2360 Project 1: Mortgages

Fabrizio Roberts
Linus Schmitz

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Contents

1 Introduction

A pair of close friends are looking to buy a house in Boulder and have asked for our help. They need to take out a mortgage of $750,000 to purchase the house and, similarly to most engineers, were shocked to learn there was more than one type of mortgage. While they have started the process of finding the best type of mortgage for them, they have some trouble with Differential Equations. Conveniently, both of the people they have tasked with finding an ideal mortgage for them are taking Differential Equations. In order to better understand mortgages, the difference between adjustable rate and fixed rate mortgages, and which mortgage is best for them, analysis will be performed with Euler's Method, Initial Value Problems, and other methods learned in Differential Equations.

2 Analysis on the Behavior of Fixed Rate Mortgages

In order to analyze fixed rate mortgages and their behavior, an initial loan must be determined. Assuming an initial loan of \$750,000 and interest rate of 3%, the first step of the analysis is to determine the effect of how often the interest compounds. Given a 5 year window, the total cost of the same amount compounded annually, semi-annually, quarterly, and monthly will be used in this comparison. Given the equation $A(t) = A(0)(1 + rn)^{nt}$ (Eq. 1), these values were calculated to be: \$869,465.56, \$870,405.62, \$870,888.11, and \$871,212.59, respectively. Growth can be seen below in Figure 1.

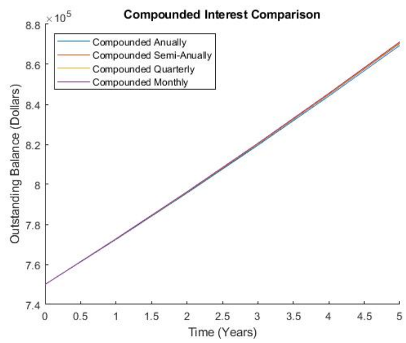

Figure 1: Annually, Semi-Annually, Quarterly, and Monthly Compound Growth

It is worth noting that as the interest compounds more often, the total cost after 5 years also increases. This trend can further be seen by calculating the total cost of the loan compounded continuously over 5 years, \$871,376. This value can be calculated by taking the limit of Eq. 1 which yields $A(t) = A(0)e^{rt}$ (Eq. 2). A visual representation of this trend is shown in Figure 2.

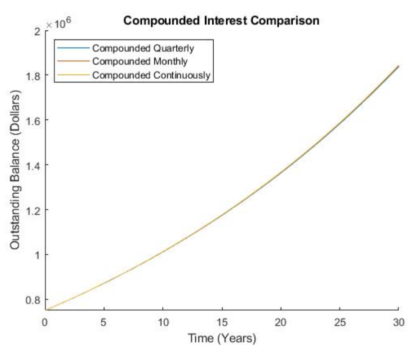

Figure 2: Quarterly, Monthly, and Continuously Compound Growth

To further understand the behavior of the mortgage, the stability of each equilibrium solution must be examined. Equilibrium solutions can be calculated by solving the initial value problem $A' = rA - 12p, A(0) = A_0$ (Eq. 3). At equilibrium $A' = 0$, therefore, the solution is $A = 12p/r$. After computing, it was determined that all values for r and p result in unstable solutions. This was determined by checking values above and below the stability solution for 100 terms of r from 0.001 to 0.1 and 100 terms of p from 10 to 200. The equilibria represent only paying off the interest on mortgages and not in fact paying off any of the initial loan.

Another method to find the equilibrium solution without a computer can be applied. By matching the initial value problem form $y' + P(x)y = Q(x)$, it is possible to analytically solve Eq. 3 by finding $P(x)$ and calculating the integrating factor. It was found that $P(x) = -r$ and $\mu = e^{-rt}$. In combination with Eq. 3, integrating yields $A(t) = 12p/r + ce^{rt})$. In order to solve for c, the initial value condition must be applied which when solved yields the general equation $A(t) = 12p/r + (A_0 - 12p/r)e^{rt}$ (Eq. 4). The derivation for Eq. 4 can be found in Appendix B. By finding the general equation, it is possible to calculate monthly contribution.

