Exercise Sheet 03
Operator Algebras

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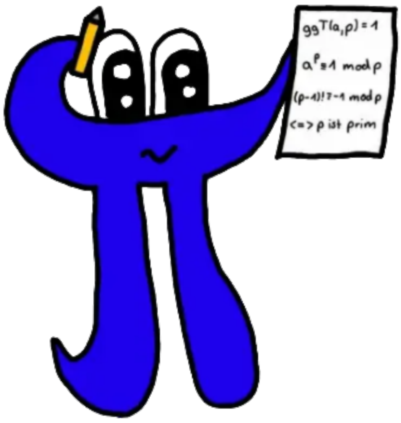

3.2

The C^*-property shows $\|a^2\| = \|a^*a\| = \|a\|^2$, and by using this as well as the C^* property again, we have for $n = 4$ that $\|a^4\| = \|a^*a^*aa\| = \|(a^2)^*(a^2)\| = \|a^2\|^2 = \|a^4\|$. Inductively, we can likewise prove $\|a^{2^k}\| = \|a\|^{2^k}$ for all $k \in \mathbb{N}$.

Now, for any $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $n + m = 2^k$ for some $k \in \mathbb{N}$. Then we have

$$\|a\|^{2^k} = \|a^{2^n}\| = \|a^n a^m\| \leq \|a^n\| \cdot \|a^m\| \leq \|a\|^n \cdot \|a^m\| \leq \|a\|^{n+m} = \|a\|^{2^k}$$

and because the first and last element are equal, we must have equality in every intermediate step. This especially proves $\|a^n\| = \|a\|^n$.

Let now $a \in \mathscr{A}$ be an arbitrary element. Then $\|a^*a \dots a^*a\| = \|(a^*a)^{\frac{n}{2}}\| = \|a^*a\|^{\frac{n}{2}} = \|a\|^n$ as proven above, because (a^*a) is self-adjunct. For non-even n (and thus even $n+1$) we can once again calculate

$$\|a\|^{n+1} = \|a^*aa^* \dots a^*a\| \leq \|a\| \cdot \|aa^* \dots a^*\| \leq \|a\| \cdot \|a\|^n = \|a\|^{n+1}$$

and therefore $\|aa^* \dots a^*\| = \|a\|^n$ by the same argument as above.

Now, for a normal $a \in \mathscr{A}$ (that is, $a^*a = aa^*$) we have

$$\|a^n\|^{\frac{1}{n}} = \left(\|a^n\|^2\right)^{\frac{1}{2n}} = \|(a^n) * a^n\|^{\frac{1}{2n}} = \|aa^*a \dots a^*\|^{\frac{1}{2n}} = \left(\|a\|^{2n}\right)^{\frac{1}{2n}} = \|a\|$$

and therefore $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \|a\|$.

Finally, we can use the fundamental theorem of continuous functional calculus. Consider for the moment $\mathscr{A}$ to be unital (if it is not, consider $\tilde{\mathscr{A}}$). Then $a \in \mathscr{A}$ is normal and $f : \mathbb{C} \to \mathbb{C}, x \mapsto |x|^2 = x \cdot \overline{x}$. f is continuous on $\mathbb{C}$ and therefore especially on $\sigma(a)$. Thus we have

$$\sigma(aa^*) = \sigma(f(a)) = f(\sigma(a)) = \{f(\lambda) \mid \lambda \in \sigma(a)\} = \{|\lambda|^2 \mid \lambda \in \sigma(a)\}.$$

As a is normal, we also have $a^*a = aa^*$ and therefore $\sigma(aa^*) = \sigma(a^*a)$.

3.6

First, to prove that $M(\mathscr{A})$ fulfills the given property. We already know that $\mathscr{A}$ is a closed, two-sided and essential ideal in $M(\mathscr{A})$. Consider the following morphism:

$$\varphi : \mathscr{B} \to M(\mathscr{A}), b \mapsto (L_b, R_b)$$

where

$$\begin{aligned} L_b &: \mathscr{A} \to \mathscr{A} a \mapsto b \cdot a \\ R_b &: \mathscr{A} \to \mathscr{A} a \mapsto a \cdot b \end{aligned}$$

defined via the multiplication in $\mathscr{B}$. Because $\mathscr{A} \trianglelefteq \mathscr{B}$, we actually have $a \cdot b, b \cdot a \in \mathscr{A}$ for all a, b and L_b, R_b are well-defined and, as they are clearly linear, φ is also well-defined. Because of $L_{ab} = L_a \circ L_b$ and $R_{ab} = R_b \circ R_a$, we have $\varphi(ab) = \varphi(a) \cdot \varphi(b)$ with the multiplication as defined in the lecture. Furthermore, $\varphi(1) = (L_1, R_1) = (\mathrm{id}, \mathrm{id})$ and φ is therefore a homomorphism. Lastly we have $\varphi(b^*) = (L_{b^*}, R_{b^*})$ and

