

Exercises to Introduction to Operator Algebras

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1 Topological Basics

Let X be a topological space, that is there exists a subset $\mathcal{O}(X) \in \mathbb{P}(X)$.

Definition 1.1 X is **Hausdorff** if for all $x, y \in X$ there exist open sets $U, V \in \mathcal{O}(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

X is **locally Hausdorff** if for all $x \in X$ there exists an open neighborhood $U \in \mathcal{O}(X)$ such that U is Hausdorff with the relative topology from X .

Example 1.2 (Snake with two heads) We consider the space $[0, 1] \cup \{1^+\}$ equipped with a topology such that both the subspace $[0, 1]$ and $[0, 1] \setminus \{1\} \cup \{1^+\}$ are isomorphic to $[0, 1]$. Then X is compact, locally Hausdorff but not Hausdorff.

Definition 1.3 X is compact if for every open cover $(U_i)_{i \in I}$ there exists a finite open subcover.

X is locally compact if for every $x \in X$ there exists a neighborhood basis of x consisting of open relatively compact subsets of X , that is for every open neighborhood U of x there exists an open neighborhood V of x such that \bar{V} is compact and $\bar{V} \subset U$.

Observation: For a locally Hausdorff X , X is locally compact if and only if for all $x \in X$ there exists an open neighborhood U of x such that \bar{U} is compact.

1.1 Results about locally compact Hausdorff spaces

Let X be Hausdorff and locally compact.

Proposition 1.4 (Uryson's Lemma) For all closed $F \subset X$ and all compact $K \subseteq X$ with $F \cap K = \emptyset$, there is a continuous function $f : X \rightarrow [0, 1]$ such that $f|_K \equiv 1$ and $f|_F \equiv 0$.

Proposition 1.5 (Tietze's extension theorem) For all $K \subseteq X$ compact and $f : K \rightarrow \mathbb{C}$ continuous, there exists an $\tilde{f} : X \rightarrow \mathbb{C}$ continuous such that $\tilde{f}|_K \equiv f$.

Proposition 1.6 (Alexandroff's compactification theorem) $\tilde{X} = X \cup \{\infty\}$ ($\infty \notin K$) is a compact Hausdorff space with $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{K^c \cup \{\infty\} \mid K \subseteq X \text{ compact}\}$.

For example, compactifying \mathbb{R} yields the unit circle \mathbb{S}^1 .

Proposition 1.7 Conversely, if Y is a compact Hausdorff space, then for all $y_0 \in Y$ the space $X = Y \setminus \{y_0\}$ is a locally compact Hausdorff space.

Proposition 1.8 More generally, if Y is a locally compact Hausdorff space and $Z \subseteq Y$ is a difference of open and closed subsets of Y (i.e. $Z = U \setminus F$ or $Z = F \setminus U$ where $U \subseteq Y$ is open and $F \subseteq Y$ is closed) then Z is locally compact.

Exercise 1.1 Let X be a locally compact Hausdorff space. The following are equivalent:

- (1) X is compact.
- (2) $C(X) = C_0(X)$ ($= C_b(X)$).
- (3) $C_0(X)$ is unital.
- (4) $1 \in C_0(X)$ where $1(x) = 1 \in \mathbb{C}$ for all $x \in X$.

PROOF:

- (1) \Rightarrow (2): Recall:

$$C_0(X) = \{f \in C(X) \mid \forall \varepsilon > 0 \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$$

If X is compact, then every closed subset of X is compact, so all sets of form $\{x \in X \mid |f(x)| \geq \varepsilon\}$ are compact, and we have $C(X) = C_0(X)$.

- (2) \Rightarrow (3): This is trivial because $C(X)$ is always unital.
- (3) \Rightarrow (4): Suppose $C_0(X)$ is unital and let $f \in C_0(X)$ be the unit. Then $f \cdot g = g$ for all $g \in C_0(X)$, that is $f(x)g(x) = 1$ for all $x \in X, g \in C_0(X)$. By Uryson's Lemma, given $x_0 \in X$, there exists a $g \in C_0(X)$ with $g(x_0) = 1$ (by looking at $K = \{x_0\}$, take any precompact open neighborhood U of x and look at $F := U^c \subseteq X$). Then we have $f(x_0) = f(x_0)g(x_0) = g(x_0) = 1$. As this is possible for every $x_0 \in X$, we have $f \equiv 1$.
- (4) \Rightarrow (1): Suppose $f = 1 \in C_0(X)$. Then choosing $\varepsilon = \frac{1}{2}$ shows that $X = \{x \in X \mid |f(x)| \geq \frac{1}{2}\}$ is compact. \square

Exercise 1.2 Let X be a locally compact Hausdorff space. Prove that $C_0(X) \simeq \{f \in C(\tilde{X}) \mid f(\infty) = 0\}$.

2 Exercise sheet 1

Exercise 2.1 (1.1)

PROOF: Case 1: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1 \alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathcal{A} \rightarrow \mathcal{M}, \lambda a \mapsto$

$\begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda} a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathcal{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to \mathbb{C} . \square

Exercise 2.2 (1.2) We consider pathological examples for $C_0(X)$.

