

Introduction to Operator Algebras

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The set of all linear bounded operators $\mathcal{L}(H) = \mathcal{B}(H)$ on a given Banach space H is a (Banach) algebra with $S \cdot T = S \circ T$. $M \subseteq \mathcal{L}$ is a Subalgebra such that $M^* \subseteq M$ where T^* is the adjoint of T . This is also a closed subspace with respect to the strong topology. This is equivalent to $M = M''$ (when $X \subseteq \mathcal{B}(H)$, $X' = \{T \in \mathcal{B}(H) \mid TS = ST \ \forall S \in X\}$)

Some topological basics

Definition 0.1

- *Topology, Open*
- *Hausdorff, locally Hausdorff*
- *compact*

Definition 0.2 A topological space X is **locally Hausdorff** if every $x \in X$ admits a compact neighborhood basis, that is for every $x \in X$ and every open set $U \ni x$ there exists an open set $V \ni x$ with \bar{V} is compact.

Corollary 0.3 If a set V is compact in any subset $U \subseteq X$, it is also compact in X .

Example 0.4 (Snake with two heads) Consider $I = [0, 1]$ with the standard topology and extend the set with an element 1^+ such that $I \cup 1^+ \setminus 1$ is isomorphic to I . Then $I \cup 1^+$ is locally Hausdorff and compact, but not Hausdorff.

Some results about locally compact Hausdorff spaces

Lemma 0.5 (Uryson's Lemma) Let X be locally compact and Hausdorff. For all $F \subseteq X$ closed and $K \subseteq X$ compact with $F \cap K = \emptyset$, there exists an $f : X \rightarrow [0, 1]$ continuous such that $f|_K \equiv 1$ and $f|_F \equiv 0$.

Theorem 0.6 (Tietze's extension theorem) Let X be locally compact, $K \subseteq X$ compact and $f : K \rightarrow \mathbb{C}$ continuous. Then there exists a continuous $\tilde{f} : X \rightarrow \mathbb{C}$ such that $\tilde{f}|_K = f$.

Theorem 0.7 (Alexandroff's compactification) If X is locally compact and Hausdorff, then $\tilde{X} \sqcup \{\infty\}$ is a compact Hausdorff space $\mathcal{O}(\tilde{X}) = \mathcal{O}(X) \cup \{K^c \cup \{\infty\} \mid K \text{ compact}\}$.

Example 0.8 Compacting the real line \mathbb{R} yields the space $\tilde{\mathbb{R}}$, which is isomorphic to the unit circle $\Pi = \mathbb{S}^1$.

Theorem 0.9 Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$, $X := Y \setminus \{y_0\}$ is locally compact (in respect to the subspace topology).

More generally, if Y is locally compact and Hausdorff, and $Z \subseteq Y$ is a difference of open and closed subsets, of Y (i.e. $Z = U \setminus F$, where U is open in Y and F is closed in Y), then Z is locally compact.

1 Algebras

Definition 1.1 An **algebra** is a (complex) vector space \mathcal{A} endowed with a bilinear and associative multiplication: $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}, (a, b) \mapsto a \cdot b$. So

- (i) $(a + \alpha b) \cdot (c + \beta d) = ac + \alpha bc + \beta ad + \alpha \beta bd$
- (ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

for all $a, b, c \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$. We say that \mathcal{A} is

- (i) **commutative**, if $ab = ba$ for all $a, b \in \mathcal{A}$ and
- (ii) **unital**, if there exists $1 = 1_{\mathcal{A}} \in \mathcal{A}$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathcal{A}$.

Example 1.2

- (i) \mathbb{C} , or more generally $\mathbb{C}^n = \mathbb{C} \oplus \dots \oplus \mathbb{C}$, is an algebra.
- (ii) Say X is any set; let $\mathbb{C}^X = \{f : X \rightarrow \mathbb{C}\}$ with point wise multiplication $(f \cdot g)(x) = f(x) \cdot g(x)$. These are commutative unital algebras (with $1(x) = 1 \in \mathbb{C}$).
- (iii) Consider the polynomials $\mathbb{C}[X] = \{\sum_{i=0}^n \lambda_i x^i \mid \lambda_i \in \mathbb{C}, n \in \mathbb{N}\}$ with the usual operations. This is a commutative unital algebra.
- (iv) Let X be a topological space and $C(X) = \{f : X \rightarrow \mathbb{C} \mid f \text{ is continuous}\} \subseteq \mathbb{C}^X$ the set of continuous functions on X . This is a commutative unital (sub)algebra (of \mathbb{C}^X).
- (v) Take any vector space A define a (trivial) multiplication $a \cdot b := 0$. This is a commutative Algebra (that is not unital unless $A = 0$).
- (vi) $M_n(\mathbb{C})$ (the complex $n \times n$ matrices) with the usual multiplication are a non-commutative (unless $n = 1$) unital algebra.
- (vii) Let V be any (complex) vector space. The set of all linear operators $L(V) := \{T : V \rightarrow V \mid T \text{ linear operator}\}$ is a unital (non-commutative for $\dim V > 1$). We observe $\mathcal{L}(\mathbb{C}^n) \simeq M_n(\mathbb{C})$.
- (viii) Let S be a semigroup (i.e. a set with an associative operation $S \times S \rightarrow S$, e.g. $(\mathbb{N}, +)$). Then $\mathbb{C}[S] = \{\sum_{s \in S} \lambda_s s \mid \lambda_s \in \mathbb{C}, |\{s : \lambda_s \neq 0\}| < \infty\}$ (the finite formal sums of elements of S) with the following product

$$\left(\sum_{s \in S'} \lambda_s s \right) \cdot \left(\sum_{t \in S} \lambda'_t t \right) := \sum_{s, t \in S} (\lambda_s \cdot \lambda'_t)(s \cdot t) \in S$$

Observe: As a vector space: $\mathbb{C}[S] \subseteq \mathbb{C}^S$. In general, this is neither commutative nor unital.

2 Normed algebras

Definition 2.1 An algebra \mathcal{A} is **normed**, if it is endowed with a (vector space) norm $\|\cdot\|: \mathcal{A} \rightarrow [0, \infty)$ satisfying $\|a \cdot b\| \leq \|a\| \cdot \|b\|$. If \mathcal{A} is unital with unit $1_{\mathcal{A}}$, we usually assume $\|1_{\mathcal{A}}\| = 1$ except for $\mathcal{A} = 0$.

Definition 2.2 A **Banach algebra** is a normed algebra that is also complete (as a metric space with respect to the distance $d(a, b) := \|a - b\|$), i.e. every Cauchy sequence converges.

Example 2.3 (i) If X is a compact space then $C(X)$ is a commutative unital Banach algebra with respect to the norm $\|f\|_{\infty} := \sup_{x \in X} |f(x)| < \infty$ (since X is compact).

(ii) If V is a normed (respectively Banach) vector space, e.g. \mathbb{C}^n or $\ell^p(\mathbb{N})$, then $\mathcal{L}(V) = \{T \in L(V) \mid T \text{ is bounded/continuous}\}$ with $\|T\| := \sup_{\|v\| \leq 1} \|T(v)\| < \infty$ is a normed Banach algebra.

(iii) If X is a topological space, then $C_b(X) = \{f \in C(X) \mid \|f\|_{\infty} < \infty\}$ (bounded continuous functions) is a Banach space.

(iv) Let X again be a topological space. Then the set of all functions **vanishing at ∞** ,

$$\begin{aligned} C_0(X) &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \exists K \subseteq X, K \text{ compact} \forall_{x \notin K} |f(x)| < \varepsilon\} \\ &= \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\} \subseteq C_b(X), \end{aligned}$$

is also a Banach algebra.

Exercise 2.1 Assume X is locally compact and Hausdorff. Prove the following are equivalent:

- (1) X is compact.
- (2) $C(X) = C_0(X)$
- (3) $C_0(X)$ is unital.
- (4) The unit function $1 \in C_b(X)$ belongs to $C_0(X)$.

PROOF: • (1) \Rightarrow (2): Recall the definition of $C_0(X)$. If X is compact, every closed subset (especially every $\{x \mid |f(x)| \geq \varepsilon\}$) is compact, so the condition of $C_0(X)$ is trivial.

• (2) \Rightarrow (3): Since $C(X)$ is unital, $C_0(X)$ is as well.

• (3) \Rightarrow (4): Suppose C_0 is unital, and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, i.e. $f(x)g(x) = g(x) \forall_{x \in X} \forall_{g \in C_0(X)}$. By Uryson's lemma, given any $x_0 \in X$, there exists $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$ and taking F as the complement of any relatively compact environment of x_0). Then $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. Doing this for every $x_0 \in X$ yields $f \equiv 1$.

• (4) \Rightarrow (1): Since $1 \in C_0(X)$, for every $\varepsilon > 0$ the set $\{x \mid |f(x)| \geq \varepsilon\}$ is compact. Choose $\varepsilon = \frac{1}{2}$. Then, $\{x \mid |f(x)| = |1| \geq \frac{1}{2}\} = X$ is compact. \square

Exercise 2.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \cong \{f \in C(X) \mid f(\infty) = 0\}$

3 Algebras

Definition 3.1 A **-algebra* is a complex algebra \mathcal{A} with an *involution* $*$: $\mathcal{A} \rightarrow \mathcal{A}$ satisfying

- (i) $(a + \lambda b)^* = a^* + \bar{\lambda}b^*$
- (ii) $(a^*)^* = a$
- (iii) $(ab)^* = b^*a^*$

for all $a, b \in \mathcal{A}$ and all $\lambda \in \mathbb{C}$.

Definition 3.2 A *normed *-algebra* is a normed algebra \mathcal{A} with an involution (such that \mathcal{A} is a *-algebra) also satisfying $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.

A *Banach*-algebra* is a complete normed *-algebra.

Definition 3.3 A *C*-algebra* is a Banach*-algebra satisfying $\|a^* \cdot a\| = \|a\|^2$.

Observation: Recall that $\|a \cdot b\| \leq \|a\| \cdot \|b\|$ in all normed algebras. Applying this to a C*-algebra we get $\|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$. If \mathcal{A} is a C*-algebra, then $\|a\|^2 = \|a \cdot a^*\| \leq \|a^*\| \cdot \|a\|$, so $\|a\| = \|a^*\|$.

Example 3.4

- (i) If X is a set, then \mathbb{C}^X is a *-algebra with $f^* = \bar{f}$ and $\mathcal{C}^\infty(X)$ is a C*-algebra.
- (ii) If X is a topological space, then $C(X) \subseteq \mathbb{C}^X$ is also a *-subalgebra and for $\{f \in C(X) \mid \text{supp}(f) = \overline{\{x \in X \mid |f(x)| \neq 0\}} \text{ compact}\}$ we have

$$C_c(X) \subseteq C_0(X) \subseteq C_b(X) \subseteq C(X) \subseteq C^\infty(X)$$

and C^∞ is a C*-algebra. C_c is a *-algebra, but not Banach in general.

If X is compact, it follows $C_c(X) = C_0(X) = C_b(X)$.

Observation: If X is locally compact and Hausdorff, then $\overline{C_c(X)} = C_0(X)$.

- (iii) Let X be a measured space (X is endowed with a σ -algebra). Then $B_\infty(X) = \{f \in C^\infty \mid f \text{ is measurable}\}$ is a C*-algebra. If μ is a measure on X (e.g. $X = \mathbb{R}^n$ and μ the Lebesgue measure) then $L^\infty(X, \mu)$ are the essentially bounded functions and

$$L^\infty(X) = \{f : X \rightarrow \mathbb{C} \mid \|f\| := \inf\{c \geq 0 \mid \mu(\{x \mid |f(x)| > c\}) = 0\}\}$$

is also a C*-algebra.

Observation: $L^2(X, \mu) = \{\mu\text{-separable function}\}$, $L^\infty(X, \mu) \xrightarrow{\mu} B(L^2(X, \mu)), f \mapsto \mu_f = \{g \mapsto f \cdot g\}$

- (iv) A non-example: Let \mathbb{D} be the unit disk and $\mathcal{A}(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid \text{analytic in } \mathbb{D}^\circ\}$

Morera's Theorem from complex analysis states that $f \in C(\mathbb{D})$ is analytic if and only if $\int_\gamma f(z)dz = 0$ for all closed and piece wise smooth paths in \mathbb{D}° . From this, it follows that $\mathcal{A}(\mathbb{D})$ is closed in $C(\mathbb{D})$, therefore a Banach algebra. It is also a Banach*-algebra with, but $f^* = \bar{f}$ (point wise) is not possible, as $z \mapsto \bar{z}$ is not analytic. Thus, we have to choose $f^*(z) = f(\bar{z})$. But $\mathcal{A}(\mathbb{D})$ is not a C*-algebra, as $\|f^*f\|_\infty \neq \|f\|_\infty^2$ for some $f \in \mathcal{A}(\mathbb{D})$.

- (v) A non-commutative example: Let H be a Hilbert space and $B(H) = \mathcal{L}(H) = \{T : H \rightarrow H \mid T \text{ bounded, continuous, linear}\}$ and $\|H\| := \sup_{\|z\| < 1} \|T(z)\| < \infty$. This is a C^* -algebra where T^* is the adjoint of T , that is $\langle T^*z, w \rangle = \langle z, Tw \rangle$ for all $z, w \in H$.

C^* -axiom: $\|T^* \cdot T\| \leq \|T\|^2$ since $\mathcal{L}(H)$ is a Banach algebra, and we also have

$$\begin{aligned} \|T\|^2 &= \sup_{\|z\| < 1} \|T(z)\|^2 = \sup_{\|z\| < 1} \langle Tz, Tz \rangle = \sup_{\|z\| < 1} \langle z, T^*Tz \rangle \\ &\leq \sup_{\|z\| < 1} \|z\| \|T^*Tz\| \leq \sup_{\|z\| < 1} \|z\| \|T^*T\| \leq \|T^*T\| \end{aligned}$$

In particular, $M_n(\mathbb{C}) \simeq \mathcal{L}(\mathbb{C}^n)$ is a unital C^* -algebra.

- (vi) To produce more examples, take any subset $S \subseteq \mathcal{L}(H)$ and take $C^*(S) \subseteq \mathcal{L}(H) = \overline{\text{span}\{S_i \mid S_i \in S \cup S^*, i \leq n \in \mathbb{N}\}}$.

Example 3.5 Let $s \in \mathcal{L}(\ell^2(\mathbb{N}))$. The shift s , defined by $s(e_i) = e_{i+1}$ for all $i \in \mathbb{N}$ (where $\{e_i\}$ is the canonical basis of the sequence space), is an isometry, that is $s^* \cdot s = \text{id}$. Since $s \cdot s^* \neq \text{id}$, it is not surjective and not a proper isometry. We define

$$T = C^*(s) = \overline{\text{span}\{s^n(s^*)^m \mid m, n \in \mathbb{N}_0\}} \subseteq \mathcal{L}(\ell^2(\mathbb{N}))$$

as the **Toeplitz algebra**.

Example 3.6 Let H be a Hilbert space and S the set of all finite rank operators on H .

Example 3.7

- (i) **Commutative:** $C_0(X)$ for a locally Hausdorff space X .
- (ii) **Non-commutative:** $\mathcal{L}(\mathfrak{H}) = \mathcal{B}(\mathfrak{H})$ for any Hilbert space \mathfrak{H} (with dimension greater 1).
- (iii) **More generally:** Take any subset $S \subseteq \mathcal{L}(\mathfrak{H})$ and construct $C^*(S) \subseteq \mathcal{L}(H)$ as

$$\overline{\text{span}\{S_1, \dots, S_n \mid S_i \in S \cap S^*\}}$$

Example 3.8 (Cuntz algebras) Take again $\mathfrak{H} = \ell^2\mathbb{N} = \{(\lambda_n)_{n \in \mathbb{N}_0} \mid \sum_{n=0}^{\infty} |\lambda_n|^2 < \infty\}$ where $\langle \lambda, \lambda' \rangle = \sum_{i \in \mathbb{N}_0} \overline{\lambda_i} \lambda'_i$ and which has the orthonormal base $(e_n)_{n \in \mathbb{N}}$ where $(e_n) = (\delta_{in})_{i \in \mathbb{N}_0}$.

