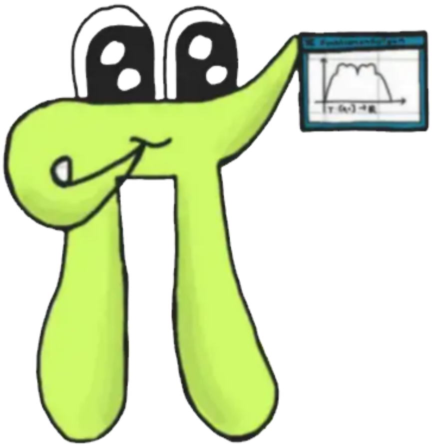
Übungsblatt 01
Operatoralgebra

Abgabe von: **Linus Mußmächer, Minona Schäfer**

May 4, 2023



1.8

We first show the prerequisites of the Stone-Weierstrass theorem.

(i) I is a C^*-subalgebra of $C_0(U)$. I is a subset of $C_0(U)$. If it were not, we would have an $f \in I, f \notin C_0(U)$ and there would exist a point $x_0 \notin U$ with $f(x_0) \neq 0$. But then x_0 would not be in $U^{\complement}$ by the definition of U. Furthermore, I is an ideal, so it is closed with respect to addition and scalar multiplication (so it is a subspace) as well as multiplication (so it is a subalgebra). As I is a closed subspace of $C_0(U)$, which is a closed subspace of the Banach space $C_0(X)$, I is Banach. The involution and its property can be inherited from $C_0(X)$.

It remains to show that I is closed with respect to this involution. For this, we use the provided hint. It is $f \in I$, then note that $f^* \in C_0(X)$ and f and f^* are both bounded. We can write $f^* f_n$ as

$$f^* f_n = f^*(ff^*)^{\frac{1}{n}} = f \cdot (\underbrace{f^{\frac{1}{n}-1}(f^*)^{\frac{1}{n}+1}}_{:=g})$$

and set $g(x) := 0$ on the zeroes of f. Then we have $g \in C_0(X)$ and also

$$|g(x)| = |f(x)^{\frac{1}{n}-1}(f^*)(x)^{\frac{1}{n}+1}| = |f(x)|^{\frac{1}{n}-1}|\overline{f(x)}|^{\frac{1}{n}+1} = |f(x)|^{\frac{2}{n}}$$

so $\{x \in X \mid |g(x)| \geq \varepsilon\} = \{x \in X \mid |f(x)| \geq \varepsilon^{\frac{n}{2}}\}$ is compact for every $\varepsilon > 0$. This shows $g \in C_0(X)$ and therefore $f^* f_n = fg \in I$ (because of the ideal property) for every

$n \in \mathbb{N}$. The limit $\lim_{n\to\infty} f^* f_n$ converges in $C_0(X)$ to f^*, as $(f^*f)^{\frac{1}{n}} = |f|^{\frac{2}{n}}$ converges to the characteristic function of the support of f, i.e. to 0 if $f(x) = 0$ and to 1 otherwise. Since all elements $f^* f_n$ of the sequence are in I and since I is closed, the limit f^* is also contained in I. This shows $I^* \subseteq I$ and therefore $I^* = I$, so I is closed in respect to the involution.

(ii) Given $x \in U$, there is $f \in I$ with $f(x) \neq 0$. Assume that such an f did not exist, then for all $f \in I$ we have $f(x) = 0$. Per Definition of $U^\complement$, this implies $x \in U^\complement$, contradicting $x \in U$.

(iii) I separates points of U. Let x, y be arbitrary points in U. As proven above, there exists a function $f \in I$ with $f(x) \neq 0$. As X is Hausdorff, there also exists an open neighborhood V of x that does not contain y and (without loss of generality) is a subset of U. Then Uryson's Lemma proves the existence of a function g that is 1 on the compact set $\{x\}$ and that is 0 on the closed set $V^\complement \supset U^\complement$. The latter condition yields $g \in C_0(U)$, so the ideal property implies $fg \in I$. Additionally, we have $(fg)(x) = f(x)g(x) = f(x) \neq 0$ and $(fg)(y) = f(x)g(x) = f(x) \cdot 0 = 0$ (since $y \in V^\complement$). So fg separates x and y.

So I is a dense subspace of $C_0(U)$ by Stone-Weierstrass. But since I is closed, we have $I = \overline{I} = C_0(U)$.

Let $U \subset V$ be open sets in X. Then we have $V^\complement \subset U^\complement$, so any function in $C_0(X)$ that is 0 outside U is also 0 outside V, and we have $C_0(U) \subset C_0(V)$. Conversely, let $U \not\subseteq V$ be open sets in X, so there exists a point $x \in U, x \notin V$. Then Uryson's Lemma shows the existence of a function f that is 1 on the compact set $\{x\}$ and 0 on the closed set $U^\complement$. Since f is 0 outside U, we have $f \in C_0(U)$. However, f is non-zero on the point x outside V, so f cannot be in $C_0(V)$. Therefore, we have $C_0(U) \not\subseteq C_0(V)$. This shows $U \subseteq V \Leftrightarrow C_0(U) \subseteq C_0(V)$.

Lastly, let I be any maximal (and therefore closed) ideal in $C_0(X)$. Then $I = C_0(U)$ for some $U \neq X$ (or $C_0(X)$ would be the whole space and thus not maximal) and $X \setminus U$ is a closed, non-empty set. If $X \setminus U$ contains only a single element, our maximal ideal is of the form $C_0(X \setminus \{x\})$ for some $x \in X \setminus U$, and we are done. If $X \setminus U$ contains more than one element, choose any fixed $x \in X \setminus U$. Then, $X \setminus \{x\} \supset X \setminus U$ and thus $C_0(X \setminus \{x\}) \supset C_0(U)$. Therefore, $C_0(U)$ cannot be a maximal ideal, as it has a super-ideal that is not yet the entire space. So all maximal ideals of $C_0(X)$ must have form $C_0(X \setminus \{x\})$.