Let $X = \{x_0\}$, e.g. $x_0 \in X$ with $\mathcal{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f : X \rightarrow \mathbb{C}$, this implies $f(x_0) \rightarrow f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall \varepsilon > 0 \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \neq 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 2.1 Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a $*$ -homomorphism between C^* -algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.

Exercise 2.3 (1.4 - Products) Let $(A_i)_{i \in I}$ be a family of C^* -algebras and define

$$\prod_{i \in I} A_i = \{a = (a_i)_{i \in I} \mid a_i \in A_i \forall i \in I \text{ and } \|a\| := \sup_{i \in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^* -axiom.

Remark 2.2 (Differences between product and direct sum)

In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall \varepsilon > 0 \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of $\prod_{i \in I} A_i$ as the closure of $\bigoplus_{i \in I}^{alg} A_i$, where

$$\bigoplus_{i \in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| = 0 \right\}.$$

For finite I , these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) $*$ -homomorphisms $\pi_j : \prod_{i \in I} A_i \rightarrow A_j$ for all $j \in I$. If B is any C^* -algebra with $*$ -homomorphisms $\varphi_j : B \rightarrow A_j$ for every $j \in I$, there is a unique $*$ -homomorphism $\varphi : B \rightarrow \prod_{i \in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi_j} & A_j \\ \downarrow \varphi & \nearrow \pi_j & \\ A & & \end{array}$$

Exercise 2.4 (1.5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a $*$ -isomorphism.

Exercise 2.5 (2.6) Let X be a locally compact Hausdorff space and $\widetilde{C_0(X)} \simeq C(\tilde{X})$ with $\tilde{X} := X \sqcup \{\infty\}$ with the topology $\mathcal{O}_{\tilde{X}} = \mathcal{O}_X \cup \{\tilde{X} \setminus K \mid K \subseteq X \text{ kompakt}\}$.

Observation: If X is already compact, then ∞ is an isolated point of \tilde{X} (i.e. $\{\infty\}$ is clopen).

If \mathcal{A} is a C^* -algebra, then $\tilde{\mathcal{A}}$ (this is not the same \sim as on the X !) is a C^* -algebra with

$$\|a + \lambda 1\|_{C^*} := \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|ab + \lambda b\|_{\mathcal{A}}$$

We check that $\tilde{\mathcal{A}}$ is a C^* -algebra.

- C^* -axiom: $\|a + \lambda 1\|_{C^*}^2 = \|(a + \lambda 1)^*(a + \lambda 1)\|_{C^*}$. We have

$$\|a + \lambda \cdot 1\|_{C^*} = \|(a^*a + \bar{\lambda}a + \lambda a^*) + |\lambda|^2 \cdot 1\|_{C^*}$$

$$= \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 \cdot b\|_{\mathcal{A}}$$

On the other hand:

$$\begin{aligned} \|a + \lambda \cdot 1\|_{C^*}^2 &:= \sup_{\|b\| \leq 1} \|ab + \lambda b\|_{\mathcal{A}}^2 \\ &= \sup_{\|b\| \leq 1} \|(ab + \lambda b)^*(ab + \lambda b)\|_{\mathcal{A}} \\ &= \sup_{\|b\| \leq 1} \|b^*a^*ab + \bar{\lambda}b^*ab + \lambda b^*a^*b + |\lambda|^2 b^*b\|_{\mathcal{A}} \\ &\leq \sup_{\|b\| \leq 1} \|b^*\|_{\mathcal{A}} \cdot \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathcal{A}} \\ &\leq \sup_{\|b\| \leq 1} \|a^*ab + \bar{\lambda}ab + \lambda a^*b + |\lambda|^2 b\|_{\mathcal{A}} \\ &= \|a + \lambda \cdot 1\|_{C^*}^2 \end{aligned}$$

- The other conditions are easy to check and are left for the student.

We still want to prove $\varphi : \widetilde{C_0(X)} \rightarrow C(\tilde{X}), f + \lambda \cdot 1 \mapsto f_\lambda$ with $f_\lambda(x) := \lambda$ for $x = \infty$ and $f_\lambda(x) = f(x) + \lambda$ otherwise. Note that once again these are not the same $\tilde{}$.

- f is well-defined: We have to check that f_λ is continuous in \tilde{X} . Take any sequence $X \ni x_i \rightarrow \infty$ in \tilde{X} . We have to show $f_\lambda(x_i) \rightarrow f_\lambda(\infty) = \lambda$. Since $f_\lambda(x_i) = f(x_i) + \lambda$ this is equivalent to $f(x_i) \rightarrow 0$. But as $f \in C_0(X)$, we have that for every $\varepsilon > 0$ the set $K_\varepsilon(f) = \{x \mid |f(x)| \geq \varepsilon\}$ is compact. Since x_i will eventually leave this compact set (or it would not diverge to ∞), we know that $f(x_i)$ eventually becomes smaller than (any) ε . So we have $f(x_i) \rightarrow 0$ and thus $f_\lambda(x_i) \rightarrow f_\lambda(\infty)$. So f_λ is continuous in ∞ . The continuity on every other point follows immediately from the continuity of f .
- φ is a *-isomorphism:

- Linearity: φ is clearly linear as we can check component-wise:

$$(f_1 + f_2)_\lambda = (f_1)_\lambda + (f_2)_\lambda$$

- Homomorphism: For every $x \in X$ we have

$$\begin{aligned} \varphi((f + \lambda \cdot 1) \cdot (g + \lambda' \cdot 1))(x) &= \varphi((fg + \lambda'f + \lambda g) + \lambda\lambda' \cdot 1)(x) \\ &= (fg + \lambda'f + \lambda g)(x) + \lambda\lambda' \\ &= (fg)(x) + \lambda'f(x) + \lambda g(x) + \lambda\lambda' \\ &= (f(x) + \lambda) \cdot (g(x) + \lambda') \\ &= (\varphi(f + \lambda) \cdot \varphi(g + \lambda'))(x). \end{aligned}$$

In the case of $x = \infty$, this equality of course also holds. Thus we have $\varphi((f + \lambda)(g + \lambda')) = \varphi(f + \lambda)\varphi(g + \lambda')$.

