Exercise Sheet 04
Operator Algebras

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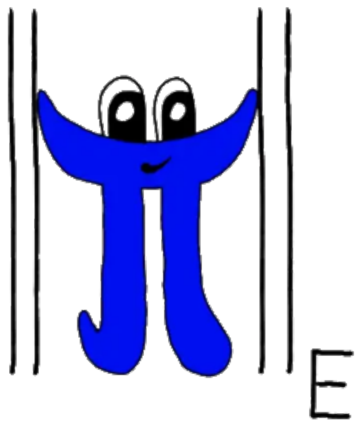
June 29, 2023



4.1

The first statement follows immediately from the fact that the canonical inclusion $\mathscr{B} \hookrightarrow \mathscr{A}$ is an injective *-homomorphism, so it is isometric as proven in the lecture.

If now $\mathscr{B}$ is a dense proper *-subalgebra of $\mathscr{A}$, assuming it could be turned into a C^*-algebra, the norm on that C^*-algebra would already have to be the norm on $\mathscr{A}$. But then the canonical inclusion is isometric and injective, so it has closed range and $\mathscr{B} \subseteq \mathscr{A}$ is closed and dense in $\mathscr{A}$. Now, however, we have $\mathscr{B} = \mathscr{A}$, a contradiction.

4.2

As hinted, the disk algebra $\mathscr{A}(\mathbb{D})$ is such an algebra. As we have $\overline{\overline{z}} = z$ for all $z \in \mathbb{D}$, the identity map is self-adjoint, but because of $i \in \mathrm{id}(\mathbb{D})$, $\mathrm{id} - i \cdot 1$ is not invertible. Thus, $i \in \sigma(\mathrm{id}) \subsetneq \mathbb{R}$.

Analogously, we can also consider the character $\varphi : \mathscr{A}(\mathbb{D}) \to \mathbb{C} : f \mapsto f(i)$. Here, we have $\varphi(\mathrm{id}^*) = \varphi(\mathrm{id}) = i \neq -i = \overline{\varphi(\mathrm{id})}$.

(i) $\Rightarrow$ (ii): Take $a \in \mathscr{A}$ be any self-adjoint element and (a) the *-subalgebra generated by a in $\mathscr{A}$. Then $\Omega((a)) \subseteq \Omega(\mathscr{A})$ (as any character of (a) can be extended to $\mathscr{A}$), so any character in $\Omega((a))$ is also symmetric. As a is self-adjoint, we have $\chi(a) = \overline{\chi(a^*)} = \overline{\chi(a)}$ and therefore $\chi(a) \in \mathbb{R}$ for any $\chi \in \Omega((a))$. As (a) is a commutative *-Banach-algebra, we have

$$\sigma_{(a)}(a) \subseteq \{\chi(a) \mid \chi \in \Omega((a))\} \cup \{0\} \subseteq \mathbb{R}.$$

Furthermore, if $a - \lambda \in (a)$ is invertible, it must also be invertible in $\mathscr{A} \supseteq (a)$, so $\rho(a)_{(a)} \subseteq \rho_{\mathscr{A}}(a)$ and therefore $\sigma_{\mathscr{A}}(a) \subseteq \sigma_{(a)}(a) \subseteq \mathbb{R}$.

(ii) $\Rightarrow$ (iii): Let $\Gamma : \mathscr{A} \to C(\Omega(\mathscr{A})), a \mapsto (\chi \mapsto \chi(a))$ be the Gelfand-transform of $\mathscr{A}$. We want to prove $\Gamma(a) = \Gamma(a^*)^*$. By the Definition of the involution on $C(\Omega(\mathscr{A}))$, this is equivalent to $\overline{\chi(a)} = \chi(a^*)$ for any χ the spectrum and $a \in \mathscr{A}$.

First, let $a \in \mathscr{A}$ self-adjoint. Then $\overline{\chi(a^*)} = \overline{\chi(a)} = \chi(a)$ as $\chi(a) \in \sigma(a) \subseteq \mathbb{R}$.

If $a \in \mathscr{A}$ is not self-adjoint, we can write $a = b + ic$ for self-adjoint elements $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$ and it follows that

$$\chi(a) = \chi(b + ic) = \chi(b) + i\chi(c) = \overline{\chi(b^*)} + i\overline{\chi(c^*)} = \overline{\chi(b^*) - i\chi(c^*)} = \overline{b^* - ic^*} = \overline{\chi(a^*)}$$

and this shows (iii).

(iii) $\Rightarrow$ (i): If Γ is a *-homomorphism, then $\Gamma(a^*) = \Gamma(a)^*$ and by the definition of the involution as discussed above this already shows $\chi(a^*) = \overline{\chi(a)}$ for every character χ.

4.3

Since the spectrum $\sigma(a)$ of a in the non-unital algebra $\mathscr{A}$ is defined as its spectrum in the unitization $\tilde{\mathscr{A}}$, the spectra in $C(\sigma(a))$ and $C_0(\sigma(a))$ have the same meaning and are not merely notationally equivalent.

Let $\Phi : C^*(a, 1) \to C(\sigma(a))$ be the isometric *-isomorphism in the fundamental theorem of functional calculus applied to $\tilde{\mathscr{A}}$ given by Gelfand. If $0 \notin \sigma(a)$, a is invertible so $1 \in C^*(a)$, and we therefore have both $C^*(a, 1) = C^*(a)$ and $C_0(\sigma(a)) = C(\sigma(a))$. Therefore, $\Phi^{-1} : C_0(\sigma(a)) \to C^*(a) \subseteq \mathscr{A}$ is already the unique isometric *-homomorphism we require, and its image is $C^*(a)$ as desired.

Now consider $0 \in \sigma(a)$ and the restriction $\Psi = \Phi|_{C^*(a)}$. This restriction retains the properties of a *-homomorphism, as well as the isometry and $a \mapsto \mathrm{id} \in C_0(\sigma(a))$. It remains to show that Ψ is still unique and $f(0) = 0$ for every element in the image of Ψ. To see the last property, notice that $\Psi(a) = \mathrm{id}$ and $\mathrm{id}(0) = 0$ as well as $\Psi(C^*(a)) = C^*(\Psi(a)) = C^*(\mathrm{id})$, so any element in the image of Ψ is composed of sums and products of id and $\overline{\mathrm{id}}$ and therefore fulfills $f(0) = 0$. So $\Psi^{-1} : C_0(\sigma(a)) \to C^*(a) \subseteq \mathscr{A}$ is our *-homomorphism.

To see the uniqueness, note that as $0 \in \sigma(a)$ we can write $\tilde{\mathscr{A}} \simeq \mathscr{A} \oplus \mathbb{C}$ and $C(\sigma(a)) \simeq C_0(\sigma(a)) \oplus \mathbb{C}$ (with a multiplication analogous to that of the unitization) and can therefore decompose Φ^{-1} into $\Phi = \Psi^{-1} \oplus \Phi^{-1}|_{\mathbb{C}}$. If there existed a second *-homomorphism $\Psi_2 : C_0(\sigma(a)) \to \mathscr{A}$ (with $\mathrm{id} \mapsto a$), then $\Phi_2 = \Psi_2 \oplus \Phi^{-1}|_{\mathbb{C}}$ were a second *-homomorphism $C_0(\sigma(a)) \oplus \mathbb{C} = C(\sigma(a)) \to \mathscr{A} \oplus \mathbb{C} = \tilde{\mathscr{A}}$ with $\mathrm{id} \mapsto a$, a contradiction.