On this algebra, define

- $S_1(e_n) = e_{2n}$.
- $S_2(e_n) = e_{2n+1}$.

We have partitioned the natural numbers into evens and odds. This defines two (proper) isometries $S_1, S_2 \in \mathcal{L}(\mathfrak{H})$, that is $S_i^* S_i = \text{id}_{\mathfrak{H}}$, to subspaces of \mathfrak{H} . Notice: $S_i^* S_j = 0$ for $i \neq j$ as well as $S_1 S_1^* + S_2 S_2^* = \text{id}_{\mathfrak{H}}$. Define $\mathcal{O}_2 = C^*(S_1, S_2) = \overline{\text{span}\{S_\alpha S_\beta^* \mid \alpha, \beta \text{ finite words in } \{1, 2\}\}}$. For example, for $\alpha = 121211$ we have $S_\alpha = S_1 S_2 S_1 S_2 S_1^2$. \mathcal{O}_2 is called the **Cuntz algebra**. More generally, one can define $\mathcal{O}_3, \mathcal{O}_4, \dots$ Cuntz algebras. Joachim Cuntz proved that these are simple C^* -algebras with additional interesting properties we will see later.

Example 3.9 (Rotation algebras) Let $\mathfrak{H} = \ell^2(\mathbb{Z})$ (bi-infinite sequences) with basis $(e_n)_{n \in \mathbb{Z}}$. Define:

- $U(e_n) := e_{n+1}$ (bilateral shift)

- $V(e_n) := \lambda^n e_n$ where $\lambda \in \mathbb{C}$ is some fixed number $|\lambda| = 1$.

This defines two *unitary* operators: $UU^* = 1 = U^*U$ and $V^*V = 1 = V^*V$. If $\exp(2\pi i\theta), \theta \in \mathbb{R}$ define $A_\theta := C^*(U, V) \subseteq \mathcal{L}(\ell^2\mathbb{N})$.

There is a special relation between U and V where $UV = \lambda VU = \exp(2\pi i\theta)VU$. From this relation, we can describe $A_\theta = \overline{\text{span}}\{\sum_{n,m \in \mathbb{Z}}^{\text{finite}} a_{n,m} U^n V^m \mid a_{n,m} \in \mathbb{C}\}$.

Furthermore, if $\theta \in \mathbb{R} \setminus \mathbb{Q}$, A_θ is simple.

Example 3.10 (C^* -algebras of groups) Let G be a (discrete) group. Look at $\mathfrak{H} = \ell^2(G) = \{(a_g)_{g \in G} \mid \sum_{g \in G} |a_g|^2 < \infty\}$ (Note: This limit will only converge if there are countably (or finitely) many non-zero parts) with ONB $(\delta_g)_{g \in G}$ where $\delta_g(h) = \delta_{gh}$. Define for each $g \in G$ an operator $\lambda_g \in \mathcal{L}(\ell^2 G)$ by $\lambda_g(\delta_h) = \delta_{gh}$. Notice that $h \mapsto gh$ is a bijection, and thus λ_g is a unitary operator with $\lambda_g^* = \lambda_{g^{-1}}$. We can now define the **reduced C^* -algebra** of the group:

$$C_R^*(G) := C_\lambda^*(G) \subseteq \mathcal{L}(\ell^2 G) = C^*(\lambda_g \mid g \in G)$$

Here, we have the relation $\lambda_g \cdot \lambda_h = \lambda_{gh}$ and thus $C_R^*(G) = \{\sum a_g \lambda_g \mid a_g \in \mathbb{C}\}$.

In general, take $U : G \rightarrow \mathcal{L}(H), g \mapsto U_g$ a **unitary representation of G** with $U_g U_h = U_{gh}$ and $U_1 = \text{id}$ as well as $U_g^{-1} = U_{g^{-1}}$. Then $C_U^*(G) := \{\sum_{g \in G} a_g U_g \mid a_g \in \mathbb{C}\} \subseteq \mathcal{L}(H)$. There exists a **universal unitary representation** $C_{\max}^*(G)$, a full C^* -algebra of G .

Remark 3.11

- (i) If G is Abelian, then $C_U^*(G)$ is also abelian (commutative). In particular, C_λ^* is abelian. Later, we will prove $C_\lambda^*(G) \simeq C(\hat{G})$ where \hat{G} is the dual of G , i.e. $\{X : G \rightarrow \mathbb{C} \text{ characters}\}$.
- (ii) For many groups, like $G = \mathbb{F}_n$ (the free groups) the reduced C^* -algebra $C_\lambda^*(G)$ is simple.

4 Homomorphisms of algebras

Definition 4.1 If \mathcal{A}, \mathcal{B} are algebras, a **homomorphism** from \mathcal{A} to \mathcal{B} is a linear map $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for any $a, b \in \mathcal{A}$.

If \mathcal{A} and \mathcal{B} are $*$ -algebras, a **$*$ -homomorphism** is a homomorphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in \mathcal{A}$.

If \mathcal{A}, \mathcal{B} are Banach algebras, then usually we want to have **continuous** homomorphisms. Even more, we usually ask for **contractive** homomorphisms $\varphi : \mathcal{A} \rightarrow \mathcal{B}$, (that is $\|\varphi\| \leq 1$).

We will be especially interested in **characters**:

Definition 4.2 A **character** of an algebra \mathcal{A} is a non-zero homomorphism $\chi : \mathcal{A} \rightarrow \mathbb{C}$.

Example 4.3 Take any subalgebra $\mathcal{A} \subseteq \mathbb{C}^X$. Take $x_0 \in X$ and set $\chi_{x_0} := \text{ev}_{x_0} : \mathcal{A} \rightarrow \mathbb{C}, f \mapsto f(x_0)$. This is not necessarily a character, but it is for example, if $\mathcal{A} = C(X)$ or $C_b(X)$ or $C_0(X)$ (if X is “nice”, like Hausdorff).

Definition 4.4 A $(*)$ -isomorphism between two $(*)$ -algebras \mathcal{A} and \mathcal{B} is a bijective $(*)$ -homomorphism $\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$.

Definition 4.5 A $(*)$ -ideal of a $*$ -algebra \mathcal{A} is a subspace $I \subset \mathcal{A}$ such that $I \cdot \mathcal{A} \subseteq I, \mathcal{A} \cdot I \subseteq I$ (if only one condition applies, we call this a **left ideal** or **right ideal**). For $*$ -ideals, we also want $I^* = I$. We notate this as $I \trianglelefteq \mathcal{A}$.

Example 4.6 If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a $(*)$ -homomorphism, then $\ker \varphi \trianglelefteq \mathcal{A}$.

Example 4.7 If $I \trianglelefteq \mathcal{A}$ for \mathcal{A} a $(*)$ -algebra

$$\mathcal{A}/I = \{a + I \mid a \in \mathcal{A}\}$$

with $(a + I) \cdot (b + I) := ab + I$ and $(a + I)^* = a^* + I$ is a $(*)$ -algebra.

Theorem 4.8 If \mathcal{A} is a Banach- $*$ -algebra, then $I \trianglelefteq \mathcal{A}$ is a closed ideal, then the quotient I/\mathcal{A} is also a Banach- $*$ -algebra.

PROOF: Later. □

5 Spectral theory

Notation 5.1 If \mathcal{A} is a unital algebra, we write

$$\text{inv}(\mathcal{A}) = \{a \in \mathcal{A} \mid a \text{ is invertible in } \mathcal{A}\} = \{a \in \mathcal{A} \mid \exists a^{-1} \in \mathcal{A} aa^{-1} = 1 = a^{-1}a\}$$

This is a group. Sometimes we also write $GL(\mathcal{A})$.

Definition 5.2 Given a unital algebra \mathcal{A} and $a \in \mathcal{A}$, we define its **spectrum** (in \mathcal{A}) as

$$\sigma_{\mathcal{A}}(a) = \sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a \notin \text{inv}(\mathcal{A})\}$$

and the resolvent of a (in \mathcal{A}) as

$$\rho_{\mathcal{A}}(a) = \rho(a) = \mathcal{A} \setminus \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in \text{inv}(\mathcal{A})\}$$

Example 5.3 (Linear Algebra) Let $\mathcal{A} = M_m(\mathbb{C})$ and $a \in \mathcal{A}$. Then we have

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \det(\lambda - a) = 0\}$$

and these are the roots of the characteristic polynomial $\det(\lambda - a)$. This is exactly the usual spectrum from linear algebra.

Example 5.4 (Functional Analysis) Let $\mathcal{A} = \mathcal{L}(\mathfrak{H})$ – where \mathfrak{H} is any Hilbert- or Banach space – and $T \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(T)$ is exactly the spectrum as defined in functional analysis.

If S is the shift in $\mathcal{L}(\ell^2\mathbb{N})$, then we have $\sigma(S) = \mathbb{D}$.

Example 5.5 Let $\mathcal{A} = \mathbb{C}[X]$. Here we have $\text{inv}(\mathcal{A}) = \{a_0 X^0 \mid a_0 \in \mathbb{C} \setminus \{0\}\}$ the constant non-zero polynomials. If $a = \sum_{k=0}^N a_k x^k \in \mathcal{A}$, then we have two cases:

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \mathbb{C} & \text{otherwise} \end{cases}$$

Example 5.6 Let $\mathcal{A} = \mathbb{C}(X) = \{p/q \mid p, q \in \mathbb{C}[X], q \neq 0\}$. Now we have $\text{inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$. If $a \in \mathcal{A}$, then

$$\sigma(a) = \begin{cases} \{a_0\} & a = a_0 \text{ (constant)} \\ \emptyset & \text{otherwise} \end{cases}$$

Example 5.7 Let $\mathcal{A} = C(X)$ for any topological space X . Then

$$\text{inv}(\mathcal{A}) = \{f \in C(X) \mid \forall_{x \in X} f(x) \neq 0\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{x \in X} f(x) = \lambda\} = \text{im}(f) = f(X).$$

Example 5.8 Let X be any topological space and consider $\mathcal{A} = C_b(X)$. Then

$$\text{inv}(C_b(X)) = \{f \in C_b(X) \mid \exists_{\varepsilon > 0} \forall_{x \in X} |f(x)| \geq \varepsilon\}$$

and

$$\sigma(f) = \{\lambda \in \mathbb{C} \mid \lambda - f \notin \text{inv}(\mathcal{A})\} = \{\lambda \in \mathbb{C} \mid \exists_{(x_n)} f(x_n) \rightarrow \lambda\} = \overline{\text{im}(f)} = \overline{f(X)}.$$

This is a compact subset of \mathbb{C} .

Theorem 5.9 (Algebraic spectral mapping theorem) Let \mathcal{A} be an algebra, $a \in \mathcal{A}$ and $p \in \mathbb{C}[X]$, $p(X) = \sum_{k=0}^n \lambda_k X^k$ and define $p(a) = \sum_{k=0}^n \lambda_k a^k$. Recall that the mapping $\mathbb{C}[X] \rightarrow \mathcal{A}$, $p \mapsto p(a)$ is a unital homomorphism.

Then $\sigma(p(a)) = p(\sigma(a))$ assuming $\sigma(a) \neq \emptyset$.

PROOF: If $p(X) = \lambda_0$ constant, this is clear (the spectrum is exactly λ_0 on both sides). Assume $p(x)$ is not constant. Fix $\mu \in \mathbb{C}$ and write

$$\mu - p(x) = \lambda_0(x - \lambda_1) \cdots (x - \lambda_n)$$

as per the fundamental theorem of algebra (note that these are not the same λ as before) with $\lambda_0 \neq 0$. Then $\mu - p(a) = \lambda_0(a - \lambda_1) \cdots (a - \lambda_n)$. Since these expressions commute, this product is invertible if and only if $(a - \lambda_i)$ is invertible for every i . So $\mu \in \sigma(p(a)) \Leftrightarrow \mu - p(a)$ is not invertible if and only if there exists an i for which $\lambda_i - a$ is not invertible, so $\lambda_i \in \sigma(a)$. But the λ_i are exactly the numbers satisfying $p(\lambda) = \mu$. Thus, μ is in $\sigma(p(a))$ if it is in the image of $\sigma(a)$ under p . Therefore, we conclude $\sigma(p(a)) = p(\sigma(a))$. \square

We now focus on invertible elements in **Banach algebras**.

Theorem 5.10 If \mathcal{A} is a unital Banach algebra and $a \in \mathcal{A}$ with $\|a\| < 1$ then $1 - a$ is invertible and $(1 - a)^{-1} = \sum_{n=0}^{\infty} a^n$.

PROOF: Observe that, since $\|a\| < 1$, we have $\sum_{n=0}^{\infty} \|a\|^n = \frac{1}{1 - \|a\|} < \infty$. This implies the (absolute) convergence of $\sum_{n=0}^{\infty} a^n$ by the characteristic property of Banach spaces. Hence, $b := \lim_{N \rightarrow \infty} \sum_{n=0}^N a^n \in \mathcal{A}$. No, if $N \in \mathbb{N}$, then

$$(1 - a) \left(\sum_{n=0}^N a^n \right) = \left(\sum_{n=0}^N a^n \right) - \left(\sum_{n=1}^{N+1} a^n \right) = 1 - a^{N+1} \rightarrow 1$$

because of $\|a\| < 1$. This yields $(1 - a)b = 1$. \square

Theorem 5.11 Let \mathcal{A} be a non-empty, non-zero unital Banach algebra. Then $\text{inv}(\mathcal{A})$ is an open subset of \mathcal{A} and the function $f : \text{inv}(\mathcal{A}) \rightarrow \mathcal{A}$, $a \mapsto a^{-1}$ is Frechet-differentiable and in particular continuous as well as $f'(a)b = -a^{-1}ba^{-1}$.

Recall from calculus that $\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$. Also recall that $f : U \xrightarrow{\text{open}} X \rightarrow Y$ with X, Y Banach spaces is **differentiable** at $x_0 \in U$ there exists an operator $D_{x_0} = f'(x_0) \in \mathcal{L}(X, Y)$ such that

$$\lim_{h \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - D_{x_0}(h)\|}{\|h\|} = 0$$

PROOF: Take $a \in \text{inv}(\mathcal{A})$. If $b \in \mathcal{A}$ such that $\|a - b\| < \|a^{-1}\|^{-1}$. From this, we have $\|ba^{-1} - 1\| = \|ba^{-1} - aa^{-1}\| = \|(b - a)a^{-1}\| \leq \|b - a\| \cdot \|a^{-1}\| < 1$. Per the previous theorem, $ba^{-1} \in \text{inv}(\mathcal{A})$. This implies that b is also invertible. This shows that $\text{inv}(\mathcal{A})$ is open.