- *-homomorphism:

$$\varphi(f + \lambda)^*(x) = \varphi(f^* + \bar{\lambda} \cdot 1)(x)$$

For $x \in X$ this follows by $\overline{f(x)} + \bar{\lambda} = f^*(x) + \bar{\lambda}$, for $x = \infty$ we have $\bar{\lambda} = \bar{\lambda}$.

- Injective: $f_\lambda(0)$ leads to $f_\lambda(x) = 0$ for all $x \in \tilde{X}$, since if $x = \infty$ then λ must be 0 and $f(x) = 0$ for all $x \in X$. Thus $f = 0$ and $\lambda = 0$.
- Surjective: Take $g \in C(\tilde{X})$ and choose $\lambda = g(\infty)$ and $f(x) := g(x) - \lambda$. and check $f \in C_0(X)$.

- We can also prove that φ is isometric for the C^* -norm:

$$\|f + \lambda \cdot 1\| := \sup_{g \in C_0(X), \|g\| \leq 1} \|fg + \lambda g\|_\infty$$

Look at

$$\begin{aligned} \|\varphi(f + \lambda 1)\| &= \sup_{x \in \tilde{X}} |f_\lambda(x)| = \max\{|\lambda|, \sup_{x \in X} |f(x) + \lambda|\} \\ &\stackrel{(*)}{=} \sup_{x \in X} |f(x) + \lambda| \end{aligned}$$

and

$$\begin{aligned} \|f + \lambda \cdot 1\|_{C^*} &= \sup_{\|g\| \leq 1} \|fg + \lambda g\|_{C_0(X)} \\ &= \sup_{|g(x)| \leq 1 \forall x} \sup_{x \in X} \|f(x)g(x) + \lambda g(x)\| \\ &= \sup_{x \in X} |f(x) + \lambda| \\ &\stackrel{(**)}{=} \sup_{x \in X} |f(x) + \lambda| \end{aligned}$$

This proof may need to be divided into two cases:

- X is not compact: We can find a net $(x_i) \subseteq X$ with $f(x_i) \rightarrow 0$ and $(*)$ follows and use a $g(x) \approx 1$ for $(**)$.
- X is compact: Choose $g \equiv 1$ for $(**)$ and think about $(*)$ later.

Exercise 2.6 (1.8) It is difficult to prove $I^* = I$. The idea is to prove $I = C_0(U)$ where $C_0(U) = \{f \in C_0(X) \mid f|_U \equiv 0\}$.

One can also prove $C_0(X)/C_0(U) \simeq C_0(F)$ (as C_0 of the subspace) where $F = U^c$.

Exercise 2.7 Prove that \mathcal{A}/\mathfrak{I} is normed algebra, and

- (i) if \mathcal{A} is Banach and $\mathfrak{I} \trianglelefteq \mathcal{A}$ is closed, then \mathcal{A}/\mathfrak{I} is Banach.
- (ii) if \mathcal{A} is unital and Banach, then \mathcal{A}/\mathfrak{I} is unital.

unital if \mathcal{A} is, Banach if \mathcal{A} is and $\mathfrak{I} \trianglelefteq \mathcal{A}$ closed.

PROOF: Consider \mathcal{A}/\mathfrak{I} with $(a + \mathfrak{I})(b + \mathfrak{I}) = ab + \mathfrak{I}$. For the norm, use $\|a + \mathfrak{I}\| = \text{dist}(a, \mathfrak{I}) = \inf_{x \in \mathfrak{I}} \|a - x\|$. This is submultiplicative. For every $\varepsilon > 0$, there exist $x, y \in \mathfrak{I}$ for which we have

$$(\varepsilon + \|a + \mathfrak{I}\|) \cdot (\varepsilon + \|b + \mathfrak{I}\|) \geq \|a + x\| \cdot \|b + y\| \geq \|(a + x)(b + y)\| \geq \|ab + \underbrace{ay + xb + xy}_{\in \mathfrak{I}}\| \geq \|ab + \mathfrak{I}\|$$

and taking the limit yields the desired result.

Result (i) follows from functional analysis, that a space is Banach if and only if the convergence of $\sum_{k=0}^{\infty} a_n$ is equivalent to the convergence of $\sum_{k=0}^{\infty} \|a_n\|$.

