

Exercises to Introduction to Operator Algebras

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Summer 2023

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1 Exercise sheet 1

Exercise 1.1 (1)

PROOF: **Case 1:** If $b_1, b_2 \in A$, then $b_i = \alpha_i a$ for certain $\alpha_i \in \mathbb{C}$. Thus, $b_1 \cdot b_2 = \alpha_1 \alpha_2 a^2 = 0$. Thus, the multiplication is trivial. From this, it immediately follows that $\varphi : \mathcal{A} \rightarrow \mathcal{M}, \lambda a \mapsto \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix}$ is an isomorphism.

Case 2: $\lambda \neq 0$, and $a^2 = \lambda a$. Let $b = \frac{1}{\lambda} a$, then $b \cdot a = a = a \cdot b$. But then, for any $c = \mu a \in \mathcal{A}$, we have $bc = \mu ba = \mu a = c = cb$, so the algebra is unital and isomorphic to \mathbb{C} . \square

Exercise 1.2 (2) We consider pathological examples for $C_0(X)$.

Let $X = \{x_0\}$, e.g. $x_0 \in X$ with $\mathcal{O}(X) = \{\{x_0\} \cup Y \mid Y \subset X\} \cup \{\emptyset\}$. X is highly non-Hausdorff unless we already have $X = \{x_0\}$. In this space, the constant sequence (x_0) converges to any $x \in X$.

For a continuous function $f : X \rightarrow \mathbb{C}$, this implies $f(x_0) \rightarrow f(x)$ for all $x \in X$, so every continuous function must already be constant. It follows that $C(X) \simeq \mathbb{C}$.

We now look at $C_0(X) = \{f \in C(X) \mid \forall_{\varepsilon > 0} \{x \in X \mid |f(x)| \geq \varepsilon\} \text{ is compact}\}$. But since all functions are constant, we can use $f(x_0)$ instead of X and $\{x \in X \mid |f(x)| \geq \varepsilon\}$ is either empty or the whole space. X is compact if and only if X is finite. From here on, assume X to be infinite. Then, only the finite subsets are compact. Thus, if we now have $f \neq 0$, there exists an $|f(x_0)| > \varepsilon > 0$ and thus $\{x \in X \mid |f(x)| \geq \varepsilon\} = X$ is not compact. This implies $C_0(X) = \{0\}$.

To find a non-compact topological space that has non-zero unital $C_0(X)$, consider $X = X_0 \sqcup X_1$ with X_0 as before and X_1 compact.

Theorem 1.1 Let $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ be a $*$ -homomorphism between C^* -algebras. Then we already have $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$.

Exercise 1.3 (4 - Products) Let $(A_i)_{i \in I}$ be a family of C^* -algebras and define

$$\prod_{i \in I} A_i = \{a = (a_i)_{i \in I} \mid a_i \in A_i \forall i \in I \text{ and } \|a\| := \sup_{i \in I} \|a_i\| < \infty\}.$$

Addition, multiplication and involution are defined coordinate-wise. We can prove that adding, multiplying and involving any bounded sequence yields another bounded sequence, so these are well-defined. We can also prove the C^* -axiom.

Remark 1.2 (Differences between product and direct sum)

In addition to the product space, we define

$$\bigoplus_{i \in I} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \forall_{\varepsilon > 0} \exists_{\text{finite } F \subseteq I} \forall_{i \notin F} \|a_i\| < \varepsilon \right\}.$$

This is a closed subspace of $\prod_{i \in I} A_i$ as the closure of $\bigoplus_{i \in I}^{alg} A_i$, where

$$\bigoplus_{i \in I}^{alg} A_i = \left\{ (a_i) \in \prod_{i \in I} A_i \mid \exists_{\text{finite } F \subseteq I} \forall_{i \notin F} \|a_i\| = 0 \right\}.$$

For finite I , these are all equal. We see that any element in the direct sum can be approximated by a sequence of elements in the algebraic sum. This direct sum is a closed two-sided ideal in the product.

The product has the following universal property: We have (surjective) $*$ -homomorphisms $\pi_j : \prod_{i \in I} A_i \rightarrow A_j$ for all $j \in I$. If B is any C^* -algebra with $*$ -homomorphisms $\varphi_j : B \rightarrow A_j$ for every $j \in I$, there is a unique $*$ -homomorphism $\varphi : B \rightarrow \prod_{i \in I} A_i$ such that $\pi_j \circ \varphi = \varphi_j$. This is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi_j} & A_j \\ \downarrow \varphi & \nearrow \pi_j & \\ A & & \end{array}$$

Exercise 1.4 (5) X is a locally compact Hausdorff space that can be written as $X = U \cup V$ with open and disjoint U, V (so U, V are clopen). We want to prove $C_0(X) \simeq C_0(U) \oplus C_0(V)$. To build this map, we map $f \mapsto (f|_U, f|_V)$. We check that this is well-defined and a $*$ -isomorphism.