Furthermore, if $\|b\| < 1$, then also $\| -b \| < 1$. Thus, $1 + b \in \text{inv}(\mathcal{A})$ and $(1 + b)^{-1} = \sum_{n=0}^{\infty} (-1)^n b^n$. Thus,

$$\|(1 + b)^{-1} - 1 + b\| = \left\| \sum_{n=0}^{\infty} (-1)^n b^n - 1 + b \right\| \leq \left\| \sum_{n=2}^{\infty} (-1)^n b^n \right\| \leq \sum_{n=2}^{\infty} \|b^n\| \leq \sum_{n=2}^{\infty} \|b\|^n = \frac{\|b\|^2}{1 - \|b\|}$$

Now let $a \in \text{inv}(\mathcal{A})$ and $c \in \mathcal{A}$ such that $\|c\| < \frac{1}{2} \|a^{-1}\|^{-1}$. Then $\|a^{-1}c\| \leq \|a^{-1}\| \|c\| \leq \frac{1}{2}$. So if $b = a^{-1}c$, then

$$\|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \leq \frac{\|a^{-1}c\|^2}{1 - \|a^{-1}c\|} < 2\|a^{-1}c\|^2$$

Now, define $U : \mathcal{A} \rightarrow \mathcal{A}, b \mapsto -a^{-1}ba^{-1}$. Then this is a linear odd operation with $\|U\| \leq \|a^{-1}\|^2$, and we have

$$\begin{aligned} \|(a + c)^{-1} - a^{-1} - U(c)\| &= \|(a + c)^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &= \|(1 + a^{-1}c)^{-1}a^{-1} - a^{-1} + a^{-1}ca^{-1}\| \\ &\leq \|(1 + a^{-1}c)^{-1} - 1 + a^{-1}c\| \cdot \|a^{-1}\| \\ &\leq 2\|a^{-1}c\|^2 \|a^{-1}\| \leq 2\|a^{-1}\|^3 \|c\|^2 \end{aligned}$$

and thus

$$\lim_{c \rightarrow 0} \frac{\|(a + c)^{-1} - a^{-1} - U(c)\|}{\|c\|} = 0 \quad \square$$

Example 5.12 If we choose $\mathcal{A} = \mathbb{C}[X]$ and the norm $\|p\| = \sup_{\lambda \in [0,1]} |p(\lambda)|$. Then $(\mathcal{A}, \|\cdot\|)$ is a normed (but not Banach) algebra. For example, we see that $\lim_{m \rightarrow 0} 1 + X/m = 1 \in \text{inv}(\mathcal{A})$, but $1 + X/m \notin \text{inv}(\mathcal{A})$ and thus $\text{inv}(\mathcal{A})$ is not open (because the complement is not closed).

Theorem 5.13 *If \mathcal{A} is a Banach algebra with unit 1, then for all $a \in \mathcal{A}$ the spectrum $\sigma(a) \subseteq \mathbb{C}$ is closed and $\sigma(a) \subseteq \overline{B(0, \|a\|)} = D(0, \|a\|) := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \|a\|\}$. Therefore, $\sigma(a)$ is compact by the Heine-Borell theorem.*

PROOF: By definition

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin \text{inv}(\mathcal{A})\}$$

is the inverse image of the closed subset $\mathcal{A} \setminus \text{inv}(\mathcal{A}) \subseteq \mathcal{A}$ by the continuous function $\lambda \mapsto \lambda - a$. Therefore, $\sigma(a)$ is closed.

Now if $|\lambda| \leq \|a\|$ then $\|\lambda^{-1}a\| < 1$. Then $1 - \lambda^{-1}a \in \text{inv}(\mathcal{A})$. Multiplying by λ yields $\lambda - a \in \text{inv}(\mathcal{A})$. Thus, $\{\lambda \in \mathbb{C} \mid |\lambda| > \|a\|\} \subseteq \rho(a)$ and thus $\sigma(a) \subseteq D(0, \|a\|)$. \square

Lemma 5.14 *Let \mathcal{A} be a unital Banach algebra and $a \in \mathcal{A}$. Then, the map $R_a : \rho(a) \subseteq \mathbb{C} \rightarrow \mathcal{A}, \lambda \mapsto (a - \lambda)^{-1}$ is Frechet-differentiable.*

PROOF: This follows from the following general result:

If $g : U \xrightarrow{\text{open}} X \rightarrow Y$ and $f : V \xrightarrow{\text{open}} Y \rightarrow Z$ for Banach spaces X, Y, Z with $g(U) \subseteq V$ are differentiable at $x_0 \in U$ or respectively $y_0 = g(x_0) \in V$, then $f \circ g$ is differentiable and $(f \circ g)'(x_0) = f'(g(x_0))g'(x_0)$. \square

Observation: For $R_a(\lambda) = (a - \lambda)^{-1}$ we get $R'_a(\lambda) = (a - \lambda)^{-2}$. We have $\mathcal{L}(\mathbb{C}, \mathcal{A}) \simeq \mathcal{A}$ by $T \mapsto T(1)$. Recall that if $f(a) = a^{-1}$ yields $f'(a)b = -a^{-1}ba^{-1}$.

Theorem 5.15 (Gelfand) *If $\mathcal{A} \neq 0$ is a unital Banach algebra and $a \in \mathcal{A}$ then $\sigma(a) \neq \emptyset$.*

PROOF: Suppose $\sigma(a) = \emptyset$. Idea: Show that $R_a : \rho(a) \subseteq \mathbb{C} \rightarrow \mathcal{A}, \lambda \mapsto (a - \lambda)^{-1} = \frac{1}{a - \lambda}$ is bounded and differentiable and achieve a contradiction by Liouville's theorem.

Claim: $\|(a - \lambda)^{-1}\| < \|a\|^{-1}$ if $|\lambda| > 2\|a\|$. Indeed, if $|\lambda| > 2\|a\|$ then $\|\lambda^{-1}a\| < \frac{1}{2}$, and in particular $1 - \lambda^{-1}a \in \text{inv}(\mathcal{A})$ and

$$\|(1 - \lambda^{-1}a)^{-1} - 1\| = \left\| \sum_{n=1}^{\infty} (\lambda^{-1}a)^{-1} \right\| \leq \sum_{n=1}^{\infty} \|\lambda^{-1}a\|^n = \frac{\|\lambda^{-1}a\|}{1 - \|\lambda^{-1}a\|} \leq 2\|\lambda^{-1}a\| < 1.$$

From here we deduce that $\|(1 - \lambda^{-1}a)^{-1}\| < 2$ and thus

$$\|(a - \lambda)^{-1}\| < \|\lambda^{-1}(\lambda^{-1}a - 1)^{-1}\| = \frac{\|(1 - \lambda^{-1}a)^{-1}\|}{|\lambda|} < \frac{2}{\lambda} < \frac{1}{\|\lambda\|}.$$

So $R_a : \mathbb{C} \rightarrow \mathcal{A}$ is bounded outside $\overline{B(0, 2\|a\|)}$. Since R_a is continuous, it is bounded on $\mathbb{C} \rightarrow \mathcal{A}$. Let $\varphi \in \mathcal{A}^*$ be a bounded linear functional in $\mathcal{L}(\mathcal{A}, \mathbb{C})$. Thus, φ is differentiable with $\varphi'(a) = \varphi$ for all $a \in \mathcal{A}$. Then $\varphi \circ R_a$ is differentiable and bounded, so it is an “integer” function. By Liouville's theorem, $\varphi \circ R_a$ is constant. Therefore, $\varphi \circ R_a(x) = \varphi \circ R_a(y)$ for all $x, y \in \mathcal{A}$. Especially, we have $\varphi((a - \lambda)^{-1}) = \varphi(a^{-1})$ for all φ . Hahn-Banach shows $(a - \lambda)^{-1} = a^{-1}$ for all λ , proving $a - \lambda = a$ for all a, λ . This is a contradiction. \square

Theorem 5.16 (Gelfand-Mazur) *If \mathcal{A} is a unital Banach algebra and every $a \neq 0$ admits an inverse (\mathcal{A} is a field), then $\mathcal{A} = \mathbb{C} \cdot 1$.*

PROOF: By the assumption, $\text{inv}(\mathcal{A}) = \mathcal{A} \setminus \{0\}$. By the previous theorem, if $a \in \mathcal{A}$ there exists some $\lambda \in \sigma(a)$, so $a - \lambda \notin \text{inv}(\mathcal{A})$, so $a - \lambda = 0$ and thus $a = \lambda \cdot 1$. \square

Corollary 5.17 *Let $\mathbb{C}(X) = \left\{ \frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{Q}[X] \right\}$ is a field, but it cannot be turned into a Banach algebra.*

Theorem 5.18 (Adjoining units - unitization of algebras) *Let \mathcal{A} be any algebra. Consider $\tilde{\mathcal{A}} = \mathcal{A} \oplus \mathbb{C}$ as a vector space. We write elements of $\tilde{\mathcal{A}}$ as $a + \lambda \cdot 1 := (a, \lambda)$. Think of $a = (a, 0)$ and $\lambda = (0, \lambda)$. Define*

$$(a + \lambda 1)(b + \lambda' 1) = (ab + \lambda'a + \lambda b) + \lambda \cdot \lambda'.$$

Ten (exercise $\tilde{\mathcal{A}}$) becomes a unital algebra with $1_{\tilde{\mathcal{A}}} = 1 = (0, 1)$.

Notice that \mathcal{A} is an ideal in $\tilde{\mathcal{A}}$.

Moreover, we get a short exact sequence

$$0 \rightarrow \mathcal{A} \hookrightarrow \tilde{\mathcal{A}} \rightarrow \mathbb{C} \rightarrow 0$$

so $1 + \lambda \mapsto \lambda$.

If \mathcal{A} is a normed algebra, then $\tilde{\mathcal{A}}$ is normed by $\|a + \lambda \cdot 1\| := \|a\| + |\lambda|$

If \mathcal{A} is Banach and closed, then so is $\tilde{\mathcal{A}}$.

If \mathcal{A} is a $*$ -algebra, then so is $\tilde{\mathcal{A}}$ with $(a + \lambda 1)^*$.

If \mathcal{A} is a (Banach) normed $*$ -algebra, then so is $\tilde{\mathcal{A}}$.

If \mathcal{A} is a C^* -algebra, in general the norm given above is not a Norm on \mathcal{A} , but $\|a + \lambda \cdot 1\| := \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|ab + \lambda b\|$ is.

Exercise 5.1 If \mathcal{A} is already unital, then $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$ as algebras by $a + \lambda \cdot 1 \mapsto (a + \lambda 1_{\mathcal{A}}, -\lambda)$.

Definition 5.19 *Re-Definition:* If \mathcal{A} is non-unital, then $\tilde{\mathcal{A}} + \mathbb{C} \cdot 1$ is a $(*-)$ Banach algebra, and we define $\sigma_{\mathcal{A}}(a) := \sigma_{\tilde{\mathcal{A}}}(a)$.

Observation: If \mathcal{A} is already unital, then for $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$ we have $\sigma_{\tilde{\mathcal{A}}}(a) = \sigma_{\mathcal{A}}(a) \cup \{0\}$.

Remark 5.20 If \mathcal{A} is a C^* -algebra, then $\tilde{\mathcal{A}}$ is a C^* -algebra.

- (i) If \mathcal{A} is unital, then $\tilde{\mathcal{A}} \simeq \mathcal{A} \oplus \mathbb{C}$ and $\|a + \lambda \cdot 1\| = \max\{\|a + \lambda \cdot 1\|, |\lambda|\}$.
- (ii) If \mathcal{A} is not unital, then $\|a + \lambda \cdot 1\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$.

6 Spectral Radius

Definition 6.1 Let \mathcal{A} be an algebra. Given $a \in \mathcal{A}$, we define:

$$\pi(a) := \sup\{|\lambda| \mid \lambda \in \sigma_{\mathcal{A}}(a)\}$$

as the **spectral radius** of a if $\emptyset \neq \sigma_{\mathcal{A}}(a)$ is bounded (e.g. if \mathcal{A} is Banach).

Observation: In a Banach algebra, we have $0 \leq \pi(a) \leq \|a\|$.

Example 6.2

- (i) Let $f \in \mathcal{A} = C_0(X)$ using $\sigma_{\mathcal{A}}(f) = \overline{f(X)}$. Thus,

$$\pi(f) = \sup\{|\lambda| \mid \lambda \in \overline{f(X)}\} = \sup_{x \in X} |f(x)| = \|f\|_{C_0(X)}$$

- (ii) Let $\mathcal{A} = M_2(\mathbb{C})$ and $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\sigma_{\mathcal{A}} = \{0\}$ and $\pi(a) = 0$, but $\|a\| = 1 \neq 0$.

Theorem 6.3 (Beurling-Gelfand) Let \mathcal{A} be a Banach algebra, then

$$\pi(a) = \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$$

PROOF: We may assume \mathcal{A} is unital (otherwise we consider $\tilde{\mathcal{A}}$). If $\lambda \in \sigma(a)$, then

$$\lambda^n \in \sigma(a^n) \Rightarrow |\lambda^n| \leq \|a^n\| \Rightarrow |\lambda| \leq \|a\|^{\frac{1}{n}} \quad \forall n \in \mathbb{N}$$

and therefore

$$\pi(a) \leq \inf_{n \in \mathbb{N}} \|a^n\|^{\frac{1}{n}} \leq \liminf_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}.$$

We prove now that $\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq \pi(a)$. Set $\Delta := B\left(0, \frac{1}{\pi(a)}\right)$. Where per convention we set $\frac{1}{\pi(a)} = \infty$ if $\pi(a) = 0$. If $\lambda \in \Delta$, then $1 - \lambda a \in \text{inv}(\mathcal{A})$ (because $|\lambda| < \frac{1}{\pi(a)}$ implies $|\lambda^{-1}| > \pi(a)$) and therefore $\lambda^{-1} \notin \sigma(a) \Rightarrow \lambda^{-1} - a \in \text{inv } A \Rightarrow 1 - \lambda a \in \text{inv}(A)$.

Now fix $\varphi \in \mathcal{A}^*$. Then $f : \Delta \rightarrow \mathbb{C}, \lambda \mapsto \varphi((1 - \lambda a)^{-1})$ is analytic, so it can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n \lambda^n, a_n = \frac{f^{(n)}(0)}{n!} \in \mathbb{C}, \lambda \in \Delta.$$

On the other hand, if

$$|\lambda| < \frac{1}{\|a\|} \leq \frac{1}{\pi(a)}$$

then $\|\lambda a\| < 1$, so

$$(1 - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n \Rightarrow f(\lambda) = \varphi((1 - \lambda a)^{-1}) = \sum_{k=0}^{\infty} \varphi(a^k) \lambda^k$$

for $|\lambda| < \frac{1}{\|a\|}$.

By uniqueness of the Taylor series expansion, it follows that

$$a_n = \varphi(a^n) \forall n \in \mathbb{N}.$$

In particular, $(\varphi(a^n) \lambda^n)$ converges to zero for all $\lambda \in \Delta$ and thus $(\varphi(a^n) \lambda^n)$ is bounded for all $\lambda \in \Delta$.

From the principle of uniform convergence, it follows that $(a^n \lambda^n)$ is bounded. So there exists an $M = M_\lambda$ such that

$$\begin{aligned} \|\lambda^n a^n\| &\leq M \forall n \in \mathbb{N} \\ \Rightarrow \|\lambda^n\|^{\frac{1}{n}} &\leq \frac{M^{\frac{1}{n}}}{|\lambda|} \forall n \in \mathbb{N}, \forall \lambda \in \Delta, \lambda \neq 0 \\ \Rightarrow \limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} &\leq \frac{1}{\lambda} \forall \lambda \in \Delta \text{ i.e. } |\lambda| < \frac{1}{\pi(a)} \end{aligned}$$

Letting $\lambda < \frac{1}{\pi(a)}$ yields $\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq \pi(a)$. □

Example 6.4 Let $A = C^1([0, 1]) = \{I \in C[0, 1] \mid \exists f'(t) \forall t \in [0, 1], t \mapsto f'(t) \text{ continuous}\}$ with $\|f\| = \|f\|_\infty + \|f'\|_\infty$.

Then \mathcal{A} is unital, commutative and a Banach algebra. Consider $x \in \mathcal{A}, x(t) = t$. We have $x^n(t) = t^n$ and

$$\begin{aligned} \|x^n\| &= \sup_{t \in [0, 1]} |t^n| + \sup_{t \in [0, 1]} |nt^{n-1}| = 1 + n \\ \pi(x) &= \lim_{n \rightarrow \infty} (1 + n)^{\frac{1}{n}} = 1 \\ \|x\| &= 2 \end{aligned}$$

Observation: $\sigma(x) = \text{im}(x) = [0, 1]$.