Now let \mathcal{A} also be unital, then \mathcal{A}/\mathfrak{I} is unital. If $\mathfrak{I} = \mathcal{A}$, the algebra is the zero-algebra. Thus, let \mathfrak{I} be a proper ideal. The fact that $1 = 1_{\mathcal{A}} + \mathfrak{I}$ is a unit is clear, but we need to prove $\|1_{\mathcal{A}} + \mathfrak{I}\| = 1$. Observe that, if $x \in \mathfrak{I} \triangleleft \mathcal{A}$ then $x \notin \text{inv}(\mathcal{A})$ and $\|1_{\mathcal{A}} + x\| \geq 1$. Because otherwise, we have $\|1_{\mathcal{A}} + x\| < 1$ and then (because \mathcal{A} is Banach) $x = a - 1_{\mathcal{A}} \in \text{inv}(\mathcal{A})$. Hence $\|1_{\mathcal{A}}\| = \inf_{x \in \mathfrak{I}} \|1_{\mathcal{A}} + x\| \geq 1$. In addition, we have $1 \leq \|1_{\mathcal{A}} + \mathfrak{I}\| = \inf_{x \in \mathfrak{I}} \|1_{\mathcal{A}} - x\| \leq \|1_{\mathcal{A}} + 0\| \leq 1$. This proves $\|1\| = \|1_{\mathcal{A}} + \mathfrak{I}\| = 1$. \square

In the following, \mathbb{D} is the **closed** unit circle.

Exercise 2.8 Consider $\chi \in \Omega(\mathcal{A})$. We have proved $\|\chi\| \leq 1$. It may happen that $\|\chi\| < 1$. We need a non-unital algebra for this, because we have $\|\chi\| = 1$ if $1 \in \mathcal{A}$.

Consider $S = (\mathbb{N}, +)$ as an additive semigroup. Then

$$\ell^1(S) = \{(a_n)_{n \in \mathbb{N}} \mid \sum_{n=0}^{\infty} |a_n| < \infty\}$$

is a unital Banach algebra with $\delta_n \cdot \delta_m = \delta_{n+m}$ for all $n, m \in \mathbb{N}$ where

$$\delta_n(m) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Observe $\ell^1(S) = \overline{\text{alg}}\{\delta_0, \delta_1\}$ because of $\delta_1^n = \delta_n$. The unit of the algebra is δ_0 . What are the characters of $\ell^1(S)$?

We can write any $a \in \ell^1(S)$ as $a = \sum_{n=0}^{\infty} a_n \delta_n$. So if $\chi \in \Omega(\ell^1(S))$ then

$$\chi(a) = \sum_{n=0}^{\infty} a_n \chi(\delta_n) \in \mathbb{C}.$$

In particular, $\chi(1) = 1$ so $\chi(\delta_0) = 1$. This leads to $\chi(\delta_n) = \chi(\delta_1^n) = \chi(\delta_1)^n = \chi(\delta_1)^n$. So if we set $z := \chi(\delta_1) \in \mathbb{C}$, we have $\chi(a) = \sum_{n=0}^{\infty} a_n z^n$. Observe $|z| = |\chi(\delta_1)| \leq \|\delta_1\| = 1$ (because the Image of a character is a subset of the spectrum, which is bounded by the norm) so z must be in \mathbb{D} . By conventioning $z^0 = 1$ for every $z \in \mathbb{C}$, we can even choose $z = 0$.

Conversely, if $z \in \mathbb{D}$, we define $\chi_z(a) := \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}$. Then $\chi_z(\delta_n) = z^n$ and

$$\chi_z(\delta_n \cdot \delta_m) = \chi(\delta_{n+m}) = z^{n+m} = z^n \cdot z^m = \chi_z(\delta_n) \cdot \chi_z(\delta_m)$$

So we get a map $\mathbb{D} \rightarrow \Omega(\ell^1(S)) \subseteq \ell^1(S)^*$, $z \mapsto \chi_z$ that is bijective and continuous. If $z_i \rightarrow z$ in \mathbb{D} , we need to prove $\chi_{z_i} \rightarrow \chi_z$ in respect to the weak *-topology. So we need to evaluate and prove $\chi_{z_i}(a) \rightarrow \chi_z(a)$, or $\sum_{n=0}^{\infty} a_n z_i^n \rightarrow \sum_{n=0}^{\infty} a_n z^n$. Partial sums would obviously converge, so χ_{z_i} converges on a dense subspace of $\ell^1(S)$. The uniform boundedness principle (if a bounded set of operators converge on a dense subset $T_i \rightarrow T$, $\sup_i \|T_i\| < \infty$, they converge everywhere) shows that the infinite sums also converge. In general, showing that an operator converges on a dense set of an algebra always shows the convergence on any point of the algebra.

Observe $\sigma(\delta_1) = \{\chi(\delta_1) \mid \chi \in \Omega(\ell^1(S))\} = \mathbb{D}$ and $\sigma(\delta_1) = \mathbb{D}$ as well.

Concerning the norm, we know that

$$|\chi_z(a)| = \left| \sum_{n=0}^{\infty} a_n z^n \right| \leq \sum_{n=0}^{\infty} |a_n| |z|^n \leq \sum_{n=0}^{\infty} |a_n| = \|a\|$$

for all $a \in \ell^1(S)$, so $\|\chi_z\| \leq 1$. For $a = (a_0, 0, 0, \dots)$ we have $|\chi_z(a)| = |a_0| = \|a\|$, so $\|\chi_z\| = 1$ for any $z \in \mathbb{C}$ (and thus for any $\chi = \chi_z \in \Omega(\ell^1(S))$).

