Exercise Sheet 03
Operator Algebras

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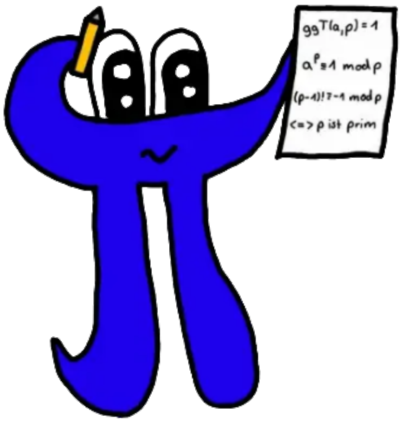

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The C^*-property shows $\|a^2\| = \|a^*a\| = \|a\|^2$, and by using this as well as the C^* property again, we have for $n = 4$ that $\|a^4\| = \|a^*a^*aa\| = \|(a^2)^*(a^2)\| = \|a^2\|^2 = \|a^4\|$. Inductively, we can likewise prove $\|a^{2^k}\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $n + m = 2^k$ for some $k \in \mathbb{N}$. Then we have

$$\|a\|^{2^k} = \|a^{2^n}\| = \|a^n a^m\| \leq \|a^n\| \cdot \|a^m\| \leq \|a\|^n \cdot \|a^m\| \leq \|a\|^{n+m} = \|a\|^{2^k}$$

and because the first and last element are equal, we must have equality in every intermediate step. This especially proves $\|a^n\| = \|a\|^n$.

Let now $a \in \mathscr{A}$ be an arbitrary element. Then $\|a^*a \dots a^*a\| = \|(a^*a)^{\frac{n}{2}}\| = \|a^*a\|^{\frac{n}{2}} = \|a\|^n$ as proven above, because (a^*a) is self-adjunct. For non-even n (and thus even $n+1$) we can once again calculate

$$\|a\|^{n+1} = \|a^*aa^* \dots a^*a\| \leq \|a\| \cdot \|aa^* \dots a^*\| \leq \|a\| \cdot \|a\|^n = \|a\|^{n+1}$$

and therefore $\|aa^* \dots a^*\| = \|a\|^n$ by the same argument as above.

Now, for a normal $a \in \mathscr{A}$ (that is, $a^*a = aa^*$) we have

$$\|a^n\|^{\frac{1}{n}} = \left(\|a^n\|^2\right)^{\frac{1}{2n}} = \|(a^n) * a^n\|^{\frac{1}{2n}} = \|aa^*a \dots a^*\|^{\frac{1}{2n}} = \left(\|a\|^{2n}\right)^{\frac{1}{2n}} = \|a\|$$

and therefore $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \|a\|$.

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