Theorem 6.5 Let $\mathcal{B} \not\subseteq \mathcal{A}$ be an inclusion of unital Banach algebras with $1 = 1_{\mathcal{A}} = 1_{\mathcal{B}}$. Then $\sigma_{\mathcal{A}}(b) \subseteq \sigma_{\mathcal{B}}(b)$ for all $b \in \mathcal{B}$ and the inclusion may be proper. If $\sigma_{\mathcal{A}}(b)$ is simply connected (not holes), then $\sigma_{\mathcal{A}}(b) = \sigma_{\mathcal{B}}(b)$.

The holes of a compact subset $K \subseteq \mathbb{C}$ are the bounded connected components of $\mathbb{C} \setminus K$. So saying that K has no holes means that $\mathbb{C} \setminus K$ is connected.

PROOF: See Murphy, 1.2.8. □

Example 6.6 Let $\mathcal{B} := A(\mathbb{D}) = \{f \in C(\mathbb{D}) \mid f \text{ analytic on } \mathbb{D}^\circ\}$ and $\mathcal{A} = C(\mathbb{S}^1)$. Then we have an embedding by $\iota : \mathcal{B} \hookrightarrow \mathcal{A}, f \mapsto f|_{\mathbb{S}^1}$.

By the principle of maximum modules, ι is an embedding of (unital) Banach algebras. Consider: $f(z) = z$ for $z \in \mathbb{D}$. (Observation: $\overline{Alg}(1, z) = A(\mathbb{D})$) Then:

$$\sigma_{A(\mathbb{D})}(f) = f(\mathbb{D}) = \mathbb{D}$$

and $\sigma_{C(\mathbb{S}^1)}(f|_{\mathbb{S}^1}) = \mathbb{S}^1$.

Definition 6.7 (Exponentials) Let \mathcal{A} be a unital Banach algebra, given $a \in \mathcal{A}$ we define

$$e^a = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

Note $\left\| \frac{a^n}{n!} \right\| \leq \frac{\|a\|^n}{n!}$, so the series converges and $\|\exp(a)\| \leq \exp(\|a\|)$.

Theorem 6.8

(i) Let \mathcal{A} be a unital Banach algebra. If $a \in \mathcal{A}$, then $f : \mathbb{R} \rightarrow \mathcal{A}, t \mapsto \exp(ta)$ is the unique solution of

$$\begin{cases} f'(t) &= af(t) \\ f(0) &= 1 \end{cases}$$

(ii) $e^a \in \text{inv}(\mathcal{A})$ and $(e^a)^{-1} = e^{-a}$.

(iii) If $a, b \in \mathcal{A}$ then $e^{a+b} = e^a \cdot e^b$ (here some commutativity is necessary).

PROOF: See Murphy, 1.2.9. □

7 Gelfand Representation for commutative Banach algebras

Idea: Given a commutative algebra \mathcal{A} , we want to represent \mathcal{A} by a homomorphism $\varphi : \mathcal{A} \rightarrow C_0(X)$ for X some locally compact Hausdorff space. We hope that φ is injective, or even isometric, or an isomorphism. But what is X , and what is φ ?

Notice that, if $\mathcal{A} = C_0(X)$ already, then for each $x \in X$ we get a character $\text{ev}_x : \mathcal{A} \rightarrow \mathbb{C}, f \mapsto f(x)$.

Definition 7.1 Given an algebra \mathcal{A} , we define

$$\hat{\mathcal{A}} = \Omega(\mathcal{A}) := \{\chi : \mathcal{A} \rightarrow \mathbb{C} \mid \chi \text{ non-zero homomorphism}\}.$$

Example 7.2

(i) For $\mathcal{A} = C_0(X)$ we get a map

$$X \rightarrow \Omega(\mathcal{A}), x \mapsto \text{ev}_x$$

that is a bijection. After we give $\Omega(\mathcal{A})$ an appropriate topology, it will also be a homomorphism.

- (ii) Let $\mathcal{A} = M_2(\mathbb{C})$ (or any $M_n(\mathbb{C})$). This is a simple algebra, so non-zero homomorphisms $\chi : \mathcal{A} \rightarrow \mathbb{C}$ do not exist (same for any \mathcal{A} with dimension > 1).

So in this case we have $\Omega(\mathcal{A}) = \emptyset$. This can also happen in commutative algebras.

- (iii) Consider

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\}$$

Then for all $a \in \mathcal{A}$ we have $a^2 = 0$, so if $\chi : \mathcal{A} \rightarrow \mathbb{C}$ is an homomorphism, then $\chi(a)^2 = \chi(a^2) = 0$, so $\chi(a) = 0$ for all $a \in \mathcal{A}$. So again, $\Omega(\mathcal{A}) = \emptyset$ (and \mathcal{A} is commutative with $\dim \mathcal{A} = 1$).

Question: Given an abstract algebra \mathcal{A} how do we possibly find its characters?

Idea: Assume that $\mathfrak{l} \triangleleft \mathcal{A}$ is a maximal ideal and \mathcal{A} is a unital Banach algebra. Then $\mathcal{A}/\mathfrak{l} \simeq \mathbb{C}$ and $\chi \in \Omega(\mathcal{A})$.

Theorem 7.3 *Let \mathcal{A} be a unital non-zero Banach algebra. If $\chi \in \Omega(\mathcal{A})$ then $\|\chi\| = \sup_{\|a\|=1} |\chi(a)| = 1$ and $\ker(\chi) \triangleleft \mathcal{A}$. So $\chi \in \mathcal{A}^*$ (the topological dual of $\Omega(\mathcal{A}) \subseteq D_{\mathcal{A}^*}(0, 1)$).*

Moreover, if \mathcal{A} is a unital Banach commutative algebra, then $\Omega(\mathcal{A}) \ni \chi \mapsto \ker(\chi) \triangleleft \mathcal{A}$ is a bijection between of characters of \mathcal{A} and maximal ideals of \mathcal{A} .

PROOF: If $a \in \mathcal{A}$ and χ a character, then $\chi(a) \in \sigma(\mathcal{A})$, because $\chi(a - \chi(a) \cdot 1) = \chi(a) - \chi(a) \cdot \chi(1) = 0$, so $a - \chi(a) \cdot 1 \in \ker(\chi) \triangleleft \mathcal{A}$ and thus $a - \chi(a) \cdot 1 \notin \text{inv}(\mathcal{A})$.

Therefore: $|\chi(a)| \leq \pi(a) \leq \|a\|$. So $\|\chi\| \leq 1$. Since $\chi(1) = 1$ and $\|1\| = 1$ we have $\|\chi\| = 1$.

Now, apply linear algebra. Then $\ker(\chi)$ is a maximal proper subspace, in particular a maximal ideal. And $\ker(\chi)$ is closed, because χ is continuous. Now assume that \mathcal{A} is commutative (in addition to unital and Banach). Then we have the mapping

$$\varphi : \Omega(\mathcal{A}) \rightarrow \text{MaxIdeals}(\mathcal{A}), \chi \mapsto \ker(\chi).$$

- φ is injective. If $\ker(\chi_1) = \ker(\chi_2)$ for $\chi_1, \chi_2 \in \Omega(\mathcal{A})$, then for every $a \in \mathcal{A}$ we have $a - \chi_1(a) \cdot 1 \in \ker(\chi_1) = \ker(\chi_2)$. Thus, $\chi_2(a - \chi_1(a) \cdot 1) = 0$ and therefore $\chi_2(a) = \chi_1(a)$ for every $a \in \mathcal{A}$.
- φ is surjective. Take $\mathfrak{l} \triangleleft \mathcal{A}$ a maximal ideal. Then $\mathfrak{l} = \bar{\mathfrak{l}}$ because $\bar{\mathfrak{l}} \neq \mathcal{A}$, otherwise $1 \in \bar{\mathfrak{l}}$ and since $\text{inv}(\mathcal{A})$ is open in \mathcal{A} , we get $\mathfrak{l} \cap \text{inv}(\mathcal{A}) \neq \emptyset$. But then we have an invertible element in the ideal \mathfrak{l} already, but this implies the contradiction $\mathfrak{l} = \mathcal{A}$. Therefore, \mathcal{A}/\mathfrak{l} is a commutative, unital Banach algebra which is simple (\mathfrak{l} is maximal).

Exercise: If $\mathfrak{l} \triangleleft \mathcal{A}$, then \mathcal{A}/\mathfrak{l} is field if and only if there exists no $\mathfrak{j} \triangleleft \mathcal{A}$ such that $\mathfrak{l} \triangleleft \mathfrak{j}$.

Thus, \mathcal{A}/\mathfrak{l} is a field and $\mathcal{A}/\mathfrak{l} \simeq \mathbb{C}$. Then the composition

$$\mathcal{A} \xrightarrow{q} \mathcal{A}/\mathfrak{l} \simeq \mathbb{C}$$

is a character with $\ker(\chi) = \mathfrak{l}$. □

Exercise 7.1 An application of Zorn's Lemma. Show that every ideal $I \triangleleft \mathcal{A}$ in a unital algebra \mathcal{A} is contained in a maximal ideal.

In particular, we can apply this to $\mathfrak{l} = 0$ in $\mathcal{A} \neq 0$ (with \mathcal{A} is unital and commutative) and thus $\Omega(\mathcal{A}) \neq \emptyset$.

Topology on $\Omega(\mathcal{A})$

We have for \mathcal{A} a Banach algebra. We can add a unit to receive $\tilde{\mathcal{A}}$, which is a Banach algebra.

Observe: If $\chi \in \Omega(\mathcal{A})$, then there exists a unique $\tilde{\chi} \in \Omega(\tilde{\mathcal{A}})$ via $\tilde{\chi}(a + \lambda \cdot 1) = \chi(a) + \lambda$. Thus, $\|\chi\| \leq \|\tilde{\chi}\| = 1$ (Note that it may still be smaller than 1. See exercises 2023-05-09).

In any case,

$$\Omega(\mathcal{A}) \subseteq D_{\mathcal{A}^*}(0, 1) = \{\varphi \in \mathcal{A}^* \mid \|\varphi\| \leq 1\}$$

and \mathcal{A}^* carries the weak *-topology (the smallest topology to make all point-evaluations continuous, that is for a net $(\varphi_i) \subset \mathcal{A}^*$ weakly converging to $\varphi \in \mathcal{A}^*$ if and only if $\varphi_i(a) \rightarrow \varphi(a)$ for all $a \in \mathcal{A}$).

Definition 7.4 Given a Banach algebra \mathcal{A} , we endow $\Omega(\mathcal{A})$ with the weak *-topology and call this the **Gelfand spectrum** of \mathcal{A} .

Proposition 7.5 $\Omega(\mathcal{A})$ is a locally compact Hausdorff space. If \mathcal{A} is unital, then $\Omega(\mathcal{A})$ is compact.

PROOF: By Banach-Alaoglu-Theorem, $D_{\mathcal{A}^*}(0, 1)$ is compact and Hausdorff with the weak *-topology. Let

$$\begin{aligned} S &:= \{\chi : \mathcal{A} \rightarrow \mathbb{C} \mid \chi \text{ hom.}\} \\ &= \Omega(\mathcal{A}) \cup \{0\} \end{aligned}$$

Then $S \subseteq D_{\mathcal{A}^*}(0, 1)$. So $\chi(ab) = \lim_{i \rightarrow \infty} K_i = \lim_{i \rightarrow \infty} \chi_i(a)\chi_i(b) = \chi(a)\chi(b)$ and therefore $x \in S$. Thus, S is a compact Hausdorff space and $\Omega(\mathcal{A}) = S \setminus \{0\}$ is relatively compact.

If \mathcal{A} is unital, then $\Omega(\mathcal{A}) \subseteq D_{\mathcal{A}^*}(0, 1)$ is closed. Then we have $(X_i) \subseteq \Omega(\mathcal{A})$ and $X_i \rightarrow X \in \mathcal{A}^*$ and thus $X \in S = \text{hom}(\mathcal{A}, \mathbb{C})$. \square

Observation: Given a Banach algebra \mathcal{A} , we have an isomorphism

$$\Omega(\tilde{\mathcal{A}}) \rightarrow \Omega(\mathcal{A}) \sqcup \{\chi_\infty\}, \varphi \mapsto \begin{cases} \varphi|_{\mathcal{A}} & \varphi|_{\mathcal{A}} \neq 0 \\ \chi_\infty & \varphi|_{\mathcal{A}} = 0 \end{cases},$$

where $\chi_\infty(a + \lambda \cdot 1) = \lambda$. Thus, $\Omega(\mathcal{A}) \sqcup \{\chi_\infty\}$ is already the unitization of $\Omega(\mathcal{A})$.

Theorem 7.6 Let \mathcal{A} be a Banach algebra. Then for every $a \in \mathcal{A}$.

$$\{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \subseteq \sigma(a)$$

If \mathcal{A} is commutative, then

- $\{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} = \sigma(a)$ in case \mathcal{A} is unital.
- $\{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} = \sigma_{\mathcal{A}}(a)$.

PROOF:

- \mathcal{A} is unital and $a \in \mathcal{A}$. $\chi(a - \chi(a) \cdot 1) = 0$, so $\chi(a) \in \sigma(a)$, so $\{\chi(a) \mid x \in \Omega(a)\} \subseteq \sigma(a)$.

Now if $\lambda \in \sigma(a)$, consider $\mathfrak{l} := (a - \lambda \cdot 1)\mathcal{A} \triangleleft \mathcal{A}$ if \mathcal{A} is commutative. By Zorns Lemma, we get $I \subseteq J \triangleleft \mathcal{A}$ with $J = \ker(\chi)$ for some $\chi \in \Omega(\mathcal{A})$. Thus we have $a - \lambda \cdot 1 \in I \subseteq J = \ker(\chi)$ so $\chi(a) = \lambda$.

- \mathcal{A} is not unital. Consider $\tilde{\mathcal{A}}$. By the first part,

$$\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(a) \supseteq \{\chi(a) \mid \chi \in \Omega(\tilde{\mathcal{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\}$$

If \mathcal{A} is commutative, by the first part again:

$$\sigma_{\mathcal{A}}(a) = \sigma_{\tilde{\mathcal{A}}}(a) = \{\chi(a) \mid \chi \in \Omega(\tilde{\mathcal{A}})\} = \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} \quad \square$$

7.1 Gelfand-Transformation

Definition 7.7 Given a Banach algebra \mathcal{A} and $a \in \mathcal{A}$, we define $\hat{a} : \Omega(\mathcal{A}) \rightarrow \mathbb{C}, \chi \mapsto \chi(a)$.

Observe that $\hat{a} \in C(\Omega(\mathcal{A}))$, because if $\chi_i \rightarrow \chi$ then we have $\hat{a}(\chi_i) = \chi_i(a) \rightarrow \chi(a) = \hat{a}(\chi)$. So we have a map $\Gamma : \mathcal{A} \rightarrow C(\Omega(\mathcal{A}))$. This map is called the **Gelfand transform** of \mathcal{A} .