Remark 2.3 (Gelfand-Representation) In general, we seek a mapping $\mathcal{A} \rightarrow C_0(X), a \mapsto \hat{a}$, taking $X = \hat{\mathcal{A}} = \Omega(\mathcal{A})$ and $\hat{a}(\chi) = \chi(a)$.

If we apply the Gelfand representation here, we have

$$\ell^1(S) \rightarrow C(\mathbb{D}), a \mapsto \hat{a} \text{ where } \hat{a}(z) = \chi_z(a) = \sum_{n=0}^{\infty} a_n z^n$$

Example 2.4 (Norms < 1) Consider

$$\ell_0^1(S) = \overline{\text{alg}}(\delta_1) = \left\{ \sum_{n=1}^{\infty} a_n \delta_n \mid a_n \in \mathbb{C} \right\} \triangleleft \ell^1(S)$$

Observe $\widehat{\ell_0^1(S)} \simeq \ell^1(S)$. Recall $\Omega(\tilde{\mathcal{A}}) = \Omega(\mathcal{A}) \sqcup \{\chi_{\infty}\}$. So we are looking for our χ_{∞} , which is $\chi_{\infty}(a_0, a_1, \dots) = a_0$ – that is χ_0 and corresponds to $z = 0$ in the unit circle. It follows $\Omega(\ell_0^1(S)) \simeq \mathbb{D} \setminus \{0\}$ and $\chi_0 \in \Omega(\ell^1(S)) \setminus \Omega(\ell_0^1(S))$.

We compute $\|\chi_z\| = \sup_{\|a\|_1 \leq 1} |\chi_z(a)|$. Consider:

$$|\chi_z(a)| = \left| \sum_{n=1}^{\infty} a_n z^n \right| = \left| z \left(\sum_{n=1}^{\infty} a_n z^{n-1} \right) \right| \leq |z| \cdot \|a\|_1$$

so because of $\chi_z(\delta_1) = z$, we have $\|\chi_z\| = |z|$, which can be smaller than 1.

Remark 2.5 Do we have $\ell^1(S) \hookrightarrow A(\mathbb{D}), a \mapsto \hat{a}$ where $\hat{a}(z) := \sum_{n=0}^{\infty} a_n z^n$?

Exercise 2.9 (02-03) Is $\mathcal{A}(\mathbb{D})$ a C^* -algebra? Consider $f(z) = \exp(iz)$, $f \in \mathcal{A}$ and notice $z^* = z$. But we have $\|f^* f\|_{\infty} \neq \|f\|_{\infty}^2$, because $f^* f = 1$ and because $f(-i) = e$, we have $\|f\|_{\infty} \geq e$ and $\|f\|_{\infty}^2 \geq e^2 > 1 = \|f^* f\|_{\infty}$. Since the $*$ -property is not fulfilled.

Remark 2.6 Talk about functoriality. If X, Y are compact Hausdorff spaces and $f : X \rightarrow Y$ is continuous then

$$f_* : C(Y) \rightarrow C(X), g \mapsto g \circ f$$

You can check that f_* is a unital $*$ -homomorphism. So we receive a functor from the compact spaces to the unital commutative C^* -algebras:

$$\begin{aligned} \text{Comp. Spaces} &\rightarrow \text{unital abelian } C^*, X \mapsto C(X) \\ \text{Hom}(X, Y) &\rightarrow \text{Hom}(C(Y), C(X)), f \mapsto f_* \end{aligned}$$

This is a contravariant function because for $f : X \rightarrow Y, g : Y \rightarrow Z$ we have $(g \circ f)_* = f_* \circ g_*$. It is also natural. If $\varphi : C(Y) \rightarrow C(X)$ is a unital $*$ -homomorphism, we get a continuous map $f : X \rightarrow Y$ by duality.

3 More multiplier algebra

We continue to look at the multiplier algebra.

$$M(\mathcal{A}) = \{\mu = (L, R) \in L(\mathcal{A}) \times L(\mathcal{A}) \mid aL(b) = R(a)b, L(ab) = L(a)b, R(ab) = aR(b)\}$$

If \mathcal{A} is a C^* -algebra, \mathcal{A} embeds into $M(\mathcal{A})$ as an essential ideal. If A embeds into a C^* -algebra B as an essential ideal, then $B \rightarrow M(\mathcal{A}), b \mapsto \mu_b$ with $(a \mapsto ba, a \mapsto ab)$ is an isomorphism.

We also define the **strict topology** on $M(\mathcal{A})$ as the smallest topology that makes the map $a \mapsto \mu \cdot a, a \mapsto \mu \cdot a$ norm-continuous on $\mathcal{A} \rightarrow \mathcal{A}$. So if $(\mu_i) \subseteq M(\mathcal{A})$ is a net, then $\mu_i \rightarrow \mu \in M(\mathcal{A})$ if and only if $\mu_i a \rightarrow \mu a$ and $a \mu_i \rightarrow a \mu$ for all $a \in \mathcal{A}$.

Remark 3.1 Writing $0 \leq a \leq 1$ in a C^* -algebra means $a \geq 0$, so $\sigma(a) \subseteq [0, \infty)$ and $a \leq 1$ means $(1 - a)$ is positive in $\tilde{\mathcal{A}}$ or $M(\mathcal{A})$ which is equivalent to $\|a\| \leq 1$.