Comparing two mortgages, a 10 year plan at a rate of 3% versus a 30 plan at a rate of 5%, Eq. 4 can be algebraically solved for p. It was found that the monthly payments for the 10 and 30 year loan were \$7,234.30 and \$4,022.55, respectively. While the monthly payment for the 30-year is almost half than that of the 10-year, the convenience of a low monthly payment comes at a cost. Simple arithmetic shows that the total amount paid after 10 years is \$868,116.58 including \$118,116.58 of interest. This can be compared to the 30-year mortgage where the total amount paid is \$1,448,119.03 in which \$698,119.03 is interest. It is worth noting that the interest on the 30 year loan is almost the amount of the original loan, \$750,000. This would indicate that the sooner the debt is paid, the less the total amount owed and overall interest will be.

A down payment is when you pay off a large amount of the debt as soon as you take it out. Interest is minimized by low initial amounts and little time so a down payment is a great way to save in the long run. The benefits of a down payment are further seen when compared to the normal growth without an initial payment. For example, what if for the 10-year and 30-year mortgages, a down payment of \$100,000 is made on the initial \$750,00 resulting in A0 being \$650,000. A down payment would cause the monthly contribution for both to change to \$6,269.73 for the 10-year and \$3486.21 for the 30-year. It is worth noting that while the difference in monthly payments are not massive, the difference quickly stacks over the course of 120 and 360 months. With the total interest costs on the 10 and 30 year being \$2,367.70 and \$505,036.49 respectively, the benefits of a down payment are obvious. By paying \$100,000 initially, the two friends could save \$115,748.88 on the 10-year mortgage and \$193,082.54 on the 30-year.

With the behavior of each of the mortgages understood, it is easy to compare the two. The advantages to having a 30 year mortgage is that you will be paying considerably less per month on the scale of almost half of what you would have to pay on a 10 year mortgage. This, however, comes with its downside as it results in far more interest paid on the initial amount. Depending upon the initial loan amount and whether or not there is a down payment this can be anywhere between 6 and nearly 250 times as much interest that would be paid on a 10 year mortgage. One the other hand, the 10 year mortgage has a much higher monthly payment: \$7,234 a month compared to \$4,022 a month with no down payment and \$6,269 a month compared to \$3,486 a month with a \$100,000 down payment. Subsequently, the 10-year mortgage has a much lower total interest paid given that the total amount of money decreases faster and as such the interest has less time to accrue. The interest paid on a 10 year mortgage is only \$118,000 versus \$698,000 on a 30 year mortgage with no down payment, which then decreases drastically when an initial down payment of \$100,000 is paid. At the new initial mortgage amount with the \$100,000 down payment the 10 year mortgage interest is only \$2,367 while the 30 year mortgage interest total remains at \$505,036.49.

3 Numerical Analysis with Euler's Method

Further analysis can still be performed, especially when it comes to variable rate mortgages which will be discussed later, however, this is not possible analytically. Numerically, software can solve any differential equation with Euler's method to some degree of accuracy. In order to prove to our friends that the software can be trusted to solve complicated differential equations, the software must be checked against known solutions. For the test, a mortgage of \$750,000 with a fixed interest rate of 5% and a monthly payment of \$4000 will be examined two ways: analytically and numerically. Since it is unlikely that Euler's method will give exactly 0 on any given step, it is worth noting that the debt is considered paid when the step becomes negative. Given the Euler's Method formula, $A_{n+1} = A_n + hf(t_n, A_n)$, besides Eq. 4 and the derivative of Eq. 4, all that is needed is a step size for approximating. Picking a step size of $h = 0.5$, it was computed that step 62 is the last positive step before becoming negative on step 63 which translates to 31 years before the loan is paid. The predicted values from Euler's Method can be seen compared to the true values in Figure 3 below.

