Exercise Sheet 01
Operator Algebras

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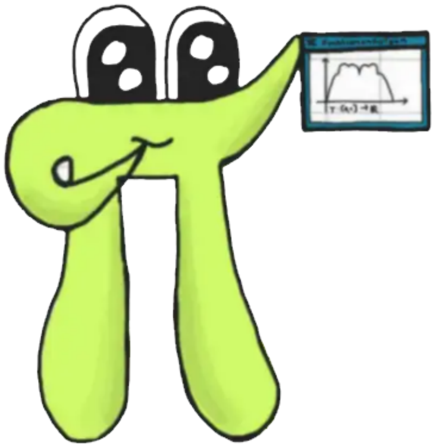

1.5

- First, we notice that if p is idempotent we have $(1-p)^2 = 1-2p+p^2 = 1-2p+p = 1-p$, so $1-p$ is also idempotent. Now consider the following two functions:

$$\begin{aligned}\varphi :&\mathscr{A} \to p\mathscr{A} \oplus (1-p)\mathscr{A}, a \mapsto pa \oplus (1-p)a\\ \psi :&p\mathscr{A} \oplus (1-p)\mathscr{A} \to \mathscr{A}, pa \oplus (1-p)b \mapsto pa + b - pb\end{aligned}$$

Concatenation of these two functions yields

$$\begin{aligned}\psi(\varphi(a)) &= \psi(pa \oplus (1-p)a) = pa + a - pa = a\\ \varphi(\psi(pa \oplus (1-p)b)) &= phi(pa + b - pb) = p(pa + b - pb) \oplus (1-p)(pa + b - pb)\\ &= p^2a + pb - p^2b \oplus pa - p^2a + b - pb - pb + p^2b\\ &= pa + pb - pb \oplus pa - pa + b - pb - pb + pb\\ &= pa \oplus (1-p)b\end{aligned}$$

showing that φ is a bijective mapping. Furthermore, φ we have

$$\begin{aligned}\varphi(a \cdot b) &= pab \oplus (1-p)ab = p^2ab \oplus (1-p)^2ab = (pa)(pb) \oplus ((1-p)a)((1-p)b)\\ &= (pa \oplus (1-p)a)(pb \oplus (1-p)b) = \varphi(a)\varphi(b)\end{aligned}$$

and φ is a homomorphism. Because $p \oplus (1-p)$ is the unit in $p\mathscr{A} \oplus (1-p)\mathscr{A}$ and $\varphi(1) = p \oplus (1-p)$, φ is also compatible with the unit.

- Consider the functions c_U and c_V, where $c_U|_U \equiv 1$ and $c_U|_V \equiv 0$ and likewise for c_V. These are continuous, idempotent and $c_U = 1 - c_V$ also holds. Note that these are in fact **not** elements of $C_0(X)$ as U, V need not necessarily be compact. However, above we have not used $p \in \mathscr{A}$ except for the fact that p admits a well-defined multiplication with elements of $\mathscr{A}$ yielding only elements of $\mathscr{A}$. Since c_U, c_V are still continuous the multiplication of $C(X)$ can be used here. By the argument above we then have $C_0(X) \simeq c_U C_0(X) \oplus c_V C_0(X) \simeq C_0(U) \oplus C_0(V)$ with

$$\varphi : C_0(X) \to C_0(U) \oplus C_0(V), f \mapsto (f \cdot c_U)|_U \oplus (f \cdot c_V)|_V = f|_U \oplus f|_V$$

an isomorphism of unital algebras. To show that this is also an isomorphism of C^*-algebras, we also have to show that φ is continuous and commutes with *. For the continuity, consider

$$\|\varphi(f)\| = \max\{\|f|_U\|, \|f|_V\|\} = \max\{\sup_{x\in U} |f(x)|, \sup_{x\in V} |f(x)|\} = \sup_{x\in X} |f(x)| = \|f\|$$

so φ is in fact even isometric (and thus bounded and continuous). Furthermore, we have

$$\varphi(f)^* = \overline{f|_U \oplus f|_V} = \overline{f|_U} \oplus \overline{f|_V} = \overline{f}|_U \oplus \overline{f}|_V = \varphi(\overline{f})$$

and φ is an isomorphism of C^*-algebras.

1.7

- Assume U is dense in X and consider $a \in C_0(X)$ with $aI = 0$ (and thus $Ia = 0$ since $C_0(X)$ is commutative). Let $x_0 \in U$ be any point in U. We want to prove that $a(x_0) = 0$. Since U is open, its complement $U^\complement$ is closed. Applying Uryson's Lemma to the compact set $\{x_0\}$ and the closed set $U^\complement$ (these sets are disjunct because of $x_0 \in U$) yields the existence of a function $f : X \to \mathbb{C}$ with $f(x_0) = 1$ and $f|_{U^\complement} \equiv 0$. Since we have $f \in C_0(U) = I$ because of the latter condition and because ideals are strongly closed with respect to multiplication, we have $af \in aI = 0$, so af is the zero function and in particular $a(x_0)f(x_0) = 0 \Rightarrow a(x_0) = 0$. Repeating this chain of reasoning for any $x_0 \in U$ shows that $a|_U \equiv 0$ and since U is dense in X and a is continuous, we have $a \equiv 0$. This shows that I is an essential ideal.

