

# Exercises to Introduction to Operator Algebras

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# 1 Exercise sheet 1

## Exercise 1.1 (1)

PROOF: **Case 1:** If  $b_1, b_2 \in A$ , then  $b_i = \alpha_i a$  for certain  $\alpha_i \in \mathbb{C}$ . Thus,  $b_1 \cdot b_2 = \alpha_1 \alpha_2 a^2 = 0$ . Thus, the multiplication is trivial. From this, it immediately follows that  $\varphi : \mathcal{A} \rightarrow \mathcal{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$  is an isomorphism.

**Case 2:**  $\lambda \neq 0$ , and  $a^2 = \lambda a$ . Let  $b = \frac{1}{\lambda} a$ , then  $b \cdot a = a = a \cdot b$ . But then, for any  $c = \mu a \in \mathcal{A}$ , we have  $bc = \mu ba = \mu a = c = cb$ , so the algebra is unital and isomorphic to  $\mathbb{C}$ .  $\square$

## Exercise 1.2 (2) We consider pathological examples for $C_0(X)$ .

Let  $X = \overline{\{x_0\}}$ , e.g.  $x_0 \in X$  with  $\mathcal{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$ .  $X$  is highly non-Hausdorff unless we already have  $X = \{x_0\}$ . In this space, the constant sequence  $(x_0)$  converges to any  $x \in X$ .

For a continuous function  $f : X \rightarrow \mathbb{C}$ , this implies  $f(x_0) \rightarrow f(x)$  for all  $x \in X$ , so every continuous function must already be constant. It follows that  $C(X) \simeq \mathbb{C}$ .

We now look at  $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$ . But since all functions are constant, we can use  $f(x_0)$  instead of  $X$  and  $\{x \in X \mid |f(x)| \geq \varepsilon\}$  is either empty or the whole space.  $X$  is compact if and only if  $X$  is finite. From here on, assume  $X$  to be infinite. Then, only the finite subsets are compact. Thus, if we now have  $f \neq 0$ , there exists an  $|f(x_0)| > \varepsilon > 0$  and thus  $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$  is not compact. This implies  $C_0(X) = \{0\}$ .

To find a non-compact topological space that has non-zero unital  $C_0(X)$ , consider  $X = X_0 \sqcup X_1$  with  $X_0$  as before and  $X_1$  compact.

**Theorem 1.1** Let  $\varphi : \mathcal{A} \rightarrow \mathcal{B}$  be a  $*$ -homomorphism between  $C^*$ -algebras. Then we already have  $\|\varphi(a)\| \leq \|a\|$  for all  $a \in \mathcal{A}$ .

## Exercise 1.3 (4 - Products) Let $(A_i)_{i \in I}$ be a family of $C^*$ -algebras and define

$$\prod_{i \in I} A_i = \{a = (a_i)_{i \in I} \mid a_i \in A_i \forall i \in I \text{ and } \|a\| := \sup_{i \in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinatewise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the  $C^*$ -axiom.

**Remark 1.2 (Differences between product and direct sum)**

In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall \varepsilon > 0 \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of  $\prod_{i \in I} A_i$  as the closure of  $\bigoplus_{i \in I}^{alg} A_i$ , where

$$\bigoplus_{i \in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists \text{finite } F \subseteq I \forall i \notin F \|a_i\| = 0 \right\}.$$

For finite  $I$ , these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following categorical universal property: We have **(surjective) \*-homomorphisms**  $\pi_j : \prod_{i \in I} A_i \rightarrow A_j$  for all  $j \in I$ . If  $B$  is any  $C^*$ -algebra with \*-homomorphisms  $\varphi_j : B \rightarrow A_j$  for every  $j \in I$ , there is a unique \*-homomorphism  $\varphi : B \rightarrow \prod_{i \in I} A_i$  such that  $\pi_j \circ \varphi = \varphi_j$ . This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi_j} & A_j \\ \downarrow \varphi & \nearrow \pi_j & \\ A & & \end{array}$$

**Exercise 1.4 (5)**  $X$  is a locally compact Hausdorff space that can be written as  $X = U \cup V$  with open and disjoint  $U, V$  (so  $U, V$  are clopen). We want to prove  $C_0(X) \simeq C_0(U) \oplus C_0(V)$ . To build this map, we map  $f \mapsto (f|_U, f|_V)$ . We check that this is well-defined and a \*-isomorphism.