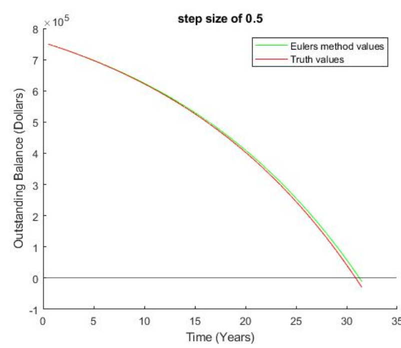

Figure 3: Euler's Method with step size of 0.5 years

While at each step, Euler's Method will stray further away from the true value, this divergence can be partially negated by a smaller step size. Not surprisingly, picking a smaller step size causes the number of steps to increase which may become a burden quickly when working by hand. Since the only limiting factor on using software for Euler's Method is the processing power of the hardware, decreasing the step size makes little difference for a large increase in accuracy. By picking a step size 50 times smaller, the solution can be computed accurately with relatively no change in complexity or processing speed. Both the predicted values at $h = 0.01$ can be seen in Figure 4. Figure 5 outlines the difference in error or the difference between the Euler approximation and the truth values for each term. While the plot itself is somewhat harder to interpret because of the horizontal scale, it is still informative on the vertical scale. It shows how at the calculated values where the mortgage is paid off, the error using a smaller step size is less than \$500 off while the error for the larger step size is more than \$18,000. This outlines the value of using a smaller step size on a numerical approximation such as Euler's method.

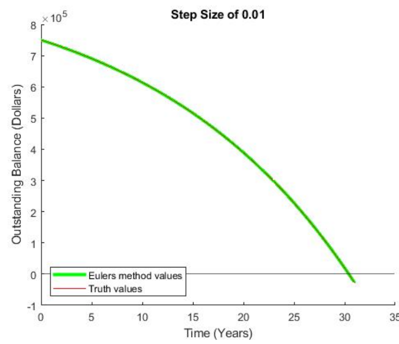

Figure 4: Euler's Method with step size of 0.01 years

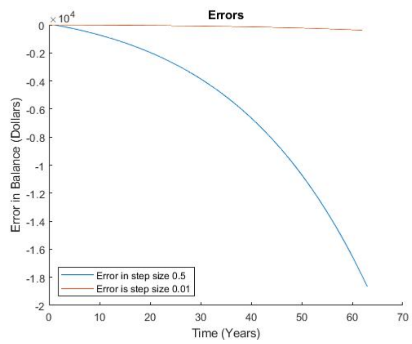

Figure 5: Euler's Method Error Comparison

While fixed rate mortgages are commonly used and fairly simple to understand, there is another type of mortgage called an adjustable rate mortgage. The difference between the two is in the name. Unlike their counterpart, the rate of an adjustable rate mortgage changes over time either by bracket (e.g. every 2 years the interest rate increases) or by function (e.g. rate is dependent on time). To fully understand the behavior of both types of adjustable rate mortgages, analysis will be performed on a mortgage with two brackets: $r = 0.03$ for 5 years and then $r = 0.03 + 0.015\sqrt{(t-5)}$ after that. Given the original loan amount, $750,000, the time to pay off the adjustable rate mortgage can be computed using Euler's Method. To showcase the importance of paying off adjustable rate mortgages early, the time taken to pay off the mortgage with a monthly payment of $4,000 versus $4,500 will be compared. Using a step size of $h = 0.01$, it was found that if contributing $4,000 a month, the debt will be fully paid in 34.97 years. For $500 more a month, however, the mortgage can be paid in 22.59 years, more than ten year less. The benefit of a $4,500 contribution is further seen when comparing the total interest paid. For the $4,000 payment, the total interest is $928,560 compared to $469,860 for the $4,000. The growth trend for both can be seen below in Figure 6.

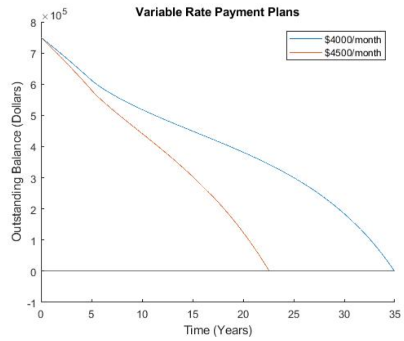

Figure 6: Comparing Monthly Contribution for an Adjustable Rate Mortgage

Compared to a fixed rate mortgage, the variable rate payment causes the time until pay off to increase by about 4.5 years for the monthly payment of \$4000. When the monthly payment is \$4500, it will decrease the amount being paid by nearly 8 years. This means it will be advantageous for the borrower to take a variable rate if they can pay \$4500 but not if they can only pay \$4000. As the monthly contribution increases, the cost benefits of a variable rate mortgage also increase.

