Exercise Sheet 02
Operator Algebras

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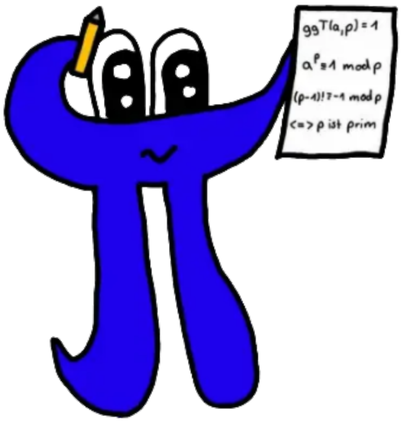


2.3 Quasi-nilpotent elements

Consider the space ℓ^2 and the operator

$$T : \ell^2 \to \ell^2, (x_0, x_1, \dots) \mapsto \left(0, \frac{x_1}{2}, \frac{x_2}{2^2}, \dots, \frac{x_k}{2^k}\right)$$

in the Banach algebra of (normed) operators on ℓ^2. Then we have $\|T\| = 1/2$ and

$$T^2(x_0, x_1, \dots) = \left(0, \frac{0}{2}, \frac{x_0}{2^{1+2}}, \frac{x_1}{2^{2+3}, \dots}\right)$$

as well as

$$T^n(x_0, x_1, \dots) = \Bigg(\underbrace{0, 0, \dots, 0}_{n \text{ zeroes}}, \frac{x_0}{2^{1+2+\dots+n}}, \frac{x_1}{2^{2+3+\dots+(n+1)}}, \dots\Bigg).$$

Then $T^n(1, 0, \dots) = (0, \dots, 0, 1/2^{\frac{n(n+1)}{2}}, 0, \dots) \neq (0, 0, \dots)$ and T is therefore not nilpotent. Furthermore, we can calculate the norm of T:

$$\|T^n(x_0, x_1, \dots)\| = \sum_{k=n}^{\infty} \left| \frac{x_{k-n}}{2^{\sum_{i=0}^{n} k+i}} \right| \leq \frac{1}{2^{\frac{n(n+1)}{2}}} \sum_{k=0}^{\infty} |x_k| = \frac{1}{2^{\frac{n(n+1)}{2}}} \|(x_0, x_1, \dots)\|$$

and equality holds for $(x_0, x_1, \dots) = (1, 0, 0)$ as seen above. Thus, $\|T^n\| = \frac{1}{2^{\frac{n(n+1)}{2}}}$. Then

$$\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{2^{\frac{n(n+1)}{2}}} \right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2^{\frac{n+1}{2}}} = 0$$

so T is quasi-nilpotent.

Assume the Banach algebra $\mathscr{A}$ is generated by a quasi-nilpotent element a. Then every element can be represented as a^k, so the algebra is commutative because $a^m a^n = a^{m+n} = a^n a^m$. Then for any $b \in \mathscr{A}$, we have

$$\lim_{n\to\infty} \|b^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{nk}\|^{\frac{1}{n}} \leq \lim_{n\to\infty} (\|a^n\|^{\frac{1}{n}})^k = 0^k = 0.$$

Now, let $\varphi \in \Omega(\mathscr{A})$ be an element of the spectrum of $\mathscr{A}$, that is, a non-zero homomorphism $\varphi : \mathscr{A} \to \mathbb{C}$. Because φ is non-zero (on a fixed element $b \in \mathscr{A}$), we have $\|\varphi\| > 0$ and for any $n \in \mathbb{N}$ the following holds:

$$|\varphi(b)| = |\varphi(b)^n|^{\frac{1}{n}} = |\varphi(b^n)|^{\frac{1}{n}} \leq \underbrace{\|\varphi\|^{\frac{1}{n}}}_{\to 1} \underbrace{\|b^n\|^{\frac{1}{n}}}_{\to 0} \to 0$$

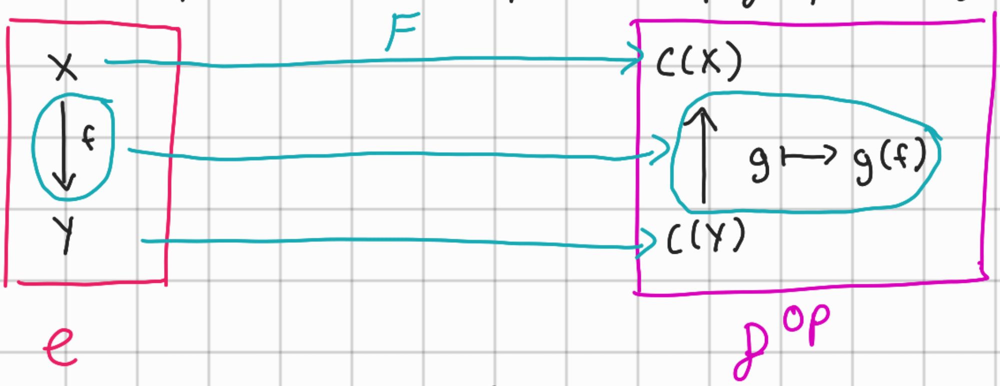
But this implies $\varphi(b) = 0$, a contradiction. Therefore, the spectrum $\Omega(\mathscr{A})$ must be empty.

