven in the lecture.

If now \mathcal{B} is a dense proper $*$ -subalgebra of \mathcal{A} , assuming it could be turned into a C^* -algebra, the norm on that C^* -algebra would already have to be the norm on \mathcal{A} . But then the canonical inclusion is isometric and injective, so it has closed range and $\mathcal{B} \subseteq \mathcal{A}$ is closed and dense in \mathcal{A} . Now, however, we have $\mathcal{B} = \mathcal{A}$, a contradiction.

4.2

As hinted, the disk algebra $\mathcal{A}(\mathbb{D})$ is such an algebra. As we have $\bar{\bar{z}} = z$ for all $z \in \mathbb{D}$, the identity map is self-adjoint, but because of $i \in \text{id}(\mathbb{D})$, $\text{id} - i \cdot 1$ is not invertible. Thus, $i \in \sigma(\text{id}) \subsetneq \mathbb{R}$.

Analogously, we can also consider the character $\varphi : \mathcal{A}(\mathbb{D}) \rightarrow \mathbb{C} : f \mapsto f(i)$. Here, we have $\varphi(\text{id}^*) = \varphi(\text{id}) = i \neq -i = \overline{\varphi(\text{id})}$.

- (i) \Rightarrow (ii): Take $a \in \mathcal{A}$ be any self-adjoint element and (a) the $*$ -subalgebra generated by a in \mathcal{A} . Then $\Omega((a)) \subseteq \Omega(\mathcal{A})$ (as any character of (a) can be extended to \mathcal{A}), so any character in $\Omega((a))$ is also symmetric. As a is self-adjoint, we have $\chi(a) = \overline{\chi(a^*)} = \overline{\chi(a)}$ and therefore $\chi(a) \in \mathbb{R}$ for any $\chi \in \Omega((a))$. As (a) is a commutative $*$ -Banach-algebra, we have

$$\sigma_{(a)}(a) \subseteq \{\chi(a) \mid \chi \in \Omega((a))\} \cup \{0\} \subseteq \mathbb{R}.$$

Furthermore, if $a - \lambda \in (a)$ is invertible, it must also be invertible in $\mathcal{A} \supseteq (a)$, so $\rho(a)_{(a)} \subseteq \rho_{\mathcal{A}}(a)$ and therefore $\sigma_{\mathcal{A}}(a) \subseteq \sigma_{(a)}(a) \subseteq \mathbb{R}$.

(ii) \Rightarrow (iii): Let $\Gamma : \mathcal{A} \rightarrow C(\Omega(\mathcal{A})), a \mapsto (\chi \mapsto \chi(a))$ be the Gelfand-transform of \mathcal{A} . We want to prove $\Gamma(a) = \Gamma(a^*)^*$. By the Definition of the involution on $C(\Omega(\mathcal{A}))$, this is equivalent to $\chi(a) = \overline{\chi(a^*)}$ for any χ the spectrum and $a \in \mathcal{A}$.

First, let $a \in \mathcal{A}$ self-adjoint. Then $\overline{\chi(a^*)} = \overline{\chi(a)} = \chi(a)$ as $\chi(a) \in \sigma(a) \subseteq \mathbb{R}$.

If $a \in \mathcal{A}$ is not self-adjoint, we can write $a = b + ic$ for self-adjoint elements $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$ and it follows that

$$\chi(a) = \chi(b + ic) = \chi(b) + i\chi(c) = \overline{\chi(b^*)} + i\overline{\chi(c^*)} = \overline{\chi(b^*) - i\chi(c^*)} = \overline{\chi(b^* - ic^*)} = \overline{\chi(a^*)}$$

and this shows (iii).

(iii) \Rightarrow (i): If Γ is a *-homomorphism, then $\Gamma(a^*) = \Gamma(a)^*$ and by the definition of the involution as discussed above this already shows $\chi(a^*) = \overline{\chi(a)}$ for every character χ .