- Proof by contraposition. Let U be non-dense in X, i.e. there exists a point $x_0 \in X$ admitting an open neighborhood $V \subset X \setminus U$. We once again use Uryson's Lemma, this time for the two disjunct sets $\{x_0\}$ (compact) and $V^\complement$ (closed), proving the existence of a function $a : X \to \mathbb{C}$ with $a|_{V^\complement} \equiv 0$ and $a(x_0) = 1$, which is therefore not equivalent to the zero function. However, for any $f \in I$ we have $a \cdot f \equiv 0$ since f is zero on $U^\complement$ and a is zero on $U \subseteq V^\complement$. Therefore, we have $aI = 0$ but $a \neq 0$ and I cannot be an essential ideal of $C_0(X)$.

1.8

We first show the prerequisites of the Stone-Weierstrass theorem.

(i) I is a C^*-subalgebra of $C_0(U)$. I is a subset of $C_0(U)$. If it were not, we would have an $f \in I, f \notin C_0(U)$ and there would exist a point $x_0 \notin U$ with $f(x_0) \neq 0$. But then x_0 would not be in $U^\complement$ by the definition of U. Furthermore, I is an ideal, so it is closed with respect to addition and scalar multiplication (so it is a subspace) as well as multiplication (so it is a subalgebra). As I is a closed subspace of $C_0(U)$, which is a closed subspace of the

Banach space $C_0(X)$, I is Banach. The involution and its property can be inherited from $C_0(X)$.

It remains to show that I is closed with respect to this involution. For this, we use the provided hint. It is $f \in I$, then note that $f^* \in C_0(X)$ and f and f^* are both bounded. We can write $f^* f_n$ as

$$f^* f_n = f^*(ff^*)^{\frac{1}{n}} = f \cdot \underbrace{(f^{\frac{1}{n}-1}(f^*)^{\frac{1}{n}+1})}_{:=g}$$

and set $g(x) := 0$ on the zeroes of f. Then we have $g \in C_0(X)$ and also

$$|g(x)| = |f(x)^{\frac{1}{n}-1}(f^*)(x)^{\frac{1}{n}+1}| = |f(x)|^{\frac{1}{n}-1}|\overline{f(x)}|^{\frac{1}{n}+1} = |f(x)|^{\frac{2}{n}}$$

so $\{x \in X \mid |g(x)| \geq \varepsilon\} = \{x \in X \mid |f(x)| \geq \varepsilon^{\frac{n}{2}}\}$ is compact for every $\varepsilon > 0$. This shows $g \in C_0(X)$ and therefore $f^* f_n = fg \in I$ (because of the ideal property) for every $n \in \mathbb{N}$. The limit $\lim_{n\to\infty} f^* f_n$ converges in $C_0(X)$ to f^*, as $(f^* f)^{\frac{1}{n}} = |f|^{\frac{2}{n}}$ converges to the characteristic function of the support of f, i.e. to 0 if $f(x) = 0$ and to 1 otherwise. Since all elements $f^* f_n$ of the sequence are in I and since I is closed, the limit f^* is also contained in I. This shows $I^* \subseteq I$ and therefore $I^* = I$, so I is closed in respect to the involution.

(ii) Given $x \in U$, there is $f \in I$ with $f(x) \neq 0$. Assume that such an f did not exist, then for all $f \in I$ we have $f(x) = 0$. Per Definition of $U^{\complement}$, this implies $x \in U^{\complement}$, contradicting $x \in U$.

(iii) I separates points of U. Let x, y be arbitrary points in U. As proven above, there exists a function $f \in I$ with $f(x) \neq 0$. As X is Hausdorff, there also exists an open neighborhood V of x that does not contain y and (without loss of generality) is a subset of U. Then Uryson's Lemma proves the existence of a function g that is 1 on the compact set $\{x\}$ and that is 0 on the closed set $V^{\complement} \supset U^{\complement}$. The latter condition yields $g \in C_0(U)$, so the ideal property implies $fg \in I$. Additionally, we have $(fg)(x) = f(x)g(x) = f(x) \neq 0$ and $(fg)(y) = f(x)g(x) = f(x) \cdot 0 = 0$ (since $y \in V^{\complement}$). So fg separates x and y.

So I is a dense subspace of $C_0(U)$ by Stone-Weierstrass. But since I is closed, we have $I = \overline{I} = C_0(U)$.

Let $U \subset V$ be open sets in X. Then we have $V^{\complement} \subset U^{\complement}$, so any function in $C_0(X)$ that is 0 outside U is also 0 outside V, and we have $C_0(U) \subset C_0(V)$. Conversely, let $U \not\subseteq V$ be open sets in X, so there exists a point $x \in U, x \notin V$. Then Uryson's Lemma shows the existence of a function f that is 1 on the compact set $\{x\}$ and 0 on the closed set $U^{\complement}$. Since f is 0 outside U, we have $f \in C_0(U)$. However, f is non-zero on the point x outside V, so f cannot be in $C_0(V)$. Therefore, we have $C_0(U) \not\subseteq C_0(V)$. This shows $U \subseteq V \Leftrightarrow C_0(U) \subseteq C_0(V)$.

Lastly, let I be any maximal (and therefore closed) ideal in $C_0(X)$. Then $I = C_0(U)$ for some $U \neq X$ (or $C_0(X)$ would be the whole space and thus not maximal) and $X \setminus U$ is a closed, non-empty set. If $X \setminus U$ contains only a single element, our maximal ideal is of the form $C_0(X \setminus \{x\})$ for some $x \in X \setminus U$, and we are done. If $X \setminus U$ contains more than one element, choose any fixed $x \in X \setminus U$. Then, $X \setminus \{x\} \supset X \setminus U$ and thus $C_0(X \setminus \{x\}) \supset C_0(U)$. Therefore, $C_0(U)$ cannot be a maximal ideal, as it has a super-ideal that is not yet the entire space. So all maximal ideals of $C_0(X)$ must have form $C_0(X \setminus \{x\})$.