$$\begin{aligned} L_{b^*}(a) &= b^*a = (a^*b)^* = R_b(a^*)^* = (R_b)^*(a) \\ R_{b^*}(a) &= ab^* = (ba^*)^* = L_b(a^*)^* = (L_b)^*(a) \\ \Rightarrow \varphi(b^*) &= (R_b^*, L_b^*) = (L_b, R_b)^* \end{aligned}$$

so φ is indeed a *-homomorphism. Since $\varphi|_{\mathscr{A}}$ reduces to the normal left- and right-multiplication on $\mathscr{A}$, it coincides with canonical inclusion map as defined in the lecture. φ therefore fulfills all conditions as given.

To conclude that the universal property is indeed correct, we need to consider the case that $\mathscr{A} \trianglelefteq \mathscr{B}$ is an essential ideal. In this case, $b\mathscr{A} = 0$ implies $b = 0$ for any $b \in \mathscr{B}$. Assume $\varphi(b) = \varphi(c)$ for any two $b, c \in \mathscr{B}$. Then we have $(L_b, R_b) = (L_c, R_c)$ and thus $ba = ca$ and $ab = ac$ for all $a \in \mathscr{A}$. This is equivalent to $b\mathscr{A} = c\mathscr{A}$ and $\mathscr{A}b = \mathscr{A}c$ or, stated differently, $(b-c)\mathscr{A} = 0$ and $\mathscr{A}(b-c) = 0$. As stated above, this implies $(b-c) = 0 \Leftrightarrow b = c$ and thus proves that φ is injective.

Next, we want to prove that any algebra $D \trianglerighteq \mathscr{A}$ that fulfills the above property (and where $\mathscr{A}$ is a closed, two-sided essential ideal in D) is already equal to $M(\mathscr{A})$.

We already know that $\mathscr{A}$ is an essential ideal in $M(\mathscr{A})$, so if D also fulfills the property above the therefore existent morphism $\varphi_D : M(\mathscr{A}) \to D$ must be injective. We may thus treat $M(\mathscr{A})$ as a subalgebra of D. In parallel, since $\mathscr{A}$ is also an essential ideal of D, the morphism $\varphi_M : D \to M(\mathscr{A})$ is also injective and we may consider $M(\mathscr{A})$ as a subalgebra of D. But then these two algebras are isomorphic to subalgebras of each other, so they must already be equal.

3.7

First, we prove $C_0(X)$ is an essential ideal.

(i) **Ideal**: Take any functions $f \in C_0(X)$ and $g \in C_b(X)$ with $\|g\|_\infty = \sup_{x\in X} |g(x)| = M \geq 0$. W.l.o.g we assume $g \neq 0$, so $M > 0$. Let $\varepsilon > 0$ be arbitrary. Then $Y = \{x \in X \mid |f(x)| \geq \frac{\varepsilon}{M}\}$ is compact (as $f \in C_0(X)$). For all $x \notin Y$ we have $|(fg)(x)| = |f(x)g(x)| \leq M \cdot |f(x)| < \varepsilon$, so $\{x \in X \mid |(fg)(x)| \leq \varepsilon\} \subseteq Y$. As this set is closed and Y is compact it is compact (for any ε), we have $fg \in C_0(X)$ and $C_0(X)$ is right ideal.

(ii) **Closed and two-sided**: As $C_b(X)$ is commutative, any one-sided ideal is also two-sided. The closedness has been proven in previous exercises.

(iii) **Essential**: Let $g \in C_b(X)$ be any function with $gC_0(X) = 0$, that is $(gf)(x) = 0$ for any $f \in C_0(X)$ and $x \in X$. Let $x_0 \in X$ be freely chosen. By Uryson's Lemma, we can find a function $f_{x_0} \in C_0(X)$ fulfilling $f_{x_0}(x_0) = 1$ and $f(x) = 0$ on the (closed complement) of an open, pre-compact set (thus actually putting f in $C_0(X)$). Then we have $g(x_0)f(x_0) = 0$, but because of $f(x_0) = 1$ we must have $g(x_0) = 0$. As x_0 was arbitrary in X, g must be 0 on the entirety of X.

Next, for any $x \in X$ and $b \in B$ pick f (as given in the exercise, but also requiring $\|f\|_\infty = 1$. The existence of such an f can be concluded by Uryson) and define F_b as given in the exercise (but with our slightly more restrictive choice of f). Then

$$|(bf)(x)| \leq \|bf\|_\infty = \|bf\|_B \leq \|b\|\|f\|_B = \|b\| \cdot 1 = \|b\||f(x)|$$

Therefore, F_b is bounded (by $\|b\|$) and well-defined.