Theorem 7.8 (Gelfand Representation) $\text{im}(\Gamma) \subseteq C_0(\Omega(\mathcal{A}))$ and $\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A}))$ is a contractive homomorphism, i.e. $\|\Gamma(a)\| \leq \pi(a) \leq \|a\|$ for every Banach algebra \mathcal{A} . If moreover \mathcal{A} is commutative, then $\|\Gamma(a)\| = \pi(a)$. Also, for all $a \in \mathcal{A}$, we have

$$\sigma(a) = \begin{cases} \text{im}(\hat{a}) & \mathcal{A} \text{ unital} \\ \text{im}(\hat{a}) \cup \{0\} & \text{otherwise} \end{cases}.$$

PROOF: If \mathcal{A} is unital, then $\Omega(\mathcal{A})$ is compact so $\text{im}(\Gamma) \subseteq C(\Omega(\mathcal{A})) = C_0(\Omega(\mathcal{A}))$. If \mathcal{A} is not unital, we use observation 7. Then we have $\Omega(\tilde{\mathcal{A}}) \simeq \Omega(\mathcal{A}) \cup \{\chi_\infty\}$ so that

$$C_0(\Omega(\mathcal{A})) \simeq \{f \in C(\Omega(\tilde{\mathcal{A}})) \mid f(x_\infty) = 0\}.$$

Now if $a \in \mathcal{A}$, then $\hat{a}(\chi_\infty) = \chi_\infty(a) = 0$.

Γ is a homomorphism: The linearity is obvious, as is the homomorphism property:

$$(\Gamma(a)\Gamma(b))(\chi) = (\hat{a} \cdot \hat{b})(\chi) = \hat{a}(\chi)\hat{b}(\chi) = \chi(a)\chi(b) = \chi(ab) = \hat{ab}(\chi) = \Gamma(ab)(\chi).$$

Γ is contractive: Given $a \in \mathcal{A}$, $\chi \in \Omega(\mathcal{A})$, we have $\hat{a}(\chi) = \chi(a) \in \sigma(a)$, so $\|\hat{a}(\chi)\| \leq \pi(a)$ yielding $\|\Gamma(a)\|_\infty = \|\hat{a}\|_\infty \leq \pi(a) \leq \|a\|$. If \mathcal{A} is commutative, we have

$$\sigma(a) = \begin{cases} \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} & 1 \in \mathcal{A} \\ \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} & \text{otherwise} \end{cases} = \begin{cases} \{\hat{a}(\chi) \mid \chi \in \Omega(\mathcal{A})\} & 1 \in \mathcal{A} \\ \{\hat{a}(\chi) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\} & \text{otherwise} \end{cases}$$

and thus

$$\|\Gamma(a)\| = \|\hat{a}\|_\infty = \sup_{\chi \in \Omega(\mathcal{A})} |\chi(a)| = \sup_{\lambda \in \sigma(a)} |\lambda| = \pi(a) \quad \square$$

As a convention, if $\Gamma(\mathcal{A}) = \{0\}$, then $C_0(\Omega(\mathcal{A})) = \{0\}$ and thus $\hat{a} = 0$ for all $a \in \mathcal{A}$.

Example 7.9

- (i) If $\mathcal{A} = M_n(\mathbb{C})$ with $n > 1$ or \mathcal{A} is any unital simple Banach algebra with $\dim \mathcal{A} > 1$, then $\Omega(\mathcal{A}) = \emptyset$ so $\Gamma \equiv 0$.

(ii) Take the commutative subalgebra

$$\mathcal{A} = \left\{ \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} \mid \lambda \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

then \mathcal{A} is not unital, commutative, Banach and $\dim \mathcal{A} = 1$. Once again, $\Omega(\mathcal{A}) = \emptyset$ and thus $\Gamma \equiv 0$.

(iii) Take

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mid \lambda, \alpha \in \mathbb{C} \right\} \subseteq M_2(\mathbb{C})$$

is a unital, commutative Banach algebra with $\dim \mathcal{A} = 2$. We have

$$\Omega(\mathcal{A}) = \{\chi_\infty\} \quad \chi_\infty : \mathcal{A} \rightarrow \mathbb{C}, \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mapsto \lambda$$

and thus

$$\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A})) = C_0(\{\chi_\infty\}) \simeq \mathbb{C}, a = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix} \mapsto \hat{a} \equiv \lambda$$

This shows that Γ is not injective, as $\dim \mathcal{A} = 2$ but $\dim \Gamma(\mathcal{A}) = 1$.

Definition 7.10 Let \mathcal{A} be a Banach algebra. We say that $a \in \mathcal{A}$ is *quasi-nilpotent* if $\pi(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = 0$. Sometimes, you will read

$$\text{Rad}(\mathcal{A}) = \{a \in \mathcal{A} \mid a \text{ quasi-nilpotent}\}$$

If $\text{Rad}(\mathcal{A}) = 0$, we say that \mathcal{A} is **semi-simple**. Notice that if $a \in \mathcal{A}$ is quasi-nilpotent, then $\Gamma(a) = \hat{a} = 0$ because $\Gamma(a) \leq \pi(a) = 0$. If \mathcal{A} is commutative, then $\ker(\Gamma) = \text{Rad}(\mathcal{A})$.

Example 7.11

(iv) $\mathcal{A} = \ell^1(\mathbb{Z}) = \{(a_n)_{n \in \mathbb{Z}} \mid \sum_{n \in \mathbb{Z}} |a_n| < \infty\}$.

Recall from exercises, that $\Omega(\ell^1(\mathbb{Z})) \simeq \mathbb{D}$ with $\mathbb{D} \rightarrow \Omega(\ell^1(\mathbb{Z})), z \mapsto \chi_z$ defined as $\chi_z(a) = \hat{a}(z) = \sum_{n=0}^{\infty} a_n z^n$.

We define a multiplication $\delta_m \cdot \delta_n = \delta_{n+m}$. Then δ_0 is the unit and δ_1 is a generator of $\mathcal{A} = \ell^1(\mathbb{Z})$.

The elements $\delta_m - (\dots, 0, 1, 0, \dots)$ form a basis for \mathcal{A} . We have $a = \sum_{n \in \mathbb{Z}} a_n \delta_n$ and for $\chi \in \mathcal{A}^*$ it follows $\chi(a) = \sum_{n \in \mathbb{Z}} a_n \chi(\delta_n)$.

We now want to calculate the spectrum. We have seen that $\chi(\delta_0) = \chi(1_{\mathcal{A}}) = 1$ and $\chi(\delta_n) = \chi(\delta_1^n) \chi(\delta_1)^n$. Therefore, χ is determined by $z = \chi(\delta_1) \in \mathbb{C}$. We know at least that $|z| = |\chi(\delta_1)| \leq \|\delta_1\| = 1$, so $z \in \mathbb{D}$. Claim: $z \in \Pi = \mathbb{S}^1$.

General fact: If $a \in \text{inv } \mathcal{A}$ for \mathcal{A} a unital Banach algebra, then $\sigma(a^{-1}) = \sigma(a)^{-1} = \{\lambda^{-1} \mid \lambda \in \sigma(a)\}$.

Observe that $\mathbb{S}^1 = \text{inv}(\mathcal{A})$ with $\delta_1^{-1} = \delta_{-1}$. So $\sigma(\delta) \subseteq \mathbb{D}$ and $\sigma(\delta_1)^{-1} = \sigma(\delta_{-1}) \subseteq \mathbb{D}$, so $\sigma(\delta_1) \subseteq \mathbb{S}^1$. So $z = \chi(\delta_1) \in \sigma(\delta_1) \subseteq \mathbb{S}^1$. Conversely, if $z \in \mathbb{S}^1$, then $\chi_z : \mathcal{A} \rightarrow \mathbb{C}, \chi_z(a) = \sum_{n \in \mathbb{Z}} a_n z^n \in \mathbb{C}$ is well-defined (as the sum converges) and is a character, as

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n z^m = \chi_z(\delta_n) \cdot \chi_z(\delta_m)$$

and checking in the basis also proves the homomorphism property for all of \mathcal{A} . Notice that $z = \chi_z(\delta_1)$. This shows the injectivity of

$$\Pi \simeq \Omega(\mathcal{A}) = \Omega(\ell^1(\mathbb{Z})), z \mapsto \chi_z, \chi(\delta_1) \leftarrow \chi$$

which is continuous and therefore a homeomorphism (isomorphism), as both spaces are compact. Notice

$$\sigma(\delta_1) = \{\chi(\delta_1) \mid \chi \in \Omega(\mathcal{A})\} = \{\chi_z(\delta_1) \mid z \in \mathbb{S}^1\} = \mathbb{S}^1$$

The Gelfand transformation is now

$$\Gamma : \mathcal{A} = \ell^1(\mathbb{Z}) \rightarrow C(\Omega(\mathcal{A})) \simeq C(\mathbb{S}^1), a \mapsto \left(\hat{a} : z \mapsto \sum_{n \in \mathbb{Z}} a_n z^n \right)$$

Γ is always a contractive algebra homomorphism, as $\|\hat{a}\|_\infty \leq \|a\|_1$. Γ is a $*$ -homomorphism where $\ell^1(\mathbb{Z})$ carries the involution $a^* = (\sum_{n \in \mathbb{Z}} a_n \delta_n)^* = \sum_{n \in \mathbb{Z}} \bar{a}_n \delta_{-n}$ because of $\delta_n^* = \delta_{-n}$. The involution of $C(\mathbb{S}^1)$ is complex conjugation. But on the unit circle, $\bar{z} = z^{-1}$, so we have a $*$ -homomorphism.

Γ is injective. If $f \in C(\mathbb{S}^1)$, we can define its “inverse Fourier-Transform”

$$\check{f}(n) = \int_{\mathbb{S}^1} f(z) z^{-n} dz = \frac{1}{2\pi} \int_0^{2\pi} f(\exp(it)) \exp(-int) dt$$

This is **not** the line integral from functional analysis, as the derivative of the path is not included. You can now check that $(\hat{a})^\sim(n) = a_n$. $g \mapsto \int_{\mathbb{S}^1} g$ is a continuous function on $C(\mathbb{S}^1)$ and we have

$$\hat{a}(z) = \sum_{n \in \mathbb{Z}} a_n z^n = \lim_{F \subseteq \mathbb{Z} \text{ finite}} \sum_{n \in F} a_n z^n = \lim_{N \rightarrow \infty} \sum_{n=-N}^N a_n z^n$$

so

$$(\hat{a})^\sim(n) = \sum_{m \in \mathbb{Z}} a_m (\hat{\delta}_m)^\sim(n)$$

Because of $\int_{\mathbb{S}^1} z^k = \delta_{k,0}$, we have

$$\int_{\mathbb{S}^1} z^m z^n dz = \delta_{n,m}$$

and using $\hat{\delta}_m(z) = z^m$ we can show $(\hat{\delta}_m)^\sim(n) = \delta_{n,m}$ and thus

$$(\hat{a})^\sim(n) = \sum_{m \in \mathbb{Z}} a_m (\hat{\delta}_m)^\sim(n) = \sum_{m \in \mathbb{Z}} a_m \delta_{m,n} = a_n$$

This shows that we can re-gain the elements of the sequence from \hat{a} , so $\Gamma : (a_n) \mapsto \hat{a}$ must be injective.

Γ has dense range because the polynomials are dense in $C(\mathbb{S}^1)$ because of Stone-Weierstraß theorem.

Γ is not isometric. If Γ was isometric, then Γ were an isometric $*$ -homomorphism with dense range. Since isometric homomorphisms have closed image, Γ were surjective and thus an isometric $*$ -isomorphism $\ell^1(\mathbb{Z}) = C(\mathbb{S}^1)$. Then $\ell^1(\mathbb{Z})$ would be a C^* -algebra with the $\ell^1(\mathbb{Z})$ -norm, and thus $\|a^*a\|_1 = \|a\|_1^2$ (with the involution as described above). Then, using the C^* -property of $C(\mathbb{S}^1)$ and isometry of Γ , we have

$$\|a^*a\|_1 = \|\Gamma(a^*a)\|_\infty = \|\Gamma(a)^*\Gamma(a)\|_\infty = \|\Gamma(a)\|_\infty^2 = \|a\|_1^2.$$

Now we only need to find $a \in \ell^1(\mathbb{Z})$ with $\|a^*a\|_1 \neq \|a\|_1^2$. Choose $a = \alpha\delta_0 + \beta\delta_1 + \gamma\delta_{-1} = \alpha + \beta\delta_1 + \gamma\delta_{-1}$ (not writing δ_0 as it is the unit).

$$a^*a = (\bar{\alpha} + \bar{\beta}\delta_{-1} + \bar{\gamma}\delta_1)(\alpha + \beta\delta_1 + \gamma\delta_{-1}) = \dots$$

and thus

$$\|a^*a\|_1 = |\alpha|^2 + |\beta|^2 + |\gamma|^2 + 2|\bar{\alpha}\beta + \alpha\bar{\gamma}| + 2|\gamma\beta|$$

while

$$\|a\|_1^2 = (|\alpha| + |\beta| + |\gamma|)^2.$$

Now choosing $\alpha = i$ and $\beta = \gamma = 1$ yields $\|a^*a\|_1 = 5$ and $\|a\|_1^2 = 9$. This shows that $\ell^1(\mathbb{Z})$ does not fulfil the $*$ -property and cannot be a C^* -algebra. This is a contradiction, so Γ cannot be isometric.

This is also a valid counterexample for the isometry directly, because a has Norm 3, but $\Gamma(a) = (z \mapsto \frac{1}{z} + i + z = 2\Re(z) + i)$ has maximum $2 + i$ with Norm $\sqrt{5} < 3$ on the unit circle. Γ is not surjective. This is complicated.

Recall: For \mathcal{A} a Banach algebra, we have a Gelfand representation

$$\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A})), a \mapsto (\hat{a} : \Omega(\mathcal{A}) \rightarrow \mathbb{C}, \chi \mapsto \chi(a))$$

where $\Omega(\mathcal{A}) = \{\chi : \mathcal{A} \rightarrow \mathbb{C} \mid \text{non-zero hom}\} \subseteq D_{\mathcal{A}^*}(0, 1)$ with the weak $*$ -topology. Γ is a contractive homomorphism, and if \mathcal{A} is commutative $\|\Gamma(a)\| = \pi(a) \leq \|a\|$ for all $a \in \mathcal{A}$.

We now want to consider commutative C^* -algebras.

Theorem 7.12 (Gelfand) *If \mathcal{A} is a commutative C^* -algebra, then $\Gamma : \mathcal{A} \rightarrow C_0(\Omega(\mathcal{A}))$ is an isometric $*$ -isomorphism.*

For this proof we require a set of lemmas.

Lemma 7.13 *If $a \in \mathcal{A}$, \mathcal{A} a C^* -algebra, with $a = a^*$ then $\pi(a) = \|a\|$.*

PROOF: Use $\pi(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}}$. Notice $\|a^2\| = \|a^*a\| = \|a\|^2$ and $\|a^4\| = \|(a^2)^*a^2\| = \|a^2\|^2 = \|a\|^4$ and likewise for all powers that are powers of 2 we have $\|a^{2^n}\| = \|a\|^{2^n}$. So $\pi(a) = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \|a\|$ is the limit of the subsequence and therefore the limit of the sequence. \square

Remark 7.14 In general, $\|a\| \neq \pi(a)$ if $a \neq a^*$ in a C^* -algebra, e.g. $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$.

But if $a^*a = aa^*$ (a is normal), then $\|a\| = \pi(a)$.

PROOF: Exercise. □

Corollary 7.15 *There exists at most one norm that makes a C^* -algebra \mathcal{A} into a C^* -algebra.*

PROOF: If \mathcal{A} is a C^* -algebra with norm $\|\cdot\|$, then for all $a \in \mathcal{A}$ we have $\|a\| = \|aa^*\|^{\frac{1}{2}}$. Note that a^*a is self-adjoint, so by the previous lemma we have

$$\|a\| = \|aa^*\|^{\frac{1}{2}} = \pi(a^*a)^{\frac{1}{2}} = \sup_{\lambda \in \sigma(a^*a)} |\lambda|^{\frac{1}{2}}$$

and this only depends on the algebra structure, not its norm. □

Corollary 7.16 *If $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ is a $*$ -homomorphism from a Banach- $*$ -algebra \mathcal{A} into a C^* -algebra \mathcal{B} then φ is contractive, i.e. $\|\varphi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$ for all $a \in \mathcal{A}$*

PROOF: Replacing \mathcal{A}, \mathcal{B} by their unitizations $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{B}}$ and extending φ to $\tilde{\varphi} : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{B}}, a + \lambda 1_{\mathcal{B}} \mapsto \varphi(a) + \lambda 1_{\mathcal{B}}$ shows that we can just assume $\mathcal{A}, \mathcal{B}, \varphi$ to be unital.