4 Conclusion

While fixed rate and adjustable rate mortgages both have their advantages and disadvantages, it is crucial to find a mortgage based on the current financial situation. For poor college students moving within two years, it does not matter if the mortgage is fully paid by the time it's sold. In that scenario, a long term fixed rate mortgage with a low monthly contribution would be ideal. However, an older professor with a stable income and family looking to settle would want to pay off the mortgage as quickly as possible with a high monthly payment adjustable rate mortgage. Simply stated, if planning on paying off the entirety of the mortgage, minimizing the total amount paid is crucial. Alternatively, if planning on reselling before paying off the mortgage, minimizing the monthly contribution is how the most money will be saved. It is worth noting before deciding, however, that the amount of mortgage left can have a serious effect on the value of the home. It may be hard to resell a home with a terrible mortgage.

5 Appendix A: Code

```matlab
%% Constants
clear
clc
close all
A0 = 750000;
r = 0.03;

%% Question 1
timesPerYear = [1 2 4 12];
A = [];
k = 1;
figure(1)
hold on
for n = timesPerYear
    for t = 0:5
            a = A0 * ( 1+ r/n) ^ (n*t);
            A(k,t+1) = a;
    end
    plot(0:5,A(k,:))
    k = k+1;
end
title("Compounded Interest Comparison");
xlabel("Time (Years)");
ylabel("Outstanding Balance (Dollars)");
legend("Compounded Anually","Compounded Semi-Anually", "Compounded Quarterly","Compounded Monthly","Location", 'northwest');
hold off
figure(2)
hold on
k = 1;
A1 = [];
for j = [ 4 12]
    for ti = 0:30
        a1 = A0 * ( 1+ r/j) ^ (j*ti);
        A1(k,ti+1) = a1;
    end
    plot(0:30, A1(k,:))
    k = k+1;
end
for tim = 0:30
    a2 = A0*exp(r*tim);
    A1(3, tim+1) = a2;
end
plot(0:30, A1(3,:))
```

```
title("Compounded Interest Comparison");
xlabel("Time (Years)");
ylabel("Outstanding Balance (Dollars)");
ylim([750000 2000000]);
legend("Compounded Quarterly","Compounded Monthly","Compounded Continuously","Location", 'northwest');
hold off

%% Question 2
%EQulibrium solutions at
%A = 12p/r

%Testing Constants
R = 0.05;
P = 100;
Atest = (12 * P / R ) - 1;
APrim = R * Atest - 12* P;
disp(APrim)

R1 = 0;
P1 = 0;

for R = linspace(0.001,0.1)
    R1 = R1 +1;
    for P = linspace(10,200)
        P1 = P1 + 1;
        Atest1 = (12 * P / R ) - 1;
        APrim1 = R * Atest1 - 12* P;
        Atest2 = (12 * P / R ) + 1;
        APrim2 = R * Atest2 - 12* P;
        if (APrim1 > 0) && (APrim2 < 0)
            Stabils(R1,P1) = 1;
            Stab1 = "Stable";
        elseif (APrim1 > 0) && (APrim2 > 0)
            Stabils(R1,P1) = 2;
            Stabl = "Semi";
        elseif (APrim1 < 0) && (APrim2 < 0)
            Stabils(R1,P1) = 2;
            Stabl = "Semi";
        else
            Stabils(R1,P1) = 3;
            Stabl = "Un";
        end
    end
    P1 = 0;
end
```

```
% 1 is stable
% 2 is semistable
% 3 is unstable

[rows, cols, StableSolutions] = find(Stabils == 3);

% All the values of the solutions are going to be unstable solutions. We
% were able to determine this by checking the values above and below the
% stability solution for 100 terms of R and 100 terms of P.
%The equilibria represent where you are only paying off the interest on the
%mortgages and not in fact paying off any of the initial loan

%% Question 3

%see below in Appendix B

%% Question 4
format bank

r = 0.03;
t = 10;
P10year = -A0*r*exp(r*t)/(12*(1-exp(r*t)));

r = 0.05;
t = 30;
P30year = -A0*r*exp(r*t)/(12*(1-exp(r*t)));

fprintf("Payment per month for a 10 year and 30 year mortgage respectivly \n")
disp(P10year)