All of this is equivalent to $\sigma(a) \subseteq [0, 1]$.

Relation with approximate units: If $(e_i) \subseteq A_{+,1}$ is an increasing net ($0 \leq e_i \leq 1$) then (e_i) is an approximate unit iff $e_i \rightarrow 1$ (strictly) in $M(\mathcal{A})$.

By definition this means $e_i a \xrightarrow{\|\cdot\|} a$, $a \cdot e_i \xrightarrow{\|\cdot\|} a$.

3.1 Non-degenerate $*$ -homomorphisms

Definition 3.2 Let $\pi : \mathcal{A} \rightarrow M(\mathcal{B})$ a $*$ -homomorphism. We say that π is **non-degenerate** if $\text{span } \pi(\mathcal{A}) \cdot B = B$.

Lemma 3.3 Let $\pi : \mathcal{A} \rightarrow M(\mathcal{B})$ be a $*$ -homomorphism. The following are equivalent:

- (i) π is non-degenerate.
- (ii) $\pi(e_i) \rightarrow 1$ (strictly) in $M(B)$ if (e_i) is some approximate unit in \mathcal{A} .
- (iii) π extends to a strictly continuous unital $*$ -homomorphism $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$.

PROOF:

(i) \Rightarrow (ii): Let (e_i) be an approximate unit. Prove $\pi(e_i) \rightarrow 1$ (strictly) in $M(\mathcal{B})$, that is $\pi(e_i)b \rightarrow b$ for all $b \in \mathcal{B}$. Since (e_i) is bounded, it is enough to show that $\pi(e_i)b \rightarrow b$ for all $b \in \pi(A)B$ as this is dense in B . But if $b = \pi(a)c$ for $a \in \mathcal{A}, c \in \mathcal{B}$, then $\pi(e_i)b = \pi(e_i)\pi(a)c = \pi(e_i a)c \rightarrow \pi(a)c = b$ because π is norm-continuous and $e_i a \rightarrow a$.

(ii) \Rightarrow (iii): We want to extend π to $\tilde{\pi} : M(A) \rightarrow M(B)$. We need $\tilde{\pi}$ such that

$$\tilde{\pi}(\mu)\pi(a) = \tilde{\pi}(\mu)\tilde{\pi}(a)\tilde{\pi}(\mu \cdot a) = \pi(\mu \cdot a).$$

Therefore, define the multiplier $\tilde{\pi}(\mu)$ just on $\pi(\mathcal{A})\mathcal{B}$ by the mappings

$$\begin{aligned} L(\pi(a)b) &= \tilde{\pi}(\mu) \cdot (\pi(a)b) = \pi(\mu a) \cdot b \in \mathcal{B} \\ R(b\pi(a)) &= (b\pi(a)) \cdot \tilde{\pi}(\mu) = b\pi(a\mu) \in \mathcal{B}. \end{aligned}$$

These morphisms are certainly linear. By (ii), notice that $\overline{\pi(\mathcal{A})\mathcal{B}} = \mathcal{B}$. So the above defines morphism on all of \mathcal{B} by continuous extension.

We need to prove: L, R are well-defined and extend to \mathcal{B} and $\mu = (L, R)$ is a multiplier of \mathcal{B} .

Claim:

$$\left\| \sum_{i=0}^n \pi(\mu \cdot a_i) \cdot b_i \right\| \leq \|\mu\| \cdot \left\| \sum_{i=0}^n \pi(a_i) b_i \right\|$$

for all sequences $(a_i) \subseteq \mathcal{A}$ and $(b_i) \subseteq \mathcal{B}$. To prove that L is well-defined compute $\tilde{\pi}\pi(a)b = \pi(\mu a)b$.

$$\pi(\mu a)b = \lim_i \pi(\mu e_i a)b = \lim_i \pi(\mu e_i)\pi(a)b$$

So L is well-defined, because we can write $L(\pi(a)b) = \lim_i \pi(\mu e_i)(\pi(a)b)$ as directly dependent on $\pi(a)b$ (so different representations of $\pi(a)b$ will yield the same result). Analogously, we have $R(b\pi(a)) = b\pi(a) \lim_i \pi(e_i \mu)$. Now we can compute

$$\|\tilde{\pi}(\mu)\pi(a)b\|^2 = \|b^* \pi(\mu a)^* \pi(\mu a)b\| = \|b^* \pi(a^* \mu^* \mu a)b\| \leq \|b^* \pi(\|\mu\|^2 a^* a)b\|$$

$$= \|\mu\|^2 \|b^* \pi(a)^* \pi(a) b\| = \|\mu\|^2 \|\pi(a) b\|^*$$

Now we show that $\tilde{\pi}$ is actually a multiplier, let $x = b_1 \pi(a_1)$ and $y = \pi(a_2) b_2$ and consider

$$xL(y) = b_1 \pi(a_1) \pi(\mu a_2) b_2 = b_1 \pi(a_1 \mu a_2) b_2 = b_1 \pi(a_1 \mu) \pi(a_2) b_2 = R(x)y$$

This proves the last multiplier property, the others follow.