2.5 Let $\mathcal{C}$ be the category of compact Hausdorff spaces with morphisms the continuous maps and let $\mathcal{D}$ be the category of unital commutative C^*-algebras with morphisms the $*$-homomorphisms.

We now define a functor $F: \mathcal{C} \to \mathcal{D}^{op}$, $X \mapsto C(X)$, which maps a compact Hausdorff space X to the set of continuous functions on X. $C(X)$ is a C^*-algebra (one could either verify this or argue that $C(X)$ inherits all its properties from the C^*-algebra $\mathbb{C}$ since all functions in $C(X)$ have $\mathbb{C}$ as their co-domain).

F is defined on Morph$(\mathcal{C})$ as the following:

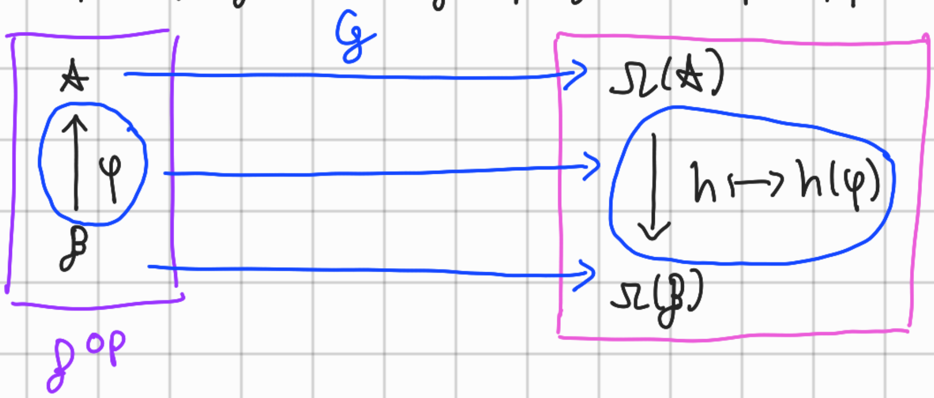
For all $f \in$ Morph$(\mathcal{C})$ we have $F: \text{Morph}(\mathcal{C}) \longrightarrow \text{Morph}(\mathcal{D}^{op})$, $\overset{X\to Y}{f} \longmapsto (\overset{Y\to\mathbb{C}}{g} \mapsto \overset{X\to Y\to\mathbb{C}}{g(f)})$



We now define a functor $G: \mathcal{D}^{op} \longrightarrow \mathcal{C}$, $\mathcal{A} \mapsto \Omega(\mathcal{A})$, which maps a unital commutative C^*-algebra to its Gelfand spectrum. Since $\mathcal{A}$ is a Banach algebra (as it is a C^*-algebra), $\Omega(\mathcal{A})$ is a compact Hausdorff space according to the lecture.

G is defined on Morph$(\mathcal{D}^{op})$ as the following:

For all $\varphi \in$ Morph$(\mathcal{D}^{op})$ we have $G: \text{Morph}(\mathcal{D}^{op}) \longrightarrow \text{Morph}(\mathcal{C})$, $\overset{\mathcal{B}\to\mathcal{A}}{\varphi} \longmapsto (\overset{\mathcal{A}\to\mathbb{C}}{h} \mapsto \overset{\mathcal{B}\to\mathcal{A}\to\mathbb{C}}{h(\varphi)})$



We know from the lecture that $\iota: X \longrightarrow \Omega(C(X))$, $x \mapsto ev_x$ with $ev_x: f \mapsto f(x)$ is a bijection and with the weak $*$-topology on $\Omega(C(X))$ a homeomorphism of compact Hausdorff spaces. So we have $id_{\mathcal{C}} \cong G \circ F$

According to the Gelfand representation theorem we have a $*$-homomorphism $\Gamma: \mathcal{A} \longrightarrow C(\Omega(\mathcal{A}))$, $a \mapsto \hat{a}$ with $\hat{a}: \varphi \longmapsto \varphi(a)$ and since $\mathcal{A}$ is unital, Γ is an isomorphism.

