Exercises to Introduction to Operator Algebras

Alcides Buss

Notes by: Linus Mußmächer

2336440

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Contents

1 Exercise sheet 1

Exercise 1.1 (1)

Proof: **Case 1**: If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1\alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathscr{A} \to \mathscr{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda} a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathscr{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to $\mathbb{C}$. $\square$

Exercise 1.2 (2) We consider pathological examples for $C_0(X)$.

Let $X = \overline{\{x_0\}}$, e.g. $x_0 \in X$ with $\mathscr{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continouus function $f : X \to \mathbb{C}$, this implies $f(x_0) \to f(x)$ for all $x \in X$, so every continouus function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon>0}\{x \in X \mid |f(x)| \geq \varepsilon\}$ is compact.$\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \not\equiv 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as befor and X_1 compact.

Theorem 1.1 *Let $\varphi : \mathscr{A} \to \mathscr{B}$ be a *-homomorphism betwenn C^*-algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathscr{A}$.*

Exercise 1.3 (4 - Products) Let $(A_i)_{i\in I}$ be a family of C^*-algebras and define

$$\prod_{i\in I} A_i = \{a = (a_i)_{i\in I} \mid a_i \in A_i \forall_{i\in I} \text{ and } \|a\| := \sup_{i\in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinatewise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^*-axiom.

Remark 1.2 (Differences between product and direct sum)
In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall_{\varepsilon > 0} \exists_{\text{finite } F \subseteq I} \forall_{i \notin F} \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of $\prod_{i \in I} A_i$ as the closure of $\bigoplus_{i \in I}^{alg} A_i$, where

$$\bigoplus_{i \in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists_{\text{finite } F \subseteq I} \forall_{i \notin F} \|a_i\| = 0 \right\}.$$

For finite I, these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following categorical universal property: We have **(surjective) *-homomorphisms** $\pi_j : \prod_{i \in I} A_i \to A_j$ for all $j \in I$. If B is any C^*-algebra with *-homomorphisms $\varphi_j \to A_j$ for every $j \in I$, there is a unique *-homomorphism $\varphi : B \to \prod_{i \in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{tikzcd} B \arrow[r, "\varphi_j"] \arrow[d, "\varphi"] & A_j \\ A \arrow[ur, "\pi_j"'] & \end{tikzcd}$$

Exercise 1.4 (5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a *-isomorphism.