You can repeat this for R . So we have a well-defined map $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$. We need to prove that this is a strictly continuous $*$ -homomorphism extending π . Calculate:

$$\tilde{\pi}(1) \pi(a) b = \pi(1 \cdot a) b = \pi(a) b$$

so $\tilde{\pi}(1) = 1$ and

$$\tilde{\pi}(\mu) \cdot \tilde{\pi}(\nu) \pi(a) b = \tilde{\pi}(\pi)(\mu) \pi(\nu a) b = \pi(\mu \cdot \nu \cdot a) b = \tilde{\pi}(\mu \cdot \nu) \pi(a) b$$

so $\tilde{\pi}(\mu) \cdot \tilde{\pi}(\nu) = \tilde{\pi}(\mu \cdot \nu)$. To prove that $\tilde{\pi}$ is strictly continuous, take $\mu_i \rightarrow \mu$ (strictly) and show

$$\tilde{\pi}(\mu_i) \pi(a) b = \pi(\mu_i a) b \xrightarrow{\|\cdot\|} \pi(\mu a) b = \tilde{\pi}(\mu) \pi(a) b$$

Now use the Cohen-Hilbert Factorization theorem:

If E is a Banach right (or left) module over some Banach algebra \mathcal{A} and \mathcal{A} has an approximate unit $(e_i) \subseteq A_1$, $\|e_i\| \leq 1$, then $\overline{\text{span } E \cdot \mathcal{A}} = E \cdot \mathcal{A} = \{x \cdot a \mid x \in E, a \in \mathcal{A}\}$.

Apply this for $E = \mathcal{B}$ (view it as a left \mathcal{A} -module via π): $a \cdot b := \pi(a) b$. This shows $\overline{\text{span } \pi(\mathcal{A}) \mathcal{B}} = \pi(\mathcal{A}) \mathcal{B}$.

(iii) \Rightarrow (i): Suppose $\mathcal{A} \rightarrow M(\mathcal{B})$ extends to $\tilde{\pi} : M(\mathcal{A}) \rightarrow M(\mathcal{B})$ (strictly continuous $*$ homomorphism).

(iv) But then $\tilde{\pi}(1) = 1$, so

$$1 = \tilde{\pi}(1) = \tilde{\pi}(\text{strict } \lim_i e_i) = \text{strict } \lim_i \pi(e_i)$$

so $\pi(e_i)$ strictly converges to 1. □

Exercise 3.1 (03-01) For any C^* -algebra \mathcal{A} and $a \in \mathcal{A}$ we have $a(a^* a)^{\frac{1}{n}} \rightarrow a$ (and $(aa^*)^{\frac{1}{n}} a \rightarrow a$).

PROOF: If $\mathcal{A} = C_0(X)$ then $a = f$ and we have $f(x)|f(x)|^{\frac{2}{n}} \rightarrow f(x)$ as $|f(x)|^{\frac{2}{n}} \rightarrow 1$ on the non-zero points of f and $\rightarrow 0$ otherwise. On a compact set, this then converges uniformly.

In general: Compute

$$\begin{aligned} \|a(a^* a)^{\frac{1}{n}} - a\|^2 &= \|(a(a^* a)^{\frac{1}{n}})^* \cdot (a(a^* a)^{\frac{1}{n}} - a)\| \\ &= \|(a^* - (a^* a)^{\frac{1}{n}} a^*) \cdot (a^* a)^{\frac{1}{n}} - a\| \\ &= \|a^* a(a^* a)^{\frac{1}{n}} - a^* a - (a^* a)^{\frac{1}{n}} a^* a(a^* a)^{\frac{1}{n}} + (a^* a)^{\frac{1}{n}} a^* a\| \\ &= \|bb^{\frac{1}{n}} - b - b^{\frac{1}{n}} b b^{\frac{1}{n}} + b^{\frac{1}{n}} b\| \end{aligned}$$

with $b = a^* a$ self-adjoint. We may then replace \mathcal{A} by $C^*(b)$, which is commutative. The above case then applies by considering b as a function via the Gelfand-Representation. □

Exercise 3.2 (04-10) $C_0(X)$ is σ -unital (there exists a countable approximate unit) if and only if X is σ -compact (X is the union of countably many compact sets).

Example 3.4 Take ω as the first uncountable ordinal. Then $X = [0, \omega)$ is not σ -compact.

If X is metrizable and locally compact, there exists a metric $d : X \times X \rightarrow \mathbb{R}$ such that the topology on X is induced by d .

If X is separable, then X is σ -compact.

PROOF: \Rightarrow : Assume $C_0(X)$ is σ -unital. Then we have a countable approximate unit $(e_n)_{n \in \mathbb{N}}$. Define $f = \sum_{n=1}^{\infty} \frac{e_n}{2^n} \in C_0(X)$ as $0 \leq e_n \leq 1$. Then f is positive ($f \geq 0$) and bounded ($f \leq 1$). f is even strictly positive ($f > 0$), that is $f(x) > 0$ for all $x \in X$ because $e_n(x) \rightarrow 1$ uniformly on compacts and $f \geq \frac{e_n}{2^n}$ for all n .

Then, defining $K_n := \{x \in X \mid f(x) \geq \frac{1}{n}\} \subseteq X$ is compact. Notice $\bigcup_{n=1}^{\infty} K_n = X$ and $K_n \subseteq K_{n+1}$.