disp(P30year)

%% Question 5
Total10Yr = P10year * 10 * 12;
Total30Yr = P30year * 30 * 12;

fprintf("Total Payments for 10 yr and 30 yr mortgages respectivly \n")
disp(Total10Yr)
disp(Total30Yr)

Interest10Year = Total10Yr - A0;
Interest30Year = Total30Yr - A0;
```

```
fprintf("Total Interests for 10 yr and 30 yr mortgages respectivly \n")
disp(Interest10Year)
disp(Interest30Year)

%% Question 6
A01 = 650000;
r = 0.03;
t = 10;
P10yearNew = -A01*r*exp(r*t)/(12*(1-exp(r*t)));

r = 0.05;
t = 30;
P30yearNew = -A01*r*exp(r*t)/(12*(1-exp(r*t)));

fprintf("Payment per month for the new 10 year and 30 year mortgage respectivly \n")
disp(P10yearNew)
disp(P30yearNew)

Interest10YearNew = (P10yearNew * 10 * 12) - A0;
Interest30YearNew = (P30yearNew * 30 * 12) - A0;

fprintf("Total Interests for 10 yr and 30 yr mortgages respectivly with downpayment \n")

disp(Interest10YearNew)
disp(Interest30YearNew)

fprintf("Difference in interest paid, ie. savings \n ")
disp(Interest10Year - Interest10YearNew)
disp(Interest30Year - Interest30YearNew)

%% Question 7
%The advantages to having a 30 year mortgage is that you will be paying
%considerably less per month on the scale of almost half of what you would
%have to pay on a 10 year motgage. This however comes with its downside as
%it results in far more interest paid on the initial amount. Depending upon
%the initial loan amount and whether or not there is a downpayment this can
%be anywhere between 6 and nearly 250 times as much interest that would be
%paid on a 10 year mortgage.
```

```
%This then translates to the 10 year mortgage as follows. The 10 year
%mortgage has a much higher monthly payment ($7234/month vs $4022/month
%with no downpayment and $6269/month vs $3486/month with a $100k
%downpayment) but subsequntly has a much lower total interest paid given
%that the total amount of money decreases faster and as such the interest
%has less time to accrew. The interest paid on a 10 year mortgage is only
%$118k versus $698k on a 30 year mortgage with no downpayment, which then
%decreases drastically when an initial downpayment of 100k is paid. At the
%new initial mortgage amount with the $100k downpayment the 10 year
%mortgage interest is only $2367 while the 30 year mortgage interest total
%remains at $505k.

%% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% SECTION 3.2

%% Question 1
%Eulers method with the following:
h = 0.5;
r = 0.05;
p = 4000;
A = zeros();
A(1) = 750000;

n = 62;

for i = 1:n
    Aprime = r*A(i) - 12*p;
    A(i+1) = A(i) + (Aprime * h);
end

%% Question 2
years = (0:n);
Atrue = (12 * p)/r + (A0 - (12*p)/r).*(exp(r.*(years)./2));

figure(4)
hold on
title("step size of 0.5")
plot((1:n+1)/2,A, 'g')
```

```
plot((1:n+1)/2,Atrue, 'r')
yline(0)
xlabel("Time (Years)");
ylabel("Outstanding Balance (Dollars)");
legend("Eulers method values", "Truth values")
hold off

%% Question 3
h1 = 0.01;
A1 = zeros();
A1(1) = 750000;
n1 = 3100;
divisor = 1/h1;

for i = 1:n1
    Aprime = r*A1(i) - 12*p;
    A1(i+1) = A1(i) + (Aprime * h1);
end

years1 = (0:n1);
Atrue1 = (12 * p)/r + (A0 - (12*p)/r).*(exp(r.*(years1)./divisor));

figure(5)
hold on
title("Step Size of 0.01")
plot((1:n1+1)/100,A1, 'g', 'LineWidth', 3)
plot((1:n1+1)/100,Atrue1, 'r')
xlabel("Time (Years)");
ylabel("Outstanding Balance (Dollars)");
yline(0)
legend("Eulers method values", "Truth values", 'Location', 'SW')
hold off

figure(6)
title("Errors")
Error = Atrue - A;
Error1 = Atrue1 - A1;
hold on
plot(1:n+1, Error)
plot((1:n1+1)/50,Error1)
xlabel("Time (Years)");
ylabel("Error in Balance (Dollars)");
legend("Error in step size 0.5", "Error is step size 0.01", "Location
    ","Southwest")
hold off