Furthermore, F_b is continuous, as for any sequence $x_n \to x_0$ in X, x_n will be in $B_1(x_0)$ for large enough n. Because $B := \overline{B_1(x_0)}$ is compact, by Uryson's Lemma we can find a $g \in C_0(X)$ with $\|g\|_\infty = 1$ and $g \equiv 1$ on B. Then $F_b(x) = (bg)(x)$ for any $x \in B$ (as the choice of f in the definition does not matter, and g fulfills all conditions). But now F_b on B coincides with $bg \in C_b(X)$, so $F_b(x_n) = (bf)(x_n) \to (bf)(x_0) = F_b(x_0)$ and F_b is continuous as our sequence was arbitrary.

Now, consider $F : B \to C_b(X), b \mapsto F_b$:

(i) **unital**: If $1_B \in B$ is the unit in B, we have $1_B b = b = b 1_B$ for any $b \in B$, therefore especially $1_B f = f = f 1_B$ for any $f \in C_0(X)$. Then $F_b(x) = (bf)(x) = f(x) = 1$ for any $x \in X$, so $F(b) = F_b \equiv 1$ on X. As the constant 1-function is the unit in $C_b(X)$, F is unital.

(ii) **Homomorphism**: Take $a, b \in B$ and calculate (for any $x \in X$ and a fitting $f \in C_0(X)$):

$$\begin{aligned}
F(a \cdot b)(x) = F_{a\cdot b}(x) &= ((ab)f)(x) && f^2 \text{ also fulfills our conditions} \\
&= ((ab)f^2)(x) && \text{Associativity} \\
&= (a(bf)f)(x) && \text{Elements } bf \text{ and } f \text{ in } C_0(X) \text{ commute} \\
&= (af(bf))(x) \\
&= ((af)(bf))(x) && \text{Multiplication of elements in } C_0(X) \\
&= (af)(x) \cdot (bf)(x) \\
&= F_a(x) \cdot F_b(x) \\
&= F(a)(x) \cdot F(b)(x)
\end{aligned}$$

as f^2 also fulfills $f^2(x) = f(x) \cdot f(x) = 1$ and $\|f^2\|_\infty \leq \|f\|_\infty^2 = 1$. This shows $F_{ab} = F_a \cdot F_b$.

(iii) ***-property**: Consider $b \in B$, and take $x \in X$ and a fitting $f \in C_0(X)$. Then

$$F(b^*)(x) = F_{b^*}(x) = (b^* f)(x) = (f^* b)^*(x) = \overline{(f^* b)(x)} = \overline{(b f^*)(x)} = \overline{F_b(x)}$$

as f^* also fulfills $f^*(x) = \overline{f(x)} = 1$ and $\|f^*\| = \|f\| = 1$.

In conclusion, $C_b(X)$ fulfills the universal property of $\mathcal{M}(C_0(X))$, so we have $C_b(X) = \mathcal{M}(C_0(X))$.

Concerning the last paragraph: The commutative, closed, two-sided C^*-algebra-ideal i *-isomorphic to $C_0(X)$ for some X, so there exists an (injective) *-isomorphism $B \to C_b(X)$. So B embeds in $C_b(X)$. But $C_b(X)$ is commutative, so B must be as well.

A sketch of the proof would work by directly proving that the multiplier algebra $\mathcal{M}(\mathscr{A})$ of any commutative algebra $\mathscr{A}$ is itself commutative: Let $(L_1, R_1), (L_2, R_2) \in \mathcal{M}(\mathscr{A})$, so L_1, L_2, R_1, R_2 are linear mappings $\mathscr{A} \to \mathscr{A}$. Then $(L_1 \cdot L_2)(a) = L_1(a) \cdot L_2(a)$ and as elements in $\mathscr{A}$ commute, L_1 and L_2 do as well. This and the equivalent result for R_1, R_2 shows

$$(L_1, R_1) \cdot (L_2, R_2) = (L_1 \cdot L_2, R_2 \cdot R_1) = (L_2 \cdot L_1, R_1 \cdot R_2) = (L_2, R_2) \cdot (L_1, R_1)$$

so $\mathcal{M}(\mathscr{A})$ is commutative. The result then follows as above: As B contains $\mathscr{A}$ as an essential, two-sided ideal the *-isomorphism $B \to \mathcal{M}(\mathscr{A})$ is injective, therefore B is embedded in $\mathcal{M}(\mathscr{A})$ and finally also commutative.