\Leftarrow : Assume $X = \bigcup_{n=1}^{\infty} K_n$ where $K_n \subseteq X$ are compact. We may assume $K_n \subseteq K_{n+1}$. From σ -compactness we may find an open, countable cover of $X = \bigcup_{n=1}^{\infty} U_n$ such that U_n is open and $\overline{U_n}$ is compact for every $n \in \mathbb{N}$. This cover can be found by covering every K_n with (finitely many) open sets and taking the union. Now we use Uryson's Lemma to find a sequence $(f_n)_{n \in \mathbb{N}}$ where $0 \leq f \leq 1$ and $f_n|_{\overline{U_n}} \equiv 1$ and $f_n \in C_0(X)$.

Take now a compact $K \subseteq X = \bigcup_{n=1}^{\infty} U_n$. As K is compact, there exists an n, m with $K \subseteq U_n \subseteq U_m$ and $f_m|_K \equiv 1$ and $m \geq n$. This shows $f_n(x) \rightarrow 1$ on compacts.

Observe: (f_n) may not be increasing. To get this, choose two covers $(U_n), (V_n)$ of X of open sets with compact closures where $U_n \subseteq \overline{U_n} \subseteq V_n \subseteq \overline{V_n} \subseteq U_{n+1}$. Now choose by Uryson $(f_n)_{n=1}^{\infty}$ with $f_n|_{\overline{U_n}} \equiv 1, f_n|_{V_n^c} \equiv 0$. Now f_n is increasing. \square

Remark 3.5 The following are equivalent:

- (i) The C^* -algebra \mathcal{A} is σ -unital (there exists a countable approximate unit $(e_n)_{n \in \mathbb{N}}$).
- (ii) There exist $a \in \mathcal{A}_{+,1}$ (write: $a > 0$), meaning that $\varphi(a) > 0$ for all $\varphi \in \mathcal{A}_+^*$ (positive linear functions) with $\|\varphi\| = 1$.
- (iii) $\overline{\mathcal{A} \cdot a} = \mathcal{A}$.
- (iv) $\overline{a\mathcal{A}} = \mathcal{A}$.

PROOF: See Diximiere (important guy). \square

Exercise 3.3 Let \mathcal{A} be a C^* -algebra, $a \in \mathcal{A}$ and $L_a := \overline{\mathcal{A}a}$.

Then L_a is the closed left ideal generated by a , and if a is a projection (self-adjoint and idempotent) we even have $L_a = \mathcal{A}a$.

Furthermore, $L_a = L_{aa^*}$.

Lastly, if \mathcal{A} is unital and $aa^* = 1$ for some a and $L_a = \mathcal{A}$. Then a is unitary.

PROOF: We prove this is the closed left ideal generated by a , that is, the smallest closed ideal containing a .

First, observe that $L_a = \overline{\mathcal{A}a}$ is a closed left ideal, because $\mathcal{A} \cdot a = \{xa \mid x \in \mathcal{A}\}$ is a left ideal. We also have $a \in L_a$, because we can use the approximate unit for x and that L_a is closed. Now if $L \subseteq \mathcal{A}$ is a closed left ideal with $a \in L$, we must have $\overline{\mathcal{A}a} \subseteq L$. This concludes the proof.

To prove that if a is a projection (idempotent would be enough), $\mathcal{A}a$ is closed, take a sequence $x_n a \rightarrow b \in \mathcal{A}$ and therefore $x_n a^2 \rightarrow ba$. So $b = ba \in \mathcal{A}a$. Thus, we could write $\mathcal{A}a = \{xa \mid x \in \mathcal{A}\} = \{x \mid x \in \mathcal{A}, xa = x\}$.

To $L_a = L_{aa^*}$, i.e. $\overline{\mathcal{A}a} = \overline{\mathcal{A}a^*a}$. Using exercise 03-01, we see $a = \lim_n a(a^*a)^{\frac{1}{n}} \in \overline{\mathcal{A}a^*a}$ (because $a^*a \in L_{a^*a}$ and therefore $(a^*a)^{\frac{1}{n}} \in L_{a^*a}$ because $(a^*a)^{\frac{1}{n}}$ is approximated by a sequence of polynomials and L_{a^*a} is closed, see Stone-Weierstraß), so $L_a \subseteq L_{a^*a}$. The other inclusion is obvious, as $a^*a \in L_a$.

For the last point, we need to prove $a^*a = 1$. We have $L_a = \overline{\mathcal{A}a} = \overline{\mathcal{A}a^*a} \stackrel{!}{=} \mathcal{A}$, i.e. $\mathcal{A}a$ must be dense in \mathcal{A} . But then the claim is already $\mathcal{A}a = \mathcal{A}$ as \mathcal{A} contains a unit 1 that is also in $\mathcal{A}a$. Otherwise, $1 \notin \mathcal{A}a$ and also $\mathcal{A}a$ does not contain any invertible elements. But the set of invertibles is non-trivially open in \mathcal{A} , a contradiction to the denseness.

So we have $\mathcal{A}a = \mathcal{A}$, meaning there exists $b \in \mathcal{A}$ with $ba = 1$, so a is left invertible. But by the assumption, there is also a right inverse. So a is invertible with inverse a^* , so it is unitary. \square