Now, if $a \in \text{inv}(\mathcal{A})$, then $\varphi(a) \in \text{inv}(\mathcal{B})$, so it follows

$$\lambda \in \rho_{\mathcal{A}}(a) \Leftrightarrow a - \lambda \in \text{inv}(\mathcal{A}) \Leftrightarrow \varphi(a) - \lambda \in \text{inv}(\mathcal{B}) \Leftrightarrow \lambda \in \rho_{\mathcal{B}}(\varphi(a))$$

so $\rho_{\mathcal{A}}(a) \subseteq \rho_{\mathcal{B}}(\varphi(a))$ and $\sigma_{\mathcal{A}}(a) \supseteq \sigma_{\mathcal{B}}(\varphi(a))$. It follows for the spectral radius: $\pi(\varphi(a)) \leq \pi(a)$. As \mathcal{B} is a C^* -algebra, this implies

$$\begin{aligned} \|\varphi(a)\|_{\mathcal{B}}^2 &= \|\varphi(a)^* \varphi(a)\|_{\mathcal{B}} = \|\varphi(a^*a)\|_{\mathcal{B}} = \pi(\varphi(a^*a)) \\ &\leq \pi(a^*a) \leq \|a^*a\|_{\mathcal{A}} \leq \|a^*\|_{\mathcal{A}} \cdot \|a\|_{\mathcal{A}} = \|a\|_{\mathcal{A}}^2 \end{aligned}$$

and therefore $\|\varphi(a)\|_{\mathcal{B}} \leq \|a\|_{\mathcal{A}}$. □

Lemma 7.17 *If \mathcal{A} is a C^* -algebra and $a \in \mathcal{A}$, then*

- (i) *If a is self-adjoint, $\sigma(a) \subseteq \mathbb{R}$.*
- (ii) *If \mathcal{A} is unital and $u \in \mathcal{U}(\mathcal{A})$ is unitary (that is, $u^*u = uu^* = 1$) then $\sigma(u) \subseteq \mathbb{S}^1$.*
- (iii) *If $a \in \text{inv}(\mathcal{A})$, then $\sigma(a^{-1}) = \sigma(a)^{-1} = \{z^{-1} \mid z \in \sigma(a)\}$.*
- (iv) *$\sigma(a^*) = \overline{\sigma(a)}$.*

PROOF: (iii) If $\lambda \in \mathbb{C}, \lambda \neq 0$ and $\lambda - a \notin \text{inv}(\mathcal{A})$. Because $\lambda - a$ is not invertible, $\lambda^{-1}(\lambda - a) = 1 - \lambda^{-1}a$ and $a^{-1}(1 - \lambda^{-1}a) = a^{-1} - \lambda^{-1}$ is also not invertible. So we have $\lambda^{-1} - a^{-1} \notin \text{inv}(\mathcal{A})$ and therefore $\sigma(a^{-1}) \subseteq \sigma(a)^{-1}$. The result follows by symmetry.

(iv) Similarly, you can prove (iv).

(ii) If $u \in \mathcal{U}(\mathcal{A})$, then $\sigma(u) \subseteq \mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ because

$$\|u\| = \|u^*u\|^{\frac{1}{2}} = \|1\|^{\frac{1}{2}} = 1.$$

So, since $u \in \mathcal{U}(\mathcal{A})$, $u^{-1} = u^* \in \mathcal{U}(\mathcal{A})$ and therefore $\sigma(u)^{-1} = \sigma(u^{-1}) \subseteq \mathbb{D}$. This implies $\|\lambda\| = 1$ for all $\lambda \in \sigma(u)$ and thus $\sigma(u) \subseteq \mathbb{S}^1$.

- (i) Assume that \mathcal{A} is unital, otherwise work in $\tilde{\mathcal{A}}$. If a is self-adjoint then $u = \exp(ia) = \sum_{n=0}^{\infty} \frac{i^n a^n}{n!} \in \mathcal{U}(\mathcal{A})$ because $\exp(ia)^* = \exp(-ia)$ and therefore $u^*u = \exp(-ia)\exp(ia) = \exp(0) = 1 = uu^*$. Because of (i) we know $\sigma(u) \subseteq \mathbb{S}^1$. Now, let $\lambda \in \sigma(u)$ and define $b = \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^n}{n!} = \exp(i(a-\lambda)) - 1$ as well as $c = \sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^{n-1}}{n!} \in \mathcal{A}$. Consider

$$\begin{aligned} \exp(ia) - \exp(i\lambda 1) &= (\exp(i(a-\lambda)) - 1) \exp(i\lambda) = b \exp(i\lambda) \\ &= \left(\sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^n}{n!} \right) \exp(i\lambda) \\ &= (a-\lambda) \left(\sum_{n=1}^{\infty} \frac{i^n (a-\lambda)^{n-1}}{n!} \right) \exp(i\lambda) \\ &= (a-\lambda) c \exp(i\lambda). \end{aligned}$$

Since $\lambda \in \sigma(a)$ and $c, (a-\lambda)$ commute, $\exp(ia) - \exp(i\lambda)$ is not invertible (or $a-\lambda$ would also be invertible) and we have $\exp(i\lambda) \in \sigma(u) \subseteq \mathbb{S}^1$. But for this to happen, we require $\lambda \in \mathbb{R}$. \square

Corollary 7.18 *If \mathcal{A} is a C^* -algebra and $\chi \in \Omega(\mathcal{A})$, then $\chi(a^*) = \overline{\chi(a)}$ for all $a \in \mathcal{A}$. So χ is a $*$ -homomorphism.*

PROOF: If $a \in \mathcal{A}$ is self-adjoint, then $\chi(a) \in \sigma(a) \subseteq \mathbb{R}$ so $\overline{\chi(a)} = \chi(a) = \chi(a^*)$.

Now, if $a \in \mathcal{A}$ is any element we can write it as $a = b + ic$ where $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$ so that b, c are self-adjoint. Now $\chi(b), \chi(c) \in \mathbb{R}$ so

$$\chi(a^*) = \chi(b - ic) = \chi(b) - i \cdot \chi(c) = \overline{\chi(b)} + i \overline{\chi(c)} = \overline{\chi(b + ic)} = \overline{\chi(a)} \quad \square$$

Corollary 7.19 *If \mathcal{A} is a commutative C^* -algebra and $\mathcal{A} \neq 0$, then $\Omega(\mathcal{A}) \neq \emptyset$.*

PROOF: If $\mathcal{A} \neq 0$ there is some self-adjoint non-zero element $a \in \mathcal{A}$ so that $\pi(a) = \|a\| \neq 0$. But $\sigma(a) \subseteq \{\chi(a) \mid \chi \in \Omega(\mathcal{A})\} \cup \{0\}$. But for this to be true there must exist a character $\chi \in \Omega(\mathcal{A})$, so $\Omega(\mathcal{A}) \neq \emptyset$. \square

PROOF (GELFAND):

- **Γ is a $*$ -homomorphism:** Consider

$$\Gamma(a)^*(\chi) = \hat{a}^*(\chi) = \overline{\hat{a}(\chi)} = \overline{\chi(a)} = \chi(a^*) = \hat{a}^*(\chi) = \Gamma(a^*)(\chi)$$

so $\Gamma(a)^* = \Gamma(a^*)$.

- **Γ is isometric:** We have

$$\|\Gamma(a)\|^2 = \|\Gamma(a)^* \Gamma(a)\| = \|\Gamma(a^* a)\| = \pi(a^* a) = \|a^* a\| = \|a\|^2$$

using our lemmas and the C^* -property.

- **Γ is surjective:** Let $\mathcal{B} := \text{im}(\Gamma) \subseteq C_0(\Omega(\mathcal{A}))$. Then \mathcal{B} is a C^* -subalgebra of $C_0(\Omega(\mathcal{A}))$. Then

- \mathcal{B} does not vanish at any point, i.e. for every point $\chi \in \Omega(\mathcal{A})$ there is a $b \in \mathcal{B}$ with $\chi(b) \neq 0$.

As $\chi \in \Omega(\mathcal{A})$ means $\chi \neq 0$, there exists an $a \in \mathcal{A}$ with $\chi(a) \neq 0$. But we can rewrite this as $b(\chi) = \hat{a}(\chi) = \chi(a) \neq 0$ for $b = \hat{a}$.

- \mathcal{B} separates points in $\Omega(\mathcal{A})$, i.e. for every $\chi_1 \neq \chi_2$ in $\Omega(\mathcal{A})$ there exists $b \in \mathcal{B}$ with $b(\chi_1) \neq b(\chi_2)$.

If $\chi_1 \neq \chi_2$ there exists $a \in \mathcal{A}$ with $\chi_1(a) \neq \chi_2(a)$. Taking $b = \hat{a}$ yields the result.

The result $\mathcal{B} = C_0(\Omega(\mathcal{A}))$ follows from the Stone-Weierstraß-theorem:

If X is a locally compact Hausdorff space and $B \subseteq C_0(X)$ is a $*$ -subalgebra satisfying

- B does not vanish on any point of X
- B separates points of \mathcal{A}

then B is dense in $C_0(X)$.

So $\text{im}(\Gamma)$ is dense and closed in $C_0(\Omega(\mathcal{A}))$, so Γ is surjective. \square

Proposition 7.20 *Conclusion: Every commutative C^* -algebra is (up to $*$ -isomorphism) of the form $C_0(X)$ for a locally compact Hausdorff space X . Let $\mathcal{A} = C_0(X)$ for a locally compact Hausdorff space X . Then $\Omega(\mathcal{A}) \simeq X$ with isomorphism*

$$\varphi : X \rightarrow \Omega(C_0(X)), x \mapsto (\text{ev}_x : C_0(X) \rightarrow \mathbb{C}, f \mapsto f(x)).$$

PROOF:

- φ is **well-defined**, because characters are never zero.
- φ is **continuous**. Take $x_i \rightarrow x$ in X . Then, for all $f \in C_0(X)$ we have $\text{ev}_{x_i}(f) \rightarrow \text{ev}_x(f)$ because f is continuous and therefore $f(x_i) \rightarrow f(x)$. This shows $\text{ev}_{x_i} \rightarrow \text{ev}_x$ in the weak $*$ -topology.
- φ is **injective**. If $x_1 \neq x_2$ there exists a function $f \in C_0(X)$ that separates them, but then $\text{ev}_{x_1}(f) \neq \text{ev}_{x_2}(f)$, so $\text{ev}_{x_1} \neq \text{ev}_{x_2}$.
- φ is **surjective**. Prove that every $\chi \in \Omega(\mathcal{A})$ is $\chi = \text{ev}_x$ for some $x \in X$.

We know that the characters of \mathcal{A} are equivalent to the ideals in $C_0(X)$, so this is equivalent to: Every maximal ideal $I \triangleleft C_0(X)$ is of the form $I = C_0(X \setminus \{x_0\}) = \{f \in C_0(X) \mid f(x_0) = 0\}$.

In Exercise 01-08 we have proven that every closed (2-sided) ideal $I \triangleleft C_0(X)$ has the form $I = C_0(U) := \{f \in C_0(X) \mid f|_{X \setminus U} \equiv 0\}$ for some open $U \subseteq X$.

See 01-08 for more details.

Take any $f \in I \triangleleft C_0(X)$. First, prove $I^* = I$. Consider $f \in I$ and

$$f_n := \sqrt[n]{f^* f} = (\overline{f} f)^{\frac{1}{n}} = |f|^{\frac{2}{n}}.$$

We have $f_n \in I$ for all n , because $g := f^* f \in I$ and $t \mapsto \sqrt[n]{t}$ is a continuous function that can be uniformly approximated by polynomials on the compact sets. It follows that $f_n = \lim g_n$ where g_n is a polynomial in $g \in I$, so $f_n \in I$. So $f^* f_n \in I$ for all n . Then

$$\begin{aligned} \|f^* - f_n f^*\|_\infty^2 &= \|(f^* - f_n f^*)(f^* - f_n f^*)\|_\infty = \|(f^* - f_n f^*)(f^* - f_n f^*)\|_\infty \\ &= \|f^* f - 2f^* f f_n + f_n^2 f^* f\|_\infty \end{aligned}$$

$$\leq \|g - g \sqrt[n]{g}\| + \|g - g \sqrt[n]{g}\| \|f_n\| \rightarrow 0,$$

because $|g(x) - g(x) \sqrt[n]{g(x)}| \rightarrow 0$ pointwise (as the n -th square root converges to the 1 on the support and 0 elsewhere) and $|g(x)| \leq \varepsilon$ everywhere except a compact set K , and on that K we have $\sup_{x \in K} |g(x)| |1 - \sqrt[n]{g(x)}| = |g(x_0)| |1 - \sqrt[n]{g(x_0)}| < \varepsilon$ for some $n \in \mathbb{N}$. We therefore have $f^* = \lim_{n \rightarrow \infty} f^* f_n \in I$ and thus $f^* = \lim_{n \rightarrow \infty} f_n f^*$. Now let $I \triangleleft C_0(X)$ closed, so $I^* = I$ and I is a C^* -subalgebra of X .

Define $U^c := \{x \in X \mid f(x) = 0 \forall f \in I\}$. This is closed (because for $x_i \rightarrow x$ in X , $x_i \in U^c$, we have $0 = f(x_i) \rightarrow f(x)$), so U is open. We claim $I = C_0(U)$.

If $f \in I$, $f|_{U^c} \equiv 0$ per Definition, so $f \in C_0(U)$. Therefore, I is a closed subideal of $C_0(U)$.

I does not vanish on U , because if there was an $x \in U$ with $f(x) = 0$ for all $f \in I$, we would have $x \in U^c$.

I separates the points of U . Take $x_1 \neq x_2$. We can choose $h \in C_0(X)$ with $h(x_1) = 1$ and $h(x_2) = 0$ (Uryson) as well as $g \in I$ with $g(x_1) \neq 0$, then $f = g \cdot h \in I$ separates x_1 from x_2 .

Stone-Weierstraß now proves $I = C_0(U)$.

Notice $U \subseteq V \subseteq X$ (open) iff $C_0(U) \subseteq C_0(V) \leq C_0(X)$ (see exercise 08-01). So we have a bijection between the opens of X and the ideals of $C_0(X)$. Especially, the maximal ideals of $C_0(X)$ correspond to the maximal open sets, that is the sets of form $X \setminus \{x_0\}$ for some x_0 , of X .

Therefore, if $\chi \in \Omega(C_0(X))$ we have $\ker \chi = C_0(X \setminus \{x_0\})$, so χ maps a function to 0 if and only if f is zero on x . This proves and $\chi = \text{ev}_x$.