```

```
%Figure 6 outlines the difference in error or the difference between the
%Euler approximation and the truth values for each term. While the plot
%itself is somewhat harder to interpret because of the horizontal scale it
%is very informative on the vertical scale. It shows how at the calculated
%values where the mortgage is paid off the error using a smaller step size
%is less than $500 off while the error for the larger step size is more
%than $18k. This outlines the value of using a smaller step size on a
%numberical approximation such as Euler's method.

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% SECTION 3.2.2

%% Question 1
htest = 0.01;
Ptest = 4000;
Atest = zeros();
Atest(1) = 750000;

maxnums = 10;
while Atest(end) >= 0
    for timestep = 1:maxnums
        if timestep <= 500
            rtest = 0.03;
        elseif timestep > 500
            rtest = 0.03 + 0.015* sqrt((timestep*htest)-5);
        end

            AprimeTest = rtest*Atest(timestep) - 12*Ptest;
            Atest(timestep+1) = Atest(timestep) + (AprimeTest * htest);
    end
    maxnums = maxnums +1;
end

figure(7)
hold on
plot((1:maxnums)/100, Atest)
yline(0);
hold off

timeToPay = length(Atest)* 0.01;
```

```
fprintf("The time to pay off the Mortgage with the new variable rate is
    %.2f years\n", timeToPay)

%% Question 2
htest = 0.01;
Ptest1 = 4500;
Atest1 = zeros();
Atest1(1) = 750000;

maxnums = 10;
while Atest1(end) >= 0
    for timestep = 1:maxnums
        if timestep <= 500
            rtest = 0.03;
        elseif timestep > 500
            rtest = 0.03 + 0.015* sqrt((timestep*htest)-5);
        end

            AprimeTest1 = rtest*Atest1(timestep) - 12*Ptest1;
            Atest1(timestep+1) = Atest1(timestep) + (AprimeTest1 *
                htest);
    end
    maxnums = maxnums +1;
end

figure(8)
hold on
plot((1:maxnums)/100, Atest1)
yline(0);
hold off

timeToPay1 = length(Atest1)* 0.01;
fprintf("The time to pay off the Mortgage with the new variable rate is
    %.2f years\n", timeToPay1)

%% Question 3
VarInterest = Ptest*timeToPay*12 - A0;
VarInterest1 = Ptest1*timeToPay1*12 -A0;
fprintf("The interest paid on $4000/month will be $%.2d while the
    interest paid on the $4500/month will be $%.2d", VarInterest,
    VarInterest1)

%% Question 4
figure(9)
title("Variable Rate Payment Plans")
hold on
```

```
plot((1:length(Atest))/100, Atest)
plot((1:length(Atest1))/100, Atest1)
yline(0)
xlabel("Time (Years)");
ylabel("Outstanding Balance (Dollars)");
hold off
legend("$4000/month", "$4500/month")
%When the monthly payment is $4000 the variable rate payment causes the
%length to pay off to increase by about 4.5 years but when the monthly
%payment is $4500 it will decrease the amount being paid by nearly 8
    years.
%This means it will be advantagious for the borrower to take a variable
%rate if they can pay $4500 but not if they can only pay $4000.
```

6 Appendix B: Equation 4 Derivation

$$A' = rA - 12p \tag{1}$$

The equation must be in Initial Value Problem form.

$$A' - rA = -12p \tag{2}$$

This allows us to determine both $P(x)$ and the integrating factor.

$$P(x) = -r, \mu = e^{-rt} \tag{3}$$

Multiplying both sides by the integrating factor yields a new equation that can be integrated.

$$Ae^{-rt} = \int -(12p/r)e^{rt} + c \tag{4}$$

Integrating the right side and multiplying both sides by e^{rt} solves for A.

$$A = 12p/r + ce^{rt} \tag{5}$$

c can be solved for with the initial value condition $A(0) = A_0$.

$$c = A_0 - 12p/r \tag{6}$$

With c calculated, it can be substituted to find the general equation.

$$A = 12p/r + (A_0 - 12p/r)e^{rt} \tag{7}$$