So we have $id_{\mathcal{D}^{op}} \cong F \circ G$.

Therefore we have a contravariant equivalence of categories.

2.9 Topological zero divisors

We consider two cases:

- First, let X be non-compact. Then we have $\inf_{x\in X} |f(x)| = 0$, because if it were $\varepsilon > 0$ we would have $X = \{x \in X \mid |f(x)| \geq \varepsilon\}$ non-compact and thus $f \notin C_0(X)$. Therefore, we need only prove $\zeta(f) = 0$.

  Choose any $\varepsilon > 0$ define $K \coloneqq \{x \in X \mid |f(x)| \geq \varepsilon\}$. Because $\inf_{x\in X} |f(x)| = 0$, there exists an $x_0 \in X$ for which $|f(x_0)| < \varepsilon$ holds (and thus $x_0 \notin K$). Because K is compact, it is closed and thus $X \setminus K$ is open. Choose an open, pre-compact neighborhood U_0 of x_0 in $X \setminus K$ and set $K' = X \setminus U_0$. The set $\{x_0\}$ is compact, and K' is closed, so Uryson's Lemma yields the existence of a function $b : X \to \mathbb{C}$ (with $\|b\| = 1$) (in $C_0(X)$) with $b(x_0) = 1$ and $b|_{K'} \equiv 0$. Then for $x \in K \subseteq K'$, we have $|(fb)(x)| = |f(x)| \cdot |b(x)| = |f(x) \cdot 0 < \varepsilon$. For $x \in K^{\complement}$, it follows that $|(fb)(x)| = |f(x) \cdot |b(x)| < \varepsilon \cdot 1 = \varepsilon$ and thus $\|fb\| < \varepsilon$. This shows $\zeta(f) = \inf_{b\in C_0(X), \|b\|=1} \|fb\| = 0$.

  So if X is not compact, $\zeta(f) = \inf_{x\in X} |f(x)| = 0$ holds and every $f \in C_0(X)$ is a topological zero divisor.

- Now, let X be a compact Hausdorff space and $f \in C_0(X)$. If f is non-invertible, we have $0 \in f(X)$ and thus $\inf_{x\in X} |f(x)| = 0$. In this case, we can argue as we did in the first point and thusly show $\zeta(f) = 0$ in much the same way.

  Consider now an invertible f with $\inf_{x\in X} |f(x)| = k > 0$. We conclude

  $$\left\| \frac{1}{f} \right\| = \sup_{x\in X} \frac{1}{|f(x)|} = \frac{1}{\inf_{x\in X} |f(x)|} = \frac{1}{k},$$

  so for any $b \in C_0(X)$ with $\|b\| = 1$ we have $\|f \cdot b\| \cdot \|\frac{1}{f}\| \geq \|f \cdot b \cdot \frac{1}{f}\| = \|b\| = 1$, so $\|f \cdot b\| \geq k$ and therefore $\zeta(f) \geq k$.

  Choose now any $\varepsilon > 0$. Then $K \coloneqq \{x \in X \mid |f(x)| \geq k + \varepsilon\}$ is compact and $K \neq X$ (or k would not be the infimum of $|f(x)|$). Just like in the first bullet point, we can choose $x_0 \in X \setminus K$ and fitting neighborhoods to get the existence of a function b fulfilling $\|b\| = 1$, $\|bf\| < k + \varepsilon$ and $b \in C_0(X)$. Therefore, $\zeta(f) \leq k$ and thus $\zeta(f) = k > 0$. This also shows that the (invertible) element f is not a topological zero divisor.

To summarize, we have proven $\zeta(f) = \inf_{x\in X} f(x)$ for any $f \in C_0(X)$, that f is a topological zero divisor in a compact space always and in a non-compact space if and only if it is invertible. It remains to show that in a commutative C^*-algebra $\mathscr{A}$, $f \in \mathscr{A}$ is a topological zero divisor if and only if $0 \in \sigma(f)$. As $\mathscr{A}$ is commutative, we can employ the Gelfand Representation (1.3.6) and conclude that $\mathscr{A}$ can be embedded in the algebra $C_0(\Omega(\mathscr{A}))$ by Γ, and $\sigma(f) = \operatorname{im} \hat{f}$ (because $\mathscr{A}$ is unital, or $\sigma(f)$ would not be defined). Then $0 \in \sigma(f) \Leftrightarrow 0 \in \operatorname{im} \hat{f} \Leftrightarrow f$ is non-invertible, and because $\Omega(\mathscr{A})$ is compact (as $\mathscr{A}$ is unital), this is equivalent to f being a topological zero divisor.