- φ is **open**. If X is compact, this is clear because $C_0(X) = C(X)$ and unital, so $\Omega(C_0(X))$ is compact and we have a bijection between two compact sets. In general, consider \tilde{X} (the compactification) and use $\widetilde{C_0(\tilde{X})} \simeq C(\tilde{X})$. So we have a homeomorphism

$$\tilde{X} \rightarrow \Omega(C(\tilde{x})) = \Omega(\widetilde{C_0(\tilde{X})}) \simeq \Omega(C_0(X)) \sqcup \{\chi_\infty\}$$

where $\infty \mapsto \chi_\infty$, so we can restrict the homeomorphism to X and are done. \square

Theorem 7.21 (Spectral inclusion for C^* -algebras) *Let $\mathcal{A} \subseteq \mathcal{B}$ be an inclusion of unital C^* -algebras with $1 = 1_{\mathcal{A}} = 1_{\mathcal{B}}$. Then for all $a \in \mathcal{A}$ we have $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$, so $\text{inv}(\mathcal{A}) = \text{inv}(\mathcal{B}) \cap \mathcal{A}$.*

PROOF: If a is self-adjoint, that is $a^* = a$, then $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$, so $\sigma_{\mathcal{A}}$ has no holes, i.e. the complement $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ is connected in \mathbb{C} . By the general result on Banach algebras $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{B}}(a)$. In particular, this implies $a \in \text{inv}(\mathcal{A}) \Leftrightarrow a \in \text{inv}(\mathcal{B})$ for all self-adjoint $a \in \mathcal{A}$.

We now prove that this holds for all $a \in \mathcal{A}$. Of course, $\text{inv}(\mathcal{A}) \subseteq \text{inv}(\mathcal{B}) \cap \mathcal{A}$. Let $a \in \mathcal{A}$ such that $a \in \text{inv}(\mathcal{B})$. Then there exists $b \in \mathcal{B}$ such that $ab = ba = 1$ and $b^* a^* = a^* b^* = 1 \Leftrightarrow bb^* a^* a = 1 = a^* a b b^*$. Therefore, $a^* a \in \text{inv} \mathcal{B} \cap \mathcal{A} \subseteq \text{inv}(\mathcal{A})$ because $a^* a$ is self adjoint. So there exists $c \in \text{algebra } \mathcal{A}$ with $ca^* a = 1 = a^* a c$ and thus $ca^* a b = ca^* = b$, so $b \in \mathcal{A}$ as it is the product of two elements $a^*, c \in \mathcal{A}$. This concludes the proof, as a is now invertible in \mathcal{A} . \square

Definition 7.22 *We say $a \in \mathcal{A}$ (for \mathcal{A} a C^* -algebra) is **normal** if $a^* a = a a^*$. This means $C^*(a)$ (the C^* -subalgebra of \mathcal{A} generated by a) is commutative. Then $C^*(a) \simeq C_0(X)$.*

Lemma 7.23 *Let $a \in \mathcal{A}$ (C^* -algebra) be a normal element. Assume that $1 \in \mathcal{A}$ (unital). Then $\Omega(C^*(a, 1)) \simeq \sigma(a)$ by homeomorphism $\chi \mapsto \chi(a)$. In general, if \mathcal{A} is possibly not unital, then $\Omega(C^*(a)) \simeq \sigma(a) \setminus \{0\}$. In particular, $\chi(a) = 0$ only if $a = 0$ but then $C^*(a)$ is just the zero space.*

PROOF: It is enough to consider the unital case.

Consider $\varphi : \Omega(C^*(a, 1)) \rightarrow \sigma(a), \chi \mapsto \chi(a)$ which is well-defined because $\chi(a) \in \sigma(a)$.

- φ is **continuous**. If $\chi_i \rightarrow \chi$ in $\Omega(C^*(a, 1))$ then this also converges point wise, so $\chi_i(a) \rightarrow \chi(a)$.
- φ is **injective**. Take $\chi_1, \chi_2 \in \Omega(C^*(a, 1))$ with $\chi_1(a) = \chi_2(a)$. Since $\chi_1(1) = 1 = \chi_2(1)$, so the two characters coincide on the generators and are thus equal by linearity and continuity.
- φ is **surjective**. We know that $\sigma(a) = \{\chi(a) \mid \chi \in \Omega(B)\}$ for all commutative unital Banach algebras B , in particular for $B = C^*(a, 1)$.

Because both spaces are compact this concludes the proof. \square

Theorem 7.24 (Fundamental theorem of continuous functional calculus)

Let \mathcal{A} be a unital C^ -algebra and $a \in \mathcal{A}$ normal. Then there exists a unique unital $*$ -homomorphism $\varphi : C(\sigma(a)) \rightarrow \mathcal{A}$ such that $\text{id}_{\sigma(a)} \mapsto a$.*

In general, if \mathcal{A} is possibly not unital, there exists a unique $$ -homomorphism $\varphi : C_0(\sigma(a)) \rightarrow \mathcal{A}$ where $C_0(\sigma(a)) := \{f \in C(\sigma(a)) \mid f(0) = 0\}$.*

Both of these morphisms are also isometric.

Notation: If $f \in C(\sigma(a))$ we write $f(a) := \varphi(a)$. Notice: If f is a polynomial in z, \bar{z} then $f(a) = \varphi(a)$ as usual.

PROOF: Consider $1 \in \mathcal{A}$ and let $\mathcal{B} = C^*(a, 1) \subseteq \mathcal{A}$. Then \mathcal{B} is commutative because a is normal (i.e. commutes with its adjoint). By Gelfand, we get an isometric $*$ -isomorphism $T : \mathcal{B} \rightarrow C(\Omega(\mathcal{B})), b \mapsto \hat{b}$. By the Lemma, $\Omega(\mathcal{B}) \equiv \sigma(a), \chi \mapsto \chi(a)$. Via this identification (homeomorphism), we have $\hat{b}(\chi) = \chi(b)$ and $\hat{a}(\chi) = \chi(a)$. So \hat{a} corresponds to $z \in C(\sigma(a)) \simeq C(\Omega(\mathcal{B}))$. Therefore, considering the inverse of T and identifying $\Omega(\mathcal{B}) \simeq \sigma(a)$ we get an isometric

$$C(\sigma(a)) \simeq C(\Omega(C^*(a, 1))) \simeq C^*(a, 1) \simeq \mathcal{A}.$$

This gives φ as defined.

The **non-unital case**: Just consider $\tilde{\mathcal{A}}$. \square

Example 7.25 Let $f(z) = \exp(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. f is a continuous function on the whole plane. If $a \in \mathcal{A}$ is normal, then $f(a) = \exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$. In general, $f(z) = \sum_{n=0}^{\infty} \lambda_n z^n$ (or $f(z) = \sum_{n=0}^{\infty} \lambda_n (z - z_0)^n$), so $f(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$ if $\sigma(a) \subseteq \text{Domain}(f)$.

Theorem 7.26 *Let \mathcal{A} be unital C^* -algebra and $a \in \mathcal{A}$ be normal. If $f \in C(\sigma(a))$, then $\sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\}$.*

Moreover, if $g \in C(\sigma(f(a)))$, then $g(f(a)) = (g \circ f)(a)$.

PROOF: Let $\mathcal{B} = C^*(a, 1) \subseteq \mathcal{A}$. \mathcal{B} is commutative and unital. Then $f(a) \in \mathcal{B}$ and $\sigma(f(a)) = \sigma_{\mathcal{B}}(f(a))$. Now notice $\chi(f(a)) = f(\chi(a))$ since both maps

$$f \mapsto \chi(f(a))$$

$$f \mapsto f(\chi(a))$$

are unital $*$ -homomorphisms that coincide on z . Therefore,

$$\sigma(f(a)) = \{\chi(f(a)) \mid \chi \in \Omega(\mathcal{B})\} = \{f(\chi(a)) \mid \chi \in \Omega(\mathcal{B})\} = f(\sigma(a)).$$

Now to prove $(g \circ f)(a) = g(f(a))$. Let $C = C^*(1, f(a)) \subseteq \mathcal{B} = C^*(1, a) \subseteq \mathcal{A}$. Let $\chi \in \Omega(\mathcal{B})$. Then $\chi_C := \chi|_C \in \Omega(C)$. So $(g \circ f)(a)$ is sensibly defined and an element of \mathcal{B} , so we can apply a character:

$$\begin{aligned} \chi((g \circ f)(a)) &= (g \circ f)(\chi(a)) = g(f(\chi(a))) = g(\chi(f(a))) = g(\chi_C(f(a))) \\ &= \chi_C(g(f(a))) = \chi(\underbrace{g(f(a))}_{\in \mathcal{B}}) \end{aligned}$$

Because the Gelfand-transform is injective, this implies $(g \circ f)(a) = g(f(a))$. \square

Proposition 7.27 *Let \mathcal{A} be a unital C^* -algebra and $u \in \mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid u^*u = 1 = uu^*\}$. If $\sigma(u) \neq \mathbb{S}^1$ there exists a self-adjoint $a \in \mathcal{A}$ with $u = \exp(ia)$.*

PROOF: The idea is to take $\log \approx \exp^{-1}$. Problem: \exp is not invertible as a complex function, because it is $2\pi i$ -periodic. We will need to restrict it. Consider the principal branch of the logarithm, $\log(z) = \log|z| + i \arg(z)$.

Given that $\sigma(u) \neq \mathbb{S}^1$, there exists an $\lambda \in \mathbb{S}^1 \setminus \sigma(u)$ and therefore also an $f_\lambda \in C(\mathbb{S}^1 \setminus \{\lambda\})$ (so some form of argument-mapping of z) such that $\exp(if_\lambda(z)) = z$. This f_λ is real-valued, continuous and analytical. Now use functional calculus: Let $a := f_\lambda(u) \in \mathcal{A}$. Since f_λ is real-valued, it is self-adjoint in the algebra, so a is also self-adjoint. By the previous theorem $\exp(ia) = \exp(if_\lambda(u)) = (\exp \circ if_\lambda)(u) = u$.

Multiplier Algebras

This is another kind of unitization. We will consider $\mathcal{A} \rightarrow M(\mathcal{A}) \ni \mu$ such that $\mu \cdot a \in \mathcal{A} \ni a \cdot \mu$ so $\mathcal{A} \trianglelefteq M(\mathcal{A})$. Remember that this was the case for the usual unitization, with Quotient \mathbb{C} . Here, the multiplier is usually much bigger, so the quotient is as well. In fact, $\mathcal{A} \times \mathbb{C}$ is the 'smallest' unitization while $M(\mathcal{A})$ is the 'largest' one.

Definition 7.28 (Multiplier, see Murphy) *Let \mathcal{A} be an algebra. A **multiplier** of \mathcal{A} is a pair $\mu = (L, R)$ where $L, R : \mathcal{A} \rightarrow \mathcal{A}$ are linear maps such that*

- (i) $L(ab) = L(a) \cdot b$ or $\mu(ab) = (\mu a)b$
- (ii) $R(ab) = a \cdot R(b)$ or $(ab)\mu = a(b\mu)$
- (iii) $a \cdot L(b) = R(a) \cdot b$ or $a(\mu b) = (a\mu)b$.

To simplify this, use the notation $\mu \cdot a := L(a)$ and $a \cdot \mu := R(a)$.

For the space of all multipliers we write $M(\mathcal{A}) = \{\mu = (L, R) \mid \mu \text{ multiplier}\}$. This is a \mathbb{C} -vector space with

$$(L_1, R_1) + (L_2, R_2) = (L_1 + L_2, R_1 + R_2) \quad \lambda(L_1, R_1) = (\lambda L_1, \lambda R_2)$$

and an algebra with

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 \cdot L_2, R_2 \cdot R_1).$$

If \mathcal{A} is a $*$ -algebra, we further define

$$(L, R)^* = (R^*, L^*) \text{ where } L^*(a) := L(a^*)^* \text{ and } R^*(a) := R(a^*)^*$$

Moreover, we have a canonical $(*)$ -homomorphism $\iota : \mathcal{A} \rightarrow M(\mathcal{A}), a \mapsto (L_a, R_a)$ where $L_a(b) = ab$ and $R_a(b) = ba$. Note: ι is always a $(*)$ -homomorphism but injective if and only if

$$\begin{aligned} \forall_{a \in \mathcal{A}} \quad a \cdot b = 0 \quad \forall_b \Rightarrow a = 0 \\ b \cdot a = 0 \quad \forall_b \Rightarrow a = 0 \end{aligned}$$

i.e. \mathcal{A} is an essential ideal of itself. This is not always true for a general algebra, consider the algebra with the 0-product $a \cdot b = 0$, but it always holds for C^* -algebras or if \mathcal{A} is unital already.

More generally this holds if \mathcal{A} is a Banach algebra with an **approximate unit**, a net $e_i \subseteq \mathcal{A}$ such that $e_i a \rightarrow a$ and $a \cdot e_i \rightarrow a$ for any $a \in \mathcal{A}$ as well as $\|e_i\|$. This is always the case for unital and C^* -algebras.

Assume ι is injective. Then \mathcal{A} is identified with an essential $(*)$ -ideal of $M(\mathcal{A})$.

Remark 7.29 (Norms on the multiplier) If \mathcal{A} is a Banach algebra with an approximate unit, we define for $\mu = (L, R) \in M(\mathcal{A})$ the norm

$$\|\mu\| := \|L\| = \|R\| < \infty.$$

PROOF: To show $\|L\|, \|R\| < \infty$ we use the Closed Graph Theorem. Say we have $(a_n) \subseteq \mathcal{A}$ with $a_n \rightarrow a$ and $L(a_n) \rightarrow b$. Take $c \in \mathcal{A}$ and consider

$$c \cdot L(a) = R(c) \cdot a = \lim_{n \rightarrow \infty} R(c) \cdot a_n = \lim_{n \rightarrow \infty} c \cdot L(a_n) = c \cdot b.$$

Because of the approximate unit (or ι injective) we have $L(a) = b$. This shows that L (and, analogously, R) are bounded. Now to prove $\|L\| = \|R\|$. Take any $a \in \mathcal{A}$ and consider

$$\|L(a)\| \stackrel{\text{approx. unit}}{=} \sup_{\|b\| \leq 1} \|bL(a)\| = \sup_{\|b\| \leq 1} \|R(b)a\| \leq \sup_{\|b\| \leq 1} \|R(b)\| \|a\| \leq \|R\| \cdot \|a\|$$

which implies $\|L\| \leq \|R\|$. By symmetry of the situation, we have $\|L\| = \|R\|$. \square

With the norm above, $M(\mathcal{A})$ becomes a Banach algebra.

Proposition 7.30 *If \mathcal{A} is a C^* -algebra then $M(\mathcal{A})$ is too.*

PROOF: Write $\mu = (L, R)$. We compute $\mu^* \mu = (R^*, L^*) \cdot (L, R) = (R^* L, R L^*)$. So $\|\mu \mu^*\| = \|R^* L\|$. Take $a \in \mathcal{A}$ with $\|a\| \leq 1$. Then

$$\|L(a)\|^2 = \|L(a)L(a)^*\| = \|L(a)L^*(a^*)\| = \|R^*(L(a))a^*\| \leq \|R^*(L(a))\| \leq \|R^* L\|$$

This shows $\|L\|^2 \leq \|R^* L\|$ and therefore $\|\mu\|^2 = \|L\|^2 \leq \|R^* L\| = \|\mu^* \mu\|$. Because $\|\mu\|^2 \geq \|\mu \mu^*\|$ is clear by submultiplicativity, the C^* -property follows. \square

Compare now $\tilde{\mathcal{A}}$ and $M(\mathcal{A})$. We have $\mathcal{A} \trianglelefteq \tilde{\mathcal{A}}$ and $\mathcal{A} \trianglelefteq M(\mathcal{A})$. When are these ideals essential?

Lemma 7.31 *Let \mathcal{A} be a C^* -algebra or Banach algebra with approximate unit. $\mathcal{A} \trianglelefteq \tilde{\mathcal{A}}$ if and only if \mathcal{A} is not unital.*

PROOF: Suppose that \mathcal{A} is unital with $1_{\mathcal{A}}$ as the unit. In this case, take $p = 1 - 1_{\mathcal{A}} \in \tilde{\mathcal{A}}$ (where $1 = (0, 1)$ is the unit in $\tilde{\mathcal{A}}$). Notice that $p \cdot \mathcal{A} = 0$, but $p \neq 0$. So \mathcal{A} is not essential in $\tilde{\mathcal{A}}$.

Suppose that \mathcal{A} is not unital. To prove: For $a + \lambda \cdot 1 \in \tilde{\mathcal{A}}$ and $(a + \lambda \cdot 1)\mathcal{A} = 0$ we have $a = 0$, $\lambda = 0$. So take any $(a + \lambda \cdot 1) \cdot b = 0$ for all $b \in \mathcal{A}$, that is $ab + \lambda b = 0$. This means $L_a(b) = -\lambda b$, that is $L_a = -\lambda \text{id}_{\mathcal{A}}$. Notice $L : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{A})$, a unital algebra with unit $\text{id}_{\mathcal{A}}$, is an injective (because ι is injective) algebra homomorphism. If $\lambda \neq 0$, then division by λ implies $\text{id}_{\mathcal{A}} \in \text{im}(L) \simeq \mathcal{A}$. But then \mathcal{A} has a unit, a contradiction. So $\lambda = 0$. Then $a \cdot b = 0$ for every b , so $a = 0$ as well. This shows that \mathcal{A} is an essential ideal of $\tilde{\mathcal{A}}$. \square

Remark 7.32 Let \mathcal{A} be a C^* -algebra or Banach algebra with approximate unit. Then \mathcal{A} is unital if and only if $M(\mathcal{A}) = \mathcal{A}$.

PROOF: One direction is simple: $M(\mathcal{A})$ is always unital, so $\mathcal{A} \simeq M(\mathcal{A})$ implies that \mathcal{A} is unital.

Let now \mathcal{A} be unital and prove that every multiplier is of the form (L_a, R_a) . Let $\mu = (L, R) \in M(\mathcal{A})$ and define $a := L(1_{\mathcal{A}})$. Then $L_a(b) = ab = L(1_{\mathcal{A}})b = L(b)$, so $L = L_a$. Analogously we can prove $R = R_a$. This shows that ι is surjective, and since it is already injective (because \mathcal{A} is either C^* or has an approximate unit) it is an isomorphism. \square

Say \mathcal{A} is a C^* -algebra (or a Banach algebra with an approximate unit) and not unital. Then $\iota : \mathcal{A} \rightarrow M(\mathcal{A}), a \mapsto \mu_a = (L_a, R_a)$ extends to a $(*)$ -embedding

$$\tilde{\iota} : \tilde{\mathcal{A}} \rightarrow M(\mathcal{A}), a + \lambda \cdot 1 \mapsto \iota(a) + \lambda \cdot \underbrace{(\text{id}, \text{id})}_{=\text{id}_{M(\mathcal{A})}}.$$

More generally: If \mathcal{B} is any C^* -algebra that contains \mathcal{A} as an essential ideal (closed), then \mathcal{B} embeds in the multiplier algebra via the following map:

$$\lambda : \mathcal{B} \rightarrow M(\mathcal{A}), b \mapsto (L_b, R_b)$$

where L_b, R_b are the usual left and right multiplication. We have $L_b(a), R_b(a) \in \mathcal{A}$ for any $a \in \mathcal{A}$ because \mathcal{A} is an ideal. The above is a universal property of the multiplier algebra. $M(\mathcal{A})$ is the largest unital C^* -algebra that contains \mathcal{A} as an essential ideal.

Example 7.33 Take $\mathcal{A} = C_0(X)$ (for a locally compact Hausdorff-space, so a commutative C^* -algebra). Then $\tilde{\mathcal{A}} = C(\tilde{X})$ where $\tilde{X} = X \sqcup \{\infty\}$. One can now show $M(\mathcal{A}) \simeq C(\beta X)$ where βX is the Stone-Cech-compactification of X . This can be proven using the universal property and the universal property of βX : βX is a compact Hausdorff space such that $X \hookrightarrow \beta X$ as a dense open topological subspace and for every other compact Hausdorff space K such that $X \rightarrow K$ via a continuous function f there exists a unique continuous extension $\beta f : \beta X \rightarrow K$.

First: Prove that $M(\mathcal{A})$ is even commutative. Then it is the continuous functions on some space, use the spectrum and compare the universal properties. For commutativity, one can show $M(C_0(X)) \simeq C_b(X)$ via the universal property.

7.2 Positive Elements of C^* -algebras

Definition 7.34 Let \mathcal{A} be a C^* -algebra. We say that $a \in \mathcal{A}$ is positive (and write $a \geq 0$) if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$.

The set of all positive elements of a given algebra we notate as \mathcal{A}_+ .

Example 7.35 Let $A = C_0(X)$ (commutative) and $f \in \mathcal{A}$. Then $f = f^*$ iff f is real (that is $f : X \rightarrow \mathbb{R}$). Since $\sigma(f) = \overline{\text{im}(f)}$ we see that $f \geq 0$ iff $f(x) \geq 0$ for all $x \in X$.

Theorem 7.36 If $a \in \mathcal{A}$ for \mathcal{A} a C^* -algebra and $a \geq 0$ then there exists a unique $b \in \mathcal{A}_+$ such that $b^2 = a$. We sometimes notate this as $b = \sqrt{a} = a^{\frac{1}{2}}$.

PROOF: Since a is positive, it is self-adjoint and therefore normal. Continuous functional calculus:

$$\varphi : C_0(\sigma(a)) \rightarrow \mathcal{A}, f \mapsto f(a)$$

Apply this to $f(x) = \sqrt{x}$. Notice that $f \in C_0(\sigma(a))$ because $\sigma(a) \subseteq [0, \infty)$. Now simply choose $b = f(a) = \sqrt{a}$. Since φ is a $*$ -homomorphism, we have $b^2 = \varphi(f)^2 = \varphi(f^2) = \varphi(\text{id}) = a$.

Reminder: Writing ' $f(a)$ ' does not mean to imply that $a \in \mathcal{A}$ can simply be plugged into the function $f : \sigma(\mathcal{A}) \rightarrow \mathbb{C}$ but is simply a different way of writing $\varphi(f) \in \mathcal{A}$.

Uniqueness: Suppose $c \in \mathcal{A}_+$ such that $c^2 = a$. Then c commutes with $c^2 = a$ and therefore c commutes with $b = \sqrt{a}$ since $b = \lim_{n \rightarrow \infty} p_n(a)$ (polynomial approximation). Then $B := C^*(b, c) \subseteq \mathcal{A}$ is a commutative C^* -algebra so $B \simeq C_0(X)$ for some locally compact Hausdorff space X . Since $a, b, c \in B = C_0(X)$ we have $a \simeq f, b \simeq g, c \simeq h \in C_0(X)$ with $f = g^2 = h^2$ where all these functions are positive. But then $f(x) = g(x)^2 = h(x)^2$ for all x . Because $g(x), h(x) \geq 0$ for all x , this shows $g(x) = h(x)$ for all x and therefore $g = h$ and $b = c$. \square

Remark 7.37 Given any self-adjoint element $a \in \mathcal{A}$ ($a^* = a$) we can write it as $a^+ - a^-$ where $a^+, a^- \geq 0$ and $a^+ \cdot a^- = 0$. Just define $f(x) = \frac{|x|+x}{2}$ and $g(x) = \frac{|x|-x}{2}$. Both are positive functions with $f \cdot g = 0$. Define $a^+ = f(a)$ and $a^- = g(a)$ (once again per continuous functional calculus), transferring the necessary properties:

$$\begin{aligned} f(a) - g(a) &= \varphi(f) - \varphi(g) = \varphi(f - g) = \varphi(\text{id}) = a \\ f(a) \cdot g(a) &= \varphi(f) \cdot \varphi(g) = \varphi(f \cdot g) = \varphi(0) = 0 \\ \sigma(f(a)) &= \sigma(\varphi(f)) \subseteq \sigma(f) = \overline{\text{im}(f)} = [0, \infty) \end{aligned}$$

Remark 7.38 If \mathcal{A} is unital C^* -algebra and $a \in \mathcal{A}$ is self-adjoint with $\|a\| \leq 1$, so $\sigma(a) \subseteq [-1, 1]$. Define

$$f(x) = x + i\sqrt{1-x^2} \quad g(x) = x - i\sqrt{1-x^2}$$

This means that $f, g \in \mathcal{UC}(\sigma(a))$ (Recall that unitaries of $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid u^*u = 1 = uu^*\}$) and $\frac{f+g}{2} = \text{id}_{\sigma(a)}$. So if we now define $u := f(a), v := g(a) \in C^*(a, 1) \subseteq \mathcal{A}$ we have $\frac{u+v}{2} = a$. In particular $\mathcal{A} = \text{span}(\mathcal{U}(\mathcal{A}))$.

Lemma 7.39 Let \mathcal{A} be a unital C^* -algebra, $a \in \mathcal{A}$ self-adjoint and $t \in \mathbb{R}_+$.

- (i) If $a \geq 0$ and $\|a\| \leq t$ then $\|a - t\| \leq t$.
- (ii) Conversely, if $\|a - t\| \leq t$ then $a \geq 0$.

PROOF: Replace \mathcal{A} by $C^*(a, 1)$ we may assume that $\mathcal{A} = C(X)$ is commutative and X compact. Let $a = f \in C(X)$ be a self-adjoint, real function and $t \geq 0$ a real number.

- (i) $f \geq 0$ and $\|f\|_\infty \leq t$ and thus $f(x) - t \in [-t, 0]$ for all $x \in X$, so $\|f - t\| \leq t$.
- (ii) Let $f \in C(X)$ be a self-adjoint real function with $\|f - t\| \leq t$, so $|f(x) - t| \leq t$ for every x . But if $f(x) < 0$ for any $x \in X$ we have $f(x) - t < -t$ and thus $|f(x) - t| > t$, a contradiction. So f must be positive. \square

Corollary 7.40 If \mathcal{A} is a C^* -algebra, then \mathcal{A}_+ is a closed subset (but not subspace!) of \mathcal{A} .

PROOF: Taking unitization, we may assume that \mathcal{A} is unital. Let $(a_n) \subseteq \mathcal{A}_+$ and $a_n \rightarrow a \in \mathcal{A}$. Then $a_n^* = a_n$ for all $n \in \mathbb{N}$ and therefore a is also self-adjoint. There also exists $t \geq 0$ with $\|a_n\| \leq t$ for all $n \in \mathbb{N}$ and by the Lemma $\|a_n - t\| \leq t$ and therefore $\|a - t\| \leq t$. Again by the Lemma $a \geq 0$. \square

Corollary 7.41 *If \mathcal{A} is a C^* -algebra and $a, b \in \mathcal{A}_+$ then $a + b \in \mathcal{A}_+$.*

PROOF: Taking unitization, we may assume that \mathcal{A} is unital. Since $a, b \geq 0$ by $t = \|a\|, \|b\|$ we have $\|a - \|a\|\| \leq \|a\|$ and $\|b - \|b\|\| \leq \|b\|$. Then

$$\|(a + b) - (\|a\| + \|b\|)\| = \|(a - \|a\|) + (b - \|b\|)\| \leq \|(a - \|a\|)\| + \|(b - \|b\|)\| \leq \|a\| + \|b\|$$

and $a + b$ is positive by the lemma. \square

Theorem 7.42 *If \mathcal{A} is a C^* -algebra and $a \in \mathcal{A}$ then $a^*a \geq 0$.*

PROOF: First, we prove that if $-a^*a \geq 0$ then $a = 0$. For this we use the following observation $\sigma(bc) \setminus \{0\} = \sigma(cb) \setminus \{0\}$ (the two sets are equal except for the zero, which may be contained in one but not the other) because for b, c in a unital algebra and $1 - bc \in \text{inv } \mathcal{A}$ iff $1 - cb \in \text{inv}(\mathcal{A})$ and if $d := (1 - bc)^{-1}$ then $(1 - cb)^{-1} = 1 + cdb$.

Therefore, if $-a^*a \in \mathcal{A}_+$ then also $-a^*a \in \mathcal{A}_+$ (notice that a, a^* are self-adjoint). Then write $a = b + c$ with $b, c \in \mathcal{A}$ self-adjoint. Then

$$a^*a + aa^* = (b - ic)(b + ic) + (b + ic)(b - ic) = b^2 + c^2 + ibc - icb + b^2 + c^2 + icb - icb = 2b^2 + 2c^2.$$

and we can write $a^*a = 2b^2 + 2c^2 - aa^*$. The squares are certainly positive and we have assumed $-aa^* \geq 0$, but then $a^*a \geq 0$. We see that $aa^* \geq 0$ as well, so the spectrum has to be zero.

Now suppose that $a \in \mathcal{A}$ arbitrarily. We show that $a^*a \geq 0$. Let $b := a^*a$. Then $b \in \mathcal{A}$ is self-adjoint with $b = b^+ - b^-$ where $b^+, b^- \geq 0$. Let $c := ab^-$. Then

$$-c^*c = -b^-a^*ab^- = -b^-(b^+ - b^-)b^- = (b^-)^3 \geq 0$$

and c must be 0 by our first result. This implies $(b^-)^3 = 0$ so $b^- = 0$. It follows that $b = b^+ \geq 0$. \square

Definition 7.43 *Let \mathcal{A} be a self-adjoint algebra and $a, b \in \mathcal{A}$. We write $a \leq b$ if $b - a \geq 0$. This turns \mathcal{A} into a poset. Because A_+ is a cone, that is $A_+ + A_+ \subseteq A_+$ and $\mathbb{R}_+ \cdot A_+ \subseteq A_+$ as well as $A_{\text{self-adjoint}} = A_+ - A_+$ and $A_+ \cap -A_+ = \{0\}$.*

Theorem 7.44 *Let \mathcal{A} be a C^* -algebra.*

- (i) $A_+ = \{a^*a \mid a \in \mathcal{A}\}$
- (ii) a, b self-adjoint and $c \in \mathcal{A}$. Then $a \leq b$ implies $c^*ac \leq c^*bc$.
- (iii) $0 \leq a \leq b$ implies $\|a\| \leq \|b\|$
- (iv) If \mathcal{A} is unital and $a, b \geq 0$ with $a \leq b$ and $a, b \in \text{inv}(\mathcal{A})$ then $b^{-1} \leq a^{-1}$.

PROOF:

- (i) It follows from the previous theorem. The fact that $a \in \mathcal{A}_+$ has a square root $a = b^2 = b^*b$ with $b \geq 0$.
- (ii) $c^*bc - c^*ac = c^*(b - a)c$ and if we set $b - a = d^*d$ for a $d \in \mathcal{A}$ we receive $c^*(b - a)c = c^*d^*dc = (dc)^*dc \geq 0$.

- (iii) We may assume $1 \in \mathcal{A}$. Notice that $b \leq \|b\| \cdot 1$ (consider the commutative case). So we have $a \leq b \leq \|b\| \cdot 1$ and therefore $a \leq \|b\| \cdot 1$ so $\|a\| \leq \|b\|$.
- (iv) Let $a, b \in \text{inv } \mathcal{A}$, $a, b \geq 0$ and $a \leq b$. We know that $\sigma(b^{-1}) = \sigma(b)^{-1} \subseteq \mathbb{R}_+$ and thus $b^{-1} \geq 0$ and Similarly $a^{-1} \geq 0$. Notice that if $c \geq 1$ (in \mathcal{A}) then $c \in \text{inv } \mathcal{A}$ (as $\sigma(c - 1) \subseteq [0, \infty)$ and thus $\sigma(c) \subseteq [1, \infty)$) and $c^{-1} \leq 1$ (think once again commutative).

Now we have $a \leq b$. Then $1 = a^{-\frac{1}{2}} a a^{-\frac{1}{2}} \leq a^{-\frac{1}{2}} b a^{-\frac{1}{2}}$. Then $(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}) = (a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}) \leq 1$ by the above, so conjugation yields $b^{-1} \leq a^{-1}$. \square